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Quantum Ising model : $\mathcal{H} = -Jg\sum_i \sigma_i^x - J\sum_{\langle ij\rangle} \sigma_i^z \sigma_j^z$

(Transverse Field Ising Model)

- Discuss the situation when $g \gg 1$ / $g \ll 1$
  - 引入有效哈密顿量 : "To eliminate the inter-group matrix element"

$$\langle i,\alpha|\mathcal{H}_{eff}|j,\alpha\rangle = E_{i\alpha}\delta_{ij} + \langle i,\alpha|\mathcal{H}_1|j,\alpha\rangle$$

Don't understand

$$+ \sum_{k,\beta\neq\alpha} \frac{\langle i,\alpha|\mathcal{H}_1|k,\beta\rangle\langle k,\beta|\mathcal{H}_1|j,\alpha\rangle}{2}\left(\frac{1}{E_{i\alpha}-E_{k\beta}} + \frac{1}{E_{j\alpha}-E_{k\beta}}\right)$$

$$\langle i,\alpha|\mathcal{H}_{eff}|j,\beta\rangle = 0 \quad (\alpha\neq\beta)$$

- First case : $g \gg 1$ expand in $\frac{1}{g}$
  - For the ground state

Let $H_I = -J\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z$ to be pertubation, the first order term gives 0

$(\langle\leftarrow|\otimes\langle\leftarrow|)\,\sigma_i^z\sigma_j^z\,(|\rightarrow\rangle\otimes|\rightarrow\rangle) = 0$

Remember σ^z is the "ladder" between $|\rightarrow\rangle$, $\langle\leftarrow|$

$$E_0^{(2)} = \sum_{k\neq 0}^{\infty} \frac{|\mathcal{H}'_{k0}|^2}{E_0^{(0)} - E_k^{(0)}} = \frac{MJ^2}{-4gJ}$$

Then, we have $E_0 = -MJg\left(1 + \frac{1}{4g^2} + \mathcal{O}\left(\frac{1}{g^3}\right)\right)$

- One particle state : $\langle i|\mathcal{H}_I|i+1\rangle = \langle i+1|\mathcal{H}_I|i\rangle = -J$

Momentum space : $|k\rangle = \sum_j e^{ikx_j}|j\rangle$

Then the Hamiltonian is diagonalised as:

$$\varepsilon_k = Jg\left[2 - (2/g)\cos(ka) + \mathcal{O}\left(\tfrac{1}{g^2}\right)\right]$$

(notice : E_0 is subtracted)

Next order correction [Didn't prove yet] :

$$\varepsilon_k = Jg\left[2 - (2/g)\cos ka + (1-\cos(2ka))/2g^2 + \mathcal{O}(1/g^3)\right]$$

Concept :

quasiparticle residue $\mathcal{A}$: overlap $|\langle k=0|\hat{\sigma}^z(k)|0\rangle|^2$

- Two particle state : [No bound states]

scattering eigenstates $\qquad e^{i(k_1x_i + k_2x_j)} + S_{k_1k_2}e^{i(k_1x_j + k_2x_i)} \quad (i<j)$

$$S_{k_1k_2} = \begin{cases} -1 & d=1 \\ \text{marginal} & d=2 \\ +1 & d>2 \end{cases}$$

- Second case : $g \ll 1$ $\quad \mathcal{H}_0 = -J\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z$; $\mathcal{H}_1 = -Jg\sum_i\sigma_i^x$
  - For the ground state

$$E_0 = -MJ\left(d + g^2/4d + \mathcal{O}(g^3)\right) \longrightarrow N_0 = 1 - g^2/8d^2 + \mathcal{O}(g^4)$$

$E_0^{(1)} = 0 \qquad E_0^{(2)} = -M\frac{J^2g^2}{4d}$

- For $d=2$
  - one particle state

Difference : No correction at order g

At order g^2 : $\varepsilon_k = J\left[8 - \frac{g^2}{4}(1 + \cos k_x + \cos k_y) + \mathcal{O}(g^3)\right]$

[Didn't prove]

- Two particle state (more particle state)

2M bound state [s wave and d wave]

Question : 3 particle bound state 12 J ?

- For $d=1$

Domain wall / kink — as the standard quaiparticle

Jordan-Wigner transformation : free fermions

- The classical Ising model

$$Z = \sum_{\{\sigma_l^z = \pm 1\}} \exp(-\mathcal{H}), \quad \mathcal{H} = -K\sum_{l=1}^{M_\tau}\sigma_l^z\sigma_{l+1}^z - h\sum_l^{M_\tau}\sigma_l^z$$

- Transfer matrix

Given PBC : $Z = \mathrm{Tr}(T_1T_2\cdots)$

$$T_1 = \begin{pmatrix} e^K & e^{-K} \\ e^{-K} & e^K \end{pmatrix} \quad T_2 = \begin{pmatrix} e^h & 0 \\ 0 & e^{-h} \end{pmatrix}$$

$$\Rightarrow Z = \mathrm{Tr}(T_1T_2)^M = \mathrm{Tr}\left(T_2^{1/2}T_1T_2^{1/2}\right)^M = \epsilon_1^M + \epsilon_2^M$$

where $\epsilon_{1,2} = e^K\cosh(h) \pm \left(e^{2K}\sinh^2(h) + e^{-2K}\right)^{1/2}$

Consider the case where $h=0$

Then $\epsilon_{1,2} = \begin{cases} 2\cosh(K) \\ 2\sinh(K) \end{cases}$ And eigenstates is $|\leftarrow\rangle$, $|\rightarrow\rangle$

$$C(l-l') = \langle\sigma_l^z\sigma_{l'}^z\rangle = \frac{\epsilon_1^{M-l'+l}\epsilon_2^{l'-l} + \epsilon_2^{M_\tau-l'+l}\epsilon_1^{l'-l}}{\epsilon_1^M + \epsilon_2^M}$$

$$\xrightarrow{M_\tau\to\infty} (\tanh(K))^{l'-l}$$

This give rise to $\frac{1}{\xi} = \frac{1}{a}\ln\coth(K)$, we'd like to consider the physics in the scale of ξ, in the regime of large K

- The scaling limit
  - Two ways to divide length into large/small
  - Reexamine the $\epsilon_{1,2}$, $\mathcal{F}$, $C(\tau)$, using correlation length
- Assertion of universality : The results of the scaling limit are not sensitive to the precise microscopic model

illustrate

Introducing second neighbor hopping

Concept : universal scaling function

Q : As the example of second hopping K_1, K_2 will both emerge in the description of ξ. why ξ can have the same behavior of $\mathcal{F}$ and correlation

- Mapping between quantum Ising Model and the classical Ising Model

classical model $\longrightarrow$ single Ising spin

$$Z \longrightarrow \mathrm{Tr}\exp(-H_Q/T) \qquad H_Q = E_0 - \frac{\Delta}{2}\hat{\sigma}^x - \tilde{h}\hat{\sigma}^z$$

- Notice that $\frac{a}{\Delta}$ serve as the role of imaginary time

Comment : H_Q is the scaling theory of H

Length $\rightarrow$ Time

quantum model $\rightarrow$ classical model ($d=1$ $D=2$)

- Reverse the former approach