Topics in Algebra solution

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Problems in Section 3.9.

1. Find the greatest common divisor of the following polynomials over F, the field of rational numbers.
a) $x^3 - 6x^2 + x + 4$ and $x^5 - 6x + 1$.

Solution. Observe that

$$
\begin{aligned}
x^5 - 6x + 1 &= (x^2 + 6x + 35)(x^3 - 6x^2 + x + 4) + 200x^2 - 65x - 139,\\
x^3 - 6x^2 + x + 4 &= \left(\frac{x}{200}\right)(200x^2 - 65x - 139) - \frac{227}{40}x^2 + \frac{339}{200}x + 4,\\
200x^2 - 65x - 139 &= \left(-\frac{8000}{227}\right)\left(-\frac{227}{40}x^2 + \frac{339}{200}x + 4\right) + \left(-\frac{1195}{227}x + \frac{447}{227}\right),\\
-\frac{227}{40}x^2 + \frac{339}{200}x + 4 &= \left(-\frac{51529}{47800}x + \frac{580212}{7140125}\right)\left(-\frac{1195}{227}x + \frac{447}{227}\right) + \frac{27417968}{7140125},\\
-\frac{1195}{227}x + \frac{447}{227} &= \left(-\frac{8532449375}{6223878736}x + \frac{3191635875}{6223878736}\right)\left(\frac{27417968}{7140125}\right) + 0.
\end{aligned}
$$

Thus the greatest common divisor of $x^3 - 6x^2 + x + 4$ and $x^5 - 6x + 1$ is 1. $\square$

b) $x^2 + 1$ and $x^6 + x^3 + x + 1$.

Solution. Note that $x^6 + x^3 + x + 1 = (x^4 - x^2 + x + 1)(x^2 + 1)$ so that their greatest common divisor is $x^2 + 1$. $\square$

2. Prove that
a) $x^2 + x + 1$ is irreducible over F, the field of integers mod 2.

Proof. Substituting $x = 0$ and $x = 1$ both to $x^2 + x + 1$ yields 1 mod 2, so that $x^2 + x + 1$ is irreducible over F. $\square$

b) $x^2 + 1$ is irreducible over the integers mod 7.

Proof. Note that for prime p, $x^2 + 1 \equiv 0 \pmod{p}$ has solution only if p is a prime of form $4k + 1$. But $7 = 4 \cdot 1 + 3$, so that $x^2 + 1 \not\equiv 0 \pmod{7}$. Hence, $x^2 + 1$ is irreducible over F. □

c) $x^3 - 9$ is irreducible over the integers mod 31.

Proof. Note that given polynomial is degree of 3. So if it was reducible, it must have at least one polynomial of degree 1 as its factor. Hence, it admits a root. Thus, assume that $x^3 \equiv 9 \pmod{31}$ for some x. By FLT, $x^{30} \equiv 1 \pmod{31}$. Consequently,

$$x^{30} \equiv 9^{10} \equiv 5 \not\equiv 1 \pmod{31},$$

which is a contradiction. Hence, $x^3 - 9$ is irreducible over F. □

d) $x^3 - 9$ is reducible over the integers mod 11.

Proof. $x = 4$ gives $4^3 = 64 \equiv 9 \pmod{11}$. Hence, $(x - 4)$ is a factor of $x^3 - 9$ in F. Thus, $x^3 - 9$ is reducible over F. □

3. Let F, K be two fields $F \subset K$ and suppose $f(x), g(x) \in F[x]$ are relatively prime in $F[x]$. Prove that they are relatively prime in $K[x]$.

Proof. As $f(x), g(x)$ are relatively prime in $F[x]$, there exists $\lambda(x), \mu(x) \in F(x)$ and an unit $k \in F[x]$ such that

$$f(x)\lambda(x) + g(x)\mu(x) = k.$$

Now merely consider the above equation as an equation in $K[x]$. Since units in $F[x]$ is also units in $K[x]$, $f(x)$ and $g(x)$ are relatively prime in $K[x]$ too. □

4. a) Prove that $x^2 + 1$ is irreducible over the field F of integers mod 11 and prove directly that $F[x]/(x^2 + 1)$ is a field having 121 elements.

Proof. Note that for a prime p, equation $x^2 + 1$ mod p admits a root only if p is a prime of form $4k + 1$. But $11 = 4 \cdot 2 + 3$, so that $x^2 + 1$ has no root in F. Thus, $x^2 + 1$ is irreducible in F. Consequently, $((x^2 + 1))$ is a maximal ideal in $F[x]$ so that $F[x]/(x^2 + 1)$ is a field. Since every element in this field is expressible in a way that;

$$\frac{F[x]}{(x^2 + 1)} = \left\{ ax + b + (x^2 + 1) \;\middle|\; a, b \in F \right\},$$

hence there are $11 \cdot 11 = 121$ distinct elements in this field. □

b) Prove that $x^2 + x + 4$ is irreducible over F, the field of integers mod 11 and prove directly that $F[x]/(x^2 + x + 4)$ is a field having 121 elements.

Proof. Since $f(x) = x^2 + x + 4$ is a polynomial of degree 2, we check if it admits a root or not. By simple calculations, $f(0) \equiv f(10) \equiv 4 \pmod{11}$, $f(1) \equiv f(9) \equiv 6 \pmod{11}$, $f(2) \equiv f(8) \equiv -1 \pmod{11}$, $f(3) \equiv f(7) \equiv 5 \pmod{11}$, $f(4) \equiv f(6) \equiv 2 \pmod{11}$, $f(5) \equiv 1 \pmod{11}$. Hence, $f(x)$ is irreducible in F. And similarly as in Problem 4, $F[x]/(f(x))$ is a field with 121 elements. $\square$

c) Prove that the fields of part a) and part b) are isomorphic.

Proof. We build a homomphism between $F[x]/(x^2+1)$ and $F[x]/(x^2+x+4)$. Suppose $\phi : F[x]/(x^2+1) \to F[x]/(x^2+x+4)$. Suppose $\phi(x) = a + bx$. Then

$$\phi(x^2+1) = \phi(x)^2 + \phi(1) = (a+bx)^2 + a = b^2x^2 + 2abx + (a^2+a)$$

must divide x^2+x+4 so that $b^2x^2 + 2abx + (a^2+a) = b^2x^2 + b^2x + 4b^2$. On solving this,

$$2ab = b^2, \quad a^2 + a = 4b^2 \pmod{11} \implies a = 3,\ b = 6.$$

Thus, $\phi(x) = 3 + 6x$. We know this yields a bijection. To check this is a homomorphism, $\phi((a+bx)+(c+dx)) = \phi((a+c)+(b+d)x) = 3(a+c)+6(b+d)x = \phi(a+bx)+\phi(c+dx)$. Also, we can check that $\phi((a+bx)(c+dx)) = \phi(a+bx)\phi(c+dx)$ similarly. Therefore, $F[x]/(x^2+1)$ and $F[x]/(x^2+x+4)$ are isomorphic. $\square$

5. Let F be the field of real numbers. Prove that $F[x]/(x^2+1)$ is a field isomorphic to the field of complex numbers.

Proof. Note that x^2+1 is irreducible in $\mathbb{R} = F$. Thus, $F[x]/(x^2+1)$ is a field, with elements of the form $a + bx + (x^2+1)$, $a, b \in F$. We now define a mapping $\phi : F[x]/(x^2+1) \to \mathbb{C}$ by $\phi(a+bx+(x^2+1)) = a+bi$. Is it well defined? Suppose $a+bx+(x^2+1) = c+dx+(x^2+1)$. Then $a-c+(b-d)x \in (x^2+1)$ so that $(a-c)+(b-d)x = 0, a = c, b = d$. Thus, $a+bi = c+di$ and hence ϕ is well defined. We check if ϕ is a homomorphism. Observe that

$$\begin{aligned}
\phi((a+bx+(x^2+1))+(c+dx+(x^2+1))) &= \phi((a+c)+(b+d)x+(x^2+1)) \\
&= (a+c)+(b+d)i = (a+bi)+(c+di) \\
&= \phi(a+bx)+\phi(c+dx), \\
\phi((a+bx)(c+dx)+(x^2+1)) &= \phi((ac-bd)x+(ad+bc)x+(x^2+1)) \\
&= (ac-bd)+(ad+bc)i = (a+bi)(c+di) \\
&= \phi(a+bx+(x^2+1))\phi(c+dx+(x^2+1)).
\end{aligned}$$

Thus, ϕ is a homomorphism. Also, it is clearly surjective. Now we consider its kernel. Suppose $\phi(a+bi+(x^2+1)) = a+bi = 0$. Then $a = 0, b = 0$ so that $\phi(a+bi+(x^2+1)) = 0 \iff \phi((x^2+1)) = 0$. Hence, ϕ is injective. Therefore, we have established an onto isomorphism between $F[x]/(x^2+1)$ and $\mathbb{C}$. $\square$

6. Define the derivative $f'(x)$ of the polynomial

$$
\begin{aligned}
f(x) &= a + 0 + a_1 x \cdots + a_n x^n \\
f'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}.
\end{aligned}
$$

Prove that if $f(x) \in F[x]$, where F is the field of rational numbers, then $f(x)$ is divisible by the square of a polynomial if and only if $f(x)$ and $f'(x)$ have a greatest common divisor $d(x)$ of positive degree.

Proof. Suppose $f(x)$ is divisible by $q(x)^2$, where $deg(q(x)) \geq 1$. Then $f(x) = k(x)q(x)^2$ for some $k(x) \in F[x]$. Consequently, $f'(x) = k'(x)q(x)^2 + 2k(x)q(x)q'(x)$ so that $q(x) \mid f'(x)$. Let $d(x)$ be the greatest common divisor of $f(x)$ and $f'(x)$. Since $deg(d(x)) \geq deg(q(x)) \geq 1$, We are done. Conversely, assume that $f(x)$ and $f'(x)$ have a greatest common divisor $d(x)$ of positive degree. Then there exists a prime(irreducible) polynomial $p(x)$ which divides both $f(x)$ and $f'(x)$. Let $f(x) = t(x)p(x)$. Then $f'(x) = t'(x)p(x) + t(x)p'(x)$, so that $p(x) \mid p'(x)t(x)$. As $deg(p(x)) > deg(p'(x))$, $p(x) \nmid p'(x)$ and since $p(x)$ is prime, $p(x) \mid t(x)$. That is, $t(x) = s(x)p(x)$ for some $s(x) \in F[x]$ Thus, $f(x) = s(x)p(x)^2$ and hence $p(x)^2 \mid f(x)$. $\square$

7. If $f(x)$ is in $F[x]$, where F is the field of integers mod p, p a prime, and $f(x)$ is irreducible over F of degree n prove that $F[x]/(f(x))$ is a field with p^n elements.

Proof. Note that $F[x]/(f(x))$ is clearly a field since $f(x)$ is irreducible over $F[x]$. Now since $F[x]$ being an Euclidean ring, division algorithm in $F[x]$ assures the uniqueness of the remainder of any polynomial on division by $f(x)$. Hence, any elements in $F[x]/(f(x))$ must be a polynomial of degree less than $n = deg(f(x))$ and vice versa, it consists of p^n elements in total. $\square$