Techniques for proving NP-Completeness

- Restriction
- Local replacement
- Component design

Restriction

- The simplest, and perhaps the most frequently applicable.
- Way:
  - show that problem Π contains a known NP-complete problem Π' as a special case,
  - Or equivalently, showing that placing some additional restrictions on instances of Π so that the resulting restricted problem is NP-complete.
- Why the above is a way of NP-completeness proof for Π?
- Examples we have shown:
  - X3C: was shown to be NP-complete by restricting its instances to 3-sets that contain one element from a set W, one from a set X and one from a set Y, where W, X, and Y are disjoint sets having the same cardinality, there by obtaining the 3DM problem.
  - Directed Hamiltonian Circuit: restricting each directed arc (u,v) to occurs only in conjunction with oppositely directed arc (v,u).

Restriction--Examples (I)

1. **Minimum Cover**

   Instance: Collection C of subset of a set S, positive integer K.

   Question: Does C contain a cover for S of size K or less, that is, a subset $C' \subseteq C$ with $|C'| \leq K$ and such that $\cup_{c \in C} c = S$?

   Proof: Restrict to X3C by allowing only instances having $|c|=3$ for all $c \in C$ and having $K=|S|/3$.

2. **Subgraph Isomorphism** (子图同构)

   Instance: Two graphs, $G=(V_G, E_G)$ and $G=(V_H, E_H)$.

   Question: Does G contain a subgraph isomorphic to H, that is, a subset $V \subseteq V_G$ and a subset $E \subseteq E_G$ such that $|V|=|V_H|$, $|E|=|E_H|$, and there is a one-to-one function f: $V_H \rightarrow V$ satisfying $\{u, v\} \in E_H \Leftrightarrow \{f(u), f(v)\} \in E_H$?

   Proof: Restrict to CLIQUE by allowing only instances for which H is a complete graph.

Restriction--Examples (II)

3. **KNAPSACK** (背包)

   Instance: A finite set U, a "size" $s(u) \in Z^+$ and a "value" $v(u) \in Z^+$ for each $u \in U$, and a size constraint $B \in Z^+$, and a value goal $K \in Z^+$.

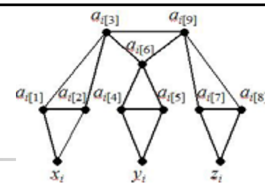
   Question: Is there a subset $U' \subseteq U$ such that $\sum_{u \in U'} s(u) \leq B$ and $\sum_{u \in U'} v(u) \geq K$?

   Proof: Restrict to PARTITION by allowing only instances in which $s(u)=v(u)$ for all $u \in U$ and $B=K=(1/2)\sum_{u \in U} s(u)$.

Local replacement

- The transformations by local replacement:
  - Pick up some aspect of the known NP-complete problem Π' to make up a collection of basic units.
  - Obtain the corresponding instance of the target problem Π by replacing each unit, in a uniform way, with a different structure.
- Examples we have shown:
  - SAT→3SAT:
    - The basic units of an instance of SAT are the clauses.
    - Each clause is replaced by a set of clauses according to the same general rule.
  - 3SAT→VC.
  - 3DM→SS.

Local replacement –An example



Partition Into Triangles (PIT)

Instance: A graph $G=(V, E)$, with $|V|=3q$ for a positive integer q. **Question**: Is there a partition of V into q disjoint sets V_1, $V_1, \ldots, V_q$ of three vertices each such that, for each $V_i=\{v_{i[1]}, v_{i[2]}, v_{i[3]}\}$, the edges $\{v_{i[1]}, v_{i[2]}\}$, $\{v_{i[1]}, v_{i[3]}\}$, and $\{v_{i[2]}, v_{i[3]}\}$ all belong to E ?

Proof: We transform X3C to PIT.

Let the set X with $|X|=3q$ and the collection C of 3-element subsets of X be an arbitrary instance of X3C.

We shall construct a graph $G=(V, E)$, with $|V|=3q'$ such that the desired partition exists for G if and only if C contains an exact cover.

The basic units of the X3C instance are the 3-element subsets in C.

The local replacement substitutes for each such subset $c_i=\{x_i, y_i, z_i\} \in C$ the collection E_i of 18 edges shown in the figure.

Thus $G=(V, E)$ is defined by

$V=X \cup \bigcup_{1\le i\le |C|}\{a_{i[j]} \mid 1\le j\le 9\}$ and $E=\bigcup_{1\le i\le |C|}E_i$.

If $C'=\{c_1, c_2, \ldots, c_q\}\subseteq C$ forms a exact cover for X, then the corresponding partition $V=V_1\cup V_2\cup\ldots V_{q'}$ of V is given by taking $\{a_{i[1]}, a_{i[2]}, x_i\}$, $\{a_{i[4]}, a_{i[5]}, y_i\}$, $\{a_{i[7]}, a_{i[8]}, z_i\}$, $\{a_{i[3]}, v_{i[6]}, a_{i[9]}\}$ whenever $c_i=\{x_i, y_i, z_i\}\in C'$ and by taking $\{a_{i[1]}, a_{i[2]}, a_{i[3]}\}$, $\{a_{i[4]}, a_{i[5]}, a_{i[6]}\}$, $\{a_{i[7]}, a_{i[8]}, a_{i[9]}\}$ whenever $c_i\notin C'$.

The converse argument is left to you.

Component design

- Examples we have shown:
  - 3SAT→VC
  - VC→HC
  - 3SAT→3DM
- Basic idea: to design certain "components".
- In general, there are three basic types of components.
- "making choice" component.
  - E.g., Selecting vertices, choose truth values for variables, ….
- "testing" component.
  - E.g., checking that each edge is covered, checking that each clause is satisfied….
- "Communication" component.
  - To join the first two types of components.