Notes on Random Quantum Circuit

Richard Liu*

*Department of Physics, National University of Singapore (NUS)

October 5, 2023

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1 Introduction to Haar Random Circuit

What is a random quantum circuit? It's the minimal quantum many-body system model of discrete spacetime that captures the generic unitary chaotic dynamics far from its ground state, which makes it possible for us to understand the process of thermalization and the many-body dynamical evolution that can arise in platforms for quantum computation/simulation. Here we only consider the two-qudit gates with q local degrees of freedoms, which is the natural representation of the nearest interaction of manybody systems.

The randomness comes from the fact that each gate is chosen from an ensemble of gates with specific probability distribution, which gives an probabilistic measure of some space. The philosophy behind is similar to the idea in random matrix theory and the SYK model. The most generic dynamics of brickwork circuit have the property of *no specifically favorable choice of basis*, which means the ensemble remains the same if we choose different basis. Mathematically, this means:

$$\mathbb{E}_{\text{observable}} f(U) = \mathbb{E}_{\text{observable}} f(VUW), \forall V, W \in \mathbf{U}(q^2)$$

If there's no restriction on the gate, i.e. U can be any unitary matrix of $\mathbf{U}(q^2)$, Mathematicians tell us that there exist a unique ensemble with *Haar measure* that suffice the requirement of left/right invariance:

$$\mathbb{E}_{U\sim\mu_H} f(U) := \int f(U) d_{\mu_H} U = \int f(VUW) d_{\mu_H} U, \forall V, W \in \mathbf{U}(q^2) \tag{1}$$

There are several properties that this measure holds:

$$\mathbb{E}_{U\sim\mu_H}[U^{\otimes k_1} \otimes U^{*\otimes k_2}] = 0, k_1 \neq k_2 \tag{2}$$

$$\mathbb{E}_{U\sim\mu_H} f(U^\dagger) = \mathbb{E}_{U\sim\mu_H} f(U) \tag{3}$$

Generically, we now consider $U \in \mathbf{U}(d)$ Thus we can define the k-moment operator $\mathcal{M}_k$ which operates on the space $\mathcal{L}\left(\left(\mathbb{C}^d\right)^{\otimes k}\right)$:

$$\mathcal{M}^{(k)}_{\mu_H}(O) := \mathbb{E}_{U\sim\mu_H}[U^{\otimes k} O U^{\dagger\otimes k}] \tag{4}$$

We can prove the k-moment operator holds the following properties:

- It is linear, trace-preserving, and self-adjoint with respect to the Hilbert-Schmidt inner product;
- For all $O \in \mathcal{L}\left(\left(\mathbb{C}^d\right)^{\otimes k}\right), B \in \mathcal{L}\left(\mathbb{C}^d\right), [\mathcal{M}_{\mu_H}^{(k)}(O), B^{\otimes k}] = 0$;
- If for $\forall B \in \mathcal{L}\left(\mathbb{C}^d\right), O \in \mathcal{L}\left(\left(\mathbb{C}^d\right)^{\otimes k}\right)$ satisfies $[\mathcal{M}_{\mu_H}^{(k)}(O), B^{\otimes k}] = 0$, then $\mathcal{M}_{\mu_H}^{(k)}(O) = O$.

Notice the commutation relationship is of vital importance to k-moment operator, we define *k-th order commutant* for given $S \subseteq \mathcal{L}\left(\mathbb{C}^d\right)$ as:

$$\text{Comm}(S, k) := \{O \in \mathcal{L}\left(\left(\mathbb{C}^d\right)^{\otimes k}\right) | [O, B^{\otimes k}] = 0\} \tag{5}$$

Apparently, $\mathcal{M}_{\mu_H}^{(k)}(\cdot)$ is the projector to $\text{Comm}(\mathbf{U}(d), k)$. To get a glimpse of the power of Haar measure, we deviate a little to derive the structure of the k-th order commutant.

Firstly, we define the permutation operators in $\mathcal{L}\left(\left(\mathbb{C}^d\right)^{\otimes k}\right)$. Given $\pi \in S_k$ an element of the symmetric group S_k, we define the permutation operator $V_d(\pi)$ as:

$$V_d(\pi) \left|\psi_1\right\rangle \otimes \cdots \otimes \left|\psi_k\right\rangle = \left|\psi_{\pi(1)}\right\rangle \otimes \cdots \otimes \left|\psi_{\pi(k)}\right\rangle, \left|\psi_i\right\rangle \in \mathbb{C}^d \tag{6}$$

Equivalently, we can write the permutation matrix as:

$$V_d(\pi) = \sum_{i_1, \ldots, i_k \in [d]^k} \left|i_{\pi(1)}, \ldots, i_{\pi(k)}\right\rangle \left\langle i_1, \ldots, i_k\right| \tag{7}$$

We can check that:

$$V_d(\sigma)V_d(\pi) = \sum_{i_1, \ldots, i_k \in [d]^k} \left|i_{\sigma(\pi(1))}, \ldots, i_{\sigma(\pi(k))}\right\rangle \left\langle i_1, \ldots, i_k\right| = V_d(\sigma\pi) \tag{8}$$

$$V_d(\pi^{-1}) = V_d(\pi)^\dagger \tag{9}$$

$$V_d(\pi)(A_1 \otimes \cdots \otimes A_k)V_d(\pi)^\dagger = A_{\pi_1} \otimes \cdots \otimes A_{\pi_k} \tag{10}$$

From equation (10), we can see that $V_d(\pi) \in \text{Comm}(\mathbf{U}(d), k)$. **Schur-Weyl duality** presented below tells us exactly how important are the permutation operators to the k-th order commutant.

(Schur-Weyl duality)

The k-th order commutant of the unitary group is the span of the permutation operators associated to S_k.

$$\text{Comm}(\mathbf{U}(d), k) = \text{Span}\left(V_d(\pi), \pi \in S_k\right) \tag{11}$$

What should be noticed here is that we only require the commutant is spaned by the permutation operators, the operators themselves are not necessarily linearly independent. We can prove that the permutation operators are linearly independent if $k \leq d$, but linearly dependent if $k > d$. Nevertheless, we can't require the permutation operators to be orthonormal.

So up to now, we've seen the permutation operators gives the structure of commutant, and the k-moment operator given by Haar measure is a projector from $\mathcal{L}\left(\mathbb{C}^d\right)$ to $\text{Comm}(\mathbf{U}(d), k)$. So the next natural question should be: can we compute the projected operator for any O? The answer is yes.

$$\mathbb{E}_{U\sim\mu_H}[U^{\otimes k}OU^{\dagger\otimes k}] = \sum_{\pi,\sigma\in S_k} \text{Wg}(\pi^{-1}\sigma, d)\text{Tr}(V_d(\sigma)^\dagger O)V_d(\pi) \tag{12}$$

It tells us that even if the gates are chosen randomly from the ensemble, utimately we can get some none-vanishing contribution as a series of permutation operators.

Detailed proof:

Since the $\mathcal{M}^{(k)}_{\mu_H}(O)$ lies inside Span $(V_d(\pi), \pi \in S_k)$, then:

$$\mathcal{M}^{(k)}_{\mu_H}(O) = \sum_{\pi\in S_k} c_\pi(O)V_d(\pi)$$

Multiply both sides by $V_d(\sigma)^\dagger$ and take the trace:

$$\begin{aligned}\sum_{\pi\in S_k} c_\pi(O)\text{Tr}(V_d(\sigma)^\dagger V_d(\pi)) &= \text{Tr}(V_d(\sigma)^\dagger \mathbb{E}_{U\sim\mu_H}[U^{\otimes k}OU^{\dagger\otimes k}]) \\ &= \text{Tr}(\mathbb{E}_{U\sim\mu_H}[U^{\otimes k}V_d(\sigma)^\dagger OU^{\dagger\otimes k}]) \quad \Longleftarrow V_d(\sigma)^\dagger \in \text{Comm}(\mathbf{U}(d), k) \\ &= \text{Tr}(V_d(\sigma)^\dagger O)\end{aligned}$$

Define the Gram matrix G with coefficients $G_{\sigma,\pi} = \text{Tr}(V_d(\sigma)^\dagger V_d(\pi)) = \text{Tr}(V_d(\sigma^{-1}\pi)) = d^{\#\text{cycle}(\sigma^{-1}\pi)}$, then the pseudo-inverse G^+ are related to Weingarten coefficients $G^+_{\pi,\sigma} = \text{Wg}(\pi^{-1}\sigma, d)$.

As an example, we can compute the second moment.

$$\mathcal{M}^{(2)}_{\mu_H}(O) = c_{\mathbb{I},O}V_d(\mathbb{I}) + c_{\mathbb{F},O}V_d(\mathbb{F})$$

Since $\mathbb{F}^{-1} = \mathbb{F}$, using what we learned in the proof we can get:

$$\begin{cases} c_{\mathbb{I},O} = \frac{Tr(\mathbb{I}O)-d^{-1}Tr(\mathbb{F}O)}{d^2-1} \\ c_{\mathbb{F},O} = \frac{Tr(\mathbb{F}O)-d^{-1}Tr(\mathbb{I}O)}{d^2-1} \end{cases} \tag{13}$$

Instead of considering k-moment operator operates on the space $\mathcal{L}\left(\left(\mathbb{C}^d\right)^{\otimes k}\right)$, we can view it as a operator on $\left(\mathbb{C}^d\right)^{\otimes 2k}$. This idea is called **vectorization formalism**. *The best way of representing this idea is using the graphic notation.*

Generically, we can define such a invertible linear map vec : $\mathcal{L}\left(\mathbb{C}^d\right) \to \mathbb{C}^{2d}$ that if $A = \sum_{ij} A_{ij}\,|i\rangle\langle j|$, then $\text{vec}(A) = \sum_{ij} A_{ij}|i\rangle \otimes |j\rangle$. We define a short notation for the map as $|A\rangle\rangle := \text{vec}(A)$. In a compact form, $|A\rangle\rangle = A\otimes I|\Omega\rangle$, where $|\Omega\rangle = \sum_i i\rangle \otimes |i\rangle$ is the vectorization of the identity matrix.

Then we can vectorize the linear superoperators (eg. k-moment operator). For any linear superoperator Φ we can decompose it as:

$$\Phi(X) = \sum_{i=1}^{d^2} A_i X B_i^\dagger \tag{14}$$

Then $|\Phi(X)\rangle\rangle = \sum_{i=1}^{d^2} A_i \otimes B_i^*|X\rangle\rangle$, which means vectorized moment operator can be defined as:

$$M^{(k)}_{\mu_H} := \text{vec}(\mathcal{M}^{(k)}_{\mu_H}) = \mathbb{E}_{U\sim\mu_H}[U^{\otimes k}\otimes U^{*\otimes k}] \tag{15}$$

With $M_{\mu_H}^{(k)}$, we can give the following properties immediately:

$$M_{\mu_H}^{(k)\,2} = M_{\mu_H}^{(k)}, M_{\mu_H}^{(k)\,\dagger} = M_{\mu_H}^{(k)} \tag{16}$$

Namely, $M_{\mu_H}^{(k)}$ is a hermitian projector satisfying left/right invariant condition.

Natually, we can derive the formula of the computation of $M_{\mu_H}^{(k)}$ using vectorized permutation operators.

$$M_{\mu_H}^{(k)} = \mathbb{E}_{U\sim\mu_H}[U^{\otimes k} \otimes U^{*\otimes k}] = \sum_{\pi,\sigma\in S_k} \mathrm{Wg}(\sigma^{-1}\pi, d)|V_d(\sigma)\rangle\rangle\langle\langle V_d(\pi)| \tag{17}$$

For second moment, we then have:

$$M_{\mu_H}^{(2)} = \frac{1}{d^2-1}\left[|\mathbb{I}\rangle\rangle\langle\langle\mathbb{I}| + |\mathbb{F}\rangle\rangle\langle\langle\mathbb{F}| - \frac{1}{d}|\mathbb{I}\rangle\rangle\langle\langle\mathbb{F}| - \frac{1}{d}|\mathbb{F}\rangle\rangle\langle\langle\mathbb{I}|\right] \tag{18}$$

We can also compute the overlap between the non-normalized states:

$$\langle\langle\mathbb{I}|\mathbb{I}\rangle\rangle = \langle\langle\mathbb{F}|\mathbb{F}\rangle\rangle = d^2 \tag{19}$$

$$\langle\langle\mathbb{I}|\mathbb{F}\rangle\rangle = \langle\langle\mathbb{F}|\mathbb{I}\rangle\rangle = d \tag{20}$$

This result will be the starting point of calculating entanglement growth.

2 Purity of the Haar Random Circuit

Now come back to the case where we have two-qudit gates with local dimension $d = q^2$. To calculate the entanglement evolution of the Haar Random Circuit, consider the purity of the quantum circuit, which is:

$$\gamma \equiv \mathrm{Tr}(\rho_A^2(t)) = e^{-S_A^{(2)}} \tag{21}$$

The idea here is simple: If we have a lower bound for $S_A^{(2)}$ then the bound is valid for von-Neumann entropy. What should be stated here is that even if we get a expectation value for γ, we can't say it's the expectation value for $S_A^{(2)}$. In fact it gives the lower bound for $S_A^{(2)}$ since the function of $-\log(\cdot)$ is convex:

$$-\log(\mathbb{E}[\gamma]) \leq \mathbb{E}[-\log(\gamma)] = \mathbb{E}[S_A^{(2)}] \leq \mathbb{E}[S_A] \tag{22}$$

where $S_A = \lim_{n\to 1}\frac{1}{1-n}\mathrm{Tr}[\rho_A^{(n)}]$ and the last inequality comes from the fact that $S_A^{(n)} \leq S_A^{(n-1)}$.

In case we are considering a brickwork circuit as shown in figure 1 and the purity of the circuit can be represented as shown in figure 2 in a four-folded manner.

Since each gate is chosen randomly from the ensemble, we consider the vectorized second moment of each gate seperately. The tracing over $\bar{A}$ and A is characterized by the boundary condition which is shown in figure 3. If the leg is within A, we contract it with $\langle\langle\mathbb{F}|$, otherwise we contract with $\langle\langle\mathbb{I}|$. It's still applicable if we divide the two leg of a single gate into different region. This is because we can rewrite the equation (18) as:

$$M_{\mu_H}^{(2)} = \frac{1}{d^2-1}\left[|\mathbb{I}\mathbb{I}\rangle\rangle\langle\langle\mathbb{I}\mathbb{I}| + |\mathbb{F}\mathbb{F}\rangle\rangle\langle\langle\mathbb{F}\mathbb{F}| - \frac{1}{d}|\mathbb{I}\mathbb{I}\rangle\rangle\langle\langle\mathbb{F}\mathbb{F}| - \frac{1}{d}|\mathbb{F}\mathbb{F}\rangle\rangle\langle\langle\mathbb{I}\mathbb{I}|\right] \tag{23}$$

What's more, we note that $\langle\langle\mathbb{F}|0^{\otimes 4}\rangle\rangle = \langle\langle\mathbb{I}|0^{\otimes 4}\rangle\rangle = 1$. Thus the boundary condition for the 4-folded brickwork circuit is totally fixed.

2.1 Emergent Ising Magnet

Since the second moment has a form as equation (23), we can map it to a honeycomb lattice shown in figure 4. Mathematically, we have:

(Rules for contraction)

- If the we assign the same spin to site σ and π on blue bonds, the contribution is $\frac{1}{q^4-1}$, otherwise the contribution will be $-\frac{1}{q^2(q^4-1)}$;
- If the we assign the same spin to site σ and π on red bonds, the contribution is q^4, otherwise it will be q^2.

Since σ and π always emerge simultaneously, we can integrate out the σ, which makes the honeycomb structure changes to triangular lattice. The integration is quite simple. Using a graphic form language, we have: figure 5. Another 4 graph can be calculated immediately using the apparently $\mathbb{Z}_2$ symmetry of the system. What can be concluded is that the actual selected spins don't matter, what really matters is **the relative spin of the triangular primitive cell**.

So up to now, calculating the purity reduced to the calculation that give all the possible configuration of the spin on the π sites, calculate the contributions by the given formula and add them up. In this sense we construct a map goes from Haar random circuit to classical two level system which we call Ising model. Before proceeding to the next step, let us reconsider the boundary conditions: the contraction with $|0^{\otimes 4}\rangle\rangle$ holds still, while for the other one, we have: figure 6. The contributions are rather the same as what we have in figure 5. So apparently, we can just add a single layer to the graph with the upper spin specified which we showed in figure 7.

Consider the circuit to be infinite large which we can never reach the left/right edges. The graph of 0 contribution tells us there will be some forbidden configurations, in other words, give 0 contribution. The only non-zero contribution to the graph is what is shown in figure 7. The areas divide by the dotted line have different spins. The contribution of all the graph is:

$$\mathbb{E}_{U\sim\mu_H}[\gamma] = (\frac{2q}{q^2+1})^{2t} \tag{24}$$

What's more, we know there is a universal bound with minimal cut description which states how much entanglement can we get via viewing the tensor legs in the evolution as where the entanglement emerges. Then we can have bounds on both sides for von Neumann entropy and the entanglement velocity:

$$2t\log(\frac{q^2+1}{2q}) \leq \mathbb{E}_{U\sim\mu_H}[S_A] \leq 2t\log q \tag{25}$$

$$2t\log_q(\frac{q^2+1}{2q}) \leq v_E \leq 2 \tag{26}$$

So apparently, if we consider large-q limit, the entanglement velocity will reach the maximum.

2.2 Away From Large q Limit

What should be noticed is that you can't get the exact purity of the system from equation (24) for finite q systems. Notice:

$$S_A^{(2)}(t) = \lim_{k\to 0}\frac{1-\mathrm{Tr}[\rho_A^2(t)]^k}{k} \tag{27}$$

For $\mathbb{E}_{U\sim\mu_H}(\text{Tr}[\rho_A^2(t)]^k)$, it's the interactions between different layers that give extra terms in addition to $(\mathbb{E}_{U\sim\mu_H}[\text{Tr}(\rho_A^2(t))])^k$. According to the reason above, in order to get the purity, we need to use the replica trick, expand in $1/q$, and take the limit $k \to 0$.

Argument:

Quantatively, in any realization of the brickwork circuit, we have:

$$S_A^{(n)}(t) = \log q(v_E^{(n)}(q)t + b\chi_A(t)t^{1/3}) \tag{28}$$

where $\chi_A(t)$ is a random variable of $O(1)$ size and the exponent $1/3$ is universal, which comes from Kardar–Parisi–Zhang(KPZ) universality.

Since I'm not an expert of KPZ universality and Weingarten coefficients, anyone who is interested in this aspect can refer to the paper by Tianci Zhou & Adam Nahum, which is exactly the third of references.

Reference

[1] Antonio Anna Mele. Introduction to Haar Measure Tools in Quantum Information: A Beginner's Tutorial.

[2] Sagar Vijay, Random Quantum Circuit of BSS2023, https://www.youtube.com/watch?v=EyxevKxAsb0

[3] Tianci Zhou, Adam Nahum. Emergent Statistical Mechanics of Entanglement in Random Unitary Circuits.

[4] Adam Nahum, Sagar Vijay, and Jeongwan Haah. Operator Spreading in Random Unitary Circuits.

Fig 1.

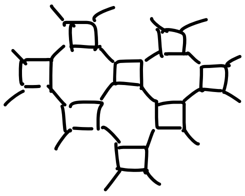

Fig 2.

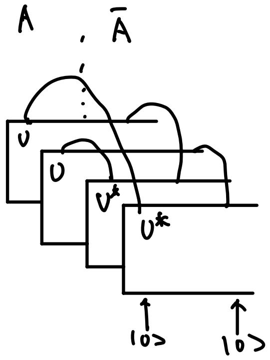

Fig 3

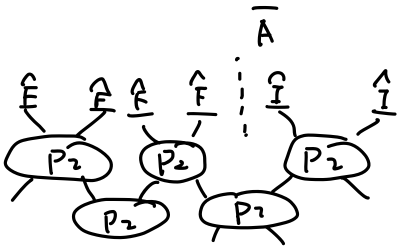

Fig 4.

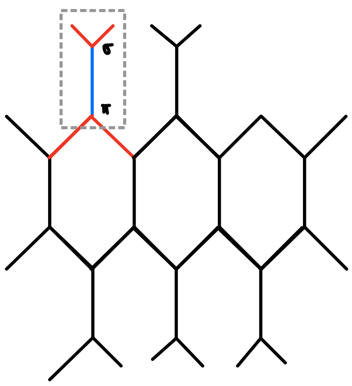

Fig 5.

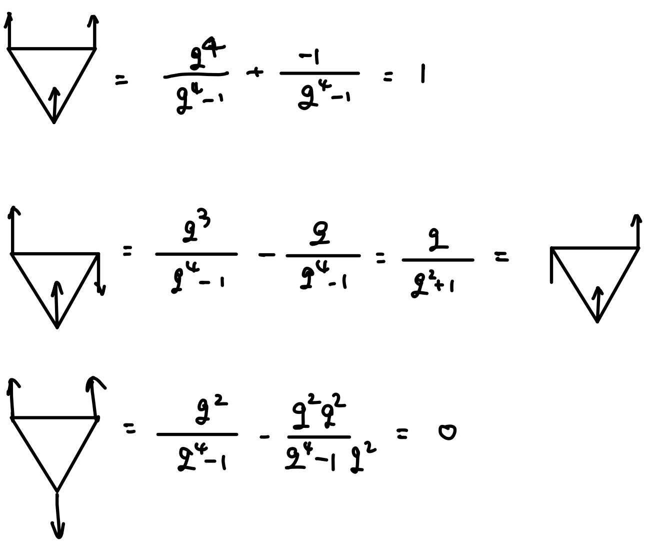

Fig 6.

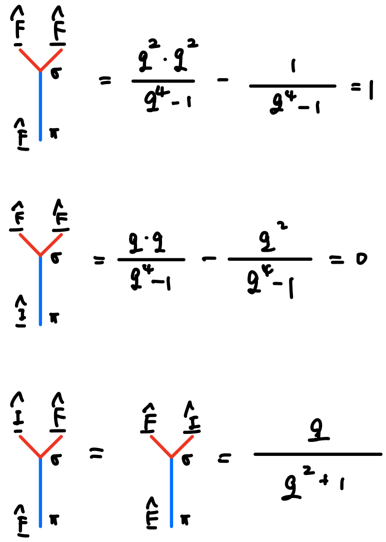

Fig 7: