Diving Deeper into Chaining and Linear Probing

COMP 480/580

17th Jan 2019

Announcements

- Assignment 1 will be released today and is, due 2 weeks from now, 31st Jan**. Submission via Canvas.**

- Tomorrow is the deadline to pick project over final exams. No email from you means you are choosing final exam by default.

- **All deadlines are 11:59pm CT on the given day.**

Refresher

- k-universal hashing family H
  - A hash function is k-universal if for any set $x_1, x_2, \ldots, x_k$
  - For h chosen uniformly from H, $h(x_1), h(x_2), \ldots, h(x_k)$ are independent random variables.
    - **OR** $\Pr\big(h(x_1) = h(x_2) = \ldots = h(x_k)\big) \leq \frac{1}{n^{k-1}}$

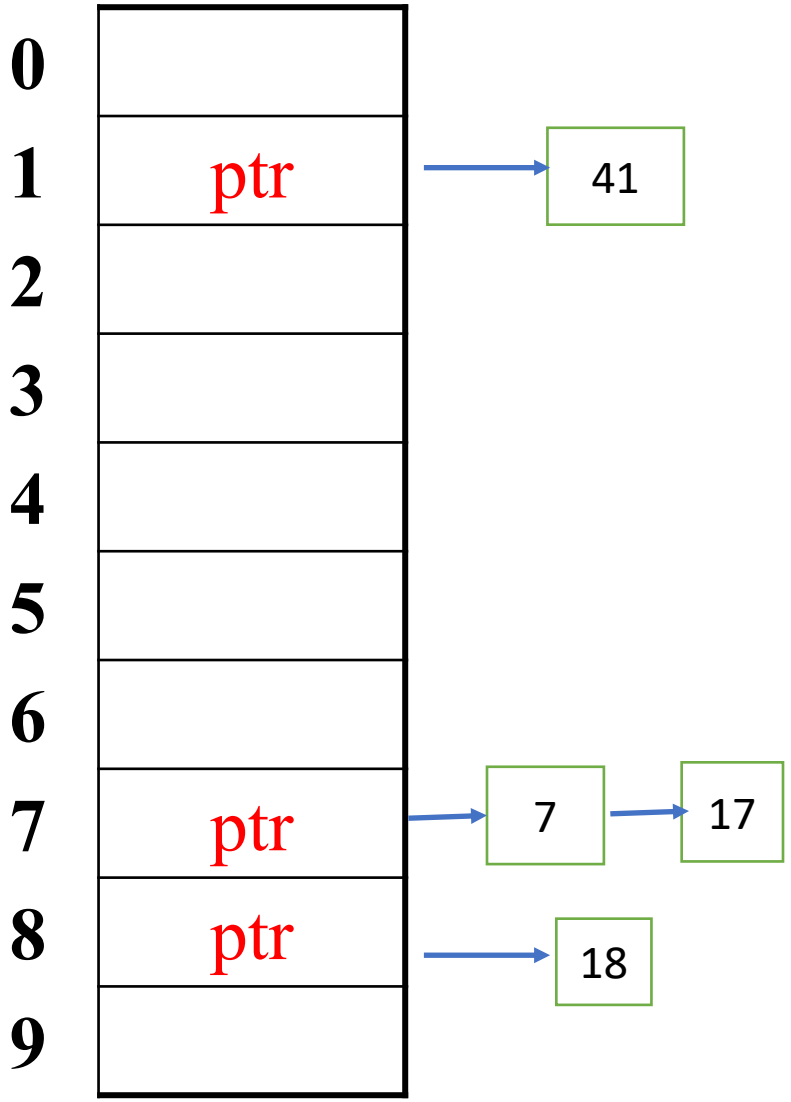
- We saw 2-universal family. Generally for k-independent family we need polynomial in k
  - h(x) = $(a_k\, x^k + a_{k-1}\, x^{k-1} + \ \ldots + a_1\, x + a_0)$ mod P mod R
  - Higher independence is harder to achieve from both computation and memory perspective.

Separate Chaining.

- key space = integers
- TableSize = 10

- **h**(K) = K mod 10

- **Insert**: 7, 41,18, 17



What we saw?

- m objects inserted into array of size n
- Expected length of chain $\leq 1 + \frac{m-1}{n}$
  - The quantity $\frac{m}{n}$ is termed as load factor denoted by α
  - Expected addition and search time $1 + \alpha$
- Worst Case: m
- Is this satisfactory? Are there better variants of chaining? How much better?

What is a good running time?

- Log(n).

- **Natural question:** What is the probability that there exist a chain of size $\geq \log(n)$. More information than expected value!!

Counting and Approximations!

Theorem: For the special case with $m=n$, with probability at least $1-1/n$, the longest list is $O(\ln n / \ln \ln n)$.

Proof:

- Let $X_{i,k}$ = Indicator of {key i hash to slot k}, $\Pr(X_{i,k}=1) = 1/n$
- The probability that a particular slot k receives $>\kappa$ keys is (letting $m=n$)

(**assuming lot if independence**)

$$\binom{n}{\kappa}\frac{1}{m^{\kappa}} = \binom{n}{\kappa}\frac{1}{n^{\kappa}} < \frac{1}{\kappa!}$$

- If we choose $\kappa = 3 \ln n / \ln \ln n$, then $\kappa! > n^2$ and $1/\kappa! < 1/n^2$.
- Thus, the probability that any n slots receives $>\kappa$ keys is $< 1/n$.

Better?

- Can we do better?
- Use two hash functions, insert at the location with smaller chain.
- **Assignment:** Using $m=n$ slots, with probability at least $1-1/n$, the longest list is $O(\log \log n)$. Exponentially better!!

Linear Probing

$$F(i) = i$$

- Probe sequence:

  0^{th} probe = h(k) mod TableSize

  1^{th} probe = (h(k) + 1) mod TableSize

  2^{th} probe = (h(k) + 2) mod TableSize

  . . .

  i^{th} probe = (h(k) + i) mod TableSize

Probing or Open Addressing

0	8
1	109
2	10
3	
4	
5	
6	
7	
8	38
9	19

Insert:
38
19
8
109
10

- **Linear Probing**: after checking spot h(k), try spot h(k)+1, if that is full, try h(k)+2, then h(k)+3, etc.

Major Milestones of Linear Probing

- In 1954, Gene Amdahl, Elaine McGraw, and Arthur Samuel invent linear probing as a subroutine for an assembler.
- In 1962, Don Knuth, in his first ever analysis of an algorithm, proves that linear probing takes expected time O(1) for lookups if the hash function is truly random (n-wise independence).
- In 2006, Anna Pagh et al. proved that 5-independent hash functions give expected constant-time lookups.
- In 2007, Mitzenmacher and Vadhan proved that 2-independence will give expected O(1)-time lookups, assuming there's some measure of randomness in the keys.

Linear Probing in Practice

- In practice, linear probing is one of the fastest general-purpose hashing strategies available.

- This is surprising – it was originally invented in 1954! It's pretty amazing that it still holds up so well.

- Why is this?
  - Low memory overhead: just need an array and a hash function.
  - **Excellent locality:** when collisions occur, we only search in adjacent locations.
  - Great cache performance: a combination of the above two factors

Analyze Expected cost of Linear Probing

- For simplicity, let's assume a load factor of $\alpha = \frac{m}{n} = ⅓$.
  - What happens to linear probing of $\alpha \geq 1$
  - Contrast with chaining.

- A region of size m is a consecutive set of m locations in the hash table.

- An element q hashes to region R if h(q) ∈ R, though q may not be placed in R.

- On expectation, a region of size 2^s should have at most ⅓ 2^s elements hash to it.

- It would be very unlucky if a region had twice as many elements in it as expected. A region of size 2^s is **overloaded** if at least ⅔ 2^s elements hash to it.

A Provable Fact

- **Theorem:** The probability that the query element q ends up between 2^s and 2^{s+1} steps from its home location is upper-bounded by
  - c · Pr[the region of size 2^s centered on h(q) is overloaded] for some fixed constant c independent of s.

- **Proof**: Set up some cleverly-chosen ranges over the hash table and use the pigeonhole principle. See Thorup's lecture notes.
  - https://arxiv.org/abs/1509.04549

Overall we can write the expectation as

- E(lookup time) $\leq$
  $O(1)\ \sum_1^{\log(n)} 2^s\ * \Pr[q\ is\ between\ 2^s\ and\ 2^{s+1}\ slot\ away\ from\ h(q)]$

$=O(1)\sum_1^{\log(n)} 2^s\ *$ Pr[the region of size 2^s centered on h(q) is overloaded]

If we can bound

Pr[the region of size 2^s centered on h(q) is overloaded]

Recall

- A region is a contiguous span of table slots, and we've chosen $\alpha = \frac{1}{3}$.
- An overloaded region has at least $\frac{2}{3} \cdot 2^s$ elements in it.
- Let the random variable B_s represent the number of keys that hash into the block of size 2^s centered on h(q) (q is our search query).
- We want to know $\Pr[\ B_s \geq \frac{2}{3} \cdot 2^S\ ]$.
- Assuming our hash functions are at least 2-independent (why?), we have
  - $E[B_s] = \frac{1}{3} \cdot 2^s$.
- $\Pr[\ B_s \geq \frac{2}{3} \cdot 2^S\ ]$ is equivalent to $\Pr[\ B_s \geq 2 \cdot E[B_s]\ ]$.
- Thus, looking up an element takes, on expectation, at least
  - $O(1)\ \sum_1^{\log(n)} 2^s\ * \Pr[B_s \geq 2\ .E[B_s]]$

Apply Markov's

- No Assumptions!!
- Pr[$B_s \geq 2 \cdot E[B_s]$] $\leq \frac{1}{2}$
- $O(1) \sum_1^{\log(n)} 2^s * \Pr[B_s \geq 2 \, . \, E[B_s]]$
- $\leq O(1) \sum_1^{\log(n)} 2^{s-1}$
- $O(n)$. **BAAAAAAD**

- (we want $\Pr[B_s \geq 2 \, . \, E[B_s]]$ to decay faster than 2^{-s} to get anything interesting)

Concentration Inequalities (Last Class)

- Markov's Inequality
  - Pr$[\ B_s - E[B_s] \geq k\ E[B_s]\ ] \leq \frac{1}{k+1}$
- Chebyshev's
  - Pr$[|B_s - E[B_s]| \geq k\ ] \leq \frac{Var[B_s]}{k^2}$
- Chernoff Bounds.
  - Pr$[\ B_s - E[B_s] \geq k\ E[B_s]\ ] \leq 2e^{-\frac{(k^2\ E[B_s])}{2+k}}$

A fair coin is tossed 200 times. How likely is it to observe at least 150 heads?

- ❖ Markov: ≤ 0.6666
- ❖ Chebyshev: ≤ 0.02
- ❖ Chernoff: ≤ 0.017

Concentration Inequalities (Last Class)

- Markov's Inequality
  - Pr$[\ B_s - E[B_s] \geq k\ E[B_s]\ ] \leq \frac{1}{k+1}$
- Chebyshev's
  - Pr$[|B_s - E[B_s]| \geq k\ ] \leq \frac{Var[B_s]}{k^2}$ $\quad (Var[B_s]\ not\ far\ from\ k \times E[B_s]^2\ )$
- Chernoff Bounds. (B_s is sum of i.i.d variables)
  - Pr$[\ B_s - E[B_s] \geq k\ E[B_s]\ ] \leq 2e^{-\frac{(k^2\ E[B_s])}{2+k}}$

More Independence → More Resources in Hash Functions

Can we bounds the variance?

- Define indicator variable:

$X_i = 1$ if i^{th} elements maps to block of size 2^s centered at h(q)

$E[X_i] = \frac{2^s}{N}$

$$B_S = \sum X_i \,,$$

$$\mathrm{E}[\mathrm{B_s}] = \mathrm{E}[\sum X_i] = \frac{1}{3}\ 2^s$$

Variance

- Var$[B_s]$ = $E[B_s^2] - (E[B_s])^2$
- E$[(\sum X_i)^2\ ]$ = $E\left[\sum X_i^2 + \sum X_i X_j\right]$
- $= \sum E[X_i^2] + \sum E[X_i X_j]$
- $= \sum E[X_i] + \sum E[X_i]E[X_j]$

(assuming 3-independence!!)

- What does 3-independence buys us

$\sum E[X_i]E[X_j] < (E[B_s])^2$ (why? … do the math)

So Var$[B_s] \leq$ E$[B_s]$

- $\Pr\left[|B_s - E[B_s]| \geq E[B_s]\right] \leq \frac{Var[B_s]}{E[B_s]^2} \leq \frac{1}{E[B_s]}$
- $\leq 3 \times 2^{-s}$
- $O(1) \sum_1^{\log(n)} 2^s * \Pr[B_s \geq 2 . E[B_s]]$
- $\leq O(1) \sum_1^{\log(n)} 1 = O(\log(n))$

The bound is tight for 3-independence

Going Beyond: 4th Moment Bound

- $\Pr[|B_s - E[B_s]| \geq a\ ] \leq \frac{4^{th} Moment[B_s]}{a^4}$

- **Assignment**: Prove that expected cost of linear probing is O(1) assuming 5-independence using the above 4th moment bounds.

In practice 2-independence suffice?

- In 2007, Mitzenmacher and Vadhan proved that 2-independence will give expected O(1)-time lookups, assuming there's some measure of randomness in the keys.
    - Assignment for Grads: Read this paper.

Cuckoo Hashing

- Worst case of both chaining and probing is O(n). Expected is O(1).
  - Includes both insertion and searching.

- Can we sacrifice insertions for worst case O(1) searching?
  - YES
  - Cuckoo Hashing

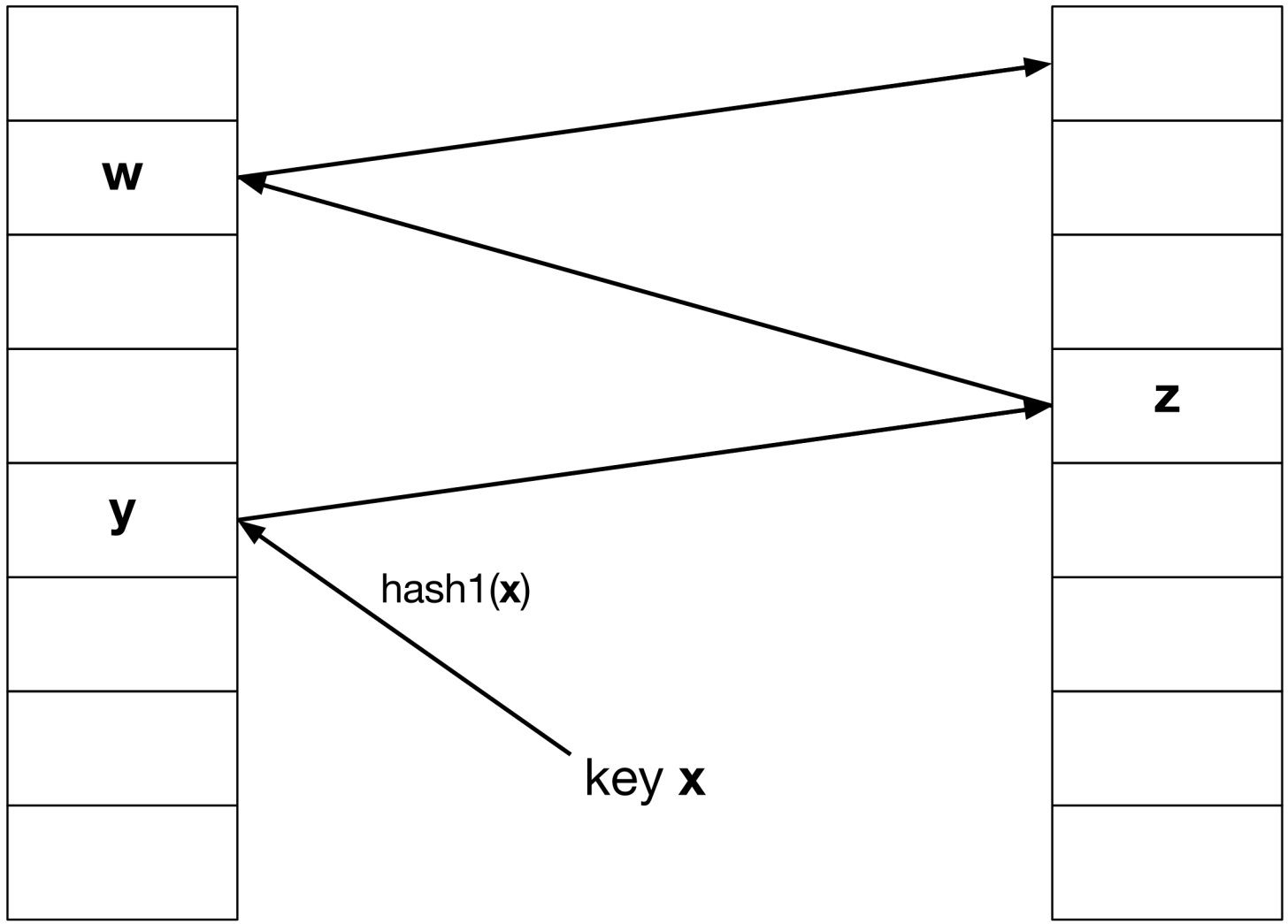


Figure 1. An example “cuckoo path”.

In practice

- In practice Cuckoo hashing is about 20-30% slower than linear probing which is the fastest of the common approaches.