Differential Geometry

——Don't make me cry

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Chapter 1

Class-1

Definition 1.1. A parametrized differentiable curve in $\mathbb{R}^3$ is a differentiable map $\alpha : I \longrightarrow \mathbb{R}^3$, where I is an interval in $\mathbb{R}$ ($t \longrightarrow \alpha(t) = (x(t), y(t), z(t))$ and the variable t is called the parameter of the curve.)

Remark 1. A function f is differentiable is equivalent to smooth.

Remark 2. α is differentiable if each x(t), y(t), z(t) is differentiable.

Remark 3. an interval I is of the form (a,b), where $a, b \in \mathbb{R} or a = -\infty, b = +\infty$

If we denote by $x'(t)$ the first derivative of x at the point t and use similar notations for the functions y and z, the vector $(x'(t), y'(t), z'(t)) = \alpha'(t) \in \mathbb{R}^3$ is called the tangent vector (or velocity vector) of the curve α at t. The image set α(I) $\in \mathbb{R}^3$ is called the trace of α, which is not a parametrized curve.

Example 1.0.1. Refer to Page 3-4 is enough, it's worth noticing that EX.3,4,5 in this book is interesting

Let's recall what we have learnt in linear algebra. We denote inner product by what?

You can refer to Inner product In the same time,review the norm and dot product in Euclidean spaces.

Let $\alpha : I \longrightarrow \mathbb{R}^3$ be a parametrized differentiable curve. For each $t \in I$ where $\alpha'(t) \neq 0$, there is a well-defined straight line, which contains the point $\alpha(t)$ and the vector $\alpha'(t)$. This line is called the tangent line to α at t. For the study of the differential geometry of a curve it is essential that there exists such a tangent line at every point. Therefore, we call any point t where $\alpha'(t) = 0$ a singular point of α and restrict our attention to curves without singular points.

Definition 1.2. A parametrized differentiable curve is said to be regular if it doesn't have singular points.

From now on we shall consider only regular parametrized differentiable curves (and, for convenience, shall usually omit the word differentiable)

Definition 1.3. Given a regular curve $\alpha : I \longrightarrow \mathbb{R}^3$, t_0 and t where $t_0 < t$ and both of them are in I. The arc length of α from t_0 to t is

$$s(t) = \int_{t_0}^{t} |\alpha'(x)|\, dx$$

Remark 4. Since α is regular, $|\alpha'(t)|$ is differentiable, we can see that s(t_1) is differentiable and $\frac{ds}{dt} = |\alpha'(t)|$

To simplify our exposition, we shall restrict ourselves to curves parametrized by arc length; we shall see later (see Sec. 1-5) that this restriction is not essential. In general, it is not necessary to mention the origin of the arc length s, since most concepts are defined only in terms of the derivatives

Then we solve Exercise 8-10

Theorem 1.1. *Let $\alpha : I \longrightarrow \mathbb{R}^3$ be a differentiable curve and let [a, b] in I be a closed interval. For every partition $a = t_0 < t_1 < \cdots < t_n = b$ of [a,b],consider the sum $\sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})| = l(\alpha, P)$, where P stands for the given partition. The norm $|P|$ of a partition P is defined as $|P| = max(t_i - t_{i-1}), i = 1, ..., n$. Geometrically, $l(\alpha, P)$ is the length of a polygon inscribed in α([a, b]) with vertices in α(ti) (see Fig. 1-12). The point of the exercise is to show that the arc length* *of $\alpha([a,\ b])$ is, in some sense, a limit of lengths of inscribed polygons. Prove that given $\epsilon > 0$ there exists $\delta > 0$ such that if $|P| < \delta$ then*

$$|\int_a^b |\alpha'(t)|dt - l(\alpha, P)| < \epsilon$$

proof. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, then $l(\alpha, P) = \sum_{i=1}^n ||\alpha(t_i) - \alpha(t_{i-1})|| = \sum_{i=1}^n \sqrt{\sum_{j=1}^3 (\alpha_j(t_i) - \alpha_j(t_{i-1})}$
Now applying the mean value theorem for a scalar function we get that there exist $t_i^j \in [t_{i-1}, t_i]$, such that

$$\alpha_j(t_i) - \alpha_j(t_{i-1}) = \alpha_j'(t_i^j)(t_i - t_{i-1})$$

So $l(\alpha, P) = \sum_{i=1}^n (\sqrt{\sum_{j=1}^3 \alpha_j'(t_i^j)^2}(t_i - t_{i-1}))$

Let $\epsilon > 0$ and $F : [a, b]^3 \to \mathbb{R}$ a function given by

$$F(x_1, x_2, x_3) = \sqrt{\sum_{j=1}^3 \alpha_j'(x_j)^2}$$

Now since α is differentiable (actually, α must be C^1 for this to hold), it follows that F is continuous and since $[a, b]^3$ is compact, it follows that F is uniformly continuous. Hence, there exists $\delta_1 > 0$ such that for all $x, y \in [a, b]^3$,

$$||x - y|| < \delta_1 \longrightarrow |F(x) - F(y)| < \frac{\epsilon}{2(b - a)}$$

Now the above sum can be written as

$$l(\alpha, P) = \sum_{i=1}^n F(t_i^1, t_i^2, t_i^3)(t_i - t_{i-1})$$

On the other hand, one integral sum for the above integral is

$$\sigma(\|\alpha'\|, P) = \sum_{i=1}^{n} \|\alpha'(t_i)\|(t_i - t_{i-1}) = \sum_{i=1}^{n} F(t_i, t_i, t_i)(t_i - t_{i-1})$$

Now we have

$$|l(\alpha, P) - \sigma(\|\alpha'\|, P)| \leq \sum_{i=1}^{n} |F(t_i^1, t_i^2, t_i^3) - F(t_i, t_i, t_i)|(t_i - t_{i-1})$$

$$< \frac{\epsilon}{2(b-a)} \sum_{i=1}^{n} (t_i - t_{i-1}) = \frac{\epsilon}{2(b-a)} \cdot (b-a) = \frac{\epsilon}{2}$$

On the other hand, applying Darboux' theorem we get that there exists $\delta_2 > 0$ such that

$$|P| < \delta_2 \Rightarrow |\sigma(\|\alpha'\|, P) - \int_a^b \|\alpha'(t)\| \, dt| < \frac{\epsilon}{2}$$

Now for $\delta = \min\{\delta_1, \delta_2\}$ and $|P| < \delta$ we have that

$$|l(\alpha, P) - \int_a^b \|\alpha'(t)\| \, dt| \leq |l(\alpha, P) - \sigma(\|\alpha'\|, P)|$$

$$+|\sigma(\|\alpha'\|, P) - \int_a^b \|\alpha'(t)\| \, dt| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

1.1 Orientation of vector space

V is a finite-dimensional vector space, dim(V) = n; Let$(e_1, \ldots, e_n), (f_1, \ldots, f_n)$ be two ordered basis of V. We say that e∼f is equivalent to $e_i = \sum_{j=1}^{n} a_{ij} f_j$ and $det(a_{ij}) > 0$

Recommended Exercise 1.1. *check $\sim$ is an equivalent relation.*

An Orientation of V is one of the equivalence class.

Remark 5. every vector space has exactly 2 orientation , but not a prefered one (Unless in $\mathbb{R}^n$ and standard basis)

Definition 1.4 (Cross product)**.** let u,v $\in \mathbb{R}^3$,define u $\wedge$ v $(u \times v)$ such that:

$$(u \wedge v) \cdot w = det(u, v, w)$$

Remark 6.

- $u \wedge v = -v \wedge u$
- $(au + bw) \wedge v = a(u \wedge v) + b(w \wedge v)$
- $u \wedge v = 0 \longleftrightarrow$ u,v are linearly dependent
- $(u \wedge v) \cdot u = 0$

We have the Lagrange inentity:

$$(u \wedge v) \cdot (x \wedge y) = \begin{vmatrix} u \cdot x & v \cdot x \\ u \cdot y & v \cdot y \end{vmatrix}$$

Recommended Exercise 1.2. *check this and use it to show that* $|u \wedge v|^2 = (|u||v| \sin\theta)^2$

Recommended Exercise 1.3. *If u,v are linearly dependent in 3-dimensional Euclidean spaces , check that u,v,u $\wedge$ v is a positive basis.*

From the formula, we can see that if u(t) and v(t) $\in \mathbb{R}^3$ are differentiable, then u $\wedge$ v is also differentiable and $\frac{d}{dt}u(t) \wedge v(t) = u'(t) \wedge v(t) + u(t) \wedge v'(t)$ (See it in appendix)

1.2 Local Geometry(of curves in $\mathbb{R}^3$)

Recall $\alpha : I \longrightarrow \mathbb{R}^3$ is a regular curve.

Definition 1.5. A reparametrization of α is a map $\beta(s) = (\alpha \circ t)(s)$, where $t : J \longrightarrow I$ and t'(s) $\neq$ 0 for all s $\in$ J.

Example 1.2.1 (reverse orientation)**.** $\beta(s) = \alpha(-s)$

proposition 1.2. Let $\alpha : I \longrightarrow \mathbb{R}^3$ be a regular curve. Then α admits a reparametrization by arc length: $\beta(s) = (\alpha \circ t)(s)$ and $|\beta'(s)| = 1$ holds for all s $\in$ J.

proof. fix $t_0 \in$ I,define s(t)=$\int_{t_0}^{t} |\alpha'(\hat{t})|d\hat{t}$.Then we have s'(t) = $|\alpha'(t)|$, so it's regular and differentiable. From the Inverse Function theorem: s has a differentiable inverse t:J$\longrightarrow I$. Define $\beta(s) = (\alpha \circ t)(s)$ and it's enough

we can see that $|\beta'(s)| = |\alpha'(t(s))||t'(s)| = |\alpha'(t(s))|\frac{1}{|s'(t(s))|} = 1$

□

From now on, we assume that α is parametrized by arc length.

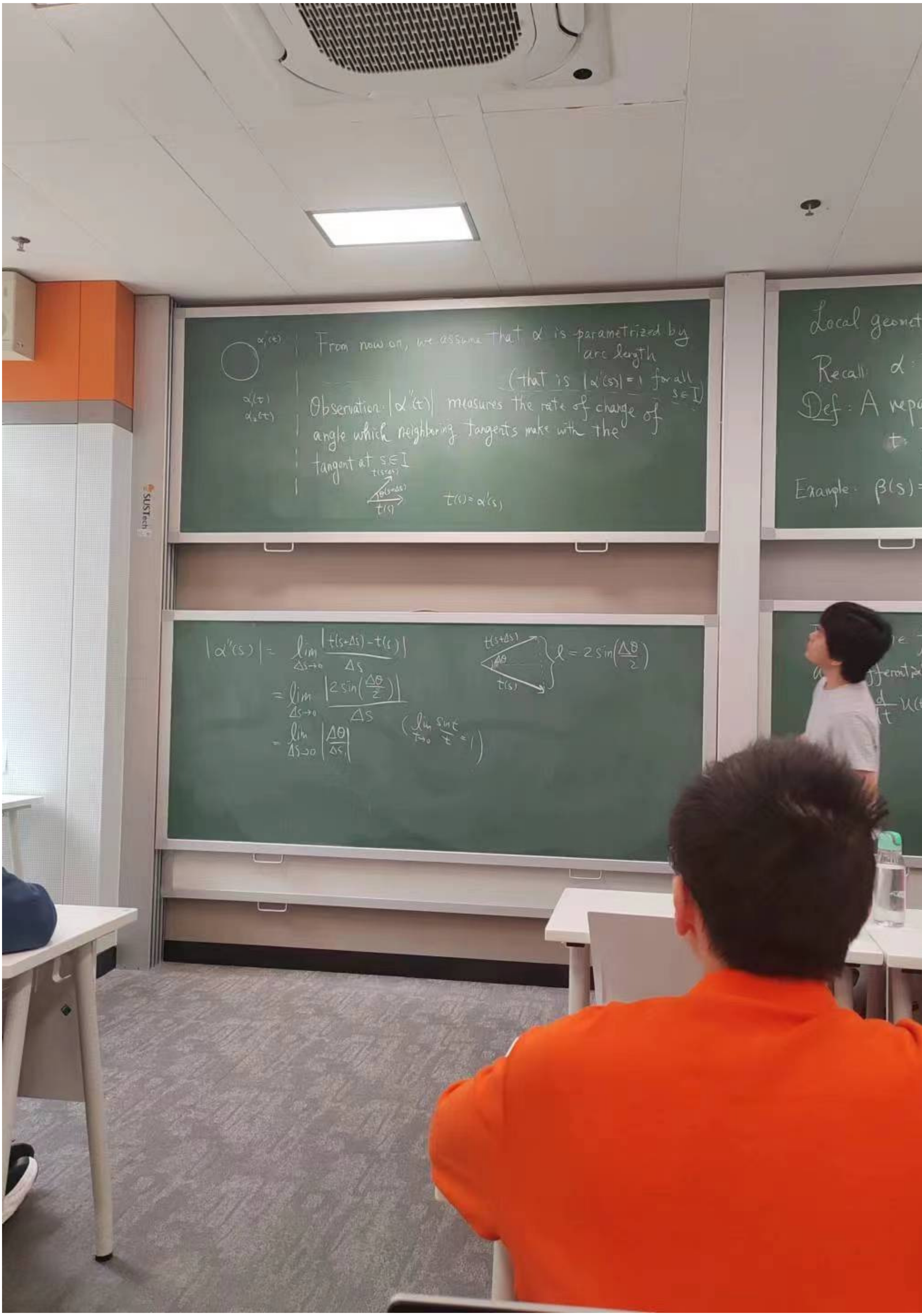
Remark 7. $|\alpha"(s)|$ measures the rate of change of angle which neighboring tangents make with the tangent at s $\in$ I

Definition 1.6. Let $\alpha : I \longrightarrow \mathbb{R}^3$ be a curve parametrized by arc length s $\in$ I. The number $|\alpha"(s)| = k(s)$ is called the curvature of α at s.

If α is a straight line, $\alpha(s) = us+v$, where u and v are constant vectors($|u| = 1$), then $k \equiv 0$. If $k \equiv 0$,then by integration,we can see it's a straight line.

Notice that: by change of orientation, the tangent vector changes its direction: If $\beta(-s) = \alpha(s)$ then
$$\frac{d\beta}{d(-s)}(-s) = -\frac{d\alpha}{ds}(s)$$
hence, we can see $\alpha"(s)$ and the curvature remain invariant under a change of orientation.

When k(s) $\neq$ 0, a unit vector n(s) in the direction $\alpha"(s)$ is well-defined by:$\alpha"(s) = k(s)n(s)$


From now on, we assume that α is parametrized by arc length
(that is |α'(s)| = 1 for all s ∈ I)
Observation: |α''(t)| measures the rate of change of angle which neighboring tangents make with the tangent at s ∈ I
t(s) = α'(s)
|α''(s)| = lim |t(s+Δs) − t(s)| / Δs
= lim |2 sin(Δθ/2)| / Δs
= lim |Δθ/Δs|
(lim sin t / t = 1)
ℓ = 2 sin(Δθ/2)
Local geomet
Recall
Def: A repa
Example: β(s) =
SUSTech

proposition 1.3. $\alpha'(s) \cdot \alpha''(s) = 0$, then we have $n(s) \cdot \alpha'(s) = 0$.

proof. since we have $\alpha'(s) \cdot \alpha'(s) = 1$, we can differentiate this. □

Thus, n(s) is normal to $\alpha'(s)$ and is called the normal vector at s.

Definition 1.7. The plane determined by the unit tangent vector and normal vector($\alpha'(s)$ and n(s)) is called the osculating plane at s.

At points where k(s) = 0; n(s) isn't defined. To proceed with the local analysis of curves, we need the osculating plane. It is therefore convenient to say that s $\in$ I is a singular point of order 1 if $\alpha''(s) = 0$(in this context, the points where $\alpha'(s) = 0$ are called singular points of order 0).

Now, we shall restrict ourselves to curves parametrized by arc length without singular points of order 1. We shall denote by t(s) = $\alpha'(s)$ the unit tangent vector of α at s. Thus, t'(s) = k(s)n(s).

The unit vector b(s) = t(s) $\wedge$ n(s) is normal to the osculating plane and will be called the binormal vector at s. Since b(s) is a unit vector, the length $|b'(s)|$ measures the rate of change of the neighboring osculating planes with the osculating plane at s; that is, $|b'(s)|$ measures how rapidly the curve pulls away from the osculating plane at s, in a neighborhood of s

Remark 8. we call the plane formed by{t(s),n(s),b(s)} the Fermet's frame

Remark 9. If we have α lies in an plane L, which means that t(s) and n(s) $\in$ L. (You can assume that Ax(t)+By(t)+Cz(t)+D=0,then differentiate both sides once, twice or more. Or you can say that t(s) and n(s) are parallel to L) As a result, we can see that, b(s) is a constant vector. then we have $b'(s) \equiv 0$.

On the other hand, if $b'(s) \equiv 0$(Or $b(s) \equiv b_0$) ,let f(s) = $(\alpha(s) - \alpha(0)) \cdot b_0$. Then $f' = \alpha' \cdot b_0 = 0 \Longleftarrow t \perp b$. similarly, we have n $\perp b \Longrightarrow f'' = 0$

Back to our discussion about b(s), the binormal vector at s. We have already talked about its geometry meaning, now let's talk about its computation. we define b(s) as t(s) $\wedge$ n(s), so we have $b'(s) = t(s) \wedge n'(s)$ (Since $t'(s) \wedge n(s) = 0$)

So we have $b'(s) \perp t(s)$. Meanwhile, notice that b(s) $\cdot$ b(s) = 1; we have b(s) $\perp$ $b'(s)$. From above, we can see $b'(s)$ is parallel to n(s).

Definition 1.8. Let $\alpha : I \longrightarrow \mathbb{R}^3$ be a curve parametrized by arc length such that $\alpha''(s) \neq 0$ for s $\in$ I. The number $\tau(s)$ defined by $b'(s) = \tau(s)n(s)$ is called the torsion of α at s.

proposition 1.4. $\tau(s) = 0 \Longleftrightarrow \alpha$ lies in a plane

proof. If α lies in a plane L, then t(s) and n(s) $\in$ L, so b(s) is a constant vector

If $\tau = 0$, then we can see $b'(s) = 0$,thus $b(s) = b(s_0) = b_0$ is a constant vector So $(\alpha'(s) \cdot b_0)' = 0$. Now,it's seen that $\alpha(s) \cdot b_0 = \alpha(s_0) \cdot b_0$ So $\alpha(s)$ is in the plane passing through $\alpha(s_0)$ and perpendicular to b_0 □

In contrast to the curvature, the torsion may be either positive or negative.

Notice that by changing orientation the binormal vector changes sign, since $b = t \wedge n$. It follows that $b'(s)$, and, therefore, the torsion, remain invariant under a change of orientation.

Let us summarize our position. To each value of the parameter s, we have associated three orthogonal unit vectors t(s), n(s), b(s). The trihedron thus formed is referred to as the Frenet trihedron at s. The derivatives $t'(s) = kn, b'(s) = \tau n$ of the vectors t(s) and b(s), when expressed in the basis {t, n, b}, yield geometrical entities (curvature k and torsion τ) which give us information about the behavior of α in a neighborhood of s.

The search for other local geometrical entities would lead us to compute n'(s). However, since n = b $\wedge$ t, we have

$$n'(s) = b'(s) \wedge t(s) + b(s) \wedge t'(s) = -\tau b - kt$$

, and we obtain again the curvature and the torsion.

For later use, we shall call the equations

$$t' = kn,$$

$$n' = -kt - \tau b,$$

$$b' = \tau n.$$

the Frenet formulas (we have omitted the s, for convenience). In this context, the following terminology is usual. The tb plane is called the rectifying plane, and the nb plane the normal plane. The lines which contain n(s) and b(s) and pass through α(s) are called the principal normal and the binormal, respectively. The inverse R = 1/k of the curvature is called the radius of curvature at s. Of course, a circle of radius r has radius of curvature equal to r, as one can easily verify.

Physically, we can think of a curve in $\mathbb{R}^3$ as being obtained from a straight line by bending (curvature) and twisting (torsion). After reflecting on this construction, we are led to conjecture the following statement, which, roughly speaking, shows that k and τ describe completely the local behavior of the curve.

Theorem 1.5 (FUNDAMENTAL THEOREM OF THE LOCAL THEORY OF CURVES.). *Given differentiable functions $k(s) > 0$ and $\tau(s)$, $s \in I$, there exists a regular parametrized curve $\alpha(s)$: $I \longrightarrow \mathbb{R}^3$ such that s is the arc length, k(s) is the curvature, and $\tau(s)$ is the torsion of α. Moreover, any other curve $\hat{\alpha}$, satisfying the same conditions, differs from α by a rigid motion; that is, there exists an orthogonal linear map ρ of $\mathbb{R}^3$, with positive determinant, and a vector c such that $\hat{\alpha} = \rho \circ \alpha + c$.*

We divide the proof into different part.

lemma (Existence and uniqueness in (linear) ODE). Let X=X(s)$\in \mathbb{R}^n$ satisfies (X:I$\longrightarrow \mathbb{R}^n$):

$$X'(s) = A(s)X(s)(s \in I) \tag{1.1}$$

where $A(s) = (a_{ij}(s))_{n\times n}$ is an n $\times$ n matrix and each entry a_{ij} is differentiable.

Then for all $s_0 \in I$ and $X_0 \in \mathbb{R}^n$, there exists unique X(s) which satisfies 1.1 and $X(s_0) = X_0$

proof. Recall we define $e^B = 1 + B + \frac{B^2}{2} + \cdots$ and log B as its power series for matrix B, we can see that $X(s) = X_0 \cdot e^{\int_{s_0}^{s} A(t)dt}$ is a solution

If X_1 and X_2 are different solutions satisfying the above requirements. then let Y $= X_1 - X_2$. we have $Y' =$ AY as well. Define $f = \frac{1}{2}|Y|^2$ we can see that $f' = Y \cdot Y' = Y \cdot AY$ Since A is differentiable,Fix s $\neq s_0$, there exists c > 0,s.t. $Y \cdot AY \in [-c|Y|^2, c|Y|^2]$ holds for all t between s_0 and s. If s$>$ s_0, we can see that:$(f' - 2cf)e^{-2ct} = (fe^{-2ct})' \leq 0$ holds for all t between s_0 and s,so $0 \leq fe^{-2cs} \leq f(s_0)e^{-2cs_0} = 0 \Longrightarrow$ f(s)=0. If s$<$ s_0, similarly we can prove it □

Theorem 1.6. *If α_1 and α_2 are two regular curves parametrized by arc length, and $\forall s, k_1(s) = k_2(s), \tau_1(s) = \tau_2(s)$. Then there is a rigid motion Mx = Px+c, where P is orthogonal,det(P)=1 and c$\in \mathbb{R}^3$, such that $M\alpha_1 = \alpha_2$*

proof. Firstly, we can see that: If there is a rigid motion Mx = Px+c, where P is orthogonal,det(P)=1 and c$\in \mathbb{R}^3$, such that M$\alpha_1 = \alpha_2$, then α_1 and α_2 have the same torsion and curvature. (Prove curvature is the same first, then use $\tau = -\frac{\alpha' \wedge \alpha'' \cdot \alpha'''}{k^2}$)

(You may need this: for a matrix M and three vectors a,b,v;$(Ma \wedge Mb) \cdot Mv = det(Ma; Mb; Mv) = det(M)det(a; b; v) = det(M) \cdot (a \wedge b) \cdot v$. Since v is arbitrary, we can see that $(Ma \wedge Mb) = det(M) \cdot (a \wedge b) \cdot v \cdot v^{-1}M^{-1}$.)

Back to the theorem, we begin to prove.

Now, assume that two curves $\alpha_1 = \alpha_1(s)$ and $\alpha_2 = \alpha_2(s)$ satisfy the conditions $k_1(s) = k_2(s)$ and $\tau_1(s) = \tau_2(s), \forall s \in I$. Let $t_1(s_0), n_1(s_0), b_1(s_0)$ and $t_2(s_0), n_2(s_0), b_2(s_0)$ be the Frenet trihedrons at $s = s_0 \in I$ of α_1 and α_2, respectively. Clearly, there is a rigid motion which takes $\alpha_2(s_0)$ into $\alpha_1(s_0)$ and $t_2(s_0), n_2(s_0), b_2(s_0)$ into $t_1(s_0), n_1(s_0), b_1(s_0)$ Thus, after performing this rigid motion on α_2, we have that $\alpha_2(s_0) = \alpha_1(s_0)$ and that the Frenet trihedrons $t_1(s), n_1(s), b_1(s)$

and $t_2(s), n_2(s), b_2(s)$ of α_1 and α_2, respectively, satisfy the Frenet equations:

$$\frac{dt_1}{ds} = kn_1, \frac{dt_2}{ds} = kn_2$$

$$\frac{dn_1}{ds} = -kt_1 - \tau b_1, \frac{dn_2}{ds} = -kt_2 - \tau b_2$$

$$\frac{db_1}{ds} = \tau n_1, \frac{db_2}{ds} = \tau n_2$$

while $t_1(s_0) = t_2(s_0), b_1(s_0) = b_2(s_0), n_1(s_0) = n_2(s_0)$

Using Frenet equations,it's obviouly that:

$$\frac{1}{2}\frac{d}{ds}(|t_1 - t_2|^2 + |b_1 - b_2|^2 + |n_1 - n_2|^2) = 0$$

holds for all s $\in$ I.

Integrating on s, we can see that $\frac{1}{2}(|t_1 - t_2|^2 + |b_1 - b_2|^2 + |n_1 - n_2|^2)$ is a constant.Since it's 0 at s=s_0,$\frac{1}{2}(|t_1 - t_2|^2 + |b_1 - b_2|^2 + |n_1 - n_2|^2) = 0$ holds for all s$\in$I. So we can see that $t_1(s) \equiv t_2(s), n_1(s) \equiv n_2(s), b_1(s) \equiv b_2(s)$ Notice that:

$$\frac{d\alpha_1}{ds} = t_1 = t_2 = \frac{d\alpha_2}{ds}$$

Then we have:$\frac{d}{ds}(\alpha_1 - \alpha_2) = 0$,which means$(\alpha_1 - \alpha_2)$ is a constant. Together with $(\alpha_1 - \alpha_2)$ is zero at s_0, we have proved $(\alpha_1 - \alpha_2) \equiv 0$

□

The left part is the existence.

proof. Let X(s)=$\begin{pmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{pmatrix}$ be the solution of

$$t' = kn,$$

$$n' = -kt - \tau b,$$

$$b' = \tau n.$$

and $e_1(s_0)=\begin{pmatrix}1\\0\\0\end{pmatrix}$,$e_2(s_0)=\begin{pmatrix}0\\1\\0\end{pmatrix}$,$e_3(s_0)=\begin{pmatrix}0\\0\\1\end{pmatrix}$,(We can use rigid motion to make sure this)(Here e_1 denotes t...)

Define a differentiable curve $\alpha : I \longrightarrow \mathbb{R}^3$ by:

$$\alpha(s) = \int_{s_0}^{s} e_1(u)du$$

Let's introduce a lemma

lemma. $e_i(s) \cdot e_j(s) = \delta_{ij}$ and $det(e_1(s); e_2(s); e_3(s)) \equiv 1$

lemma's proof. Define $g_{ij} = e_i(s) \cdot e_j(s)$ for all i,j $\in [1, 3]$ Let Y= $\begin{pmatrix}g_{11}\\g_{12}\\g_{13}\\g_{21}\\g_{22}\\g_{23}\\g_{31}\\g_{32}\\g_{33}\end{pmatrix} \in \mathbb{R}^9$. By computing, we can see that: $Y' = BY$ holds for some differentiable matrix B. and $Y_{ij}(s_0) = \delta_{ij}$. Notice that:$Y_{ij}(s) \equiv \delta_{ij}$ is still a solution. By the ODE theorem, we can see that $g_{ij}(s) = Y_{ij}(s) \equiv \delta_{ij}$

Meanwhile, notice that $det(e_1(s); e_2(s); e_3(s)) = \pm 1$ and its value at s_0 is 1,we can prove the second part. □

Back to our proof, now we can see

- α is parametrized by arc length,since$|\alpha'(s)| = |e_1(s)| \equiv 1$.
- $\alpha'' = e_1' = ke_2 \Longrightarrow e_2$ is the normal vector while k is the curvature
- $(e_1 \wedge e_2)' = \tau e_3 \wedge e_1 = \tau e_2 \Longrightarrow b' \cdot n = (e_1 \wedge e_2)' \cdot e_2 = \tau$ is the torsion of α.

Till now, we have constructed a description of curves in 3-dimensional Euclidean spaces. □

Remark 10. In the particular case of a plane curve $\alpha : I \longrightarrow \mathbb{R}^3$, it is possible to give the curvature k a sign. For that, let $\{e_1, e_2\}$ be the natural basis (see of $\mathbb{R}^2$ and define the normal vector n(s), s $\in$ I , by requiring the basis $\{t(s), n(s)\}$ to have the same orientation as the basis $\{e_1, e_2\}$. The curvature k is then defined by:

$$\frac{dt}{ds} = kn$$

and might be either positive or negative. It is clear that $|k|$ agrees with the previous definition and that k changes sign when we change either the orientation of α or the orientation of $\mathbb{R}^2$

1.3 the global theory of Plane Curve

Definition 1.9. A closed plane curve is a regular parametrized curve $\alpha : [a, b] \longrightarrow \mathbb{R}^2$ such that α and all its derivatives agree at a and b; that is,

$$\alpha^i(a) = \alpha^i(b), i = 0, 1, 2, \cdots$$

The curve α is simple if it has no further self-intersections; that is, if $t_1, t_2 \in [a, b), t_1 \neq t_2$, then $\alpha(t_1) \neq \alpha(t_2)$

In this part, we usually consider the curve $\alpha : [0, l] \longrightarrow \mathbb{R}^2$ parametrized by arc length s; hence, l is the length of α. Sometimes we refer to a simple closed curve

C, meaning the trace of such an object. The curvature of α will be taken with a sign

We assume that a simple closed curve C in the plane bounds a region of this plane that is called the interior of C. Whenever we speak of the area bounded by a simple closed curve C, we mean the area of the interior of C. We assume further that the parameter of a simple closed curve can be so chosen that if one is going along the curve in the direction of increasing parameters, then the interior of the curve remains to the left. Such a curve will be called positively oriented.

In fact, in topology, we have:

Theorem 1.7. *A curve in plane or spherical surface is called a Jordan-curve $\Longleftrightarrow$ it's hemeomorphic to S^1 (Simple,closed curve). If J is a Jordan-curve in $\mathbb{R}^2$, then $\mathbb{R}^2 \backslash J$ has two connected components.They both have J as boundary.*

Now let's talk about our first problem:Of all simple closed curves in the plane with a given length l, which one bounds the largest area? It's also called isoperimetric problem.

Let's recall: for the area A bounded by a positively oriented simple closed curve α(t) = (x(t), y(t)), where t $\in$ [a, b] is an arbitrary parameter:

$$A = -\int_a^b y(t)x'(t)\,dt = \int_a^b x(t)y'(t)\,dt = \frac{1}{2}\int_a^b (xy' - yx')\,dt \quad \textbf{(1)}$$

Notice that the second formula is obtained from the first one by observing that

$$\begin{aligned}\int_a^b xy'dt &= \int_a^b (xy)'\,dt - \int_a^b x'y\,dt = [xy(b) - xy(a)] - \int_a^b x'y\,dt \\ &= -\int_a^b x'y\,dt,\end{aligned}$$

Theorem 1.8 (The Isoperimetric Inequality)**.** *Let C be a simple closed plane curve with length l, and let A be the area of the region bounded by C. Then*

$$l^2 - 4\pi A \geq 0$$

and equality holds if and only if C is a circle

proof. Let E and E' be two parallel lines which do not meet the closed curve C, and move them together until they first meet C. We thus obtain two parallel tangent lines to C, L and L' , so that the curve is entirely contained in the strip bounded by L and L' . Consider a circle S^1 which is tangent to both L and L' and does not meet C. Let O be the center of S^1 and take a coordinate system with origin at O and the x axis perpendicular to L and L'. Parametrize C by arc length, α(s) = (x(s), y(s)), so that it is positively oriented and the tangency points of L and L' are s = 0 and $s = s_1$,respectively.

We can assume that the equation of S^1 is

$$\bar{\alpha}(s) = (\bar{x}(s),\ \bar{y}(s)) = (x(s), \bar{y}(s)), s \in [0, l].$$

(Consider the reparametrization:$\bar{\alpha} : [0, l] \longrightarrow \mathbb{R}^2$,where $\bar{y}(s) =$

$$\begin{array}{r} \sqrt{r^2 - x^2(s)}, if x \in [0, s_1] \\ -\sqrt{r^2 - x^2(s)}, if x \in [s_1, l] \end{array}$$

) Let 2r be the distance between L and L'. Denoting by $\bar{A}$ the area bounded by S^1, we have

$$A = \int_0^l xy'\, ds, \quad \bar{A} = \pi r^2 = -\int_0^l \bar{y}x'\, ds.$$

Thus,

$$\begin{aligned} A + \pi\, r^2 &= \int_0^l (xy' - \bar{y}x')\, ds \leq \int_0^l \sqrt{(xy' - \bar{y}x')^2}\, ds \\ &\leq \int_0^l \sqrt{(x^2 + \bar{y}^2)((x')^2 + (y')^2)}\, ds = \int_0^l \sqrt{\bar{x}^2 + \bar{y}^2}\, ds \\ &= lr, \end{aligned}$$

Use AM-GM Inequality, we can see that $lr \geq 2\sqrt{A\pi r^2}$, then we have prooved the Inequality. When the equality holds, we can see that $A = \pi r^2$,thus $l = 2\pi r$ From AM-GM Inequality, we have:

$$(x, \bar{y}) = \lambda(y', -x')$$

Thus, $x = \pm ry'$. Since r does not depend on the choice of the direction of L, we can interchange x and y in the last relation and obtain $y = \pm rx'$. Thus, $x^2 + y^2 = r^2((x')^2 + (y')^2) = r^2$ and C is a circle. (Notice that: $\lambda = \frac{x}{y'} = \frac{\bar{y}}{x'} = \frac{\sqrt{x^2+\bar{y}^2}}{\sqrt{(y')^2+(x')^2}} = \pm r$.) $\square$

Remark 11. It is easily checked that the above proof can be applied to C^1 curves

Remark 12. It is convenient to remark that the theorem holds for piecewise C^1 curves, that is, continuous curves that are made up by a finite number of C^1 arcs. These curves can have a finite number of corners, where the tangent is discontinuous

Chapter 2

Regular Surfaces

Definition 2.1. A subset S $\in R^3$ is a regular surface if, for each p in S, there exists a neighborhood V in R^3 and a map $x : U \longrightarrow V \bigcap S$ of an open set U $in R^2$ onto $V \bigcap S \in R^3$ such that:

- I,x is differentiable. This means that if we write

  $$\mathbf{x}(\mathbf{u}, \mathbf{v}) = (\mathbf{x}(\mathbf{u}, \mathbf{v}), \mathbf{y}(\mathbf{u}, \mathbf{v}), \mathbf{z}(\mathbf{u}, \mathbf{v})), \quad (\mathbf{u}, \mathbf{v}) \in \mathbf{U},$$

  the functions x(u, v), y(u, v), z(u, v) have continuous partial derivatives of all orders in U.

- II,x is a homeomorphism. Since x is continuous by condition 1, $this$ means has an inverse $\mathbf{x}^{-1}$:V$\cap S$ $\rightarrow$U which is continuous.

- (The regularity condition.)III,For each q $\in$ U, the differential $dx_q : R^2 \longrightarrow R^3$ is one to one.

Recall: $F : U \subset R^n \longrightarrow R^m$ differentiable. To each p $\in$ U, one define a map: $dF_p : R^n \longrightarrow R^m$ as follows: for each v $\in R^n$, choose a parametrized curve α:$(-\epsilon, \epsilon) \longrightarrow U$ so that $\alpha(0) = p, \alpha'(0) = v$.Define $\beta = F \circ \alpha$ and $dF_p(v) = \beta'(0)$

Remark 13. $dF_p(v)$ doesn't depend on the choice of α. To see this, just compute directly. Maybe I will put the general edition in appendix. Also, you may ask whether α in the recall part exists. That's right. See appendix

Now we only prove what we want. Let F(u,v)=(x(u,v),y(u,v),z(u,v)),$\alpha(t) = (a(t), b(t))$,then we have $F \circ \alpha(t) = (x(a(t), b(t)), y(a(t), b(t)), z(a(t), b(t)))$,we have $\beta'(0) = (\frac{\partial x}{\partial u}(\alpha(0))a'(0)+\frac{\partial x}{\partial v}(\alpha(0))b'(0), \frac{\partial y}{\partial u}(\alpha(0))a'(0)+\frac{\partial y}{\partial v}(\alpha(0))b'(0), \frac{\partial z}{\partial u}(\alpha(0))a'(0)+ \frac{\partial z}{\partial v}(\alpha(0))b'(0)))$ So we can see that $dF_p(v) = \begin{pmatrix} \frac{\partial x}{\partial u}(p) & \frac{\partial x}{\partial v}(p) \\ \frac{\partial y}{\partial u}(p) & \frac{\partial y}{\partial v}(p) \\ \frac{\partial z}{\partial u}(p) & \frac{\partial z}{\partial v}(p) \end{pmatrix} v$

Now, let's back to our definition.Condition 3 of Def 2.1 may now be expressed by requiring the two column vectors of this matrix to be linearly independent; or, equivalently, that the vector product $\partial \mathbf{x}/\partial u \wedge \partial \mathbf{x}/\partial v \neq 0$; or, in still another way, that one of the minors of order 2 of the matrix of $d\mathbf{x}_q$, that is, one of the Jacobian determinants:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}, \quad \frac{\partial(y, z)}{\partial(u, v)}, \quad \frac{\partial(x, z)}{\partial(u, v)},$$

be different from zero at q.

Remark 14. Definition 2.1 deserves a few comments. First, in contrast to our treatment of curves in Chapter 1, we have defined a surface as a subset S of R^3, and not as a map. This is achieved by covering S with the traces of parametrizations which satisfy conditions 1, 2, and 3.

Remark 15. Condition 1 is very natural if we expect to do some differential geometry on S. The one-to-oneness in condition 2 has the purpose of preventing self-intersections in regular surfaces. This is clearly necessary if we are to speak about, say, the tangent plane at a point p *in* S. The continuity of the inverse in condition 2 has a more subtle purpose which can be fully understood only in the next section. For the time being, we shall mention that this condition is essential to proving that certain objects defined in terms of a parametrization do not depend on this parametrization but only on the set S itself. Finally, as we shall show in Sec. 2.4, condition 3 will guarantee the existence of a "tangent plane" at all points of S.

Example 2.0.1. The unit sphere S^2 is a regular surface

proof. We first verify that the map $\mathbf{x}_1{:}U \subset R^2 \to R^3$ given by

$$\mathbf{x}_1(x, y) = (x,\ y, +\sqrt{1-(x^2+y^2)}\ ), \quad (x,\ y) \in U,$$

where $R^2 = \{(x, y, z) \in R^3; z = 0\}$ and $U = \{(x, y) \in R^2; x^2 + y^2 < 1\}$, is a parametrization of S^2. Observe that $\mathbf{x}_1(U)$ is the (open) part of S^2 above the xy plane.

Since $x^2 + y^2 < 1$, the function $+\sqrt{1-(x^2+y^2)}$ has continuous partial derivatives of all orders. Thus, $\mathbf{x}_1$ is differentiable and condition 1 holds.

Condition 3 is easily verified, since

$$\frac{\partial(x, y)}{\partial(x, y)} \equiv 1.$$

To check condition 2, we observe that $\mathbf{x}_1$ is one-to-one and that $\mathbf{x}_1^{-1}$ is the restriction of the (continuous) projection $\pi(x, y, z) = (x, y)$ to the set $\mathbf{x}_1(U)$. Thus, $\mathbf{x}_1^{-1}$ is continuous in $\mathbf{x}_1(U)$. We shall now cover the whole sphere with similar parametrizations as follows. We define $\mathbf{x}_2{:}U \subset R^2 \to R^3$ by

$$\mathbf{x}_2(x, y) = (x, y, -\sqrt{1-(x^2+y^2)}\ ),$$

check that $\mathbf{x}_2$ is a parametrization, and observe that $\mathbf{x}_1(U) \cup \mathbf{x}_2(U)$ covers S^2 minus the equator

$$\{(x, y, z) \in R^3; x^2 + y^2 = 1, z = 0\}.$$

Then, using the xz and zy planes, we define the parametrizations

$$
\begin{aligned}
\mathbf{x}_3(x, z) &= (x, +\sqrt{1-(x^2+z^2)}\,,\ z),\\
\mathbf{x}_4(x, z) &= (x, -\sqrt{1-(x^2+z^2)}\,, z),\\
\mathbf{x}_5(y, z) &= (+\sqrt{1-(y^2+z^2)}\,, y, z),\\
\mathbf{x}_6(y, z) &= (-\sqrt{1-(y^2+z^2)}\,,\ y, z),
\end{aligned}
$$

which, together with $\mathbf{x}_1$ and $\mathbf{x}_2$, cover S^2 completely (Fig. 2-4) and show that S^2 is a regular surface.

$\square$

It is convenient to relate parametrizations to the geographical coordinates on S^2. Let $V = \{(\theta, \varphi); 0 < \theta < \pi, 0 < \varphi < 2\pi\}$ and let $\mathbf{x}: V \to R^3$ be given by

$$\mathbf{x}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

Clearly, $\mathbf{x}(V) \subset S^2$. We shall prove that x is a parametrization of $S^2.\theta$ is usually called the *colatitude* (the complement of the latitude) and φ the *longitude* .

It is clear that the functions $\sin\theta\cos\varphi$, $\sin\theta\sin\varphi, \cos\theta$ have continuous partial derivatives of all orders; hence, x is differentiable. Moreover, in order that the Jacobian determinants

$$
\begin{aligned}
\frac{\partial(x, y)}{\partial(\theta, \varphi)} &= \cos\theta\ \sin\theta,\\
\frac{\partial(y, z)}{\partial(\theta, \varphi)} &= \sin^2\theta\cos\varphi,\\
\frac{\partial(x, z)}{\partial(\theta, \varphi)} &= -\sin^2\theta\sin\varphi
\end{aligned}
$$

vanish simultaneously, it is necessary that

$$\cos^2\theta\sin^2\theta + \sin^4\theta\cos^2\varphi + \sin^4\theta\sin^2\varphi = \sin^2\theta = 0.$$

This does not happen in V, and so conditions 1 and 3 of Def. 1 are satisfied. Next, we observe that given $(x, y, z) \in S^2 - C$,where C is the semicircle

$$C = \{(x, y, z) \in S^2;\ y = 0,\ x \geq 0\},$$

θ is uniquely determined by $\theta = \cos^{-1} z$, since $0 < \theta < \pi$. By knowing θ, we find sin φ and cos φ from $x = \sin\theta\cos\varphi, \quad y = \sin\theta\sin\varphi$ and this determines φ uniquely $(0 < \varphi < 2\pi)$. It follows that x has an inverse $\mathbf{x}^{-1}$. To complete the verification of condition 2, we should prove that $\mathbf{x}^{-1}$ is continuous. However, since we shall soon prove that this verification is not necessary provided we already know that the set S is a regular surface,

We remark that $\mathbf{x}(V)$ only omits a semicircle of S^2 (including the two poles) and that S^2 can be covered with the coordinate neighborhoods of two parametrizations this type.

From this example, you may feel that checking these conditions is tiring. Yeah, so we have some propositions to simplify this.

proposition 2.1. If f:$U \to \mathbb{R}$ is a differentiable function in an open set U of R^2, then the graph of f, that is, the subset of R^3 given by (x, y, f(x, y)) for (x, y) $\in$ U, is a regular surface.

proof. It suffices to show that the map $\mathbf{x}$:$U \to R^3$ given by

$$\mathbf{x}(u, v) = (u, v, f(u, v))$$

is a parametrization of the graph whose coordinate neighborhood covers every point of the graph. Condition 1 is clearly satisfied, and condition 3 also offers no difficulty since $\partial(x, y)/\partial(u, v) \equiv 1$. Finally, each point (x,y,z) of the graph is the image under x of the unique point $(u, v) = (x, y) \in U$.$\mathbf{x}$ is therefore one-to-one, and since $\mathbf{x}^{-1}$ is the restriction to the graph of f of the (continuous) projection of R^3 onto the xy plane, $\mathbf{x^{-1}}$ is continuous. $\square$

In topological language, we usually use submersion and immersion to describe the differential between two manifolds. Now we give a simplified version

Definition 2.2. Given a differentiable map $F : U \subset R^n \to R^m$ defined in an open set U of R^n we say that p $\in$ U is a critical point of F if the differential $dF_p : R^n \to R^m$ is not a surjective (or onto) mapping. The image $F(p) \in R^m$ of a critical point is called a critical value of F. A point of R^m which is not a critical value is called a regular value of F.

Remark 16. Any point a $\notin$ F(U) is a regular value

If $f : U \subset R^3 \to R$ is a differentiable function, then df_p applied to the vector(1,0,0)is obtained by calculating the tangent vector at $f(p)$ to the curve:

$$x \longrightarrow f(x, y_0, z_0)$$

It follows that

$$df_p(1,0,0) = \frac{\partial f}{\partial x}(x_0, y_0, z_0) = f_x$$

and analogously that

$$df_p(0,1,0) = f_y, \quad df_p(0,0,1) = f_z.$$

We conclude that the matrix of df_p in the basis (1,0,0),(0,1,0),(0,0,1) is given by

$$df_p = (f_x,\ f_y,\ f_z).$$

Note, in this case, that to say that df_p is not surjective is equivalent to saying that $f_x = f_y = f_z = 0$ at p. Hence, $a \in f(U)$ is a regular value of f:$U \subset R^3 \to R$ if and only if f_x, f_y,andf_z do not vanish simultaneously at any point in the inverse image

$$f^{-1}(a) = \{(x, y, z) \in U\colon f(x, y, z) = a\}.$$

It's obviouly that you can repeat the process in R^n instead of R^3.More generally,we have:

Theorem 2.2. *Given M is a smooth manifold,for a real-valued function f : $M \to \mathbb{R}$, a point p in M is a critical point if and only if relative to some chart $(U, x^1, \ldots, x^n)$ containing p, all the partial derivatives satisfy:*

$$\frac{\partial f}{\partial x^j}(p) = 0, \quad j = 1, \ldots, n.$$

The proof needs more clear and general definition of partial derivatives between manifolds, which is covered in appendix.

Now we have:

proposition 2.3. If $f : U \subset R^3 \to R$ is a differentiable function and a $\in$ f(U) is a regular value of f, then f^{-1}(a) is a regular surface in R^3. .

proof. Let p = (x_0, y_0, z_0) be a point of f^{-1} (a). Since a is a regular value of f, it is possible to assume, by renaming the axis if necessary, that $f_z \neq 0$ at p. We define a mapping $F : U \subset R^3 \to R$ by:

$$F(x, y, z) = (x, y, f(x, y, z))$$

and we indicate by (u, v, t) the coordinates of a point in R^3 where F takes its values. The differential of F at p is given by

$$dF_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{pmatrix},$$

where

$$\det(dF_p) = f_z \neq 0.$$

We can therefore apply the inverse function theorem, which guarantees the existence of neighborhoods V of p and W of $F(p)$ such that F:$V \to W$ is invertible and the inverse F^{-1}:$W \to V$ is differentiable. It follows that the coordinate functions of F^{-1}, i.e., the functions

$$x = u, \quad y = v, \quad z = g(u, v, t), \quad (u, v, t) \in W,$$

are differentiable. In particular, z = g(u, v, a) = h(x, y) is a differentiable function defined in the projection of V onto the xy plane. Since

$$F(f^{-1}(a) \cap V) = W \cap \{(u, v, t); t = a\},$$

we conclude that the graph of h is $f^{-1}(a) \cap V$. By Prop 2.1, $f^{-1}(a) \cap V$ is a coordinate neighborhood of p. Therefore, every $p \in f^{-1}(a)$ can be covered . by a coordinate neighborhood, and so $f^{-1}(a)$ is a regular surface. □

Generally, we have:

Theorem 2.4. *Let $F : N \to M$ be a C^∞ map of manifolds, with $dimN = n$ and $dimM = m$. Then a nonempty regular level set $F^{-1}(c)$, where $c \in M$, is a regular submanifold of N of dimension equal to n-m*

A surface $S \subset R^3$ is said to be connected if any two of its points can be joined by a continuous curve in S. In the definition of a regular surface we made no restrictions on the connectedness of the surfaces, and the following example shows that the regular surfaces may not be connected.

Example 2.0.2. The hyperboloid of two sheets $-x^2 - y^2 + z^2 = 1$ is a regular surface, since it is given by $S = f^{-1}(0)$, where 0 is a regular value of $f(x, y, z) = -x^2 - y^2 + z^2 - 1$. Note that the surface S is not connected; that is, given two points in two distinct sheets ($z > 0$ and $z < 0$) it is not possible to join them by a continuous curve $\alpha(t) = (x(t), y(t), z(t))$ contained in the surface; otherwise, z changes sign and, for some t_0, we have $z(t_0) = 0$,which means that $\alpha(t_0) \notin S$.

Starting from this example,we can conclude that:

proposition 2.5. If f: $S \subset R^3 \to R$ is a nonzero continuous function defined on a connected surface S, then f does not change sign on S.

proof. Assume, by contradiction, that $f(p) > 0$ and $f(q) < 0$ for some points p, q ∈ S. Since S is connected, there exists a continuous curve $\alpha : [a, b] \to S$ with $\alpha(a) = p, \alpha(b) = q$. By applying the intermediate value theorem to the continuous function $f \circ \alpha : [a, b] \to R$, we find that there exists c ∈ (a, b) with $f \circ \alpha(c) = 0$; that is, f is zero at α(c), a contradiction. □

Since we have shown that any graph of a differentiable function is a regular surface, a normal question is that given a regular surface, cam we find a differentiable function whose graph is this surface. The following proposition provides a local converse of this; that is, any regular surface is locally the graph of a differentiable function.

proposition 2.6. Let $S \subset R^3$ be a regular surface and p ∈ S. Then there exists a neighborhood V of p in S such that V is the graph of a differentiable function which has one of the following three forms: z = f(x, y), y = g(x, z), x = h(y, z).

proof. Let x: $U \subset R^2 \to S$ be a parametrization of S in p, and write $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)), (u, v) \in U$. By condition 3 of Def. 1, one of the Jacobian determinants

$$\frac{\partial(x, y)}{\partial(u, v)}, \quad \frac{\partial(y, z)}{\partial(u, v)}, \quad \frac{\partial(z, x)}{\partial(u, v)}$$

is not zero at $\mathbf{x}^{-1}(p) = q$. Suppose first that $(\partial(x, y)/\partial(u, v))(q) \neq 0$, and consider the map $\pi \circ \mathbf{x} : U \to R^2$, where π is the projection $\pi(x, y, z) = (x, y)$. Then $\pi \circ \mathbf{x}(u, v) = (x(u, v), y(u, v))$, and, since $(\partial(x, y)/\partial(u, v))(q) \neq 0$, we can apply the inverse function theorem to guarantee the existence of neighborhoods V_1 of q, V_2 of $\pi \circ \mathbf{x}(q)$ such that $\pi \circ \mathbf{x}$ maps V_1 diffeomorphically onto V_2. It follows that π restricted to $\mathbf{x}(V_1) = V$ is one-to-one and that there is a differentiable inverse $(\pi \circ \mathbf{x})^{-1} : V_2 \to V_1$. Observe that, since x is a homeomorphism, V is a neighborhood of p in S. Now, if we compose the map $(\pi \circ \mathbf{x})^{-1}$: $(x, y) \to (u(x, y), v(x, y))$ with the function $(u, v) \to z(u, v)$, we find that V is the graph of the differentiable function $z = z(u(x, y), v(x, y)) = f(x, y)$, and this settles the first case.

The remaining cases can be treated in the same way, yielding $x = h(y, z)$ and

$y = g(x, z)$. □

Example 2.0.3. The one-sheeted cone C, given by

$$z = +\sqrt{x^2 + y^2}, \quad (x, y) \in R^2,$$

is not a regular surface. Observe that we cannot conclude this from the fact alone that the "natural" parametrization

$$(x, y) \to (x, y, +\sqrt{x^2 + y^2})$$

is not differentiable; there could be other parametrizations satisfying Def 2.1. To show that this is not the case, we use Prop 2.6 If C were a regular surface, it would be, in a neighborhood of $(0, 0, 0) \in C$, the graph of a differentiable function having one of three forms: $y = h(x, z), x = g(y, z), z = f(x, y)$. The two first forms can be discarded by the simple fact that the projections of C over the xz and yz planes are not one-to-one. The last form would have to agree, in a neighborhood of $(0, 0, 0)$, with $z = +\sqrt{x^2 + y^2}$. Since $z = +\sqrt{x^2 + y^2}$ is not differentiable at (0,0), this is impossible.

proposition 2.7. Let p $\in$ S be a point of a regular surface S and let $x : U \subset R^2 \to R^3$ be a map with p $\in x(U) \subset S$ such that conditions 1 and 3 of Def 2.1 hold. Assume that x is one-to-one. Then x^{-1} is continuous.

proof. Write $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)), (u, v) \in U$, and let q $\in$ U. By conditions 1 and 3, we can assume, interchanging the coordinate axis if necessary, that $\partial(x, y)/\partial(u, v)(q) \neq 0$.Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection $\pi(x, y, z) = (x, y)$. By the inverse function theorem, we obtain neighborhoods V_1 of q in U and V_2 of $\pi \circ \mathbf{x}(q)$ in $\mathbb{R}^2$ such that $\pi \circ \mathbf{x}$ applies V_1 diffeomorphically onto V_2.

Assume now that x is bijective. Then, restricted to $\mathbf{x}(V_1), \mathbf{x}^{-1} = (\pi \circ \mathbf{x})^{-1}$ $\circ\pi$.Thus x^{-1}, as a composition of continuous maps, is continuous. □

Example 2.0.4. A parametrization for the torus T can be given by

$$\mathbf{x}(u,v) = ((r\cos u + a)\cos v,\ (r\cos u + a)\sin v, r\sin u),$$

where $0 < u < 2\pi, 0 < v < 2\pi$. (The torus T is a "surface" generated by rotating a circle S^1 of radius r about a straight line belonging to the plane of the circle and at a distance $\alpha > r$ away from the center of the circle.) Condition l of Def 2.1 is easily checked, and condition 3 reduces to a straightforward computation, which is left as an exercise. Since we know that T is a regular surface, condition 2 is equivalent, by Prop. 4, to the fact that x is one-to-one.

To prove that x is one-to-one, we first observe that $\sin u = z/r$; also, if $\sqrt{x^2+y^2} \leq \alpha$, then $\pi/2 \leq u \leq 3\pi/2$, and if $\sqrt{x^2+y^2} \geq \alpha$, then either $0 < u \leq \pi/2$ or $3\pi/2 \leq u < 2\pi$. Thus, given (x, y, z), this determines u $0 < u < 2\pi$, uniquely. By knowing u, x,and y we find cos v and $\sin v$. This determines v uniquely, $0 < v < 2\pi$ .Thus, x is one-to-one.

It is easy to see that the torus can be covered by three such coordinate neighborhoods.

Now we introduce an important proposition.

proposition 2.8. Let p be a point of a regular surface S, and let $\mathbf{x}$:$\mathcal{U} \subset \mathbb{R}^2 \to \mathcal{S}, \mathbf{y}$:$\mathcal{V} \subset \mathbb{R}^2 \to \mathcal{S}$ be two parametrizations of S such that p $\in \mathbf{x}(\mathcal{U}) \cap \mathrm{y}(\mathcal{V}) = \mathcal{W}$.Then the "change of coordinates" $\mathbf{h} = \mathbf{x}^{-1} \circ \mathbf{y} : \mathbf{y}^{-1}(\mathbf{W}) \to \mathbf{x}^{-1}(\mathbf{W})$ is α diffeomorphism; that is, h is differentiable and has a differentiable inverse h^{-1}.

What does this mean? Let's see together. if x and y are given by

$$\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v)), \qquad (u,v) \in U,$$
$$\mathbf{y}(\xi,\eta) = (x(\xi,\eta), y(\xi,\eta), z(\xi,\eta)), \qquad (\xi,\eta) \in V,$$

then the change of coordinates h, given by

$$u = u(\xi,\eta), \quad v = v(\xi,\eta), \quad (\xi,\eta) \in \mathbf{y}^{-1}(W),$$

has the property that the functions u and v have continuous partial derivatives of all orders, and the map h can be inverted, yielding

$$\xi = \xi(u, v), \quad \eta = \eta(u, v), \quad (u, v) \in \mathbf{x}^{-1}(W),$$

where the functions ξ and η also have partial derivatives of all orders. Since

$$\frac{\partial\,(u, v)}{\partial\,(\xi, \eta)} \cdot \frac{\partial\,(\xi, \eta)}{\partial\,(u, v)} = 1,$$

this implies that the Jacobian determinants of both h and h^{-1} are nonzero everywhere.

proof. $h = x^{-1} \circ \mathbf{y}$ is a homeomorphism, since it is composed of homeomorphisms. It is not possible to conclude, by an analogous argument, that h is differentiable, since x^{-1} is defined in an open subset of S, and we do not yet know what is meant by a differentiable function on S.

We proceed in the following way. Let $r \in \mathbf{y}^{-1}(W)$ and set $q = h(r)$. Since $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$ is a parametrization, we can assume, by renaming the axis if necessary, that:

$$\frac{\partial(x, y)}{\partial(u, v)}(q) \neq 0.$$

We extend $\mathbf{x}$ to a map $F{:}U \times R \to R^3$ defined by

$$F(u, v, t) = (x(u, v), y(u, v), z(u, v) + t), \quad (u, v) \in U, t \in R.$$

It is clear that F is differentiable and that the restriction $F|_{U\times\{0\}} = \mathbf{x}$. Calculating the determinant of the differential dF_q, we obtain

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & 0 \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & 0 \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & 1 \end{vmatrix} = \frac{\partial(x,y)}{\partial(u,v)}(q) \neq 0.$$

It is possible therefore to apply the inverse function theorem, which guarantees the existence of a neighborhood M of $\mathbf{x}(q)$ in R^3 such that F^{-1} exists and is differentiable in M.

By the continuity of y, there exists a neighborhood N of r in V such that $\mathbf{y}(N) \subset M$ and $N \subset \mathbf{y}^{-1}(W)$. Notice that, restricted to N, $h|_N = F^{-1} \circ \mathbf{y}|_N$ is a composition of differentiable maps. Thus, we can apply the chain rule for maps and conclude that h is differentiable at r .Since r is arbitrary, h is differentiable on $\mathbf{y}^{-1}(W)$.

Exactly the same argument can be applied to show that the map h^{-1} is differentiable, and so h is a diffeomorphism.

□

Remark 17. Since S may be a close or other kinds of subsets of R^3, x^{-1} may be not defined on a open subset of R^3, so we use F, which is defined on an open subset of R^3 to prove.

Remark 18. we may use to projection π to make sure: $h|_N = \pi \circ F^{-1} \circ \mathbf{y}|_N$. Also notice that if we know x(u,v), y(u,v) in M, then we can see that u,v is uniquely confirmed, which has nothing to do with t.

Remark 19. I'm so foolish that I spend almost half day understanding this proposition. Here I'm going to introduce another way of this definition. However, it's more about manifolds. Since there is only one differential structure on 2-dimensional smooth manifold, any two charts(parametrizations) is compatible to each other. If I have time, I will write it in appendix.

We shall now give an explicit definition of what is meant by a differentiable function on a regular surface.

Definition 2.3. Let f: $\mathcal{V} \subset \mathcal{S} \to \mathbb{R}$ be a function defined in an open subset $\mathcal{V}$ of a regular surface $\mathcal{S}$. Then f is said to be differentiable at p $\in \mathcal{V}$ if, for some parametrization $\mathbf{x}:\mathcal{U} \subset \mathbb{R}^2 \to \mathcal{S}$ with p $\in \mathbf{x}(\mathcal{U}) \subset \mathcal{V}$, the composition f $\circ \mathbf{x}:\mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $\mathrm{x}^{-1}(\mathfrak{p})$. f is differentiable in V if it is differentiable at all points of V.

It follows immediately from the above proposition that the definition given does not depend on the choice of the parametrization x. In fact, if $\mathbf{y}:V \subset R^2 \to S$ is another parametrization with $p \in \mathbf{y}(V)$, and if $h = \mathbf{x}^{-1} \circ \mathbf{y}$, then $f \circ \mathbf{y} = f \circ \mathbf{x} \circ h$ is also differentiable, whence the asserted independence.

Remark 20. The proof of the above proposition makes essential use of the fact that the inverse of a parametrization is continuous. Since we need it to be able to define differentiable functions on surfaces (a vital concept), we cannot dispose of this condition in the definition of a regular surface

Remark 21. We shall frequently make the notational abuse of indicating f and $f \circ \mathbf{x}$ by the same symbol f(u, v), and say that f(u, v) is the expression of f in the system of coordinates $\mathbf{x}$. This is equivalent to identifying x(U) with U and thinking of (u, v), indifferently, as a point of U and as a point of x(U) with coordinates (u, v). From now on, abuses of language of this type will be used without further comment.

Example 2.0.5. Let S be a regular surface and $V \subset R^3$ be an open set such that $S \subset V$. Let $f:V \subset R^3 \to R$ be a differentiable function. Then the restriction of f to S is a differentiable function on S. In fact, for any $p \in S$ and any parametrization $\mathbf{x}:U \subset R^2 \to S$ in p, the function $f \circ \mathbf{x}:U \to R$ is differentiable. In particular, the following are differentiable functions:

- The *height* function relative to a unit vector $v \in R^3, h:S \to R$, given by $h(p) = p \cdot v, p \in S$, where the dot denotes the usual inner product in $R^3.h(p)$ is the height of $p \in S$ normal to v and in $R^3.h(p)$ is the height of $p \in S$ relative to a plane normal to v and passing through the origin of R^3.

- The square of the distance from a fxed point $p_0 \in R^3$, $\quad f(p) = |p - p_0|^2$, $p \in S$. The need for taking the square comes from the fact that the distance $|p - p_0|$ is not differentiable at $p = p_0$.

The definition of differentiability can be easily extended to mappings between surfaces. A continuous map $\varphi{:}V_1 \subset S_1 \to S_2$ of an open set V_1 of a regular surface S_1 to a regular surface S_2 is said to be $differentiable$ at p$\in V_1$ if, given parametrizations

$$\mathbf{x}_1{:}U_1 \subset R^2 \to S_1 \quad \mathbf{x}_2{:}U_2 \subset R^2 \to S_2,$$

with $p \in \mathbf{x}_1(U)$ and $\varphi(\mathbf{x}_1(U_1)) \subset \mathbf{x}_2(U_2)$, the map

$$\mathbf{x}_2^{-1} \circ \varphi \circ \mathbf{x}_1 : U_1 \to U_2$$

is differentiable at $q = \mathbf{x}_1^{-1}(p)$.

Remark 22. these definitions can be extended to apply in more general cases, just like general linear group....(all snooth manifolds)

To check this definition is well-defined, just use pur proposition.

We should mention that the natural notion of equivalence associated with differentiability is the notion of diffeomorphism. Two regular surfaces S_1 and S_2 are $diffeomorphic$ if there exists a differentiable map $\varphi{:}S_1 \to S_2$ with a differentiable inverse $\varphi^{-1}{:}S_2 \to S_1$. Such a φ is called a $diffeomorphism$ from S_1 to S_2. The notion of diffeomorphism plays the same role in the study of regular surfaces that the notion of isomorphism plays in the study of vector spaces or the notion of congruence plays in Euclidean geometry. In other words, from the point of view of differentiability, two diffeomorphic surfaces are indistinguishable.

Remark 23. from the definition of diffeomorphism, we can see that it's stronger than homeomorphism.

Back to some meaningful examples,

Example 2.0.6. If $\mathbf{x}$: $U \subset R^2 \to S$ is a parametrization, $\mathbf{x^{-1}}{:}\mathbf{x}(U) \to R^2$ is differentiable. In fact, for any $p \in \mathbf{x}(U)$ and any parametrization y: $V \subset R^2 \to S$

in p, we have that $\mathbf{x}^{-1} \circ \mathbf{y}{:}\mathbf{y}^{-1}(W) \to \mathbf{x}^{-1}(W)$, where

$$W = \mathbf{x}(U) \cap \mathbf{y}(V),$$

is differentiable. This shows that U and $\mathbf{x}(U)$ are diffeomorphic(i.e., every regular surface is locally diffeomorphic to a plane).

Example 2.0.7. Let S_1 and S_2 be regular surfaces. Assume that $S_1 \subset V \subset R^3$, where V is an open set of R^3, and that $\varphi{:}V \to R^3$ is a differentiable map such that $\varphi(S_1) \subset S_2$. Then the restriction $\varphi|_{S_1}{:}S_1 \to S_2$ is a differentiable map. In fact, given $p \in S_1$ and parametrizations $\mathbf{x}_1{:}U_1 \to S_1, \mathbf{x}_2{:}U_2 \to S_2$, with $p \in \mathbf{x}_1(U_1)$ and $\varphi(\mathbf{x}_1(U_1)) \subset \mathbf{x}_2(U_2)$, we have that the map

$$\mathbf{x}_2^{-1} \circ \varphi \circ \mathbf{x}_1 : U_1 \to U_2$$

is differentiable. The following are particular cases of this general example:

- Let S be symmetric relative to the xy plane; that is, if $(x, y, z) \in S$, then also $(x, y, -z) \in S$. Then the map $\sigma{:}S \to S$ which takes $p \in S$ into its symmetrical point, is differentiable, since it is the restriction to S of $\sigma : R^3 \to R^3, \sigma(x, y, z) = (x, y, -z)$. This, of course, generalizes to surfaces symmetric relative to any plane of R^3.

- Let $R_{z,\theta}{:}R^3 \to R^3$ be the rotation of angle θ about the z axis, and let $S \subset R^3$ be a regular surface invariant by this rotation; i.e., if $p \in S$, $R_{z,\theta}(p) \in S$. Then the restriction $R_{z,\theta}{:}S \to S$ is a differentiable map.

- Let $\varphi{:}R^3 \to R^3$ be given by $\varphi(x, y, z) = (xa, yb, zc)$,where α, b,and c are nonzero real numbers. φ is clearly differentiable, and the restriction $\varphi|S^2$ is a differentiable map from the sphere

$$S^2 = \{(x, y, z) \in R^3; x^2 + y^2 + z^2 = 1\}$$

into the ellipsoid:

$$\left\{(x,y,z) \in R^3; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}$$

There is an important theorem to help you understand.

Theorem 2.9. *Given three regular surfaces S_1, S_2, S_3 and let f:$S_1 \to S_2$ is differentiable at p, g:$S_2 \to S_3$ is differentiable at f(p), then $g \circ f : S_1 \to S_3$ is differentiable at p.*

proof. There exists parametrizations $\mathbf{x_1} : U_1 \subset R^2 \to S_1$,$\mathbf{x_2} : U_2 \subset R^2 \to S_2$ with p $\in \mathbf{x}_1(U_1)$ and $f(\mathbf{x}_1(U_1)) \subset \mathbf{x}_2(U_2)$, such that

$$\mathbf{x}_2^{-1} \circ f \circ \mathbf{x}_1 : U_1 \to U_2$$

is differentiable at $q = \mathbf{x}_1^{-1}(p)$

Similarly, there exists parametrizations $\mathbf{x} : U \subset R^2 \to S_2$,$\mathbf{x_3} : U_3 \subset R^2 \to S_3$ with f(p) $\in \mathbf{x}(U)$ and $g(\mathbf{x}(U)) \subset \mathbf{x}_3(U_3)$, such that

$$\mathbf{x}_3^{-1} \circ g \circ \mathbf{x} : U \to U_3$$

is differentiable at $w = \mathbf{x}^{-1}(f(p))$

Notice that:$f(p) \in \mathbf{x}(U) \bigcap \mathbf{x}_2(U_2) \neq \emptyset$, so $\mathbf{x}^{-1} \circ \mathbf{x}_2$ is differentiable in $\mathbf{x}_2^{-1}(\mathbf{x}(U) \bigcap \mathbf{x}_2(U_2))$. Finally we can get $\mathbf{x}_3^{-1} \circ (g \circ f) \circ \mathbf{x}_1 = (\mathbf{x}_3^{-1} \circ g \circ \mathbf{x}) \circ (\mathbf{x}^{-1} \circ \mathbf{x}_2) \circ (\mathbf{x}_2^{-1} \circ f \circ \mathbf{x}_1)$ is differentiable. $\square$

Solution for some Exercises

Solution for 1.2

- 1, Notice that $(\sin t, \cos t)$ satisfies the requirement.
- 2,We have $|\alpha(t)|$ is minimized, so $\alpha(t) \cdot \alpha(t)$ is minimized. Then, we have its derivation 2 $\alpha(t)\alpha'(t) = 0$ holds for t_0. Since $\alpha(t)$ and $\alpha'(t)$ are all unequal to 0 at t_0, we can finish the proof.
- 3, we have $\alpha(t)$ is a constant vector. Integrating it.
- 4, For any t $\in$ I, we have $\alpha(t) = \alpha(0) + \int_0^t \alpha'(x)dx$, then use dot product by v.
- 5,Just like 2.

Solution for 1.3

- 1,compute the dot and wedge product of $(3, 6t, 6t^2)$ and (x,0,x)
- 2,(a)(t - $\sin t$,1- $\cos t$) and t = 2πn, where n $\in \mathbb{Z}$.
- (b) 8
- 3 (a) compute BO and CO is enough
- (b) compute

- (c) Obviously
- 4, $\sin t$ is obviouly differentiable.so as $\cos t$. Also notice that derivation on $\log(\tan(t/2))$ is a combination of sint and cost.
- (b) is obviously (Pay attetion to the x-coordinate of α and what t means)
- 5, (a)compute $\alpha'(t)$ and $\alpha(0)$ is enough.
- (b) Just compute.
- (c) we can see that

  $$\alpha(t) = (\frac{-3at}{1-t^3}, \frac{3at^2}{1-t^3})$$

  , where t $\in (-\infty, 1)$

Chapter 3

Appendix

3.1 Tensor product and wedge product

3.2 Topological manifolds