Part A

2. (a) First, I assume $\theta = (\theta_1, \theta_2, \dots, \theta_N)$ represents all students' ability where $\theta_i \in \theta$, also assume $\beta = (\beta_1, \beta_2, \dots, \beta_M)$ represents all probabilities that all questions are correctly answered by students.

Thus, I can get $L(C|\theta,\beta) = \prod_i^N \prod_j^M P(C_{ij} | \theta_i, \beta_j)$

Since $P(C_{ij}=1 | \theta_i, \beta_j) = \frac{\exp(\theta_i - \beta_j)}{1+\exp(\theta_i - \beta_j)}$

Thus,

$$L(C|\theta,\beta) = \prod_i^N \prod_j^M \left(\frac{\exp(\theta_i - \beta_j)}{1+\exp(\theta_i - \beta_j)}\right)^{C_{ij}} \left(1 - \frac{\exp(\theta_i - \beta_j)}{1+\exp(\theta_i - \beta_j)}\right)^{(1-C_{ij})}$$

$$= \prod_i^N \prod_j^M \left(\frac{\exp(\theta_i - \beta_j)}{1+\exp(\theta_i - \beta_j)}\right)^{C_{ij}} \left(\frac{1}{1+\exp(\theta_i - \beta_j)}\right)^{(1-C_{ij})}$$

Then, take the log of $L(C|\theta,\beta)$ to get

$$l(C|\theta,\beta) = \log\left(\prod_{i=1}^N \prod_{j=1}^M \left(\frac{\exp(\theta_i - \beta_j)}{1+\exp(\theta_i - \beta_j)}\right)^{C_{ij}} \left(\frac{1}{1+\exp(\theta_i - \beta_j)}\right)^{(1-C_{ij})}\right)$$

$$= \sum_{i=1}^N \sum_{j=1}^M C_{ij} \log\left(\frac{\exp(\theta_i - \beta_j)}{1+\exp(\theta_i - \beta_j)}\right) + (1-C_{ij}) \log\left(\frac{1}{1+\exp(\theta_i - \beta_j)}\right)$$

$$= \sum_{i=1}^N \sum_{j=1}^M C_{ij} \log\left(\frac{\exp(\theta_i - \beta_j)}{1+\exp(\theta_i - \beta_j)}\right) + \log\left(\frac{1}{1+\exp(\theta_i - \beta_j)}\right) - C_{ij} \log\left(\frac{1}{1+\exp(\theta_i - \beta_j)}\right)$$

$$= \sum_{i=1}^N \sum_{j=1}^M C_{ij} \left(\log(\exp(\theta_i - \beta_j))\right) + \log\left(\frac{1}{1+\exp(\theta_i - \beta_j)}\right)$$

$$= \sum_{i=1}^N \sum_{j=1}^M C_{ij} (\theta_i - \beta_j) - \log(1+\exp(\theta_i - \beta_j))$$

Now, take the derivative of $l(C|\theta,\beta)$ of θ_i to get the log-likelihood

$$\frac{\partial l}{\partial \theta_i} = \frac{\partial}{\partial \theta_i} \sum_{i=1}^N \sum_{j=1}^M C_{ij} (\theta_i - \beta_j) - \log(1+\exp(\theta_i - \beta_j))$$

$$= \sum_{j=1}^M \frac{\partial}{\partial \theta_i} \sum_{i=1}^N C_{ij} (\theta_i - \beta_j) - \log(1+\exp(\theta_i - \beta_j))$$

$$= \sum_{j=1}^M C_{ij} - \frac{\exp(\theta_i - \beta_j)}{1+\exp(\theta_i - \beta_j)}$$

Take the derivative of $l(C|\theta,\beta)$ by β_j to get the log-likelihood:

$$\frac{\partial l}{\partial \beta_j} = \frac{\partial}{\partial \beta_j} \sum_{i=1}^{N} \sum_{j=1}^{M} c_{ij} (\theta_i - \beta_j) - \log (1 + \exp(\theta_i - \beta_j))$$

$$= \sum_{i=1}^{N} \frac{\partial}{\partial \beta_j} \sum_{j=1}^{M} c_{ij} (\theta_i - \beta_j) - \log (1 + \exp(\theta_i - \beta_j))$$

$$= \sum_{i=1}^{N} -c_{ij} + \frac{\exp(\theta_i - \beta_j)}{1 + \exp(\theta_i - \beta_j)}$$