Peierls paths and the periodicity of Hofstadter butterfly

Guan Yifei

To consider the periodicity of the Hofstadter butterfly, we shell start from a general form of tight-binding model where

$$\hat{H} = \sum t_{ij} c_i^\dagger c_j$$

The operators $c_i^\dagger/c_j$ create/annihate a state on site i/j. t_{ij} indicates the hopping integral. With a finite magnetic field, the Peierls substitution gives a phase modulation: $t_{ij} \to e^{i\phi_{ij}} t_{ij}$.

In the most general case, ont should have different loops formed by the Peierls paths. The Hamiltonian returns to the same eigenvalue up to a gauge transformation: $\hat{H} \sim G^\dagger \hat{H} G$. Such a condition requires the flux quanta of all the loops be multiplys of 2π. To determine the periodicity of Hofstadter butterfly, it then needs to find the series of the area of the loops.

Such a problem of calculating the loops is related to the lattice structures, and consequently, the space groups. In the following derivations, we list the cases and discuss the perodicity of the Hofstadter spectrum.

The sites are on rational positions

This is a super-simple case: in the unit of lattice vectors, all the sites are on rational positions: $r_i = \frac{p}{q}$. The denominator q does not have to be the same since we can multiply different qs.

In number theory, there is a theorem sometimes referred as Bezout's theorem(which is a special case of the Bezout theorem in algebraic curves):

For two integer numbers p, q the equation holds:

$$px + qy = N \times \gcd(p, q)$$

since the expression of areas can be written as:

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

which is in the same form, the series of loop areas is then determined by the series of rational site positions.

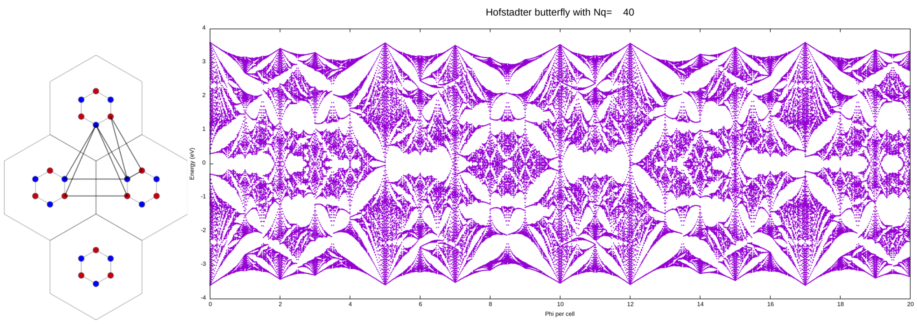

Fig. 1: Graphene lattice with irrational Kekule distortion

Consideration by the point groups

In crystal lattices the conditions above can be alternatively considered with point groups. However, in a general case where one may have irrational site positions, the Bezout method does not hold and the Hofstadter butterfly never repeat itself exactly. This can be seen in a Kekule distorted graphene lattice.