Math 3406 HW8

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Problem 1. Given a 3×3 matrix A with eigenvalues 0, 3, and 5, and corresponding eigenvectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ respectively.

a) Are the eigenvectors linearly independent?

Yes, the eigenvectors corresponding to distinct eigenvalues of a matrix are linearly independent. Since the eigenvalues 0, 3, and 5 are distinct, the corresponding eigenvectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly independent.

b) Give a basis for the null space and a basis for the column space of A.

Basis for the null space: The null space of A consists of all vectors $\vec{x}$ such that $A\vec{x} = 0$. Since 0 is an eigenvalue of A, its corresponding eigenvector $\vec{u}$ is in the null space of A. Therefore, $\{\vec{u}\}$ forms a basis for the null space of A.

Basis for the column space: The column space of A can be spanned by the eigenvectors corresponding to the non-zero eigenvalues. Thus, $\{\vec{v}, \vec{w}\}$ forms a basis for the column space of A.

c) Find all the solutions of $A\vec{x} = \vec{v} + \vec{w}$.

A particular solution to the equation can be expressed as:

$$\vec{x}_p = \frac{1}{3}\vec{v} + \frac{1}{5}\vec{w}$$

This utilizes the property that $A\vec{v} = 3\vec{v}$ and $A\vec{w} = 5\vec{w}$, thus:

$$A\left(\frac{1}{3}\vec{v} + \frac{1}{5}\vec{w}\right) = \frac{1}{3}A\vec{v} + \frac{1}{5}A\vec{w} = \frac{1}{3}(3\vec{v}) + \frac{1}{5}(5\vec{w}) = \vec{v} + \vec{w}$$

To find all solutions, we add to this particular solution any vector in the null space of A. The null space is spanned by the eigenvector corresponding to the eigenvalue 0, which is $\vec{u}$. Therefore, the general solution can be written as:

$$\vec{x} = c\vec{u} + \frac{1}{3}\vec{v} + \frac{1}{5}\vec{w}$$

where c is any scalar in $\mathbb{R}$. This accounts for the particular solution and includes the contribution from the null space of A, represented by any scalar multiple of $\vec{u}$.

d) Show that $A\vec{x} = \vec{u}$ has no solution.

Since $\vec{u}$ is an eigenvector corresponding to the eigenvalue 0, it means that $A\vec{u} = 0$. Since $\vec{u}$ corresponds to the eigenvalue 0, it is only mapped to the zero vector, and there is no solution to $A\vec{x} = \vec{u}$ except in the trivial case where $\vec{u} = 0$, which contradicts the fact of $\vec{u}$ being an eigenvector. If $A\vec{x} = \vec{u}$, then $\vec{u}$ would be in the column space.

Problem 2. Find the rank and the four eigenvalues of the matrices A and B, where:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Since all columns of A are equal, we can row reduce matrix A into one row, and so the **rank** of A is 1. This implies that 0 is an eigenvalue of A, and the eigenvectors are in the nullspace of A. Since elements of the nullspace of A are of the form

$$x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

we see that these eigenvectors are:

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent eigenvectors corresponding to the eigenvalue 0. To find the rest eigenvalue of A other than 0, we solve

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= (1-\lambda)^4 - 6(1-\lambda)^2 + 8(1-\lambda) - 3 \\ &= \lambda^3(\lambda - 4). \end{aligned}$$

Therefore, the nonzero eigenvalue of A is 4; the corresponding eigenvector is in the nullspace of

$$A - 4I = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}.$$

This matrix reduces to

$$\begin{bmatrix} -3 & 0 & 0 & 3 \\ 0 & -\frac{8}{3} & 0 & \frac{8}{3} \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the nullspace is the line containing the vector

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

meaning that this is the eigenvector associated with the eigenvalue 4. Therefore, the four eigenvalues of the matrices A is given as

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

For matrix B, notice that the first two columns are linearly independent, but that the third and fourth columns are repeats of the first two columns, so after row reduce, the matrix has **rank** 2. Therefore, one eigenvalue is 0; the corresponding eigenvectors will be the elements of the nullspace of B. Since B reduces to

$$B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we see that

$$\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

forms a basis for the nullspace of B, meaning that these two vectors are eigenvectors corresponding to the eigenvalue 0.

The other two eigenvalues will come from solving

$$\begin{aligned} 0 &= \det(B - \lambda I) \\ &= \det\left(\begin{bmatrix} -\lambda & 1 & 0 & 1 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 1 \\ 1 & 0 & 1 & -\lambda \end{bmatrix} \right) \\ &= -\lambda^3(\lambda + 2)(\lambda - 2). \end{aligned}$$

Therefore, the nonzero eigenvalues of B are 2 and -2. For the eigenvalue 2, the corresponding eigenvector will be any vector in the nullspace for

$$B - 2I = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix}.$$

Which row reduced to

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

so the nullspace is the line containing the vector

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

meaning that this is the eigenvector associated with the eigenvalue 2.
For the eigenvalue -2, the corresponding eigenvector will be any vector in the nullspace for

$$B - (-2)I = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}.$$

Which row reduced to

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

so the nullspace is the line containing the vector

$$\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

meaning that this is the eigenvector associated with the eigenvalue -2. Therefore, the four eigenvalues of the matrices A is given as

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Problem 3. Consider the three-term recursion

$$G_{k+2} = \frac{1}{2}G_{k+1} + \frac{1}{2}G_k.$$

a) Write this recursion as $\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = A \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$, what is A?

$$\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}G_{k+1} + \frac{1}{2}G_k \\ G_{k+1} + 0G_k \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}.$$

b) Find the eigenvalues and eigenvectors of A.

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} \frac{1}{2} - \lambda & \frac{1}{2} \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

$$\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0$$

$$(\lambda - 1)(\lambda + \frac{1}{2}) = 0$$

So the eigenvalues are:

$$\lambda = 1, -\frac{1}{2}.$$

To find the eigenvector using the eigenvalue $\lambda = 1$, we have:

$$(A - \lambda I)\vec{v} = 0$$

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector corresponding to the eigenvalue $\lambda_1 = 1$.

The eigenvector using the eigenvalue $\lambda = -\frac{1}{2}$.

$$(A + \frac{1}{2}I)\vec{v} = 0$$

$$\begin{bmatrix} 1 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence

$$\vec{v}_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

is an eigenvector corresponding to the eigenvalue $\lambda_2 = -\frac{1}{2}$.

c) Compute $\lim_{n\to\infty} A^n$.

As we did in class, we rewrite matrix A in the form $A = VDV^{-1}$, where V is consists of $\begin{bmatrix}\vec{v_1}, \vec{v_2}\end{bmatrix}$, and D is the diagonal matrix. So we have:

$$V = \begin{bmatrix}1 & -\frac{1}{2}\\ 1 & 1\end{bmatrix}, D = \begin{bmatrix}1 & 0\\ 0 & -\frac{1}{2}\end{bmatrix}, V^{-1} = \begin{bmatrix}\frac{2}{3} & \frac{1}{3}\\ -\frac{2}{3} & \frac{2}{3}\end{bmatrix}$$

$$A^n = VDV^{-1}VDV^{-1}\cdots VDV^{-1} = VD^nV^{-1}$$

$$\lim_{n\to\infty} A^n = V \lim_{n\to\infty} D^nV^{-1} = \begin{bmatrix}1 & -\frac{1}{2}\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{2}{3} & \frac{1}{3}\\ -\frac{2}{3} & \frac{2}{3}\end{bmatrix} = \begin{bmatrix}\frac{2}{3} & \frac{1}{3}\\ \frac{2}{3} & \frac{1}{3}\end{bmatrix}$$

d) If $G_0 = 0$ and $G_1 = 1$, find $\lim_{n\to\infty} G_n$. By definition,

$$\begin{bmatrix}G_{n+2}\\ G_{n+1}\end{bmatrix} = A\begin{bmatrix}G_{n+1}\\ G_n\end{bmatrix} = A^2\begin{bmatrix}G_n\\ G_{n-1}\end{bmatrix} = \cdots\cdots = A^{n+1}\begin{bmatrix}G_1\\ G_0\end{bmatrix}$$

$$\lim_{n\to\infty}\begin{bmatrix}G_{n+1}\\ G_n\end{bmatrix} = \lim_{n\to\infty} A^n\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}\frac{2}{3} & \frac{1}{3}\\ \frac{2}{3} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\\ \frac{2}{3}\end{bmatrix}$$

This implies $G_n \to \frac{2}{3}$ as $n \to \infty$.

Problem 4. Find the eigenvalues and eigenvectors of the Hermitian matrix

$$S = \begin{bmatrix}1 & 1+i\\ 1-i & 2\end{bmatrix}.$$

We start from forming a new matrix by subtracting λ from the diagonal entries of the given matrix:

$$\begin{bmatrix}1-\lambda & 1+i\\ 1-i & 2-\lambda\end{bmatrix}$$

The determinant of the obtained matrix is

$$\det\left(\begin{bmatrix}1-\lambda & 1-i\\ 1+i & 2-\lambda\end{bmatrix}\right) = (1-\lambda)\cdot(2-\lambda) - (1+i)\cdot(1-i) = \lambda^2 - 3\lambda$$

Solve the equation $\lambda(\lambda$ - 3) = 0. The roots are $\lambda_1 = 3$, $\lambda_2 = 0$.
Next, find the eigenvectors.
For $\lambda = 3$:

$$\begin{bmatrix}1-\lambda & 1+i\\ 1-i & 2-\lambda\end{bmatrix} = \begin{bmatrix}-2 & 1+i\\ 1-i & -1\end{bmatrix}$$

After row reduce:

$$\begin{bmatrix}1 & -\frac{1}{2}-\frac{1}{2}i\\ 0 & 0\end{bmatrix}$$

The eigenvector of this matrix is $\begin{bmatrix} 0.5+0.5i \\ 1 \end{bmatrix}$

For $\lambda = 0$:

$$\begin{bmatrix} 1-\lambda & 1+i \\ 1-i & 2-\lambda \end{bmatrix} = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$$

After row reduce:

$$\begin{bmatrix} 1 & 1+i \\ 0 & 0 \end{bmatrix}$$

The eigenvector of this matrix is $\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$.

Problem 5. A 3×3 matrix B is known to have the eigenvalues 0, 1, 2. Is this information enough to find

a) The rank of B. If yes, what is it?

B has 0 as an eigenvalue and is therefore singular and not invertible. Since B is a three by three matrix, this means that its rank can be at most 2. Since B has two distinct nonzero eigenvalues, its rank is exactly 2.

b) The determinant of $B^T B$? If yes what is it?

Since B is singular, $\det(B) = 0$. Thus $\det(B^T B) = \det(B^T)\det(B) = 0$.

c) The eigenvalues of $B^T B$? If yes what are they?

c) There is not enough information to find the eigenvalues of $B^T B$. For example:

$$\text{If } B = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 2 \end{bmatrix} \text{ then } B^T B = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 4 \end{bmatrix}.$$

$$\text{If } B = \begin{bmatrix} 0 & 1 & \\ & 1 & \\ & & 2 \end{bmatrix} \text{ then } B^T B = \begin{bmatrix} 0 & & \\ & 2 & \\ & & 4 \end{bmatrix}.$$

d) The eigenvalues of $(B^2 + I)^{-1}$? If yes what are they?

If $p(t)$ is a polynomial and if x is an eigenvector of A with eigenvalue λ, then $p(A)x = p(\lambda)x$. We also know that if λ is an eigenvalue of A then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. Hence the eigenvalues of $(B^2+I)^{-1}$ are $\frac{1}{0^2+1}$, $\frac{1}{1^2+1}$ and $\frac{1}{2^2+1}$, or 1, $\frac{1}{2}$ and $\frac{1}{5}$.

Problem 6. Find the eigenvalues and eigenvectors for the symmetric matrix

$$A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}.$$

The matrix is symmetric. Is it diagonalizable?

Start from forming a new matrix by subtracting λ from the diagonal entries of the given matrix:

$$\begin{bmatrix} -\lambda+i & 1 \\ 1 & -\lambda-i \end{bmatrix}.$$

The determinant of the obtained matrix is

$$\det\left(\begin{bmatrix} -\lambda+i & 1 \\ 1 & -\lambda-i \end{bmatrix}\right) (-\lambda+i)\cdot(-\lambda-i)-(1)\cdot(1)=\lambda^2$$

The determinant of the obtained matrix is λ^2. The roots are $\lambda_1 = 0, \lambda_2 = 0$. Then, we find the eigenvectors. For $\lambda = 0$,

$$\begin{bmatrix} -\lambda+i & 1 \\ 1 & -\lambda-i \end{bmatrix} = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}.$$

After row reduce:

$$\begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

So, the eigenvector of this matrix is $\begin{bmatrix} i \\ 1 \end{bmatrix}$.

Since A has a repeated eigenvalues of 0, this matrix is not diagonalizable.

Problem 7. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 110 & 55 & -164 \\ 42 & 21 & -62 \\ 88 & 44 & -131 \end{bmatrix}.$$

(Hint: Look at the numbers and think.)

We observe that the sum of elements in each row are all equal to 1:

$$(110 + 55 - 164) = 1$$
$$(42 + 21 - 62) = 1$$
$$(88 + 44 - 131) = 1$$

which means that

$$A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

which means that $\lambda_1 = 1$ is an eigenvalue of A. Similarly, we can see that elements in the first column are twice the elements in the second column so

$$A\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and so

$$A = 0$$

So, $\lambda_2 = 0$ is also an eigenvalue of A. We can also claim that since the columns of A are not linearly independent, so there exists an eigenvalue of 0. Now, the sum of all eigenvalues of A is equal to the trace of A which means the third eigenvalue of A is

$$\begin{aligned}\lambda_3 &= \mathrm{tr}(A) - (\lambda_1 + \lambda_2)\\ &= (110 + 21 - 131) - 1\\ &= -1\end{aligned}$$

Hence, eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -1$.

Problem 8. Let A be a complex $n \times n$ matrix and assume that for all $\vec{z} \in \mathbb{C}^n$, $\vec{z}^* A \vec{z} = 0$

a) Show that A is the zero matrix. (Hint: Start expanding $0 = (\vec{z}+\vec{w})^* A(\vec{z}+\vec{w})$ and $0 = (\vec{z}+i\vec{w})^* A(\vec{z}+i\vec{w})$ to show that $\vec{z}^* A\vec{w} = 0$ for all $\vec{z}, \vec{w} \in \mathbb{C}^n$.)

1. Expanding $0 = (\vec{z} + \vec{w})^* A(\vec{z} + \vec{w})$:

$$0 = (\vec{z} + \vec{w})^* A(\vec{z} + \vec{w}) = \vec{z}^* A\vec{z} + \vec{z}^* A\vec{w} + \vec{w}^* A\vec{z} + \vec{w}^* A\vec{w}$$

Since $\vec{z}^* A\vec{z} = 0$ and $\vec{w}^* A\vec{w} = 0$, we get:

$$0 = \vec{z}^* A\vec{w} + \vec{w}^* A\vec{z}$$

2. Expanding $0 = (\vec{z} + i\vec{w})^* A(\vec{z} + i\vec{w})$

$$0 = (\vec{z} + i\vec{w})^* A(\vec{z} + i\vec{w}) = \vec{z}^* A\vec{z} + i\vec{z}^* A\vec{w} - i\vec{w}^* A\vec{z} + (i\vec{w})^* A(i\vec{w})$$

Since $(i\vec{w})^* = -i\vec{w}^*$ and $\vec{z}^* A\vec{z} = 0$ (also, $(i\vec{w})^* A(i\vec{w}) = -\vec{w}^* A\vec{w} = 0$), we get:

$$0 = i\vec{z}^* A\vec{w} - i\vec{w}^* A\vec{z}$$

By comparing the two expansions, we see that both $\vec{z}^* A\vec{w}$ and $\vec{w}^* A\vec{z}$ must be zero. Since $\vec{z}$ and $\vec{w}$ are arbitrary vectors in $\mathbb{C}^n$, this implies that $A\vec{w} = 0$ for any $\vec{w} \in \mathbb{C}^n$, and thus A must be the zero matrix.

b) Show by example that if A is a real matrix such that $\vec{x}^T A\vec{x} = 0$ for all $\vec{x} \in \mathbb{R}^n$ then A is not necessarily the zero matrix.

To show by example that if A is a real matrix such that $\vec{x}^T A\vec{x} = 0$ for all $\vec{x} \in \mathbb{R}^n$ then A is not necessarily the zero matrix, consider the matrix:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Now, take an arbitrary vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, then:

$$\vec{x}^T A \vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} = -x_1 x_2 + x_2 x_1 = 0$$

Thus, $\vec{x}^T A \vec{x} = 0$ for all $\vec{x} \in \mathbb{R}^2$, yet A is clearly not the zero matrix. This shows that the property doesn't necessarily hold for real matrices, contrasting with complex matrices as shown in part a).