INSTITUTE OF TECHNOLOGY

TCSS 343 - Assignment 1

Version 1.0

December 30, 2016

1 GUIDELINES

Homework should be electronically submitted to the instructor by midnight on the due date. A submission link is provided on the course Canvas Page. The submitted document should be typeset using any common software and submitted as a PDF. We strongly recommend using LaTeX to prepare your solution. You could use any LaTeX tools such as Overleaf, ShareLatex, TexShop etc. Scans of handwritten/hand drawn solutions are acceptable, but you will be docked 1 point per problem if the handwritting is unclear.

Each problem is worth a total of 20 points except the challenge problem which is worth 10 points. Solutions receiving full points must be correct (no errors or omissions), clear (stated in a precise and concise way), and have a well organized presentation. Show your work as partial points will be awarded to rough solutions or solutions that make partial progress toward a correct solution.

Remember to cite all sources you use other than the text, course material or your notes.

2 Problems

2.1 Understand

(6 points) 1. Prove the following theorems. Use a **direct proof** to find constants that satisfy the definition of $\Theta(n^2)$ *or* use the **limit test**. Make sure your proof is complete, concise, clear and precise.

Theorem 1. $3n^2 - 2n - 4 \in \Theta(n^2)$

Theorem 2. $\log(n^2+1) \in \Theta(\log(n))$

Theorem 3. $2^{n+2} + 2 \in \Theta(2^n)$

(8 points) 2. Find a closed for expression for these sum where c is a constant:

a)

$$\sum_{i=1}^{n}(n+i+c)$$

b)

$$\sum_{i=1}^{n}\left(\sum_{j=i}^{n} c\right)$$

c)

$$\sum_{i=1}^{n}\frac{2^i}{2^n}$$

d)

$$\sum_{i=\lfloor \frac{n}{2} \rfloor}^{n} i$$

(6 points) 3. Express the worst case run time of these pseudo-code functions as summations. You do not need to simplify the summations.

a)
```
function(n)
   let A be an empty stack
   for int i from 1 to n
      A.push(i)
   endfor
endfunction
```

b)
```
function(A[1...n] a list of n integers)
   for int i from 1 to n
      find and remove the minimum integer in A
   endfor
endfunction
```

c)
```
function(H[1...n] a min-heap of n integers)
   for int i from 1 to n
      find and remove the minimum integer in H
   endfor
endfunction
```

Grading You will be docked points for errors in your math, disorganization, unclarity, or incomplete proofs.

2.2 Explore

(3 points) 1. Prove using the definition of $\Theta(\log_2(n))$ or the limit test the following theorem.

Theorem 4. *Let $d > 1$ be a real number.*

$$\log_d(n) \in \Theta(\log_2(n))$$

(3 points) 2. Prove using the definition of $\Theta(n)$ or the limit test the following theorem.

Theorem 5. *Let $f(n) \in \Theta(n)$ and let $g(n) \in \Theta(n)$ then $f(n) + g(n) \in \Theta(n)$.*

(3 points) 3. Prove using the definition of $\Theta(h(n))$ or the limit test the following theorem.

Theorem 6. *Let $f(n) \in \Theta(g(n))$ and let $g(n) \in \Theta(h(n))$ then $f(n) \in \Theta(h(n))$.*

(10 points) 1. Place these functions in order from slowest asymptotic growth to fastest asymptotic growth. You will want to simplify them algebraically before comparing them. You do not need to prove any relationships.

$$\begin{aligned}
f_0(n) &= 6n^2 + 12n - 4 \\
f_1(n) &= 2^{2n} \\
f_2(n) &= \left(\frac{n}{\log_2 n}\right)^2 \\
f_3(n) &= 3^{\log_2 n} \\
f_4(n) &= 3^n \\
f_5(n) &= \log_2(n \cdot n^n) \\
f_6(n) &= \log_2 n + 3 \\
f_7(n) &= \log_2\left(\log_2 n + 3\right) \\
f_8(n) &= 2^{\sqrt{n}} \\
f_9(n) &= 10^9 + 25^2
\end{aligned}$$

Grading You will be docked points for functions in the wrong order and for disorganization, unclarity, or incomplete proofs.

2.3 Expand

(5 points) 1. Prove the theorem below using the techniques of **binding the term** and **splitting the sum** to find a tight bound for the sum. Make sure your proof is complete, concise, clear and precise.

Theorem 7.

$$\sum_{i=1}^{n} i^5 \in \Theta\left(n^6\right)$$

(5 points) 2. Prove the theorem below using the techniques of **binding the term** and **splitting the sum** to find a tight bound for the sum. Make sure your proof is complete, concise, clear and precise.

Theorem 8.

$$\sum_{i=1}^{\log_2 n} i \in \Theta\left((\log_2 n)^2\right)$$

(5 points) 1. Prove the theorem below using the techniques of **binding the term** and **splitting the sum** to find a tight bound for the sum. Make sure your proof is complete, concise, clear and precise.

Theorem 9.

$$\sum_{i=1}^{n} i^d \in \Theta\left(n^{d+1}\right)$$

(5 points) 2. Prove the theorem below using the techniques of **binding the term** and **splitting the sum** to find a tight bound for the sum. Make sure your proof is complete, concise, clear and precise.

Theorem 10.

$$\sum_{i=1}^{\sqrt{n}} \sqrt{i} \in \Theta\left(n^{3/4}\right)$$

Grading You will be docked points for errors in your math, disorganization, unclarity, or incomplete proofs.

2.4 CHALLENGE

In this problem you will prove there is a function that is in $O(n^3)$ and $\Omega(n)$ but is not in $\Theta(n^d)$ for any $1 \le d \le 3$.

(2 points) 1. State a function $f(n)$ that is in $O(n^3)$ and $\Omega(n)$ but is not in $\Theta(n^d)$ for any $1 \le d \le 3$.

(2 points) 2. Prove that $f(n) \in O(n^3)$.

(2 points) 2. Prove that $f(n) \in \Omega(n)$.

(4 points) 3. Prove that $f(n) \notin \Theta(n^d)$ for any $1 \le d \le 3$.

Grading Correctness and precision are of utmost importance. Use formal proof structure for big-Oh, big-Omega and big-Theta bounds. You will be docked points for errors in your math, disorganization, unclarity, or incomplete proofs.