Math 4108 HW1

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Problems in the Section 1

1.1 Many of the standard techniques of classical algebra are consequences of the axioms of a ring. The exceptions are those depending on commutativity of multiplication (R7) and divisibility (R10). Let R be a ring.

(i) Show that, for all a in R,

$$0a = 0a = 0$$

Prove:

$$\begin{gathered} a0 = a(0+0) \\ a0 = a0 + a0 \\ a0 + (-a0) = (a0 + a0) + (-a0) \\ a0 + (-a0) = a0 + (a0 + (-a0)) \\ 0 = a0 + 0 \\ a0 = 0 \end{gathered}$$

(ii) Show that, for all a, b in R, $a(-b) = (-a)b = -ab, (-a)(-b) = ab$

Prove:

$$\begin{aligned} (-a)b &= (-a)b + 0 \\ &= (-a)b + (ab + (-ab)) \\ &= ((-a)b + ab) + (-ab) \\ &= ((-a + a)b) + (-ab) \\ &= (0b) + (-ab) \\ &= (0b + 0) + (-ab) \\ &= (0b + (0b + (-0b))) + (-ab) \\ &= ((0b + 0b) + (-0b)) + (-ab) \\ &= ((0 + 0)b + (-0b)) + (-ab) \\ &= (0b + (-0b)) + (-ab) \\ &= 0 + (-ab) \\ &= -ab \end{aligned}$$

1.2 What difference does it make if the stipulation that $1 \neq 0$ is omitted from Axiom (R7)? Remark: (R7) the commutative law for multiplication:

$$ab = ba(a, b \in R).$$

The stipulation that $1 \neq 0$ is usually included in this axiom to distinguish between rings that contain a multiplicative identity (denoted as 1) and those that do not.

If the stipulation $1 \neq 0$ is omitted from Axiom (R7), it means that the ring under consideration may not necessarily have a multiplicative identity. In such a case, a more general algebraic structure known as a rng (pronounced "rung"), which is like a ring but without the requirement of a multiplicative identity.

Omitting the stipulation allows for a broader consideration of algebraic structures that do not necessarily have a multiplicative identity.

1.3. Axiom (R7) ensures that a field has at least two elements. Show that there exists a field with exactly two elements.