CSCE 222 Discrete Structures for Computing – Fall 2023
Hyunyoung Lee

**Problem Set 4**

**Due dates:** Electronic submission of *yourLastName-yourFirstName-hw4.tex* and *yourLastName-yourFirstName-hw4.pdf* files of this homework is due on **Friday, 10/13/2023 before 11:59 p.m.** on `https://canvas.tamu.edu`. You will see two separate links to turn in the .tex file and the .pdf file separately. Please do not archive or compress the files. **If any of the two files are missing, you will receive zero points for this homework.**

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**Resources.** (All people, books, articles, web pages, etc. that have been consulted when producing your answers to this homework)

On my honor, as an Aggie, I have neither given nor received any unauthorized aid on any portion of the academic work included in this assignment. Furthermore, I have disclosed all resources (people, books, web sites, etc.) that have been used to answer this homework.

**Electronic signature:** **(Manas Navale)**

Total $100 + 5$ (bonus) points.

The intended formatting is that this first page is a cover page and each problem solved on a new page. You only need to fill in your solution between the `\begin{solution}` and `\end{solution}` environment. Please do not change this overall formatting.

**Make sure that you strictly follow the structure of induction proof as shown in the lecture notes and how I solved in my videos.**

**Checklist:**
- ☐ Did you type in your name and UIN?
- ☐ Did you disclose all resources that you have used? (This includes all people, books, websites, etc. that you have consulted)
- ☐ Did you sign that you followed the Aggie Honor Code?
- ☐ Did you solve all problems?
- ☐ Did you submit both the .tex and .pdf files of your homework to each correct link on Canvas?

**Problem 1.** (15 points) Section 4.1, Exercise 4.3

**Solution.**

Base case:

$$S(1) = 1^2 = 1$$

and Also

$$1 * (1 + 1) * 3/6$$

meaning the base case is satisfied. Induction Hypothesis would be true for all values of $n < m$ By definition

$$S(m) = S(m - 1) + m^2$$

and then by the inductive step

$$S(m - 1) = (m - 1) * (m - 1 + 1) * (2(m - 1) + 1)/6$$

Since induction step allows us to assume that the step is true for all values of $n < m$. Specifically n = m - 1 We have:

$$S(m) = \frac{(m-1) \cdot (m) \cdot (2m-1)}{6} + m^2$$

By rearranging terms, we get:

$$S(m) = \frac{m}{6}\left((m-1) \cdot (2m-1) + 6m\right)$$

That is,

$$S(m) = \frac{m}{6}\left(2m^2 - 2m - m + 1 + 6m\right)$$

Simplifying further:

$$S(m) = \frac{m}{6}\left(2m^2 + 3m + 1\right)$$

And finally,

$$S(m) = \frac{m}{6}(m + 1)(2m + 1)$$

Therefore, the hypothesis is true for $m$.

**Problem 2.** (15 points) Section 4.1, Exercise 4.4

**Solution.**

Base case: For $n = 1$, we have:

$$1^3 = \left(\frac{1(1+1)}{2}\right)^2$$

This simplifies to $1 = 1$, which is true.

Inductive Hypothesis: Assume that the formula holds for some positive integer $k$:

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$$

Simplify:

$$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

Factor:

$$\frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} = \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

Factor Out a Common Factor:

$$(k+1)^2\left(\frac{k^2 + 4(k+1)}{4}\right)$$

Simplify the Expression Inside the Parentheses:

$$(k+1)^2\left(\frac{k^2 + 4k + 4}{4}\right) = (k+1)^2\left(\frac{(k+2)^2}{4}\right)$$

Further Simplify:

$$(k+1)^2\left(\frac{(k+2)^2}{4}\right) = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

Based on this we have shown that the sum of the cubes of the first $(n)$ natural numbers is equal to $\left(\frac{n(n+1)}{2}\right)^2$.

**Problem 3.** (15 points) Section 4.1, Exercise 4.5

**Solution.**

Base Case: For $n = 1$, we have:

$$1^2 = \frac{1}{3}(4(1^3) - 1) = \frac{1}{3}(4 - 1) = \frac{1}{3}(3) = 1$$

This is true for the base case.

Inductive Hypothesis: Assume that the formula holds for some positive integer $k$:

$$1^2 + 3^2 + 5^2 + \ldots + (2k - 1)^2 = \frac{1}{3}(4k^3 - k)$$

Inductive Step We want to prove for $(k + 1)$: Assume Hypothesis:

$$1^2 + 3^2 + 5^2 + \ldots + (2k - 1)^2 + (2(k + 1) - 1)^2 = \frac{1}{3}(4(k + 1)^3 - (k + 1))$$

Simplify:

$$1^2 + 3^2 + 5^2 + \ldots + (2k - 1)^2 + (2(k + 1) - 1)^2 = \frac{1}{3}(4k^3 - k) + (2(k + 1) - 1)^2$$

Factor:

$$\frac{1}{3}(4k^3 - k) + (2(k + 1) - 1)^2 = \frac{1/3}{(}4k^3 - k) + (2k + 2 - 1)^2$$

Simplify:

$$\frac{1}{3}(4k^3 - k) + (2k + 1)^2 = \frac{1}{3}(4k^3 - k) + 4k^2 + 4k + 1$$

Conclude: By combining terms, we get

$$\frac{1}{3}(4k^3 - k + 9k^2 + 12k + 3)$$

Factor and simplify further:

$$\frac{(k + 1)(4k^2 + 5k + 3)}{3}$$

Simplify:

$$\frac{(k + 1)(4k^2 + 5k + 3)}{3}$$

Factor the quadratic expression:

$$\frac{(k + 1)(4k + 3)(k + 1)}{3}$$

Simplify the right side to get

$$\frac{4(k+1)(k+1)(k+1)}{3}$$

By mathematical induction, we have shown that the sum of the squares of the first $(n)$ odd positive integers is given by $((1/3) * (4n^3 - n))$.

**Problem 4.** (20 points) Section 4.1, Exercise 4.6

**Solution.**

Base Case: For $n = 1$, we have:

$$22^1 - 1 = 22 - 1 = 21$$

Since 21 is divisible by 3, the formula holds for the base case. Inductive Hypothesis: Assume that the formula holds for some positive integer $k$:

$$22^k - 1 \text{ is divisible by 3.}$$

Inductive Step: Prove: $22^{k+1}-1$ is divisible by 3. Start with the left side and use the inductive hypothesis:

$$22^{k+1} - 1 = 22 \cdot 22^k - 1 = 21 \cdot 22^k + 22^k - 1$$

Factor out $22^k$ from the first term:

$$21 \cdot 22^k + 22^k - 1 = 22^k(21 + 1) - 1$$

Simplify:

$$22^k(22) - 1 = 22(22^k) - 1$$

Since $22^k - 1$ is divisible by 3 (by the inductive hypothesis), express it as $3m$ for some positive integer $m$:

$$22(22^k) - 1 = 22(3m) = 3(22m)$$

Since $22m$ is also a positive integer, we can see that $22^{k+1} - 1$ is divisible by 3.

By mathematical induction, we've shown that $22^n - 1$ is divisible by 3 for all positive integers $n$.

**Problem 5.** (20 points) Section 4.3, Exercise 4.15

**Solution.**

Base Case: For $n = 1$, we have:

$$f_2 = 1, \text{ and } f_3 - 1 = 2 - 1 = 1.$$

So, the formula holds for the base case.

Inductive Hypothesis: Assume: $f_2 + f_4 + \ldots + f_{2k} = f_{2k+1} - 1$.

Inductive Step: Prove: $f_2 + f_4 + \ldots + f_{2(k+1)} = f_{2(k+1)+1} - 1$. Start with the left side and use the inductiv

$$f_2 + f_4 + \ldots + f_{2k} + f_{2(k+1)} = f_{2k+1} - 1 + f_{2(k+1)}.$$

Use the Fibonacci sequence property:

$$f_{2k+1} - 1 + f_{2(k+1)} = f_{2k+1} + f_{2k} - 1.$$

Simplify:

$$f_{2k+1} + f_{2k} = f_{2k-1} + f_{2k-2} + f_{2k}.$$

Apply the Fibonacci sequence property:

$$f_{2k-1} + f_{2k-2} + f_{2k} = f_{2k-1} + f_{2k-1} - 1 = 2f_{2k-1} - 1.$$

Apply the Fibonacci sequence property again:

$$2f_{2k-1} = f_{2k}.$$

So, we have:

$$f_{2k} - 1 = f_{2k} - 1.$$

This proves that the formula holds for $(k + 1)$. So by mathematical induction, we have shown that the sum of even Fibonacci numbers $(f_2 + f_4 + \ldots + f_{2n})$ is indeed equal to $f_{2n+1} - 1$.

**Problem 6.** (20 points) Section 4.6, Exercise 4.31

**Solution.**

Base Case: For $n = 4$, we have:

$$n = 4 \text{ or } n - 3 = 1$$

$$f_4 = 4(4-1)(4-2)f_1$$
$$f_4 = 4 * 3 * 2 * 1 = 4!$$
$$f_4 = 4!$$

Inductive Hypothesis: Assume given relation is true for $n = k$

$$f_k = k(k-1)(k-2)f_{k-3}$$

$$f_{k-1} = (k-1)(k-2)(k-3)f_{k-4}$$

all are true

Inductive Step: To prove relation is true for $n = k + 1$ or

$$f_{k+1} = (k+1)(k)(k-1)f_{k-2}$$

From:

$$f_{k-2} = (k-2)(k-3)(k-4)f_{k-5}$$
$$f_{k-5} = (k-5)(k-6)(k-7)f_{k-8}$$
$$f_4 = 4(4-1)(4-2)f_1$$

Putting all of the values in $f_{k+1} = (k+1)(k)(k-1)f_{k-2}$ proves

$$f_{k+1} = (k+1)!$$