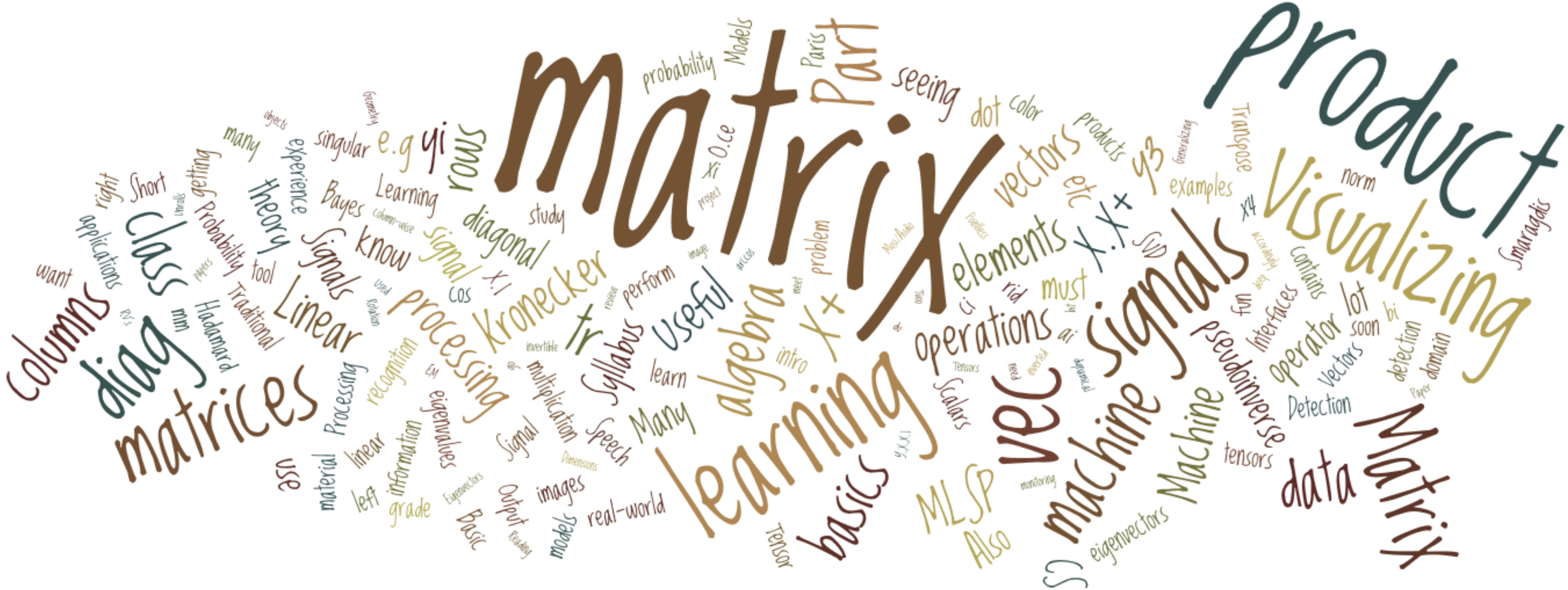


CS598PS – Machine Learning for Signal Processing

Introduction &
Linear Algebra Basics

30 August 2017

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

What do I need to know?

- I won't assume you know machine learning or signal processing
  - You are here to learn, not to know!
- Be comfortable with the basics
  - Some linear algebra, some probability
    - If you are rusty that's ok, you are here to learn

What is this?

- Is it a signal processing class?
  - What is signal processing?
- Is it a machine learning class?
  - What is machine learning?

Signal Processing

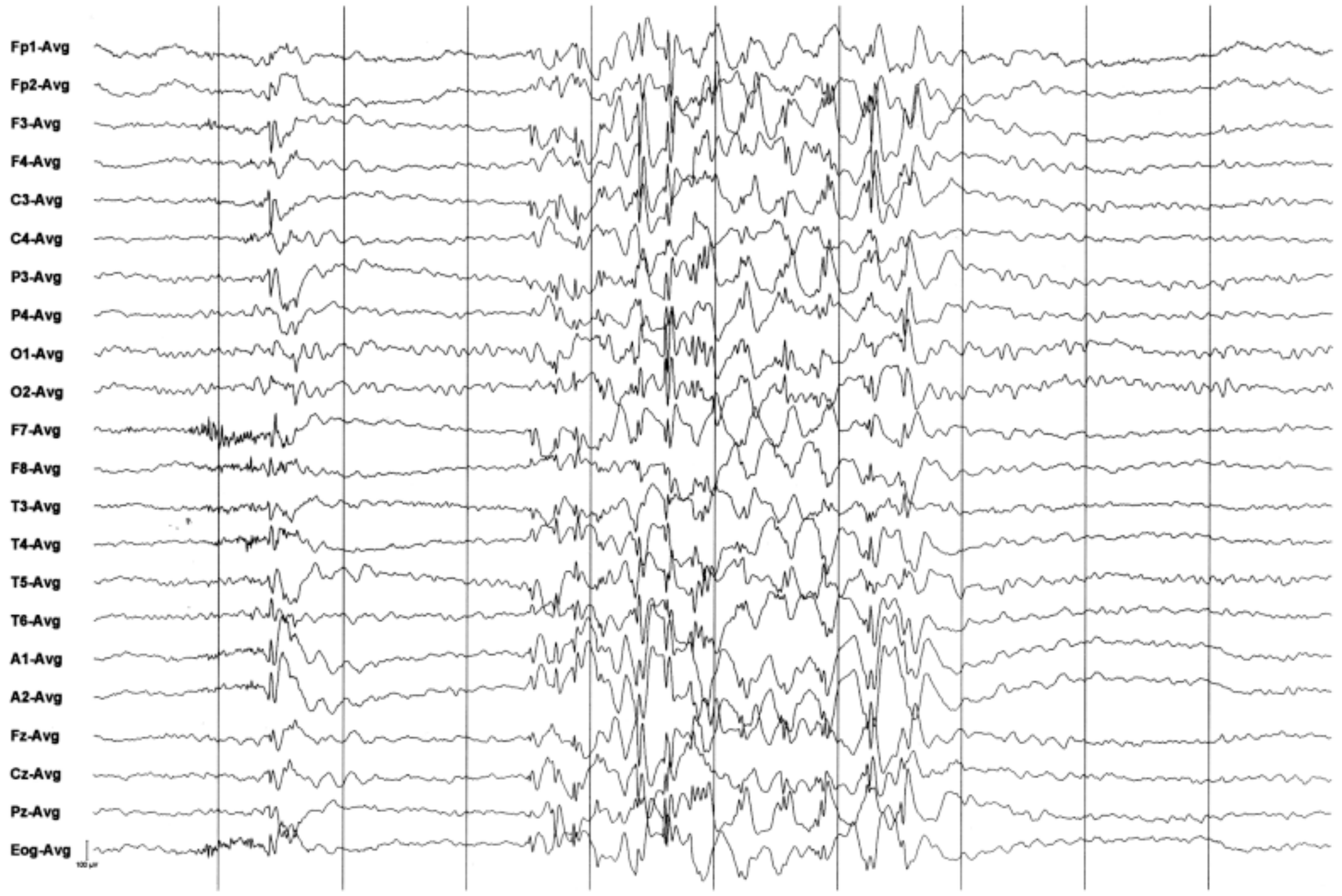
- The study of capturing, processing and manipulating “signals”
- What is a signal?

Signals (as far as this class goes)

- "Structured" collections of measurements that convey information

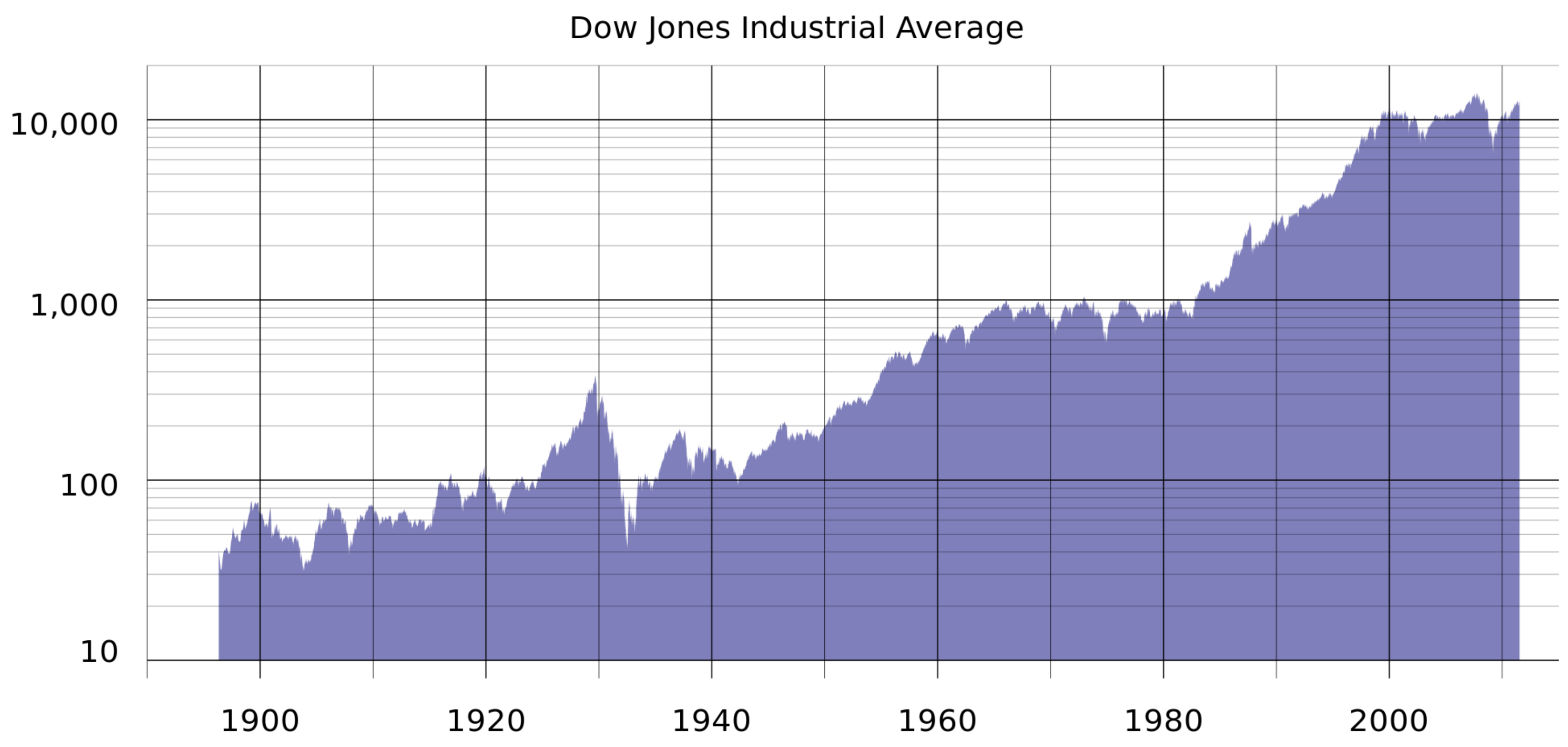
Images

Sounds



Brains!!

Stocks



Machine Learning

- The study of discovering and extracting information from "data"

- For data we will use signals

Why bother?

- Traditional signal processing doesn't really care about its input's content
- Traditional machine learning is not signals-friendly

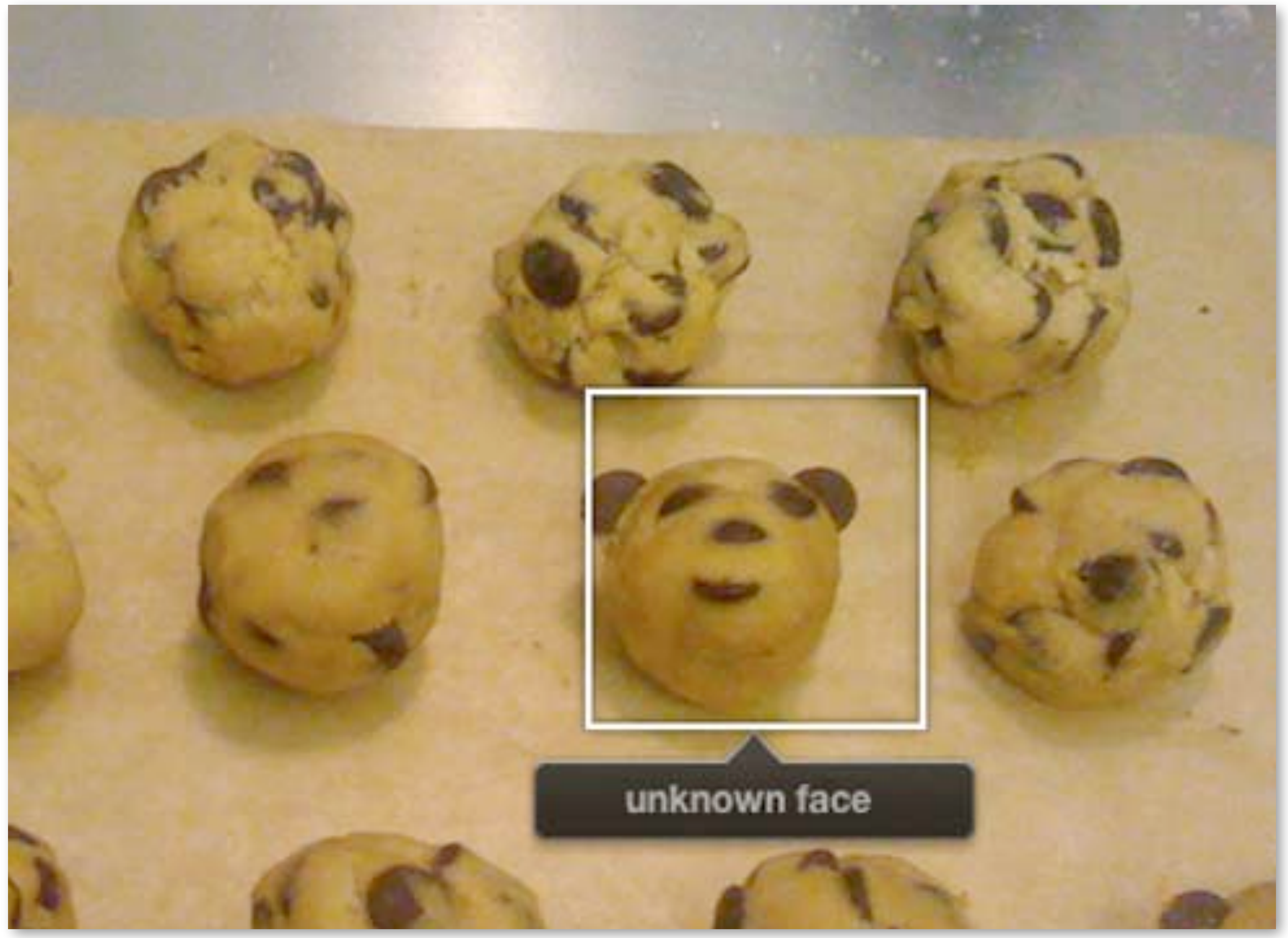
MLSP

- MLSP combines both disciplines to perform learning on signals data

- Many examples of MLSP in the real-world

Face recognition

- Found in cameras and photo software

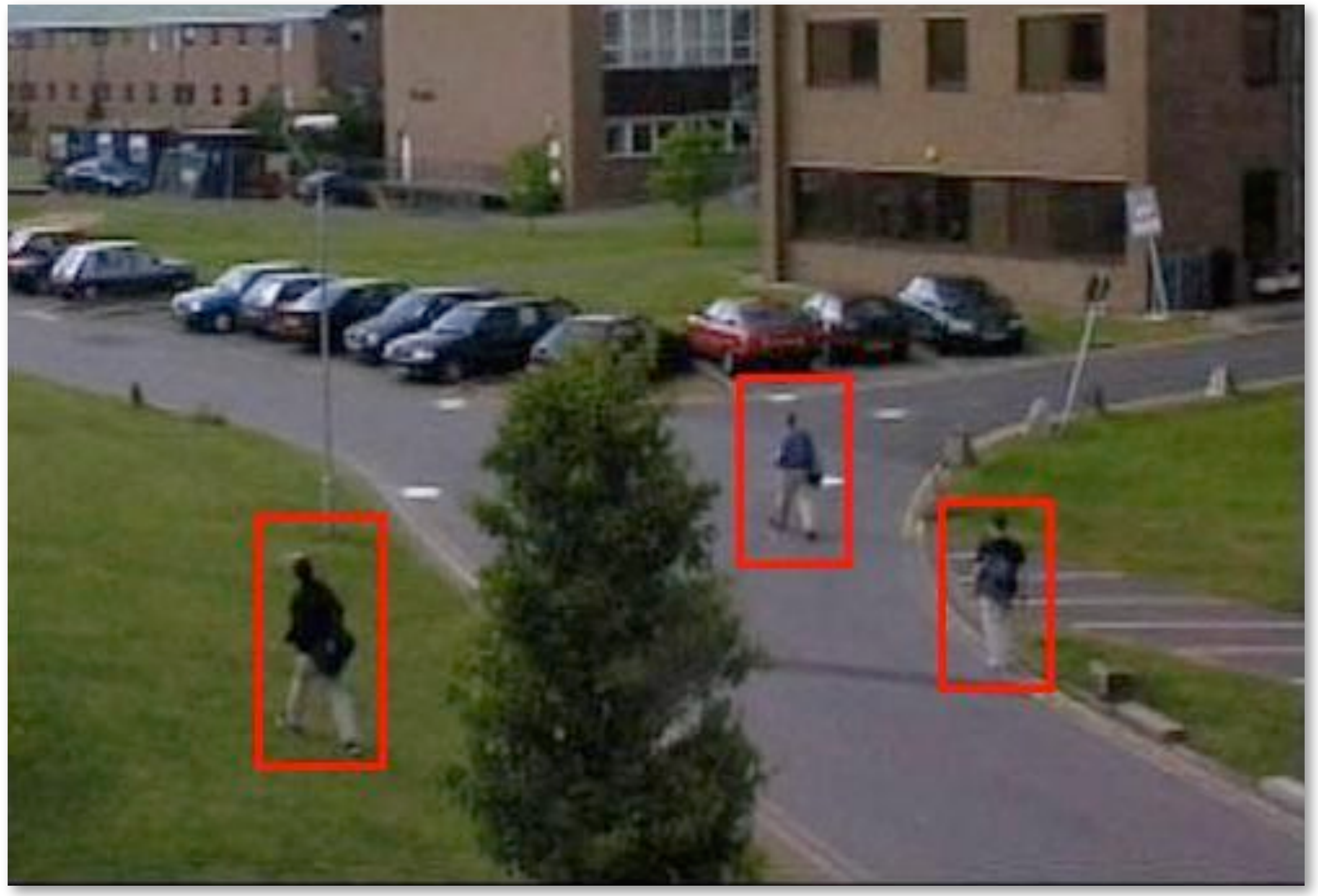


Speech recognition

Surveillance

- Detection of specified objects / activity

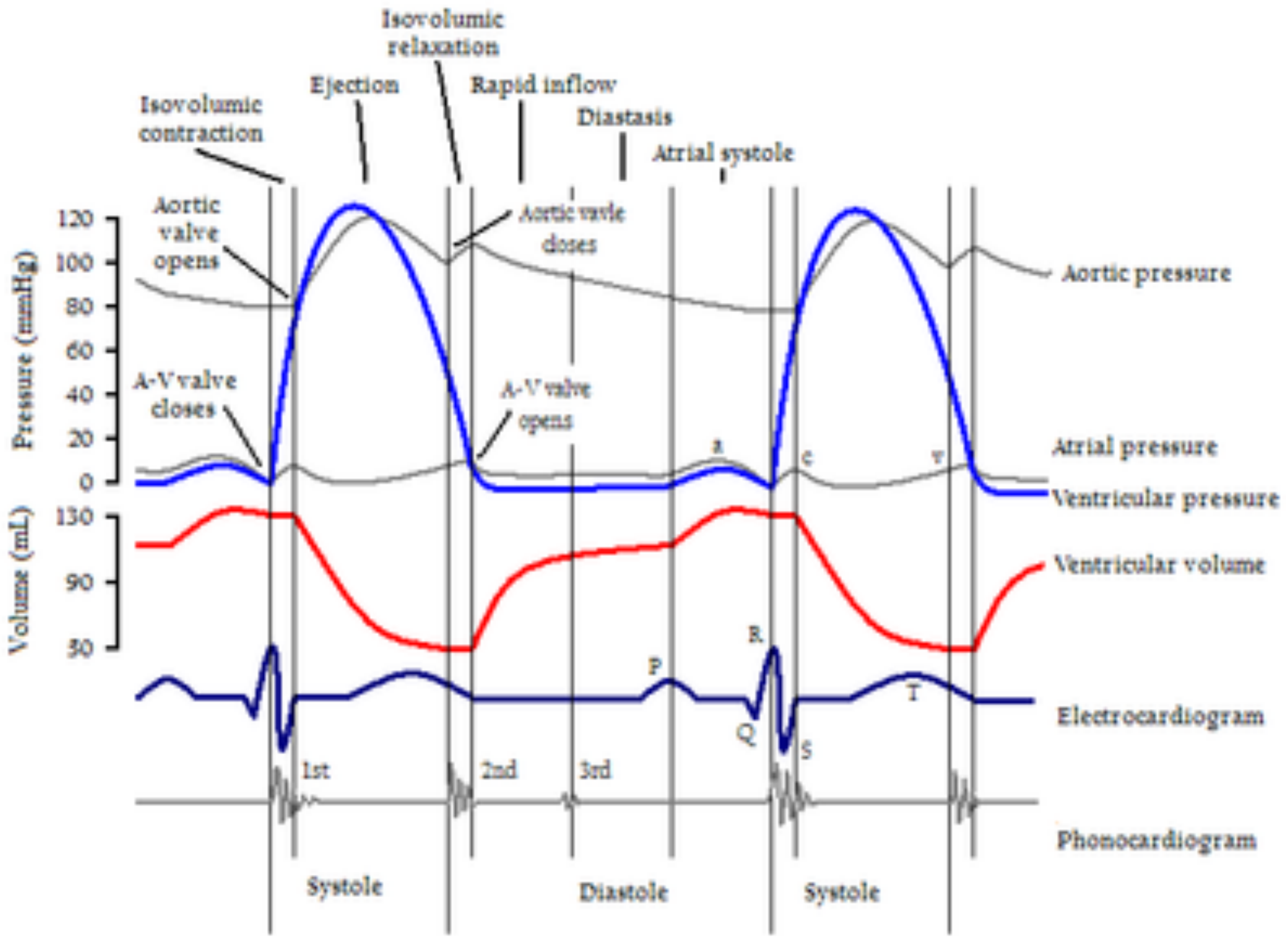
Gunshot detection



Pedestrian detection

Bio-signals

- Interpreting our body's data



Heartbeat processing

Brain Machine Interfaces

Many other applications

- Biometrics
- Music Information Retrieval
- Robotics
- Gesture-based UIs
- Condition monitoring (power grids, automotive, ...)
- Financial data mining
- Many more ...

About this course

- Heavy on practical applications
  - Please bring your own domain problem in class!
- Won't go excruciatingly deep on theory
  - We'll skip convergence proofs, etc.
    - Many more courses here that cover all that
- Our objective is intuition and real-world experience

Syllabus: the basics

- Covering the basics:
  - Part 1: Linear Algebra and Probability
  - Part 2: Signals Theory
  - Part 3: Representations and Features

Syllabus: machine learning review

- Elements of machine learning
  - Part 4: Unsupervised learning basics
  - Part 5: Detection and classification
  - Part 6: Time-series and dynamical models

Syllabus: The fun stuff

- Applications and theory:
  - Matrix Factorizations, Bag Models
  - Manifolds and Embedding
  - Graphical Models
  - Deep Learning
  - ICA, MIMO models, Arrays, Sensor Fusion
  - Compressive Sensing and Sparsity
  - Computer Vision, Speech Recognition
  - Music/Audio Informatics
  - Bio/Brain-signals
  - …

Your part

- Problem sets: 40% of the grade
  - One every two weeks, will be mostly machine problems
- Final project: 50% of the grade
  - Mid-semester: Proposal due
  - Last 1-2 weeks: Presentations and/or posters
- Remaining 10% of grade
  - Show your face in class, ask questions, make sure I know who you are!

The final project is "conference style"

- Teams of 2-3 students (no more than 3, no less than 2)
  - Start making friends now!
- Mid-term: Abstract submission
  - Short abstract describing the problem you want to solve and how you plan to
- Week ~13: Paper Submission
  - 4-6 page paper
  - Peer reviewed by all of you
  - All papers will be accepted!

Communications

- We have a course page:
  - http://courses.engr.illinois.edu/cs598ps
  - Will have lectures, problem sets, data, links, etc.

- We have a piazza.com page
  - Look for CS598PS
  - Use for questions, discussions, finding project-mates, etc.
  - I will be using it for announcements

Who am I?

- Instructor: Paris Smaragdis (CS & ECE)
  - paris@illinois.edu
  - Office: Siebel Center 3231
    - Send me email of you want to meet
- Interested in machine perception, computational audition
- Previously chaired the IEEE MLSP Technical Committee
- Plenty of commercialized experience on the subject

Who are the TAs?

- Cem Subakan
  - subakan2@illinois.edu
- Zesheng Wang
  - zwang180@illinois.edu

- TA office hours at Siebel 0207
  - Wednesdays 5:00-6:00PM

Who are you?

- Name, department, advisor, ugrad?
  - What are your interests in this area?
    - Hint: you will need a project-mate (take notes!)

Final administrative note

- As with every year, this class is oversubscribed
  - If you don't think you will stay until the end please consider dropping the course so that waitlisted students can register
- If you are not formally registered yet, send me your UID
  - I need this for the email list so you get all the announcements

Linear algebra refresher

- Linear algebra is the math tool du jour
  - Compact notation
  - Convenient set operations
  - Used in all modern texts
  - Interfaces well with MATLAB, numpy, R, TensorFlow, etc.

- We will use <u>a lot</u> of it!!

Scalars, Vectors, Matrices, Tensors

Scalar

$$x$$

Vector

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

Matrix

$$\mathbf{X} = \begin{bmatrix} \mathbf{x} & \cdots & \mathbf{x}_N \end{bmatrix} = \begin{bmatrix} x_{1,1} & & x_{N,1} \\ \vdots & \cdots & \vdots \\ x_{1,M} & & x_{N,M} \end{bmatrix}$$

1st dimension

2nd dimension

Tensor

$$\mathbf{X} = \{\mathbf{X}_1, \ldots \mathbf{X}_K\} = \begin{bmatrix} x_{1,1,1} & & x_{N,1,1} \\ \vdots & \cdots & \vdots \\ x_{1,M,1} & & x_{N,M,1} \end{bmatrix} \; \ddots \; \begin{bmatrix} x_{1,1,K} & & x_{N,1,K} \\ \vdots & \cdots & \vdots \\ x_{1,M,K} & & x_{N,M,K} \end{bmatrix}$$

3rd dimension

1st dimension

2nd dimension

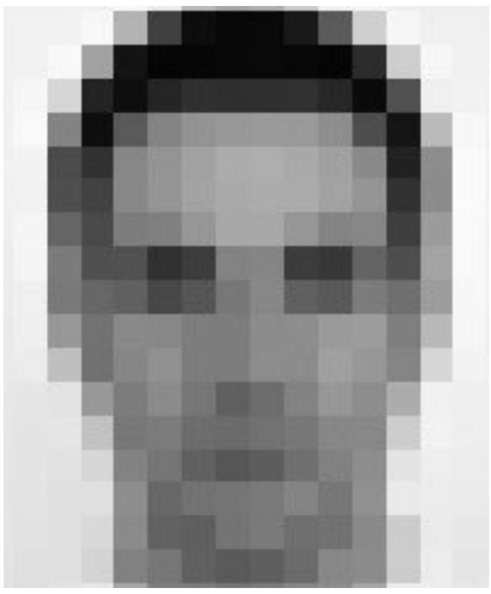
How will we see these?

- 1D signals (e.g. sounds) will be vectors

$$\mathbf{x}^\top = \begin{bmatrix} x(0) & \cdots & x(T) \end{bmatrix} =$$

- 2D signals (e.g. images) will be matrices

$$\mathbf{X} = \begin{bmatrix} x_{1,1} & & x_{N,1} \\ \vdots & \cdots & \vdots \\ x_{1,M} & & x_{N,M} \end{bmatrix} =$$



- 3D data will be videos, etc …

Element-wise operations

- Addition/subtraction

$$\mathbf{a} \pm \mathbf{b} = \mathbf{c} \rightarrow a_i \pm b_i = c_i$$

- Multiplication (Hadamard product)

$$\mathbf{a} \odot \mathbf{b} = \mathbf{c} \rightarrow a_i \, b_i = c_i$$

  - Other times denoted as $\mathbf{a} \circ \mathbf{b}$ or `a.*b`

- No named operator for element-wise division
  - Just use Hadamard with inverted elements

Transpose

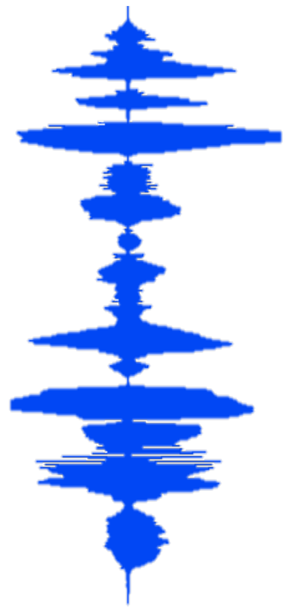
- Transpose
  - Change rows to columns (and vice versa)

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \mathbf{x}^\top = \begin{bmatrix} x_1 & \cdots & x_N \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix}, \quad \mathbf{x}^\top = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

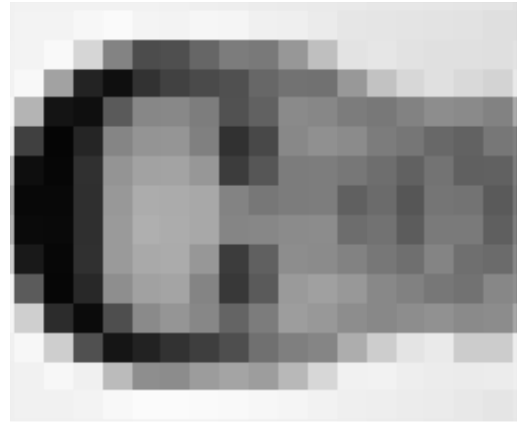
- Hermitian (conjugate transpose)
  - Notated as $\mathbf{X}^{\mathrm{H}}$
    - MATLAB note: `x'` is Hermitian transpose, `x.'` is transpose

Visualizing transposition

- Mostly pointless for 1D signals

$\mathbf{x} =$ , $\mathbf{x}^\top =$ 

- Swap dimensions for 2D signals

$\mathbf{x} =$ , $\mathbf{x}^\top =$ 

Reshaping operators

- The vec operator
  - Unrolls elements column-wise
  - Useful for getting rid of matrices/tensors

$$\text{vec}(\mathbf{x}) = \text{vec}\begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

- The vec-transpose
  - For (p) of an $m \times n$ matrix make each column a $p \times (m/p)$ matrix
  - Useful for inverting vec and getting rid of tensors $\mathbf{X} = \text{vec}(\mathbf{X})^{(M)}, \mathbf{X} \in \mathbb{R}^{M \times N}$

$$\mathbf{x}^{(2)} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \\ x_{41} & x_{42} \\ x_{51} & x_{52} \\ x_{61} & x_{62} \end{bmatrix}^{(2)} = \begin{bmatrix} x_{11} & x_{31} & x_{51} \\ x_{21} & x_{41} & x_{61} \\ x_{12} & x_{32} & x_{52} \\ x_{22} & x_{42} & x_{62} \end{bmatrix}$$

Trace and diag

- Matrix trace
  - Sum of diagonal elements

$$\mathrm{tr}(\mathbf{X}) = \mathrm{tr}\begin{bmatrix} x_{11} & \cdots & x_{1N} \\ \vdots & \ddots & \vdots \\ x_{N1} & \cdots & x_{NN} \end{bmatrix} = \sum_i x_{ii}$$

- The diag operator

$$\mathrm{diag}(\mathbf{x}) = \mathrm{diag}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}$$

$$\mathrm{diag}^{-1}\begin{bmatrix} x_1 & a \\ b & x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

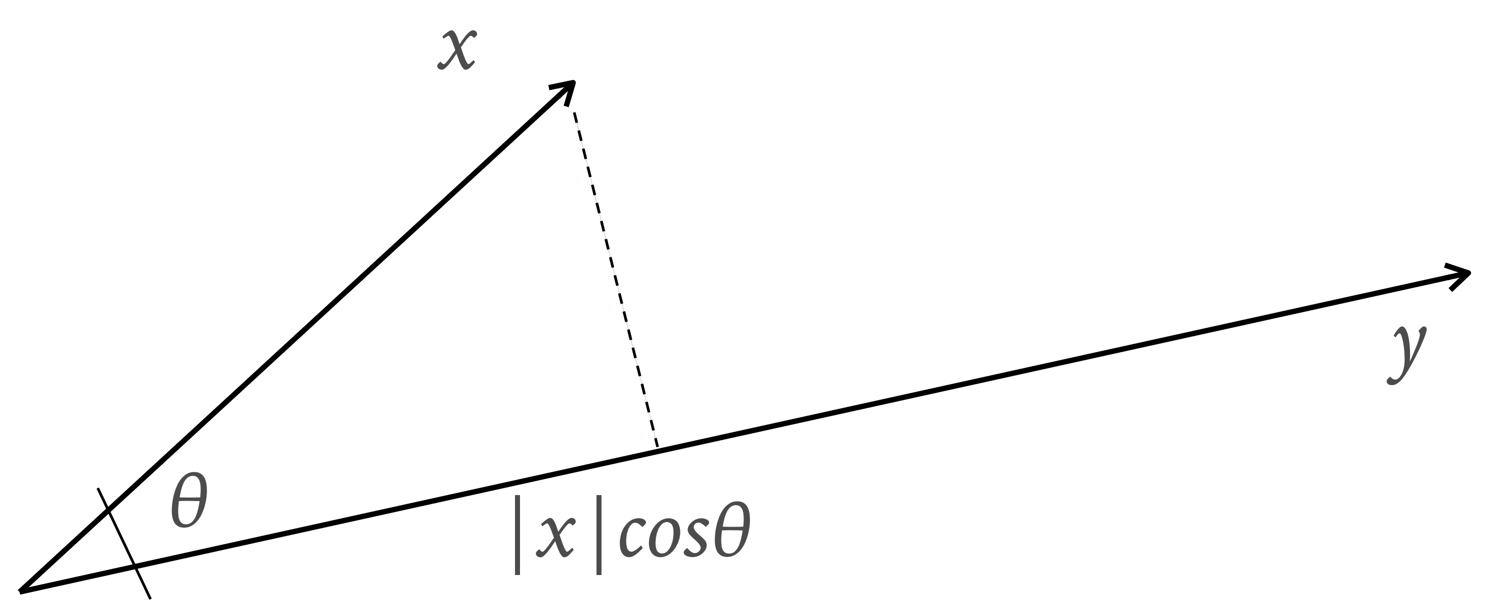
The dot product

- Dot product
  - Shorthand for multiply and accumulate

$$\mathbf{x}^\top \cdot \mathbf{y} = \sum_i x_i \cdot y_i = |\mathbf{x}| \cdot |\mathbf{y}| \cos\theta$$

- Geometry
  - For unit vectors:

$$\theta = \arccos(\mathbf{x}^\top \cdot \mathbf{y})$$



  - Great tool for checking out similarity

Matrix-vector product

- Generalizing the dot product

$$\mathbf{X}\cdot\mathbf{y} = \begin{bmatrix} \mathbf{x}_1^\top \\ \mathbf{x}_2^\top \\ \mathbf{x}_3^\top \end{bmatrix} \cdot \mathbf{y} = \begin{bmatrix} \mathbf{x}_1^\top \cdot \mathbf{y} \\ \mathbf{x}_2^\top \cdot \mathbf{y} \\ \mathbf{x}_3^\top \cdot \mathbf{y} \end{bmatrix} = \begin{bmatrix} \sum x_{1,i} \cdot y_i \\ \sum x_{2,i} \cdot y_i \\ \sum x_{2,i} \cdot y_i \end{bmatrix}$$

  - $\mathbf{X}$ must have as many columns a $\mathbf{y}$ has elements
    - Non-commutative!

- Useful for computing multiple dot products
  - Pack all vectors that you want to multiply in a matrix

Matrix-matrix product

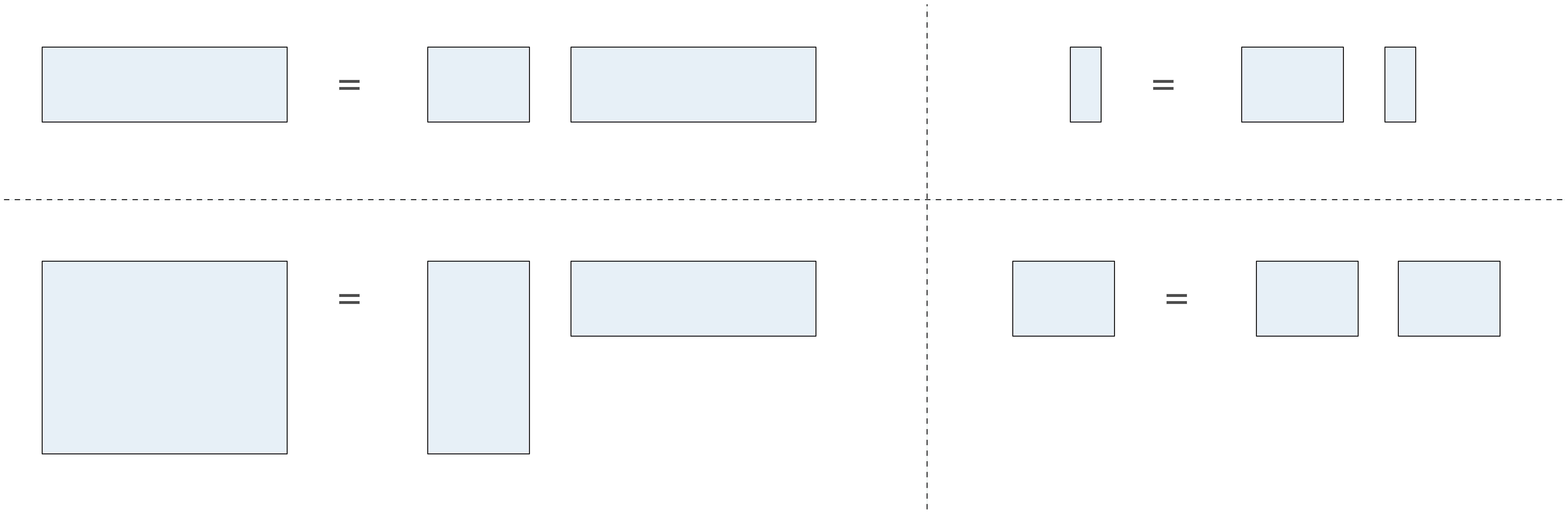
- Between two matrices:

$$\mathbf{X}\cdot\mathbf{Y} = \begin{bmatrix} \mathbf{x}_1^\top \\ \mathbf{x}_2^\top \\ \mathbf{x}_3^\top \end{bmatrix} \cdot \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1^\top\cdot\mathbf{y}_1 & \mathbf{x}_1^\top\cdot\mathbf{y}_2 & \mathbf{x}_1^\top\cdot\mathbf{y}_3 \\ \mathbf{x}_2^\top\cdot\mathbf{y}_1 & \mathbf{x}_2^\top\cdot\mathbf{y}_2 & \mathbf{x}_2^\top\cdot\mathbf{y}_3 \\ \mathbf{x}_3^\top\cdot\mathbf{y}_1 & \mathbf{x}_3^\top\cdot\mathbf{y}_2 & \mathbf{x}_3^\top\cdot\mathbf{y}_3 \end{bmatrix}$$

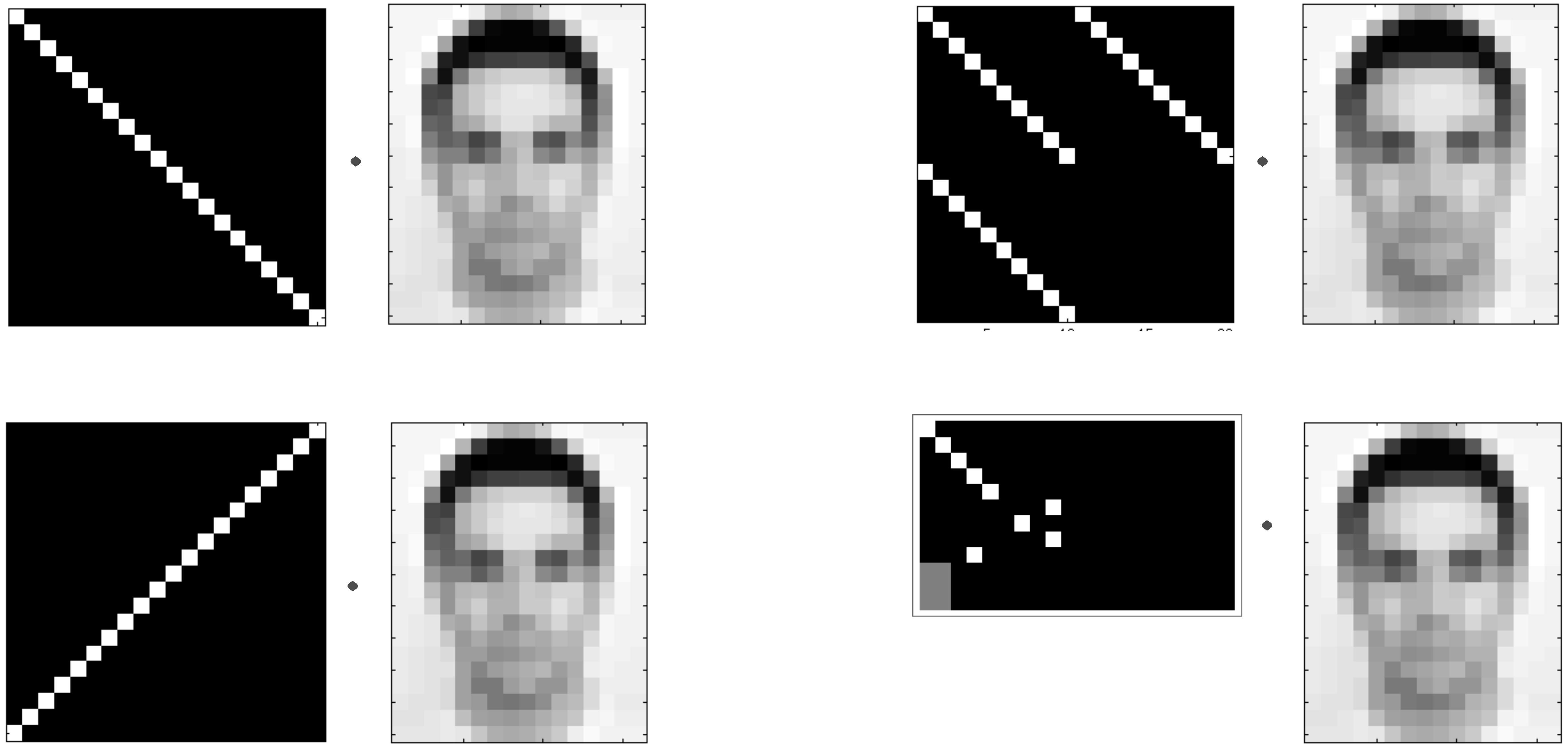
  - $\mathbf{X}$ must have as many columns as $\mathbf{Y}$ has rows
    - $(M\times N) = (M\times K)\cdot(K\times N)$
    - Remember this doesn’t commute!
- All linear operations can be represented as a matrix product
  - We’ll be seeing that a lot!

Matrix products

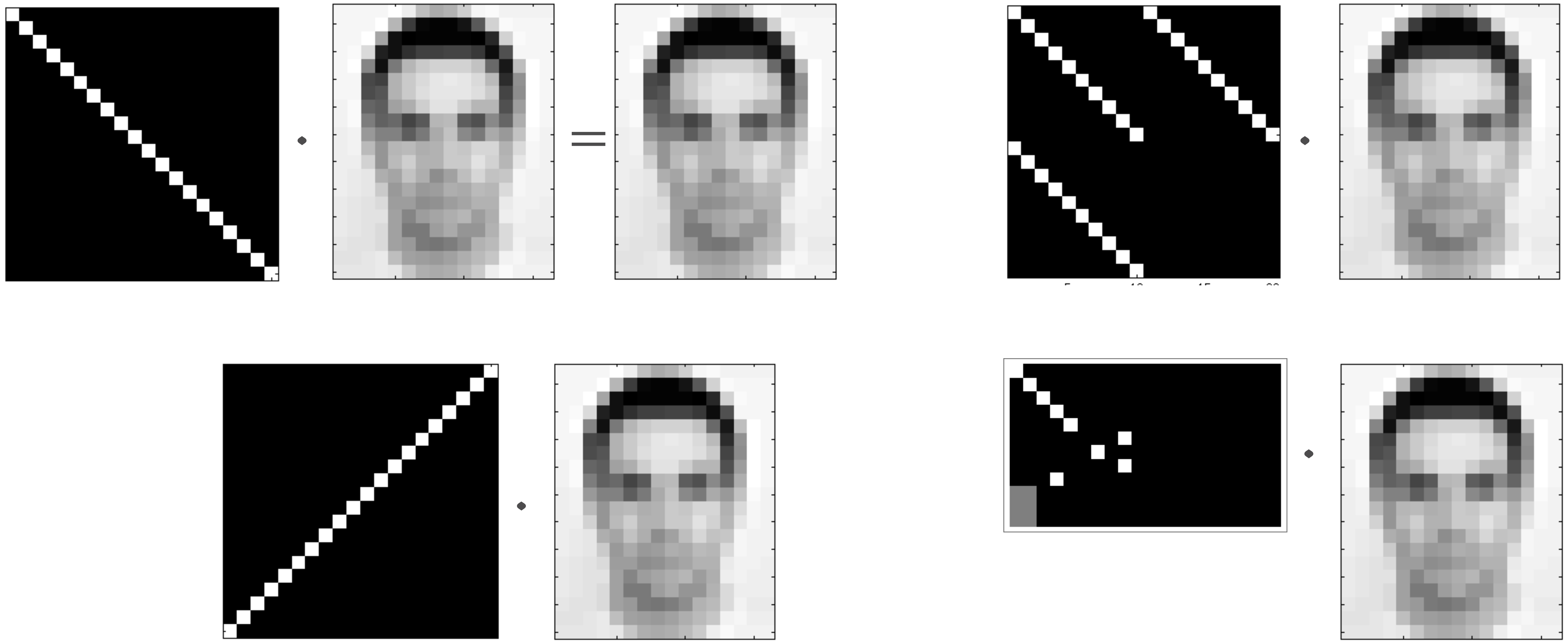
- Output rows == left matrix rows
- Output columns == right matrix columns



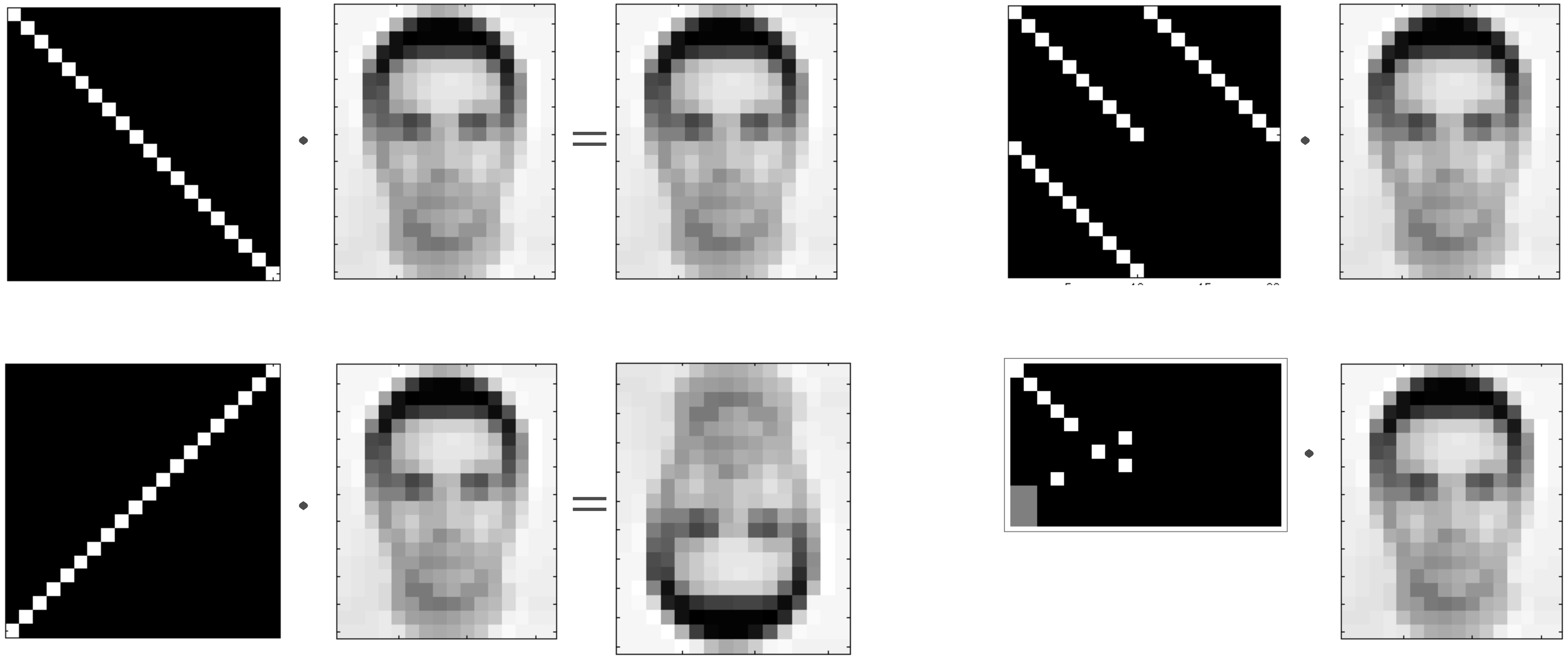
Visualizing the matrix product



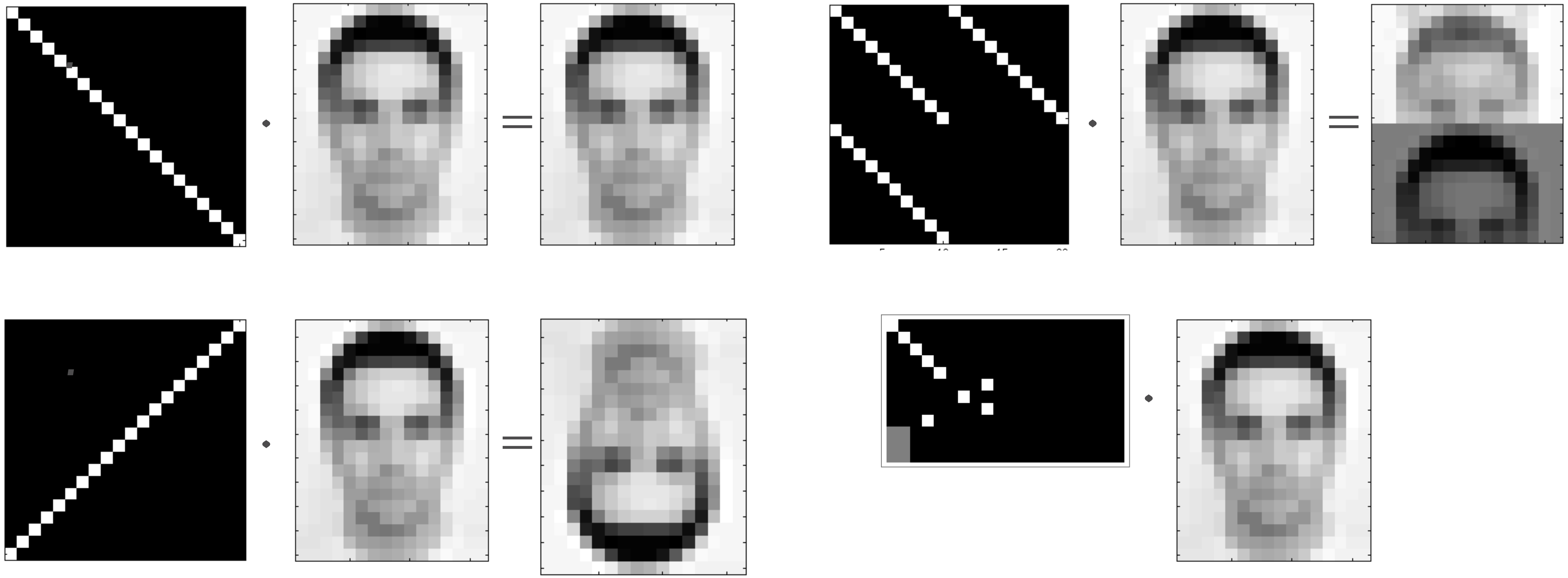
Visualizing the matrix product



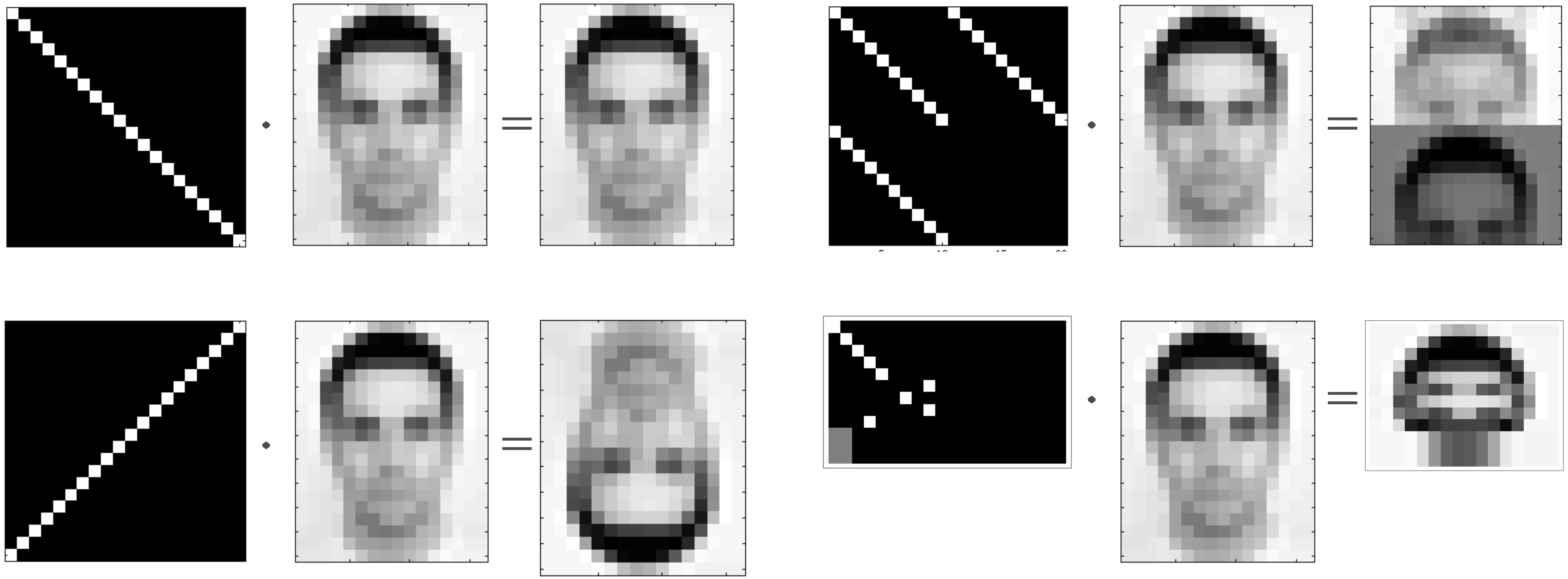
Visualizing the matrix product



Visualizing the matrix product



Visualizing the matrix product



Norms

- 2-norm:

$$\left\|\mathbf{x}\right\| = \sqrt{\sum x^2_i}$$

- p-norms:

$$\left\|\mathbf{x}\right\|_p = \left(\sum_i \left|x_i\right|^p\right)^{1/p}$$

- Frobenius norm:

$$\left\|\mathbf{X}\right\|_F = \sqrt{\operatorname{tr}\left(\mathbf{X}^\top \cdot \mathbf{X}\right)} = \sqrt{\operatorname{tr}\left(\begin{bmatrix} \mathbf{x}_1^\top \\ \mathbf{x}_2^\top \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}\right)} = \sqrt{\operatorname{tr}\left(\begin{bmatrix} \mathbf{x}_1^\top \cdot \mathbf{x}_1 & \mathbf{x}_1^\top \cdot \mathbf{x}_2 \\ \mathbf{x}_2^\top \cdot \mathbf{x}_1 & \mathbf{x}_2^\top \cdot \mathbf{x}_2 \end{bmatrix}\right)}$$

Kronecker product

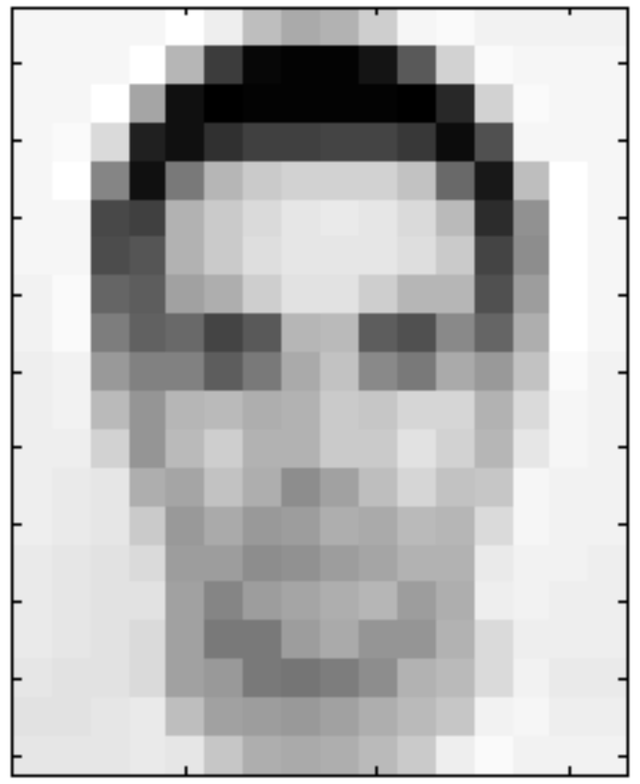
- A bit more complex
  - Replicate and multiply right matrix with each scalar of left matrix

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \otimes \mathbf{Y} = \begin{bmatrix} x_{11} \cdot \mathbf{Y} & x_{12} \cdot \mathbf{Y} \\ x_{21} \cdot \mathbf{Y} & x_{22} \cdot \mathbf{Y} \end{bmatrix}$$

- Useful result:

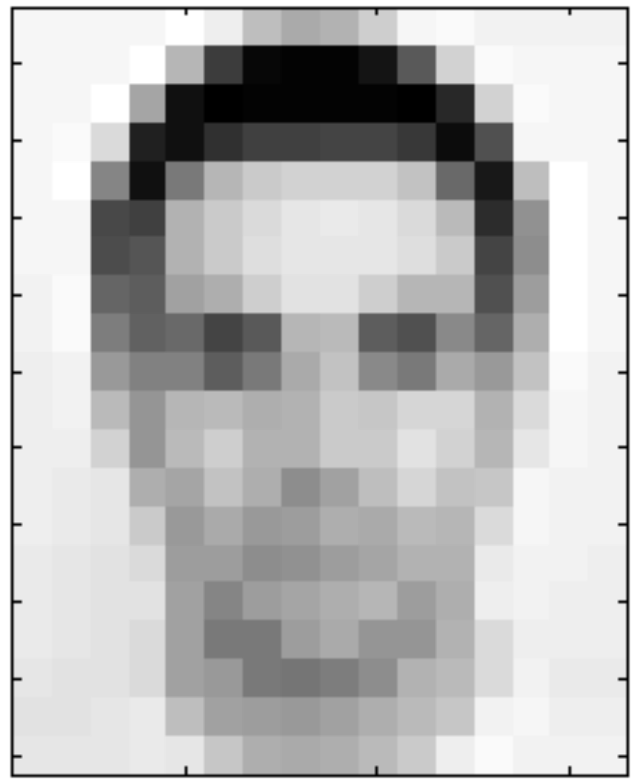
$$\operatorname{vec}(\mathbf{X} \cdot \mathbf{Y} \cdot \mathbf{Z}) = (\mathbf{Z}^\top \otimes \mathbf{X}) \operatorname{vec}(\mathbf{Y})$$

Visualizing Kronecker

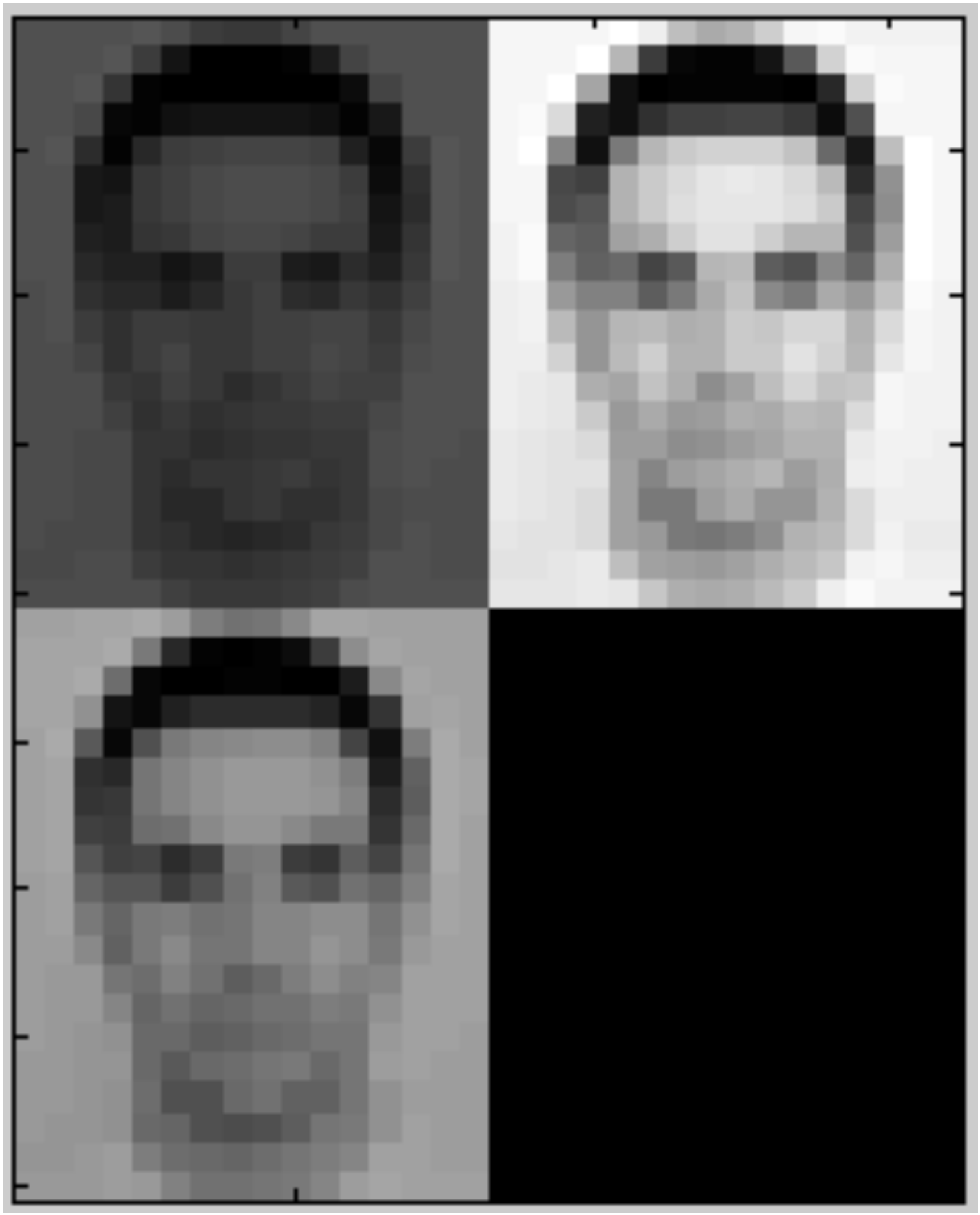
$$\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} \otimes$$  $$= ?$$

Visualizing Kronecker

$$\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} \otimes$$

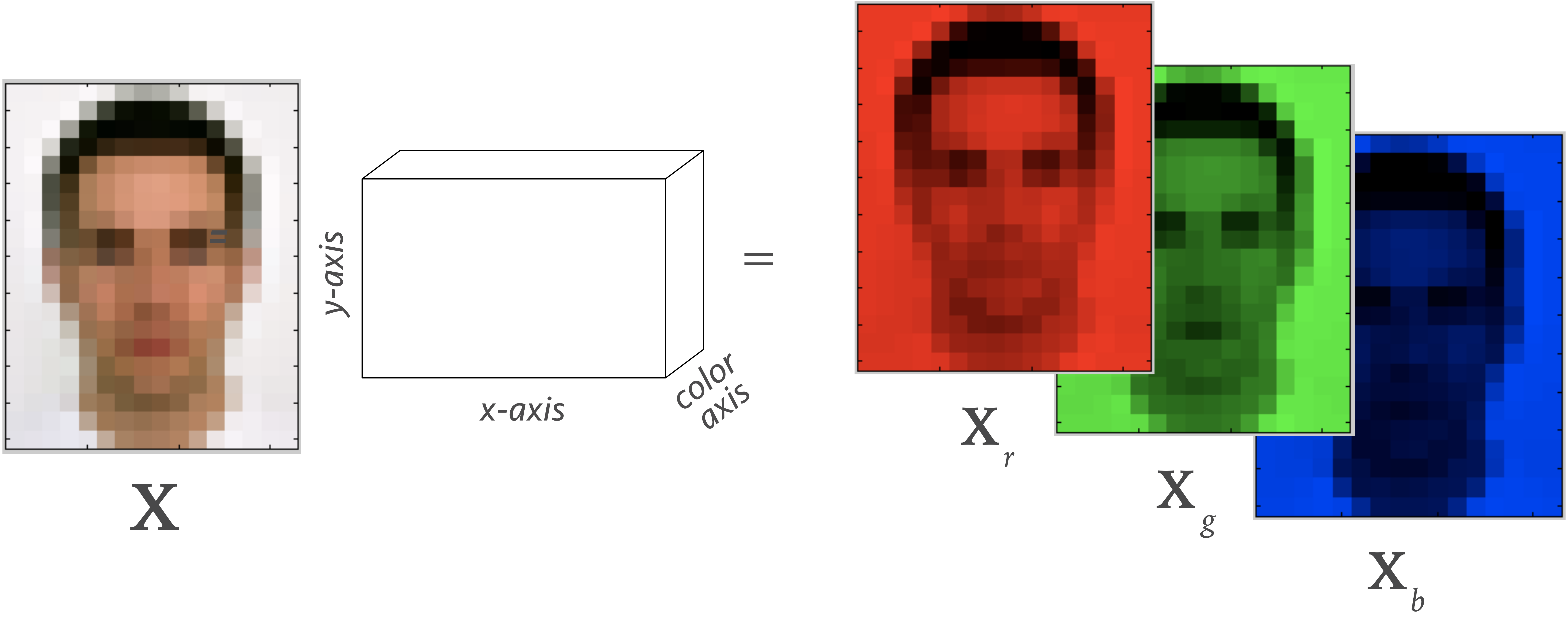


$$=$$



Dealing with tensors

- Using Kronecker products and the vec operator we can perform *multilinear transforms*
  - Tensor example with RGB images:



Mixing the colors

- Define color, horizontal, and vertical mixing

$$\Big(\underbrace{\mathbf{C}^\top}_{3^{rd}\ dim} \otimes \underbrace{\mathbf{H}^\top}_{2^{nd}\ dim} \otimes \underbrace{\mathbf{V}^\top}_{1^{st}\ dim}\Big)\cdot \mathrm{vec}(\mathbf{X})$$

- Example: color mixing

$$\left(\mathrm{diag}\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \otimes \mathbf{I} \otimes \mathbf{I}\right)\mathrm{vec}(\mathbf{X}) =$$

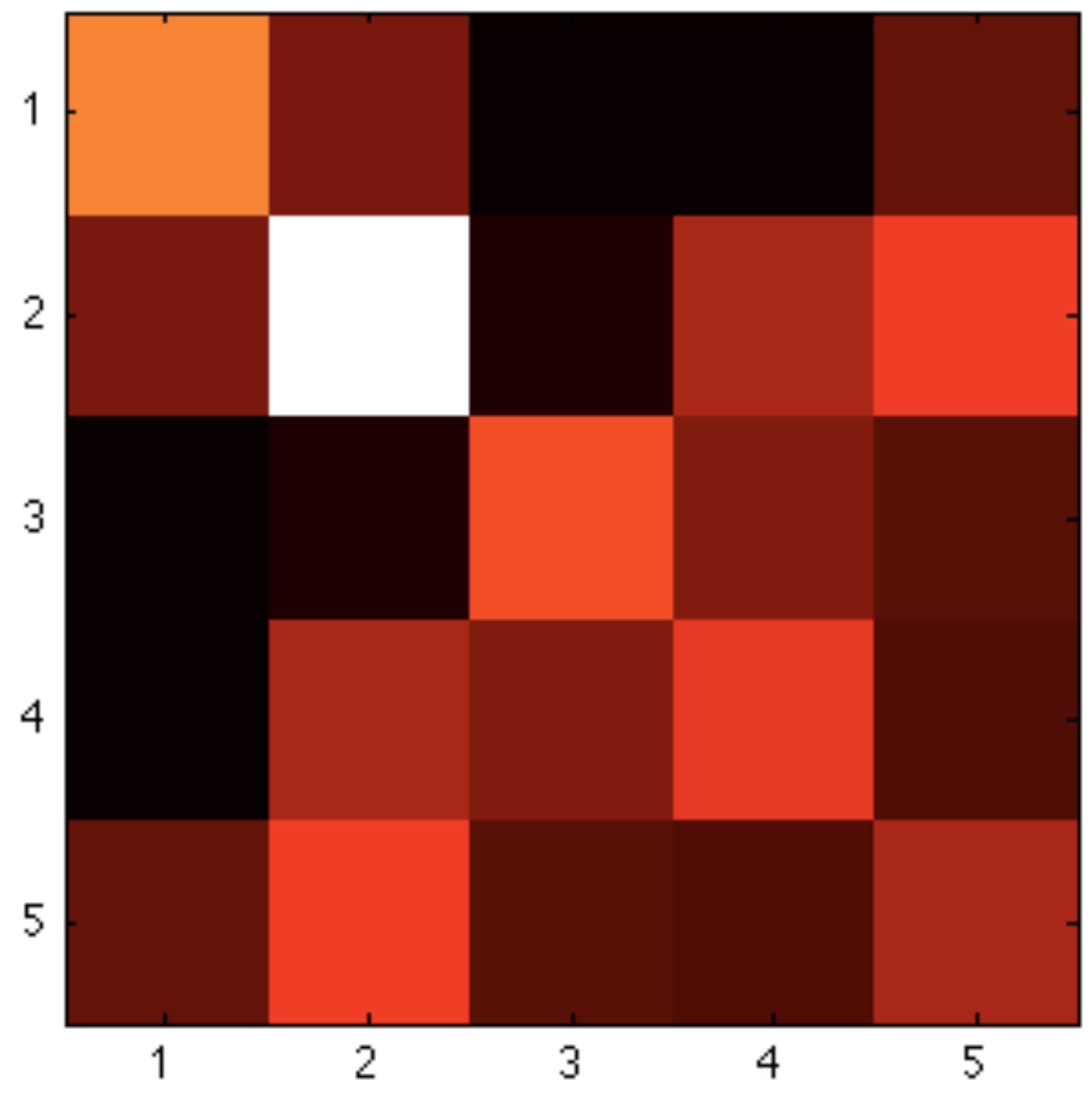
  - Caution: Dimensions will quickly get out of hand this way

Special matrices

- Symmetric:

$$\mathbf{X} = \mathbf{X}^\top \Rightarrow x_{ij} = x_{ji}$$

- Positive definite
  - Is so if: $\mathbf{y}^\top \cdot \mathbf{X} \cdot \mathbf{y} > 0, \ \forall \ \mathbf{y}$
  - Also symmetric



- Orthonormal:

$$\mathbf{X}^\top \cdot \mathbf{X} = \mathbf{X} \cdot \mathbf{X}^\top = \mathbf{I}$$

Matrix inverse

- "Undoes" a matrix multiplication
  - Only for square matrices
  - Not all matrices are invertible
    - must be a full-rank matrix

$$\mathbf{X}^{-1} \cdot \mathbf{X} = \mathbf{I}$$

$$\mathbf{X}^{-1} \cdot \mathbf{X} \cdot \mathbf{Y} = \mathbf{Y}$$

$$\mathbf{Y} \cdot \mathbf{X} \cdot \mathbf{X}^{-1} = \mathbf{Y}$$

- Remember, in matrix multiplication order matters

$$\mathbf{X} \cdot \mathbf{Y} \cdot \mathbf{X}^{-1} \neq \mathbf{Y}$$

Matrix pseudoinverse

- Also known as Moore-Penrose (or MP) pseudoinverse
  - For an $m \times n$ matrix $\mathbf{X}$, pseudoinverse is $n \times m$ matrix $\mathbf{X}^+$

$$
\begin{aligned}
&\mathbf{X} \cdot \mathbf{X}^+ \cdot \mathbf{X} = \mathbf{X} \\
&\mathbf{X}^+ \cdot \mathbf{X} \cdot \mathbf{X}^+ = \mathbf{X}^+ \\
&(\mathbf{X} \cdot \mathbf{X}^+)^\top = \mathbf{X} \cdot \mathbf{X}^+ \\
&(\mathbf{X}^+ \cdot \mathbf{X})^\top = \mathbf{X}^+ \cdot \mathbf{X}
\end{aligned}
$$

  - We'll be seeing this operation a lot, it's essentially least squares

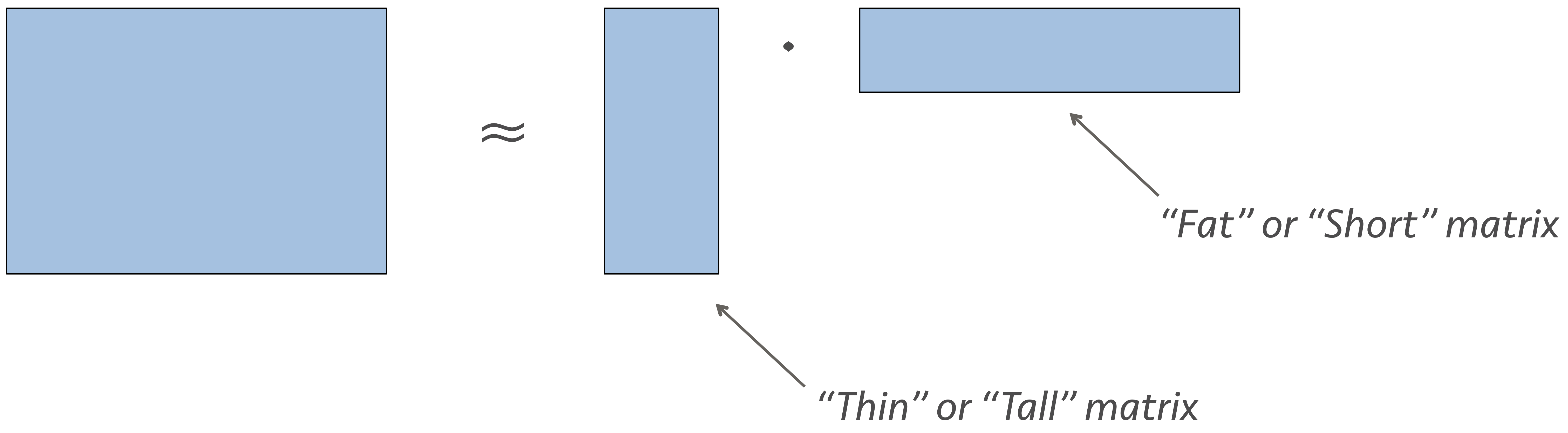
$$\mathbf{A} \cdot \mathbf{x} = \mathbf{y} \rightarrow \mathbf{x} = \mathbf{A}^+ \cdot \mathbf{y}$$

Low rank approximations

- Use smaller matrices to describe a large matrix

$$\mathbf{Y} \approx \mathbf{A} \cdot \mathbf{X}$$

  - With $\mathbf{Y}$ being $m \times n$, $\mathbf{A}$ being $m \times r$, $\mathbf{X}$ being $r \times n$, and $r < m$



Eigenanalysis

- Eigenvectors and eigenvalues
  - For an $n \times n$ matrix $\mathbf{X}$

$$\mathbf{X} \cdot \mathbf{V} = \mathbf{V} \cdot \mathbf{L}$$
$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_N \end{bmatrix}$$
$$\mathbf{L} = \mathrm{diag}\begin{bmatrix} \lambda_1 & \cdots & \lambda_N \end{bmatrix}$$

  - $\mathbf{V}$ is $m \times n$ and contains the eigenvectors $\mathbf{v}_i$
    - It will be an orthogonal matrix for positive (semi-)definite matrix inputs
  - $\mathbf{L}$ is $n \times n$ contains the eigenvalues λ_i

The Singular Value Decomposition (SVD)

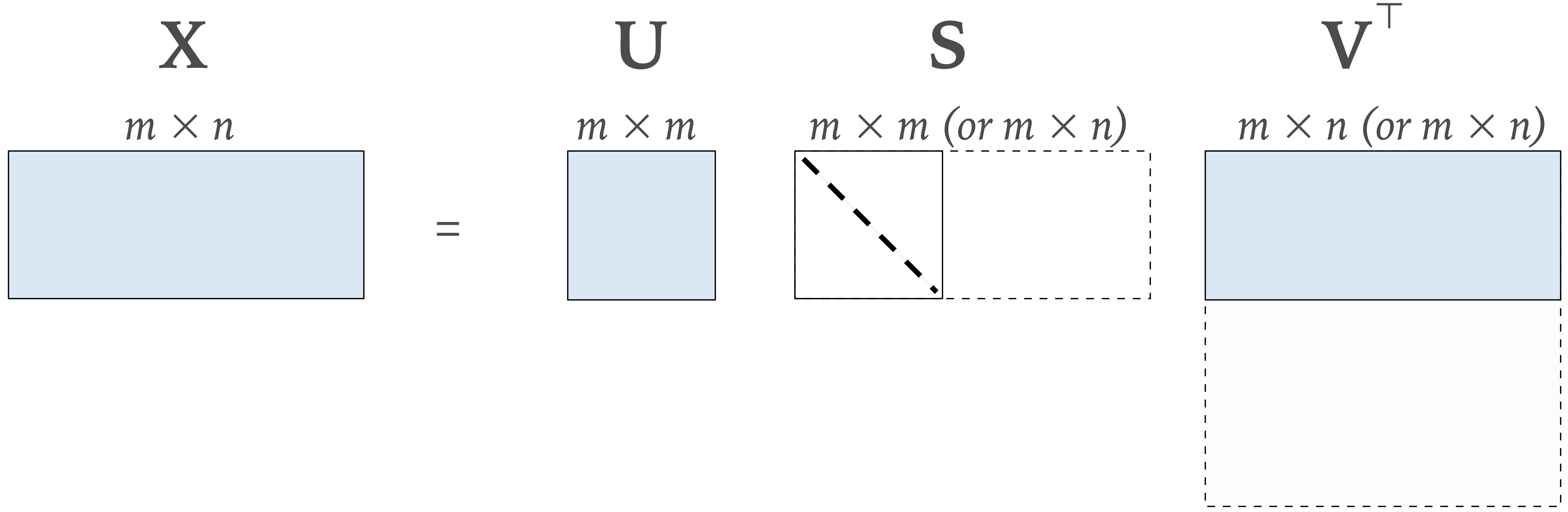
- Similar decomposition to eigenanalysis
  - For a matrix $\mathbf{X}$

$$\mathbf{X} = \mathbf{U} \cdot \mathbf{S} \cdot \mathbf{V}^\top \quad \text{where} \quad \begin{array}{l} \mathbf{U}^\top \cdot \mathbf{U} = \mathbf{I} \\ \mathbf{V}^\top \cdot \mathbf{V} = \mathbf{I} \\ \mathbf{S} = \mathrm{diag}(\sigma_i) \end{array}$$

  - $\mathbf{U}$, $\mathbf{V}$ are orthonormal
    - Contain the left and right singular vectors of $\mathbf{X}$
  - $\mathbf{S}$ is diagonal
    - Contains on its diagonal the singular values σ_i

Visualizing the SVD

- Comes in two versions, *full* and *economy*
  - The only difference is the size of the matrices
    - The numerical approximation is the same
  - By truncating the columns of $\mathbf{S}$ we can make a low-rank approx.



Recap

- What's this class about?
  - Signals, learning, fun, etc …
- Linear algebra basics
  - Algebraic operations, norms, decompositions, form manipulations

Finale

- Skim through this material for now
  - We'll be seeing it in context soon
    - e.g., what are the eigenvectors of image matrices?

- Reading material
  - Old and new algebra useful for statistics
    - http://research.microsoft.com/en-us/um/people/minka/papers/matrix/
  - The Matrix Cookbook
    - http://www2.imm.dtu.dk/pubdb/views/publication_details.php?id=3274

Questions?

- Slides will be online soon after all lectures at:
  - http://courses.engr.illinois.edu/cs598ps/

Next lecture

- Review of:
  - Probability theory
    - Bayes theorem, probability rules, Bayes nets
    - Basic distributions, transformations of RV's
  - Statistics
    - Basic measures, independence, information
  - Parameter estimation intro
    - Maximum likelihood, MAP, Bayesian, EM intro