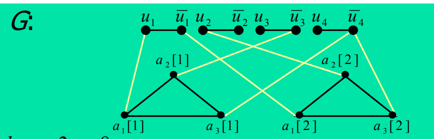
Algorithms

VC$\in$ NPC (I)

3SAT :
$U = \{u_1, u_2, u_3, u_4\}$
$C = \{\{u_1, \bar{u}_3, \bar{u}_4\}, \{\bar{u}_1, u_2, \bar{u}_4\}\}$

$\Rightarrow$ *G*:



$k=n+2m=8$

- **VC$\in$NP:** A nondeterministic Alg. needs only to guess a subset of vertices and check in polynomial time whether that subset contains at least one endpoint of every edge and has the appropriate size.
- **The transformation from 3SAT to VC:** Let $U=\{u_1, u_2, \ldots, u_n\}$ and $C=\{c_1, c_2, \ldots, c_m\}$ be any instance of 3SAT, the corresponding instance ($G=(V, E)$, k) of VC is constructed as follows.

For each variable $u_i \in U$ there is a truth-setting component $T_i = (V_i, E_i)$ with $V_i = \{u_i, \bar{u}_i\}$ and $E_i = \{\{u_i, \bar{u}_i\}\}$

For each clause $c_j \in C$ there is a satisfaction setting component $S_j = (V_j^{'}, E_j^{'})$ where
$V_j^{'} = \{a_1[j], a_2[j], a_3[j]\}$ and
$E_j^{'} = \{\{a_1[j], a_2[j]\}, \{a_2[j], a_3[j]\}, \{a_3[j], a_1[j]\}\}$

For each clause $c_j=\{x_j, y_j, z_j\}$ there is a set of communication edges to reflect the relationship between the variables and the clauses
$E_j^{''} = \{\{a_1[j], x_j\}, \{a_2[j], y_j\}, \{a_3[j], z_j\}\}$

Finally, let $k=n+2m$.

VC$\in$ NPC (II)

3SAT :
$U = \{u_1, u_2, u_3, u_4\}$
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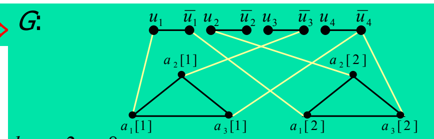

$k=n+2m=8$

- **Polynomial time:** It is easy to see how the construction can be accomplished in polynomial time.
- **(U,C) is YES to SAT $\Leftrightarrow$ (G,k) is YES to VC:** that is, C is satisfiable $\Leftrightarrow$ G has a vertex cover of size k or less.

$\Leftarrow$ Suppose that $V' \subseteq V$ is a vertex cover for G with $|V'| \le k$.

- V' must contain at least one vertex from each T_i and at least two vertices from each S_j,
- giving a total of at least $|V'|=n+2m=k$ vertices.
- V' must contain exactly one vertex from each T_i and exactly two vertices from each S_j.
- We merely set $t(u_i)=T$ if $u_i \in V'$, and F otherwise.
- To see that t satisfies each $c_j \in C$: Consider the 3 edges in E_j'', only two of them can be covered by vertices from $V_j' \cap V'$, so one of them must be covered by a vertex from V_i that belongs to V'. But that implies that the corresponding literal, either u_i or $\bar{u}_i$, from clause c_j is true under t, and hence clause c_j is satisfied by t.
- It follows that C is satisfiable.

$\Rightarrow$ Conversely, suppose that $t: U \rightarrow \{T,F\}$ satisfies every $c_j \in C$. Consider the $V' \subseteq$V that

- Includes u_i if $t(u_i)=T$ and $\bar{u}_i$ if $t(u_i)=F$. This ensures
  - ✓ that the edge in each E_i is covered, and
  - ✓ that at least one of the three edges from each E_j'' is covered.
- Include in V' the endpoints from S_j of the other two edges in E_j''. V' covers the edges in $E_j'' \cup E_j''$.
- V' is the desired vertex cover of $n+2m=k$ vertices.

CLIQUE$\in$ NPC

- **VERTEX COVER (VC)**
- **Instance**: A graph $G=(V, E)$ and a positive integer k.
- **Question**: Is there a vertex cover of size k or less for G?

- **CLIQUE**
- **Instance**: A graph $G=(V, E)$ and a positive integer k.
- **Question**: Does G contain a clique of size k or more?

- Despite the fact that VC and CLIQUE are independently useful for proving NP-completeness, they are really just different ways of looking at the the same problem.
- To see the above, it is convenient to consider them in conjunction with a third problem, called INDEPENDENT SET (IS).

- **INDEPENDENT SET (IS)**
- **Instance**: A graph $G=(V, E)$ and a positive integer k.
- **Question**: Is there an independent set of size k or more in G?

- The following relationships between independent sets, cliques, and vertex covers are easy to verify.

For any graph $G=(V,E)$ and subset $V' \subseteq V$, the following statements are equivalent.
1) V' is a vertex cover for G.
2) $V - V'$ is an independent set for G.
3) $V - V'$ is a clique in the *complement* G^c of G, where $G^c=(V,E^c)$ with $=\{\{u,v\} \mid u,v \in V \text{ and } \{u,v\} \notin E\}$.

- Thus we see that, in a strong sense, these three problems might be regarded simply as "different versions" of one another.
- The relationships make it a trivial matter to transform any one of the problems to either of the others.