Reference: David Tong, Chapter 2 & Peskin, Chapter 2

A. Zee, Chapter 1 & Nagaosa, Chapter 2 & Atland, Chapter 4

Content: 1. Canonical Quantization

- Scalar field
- Complex scalar field
- Invariant and conservation

2. Causality and Dynamics

- Heisenberg picture
- Causality and propagator
- Recovery of the known

3. Path Integral Approach

- Quantum Mechanics
- Mathematical foundation
- Quantum field theory

4. Application

- Simplified casimir effect

Part 1.

Quantum field theory is an operator valued function.

$$\begin{cases} [\phi_a(\vec{x}), \phi_b(\vec{y})] = [\pi^a(\vec{x}), \pi^b(\vec{y})] = 0 \\ [\phi_a(\vec{x}), \pi^b(\vec{y})] = i\delta(\vec{x}-\vec{y})\,\delta_a^b \end{cases}$$

Treating time/space on different footing

Free field: each degree of freedom evolves independently

(easy to obtain the spectrum ; Lagrangians usually quadratic)

KG field : $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0 \xrightarrow[\text{Transformation}]{\text{Fourier}} \Big[\frac{\partial^2}{\partial t^2} + \underbrace{\omega^2(\vec{p})}_{p^2+m^2}\Big]\phi(\vec{p},t) = 0$$

free field

General Solution :

$$\begin{cases} \phi(\vec{p},t) = \phi_1(\vec{p})e^{-i\omega_p t} + \phi_2(\vec{p})e^{i\omega_p t} \\ \underline{\phi(\vec{x},t)} = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}}\,\phi(\vec{p},t) \end{cases}$$

Hermitian $\longrightarrow \phi(\vec{p},t) = \phi^\dagger(-\vec{p},t) \longrightarrow \phi_2(\vec{p}) = \phi_1^\dagger(-\vec{p})$

$$\Rightarrow \begin{cases} \phi(\vec{p},t) = \phi_1(\vec{p})\,e^{-i\omega_p t} + \phi_1^\dagger(-\vec{p})\,e^{i\omega_p t} \\ \pi(\vec{p},t) = -i\omega_p\,\phi_1(\vec{p})\,e^{-i\omega_p t} + i\omega_p\,\phi_1^\dagger(-\vec{p})\,e^{i\omega_p t} \end{cases}$$

Define $a_p = \sqrt{2\omega_p}\,\phi_1(\vec{p})$, the commutation relations become:

$$\begin{cases} [a_p, a_q] = [a_p^\dagger, a_q^\dagger] = 0 \\ [a_p, a_q^\dagger] = (2\pi)^3\delta(\vec{p}-\vec{q}) \end{cases} \longrightarrow \text{Recall simple harmonic oscillator}$$

Ultimatly, we have:

$$\begin{cases} \phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left[a_p e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{-i\vec{p}\cdot\vec{x}}\right] \\ \pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_p}{2}}\left[a_p e^{i\vec{p}\cdot\vec{x}} - a_p^\dagger e^{-i\vec{p}\cdot\vec{x}}\right] \end{cases}$$

$$\text{or} \begin{cases} \phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left[a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right] \\ \pi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_p}{2}}\left[a_p e^{ip\cdot x} - a_p^\dagger e^{-ip\cdot x}\right] \end{cases}$$

$\longleftarrow$ Discuss Later

Caculate the Hamiltonian:

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_p \left[a_p^\dagger a_p + \frac{1}{2}(2\pi)^3 \delta(0)\right]$$

Two types of infinities: $\underline{\text{infra-red divergence}}$ / $\underline{\text{ultra-violet divergence}}$

$$(2\pi)^3 \delta(0) = \lim_{L\to\infty} \int_{-L/2}^{L/2} d^3x \, e^{i\vec{x}\cdot\vec{p}} \Big|_{\vec{p}=0} = V$$

ground state energy
cut-off at high momentum
$\downarrow$
cosmological constant ?

Normal Ordering: $:H: = \int \frac{d^3p}{(2\pi)^3} \omega_p a_p^\dagger a_p$

spectrum

Complex scalar field: $\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - M^2 \psi^* \psi$

EoM: $\begin{cases} \partial_\mu \partial^\mu \phi + M^2 \phi = 0 \\ \partial_\mu \partial^\mu \phi^* + M^2 \phi^* = 0 \end{cases}$ No Hermitian requirement, we get:

$$\begin{cases} \phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(b_p e^{i\vec{p}\cdot\vec{x}} + c_p^\dagger e^{-i\vec{p}\cdot\vec{x}}\right) \\ \pi^\dagger(x) = \dot{\phi} = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{\omega_p}{2}} \left(b_p e^{i\vec{p}\cdot\vec{x}} - c_p^\dagger e^{-i\vec{p}\cdot\vec{x}}\right) \end{cases}$$

Commutation Relation gives: (Non-trival part)

$$[b_p, b_q^\dagger] = [C_p, C_q^\dagger] = (2\pi)^3 \delta(\vec{p}-\vec{q})$$

Calculate the Hamiltonian:

$$H = \int d^3x \; \pi\pi^\dagger + \pi^\dagger\pi - \pi\pi^\dagger + |\nabla\phi|^2 + M^2\phi^\dagger\phi$$

$$= \int d^3x \frac{d^3p\, d^3q}{(2\pi)^6} \left(-\frac{\sqrt{\omega_p\omega_q}}{2}\right)(b_p - c_{-p}^\dagger)(b_q^\dagger - c_{-q}) e^{i(\vec{p}+\vec{q})\cdot x} + \frac{\vec{p}\cdot\vec{q}+M^2}{2\sqrt{\omega_p\omega_q}}(b_p + c_{-p}^\dagger)(b_q^\dagger + c_{-q}) e^{i(\vec{p}+\vec{q})\cdot x}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{2}\left[(b_p + C_{-p}^\dagger)(b_{-p}^\dagger + C_p) - (b_p - c_{-p}^\dagger)(b_{-p}^\dagger - C_p)\right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \omega_p (b_p^\dagger b_p + c_p^\dagger c_p)$$

Particles and invariant

Def: the vacuum: $a_{\vec{p}}|0\rangle = 0$, $\forall \vec{p}$, $\langle 0|0\rangle = 1$

For scalar field, we have: $[H, a_p^\dagger] = \omega_p a_p^\dagger$, $[H, a_p] = -\omega_p a_p$

The energy eigenstates $|\vec{p}\rangle = a_p^\dagger|0\rangle$ $\quad H|\vec{p}\rangle = \omega_p|\vec{p}\rangle$

$\longrightarrow$ Creat multi-particles states in Fock space.

BUT:

(a) the eigenstates is not particles:

$$\langle\vec{p}|\vec{p}\rangle = \langle 0|a_p a_p^\dagger|0\rangle = (2\pi)^3\delta(0), \quad \langle 0|\phi(x)\phi^\dagger(x)|0\rangle = \langle x|x\rangle = \delta(0)$$

They are operator valued distributions! We should create wave packets!

(b) Does the distributions Lorentz invariant?

We prefer under Lorentz transformation, $|p\rangle \longrightarrow |p'\rangle = U(\Lambda)|p\rangle$

where $U(\Lambda)$ is unitary $\longrightarrow$ relativistic normalization

Construct invariant!

① $I = \int \frac{d^3p}{(2\pi)^3} |p\rangle\langle p|$ ② $\int d^4p\, \delta(p_0^2 - p^2 - m^2)\big|_{p_0>0} = \int \frac{d^3p}{2p_0}\Big|_{p_0=E_p} = \int \frac{d^3p}{2E_p}$

$\Rightarrow$ relativistically normalized state is $\sqrt{2E_p}\,|p\rangle$, the creation operator $a^\dagger(p) = \sqrt{2E_p}\,a_p^\dagger$

Conservation law: Classical / Quantum?

Translation symmetry of scalar field gives:

$$p^i = -\int d^3x\, \dot{\phi}\, \partial^i \phi$$

$$= -\int d^3x \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}(-i)\sqrt{\omega_q/\omega_p}\,\frac{1}{2}(a_q e^{i\vec{q}\cdot\vec{x}} - a_q^\dagger e^{-i\vec{q}\cdot\vec{x}})(ip^i a_p e^{i\vec{p}\cdot\vec{x}} - ip^i a_p^\dagger e^{-i\vec{p}\cdot\vec{x}})$$

$$= -\int \frac{d^3x\, d^3p\, d^3q}{2(2\pi)^6} p^i \sqrt{\omega_q/\omega_p}\Big[a_q a_p e^{i(\vec{q}+\vec{p})\cdot\vec{x}} + a_q^\dagger a_p^\dagger e^{-i(\vec{q}+\vec{p})\cdot\vec{x}} - a_q a_p^\dagger e^{i(\vec{q}-\vec{p})\cdot\vec{x}} - a_q^\dagger a_p e^{i(\vec{p}-\vec{q})\cdot\vec{x}}\Big]$$

$$= -\int \frac{1}{2}\frac{d^3p}{(2\pi)^3} p^i \big[\underbrace{a_p a_{-p} + a_{-p}^\dagger a_p^\dagger}_{\text{cancel}} - \underbrace{a_p a_p^\dagger - a_p^\dagger a_p}_{\text{normal ordering}}\big]$$

$$= \int \frac{d^3p}{(2\pi)^3} p^i a_p^\dagger a_p \qquad [P^i, H] = 0\,!$$

$\Rightarrow$ $|p\rangle$ is the moment eigenstates

Rotation symmetry: for scalar field,

$$Q^{ij} = \int d^3x\,(x^i T^{0j} - x^j T^{0i}) = \int d^3x\,(x^i \partial^0\phi\, \partial^j\phi - x^j \partial^0\phi\, \partial^i \phi)$$

$$= \int d^3x \frac{d^3p\, d^3q}{2(2\pi)^6}(x^i p^j - x^j p^i)\sqrt{\frac{\omega_q}{\omega_p}}\, e^{i(\vec{p}+\vec{q})\cdot\vec{x}}(a_p + a_{-p}^\dagger)(a_q - a_{-q}^\dagger)$$

$$= \int d^3x \frac{d^3p\, d^3q}{2(2\pi)^6}(x^i p^j - x^j p^i)\sqrt{\frac{\omega_q}{\omega_p}}\, e^{i(\vec{p}+\vec{q})\cdot\vec{x}}(a_p a_q + a_{-p}^\dagger a_{-q}^\dagger - a_{-p}^\dagger a_q - a_p a_{-q}^\dagger)$$

$Q^{ij}|p=0\rangle = ?$

Number operator $N = \int \frac{d^3p}{(2\pi)^3} a_p^\dagger a_p$ for $[N, H] = 0 \xrightarrow[\text{dynamic}]{}$ "conserved!"

For complex scalar field, we have classical conserved charge

$$\begin{cases} Q = i\int d^3x\,(\pi\psi - \psi^*\pi^*) = \int \frac{d^3p}{(2\pi)^3}(c_p^\dagger c_p - b_p^\dagger b_p) \\ [Q, H] = \left[\int \frac{d^3p}{(2\pi)^3}(c_p^\dagger c_p - b_p^\dagger b_p),\ \int \frac{d^3q}{(2\pi)^3}\omega_q (b_q c_q + b_q^\dagger c_q^\dagger)\right] = 0 \end{cases}$$

Notice: Q is still conserved in interacting theory, N_c/N_b is conserved in free field.

Heisenberg picture:

To present Lorentz invariant, we introduce Heisenberg picture:

$$\frac{dO_H}{dt} = i[H, O_H] \longrightarrow \text{still apply in QFT}$$

It's straight forward to calculate:

$$\begin{cases} \dot{\phi} = i[\mathcal{H}, \phi] = \pi(x) \\ \dot{\pi} = \nabla^2\phi - m^2\phi \end{cases} \Rightarrow \partial_\mu \partial^\mu \phi + m^2 \phi = 0 \quad \text{KG equation!}$$

As usual, we have $a_p^\dagger, a_p$ evolve

$$\begin{cases} e^{i\mathcal{H}t} a_p^\dagger e^{-i\mathcal{H}t} = e^{-iE_p t} a_p^\dagger \\ e^{i\mathcal{H}t} a_p e^{i\mathcal{H}t} = e^{iE_p t} a_p^\dagger \end{cases} \Rightarrow \phi(x, 0) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x})$$

Causality: the measure of one place will influence another one?

Define $\Delta(x-y) = [\phi(x), \phi(y)]$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{\sqrt{4E_pE_q}} [(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}), (a_q e^{iq\cdot y} + a_q^\dagger e^{-iq\cdot y})]$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{\sqrt{4E_pE_q}} \left([a_p, a_q^\dagger] e^{ip(x-y)} + [a_p^\dagger, a_q] e^{-ip(x-y)}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} [e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}]$$

Check:
(a) Lorentz invariant!
(b) Doesn't vanish for time-like seperation!
(c) vanish for space-like seperation!
$\downarrow$
equal time seperation

Propagator: the road to path integral!

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)} \equiv D(x-y) \longrightarrow \text{propagator!}$$

for $(x-y)^2 < 0$ (time-like!)

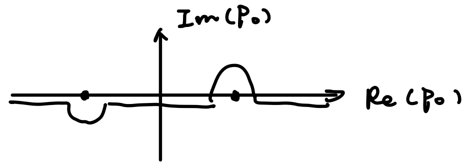
we can know: $D(x-y) \sim e^{-m|x-y|}$

Define Feynman propagator:

$$\Delta_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle = \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & x^0 < y^0 \end{cases}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip(x-y)} \longrightarrow \text{to show this, use "}i\epsilon\text{ prescription"!}$$

the contour is



As Green's function: If stay away from singularities,

$$(\partial_t^2 - \nabla^2 + m^2)\Delta_F(x-y) = -i\delta^4(x-y)$$

Retarded / advanced Green's function :

$$\Delta_R(x-y) = \begin{cases} D(x-y) - D(y-x) & x^0 > y_0 \\ 0 & x^0 < y_0 \end{cases}$$

Recovery of the Known :

we decompose the field : $\psi(\vec{x}, t) = e^{-imt} \tilde{\psi}(x, t)$

then KG equation becomes : $e^{-imt} [\ddot{\tilde{\psi}} - 2im\dot{\tilde{\psi}} - \nabla^2 \tilde{\psi}] = 0$

Non-relativistic limit is $|\vec{p}| << m$, $\ddot{\tilde{\psi}}$ omit !

$$i \frac{\partial \tilde{\psi}}{\partial t} = -\frac{1}{2m} \nabla^2 \tilde{\psi}$$ No probability problem !

For a complex scalar field ,

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \qquad \phi = e^{-imt} \tilde{\phi} \qquad \phi^* = e^{imt} \tilde{\phi}^*$$

$$\Rightarrow \mathcal{L} = i\psi^* \dot{\psi} - \frac{1}{2m} \nabla\psi^* \nabla\psi = \frac{i}{2}(\phi^* \dot{\phi} - \phi \dot{\phi}^*) - \frac{1}{2m} \nabla\psi^* \nabla\psi$$

$$\partial_t \phi^* \partial^t \phi = (im\phi^* + \dot{\tilde{\phi}}^*)(-im\phi + \dot{\phi})$$

$$\approx m^2 \phi \phi^* + im(\phi^* \dot{\phi} - im \phi \dot{\phi}^*)$$

$$j^\mu = (-\psi^* \psi , \frac{i}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*))$$

Then we quantise this field as before , we find

$$\psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} a_p e^{i\vec{p}\cdot\vec{x}} , \quad [a_p, a_q^\dagger] = (2\pi)^3 \delta(\vec{p}-\vec{q}) , \quad \psi^\dagger(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} a_p^\dagger e^{-i\vec{p}\cdot\vec{x}}$$

$$\mathcal{H} = \frac{1}{2m} \nabla\psi^* \nabla\psi = \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{2m} a_p^\dagger a_p$$

$$\downarrow$$

$$[a_p , \psi^\dagger(\vec{x})] = e^{-i\vec{p}\cdot\vec{x}}$$

What should be noticed here :

1) No anti-particle $\longrightarrow$ requirement of relativity !

2) $\int d^3x \, \psi^* \psi$ is conserved $\longrightarrow$ particle number !

3) No non-relativistic limit of a real scalar field !

Construct $\begin{cases} X = \int d^3x \, \vec{x} \, \psi^\dagger(x) \psi(x) \\ P = \int d^3p/(2\pi)^3 \, \vec{p} \, a_p^\dagger a_p \end{cases}$ assume $|\varphi\rangle = \int d^3x \, \varphi(\vec{x}) |\vec{x}\rangle$

$$P^i |\varphi\rangle = \int \frac{d^3x \, d^3p}{(2\pi)^3} \varphi(x) p^i a_p^\dagger a_p \varphi(\vec{x}) \psi^+(x) |0\rangle$$

$$= \int \frac{d^3x \, d^3p}{(2\pi)^3} \varphi(x) p^i a_p^\dagger e^{-i\vec{p}\cdot\vec{x}} |0\rangle$$

$$= \int \frac{d^3x \, d^3p}{(2\pi)^3} \varphi(x) (i \frac{\partial}{\partial x^i} e^{-i\vec{p}\cdot\vec{x}}) a_p^\dagger |0\rangle$$

$$= \int \frac{d^3x \, d^3p}{(2\pi)^3} e^{-i\vec{p}\cdot\vec{x}} (-i \frac{\partial}{\partial x^i} \varphi(x)) a_p^\dagger |0\rangle$$

$$= \int \frac{d^3x}{(2\pi)^3} (-i \frac{\partial \varphi}{\partial x^i}) |x\rangle$$

$$\Rightarrow [x^i , p^j] = i\delta^{ij}$$

Dynamic of $\varphi(x,t)$:

$$\dot{\psi}^\dagger = i[\mathcal{H}, \psi^\dagger] = i\int \frac{d^3p}{(2\pi)^3}\frac{p^2}{2m}[a_p^\dagger a_p, \psi^\dagger] = i\int \frac{d^3p}{(2\pi)^3}\frac{p^2}{2m} a_p^\dagger e^{-i\vec{p}\cdot\vec{x}}$$

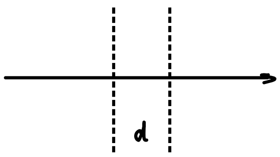
$$\varphi(x,t)\psi_x^\dagger = \varphi(x)\psi^\dagger(x,t)$$

$$\frac{\partial\varphi}{\partial t}\psi^\dagger = \varphi\dot{\psi}^\dagger = i\int\frac{d^3p}{(2\pi)^3}\varphi\left(-\frac{1}{2m}\right)\nabla^2 e^{-i\vec{p}\cdot\vec{x}}\, a_p^\dagger$$

$$= \frac{i}{2m}\nabla^2\varphi\,\psi^\dagger$$

Then $i\frac{\partial\varphi}{\partial t} = -\frac{1}{2m}\nabla^2\varphi \longrightarrow$ Schrödinger equation !

Casimir effect : vacuum fluctuation $\longleftarrow$ simplification $\begin{cases} 1+1\,d \text{ scalar field} \\ \text{massless}\end{cases}$



Q: What does it mean "$\phi(x) = 0$" ?

The plates quantize the momentum: $p_x = \frac{n\pi}{d}$

$$\text{Then } E_{1+1}(d) = \frac{\pi}{2d}\sum_{n=1}^{\infty} n\, \underbrace{e^{-an\pi/d}}_{} \longrightarrow \text{UV cut-off}$$

$$= \frac{d}{2\pi a^2} - \frac{\pi}{24d} + \mathcal{O}(a^2)$$

$$\Rightarrow E_{total} = E_{1+1}(d) + \underbrace{E_{1+1}(L-d)}_{\text{infra-red}} = \frac{L}{2\pi a^2} - \frac{\pi}{24}\left(\frac{1}{d} + \frac{1}{L-d}\right) + \mathcal{O}(a^2) \longrightarrow \frac{\partial E}{\partial d} = \frac{\pi}{24d^2} + \cdots$$

In 3+1 dimensions, $\frac{1}{A}\frac{\partial E}{\partial d} = \frac{\pi^2}{480 d^4}$