Topics in Algebra solution

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November 6, 2020

Problems in the Section 2.10.

1. Find the orbits and cycles of the following permutations:
a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 1 & 6 & 7 & 9 & 8 \end{pmatrix}$.

Solution. $\text{Orb}(1) = \{1,2,3,4,5\}$, $\text{Orb}(6) = \{6\}$, $\text{Orb}(7) = \{7\}$, $\text{Orb}(8) = \{8,9\}$. Cycles: $(1,2,3,4,5),(8,9)$. $\square$

b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 1 & 2 \end{pmatrix}$.

Solution. $\text{Orb}(1) = \{1,6,2,5\}$, $\text{Orb}(3) = \{3,4\}$. Cycles: $(1,6,2,5),(3,4)$. $\square$

2. Write the permutations in Problem 1 as the product of disjoint cycles.

Solution. For a), $(1,2,3,4,5)(8,9)$. For b), $(1,6,2,5)(3,4)$. $\square$

3. Express as the product of disjoint cycles:
a) $(1,2,3)(4,5)(1,6,7,8,9)(1,5)$.

Solution. $(1,2,3)(4,5)(1,6,7,8,9)(1,5) = (1,2,3,6,7,8,9,5,4)$. $\square$

b) $(1,2)(1,2,3)(1,2)$.

Solution. $(1,2)(1,2,3)(1,2) = (1,3,2)$. $\square$

4. Prove that $(1,2,\cdots,n)^{-1} = (n,n-1,n-2,\cdots,2,1)$.

Proof. Let $\sigma = (1,2,\cdots,n)$. Considering σ as a bijective function,

$$\sigma \cdot (n,n-1,n-2,\cdots,2,1)(k) = (n,n-1,n-2,\cdots,2,1) \cdot \sigma(k) = k \quad \text{for all } 1 \le k \le n.$$

Therefore, $\sigma^{-1} = (1,2,\cdots,n)^{-1} = (n,n-1,n-2,\cdots,2,1)$. $\square$

5. Find the cycle structure of all the powers of $(1, 2, \cdots, 8)$.

Solution. Note that

$$\begin{aligned}
(1,2,\cdots,8)^1 &= (1,2,\cdots,8),\\
(1,2,\cdots,8)^2 &= (1,3,5,7)(2,4,6,8),\\
(1,2,\cdots,8)^3 &= (1,4,7,2,5,8,3,6),\\
(1,2,\cdots,8)^4 &= (1,5)(2,6)(3,7)(4,8),\\
(1,2,\cdots,8)^5 &= (1,6,3,8,5,2,7,4),\\
(1,2,\cdots,8)^6 &= (1,7,5,3)(2,8,6,4),\\
(1,2,\cdots,8)^7 &= (8,7,6,5,4,3,2,1),\\
(1,2,\cdots,8)^8 &= id.
\end{aligned}$$

□

6. a) What is the order of an n-cycle?

Proof. Order of n-cycle is n. Let $\sigma = (a_1, a_2, \cdots, a_n)$ be an n-cycle. Then

$$\sigma^i(a_k) = a_{(k+i) \mod n}, \quad (1 \leq i)$$

so that $\sigma^i(a_k) = a_k \iff a_k = a_{(k+i) \mod n}, i \equiv 0 \pmod{n}$. If i was to be the order of σ, it is must that $i = n$. Therefore, the order of an n-cycle is n. □

b) What is the order of the product of the disjoint cycles of lengths $m_1, m_2, \cdots, m_k$?

Proof. We prove that the order of given permutation is $\operatorname{lcm}(m_1, m_2, \cdots, m_k)$. Let $s = \operatorname{lcm}(m_1, m_2, \cdots, m_k)$. Denote the disjoint cycles of lengths m_i by $\sigma_i, 1 \leq i \leq k$. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_k$. Suppose n is the order of σ. Using the fact that each σ_i are disjoint(and also commutative), we have

$$\begin{aligned}
e = \sigma^n = (\sigma_1\sigma_2\cdots\sigma_k)^n = \sigma_1^n\sigma_2^n\cdots\sigma_k^n &\implies \sigma_i^n = e,\\
\implies m_i \mid n, \quad \operatorname{lcm}(m_1, m_2, \cdots, m_k) &= s \leq n.
\end{aligned}$$

Moreover,

$$\sigma^s = (\sigma_1\sigma_2\cdots\sigma_k)^s = \sigma_1^s\sigma_2^s\cdots\sigma_k^s = e$$

so that $n \leq s$. This proves that $n = s = \operatorname{lcm}(m_1, m_2, \cdots, m_k)$. □

c) How do you find the order of a given permutation?

Solution. Express the permutation into product of disjoint cycles, find the length(order) of each cycles and compute their least common multiple. This yields the order of given permutation. □

7. Compute $a^{-1}ba$, where
1) $a = (1,3,5)(1,3), b = (1,5,7,9)$.

Solution. $a^{-1}ba = (2,7,9,3)$. □

2) $a = (5,7,9), b = (1,2,3)$.

Solution. $a^{-1}ba = (1,2,3)$. □

8. a) Given the permutation $x = (1,2)(3,4), y = (5,6)(1,3)$, find a permutation a such that $a^{-1}xa = y$.

Solution. Let $a = (4,3,1,5)(2,6)$. By a simple calculation, $a^{-1}xa = y$. □

b) Prove that there is no a such that $a^{-1}(1,2,3)a = (1,3)(5,7,8)$.

Proof. Note that $sgn(a^{-1}(1,2,3)a) \equiv 0 \pmod 2$ while $sgn((1,3)(5,7,8)) \equiv 1 \pmod 2$. Thus, there is no a satisfying the given relation. □

c) Prove that there is no permutation a such that $a^{-1}(1,2)a = (3,4)(1,5)$.

Proof. Note that $sgn(a^{-1}(1,2)a) \equiv 1 \pmod 2$ while $sgn((3,4)(1,5)) \equiv 0 \pmod 2$. Thus, there is no a satisfying the given relation. □

9. Determine for what m an m-cycle is an even permutation.

Solution. Note that every permutation is the product of transpositions. Let $\sigma = (a_1, a_2, \cdots, a_m)$ be an m-cycle. Then

$$\sigma = (a_1, a_2, \cdots, a_m) = (a_1, a_2)(a_1, a_3) \cdots (a_1, a_m)$$

where RHS is product of $m-1$ transpositions. Thus, whenever $m-1$ is even, that is, m is odd, the given m-cycle is an even permutation. □

10. Determine which of the following are even permutations.
a) $(1,2,3)(1,2)$.

Solution. It is an odd permutation. □

b) $(1,2,3,4,5)(1,2,3)(4,5)$.

Solution. It is an odd permutation. □

c) $(1,2)(1,3)(1,4)(2,5)$.

Solution. It is an even permutation. $\square$

11. Prove that the smallest subgroup of S_n containing $(1,2)$ and $(1,2,\cdots,n)$ is S_n. (In other words, these generate S_n.)

Proof. Note that

$$\begin{aligned}(n-1,n) = (1,2,\cdots,n)(1,2)^{n-2}, (n-2,n-1,n) = (1,2,\cdots,n)(1,2)^{n-3}\\ \implies (n-2,n-1,n)(n-1,n) = (n-2,n).\end{aligned}$$

Repeating the similar process by interchanging $(n-1,n)$ by $(n-2,n)$, we can generate $(1,n),(2,n),\cdots,(n-1,n)$. Let (i,j) be an arbitrary transposition. Since every permutation is a product of transpositions, it is enough to show that (i,j) can be generated. Observe that

$$(i,j) = (i,n)(j,n)(i,n),$$

so that (i,j) can be generated with $(1,2)$ and $(1,2,\cdots,n)$. Therefore, $((1,2),(1,2,\cdots,n)) = S_n$. $\square$

12. Prove that for $n \geq 3$ the subgroup generated by the 3-cycles is A_n.

Proof. Let H be the set of all 3-cycles. Clearly $H \subset A_n$. It is enough to show that the permutations $(i,j)(k,l),(i,j)(j,k) \in A_n$ are in H. Note that

$$(i,j)(k,l) = (i,k,j)(k,j,l), \quad (i,j)(j,k) = (i,k,j)$$

so that $A_n \subset H$. Therefore, $H = A_n$. $\square$

13. Prove that if a normal subgroup of A_n contains even a single 3-cycle it must be all of A_n.

Proof. It is enough to show that such N contains every possible 3-cycles of A_n. The case for S_3 is trivial since $(1,2,3) \in N, \implies (1,3,2) = (1,2,3)^2 \in N$. Now consider the case $n \leq 4$. Suppose $(a,b,c) \in N$. We show that (a,b,d) is in N for any $d \neq a,b,c$. Observe that

$$(a,b,d) = (a,b)(c,d)(a,b,c)^2[(a,b)(c,d)]^{-1} \in N.$$

This implies that, a 3-cycle with at most one element interchanged from a,b,c is still in N. Now we consider the case (a,d,e) where $d,e \neq a,b,c$. Note that

$$(a,d,e) = (a,b,d)^2(a,b,e) \in N$$

so that a 3-cycle with at most two elements different from a, b, c is in N. Finally, we are left with case (d, e, f) where $d, e, f \neq a, b, c$. Note that $(a, d, e) \in N$, $(a, d, f) \in N$. Consequently,

$$(d, e, f) = ((a, d, f)(a, e, d))^2 = (d, f, e)^2 \in N.$$

Therefore, we can conclude that every 3-cycles is contained in the normal subgroup N. Applying the Problem 12, we have that $N = A_n$. Hence proved. □

14. Prove that A_5 has no normal subgroups $N \neq (e), A_5$.

Proof. We introduce two proofs, one with most elementary approach and the other one using the conjugacy class.
(Elementary proof) There are 5 cases that the normal subgroup N of A_n can have:
1) If N contains a 3 cycle: Apply the Problem 13. Then we have $N = A_5$.
2) If N contains a product of disjoint cycles, with at least one has length greater than 3. That is, $\sigma = \mu(a_1, a_2, a_3, \cdots, a_r) \in N$. In this case, note that

$$\sigma^{-1}[(a_1, a_2, a_3)\sigma(a_1, a_2, a_3)^{-1}] = (a_1, a_3, a_4) \in N,$$

so that applying the Problem 12 again, we obtain $N = A_5$.
3) If N contains a product of disjoint cycles where $\sigma = \mu(a_4, a_5, a_6)(a_1, a_2, a_3)$. Observe that

$$\sigma^{-1}[(a_1, a_2, a_4)\sigma(a_1, a_2, a_4)^{-1}] = (a_1, a_4, a_2, a_3, a_5) \in N$$

so that this is the same case of case 2). Hence, $N = A_5$ again.
4) If N contains a product of disjoint cycles where $\sigma = \mu(a_1, a_2, a_3)$ where μ is product of disjoint transpositions. Observe that

$$\sigma^2 = \mu^2(a_1, a_2, a_3)^2 = (a_1, a_3, a_2) \in N$$

so that applying Problem 12, we have $N = A_5$.
5) If N contains a permutations of the form $\sigma = \mu(a_3, a_4)(a_1, a_2)$ where μ is product of even number of disjoint transpositions. Note that

$$\sigma^{-1}[(a_1, a_2, a_3)\sigma(a_1, a_2, a_3)^{-1}] = (a_1, a_4)(a_2, a_3) \in N.$$

Since $n = 5$, there exists $a_5 \neq a_i, i = 1, 2, 3, 4$. Let $\tau = (a_1, a_3, a_5)$. Observe that

$$\tau^{-1}(a_1, a_4)(a_2, a_3)\tau(a_1, a_4)(a_2, a_3) = (a_1, a_4, a_2, a_5, a_3) \in N$$

and similarly with the case 2), we have $N = A_5$.
Therefore, we can conclude that A_5 has no normal subgroups other than (e) or A_5 itself(A_5

is simple).

(Proof using conjugacy class) Note that the conjugacy class sizes of A_5 are: 1,12,12,20,15. Since any non-trivial normal subgroup must contain conjugacy class of size 1(the identity) and one or more other conjugacy class, the possible order of such normal group is given by the summations of the conjugacy sizes. By simple calculations, the possible candidates for the order of the normal subgroups are: 13,16,21,25,28,33,36,40,45,48,60. Now applying the Lagrange's theorem, the only possible non-trivial normal subgroup is A_5 itself. Hence A_5 is simple. $\square$

15. Assuming the result of Problem 14, prove that any subgroup of A_5 has order at most 12.

Proof. Suppose there exists a subgroup H of order 30. But since $[A_5 : H] = 2$, H is normal in A_5, contradiction to the fact that A_5 is simple. Suppose there is a subgroup H of order 20. Then $[G : H] = 3, 60 \nmid 3! = 6$ so that H contains a non-trivial normal subgroup of A_5. But this is also a contradiction. Similarly for H of order 15. Since $[G : H] = 4$ and $60 \nmid 4! = 24$, A_5 admits a non-trivial normal subgroup, hence a contradiction. Also note that $A_4 \subset A_5$, where $o(A_4) = 12$. Therefore, any subgroup of A_5 has order at most 12. $\square$

16. Find all the normal subgroups in S_4.

Solution. There are 4 normal subgroups of S_4: The whole group, trivial group (e), A_4 and normal Klein-4 group $= \{id, (12)(34), (13)(24), (14)(23)\}$. $\square$

17. If $n \geq 5$ prove that A_n is the only nontrivial normal subgroup in S_n.

Proof. We introduce some useful lemmas:

Lemma. 1. The commutator subgroup of $S_n, n \geq 3$ is A_n.
$\Rightarrow$ Note that every elements of the form $\sigma\tau\sigma^{-1}\tau^{-1}$ where $\sigma, \tau \in S_n$ is even permutations. Thus, $S'_n \subset A_n$. Now, observe that

$$(1,3,2)(1,2)(1,3,2)^{-1}(1,2)^{-1} = (1,2,3) \in S'_n.$$

Since every commutator subgroup is a normal subgroup, and $S'_n \subset A_n$, applying the result of Problem 13 we have $S'_n = A_n$. Hence proved.

Lemma. 2. The symmetric group $S_n, n \geq 3$ has trivial center. That is, $Z(S_n) = (e)$.
$\Rightarrow$ Let $\sigma \neq id, \in S_n$. Suppose σ maps i to j. Since $n \geq 3$, we can choose $k \neq i, j$ and a permutation $\tau \in S_n$ such that τ maps k into i. Then clearly $\tau\sigma\tau^{-1}$ maps k into j so that $\tau\sigma\tau^{-1} \neq \sigma$. This shows that no element $\sigma \in S_n$ lies in the center $Z(S_n)$. Therefore, $Z(S_n) = (e)$.

Lemma. 3. Suppose H is a normal subgroup of G and $H \cap G' = (e)$, where G' is the commutator subgroup of G. Then $H \subset Z(G)$.
$\Rightarrow$ Let $h \in H$ and $g \in G$. Since $ghg^{-1} \in H$ and $h^{-1} \in H$, so does $ghg^{-1}h^{-1} \in H$. But $ghg^{-1}h^{-1} \in G'$ also. Since $H \cap G' = (e)$, it is must that $ghg^{-1}h^{-1} = e \iff gh = hg$. Therefore, $h \in Z(G)$ and hence $H \subset Z(G)$.

Lemma. 4. For $n \geq 5$, the alternating group A_n is simple.
$\Rightarrow$ We know that A_5 is simple. Now, for the sake of contradiction, assume that there is a nontrivial normal subgroup N of $A_n, n \geq 6$. Let $\sigma \in N$ which is the permutation with maximal number of fixed points, that is, $i\sigma = i$. We prove that σ is 3-cycle or an identity. Suppose then decompose $J_n = \{1, 2, \cdots, n\}$ into the orbits of σ. Suppose σ has only two elements for each orbits(except for fixed points). Then since σ is of even permutation, it must have at least two such (distinct) orbits. Then σ can be represented as

$$\sigma = \mu(i, j)(k, l)$$

where μ is disjoint with $(i, j), (k, l)$. Now consider $\tau = (k, l, m)$ where $m \neq i, j, k, l$. Then

$$\sigma' = \tau\sigma\tau^{-1}\sigma^{-1}$$

fixes i, j and possibly the rest of $t \in J_n - \{i, j, k, l, m\}$. Therefore, σ' fixes at least one more elements in J_n then σ, contradicting that σ is the permutation with maximal number of fixed points. Hence, N admits a 3-cycle and consequently, $N = A_n$. Thus, A_n is simple for $n \geq 5$.(In fact, this proof is a generalization of the elementary method used for Problem 15)

Now suppose there is a non-trivial normal subgroup H of S_n. Since the intersection of two normal subgroups is also normal, $H \cap A_n$ is normal. But as A_n is simple by lemma 4, $H \cap A_n = (e)$. Now applying the lemma 1 and lemma 3 , $H \subset Z(S_n)$. Applying lemma 2 now, $H \subset Z(S_n) = (e)$. Thus, $H = (e)$, a contradiction. Hence A_n is the only nontrivial normal subgroup of S_n. $\square$

18. Find the permutation representation of a cyclic group of order n.

Solution. Let G be the cyclic group of order n. Let $a \in G$ be the generator. Then G can be written in a cycle form if we represent G by $\{1, 2, \cdots, n\}$ in the following way:

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ e & a & \cdots & a^{n-1} \end{pmatrix}.$$

Using the notation of the theorem 2.9.1 we have:

$$\begin{aligned}
&\tau_e = id, \\
&\tau_a = (1, 2, \cdots, n), \\
&\tau_{a^2} = (1, 2, \cdots, n)^2, \\
&\tau_{a^k} = (1, 2, \cdots, n)^k, \\
&\tau_{a^{n-1}} = (n, n-1, \cdots, 1).
\end{aligned}$$

Hence, the permutation representation of G is given by $((1, 2, \cdots, n))$. $\square$

19. Let G be the group $\{e, a, b, ab\}$ of order 4, where $a^2 = b^2 = e$, $ab = ba$. Find the permutation representation of G.

Solution. We shall write G is a cycle-form by representing G by $\{1, 2, 3, 4\}$ in the following way:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ e & a & b & ab \end{pmatrix}.$$

Using the notation of the theorem 2.9.1 we have:

$$\begin{aligned}
&\tau_e = id, \\
&\tau_a = (1, 2)(3, 4), \\
&\tau_b = (1, 3)(2, 4), \\
&\tau_{ab} = (1, 4)(2, 3).
\end{aligned}$$

Hence, the permutation representation of G is given by V_4, where

$$V_4 = \{id, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}.$$

$\square$

20. Let G be the group S_3. Find the permutation representation of S_3.(Note: This gives an isomorphism of S_3 into S_6.

Solution. Note that $S_3 = \{e, y, y^2, x, xy, xy^2\}$. We represent G in a cycle-form by representing it by $\{1, 2, 3, 4, 5, 6\}$ in the following way:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ e & y & y^2 & x & xy & xy^2 \end{pmatrix}.$$

Using the notation of the theorem 2.9.1 we have:

$$\begin{aligned}
&\tau_e = id,\\
&\tau_y = (1,2,3)(4,5,6),\\
&\tau_{y^2} = (1,3,2)(4,6,5),\\
&\tau_x = (1,4)(2,6)(3,5),\\
&\tau_{xy} = (1,5)(2,4)(3,6),\\
&\tau_{xy^2} = (1,6)(2,5)(3,4).
\end{aligned}$$

Hence, the permutation representation of G is given by:

$$S_3 = \{id, (1,2,3)(4,5,6), (1,3,2)(4,6,5), (1,4)(2,6)(3,5), (1,5)(2,4)(3,6), (1,6)(2,5)(3,4)\}$$

in S_6. □

21. Let G be a group $\{e, \theta, a, b, c, \theta a, \theta b, \theta c\}$, where $a^2 = b^2 = c^2 = \theta$, $\theta^2 = e$, $ab = \theta ba = c$, $bc = \theta cb = a$, $ca = \theta ac = b$.
a) Show that θ is in the center Z of G, and that $Z = \{e, \theta\}$.

Proof. Note that $\theta a = a^3 = a \cdot a^2 = a\theta$, $(\theta a)\theta = a^5 = a^2 \cdot (a^2 \cdot a) = \theta(\theta a)$. This holds for b, c also. Hence, $\theta \in Z$. But note that in general, $ab = c \neq \theta c = ba$. Hence, $Z = \{e, \theta\}$. □

b) Find the commutator subgroup of G.

Proof. Note that $aba^{-1}b^{-1} = ab(\theta a)(\theta b) = ab(ab) = c^2 = \theta$. Also, $a(\theta b)a^{-1}(\theta b)^{-1} = a(\theta b)(\theta a)b = abab = \theta$. And $aea^{-1}e = e$. This holds even if we change the role a, b into any two other elements of a, b, c. Hence, the commutator group G' is exactly Z. □

c) Show that every subgroup of G is normal.

Proof. There are only 4 non-trivial subgroups of G: Z, $\{e, \theta, a, \theta a\}$, $\{e, \theta, b, \theta b\}$, $\{e, \theta, c, \theta c\}$. Z is clearly normal in G, and the rest are of index 2 in G. Therefore normal. Thus, every subgroup of G is normal. □

d) Find the permutation representation of G.

Solution. We represent G in a cycle-form by representing it by $\{1, 2, 3, 4, 5, 6, 7, 8\}$ in the following way:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ e & \theta & a & b & c & \theta a & \theta b & \theta c \end{pmatrix}.$$

Using the notation of the theorem 2.9.1 we have:

$$\begin{aligned}
&\tau_e = id,\\
&\tau_\theta = (1,2)(3,6)(4,7)(5,8),\\
&\tau_a = (1,3,2,6)(4,8,7,5),\\
&\tau_b = (1,4,2,7)(3,5,6,8),\\
&\tau_c = (1,5,2,8)(3,7,6,4),\\
&\tau_{\theta a} = (1,6,2,3)(4,5,7,8),\\
&\tau_{\theta b} = (1,7,2,4)(3,8,6,5),\\
&\tau_{\theta c} = (1,8,2,5)(3,4,6,7).
\end{aligned}$$

Hence, the permutation representation of G is given by:

$$\begin{aligned}
G = \{&id, (1,2)(3,6)(4,7)(5,8), (1,3,2,6)(4,8,7,5), (1,4,2,7)(3,5,6,8),\\
&(1,5,2,8)(3,7,6,4), (1,6,2,3)(4,5,7,8), (1,7,2,4)(3,8,6,5), (1,8,2,5)(3,4,6,7)\}.
\end{aligned}$$

in S_8. $\square$

22. Let G be the dihedral group of order $2n$ (see Problem 27, Section 2.6). Find the permutation representation of G.

Solution. Recall that $D_{2n} = \{e, y, \cdots, y^{n-1}, x, xy, \cdots, xy^{n-1}.\}$. We represent G in a cycle-from by representing it by $\{1, 2, \cdots, n, n+1, \cdots, 2n\}$ in the following way:

$$\begin{pmatrix} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & 2n \\ e & y & \cdots & y^{n-1} & x & xy & \cdots & xy^{n-1} \end{pmatrix}.$$

Using the notation of the theorem 2.9.1 we have:

$$\begin{aligned}
&\tau_e = id,\\
&\tau_y = (1,2,\cdots,n)(n+1,n+2,\cdots,2n),\\
&\tau_{y^2} = (1,2,\cdots,n)^2(n+1,n+2,\cdots,2n)^2,\\
&\vdots\\
&\tau_{y^{n-1}} = (n,n-1,\cdots,2,1)(2n,2n-1,\cdots,n+2,n+1),\\
&\tau_x = (1,n+1)(2,2n)(3,2n-1)\cdots(n-1,n+3)(n,n+2),\\
&\tau_{xy} = \tau_x\tau_y = [(1,n+1)(2,2n)\cdots(n,n+2)]\cdot[(1,2,\cdots,n)(n+1,n+2,\cdots,2n)]\\
&= (1,n+2)(2,n+1)(3,2n)(4,2n-1)\cdots(n-1,n+4)(n,n+3),\\
&\tau_{xy^2} = \tau_x\tau_{y^2} = [(1,n+1)(2,2n)\cdots(n,n+2)]\cdot[(1,2,\cdots,n)^2(n+1,n+2,\cdots,2n)^2]\\
&= (1,n+3)(2,n+2)(3,n+1)(4,2n)(5,2n-1)\cdots(n-1,n+5)(n,n+4),\\
&\vdots\\
&\tau_{xy^{n-1}} = (1,2n)(2,2n-1)\cdots(n-1,n+2)(n,n+1).
\end{aligned}$$

Here we used the fact that $\tau_{xy} = \tau_x \cdot \tau_y$. Now the group of collection of all above permutations is the permutation representation of G in S_{2n}. □

23. Show that if G is an abelian group, then the permutation representation of G coincides with the second permutation representation of G(i.e., in the notation of the previous section, $\lambda_g = \tau_g$ for all $g \in G$.

Proof. It is enough to show that $x\lambda_g = x\tau_g$ for all $x \in G$ for each $g \in G$. Since G is abelian, for each fixed $g \in G$, $x\lambda_g = gx = xg = x\tau_g$ for all $x \in G$. Hence, the permutation representation of G coincides with the second permutation representation of G. □

24. Find the second permutation representation of S_3. Verify directly from the permutations obtained here and in Problem 20 that $\lambda_a\tau_b = \tau_b\lambda_a$ for all $a, b \in S_3$.

Solution. Note that $S_3 = \{e, y, y^2, x, xy, xy^2\}$. We represent G in a cycle-form by representing it by $\{1,2,3,4,5,6\}$ in the following way:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ e & y & y^2 & x & xy & xy^2 \end{pmatrix}.$$

Using the $\lambda_g = gx$ for the imbedding into S_6, we have

$$\begin{aligned}
&\lambda_e = id, \\
&\lambda_y = (1,2,3)(4,6,5), \\
&\lambda_{y^2} = (1,3,2)(4,5,6), \\
&\lambda_x = (1,4)(2,5)(3,6), \\
&\lambda_{xy} = (1,5)(2,6)(3,4), \\
&\lambda_{xy^2} = (1,6)(2,4)(3,5).
\end{aligned}$$

Hence, the permutation representation of G is given by:

$S_3 = \{id, (1,2,3)(4,6,5), (1,3,2)(4,5,6), (1,4)(2,5)(3,6), (1,5)(2,6)(3,4), (1,6)(2,4)(3,5)\}$

in S_6. Now we check if $\lambda_a \tau_b = \tau_b \lambda_a$ for all $a, b \in S_3$. It is enough to show that the above equation holds for generators x and y. Observe that

$$\begin{aligned}
\lambda_x \tau_y &= (1,4)(2,5)(3,6) \cdot (1,2,3)(4,5,6) = (1,5,3,4,2,6) \\
&= (1,2,3)(4,5,6) \cdot (1,4)(2,5)(3,6) = \tau_y \lambda_x, \\
\lambda_y \tau_x &= (1,2,3)(4,6,5) \cdot (1,4)(2,6)(3,4) = (1,6,3,4,2,5) \\
&= (1,4)(2,6)(3,4) \cdot (1,2,3)(4,6,5) = \tau_x \lambda_y, \\
\lambda_x \tau_x &= (1,4)(2,6)(3,5) \cdot (1,4)(2,5)(3,6) = (2,3)(5,6) \\
&= (1,4)(2,5)(3,6) \cdot (1,4)(2,6)(3,5) = \tau_x \lambda_x, \\
\lambda_y \tau_y &= (1,2,3)(4,5,6) \cdot (1,2,3)(4,6,5) = (1,3,2) \\
&= (1,2,3)(4,6,5) \cdot (1,2,3)(4,5,6) = \tau_y \lambda_y,
\end{aligned}$$

so that $\lambda_a \tau_b = \tau_b \lambda_a$ for all $a, b \in S_3$. □

25. Find the second permutation representation of the group G defined in Problem 21.

Solution. We represent G in a cycle-form by representing it by $\{1,2,3,4,5,6,7,8\}$ in the following way:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ e & \theta & a & b & c & \theta a & \theta b & \theta c \end{pmatrix}.$$

Using the $\lambda_g = gx$ for the imbedding into S_8, we have

$$\begin{aligned}
\lambda_e &= id, \\
\lambda_\theta &= (1,2)(3,6)(4,7)(5,8), \\
\lambda_a &= (1,3,2,6)(4,5,7,8), \\
\lambda_b &= (1,4,2,7)(3,8,6,5), \\
\lambda_c &= (1,5,2,8)(3,4,6,7), \\
\lambda_{\theta a} &= (1,6,2,3)(4,8,7,5), \\
\lambda_{\theta b} &= (1,7,2,4)(3,5,6,8), \\
\lambda_{\theta c} &= (1,8,2,5)(3,7,6,4).
\end{aligned}$$

Hence, the permutation representation of G is given by:

$$\begin{aligned}
G = \{&id, (1,2)(3,6)(4,7)(5,8), (1,3,2,6)(4,5,7,8), (1,4,2,7)(3,8,6,5), \\
&(1,5,2,8)(3,4,6,7), (1,6,2,3)(4,8,7,5), (1,7,2,4)(3,5,6,8), (1,8,2,5)(3,7,6,4)\}
\end{aligned}$$

in S_8. $\square$

26. Find the second permutation representation of the dihedral group of order $2n$.

Solution. Recall that $D_{2n} = \{e, y, \cdots, y^{n-1}, x, xy, \cdots, xy^{n-1}.\}$. We represent G in a cycle-from by representing it by $\{1, 2, \cdots, n, n+1, \cdots, 2n\}$ in the following way:

$$\begin{pmatrix} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & 2n \\ e & y & \cdots & y^{n-1} & x & xy & \cdots & xy^{n-1} \end{pmatrix}.$$

Using the $\lambda_g = gx$ for the imbedding into S_{2n}, we have

$$\begin{aligned}
&\lambda_e = id, \\
&\lambda_y = (1,2,\cdots,n)(2n, 2n-1, \cdots, n+2, n+1), \\
&\lambda_{y^2} = (1,2,\cdots,n)^2(2n, 2n-1, \cdots, n+2, n+1)^2, \\
&\vdots \\
&\lambda_{y^{n-1}} = (n, n-1, \cdots, 2, 1)(n+1, n+2, \cdots, 2n-1, 2n), \\
&\lambda_x = (1, n+1)(2, n+2)\cdots(n, 2n), \\
&\lambda_{xy} = [(1,2,\cdots,n)(2n, 2n-1, \cdots, n+2, n+1)] \cdot [(1, n+1)(2, n+2)\cdots(n, 2n)], \\
&\lambda_{xy^2} = [(1,2,\cdots,n)^2(2n, 2n-1, \cdots, n+2, n+1)^2] \cdot [(1, n+1)(2, n+2)\cdots(n, 2n)], \\
&\vdots \\
&\lambda_{xy^{n-1}} = (n, n-1, \cdots, 2, 1)(n+1, n+2, \cdots, 2n-1, 2n) \cdot (1, n+1)(2, n+2)\cdots(n, 2n).
\end{aligned}$$

Here we used the fact that $\lambda_{xy} = \lambda_y \lambda_x$ to compute the permutations. Now the group of collection of all above permutations is the permutation representation of G in S_{2n}. $\square$

27. Let $G = (a)$ be a cyclic group of order 8 and let $H = (a^4)$ be its subgroup of order 2. Find the coset representation of G by H.

Solution. Let S be the set of right cosets of H in G. That is, $S = \{H, Ha, Ha^2, Ha^3\}$. From the mapping $\tau_g : S \to S$ defined as $(Ha)g = H(ag)$, we can rewrite it as in the cycle-form if we represent S by $\{1, 2, 3, 4\}$ in the following way:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ H & Ha & Ha^2 & Ha^3 \end{pmatrix}.$$

Therefore, $t_a = (1, 2, 3, 4)$. From the fact that $t_{a^i} = (t_a)^i$,

$$\begin{aligned} \tau_e &= \tau_{a^4} = id, \\ \tau_a &= \tau_{a^5} = (1, 2, 3, 4), \\ \tau_{a^2} &= \tau_{a^6} = (1, 3)(2, 4), \\ \tau_{a^3} &= \tau_{a^7} = (1, 4, 3, 2). \end{aligned}$$

Thus, the coset representation of G is given by : $\{id, (1, 2, 3, 4), (1, 3)(2, 4), (1, 4, 3, 2)\}$. $\square$

28. Let G be the dihedral group of order $2n$ generated by elements a, b such that $a^2 = b^n = e$, $ab = b^{-1}a$. Let $H = \{e, a\}$. Find the coset representation of G by H.

Solution. Let S be the set of right cosets of H in G. That is, $S = \{H, Hb, Hb^2, \cdots, Hb^{n-1}\}$. From the mapping $\tau_g : S \to S$ defined as $(Ha)g = H(ag)$, we can rewrite it as in the cycle-form if we represent S by $\{1, 2, \cdots, n\}$ in the following way:

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ H & Hb & \cdots & Hb^{n-1} \end{pmatrix}.$$

Therefore, we have the following:

$$\begin{aligned} \tau_e &= id, \\ \tau_b &= (1, 2, \cdots, n), \\ \tau_{b^2} &= (1, 2, \cdots, n)^2, \\ &\vdots \\ \tau_{b^{n-1}} &= (n, n-1, \cdots, 2, 1). \end{aligned}$$

For τ_{ab^i}, the coset representation differs with the parity of n. If n is odd,

$$\begin{aligned} \tau_a &= (2, n)(3, n-1) \cdots ((n-1)/2, (n+1)/2), \\ \tau_{ab} &= \tau_a \tau_b = [(2, n)(3, n-1) \cdots ((n-1)/2, (n+1)/2)] \cdot (1, 2, \cdots, n), \\ \tau_{ab^2} &= \tau_a \tau_b^2 = [(2, n)(3, n-1) \cdots ((n-1)/2, (n+1)/2)] \cdot (1, 2, \cdots n)^2, \\ &\vdots \\ \tau_{ab^{n-1}} &= (2, n)(3, n-1) \cdots ((n-1)/2, (n+1)/2) \cdot (n, n-1, \cdots, 2, 1). \end{aligned}$$

If n is even,

$$\begin{aligned}
&\tau_a = (2,n)(3,n-1)\cdots(n/2-1,n/2+1),\\
&\tau_{ab} = \tau_a\tau_b = [(2,n)(3,n-1)\cdots(n/2-1,n/2+1)]\cdot(1,2,\cdots,n),\\
&\tau_{ab^2} = \tau_a\tau_b^2 = [(2,n)(3,n-1)\cdots(n/2-1,n/2+1)]\cdot(1,2,\cdots,n)^2,\\
&\vdots\\
&\tau_{ab^{n-1}} = [(2,n)(3,n-1)\cdots(n/2-1,n/2+1)]\cdot(n,n-1,\cdots,2,1).
\end{aligned}$$

Hence we have established the coset representation of the dihedral group D_{2n}. $\square$

29. Let G be the group of Problem 21 and let $H = \{e, \theta\}$. Find the coset representation of G by H.

Solution. Let S be the set of right cosets of H in G. That is, $S = \{H, Ha, Hb, Hc\}$. From the mapping $\tau_g : S \to S$ defined as $(Hx)g = H(xg)$, we can rewrite it as in the cycle-form if we represent S by $\{1,2,3,4\}$ in the following way:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ H & Ha & Hb & Hc \end{pmatrix}.$$

Therefore, we have the following:

$$\begin{aligned}
&\tau_e = \tau_\theta = id,\\
&\tau_a = \tau_{\theta a} = (1,2)(3,4),\\
&\tau_b = \tau_{\theta b} = (1,3)(2,4),\\
&\tau_c - \tau_{\theta c} = (1,4)(2,3).
\end{aligned}$$

Thus, the coset representation of G is given by: $\{id, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$. $\square$

30. Let G be S_n, the symmetric group of order n, acting as permutations on the set $\{1,2,\cdots,n\}$. Let $H = \{\sigma \in G : n\sigma = n\}$.
a) Prove that H is isomorphic to S_{n-1}.

Proof. Note that every elements in H is a permutation that fixes n. Hence, we can regard $\sigma \in H$ as the permutation of the set $\{1,2,\cdots,n-1\}$. Moreover, every elements in S_{n-1}, when regarded as an element in S_n, fixes n trivially. H being a subgroup of S_n, $H \simeq S_{n-1}$. $\square$

b) Find a set of elements $a_1,\cdots,a_n \in G$ such that $Ha_1,\cdots,Ha_n$ give all the right cosets of H in G.

Solution. Let $\sigma = (1, 2, \cdots, n) \in S_n$. Note that if $H\sigma^i = H\sigma^j$, $0 \le i < j \le n$ implies $\sigma^{j-i} \in H$. But σ^k does not fix n for any $k \not\equiv 0 \pmod{n}$ but $i = j$. Therefore, $S = \{H, H\sigma, H\sigma^2, \cdots, H\sigma^{n-1}\}$ is the set of right cosets of H in G. $\square$

c) Find the coset representation of G by H.

Solution. Let S be the set of right cosets of H in G. From the mapping $\tau_g : S \to S$ defined as $(Hx)g = H(xg)$, we can rewrite it as in the cycle-form if we represent S by $\{1, 2, \cdots, n\}$ in the following way:

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ H & H\sigma & \cdots & H\sigma^{n-1} \end{pmatrix}.$$

Therefore, we have the following: For any $\mu \in H \simeq S_{n-1}$,

$$\begin{aligned} &\tau_\mu = id, \\ &\tau_{\mu\sigma} = (1, 2, \cdots, n), \\ &\tau_{\mu\sigma^2} = (1, 2, \cdots, n)^2, \\ &\vdots \\ &\tau_{\mu\sigma^{n-1}} = (n, n-1, \cdots, 2, 1). \end{aligned}$$

Therefore, the coset representation of G by H is given by: $\langle (1, 2, \cdots, n) \rangle$. $\square$