- **VERTEX COVER (VC)**
- **Instance**: A graph $G=(V, E)$ and a positive integer k.
- **Question**: Is there a vertex cover of size k or less for G?

$\Rightarrow$

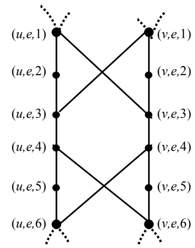
- **Hamiltonian Circuit (HC)**
- **Instance**: A graph $G=(V, E)$.
- **Question**: Does G contain a Hamiltonian circuit?

HC∈ NPC (I)

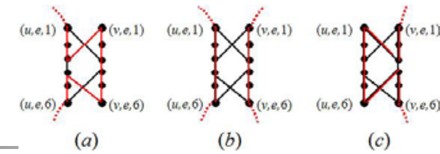
- **HC∈ NP:** it is easy to see this.
- **The Transformation (from VC):** let $<G=(V, E), k>$ be any instance of VC, then the corresponding instance $G'=(V', E')$ is constructed as follows.

Only $(u,e,1)$, $(u,e,1)$, $(u,e,1)$, and $(u,e,1)$ from this component will be involved in further additional edges.

First: for each edge $e=\{u,v\}\in E$ there is a cover-testing component $T_e=(V'_e, E'_e)$, with 12 vertices
$V'_e=\{(u,e,i), (v,e,i) | 1\le i\le 6\}$
and 14 edges
$E'_e=\{\{(u,e,i), (u,e,i+1)\}, \{(v,e,i), (v,e,i+1)\} | 1\le i\le 5\}$
$\cup \{\{(u,e,3), (u,e,1)\}, \{(v,e,3), (v,e,1)\}\}$
$\cup \{\{(u,e,6), (u,e,4)\}, \{(v,e,6), (v,e,4)\}\}$



HC∈ NPC (II)



Claim: Any HC of G' will have to be one of the three configurations shown in the above figure.

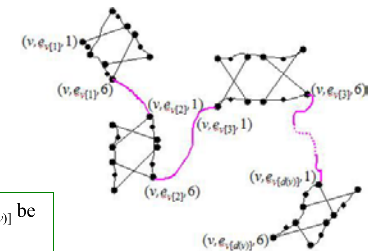
E.g: If an HC circuit "enters" this component at $(u,e,1)$, it will have to "exit" at $(u,e,6)$ and visit either all 12 vertices or just the 6 vertices (u,e,i) .

$E'_v=\{\{(v, e_{v[1]},6), (v, e_{v[i+1]},1)\} | 1\le i<d(v)\}$.

The idea behind the configurations:

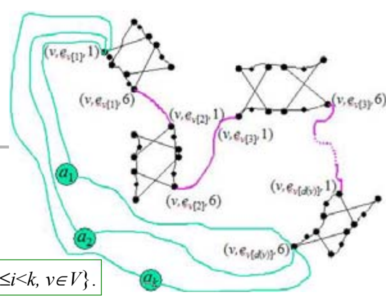
- (a)—u belongs the cover but v does not,
- (b)—both u and v belongs the cover,
- (c)—v belongs the cover but u does not.

Second: for each $v\in V$, let $e_{v[1]}, e_{v[2]}, \ldots, e_{v[d(v)]}$ be the edges incident on v. All the cover-testing components corresponding to these edges are joined together by the following edges.



This creates a "path" for each v.

HC∈ NPC (III)



Finally: Add k vertices $a_1, a_2, \ldots, a_k$ (called selectors), and connect everyone of them to the first and the last vertices of each path. That is,

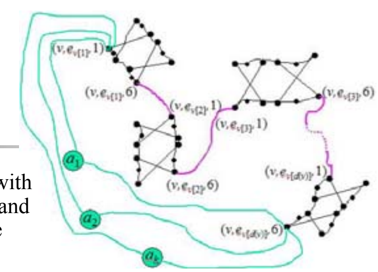
$E''=\{\{(a_i, \{(v, e_{v[1]},1)\}, \{(a_i, \{(v, e_{v[d(v)]},6)\} | 1\le i<k, v\in V\}$.

Thus, the instance of HC is $G'=(V', E')$, where
$V'=\{a_i | 1\le i\le k\}+(\cup_{e\in E}V'_e)$
and
$E'=(\cup_{e\in E}E'_e)+(\cup_{v\in V}E'_v)+E''$

- **Polynomial time:** it is not hard to see that $G'=(V', E')$ can be constructed from $<G=(V, E), k>$ in polynomial time.

- ***G'* has a HC if and only *G* has a vertex cover of k or less.**
- Proof: the next two slides

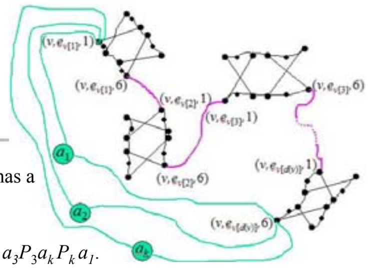
HC∈ NPC (IV)



$\Leftarrow$: Suppose $V^*\subseteq V$ is a vertex cover for G with $|V^*|\le k$. W.L.O.G, we can assume $|V^*|=k$ and $V^*=\{v_1, v_2, \ldots, v_k\}$. An HC C in G' can be constructed as follows.

- $C=a_1P_1a_2P_2\, a_3P_3a_kP_k\, a_1$, where P_i is the path passing through the "Path" for $v_i\in V^*$, i.e, P_i passes all the cover-testing components T_e corresponding to edges $e=\{v_i, u\}$ incident on v_i. The way that P_i pass T_e depends on whether u belongs to V^*.
- If $u\notin V^*$, P_i visits all the 12 vertices in T_e, taking configuration (a) or (c).
- If $u\in V^*$, P_i visits the 6 vertices $\{(v,e,i) | 1\le i\le 6\}$ in T_e, taking configuration (b).
- Circuit C visits each of the k selectors $a_1, a_2, \ldots, a_k$ exactly once, visits each of the other vertices at least once because V^* is a vertex cover for G.
- But the way for P_i passing each cover-testing component guarantees that C passes each vertex at most once.
- Circuit C is an HC for G'.

HC∈ NPC (V)



⇒ : Suppose C is an HC in G'. We prove that G has a vertex cover V^* of no more than k vertices.

- W.L.O.G, we can assume that $C=a_1P_1a_2P_2\ a_3P_3a_kP_ka_1$.
- Then, because of the previous mentioned restrictions on the way in which an HC can pass through a cover-testing component, P_i must pass through a set of cover-testing components corresponding to exactly those edges that are incident on some particular vertices $v_i \in V^*$.
- Let $V^*=\{v_1, v_2,\ldots, v_k\}$, then V^* is a vertex cover of k vertices.

Variants of HC problems

- The following two variants of HC are also NP-complete.
  - **Hamiltonian Path (HP)**
  - **Instance**: A graph $G=(V, E)$.
  - **Question**: Does G contain a Hamiltonian path?

  - **Hamiltonian Path Between Two Points**
  - **Instance**: A graph $G=(V, E)$, and two specified vertices $a,b\in V$.
  - **Question**: Does G contain a Hamiltonian path between a and b?
- The next famous problem is NP-complete.
  - **Traveling Salesman (TS)**
  - **Instance**: A complete graph $G=(V, E)$ with a weight function $d: E\rightarrow Z^+$ that assigns each edge a positive integer (size), and positive integer K.
  - **Question**: Does G contain a Hamiltonian circuit of size K or less? Here the size of a Hamiltonian circuit is defined to be the sum of the sizes of the edges in the circuit.

- **3-Satisfiability (3SAT)**
- **Instance**: Collection C of clauses on a set U of Boolean variables such that $|c|=3$ for all $c\in C$.
- **Question**: Is there a truth assignment for U that satisfies C?

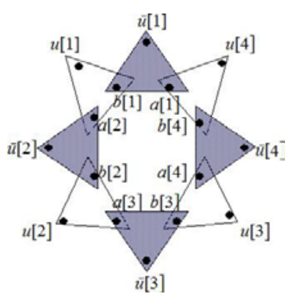
⇓

- **3-Dimensional Matching (3DM)**
- **Instance**: A set $M\subseteq W\times X\times Y$, where W, X, and Y are disjoint sets having the same number of q elements.
- **Question**: Does M contain a matching M'?

3DM∈ NPC (I)

- **The Transformation** (from 3SAT): for a given instance <U,C> of 3SAT with $C=\{c_1, c_2,\ldots, c_m\}$ and $U=\{u_1, u_2,\ldots, u_n\}$, we construct W, X, Y, and M in three steps.

First: for each variable $u_i\in U$, create a truth-setting component T_i structured according to m.
$T_i = T_i^t \cup T_i^f$, where
$T_i^t=\{(\bar{u}_i[j], a_i[j], b_i[j])\ |1\le j\le m\}$
and
$T_i^f=\{(u_i[j], a_i[j+1], b_i[j])\ |1\le j\le m\} \cup \{(u_i[m], a_i[1], b_i[m])\}$

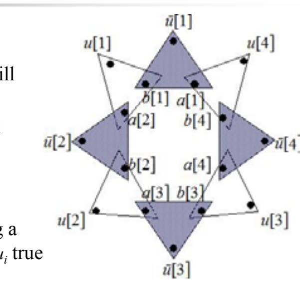


The figure illustrates the truth-setting component T_i when m=4 (subscripts have been deleted for simplicity).

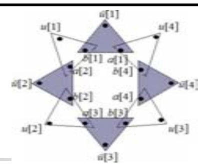
3DM∈ NPC (II)

Comments on the truth-setting component T_i :

- None of the internal elements $\{a_i[j], b_i[j]\ |1\le j\le m\}$ will appear in any triples outside of T_i.
- Thus, in order to form a matching $M'\subseteq M$, either all the sets of T_i^t (the shaded sets) or all the sets of T_i^f (the unshaded sets) must be chosen, leaving uncovered all the $u_i[j]$ or all the $\bar{u}_i[j]$, respectively
- Hence, we can think of the component T_i as forcing a matching $M'\subseteq M$ to make a choice between setting u_i true and setting u_i false.
- So, in general, a matching $M'\subseteq M$ specifies a truth assignment for U, with the variable u_i being set true if and only if $M'\cap T_i = T_i^t$.

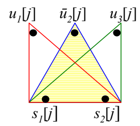


3DM∈ NPC (III)



Second: for each variable $c_j \in C$, create a satisfaction-testing component S_j. The component involves two "internal" elements, $s_1[j] \in X$ and $s_2[j] \in Y$, and external elements from $\{u_i[j], \bar{u}_i[j] \mid 1 \le i \le n\}$, determined by which literals occur in c_j. The set of triples making up this component is defined as

$S_j = \{(u_i[j], s_1[j], s_2[j]) \mid u_i \in c_j\}$
$\cup \{(\bar{u}_i[j], s_1[j], s_2[j]) \mid \bar{u}_i \in c_j\}$



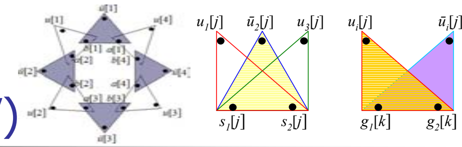
The figure illustrates the satisfaction-testing component S_j when $c_j = \{u_1, \bar{u}_2, u_3\}$.

Comments on S_j : Since $s_1[j]$ and $s_2[j]$ are "internal",

- Any matching $M' \subseteq M$ will have to contain exactly one triple from C_j.
- This can only be done if some $u_i[j]$ (or $\bar{u}_i[j]$) for a literal $u_i \in c_j$ (or $\bar{u}_i \in c_j$) does not occur in the triples in $M' \cap T_i$.
- Which will be the case if and only if the truth setting determined by M' satisfies clause c_j.

2007/11/4 CS,OUC 9

3DM∈ NPC (IV)



Third: construct a garbage-collection component G, involving "internal" elements $g_1[k] \in X$ and $g_2[k] \in Y$, $1 \le k \le m(n-1)$, and external elements of the form $u_i[j]$ and $\bar{u}_i[j]$ from W. It consists of the following set of triples:

$G = \{(u_i[j], g_1[k], g_2[k]), (\bar{u}_i[j], g_1[k], g_2[k]) \mid$
$1 \le i \le n,\ 1 \le j \le m,\ 1 \le k \le m(n-1)\}$.

Comments on G:

- Each pair $s_1[k]$, $g_2[k]$ must be matched with a unique $u_i[j]$ or $\bar{u}_i[j]$ that does not occur in any triples of M'-G.
- There are exactly $m(n-1)$ such uncovered elements $u_i[j]$ and $\bar{u}_i[j]$, and
- G is structured to insure that they can always be covered by choosing $M' \cap G$ appropriately.

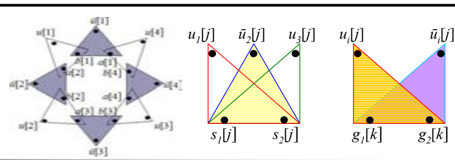
Thus, whenever a subset of M-G satisfies all the constraints imposed by the truth-setting and satisfaction-testing components, the subset can be extended to a matching of M.

Summarization:

$W = \{u_i[j], \bar{u}_i[j] \mid 1 \le i \le n, 1 \le j \le m\}$;
$X = A \cup S_1 \cup G_1$, where
$A = \{a_i[j] \mid 1 \le i \le n, 1 \le j \le m\}$
$S_1 = \{s_1[j] \mid 1 \le j \le m\}$
$G_1 = \{g_1[k] \mid 1 \le k \le m(n-1)\}$;
$Y = B \cup S_2 \cup G_2$, where
$B = \{b_i[j] \mid 1 \le i \le n, 1 \le j \le m\}$
$S_2 = \{s_2[j] \mid 1 \le j \le m\}$
$G_2 = \{g_2[k] \mid 1 \le k \le m(n-1)\}$;
$M = (\cup_{1 \le i \le n} T_i) \cup (\cup_{1 \le j \le m} S_j) \cup G$.

2007/11/4 CS,OUC 10

3DM∈ NPC (V)



The Proof of that there is a $t: U \rightarrow \{T,F\}$ that satisfies C ⇔ there is a matching M' for M.

⇒ If $t: U \rightarrow \{T,F\}$ satisfies C , we can construct a matching $M' \subseteq M$ as follows:

- For each clause $c_j \in C$, let $z_j \in \{u_i[j], \bar{u}_i[j] \mid 1 \le i \le n, 1 \le j \le m\} \cap c_j$ be a literal that is set true by t. We then set

  $M' = (\cup_{t(u_i)=T} T_i^t) \cup (\cup_{t(u_i)=F} T_i^f)$
  $\cup \{(z_j[j], s_1[j], s_2[j]) \mid 1 \le j \le m\}$
  $\cup G'$.

  where G' is an appropriately chosen subset of G that includes all the $g_1[k]$, $g_2[k]$, and remaining $u_i[j]$ and $\bar{u}_i[j]$.
- It is easy to verify that such a G' can always be chosen and that the resulting set M' is a matching.

⇐ From the comments made during the description of M, it follows immediately that M cannot contain a matching unless C is satisfiable.

2007/11/4 CS,OUC 11

A variant 3DM

Next problem is NP-complete.

- **Exact Cover by 3-Sets (X3C)**
- **Instance**: A finite set X with $|X|=3q$ and a collection C of 3-element subsets of X.
- **Question**: Does C contain a an exact cover for X, that is, a sub-collection $C' \subseteq C$ such that every element of X occurs in exactly one member of C'?

Note that:

- Every instance is 3DM can be viewed as an instance of X3C.
- Thus 3DM is a restricted version of X3C.
- The NP-completeness of X3C follows by a trivial transformation from 3DM.

2007/11/4 CS,OUC 12

3DM⇒

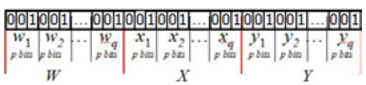
- **Subset Sum (SS)**
- **Instance**: A finite set $A = \{a_1, a_2, \ldots, a_n\} \subseteq Z^+$ and $B \in Z^+$.
- **Question**: Does A contain a subset A' such that $\sum_{a \in A'} a = B$?

SS∈ NPC

The Transformation (from 3DM to SS):

- Let the sets W, X, Y, with $|W|=|X|=|Y|=q$, and $M \subseteq W \times X \times Y$ be an arbitrarily instance of 3DM.
- Let the elements of these sets be denoted by $W=\{w_1, w_2, \ldots, w_q\}$, $X=\{x_1, x_2, \ldots, x_q\}$, $Y=\{y_1, y_2, \ldots, y_q\}$, and $M=\{m_1, m_2, \ldots, m_k\}$, where $k=|M|$ and $m_i=(w_{f(i)}, x_{g(i)}, y_{h(i)})$.
- Let $p=\lceil \log 2(k+1) \rceil$.
- We create $A = \{a_1, a_2, \ldots, a_k\}$ by setting

$$a_i = 2^{p(3q - f(i))} + 2^{p(2q - g(i))} + 2^{p(q - f(i))}.$$



- We create $B = \sum_{0 \le j \le 3q-1} 2^{pj}$.
- A and B is the constructed instance of SS.

Polynomial time: Since each a_k can be expressed in binary with no more than 3pq bits, it is clear to see this.

(W, X, Y, M) is yes for 3DM ⇔ (A, B) is yes for SS :

- If we sum up all the entries in any p-bit zone overall $\{a_k \mid 1 \le i \le k\}$, the total can never exceed $k=2^p-1$.
- Hence, in adding up $\sum_{a \in A'} a$ for any subset $A' \subseteq A$, there will never be any "carries" from one p-bit zone to the next.
- So, any subset $A' \subseteq A$ will satisfy $\sum_{a \in A'} a = B$ if and only $M' = \{m_i \mid a_i \in A'\}$ is a matching for M.

- **PARTITION**
- **Instance**: A finite set $A = \{a_1, a_2, \ldots, a_n\} \subseteq Z^+$.
- **Question**: Does A contain a subset A' such that $\sum_{a \in A'} a = \sum_{a \in A-A'} a$?

⇐

- **Subset Sum (SS)**
- **Instance**: A finite set $A = \{a_1, a_2, \ldots, a_n\} \subseteq Z^+$ and $B \in Z^+$.
- **Question**: Does A contain a subset A' such that $\sum_{a \in A'} a = B$?

PARTITION∈ NPC

The Transformation: for a given instance $<A,B>$ of SS with $A=\{a_1, a_2, \ldots, a_n\}$, we simply add two elements b_1 and b_2 to A to form the instance of PARTITION, where

$$b_1 = 2\left(\sum_{a \in A} a\right) - B \text{ and } b_2 = \left(\sum_{a \in A} a\right) + B.$$

(A, B) is yes for 3DM ⇔ $A \cup \{b_1, b_2\}$ is yes for PARTITION :

⇐ Suppose that there is a subset $A' \subseteq A \cup \{b_1, b_2\}$ that satisfies $\sum_{a \in A'} a = \sum_{a \in (A \cup \{b1, b2\}) - A'} a$.

- Then both of these sums must be equal to $2\sum_{a \in A'} a$,
- and one of the two sets, A' or $A \cup \{b_1, b_2\} - A'$, contains b_1 but not b_2.
- It follows that the remaining elements of that set form a subset of A and the elements sum to B.

⇒ Suppose that there is a subset $A' \subseteq A$ that satisfies $\sum_{a \in A'} a = B$.

- Then the subset $A' \cup \{b_1\}$ of $A \cup \{b_1, b_2\}$ satisfies
- $\sum_{a \in A' \cup \{b1\}} a = 2\sum_{a \in A'} a = \sum_{a \in (A \cup \{b1, b2\}) - (A' \cup \{b1\})} a.$