Reference：大黄书，Chapter 4 & 2208.05180 & Ginsparg

Conformal Transformation：

an invertible mapping satisfying: $g_{\mu\nu}(x) \longrightarrow g'_{\mu\nu}(x) = \Lambda(x)\, g_{\mu\nu}(x)$

preserve angle, locally "rotation" + "dilation"

If the transformation is infinitely small: $x'^{\mu} = x^{\mu} + \epsilon^{\mu}$

$$g'_{\mu\nu} = g_{\mu\nu} - \underbrace{(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)}_{= f(x)\, g_{\mu\nu}}$$

Then we have the properties:

1). [Trace] $f(x) = \frac{2}{d} \partial_\rho \epsilon^\rho$

Take $g_{\mu\nu} = \eta_{\mu\nu}$,

2) [Partial + Permute]: $2\partial_\mu \partial_\nu \epsilon_\rho = \eta_{\mu\rho}\partial_\nu f + \eta_{\nu\rho}\partial_\mu f - \eta_{\mu\nu}\partial_\rho f$

3) [contract $\eta^{\mu\nu}$] $2\partial^2 \epsilon_\mu = (2-d)\partial_\mu f$

4) [∂^2(original) = ∂^ν(3)] $(2-d)\partial_\mu \partial_\nu f = 2\partial_\rho \partial^\rho \partial_\nu \epsilon_\mu = \eta_{\mu\nu}\partial^2 f$

5) [contract $\eta^{\mu\nu}$] $(d-1)\partial^2 f = 0$

if $d=1$: (5) means no constrain on f

if $d \geq 3$:

(4)+(5) imply $\partial_\mu \partial_\nu f = 0 \rightarrow f = A + B_\mu x^\mu$

(2) imply $\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho$

original + (1) gives $b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} b^\lambda{}_\lambda \eta_{\mu\nu} \rightarrow b_{\mu\nu} = \underbrace{\text{antisymmetric}}_{\text{rotation}} + \underbrace{\text{trace}}_{\text{global dialation}}$

(2) imply $c_{\mu\nu\rho} = \eta_{\mu\rho} b_\nu + \eta_{\mu\nu} b_\rho - \eta_{\nu\rho} b_\mu$ $\quad (b_\mu \equiv \frac{1}{d} c^\sigma{}_{\sigma\mu})$

$\therefore \epsilon'_\mu = c_{\mu\nu\rho} x^\nu x^\rho = 2(\vec{b}\cdot\vec{x})x_\mu - x^2 b_\mu \longrightarrow$ special conformal tranformation

For finite transformation:

translation $\quad x'^\mu = x^\mu + a^\mu$

dilation $\quad x'^\mu = \alpha x^\mu$

rigid rotation $\quad x'^\mu = M^\mu{}_\nu x^\nu$

SCT $\quad x'^\mu = \dfrac{x^\mu - b^\mu x^2}{1 - 2\vec{b}\cdot\vec{x} + b^2 x^2}$ or $\dfrac{x'^\mu}{x'^2} = \dfrac{x^\mu}{x^2} - b^\mu$

easy to verify, see stackexchange "SCT"!

for SCT: $x'^2 = \dfrac{x^2}{1 - 2\vec{b}\cdot\vec{x} + b^2 x^2} = \dfrac{x^2}{\Lambda^2}$

Generators: $iG_a \Phi = \dfrac{\delta x^\mu}{\delta w_a}\partial_\mu \Phi - \dfrac{\delta \mathcal{F}}{\delta w_a}$, and suppose $\mathcal{F}(\Phi) = \Phi$

translation: $P_\mu = -i\partial_\mu$

dilation: $D = -ix^\mu \partial_\mu$ $\qquad$ SCT: $K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)$

rotation: $L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$

Non-trivial commutation relation:

$$\boxed{\begin{aligned}
&[D, P_\mu] = iP_\mu \\
&[D, K_\mu] = -iK_\mu \\
&[K_\mu, P_\nu] = 2i(\eta_{\mu\nu} D - L_{\mu\nu}) \\
&[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu) \\
&[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu) \\
&[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho})
\end{aligned}}$$

$$\left(\begin{array}{c|c|c} & D & \\ \hline D & & \\ \hline \frac{1}{2}(P-K) & \frac{1}{2}(P+K) & L_{\mu\nu} \end{array}\right)$$

In order to simplify the commutation relation, we put:

$$J_{\mu\nu} = L_{\mu\nu} \qquad J_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu) \qquad J_{-1,0} = D \qquad J_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu)$$

$J_{\mu\nu}$ is antisymmetric, and obey $SO(d+1,1)$ commutation relation:

$$[J_{ab}, J_{cd}] = i(\eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac})$$

if spacetime metric is Euclidian

Generally, we have $SO(p,q) \xrightarrow{\text{map}} SO(p+1, q+1)$

conformal invariant:

the first 3 means invariant of type $\frac{|x_i - x_j|}{|x_k - x_m|}$

for SCT:

$$(x_1' - x_2')^2 = \left(\frac{x_1 - b x_1^2}{\Lambda_1^2} - \frac{x_2 - b x_2^2}{\Lambda_2^2}\right)^2$$

$$\Lambda_i^2 = 1 - 2b\cdot x_i + b^2 x_i^2 \qquad = \frac{x_1^2}{\Lambda_1^2} + \frac{x_2^2}{\Lambda_2^2} - 2\frac{1}{\Lambda_1^2\Lambda_2^2}(x_1 x_2 - x_1 b\, x_2^2 - x_2 \cdot b\, x_1^2 + b^2 x_1^2 x_2^2)$$

$$= \frac{1}{\Lambda_1^2 \Lambda_2^2}\left[x_1^2 \Lambda_2^2 + x_2^2 \Lambda_1^2 - 2(x_1 x_2 - x_1 b\, x_2^2 - x_2 \cdot b\, x_1^2 + b^2 x_1^2 x_2^2)\right]$$

$$\text{numerator} = x_1^2 + x_2^2 - 2(b\cdot x_2) x_1^2 \underline{+2b^2 x_1^2 x_2^2} - 2b\cdot x_1\, x_2^2 - 2x_1 \cdot x_2 + 2x_1 \cdot b\, x_2^2 + 2x_2 \cdot b\, x_1^2 \underline{-2b^2 x_1^2 x_2^2}$$

$$= x_1^2 + x_2^2 - 2x_1 \cdot x_2$$

$$\Rightarrow |x_1' - x_2'| = \frac{|x_1 - x_2|}{\Lambda_1 \Lambda_2}$$

So possible invariant: $\frac{|x_1 - x_2||x_3 - x_4|}{|x_1 - x_3||x_2 - x_4|}$ (simplest form)

Classical version:

new generator T_g: $\phi'(x') = (1 - i\omega_g T_g)\phi(x)$, $\phi(x)$ is multicomponent field

for $F(\phi) \neq \phi$, there will be additional term in the algebra.

Consider Lorentz group first:

$$L_{\mu\nu}\phi(0) = S_{\mu\nu}\phi(0) \xrightarrow{\text{translate}} e^{ix^\rho P_\rho} L_{\mu\nu} e^{-ix^\rho P_\rho} = S_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu$$

Root: $[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu)$

then we have: $\begin{cases} P_\mu \phi(x) = -i\partial_\mu \phi(x) \\ L_{\mu\nu}\phi(x) = i(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi(x) + S_{\mu\nu}\phi(x) \end{cases}$

Similarly, we can remove translation from the algebra

At $x=0$, $\begin{cases} L_{\mu\nu} \to S_{\mu\nu} \\ D \to \tilde{\Delta} \\ K_\mu \to k_\mu \end{cases}$, then: $\boxed{\begin{aligned} &[\tilde{\Delta}, k_\mu] = -ik_\mu \\ &[k_\rho, S_{\mu\nu}] = i(\eta_{\rho\mu}k_\nu - \eta_{\rho\nu}k_\mu) \\ &[S_{\mu\nu}, S_{\rho\sigma}] = i(\eta_{\nu\rho}S_{\mu\sigma} + \eta_{\mu\sigma}S_{\nu\rho} - \eta_{\mu\rho}S_{\nu\sigma} - \eta_{\nu\sigma}S_{\mu\rho}) \end{aligned}}$

After translation:

$$\begin{cases} D\phi(x) = (\tilde{\Delta} - ix^\nu\partial_\nu)\phi(x) \\ K_\mu\phi(x) = (k_\mu + 2x_\mu\tilde{\Delta} - 2x^\nu S_{\mu\nu} - 2ix_\mu x^\nu\partial_\nu + ix^2\partial_\mu)\phi(x) \end{cases}$$

if $\phi(x)$ is a irreducible representation of the Lorentz group,

$[\tilde{\Delta}, S_{\mu\nu}] = 0 \xrightarrow{\text{Schur's Lemma}} \tilde{\Delta} = -i\Delta$

Notice: under global dialation, $x^\mu \to (1+\delta a)x^\mu = e^{\delta a}x^\mu$

$\phi'(x') = \phi(x)(1 - i\delta a\cdot(-i\Delta)) = \phi(x)(1-\Delta\delta a) = (e^{\delta a})^{-\Delta}$ $\Bigg\} \Rightarrow \begin{cases} x' = \lambda x \\ \phi'(x') = \lambda^{-\Delta}\phi(x) \end{cases}$

Then we can define quasi-primary field:

$$\phi(x) \longrightarrow \phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d}\phi(x)$$

Energy-momentum tensor

For symmetric $T_{\mu\nu}$,

$$\delta S = \int d^dx\, T^{\mu\nu}\partial_\mu\epsilon_\nu = \frac{1}{2}\int d^dx\, T^{\mu\nu}(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) = \frac{1}{d}\int d^dx\, T^\mu{}_\mu \partial_\rho\epsilon^\rho$$

if $\underline{T^\mu{}_\mu = 0}$ imply S is invariant under conformal transformation.

$\hookrightarrow$ Scale invariance may imply traceless

For a generic field with scale invariance, $x'^\mu = (1+\alpha)x^\mu$ $F(\phi) = (1-\alpha\Delta)\phi$:

$$j_D^\mu = x^\nu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}\right] + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi = -\mathcal{L}x^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}x^\nu\partial_\nu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi$$

$$\Rightarrow j_D^\mu = T_c{}^\mu{}_\nu x^\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi \quad [\text{Hypothesis}: \partial_\mu j_D^\mu = 0]$$

Appendix. Belinfante Tensor [Chapter 2.5]

Canonically, $T_c^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi \longrightarrow$ generically not symmetric

Introduce Belinfante energy-moment tensor

$T_B^{\mu\nu} = T_c^{\mu\nu} + \partial_\rho B^{\rho\mu\nu}$, where $B^{\rho\mu\nu} = -B^{\mu\rho\nu} \longrightarrow$ $\boxed{\text{Additional term doesn't affect classical conservation nor the Ward identity}}$

make $T_B^{\mu\nu} = T_B^{\nu\mu}$

Consider current associated with Lorentz transformation:

$j^{\mu\nu\rho} = T_c^{\mu\nu}x^\rho - T_c^{\mu\rho}x^\nu + \frac{1}{2}i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}S^{\nu\rho}\phi$

If $j^{\mu\nu\rho} = T_B^{\mu\nu}x^\rho - T_B^{\mu\rho}x^\nu$, by $\begin{cases} \partial_\mu j^{\mu\nu\rho} = 0 \\ \partial_\mu T_B^{\mu\nu} = 0 \end{cases} \Rightarrow T_B^{\mu\nu}\delta^\rho_\mu - T_B^{\mu\rho}\delta^\nu_\mu = 0 \Leftrightarrow T_B^{\rho\nu} = T_B^{\nu\rho}$ symmetric!

By inspection, $B^{\mu\rho\nu} = \frac{1}{4} i \left[\frac{\partial L}{\partial(\partial_\mu \phi)} S^{\nu\rho} + \frac{\partial L}{\partial(\partial_\rho \phi)} S^{\mu\nu} + \frac{\partial L}{\partial(\partial_\nu \phi)} S^{\mu\rho} \right] \Phi$

$S^{\mu\nu} = -S^{\nu\mu}$ ensure antisymmetry in the first 2 indices

$$\begin{cases} B^{\mu\rho\nu} - B^{\mu\nu\rho} = \frac{1}{2} i \frac{\partial L}{\partial(\partial_\mu \phi)} S^{\nu\rho} \phi \\ \partial_\mu j^{\mu\nu\rho} = 0 \iff T_c^{\rho\nu} - T_c^{\nu\rho} = -\frac{1}{2} i \partial_\mu \left[\frac{\partial L}{\partial(\partial_\mu \phi)} S^{\nu\rho} \phi \right] = \partial_\mu B^{\mu\nu\rho} - \partial_\mu B^{\mu\rho\nu} \end{cases}$$

Check the expression of $j^{\mu\nu\rho}$!

Define : viral $V^\mu = \frac{\delta L}{\delta(\partial^\rho \phi)} (\eta^{\mu\rho} \Delta + i S^{\mu\rho}) \phi$ Assuming $V^\mu = \partial_\alpha \sigma^{\alpha\mu}$

Then define : $\sigma_+^{\mu\nu} = \frac{1}{2} (\sigma^{\mu\nu} + \sigma^{\nu\mu})$

$$X^{\lambda\rho\mu\nu} = \frac{2}{d-2} \left\{ \eta^{\lambda\rho} \sigma_+^{\mu\nu} - \eta^{\lambda\mu} \sigma_+^{\rho\nu} - \underline{\eta^{\rho\nu} \sigma^{\lambda\mu}} + \eta^{\mu\nu} \sigma_+^{\lambda\rho} + \frac{1}{d-1} (\eta^{\lambda\rho} \eta^{\mu\nu} - \eta^{\lambda\mu} \eta^{\rho\nu}) \sigma_{+\ \alpha}^{\alpha} \right\}$$

consider $T^{\mu\nu} = \underbrace{\underbrace{T_c^{\mu\nu} + \partial_\rho B^{\rho\mu\nu}}_{\text{Belinfante}} + \frac{1}{2} \partial_\lambda \partial_\rho X^{\lambda\rho\mu\nu}}_{\text{traceless}}$

Not check at all $\leftarrow$ For $X^{\lambda\rho\mu\nu}$: (1) $\partial_\mu \partial_\lambda \partial_\rho X^{\lambda\rho\mu\nu} = 0 \longrightarrow$ not affect conservation

(2) $X^{\lambda\rho\mu\nu} - X^{\lambda\rho\nu\mu} = \frac{2}{(d-2)(d-1)} \sigma_{+\ \alpha}^{\alpha} (\eta^{\lambda\mu} \eta^{\rho\nu} - \eta^{\lambda\nu} \eta^{\rho\mu})$

$\partial_\lambda \partial_\rho (\ ) = 0 \iff$ Not spoil the symmetry of Belinfante tensor

(3) $\frac{1}{2} \partial_\lambda \partial_\rho X^{\lambda\rho\mu}{}_\mu = \partial_\mu V^\mu$, $\partial_\rho B^{\rho\mu}{}_\mu = \frac{1}{2} i \partial_\rho \left(\frac{\partial L}{\partial(\partial_\mu \phi)} S^{\mu\rho} \phi \right)$

$\Rightarrow T^\mu{}_\mu = T_c{}^\mu{}_\mu + \Delta \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi)} \Phi \right) = \partial_\mu j_D^\mu = 0$

In a nutshell, scale invariance and $V^\mu = \partial_\alpha \sigma^{\alpha\mu}$ imply the existence of traceless $T^{\mu\nu}$

We hope the $T^{\mu\nu}$ in $d=2$ still holds, in quantum version, the vaccum expectation of $(T^\mu{}_\mu)^2$ vanish in $d=2$ if conformal invariance is presented.

Quantum version: mainly focus of conformal invariance on two / three point correlation function of quasi-primary fields.

"field" : ϕ, $\partial_\mu \phi$, $\cdots \longrightarrow$ local quantities

consider spinless field :

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \left| \frac{\partial x'}{\partial x} \right|_{x'=x_1}^{\Delta_1/d} \left| \frac{\partial x'}{\partial x} \right|_{x'=x_2}^{\Delta_2/d} \langle \phi_1(x_1') \phi_2(x_2') \rangle$$

for scale transformation $x \to \lambda x$ for rotation & translation

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \lambda^{\Delta_1+\Delta_2} \langle \phi_1(\lambda x_1) \phi_2(\lambda x_2) \rangle = f(|x_1 - x_2|) = \lambda^{\Delta_1+\Delta_2} f(\lambda |x_1 - x_2|)$$

$\Rightarrow f(x) = \lambda^{\Delta_1+\Delta_2} f(\lambda x) \Rightarrow \frac{f(x)}{\lambda^{\Delta_1+\Delta_2+1}} (-\Delta_1 - \Delta_2) = x f'(\lambda x)$

$\Rightarrow f'(x) / f(x) = -(\Delta_1 + \Delta_2) \frac{1}{x} \Rightarrow f(x) = \frac{C_{12}}{x^{\Delta_1+\Delta_2}}$

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}$$

for SCT: $\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{\Lambda^{3d}}$

$$\Rightarrow \langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}} = \frac{1}{\Lambda_1^{2\Delta_1}\Lambda_2^{2\Delta_2}} (\Lambda_1\lambda_2)^{\Delta_1+\Delta_2} \frac{C_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}} \qquad (\gamma = \Lambda^2)$$

4.56 Not check yet! $\Rightarrow \begin{cases} \Delta_1=\Delta_2 \\ C_{12}=0 \end{cases}$, $\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \begin{cases} 0 & \Delta_1 \neq \Delta_2 \\ \frac{C_{12}}{|x_1-x_2|^{2\Delta}} & \Delta_1=\Delta_2=\Delta \end{cases}$

Then for 3-point functions, similarly rotation, translation, dialation give:

$$\langle \phi_1\phi_2\phi_3\rangle = \frac{C_{123}^{(abc)}}{x_{12}^{a}\, x_{23}^{b}\, x_{13}^{c}} \qquad a+b+c = \Delta_1+\Delta_2+\Delta_3$$

for SCT: $\frac{1}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}} \gamma_1^{\frac{a+c}{2}} \gamma_2^{\frac{a+b}{2}} \gamma_3^{\frac{b+c}{2}} = 1$

$\boxed{\text{the result may require the correlation to be analytic}}$

$$\Rightarrow \begin{cases} a = \Delta_1+\Delta_2-\Delta_3 \\ b = \Delta_2+\Delta_3-\Delta_1 \\ c = \Delta_3+\Delta_1-\Delta_2 \end{cases} \Rightarrow \langle \phi_1\phi_2\phi_3\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}\, x_{23}^{\Delta_2+\Delta_3-\Delta_1}\, x_{13}^{\Delta_3+\Delta_1-\Delta_2}}$$

For more point function, it's no more elegant!

Ward Identities: Consequence of a symmetry of the action and the measure on correlation.

Appendix [Chapter 2.4.3 & 2.4.4]

Consider a correlation function $\langle \phi(x_1)\cdots\phi(x_n)\rangle = \frac{1}{Z}\int [d\phi]\,\phi(x_1)\cdots\phi(x_n)\, e^{-S[\phi]}$

Consequence of symmetry $S[\phi]$ invariant

$\hookrightarrow \langle \phi(x_1')\cdots\phi(x_n')\rangle = \langle F[\phi(x_1)]\cdots F[\phi(x_n)]\rangle \quad (*)$

Besides, infinitesimal transformation gives $\phi'(x) = \phi(x) - i\omega_a G_a \phi(x)$

$$\langle X\rangle = \frac{1}{Z}\int \underset{[d\phi]}{\underset{\parallel}{[d\phi']}} (X+\delta X) \exp -\left\{ S[\phi] + \int dx\, \partial_\mu j^\mu_a\, \omega_a(x)\right\}$$

Expand to the first order of $\omega_a(x)$ $\longrightarrow \langle \delta X\rangle = \int dx\, \partial_\mu \langle j^\mu_a(x) X\rangle\, \omega_a(x)$

However, $\delta X = -i\left[\sum_{i=1}^{n} \phi(x_1)\cdots G_a\phi(x_i)\cdots\phi(x_n)\right]\omega_a(x_i)$

$$= -i\int dx\, \omega_a(x)\left[\sum_{i=1}^{n}\phi(x_1)\cdots G\phi(x_i)\cdots\phi(x_n)\,\delta(x-x_i)\right]$$

$$\Rightarrow \frac{\partial}{\partial x^\mu}\langle j^\mu_a(x)\,\phi(x_1)\cdots\phi(x_n)\rangle = -i\sum_{i=1}^{n}\langle \phi(x_1)\cdots G\phi(x_i)\,\phi(x_n)\rangle\,\delta(x-x_i)$$

[Ward identity for the current j^μ_a]

Integral gives:

$\delta_\omega X = -i\omega_a \sum_{i=1}^{n}\langle \phi(x_1)\cdots G\phi(x_i)\cdots\phi(x_n)\rangle = 0 \longrightarrow$ infinite version of $(*)$

The Ward identity of translation symmetry:

$\partial_\mu\langle T^\mu{}_\nu X\rangle = -\sum_i \delta(x-x_i)\frac{\partial}{\partial x_i^\nu}\langle X\rangle$, this equation still holds after modification

Consider Lorentz (Rotation) symmetry: $j^{\mu\nu\rho} = T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu$

the ward identity gives

substitute

$$\partial_\mu\langle (T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu) X\rangle = \sum_i \delta(x-x_i)\left[(x_i^\nu\partial_i^\rho - x_i^\rho\partial_i^\nu)\langle X\rangle - iS_i^{\nu\rho}\langle X\rangle\right]$$

$$\Rightarrow \langle (T^{\rho\nu} - T^{\nu\rho}) X \rangle = -i \sum_i \delta(x-x_i)\, S_i^{\nu\rho}(X)$$

Notice: When $x = x_i$, $\langle (T^{\rho\nu} - T^{\nu\rho}) X \rangle$ generically non-zero

Consider scale invariance: $j_D^{\mu} = T^{\mu}{}_{\nu} x^{\nu}$ ($T^{\mu\nu}$ symmetric and traceless)

$$\partial_\mu \langle T^{\mu}{}_{\nu} x^{\nu} X \rangle = -\sum_i \delta(x-x_i) \left\{ x_i^{\nu} \frac{\partial}{\partial x_i^{\nu}} \langle X \rangle + \Delta_i \langle X \rangle \right\}$$

$$\Rightarrow \langle T^{\mu}{}_{\mu} X \rangle = -\sum_i \delta(x-x_i)\, \Delta_i \langle X \rangle$$

$T_{\mu\nu}$ in two dimensions

Consider two-point function of energy-momentum tensor (Schwinger function)

$S_{\mu\nu\rho\sigma}(x) = \langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle$ $T_{\mu\nu}$ is symmetric and conserved

$$S_{\mu\nu\rho\sigma} = S_{\nu\mu\rho\sigma} = S_{\nu\mu\sigma\rho} = S_{\mu\nu\sigma\rho}$$

Translation symmetry: $S_{\mu\nu\rho\sigma}(x) = \langle T_{\mu\nu}(0) T_{\rho\sigma}(-x) \rangle = \langle T_{\rho\sigma}(-x) T_{\mu\nu}(0) \rangle = S_{\rho\sigma\mu\nu}(-x)$

May come from normal ordering

Moreover, parity gives: $S_{\mu\nu\rho\sigma}(x) = S_{\rho\sigma\mu\nu}(x)$

Scale invariance gives: $S_{\mu\nu\rho\sigma}(\lambda x) = \lambda^{-4} S_{\mu\nu\rho\sigma}(x)$

The general solution gives:

Not check yet! ←

$$S_{\mu\nu\rho\sigma}(x) = (x^2)^{-4} \{ A_1 g_{\mu\nu} g_{\rho\sigma} (x^2)^2 + A_2 (g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}) (x^2)^2 + A_3 (g_{\mu\nu} x_\rho x_\sigma + g_{\rho\sigma} x_\mu x_\nu) x^2 + A_4 x_\mu x_\nu x_\rho x_\sigma \}$$

because $\partial^\mu T_{\mu\nu} = 0 \Rightarrow \partial^\mu S_{\mu\nu\rho\sigma}(x) = 0$

$$\Rightarrow A_1 = 3A \quad A_2 = -A \quad A_3 = -4A \quad A_4 = 8A$$

$$\Rightarrow S_{\mu\nu\rho\sigma}(x) = \frac{A}{(x^2)^4} \{ (3 g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})(x^2)^2 - 4x^2 (g_{\mu\nu} x_\rho x_\sigma + g_{\rho\sigma} x_\mu x_\nu) + 8 x_\mu x_\nu x_\rho x_\sigma \}$$

then $S^{\mu}{}_{\mu}{}^{\sigma}{}_{\sigma}(x) = \frac{A^2}{(x^2)^4} \{ [3\times2\times2 - 2 - 2 - 4\times(2+2) + 8](x^2)^2 \} = 0$

especially, $S^{\mu}{}_{\mu}{}^{\sigma}{}_{\sigma}(0) = \langle T^{\mu}{}_{\mu}{}^{2}(0) \rangle = 0$