Degree Spectra and Computable Dimension in Algebraic Structures

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Abstract

Whenever a structure with a particularly interesting computability-theoretic property is found, it is natural to ask whether similar examples can be found within well-known classes of algebraic structures, such as groups, rings, lattices, and so forth. One way to give positive answers to this question is to adapt the original proof to the new setting. However, this can be an unnecessary duplication of effort, and lacks generality. Another method is to code the original structure into a structure in the given class in a way that is effective enough to preserve

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the property in which we are interested. In this paper, we show how to transfer a number of computability-theoretic properties from directed graphs to structures in the following classes: symmetric, irreflexive graphs; partial orderings; lattices; rings (with zero-divisors); integral domains of arbitrary characteristic; commutative semigroups; and 2-step nilpotent groups. This allows us to show that several theorems about degree spectra of relations on computable structures, nonpreservation of computable categoricity, and degree spectra of structures remain true when we restrict our attention to structures in any of the classes on this list. The codings we present are general enough to be viewed as establishing that the theories mentioned above are computably complete in the sense that, for a wide range of computability-theoretic nonstructure type properties, if there are any examples of structures with such properties then there are such examples which are models of each of these theories.

1 Introduction

With the formalization of the notions of algorithm and computable function in the first half of the twentieth century and the subsequent development of computability theory, there has been increasing interest in recent decades in investigating the effective content of mathematics. In this paper, we are concerned with the connection between two branches of this effective mathematics program, computable model theory and computable algebra. We assume the reader is familiar with basic concepts of computability theory, model theory, and algebra; standard references are [38], [21], and [27], respectively.

One of the main concerns of computable model theory is the study of computability-theoretic properties of countable structures. We will always assume we are working with computable languages. Let us for the moment focus on computable structures.

1.1. Definition. A structure $\mathcal{A}$ is *computable* if both its domain $|\mathcal{A}|$ and the atomic diagram of $\langle \mathcal{A}, a \rangle_{a \in |\mathcal{A}|}$ are computable. If, in addition, the existential diagram of $\langle \mathcal{A}, a \rangle_{a \in |\mathcal{A}|}$ is computable then $\mathcal{A}$ is *1-decidable*, while if the full first-order diagram of $\langle \mathcal{A}, a \rangle_{a \in |\mathcal{A}|}$ is computable then $\mathcal{A}$ is *decidable*.

An isomorphism from a structure $\mathcal{M}$ to a computable structure is called a *computable presentation* of $\mathcal{M}$. We often abuse terminology and refer to the image of a computable presentation as a computable presentation. If $\mathcal{M}$ has a computable presentation then it is *computably presentable*.

Whenever a structure with a particularly interesting computability-theoretic property is found, it is natural to ask whether similar examples can be found within well-known classes of algebraic structures, such as groups, rings, lattices, and so forth. As an example, let us consider the computable dimension of computable structures, which is a special case of the following definition.

1.2. Definition. Given a (Turing) degree $\mathbf{d}$, the $\mathbf{d}$-*computable dimension* of a computably presentable structure $\mathcal{M}$ is the number of computable presentations of $\mathcal{M}$ up to $\mathbf{d}$-computable isomorphism. If $\mathcal{M}$ has $\mathbf{d}$-computable dimension 1 then it is $\mathbf{d}$-*computably categorical.*

For an ordinal α, $\mathbf{0}^{\alpha}$-computably categorical structures are usually called $\Delta^0_{\alpha+1}$-*categorical* structures. An equivalent definition, which also works for limit ordinals, is that a computably presentable structure is Δ^0_α-categorical if any two of its computable presentations are isomorphic via a Δ^0_α isomorphism.

It is easy to construct computable structures with computable dimension 1 or ω. Indeed, most familiar structures and even all members of many classes of familiar structures have computable dimension 1 or ω. For example, Nurtazin [33] showed that all decidable structures fall into this category. Goncharov [8] later extended this result to 1-decidable structures, and there have been several other familiar algebraic classes of structures for which similar results have been proved.

1.3. Theorem. (Goncharov; Goncharov and Dzgoev; Metakides and Nerode; Nurtazin; LaRoche; Remmel) *All structures in each of the following classes have computable dimension* 1 *or* ω*: algebraically closed fields, real closed fields, Abelian groups, linear orderings, Boolean algebras, and* Δ^0_2*-categorical structures.*

The result for algebraically closed and real closed fields is implied by the results in [33]; the result for algebraically closed fields was also independently proved in [31]. The result for Abelian groups appears in [10], that for linear orderings independently in [13] and [36], and that for Δ^0_2-categorical structures in [11]. The result for Boolean algebras appears in full in [12], though it is implicit in earlier work of Goncharov and, independently, in [28].

In most cases, these results were proved via structure theorems, that is, theorems that connect computability-theoretic properties of structures in the relevant classes to their structural properties. For example, a linear ordering with finitely many pairs of adjacent elements is computably categorical, while one with infinitely many such pairs has infinite computable dimension. The methods in this paper can be seen as the development of a nonstructure theory for computable model theory, in the same sense that, for instance, the study of Borel completeness provides such a theory for descriptive set theory. We will comment on this further below.

In light of results such as those mentioned above, an important question early in the development of computable model theory was whether there exist computable structures with finite computable dimension greater than 1. This question was answered positively by Goncharov [9].

1.4. Theorem. (Goncharov) *For each $n > 0$ there is a computable structure with computable dimension n.*

Further investigation led to examples of computable structures with finite computable dimension greater than 1 in several classes of algebraic structures. In each case, the proof consisted of coding families of computably enumerable (c.e.) sets with a finite number of computable enumerations (up to a suitable notion of computable equivalence of enumerations) in a sufficiently effective way.

1.5. Theorem. (Goncharov; Goncharov, Molokov, and Romanovskii; Kudinov) *For each $n > 0$ there are structures with computable dimension n in each of the following classes: graphs, lattices, partial orderings, 2-step nilpotent groups, and integral domains.*

The results for partial orderings and (implicitly) graphs appear in [9], and the result for lattices is an easy consequence of the results in that paper. The result for 2-step nilpotent groups (which improves a result in [10]) appears in [15], and that for integral domains in [26].

We now turn to the other computability-theoretic properties of structures with which this paper is concerned and give examples of the kinds of results we would like to show remain true when we restrict our attention to certain classes of algebraic structures. For more thorough treatments of these and related results, see [24] or the articles in [6].

One way to understand the differences between noncomputably isomorphic computable presentations of a structure $\mathcal{M}$ is to compare (from a computability-theoretic point of view) the images in these presentations of additional relations on the domain of $\mathcal{M}$, that is, relations that are not the interpretation in $\mathcal{M}$ of relation symbols in the language of $\mathcal{M}$. The study of relations on computable structures began with the work of Ash and Nerode [2], who were concerned with relations that maintain some degree of effectiveness in different computable presentations of a structure.

1.6. Definition. Let U be a relation on the domain of a computable structure $\mathcal{A}$ and let $\mathfrak{C}$ be a class of relations. U is *intrinsically* $\mathfrak{C}$ on $\mathcal{A}$ if the image of U in any computable presentation of $\mathcal{A}$ is in $\mathfrak{C}$.

A different way to approach the study of relations on computable structures, introduced by Harizanov and Millar, is to look at the (Turing) degrees of the images of a relation in different computable presentations of a structure.

1.7. Definition. Let U be a relation on the domain of a computable structure $\mathcal{A}$. The *degree spectrum* of U on $\mathcal{A}$, $\mathrm{DgSp}_{\mathcal{A}}(U)$, is the set of degrees of the images of U in all computable presentations of $\mathcal{A}$.

It is easy to give examples of relations on computable structures whose degree spectra are singletons or infinite. Harizanov [16] was the first to give an example of an intrinsically Δ^0_2 relation with a two-element degree spectrum that includes $\mathbf{0}$. This was improved by Khoussainov and Shore [23] as follows. (The $n = 2$ case is also due to Goncharov and Khoussainov [14].)

1.8. Theorem. (Khoussainov and Shore) *For each $n > 0$ there exists an intrinsically c.e. relation U on the domain of a computable structure $\mathcal{A}$ of computable dimension n such that $\mathrm{DgSp}_{\mathcal{A}}(U)$ consists of n distinct c.e. degrees, including $\mathbf{0}$.*

The following result was also proved in [23].

1.9. Theorem. (Khoussainov and Shore) *For each computable partial ordering $\mathcal{P}$ there exists an intrinsically c.e. relation U on the domain of a computable structure $\mathcal{A}$ such that $\langle \mathrm{DgSp}_{\mathcal{A}}(U), \leqslant_{\mathrm{T}} \rangle \cong \mathcal{P}$. If $\mathcal{P}$ has a least element then we can pick U and $\mathcal{A}$ so that $\mathbf{0} \in \mathrm{DgSp}_{\mathcal{A}}(U)$.*

Later, Hirschfeldt [18] and Khoussainov and Shore [25] independently obtained the following strengthenings of these results.

1.10. Theorem. (Hirschfeldt; Khoussainov and Shore) *Let $\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}$ be c.e. degrees. There exists an intrinsically c.e. relation U on the domain of a computable structure $\mathcal{A}$ of computable dimension n such that $\mathrm{DgSp}_{\mathcal{A}}(U) = \{\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}\}$.*

1.11. Theorem. (Hirschfeldt) *Let $\{A_i\}_{i\in\omega}$ be a uniformly c.e. collection of sets. There exists an intrinsically c.e. relation U on the domain of a computable structure $\mathcal{A}$ such that $\mathrm{DgSp}_{\mathcal{A}}(U) = \{\deg(A_i) \mid i \in \omega\}$.*

A related issue is the question of what happens to the computable dimension of a computably categorical structure when it is expanded by finitely many constants. Millar [32] showed that, with a relatively small additional amount of decidability, computable categoricity is preserved under expansion by finitely many constants.

1.12. Theorem. (Millar) *If $\mathcal{A}$ is computably categorical and 1-decidable then any expansion of $\mathcal{A}$ by finitely many constants remains computably categorical.*

However, preservation of categoricity does not hold in general, as was shown by Cholak, Goncharov, Khoussainov, and Shore [4].

1.13. Theorem. (Cholak, Goncharov, Khoussainov, and Shore) *For each $k > 0$ there exists a computably categorical structure $\mathcal{A}$ and an $a \in |\mathcal{A}|$ such that $\langle \mathcal{A}, a \rangle$ has computable dimension k.*

In fact, as shown by Hirschfeldt, Khoussainov, and Shore [19], not even finite computable dimensionality is always preserved under expansion by a constant.

1.14. Theorem. (Hirschfeldt, Khoussainov, and Shore) *There is a computably categorical structure $\mathcal{A}$ and an $a \in |\mathcal{A}|$ such that $\langle \mathcal{A}, a \rangle$ has computable dimension ω.*

Another important topic in computable model theory is studying the complexity of the isomorphisms between different computable presentations of a structure. We say that a computable structure is *strictly Δ^0_α-categorical* if it is Δ^0_α-categorical but not Δ^0_β-categorical for any $\beta < \alpha$. The following result was proved by Ash in [1].

1.15. Theorem. (Ash) *For each computable limit ordinal δ (including $\delta = 0$) and each $n \in \omega$, there is a strictly $\Delta^0_{\delta+2n}$-categorical well-ordering.*

The work of computable model theory is not restricted to computable structures, of course. When a countable structure is not computably presentable, it is of interest to find out just how far from being computably presentable it is. One way to measure this is to look at the degrees of presentations of the structure.

1.16. Definition. Let $\mathbf{d}$ be a degree. A structure $\mathcal{A}$ with computable domain is $\mathbf{d}$-*computable* if the atomic diagram of $\langle \mathcal{A}, a \rangle_{a \in |\mathcal{A}|}$ is $\mathbf{d}$-computable. The *degree* of $\mathcal{A}$, $\deg(\mathcal{A})$, is the least degree $\mathbf{d}$ (which always exists) such that $\mathcal{A}$ is $\mathbf{d}$-computable.

An isomorphism from a structure $\mathcal{M}$ to a ($\mathbf{d}$-computable) structure with computable domain is called a *($\mathbf{d}$-computable) presentation* of $\mathcal{M}$. We often abuse terminology and refer to the image of a (**d**-computable) presentation as a (**d**-computable) presentation. In particular, when we refer to the degree of a presentation, we always mean the degree of the image, rather than that of the isomorphism. If $\mathcal{M}$ has a **d**-computable presentation then it is **d**-*computably presentable*.

Every countable structure is isomorphic to a structure with computable domain. Therefore, whenever we mention a countable structure we assume that it has computable domain, so that it may be thought of as a presentation of itself.

1.17. Definition. The *degree spectrum* of a countable structure $\mathcal{A}$, $\mathrm{DgSp}(\mathcal{A})$, is the set of degrees of presentations of $\mathcal{A}$.

Any set that is computable in every nonzero degree is in fact computable, but as shown independently by Slaman [37] and Wehner [39], the analogous fact is not true of structures.

1.18. Theorem. (Slaman; Wehner) *There is a structure $\mathcal{A}$ that has presentations of every degree except* $\mathbf{0}$. *(In other words,* $\mathrm{DgSp}(\mathcal{A}) = \mathcal{D} - \{\mathbf{0}\}$*.)*

In the original proofs of Theorems 1.8–1.11, 1.13, 1.14, and 1.18, the structures in question were directed graphs, and the relations mentioned in Theorems 1.8–1.11 were unary. It is natural to ask, in the spirit of what was done for structures of finite computable dimension, for which theories these theorems remain true if we require that $\mathcal{A}$ be a model of the given theory. Our main result gives a partial answer to this question.

The following condition on a theory T is clearly sufficient for the theorems mentioned in the previous paragraph, as well as other similar results, to remain true when we restrict our attention to models of T.

1.19. Definition. A theory T is *complete with respect to effective dimensions, expansion by constants, and degree spectra of structures and relations* if for every countable graph $\mathcal{G}$ there is an $\mathcal{A} \vDash T$ with the following properties.

1. $\mathrm{DgSp}(\mathcal{A}) = \mathrm{DgSp}(\mathcal{G})$.
2. If $\mathcal{G}$ is computably presentable then the following hold.
   - (a) For any degree $\mathbf{d}$, $\mathcal{A}$ has the same $\mathbf{d}$-computable dimension as $\mathcal{G}$.
   - (b) If $x \in |\mathcal{G}|$ then there exists an $a \in D(\mathcal{A})$ such that $\langle \mathcal{A}, a \rangle$ has the same computable dimension as $\langle \mathcal{G}, x \rangle$.

(c) If $S \subseteq |\mathcal{G}|$ then there exists a $U \subseteq D(\mathcal{A})$ such that $\mathrm{DgSp}_{\mathcal{A}}(U) = \mathrm{DgSp}_{\mathcal{G}}(S)$ and if S is intrinsically c.e. then so is U.

The terminology adopted in Definition 1.19 suggests that a theory satisfying this definition should still satisfy it if "every countable graph $\mathcal{G}$" is replaced by "every countable structure $\mathcal{G}$". This is indeed the case, since it is not hard to code a countable structure into a countable graph in a highly effective way. We give such a coding in Appendix A.

We can now state our main result.

1.20. Theorem. *Let T be any of the following theories: symmetric, irreflexive graphs; partial orderings; lattices; rings (with zero-divisors); integral domains of arbitrary characteristic; commutative semigroups; and 2-step nilpotent groups. Then T is complete with respect to effective dimensions, expansion by constants, and degree spectra of structures and relations. In particular, Theorems 1.8–1.11, 1.13–1.15, and 1.18 remain true if we require that $\mathcal{A} \models T$. Furthermore, Theorems 1.8–1.11 remain true if we also require that U be a submodel of $\mathcal{A}$.*

Notice that, by Theorem 1.3, this result cannot be extended from partial orderings to linear orderings, from lattices to Boolean algebras, or from commutative semigroups and 2-step nilpotent groups to Abelian groups. A natural open question is what is the situation for fields. It is not even known whether there exist fields of finite computable dimension greater than 1. Of course, some of the theorems mentioned in our main result do not involve finite computable dimension, and thus could in principle still hold for some of the classes mentioned in Theorem 1.3. For instance, in the case of linear orderings, Hirschfeldt [17] has shown that Theorem 1.9 does not hold, but whether Theorem 1.18 holds is still an open question (see [5] for a discussion of this question).

The rest of this paper is dedicated to the proof of Theorem 1.20. Most of the cases are handled by coding computable graphs with the desired properties into models of the given theories in a way that is effective enough to preserve these properties. This approach is much simpler and more general than attempting to adapt the original proofs of the relevant theorems. Furthermore, our codings are sufficiently effective to make similar results that might be proved for graphs in the future carry over to the classes of structures mentioned in Theorem 1.20 without additional work.

As mentioned above, our results fit into a framework that has become important in several areas of mathematical logic and theoretical computer science, namely the study of dichotomies between structure theory, represented here by results such as Theorem 1.3, and nonstructure theorems, the latter often proved by coding structures that are known to be “as complicated as possible” into the particular structures being studied. In model theory, interpretations of various kinds have long been used to transfer model-theoretic properties from classes of structures where they are easy to determine to other classes in which they are less obvious. One example is Mekler [30]. In descriptive set theory, the study of Borel reducibilities and Borel completeness has received much attention in recent years, as, for example, in Friedman and Stanley [7], Hjorth and Kechris [20], and Camerlo and Gao [3]. A recent survey is Kechris [22]. Another example of the use of codings to show that certain phenomena that can occur in general already occur in what would seem to be a much more restricted setting is the work of Peretyat'kin [34] on finitely axiomatizable theories, which touches on both classical and computable model theory. Probably best-known of all, of course, is the use of highly effective reducibilities in complexity theory to show that certain problems are complete for various complexity classes, which is modeled on the use of reducibilities to prove index set results in computability theory.

In all the examples mentioned above, uncovering the correct notions of reducibility is essential. In Sections 2 and 4, we present sufficient conditions for a coding of a graph into a structure to be effective enough for our purposes. These conditions will be useful in all cases except that of nilpotent groups, in which, instead of coding graphs, we code rings into nilpotent groups. Even in this case, the properties of the coding that must be verified are very similar to those in our general conditions.

Section 3 deals with undirected graphs, partial orderings, and lattices, Section 4 with rings, Section 5 with integral domains and commutative semigroups, and Section 6 with 2-step nilpotent groups.

2 A Sufficient Condition

In this section, we give a sufficient condition for a coding of a graph into a structure to be effective enough for our purposes. This condition is far from being the most general one we could give, but it is sufficient for our needs. It corresponds to an especially effective version of interpretations of theories (in the standard model-theoretic sense) in which equality is interpreted as equality. In Section 4, we will present a generalization of this condition which corresponds to interpretations in which equality is interpreted as an equivalence relation. (See chapter 5 of [21] for more on interpretations of theories.)

We begin with two definitions.

2.1. Definition. A relation U on a structure $\mathcal{M}$ is *invariant* if for every automorphism $f : \mathcal{M} \cong \mathcal{M}$ we have $f(U) = U$.

Because we will be dealing with arbitrary presentations, rather than only computable ones, we will need to consider the relativized version of the notion of intrinsic computability.

2.2. Definition. A relation U on the domain of a structure $\mathcal{A}$ is *relatively intrinsically computable* if for every presentation $f : \mathcal{A} \cong A$, $f(U)$ is computable in $\deg(A)$.

Now let $\mathcal{G}$ be a countable directed graph and let $\mathcal{A}$ be a countable structure. Assume that there exist relatively intrinsically computable, invariant relations $D(x)$ and $R(x, y)$ on the domain of $\mathcal{A}$ and a map $G \mapsto A_G$ from the set of presentations of $\mathcal{G}$ to the set of presentations of $\mathcal{A}$ with the following properties.

(P0) For every presentation G of $\mathcal{G}$, $\deg(A_G) = \deg(G)$.

(P1) For every presentation G of $\mathcal{G}$ there is a $\deg(G)$-computable map $g_G : D(A_G) \xrightarrow[\text{onto}]{1\text{–}1} |G|$ such that for every $x, y \in D(A_G)$, $R^{A_G}(x, y) \Leftrightarrow E^G(g_G(x), g_G(y))$.

(P2) If $f : D(\mathcal{A}) \xrightarrow[\text{onto}]{1\text{–}1} D(\mathcal{A})$ is such that $R(x, y) \Leftrightarrow R(f(x), f(y))$ then f can be extended to an automorphism of $\mathcal{A}$.

(P3) For every presentation G of $\mathcal{G}$ there exists a $\deg(G)$-computable set of existential formulas $\varphi_0(\vec{a}, \vec{b}_0, x), \varphi_1(\vec{a}, \vec{b}_1, x), \ldots$ such that $\vec{a}$ is a tuple of elements of $|A_G|$, for each $i \in \omega$, $\vec{b}_i$ is a tuple of elements of $D(A_G)$, each $x \in |A_G|$ satisfies some φ_i, and no two elements of $|A_G|$ satisfy the same φ_i. (Such a set of formulas is known as a $\deg(G)$-*computable defining family* for $\langle A_G, a\rangle_{a \in D(A_G)}$.)

We wish to show that the following hold.

1. $\mathrm{DgSp}(\mathcal{A}) = \mathrm{DgSp}(\mathcal{G})$.

2. If $\mathcal{G}$ is computably presentable then

   (a) For any degree $\mathbf{d}$, $\mathcal{A}$ has the same $\mathbf{d}$-computable dimension as $\mathcal{G}$.

   (b) if $x \in |\mathcal{G}|$ then there exists an $a \in D(\mathcal{A})$ such that $\langle \mathcal{A}, a\rangle$ has the same computable dimension as $\langle \mathcal{G}, x\rangle$.

(c) if $S \subseteq |\mathcal{G}|$ then there exists a $U \subseteq D(\mathcal{A})$ such that $\mathrm{DgSp}_{\mathcal{A}}(U) = \mathrm{DgSp}_{\mathcal{G}}(S)$ and if S is intrinsically c.e. then so is U.

We begin with a series of lemmas.

2.3. Lemma. *Let A and G be presentations of $\mathcal{A}$ and $\mathcal{G}$, respectively, of the same degree $\mathbf{d}$ and let $f : A \cong A_G$. Then f is $(\mathbf{d} \vee \deg(f \restriction D(A)))$-computable.*

Proof. It is enough to show that f^{-1} is $(\mathbf{d} \vee \deg(f \restriction D(A)))$-computable. Given $x \in |A_G|$, find an $i \in \omega$ such that $A_G \vDash \varphi_i(\vec{a}, \vec{b}_i, x)$, where $\varphi_i(\vec{a}, \vec{b}_i, x)$ is as in (P3). By definition, x is the only element of $|A_G|$ that satisfies φ_i. Thus there exists a unique $y \in |A|$ such that $A \vDash \varphi_i(f^{-1}(\vec{a}), f^{-1}(\vec{b}_i), y)$, and $f^{-1}(x) = y$. Since both A_G and A are $\mathbf{d}$-computable, f^{-1} is $(\mathbf{d} \vee \deg(f \restriction D(A)))$-computable. □

2.4. Lemma. *Let A and G be presentations of $\mathcal{A}$ and $\mathcal{G}$, respectively, of the same degree $\mathbf{d}$. Suppose that there exists a map $f : D(A) \xrightarrow[\mathit{onto}]{\mathit{1–1}} D(A_G)$ such that, for each $x, y \in D(A)$, $R^A(x, y) \Leftrightarrow R^{A_G}(f(x), f(y))$. Then f can be extended to a $(\mathbf{d} \vee \deg(f))$-computable isomorphism $\hat{f} : A \cong A_G$.*

Proof. Since A and A_G are both presentations of $\mathcal{A}$, there exists an isomorphism $h : A \cong A_G$. Let $d \equiv h \restriction D(A)$. Then $c \equiv f \circ d^{-1}$ is a one-to-one map from $D(A_G)$ onto itself such that, for each $x, y \in D(A_G)$, $R^{A_G}(x, y) \Leftrightarrow R^{A_G}(c(x), c(y))$. So, by (P2), c can be extended to $\hat{c} : A_G \cong A_G$. Now let $\hat{f} \equiv \hat{c} \circ h$. Then $\hat{f} : A \cong A_G$ and $\hat{f} \restriction D(A) \equiv f \circ d^{-1} \circ d \equiv f$. Lemma 2.3 implies that $\hat{f}$ is $(\mathbf{d} \vee \deg(f))$-computable. □

2.5. Lemma. *If G and G' are computable presentations of $\mathcal{G}$ and $h : G \cong G'$ is an isomorphism then there exists a $\deg(h)$-computable isomorphism $\hat{f} : A_G \cong A_{G'}$ such that $\hat{f} \restriction D(A_G) \equiv g_{G'}^{-1} \circ h \circ g_G$.*

Proof. Let $f : D(A_G) \xrightarrow[\text{onto}]{\text{1–1}} D(A_{G'})$ be defined by $f \equiv g_{G'}^{-1} \circ h \circ g_G$. Clearly, f is $\deg(h)$-computable. Furthermore, for each $x, y \in D(A_G)$, $R^{A_G}(x, y) \Leftrightarrow E^G(g_G(x), g_G(y)) \Leftrightarrow E^{G'}(h \circ g_G(x), h \circ g_G(y)) \Leftrightarrow R^{A_{G'}}(f(x), f(y))$. So, by Lemma 2.4, there exists a $\deg(h)$-computable isomorphism $\hat{f} : A_G \cong A_{G'}$ extending f. □

For any presentation A of $\mathcal{A}$, let $\widetilde{G}_A$ be the graph whose domain is $D(A)$, with an edge between x and y if and only if $R^A(x, y)$. Clearly, there exist a $\deg(A)$-computable map h_A and a $\deg(A)$-computable graph G_A such that $h_A : \widetilde{G}_A \to G_A$ is a $\deg(A)$-computable presentation of $\widetilde{G}_A$. If A is computable then we take $G_A = \widetilde{G}_A$ and let h_A be the identity. In any case, it is easy to check that G_A is a $\deg(A)$-computable presentation of $\mathcal{G}$.

2.6. Lemma. *If A and A' are computable presentations of $\mathcal{A}$ and $f : A \cong A'$ is an isomorphism then $f \upharpoonright D(A)$ is a* $\deg(f)$*-computable isomorphism from G_A to $G_{A'}$.*

Proof. Since $E^{G_A}(x,y) \Leftrightarrow R^A(x,y) \Leftrightarrow R^{A'}(f(x),f(y)) \Leftrightarrow E^{G_{A'}}(f(x),f(y))$, $|G_A| = D(A)$, and $|G_{A'}| = D(A')$, $f \upharpoonright D(A)$ is an isomorphism from G_A to $G_{A'}$. $\square$

2.7. Lemma. *If G is a computable presentation of $\mathcal{G}$ then g_G is a computable isomorphism from G_{A_G} to G.*

Proof. If $x,y \in |G_{A_G}|$ then $E^{G_{A_G}}(x,y) \Leftrightarrow R^{A_G}(x,y) \Leftrightarrow E^G(g_G(x),g_G(y))$. Thus g_G is a computable isomorphism from G_{A_G} to G. $\square$

2.8. Lemma. *If A is a presentation of $\mathcal{A}$ then there exists a* $\deg(A)$*-computable isomorphism $f : A \cong A_{G_A}$ such that $f \upharpoonright D(A) \equiv g_{G_A}^{-1} \circ h_A$.*

Proof. The map $g_{G_A}^{-1} \circ h_A$ is $\deg(A)$-computable, and for each $x,y \in D(A)$, $R^A(x,y) \Leftrightarrow E^{G_A}(x,y) \Leftrightarrow R^{A_{G_A}}(g_{G_A}^{-1} \circ h_A(x), g_{G_A}^{-1} \circ h_A(y))$. So, by Lemma 2.4, $g_{G_A}^{-1} \circ h_A$ can be extended to a $\deg(A)$-computable isomorphism from A to A_{G_A}. $\square$

We are now ready to show that 1 and 2.a–c above hold.

2.9. Proposition. $\mathrm{DgSp}(\mathcal{A}) = \mathrm{DgSp}(\mathcal{G})$.

Proof. For any presentation G of $\mathcal{G}$, $\deg(A_G) = \deg(G)$, so $\mathrm{DgSp}(\mathcal{A}) \supseteq \mathrm{DgSp}(\mathcal{G})$. On the other hand, by Lemma 2.8, for any presentation A of $\mathcal{A}$, $\deg(G_A) = \deg(A_{G_A}) = \deg(A)$, so $\mathrm{DgSp}(\mathcal{A}) \subseteq \mathrm{DgSp}(\mathcal{G})$. $\square$

Now assume that $\mathcal{G}$ is computably presentable.

2.10. Proposition. *For any degree $\mathbf{d}$, $\mathcal{A}$ has the same $\mathbf{d}$-computable dimension as $\mathcal{G}$.*

Proof. Let G and G' be computable presentations of $\mathcal{G}$ that are not $\mathbf{d}$-computably isomorphic. By Lemma 2.7, G_{A_G} and $G_{A_{G'}}$ are not $\mathbf{d}$-computably isomorphic. Thus, by Lemma 2.6, A_G and $A_{G'}$ are not $\mathbf{d}$-computably isomorphic. So the $\mathbf{d}$-computable dimension of $\mathcal{A}$ is at least the same as that of $\mathcal{G}$.

Now let A and A' be $\mathbf{d}$-computable presentations of $\mathcal{A}$ that are not computably isomorphic. By Lemma 2.8, A_{G_A} and $A_{G_{A'}}$ are not $\mathbf{d}$-computably isomorphic. Thus, by Lemma 2.5, G_A and $G_{A'}$ are not $\mathbf{d}$-computably isomorphic. So the $\mathbf{d}$-computable dimension of $\mathcal{G}$ is at least the same as that of $\mathcal{A}$. $\square$

2.11. Proposition. *Let $x \in |\mathcal{G}|$. There exists an $a \in D(\mathcal{A})$ such that $\langle \mathcal{A}, a\rangle$ has the same computable dimension as $\langle \mathcal{G}, x\rangle$.*

Proof. Let $f : \mathcal{G} \cong G$ be a computable presentation of $\mathcal{G}$, let $h : \mathcal{A} \cong A_G$ be an isomorphism, and let $a = h^{-1} \circ g_G^{-1} \circ f(x)$. By Lemma 2.5, for every computable presentation $f' : \mathcal{G} \cong G'$ of $\mathcal{G}$ there exists an isomorphism $k : \mathcal{A} \cong A_{G'}$ such that $a = k^{-1} \circ g_{G'}^{-1} \circ f'(x)$. The rest of the proof is similar to the proof of Proposition 2.10.

Let $\langle G, x^G\rangle$ and $\langle G', x^{G'}\rangle$ be computable presentations of $\langle \mathcal{G}, x\rangle$ that are not computably isomorphic. By Lemma 2.7, $\langle G_{A_G}, g_G^{-1}(x^G)\rangle$ and $\langle G_{A_{G'}}, g_{G'}^{-1}(x^{G'})\rangle$ are not computably isomorphic. Thus, by Lemma 2.6, $\langle A_G, g_G^{-1}(x^G)\rangle$ and $\langle A_{G'}, g_{G'}^{-1}(x^{G'})\rangle$ are not computably isomorphic. So the computable dimension of $\langle \mathcal{A}, a\rangle$ is at least the same as that of $\langle \mathcal{G}, x\rangle$.

Now let $\langle A, a^A\rangle$ and $\langle A', a^{A'}\rangle$ be computable presentations of $\langle \mathcal{A}, a\rangle$ that are not computably isomorphic. By Lemma 2.8, $\langle A_{G_A}, g_{G_A}^{-1}(a^A)\rangle$ and $\langle A_{G_{A'}}, g_{G_{A'}}^{-1}(a^{A'})\rangle$ are not computably isomorphic. Thus, by Lemma 2.5, $\langle G_A, a^A\rangle$ and $\langle G_{A'}, a^{A'}\rangle$ are not computably isomorphic. So the computable dimension of $\langle \mathcal{G}, x\rangle$ is at least the same as that of $\langle \mathcal{A}, a\rangle$. ☐

2.12. Proposition. *Let $S \subseteq |\mathcal{G}|$. There exists a $U \subseteq D(\mathcal{A})$ such that* $\mathrm{DgSp}_{\mathcal{A}}(U) = \mathrm{DgSp}_{\mathcal{G}}(S)$ *and if S is intrinsically c.e. then so is U.*

Proof. Let $f : \mathcal{G} \cong G$ be a computable presentation of $\mathcal{G}$, let $h : \mathcal{A} \cong A_G$ be an isomorphism, and let $U = h^{-1} \circ g_G^{-1} \circ f(S)$. Clearly, if S is intrinsically c.e. then so is U.

By Lemma 2.5, for every computable presentation $f' : \mathcal{G} \cong G'$ of $\mathcal{G}$ there exists an isomorphism $k : \mathcal{A} \cong A_{G'}$ such that $U = k^{-1} \circ g_{G'}^{-1} \circ f'(S)$, which implies that $\mathrm{DgSp}_{\mathcal{G}}(S) \subseteq \mathrm{DgSp}_{\mathcal{A}}(U)$.

On the other hand, we claim that for every computable presentation $k : \mathcal{A} \cong A$ of $\mathcal{A}$ there exists an isomorphism $m : \mathcal{G} \cong G_A$ such that $S = m^{-1} \circ k(U)$. Indeed, let f, G, and h be as above. Then $S = f^{-1} \circ g_G \circ h(U)$, so if we let $m = k \circ h^{-1} \circ g_G^{-1} \circ f$ then $m^{-1} \circ k(U) = f^{-1} \circ g_G \circ h \circ k^{-1} \circ k(U) = f^{-1} \circ g_G \circ h(U) = S$. Furthermore, it is not hard to check that $m : \mathcal{G} \cong G_A$. This establishes our claim, which implies that $\mathrm{DgSp}_{\mathcal{A}}(U) \subseteq \mathrm{DgSp}_{\mathcal{G}}(S)$. ☐

We conclude from the previous four propositions that, given a theory T, if for every countably infinite directed graph $\mathcal{G}$ we can find $\mathcal{A} \vDash T$, D, and R satisfying properties (P0)–(P3), then T is complete with respect to effective dimensions, expansion by constants, and degree spectra of structures and relations. But it is not actually necessary that we be able to code every countably infinite graph into a model of T, as long as we can code enough such graphs.

2.13. Proposition. *Let T be a theory and let P be a theory of directed graphs that is complete with respect to effective dimensions, expansion by constants, and degree spectra of structures and relations. If for every countably infinite $\mathcal{G} \models P$ we can find $\mathcal{A} \models T$, D, and R satisfying properties (P0)–(P3) then T is complete with respect to effective dimensions, expansion by constants, and degree spectra of structures and relations.*

3 Simple Codings

It is usually easier to code symmetric, irreflexive graphs into structures rather than arbitrary directed graphs. Thus we begin this section by showing that the theory of symmetric, irreflexive graphs is complete with respect to effective dimensions, expansion by constants, and degree spectra of structures and relations. We then show how to apply this result to partial orderings and lattices.

3.1 Undirected Graphs

Let G be a countably infinite directed graph with edge relation E.

Let G' be a presentation of G. The $\deg(G')$-computably presentable symmetric, irreflexive graph $H_{G'} = \langle |H_{G'}|, F\rangle$ is defined as follows.

1. $|H_{G'}| = \{a, a', b\} \cup \{c_i, d_i, e_i : i \in |G'|\}$.
2. $F(x, y)$ holds only in the following cases.
   (a) $F(a, a')$ and $F(a', a)$.
   (b) For all $i \in |G'|$,
      i. $F(a, c_i)$ and $F(c_i, a)$,
      ii. $F(b, e_i)$ and $F(e_i, b)$,
      iii. $F(c_i, d_i)$ and $F(d_i, c_i)$.
      iv. $F(d_i, e_i)$ and $F(e_i, d_i)$.
   (c) If $E^{G'}(i, j)$ then $F(c_i, e_j)$ and $F(e_j, c_i)$.

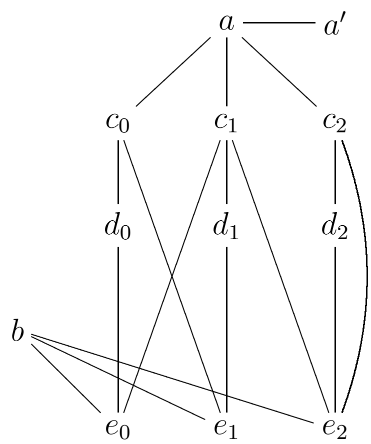

Figure 1: H_G

As an example, Figure 1 shows the graph H_G in the case in which $E(0,1)$, $E(1,0)$, $E(1,2)$, $E(2,2)$, $\neg E(0,0)$, $\neg E(0,2)$, $\neg E(1,1)$, $\neg E(2,0)$, and $\neg E(2,1)$.

Fix a $\deg(G')$-computable presentation of $H_{G'}$ for which the map $g_{G'} : c_i \mapsto i$ is $\deg(G')$-computable and identify $H_{G'}$ with this presentation.

It is easy to see that, for any presentation G' of G, $H_G \cong H_{G'}$. Now let a, a', and b be as in the definition of H_G and define

$$D(x) = \{x \in |H_G| : x \neq a' \wedge F(a,x)\}$$

and

$$R(x,y) = \{(x,y) : D(x) \wedge D(y) \wedge \exists d, e(F(b,e) \wedge F(y,d) \wedge F(d,e) \wedge F(x,e))\}.$$

Clearly, D is relatively intrinsically computable, and so is R, since, for $x, y \in D(H_G)$,

$$\exists d, e(F(b,e) \wedge F(y,d) \wedge F(d,e) \wedge F(x,e)) \Leftrightarrow \\ \neg\exists d, e(F(b,e) \wedge F(y,d) \wedge F(d,e) \wedge \neg F(x,e)).$$

Furthermore, for any presentation G' of G, $D(H_{G'}) = \mathrm{dom}(g_{G'})$ and

$$R^{H_{G'}}(x,y) \Leftrightarrow E^{G'}(g_{G'}(x), g_{G'}(y)).$$

To see that D and R are invariant, it is enough to notice that a is the only element of H_G that satisfies the formula

$$\exists^\infty y(F(x,y)) \wedge \exists z(F(x,z) \wedge \forall w(F(w,z) \to w = x)),$$

a' is the only element of H_G that satisfies

$$F(x,a) \wedge \forall y(F(x,y) \rightarrow y = a)$$

and $x = b$ is the only element of H_G that satisfies

$$\exists^{\infty} y(F(x,y)) \wedge \neg F(a,x) \wedge \neg \exists z(F(a,z) \wedge F(x,z)).$$

If $f : D(H_G) \xrightarrow[\text{onto}]{1\text{–}1} D(H_G)$ is such that $R(x,y) \Leftrightarrow R(f(x), f(y))$ then we can extend f as follows. Let a, a', b, d_i, and e_i be as in the definition of H_G. Let $f(a) = a$, $f(a') = a'$, $f(b) = b$, $f(d_i) = d_{g_G \circ f(i)}$, and $f(e_i) = e_{g_G \circ f(i)}$. It can be easily verified that this extended map is an automorphism of H_G.

Finally, given a presentation G' of G, let a, a', and b be as in the definition of $H_{G'}$ and consider the $\deg(G')$-computable set of formulas

$$\{x = a, x = a', x = b\} \cup \{x = c : c \in D(H_{G'})\} \cup \{x \neq a \wedge F(c,x) \wedge \\ \neg F(b,x) : c \in D(H_{G'})\} \cup \{F(b,x) \wedge \exists d(F(c,d) \wedge F(d,x)) : c \in D(H_{G'})\}.$$

Clearly, every $x \in |H_{G'}|$ satisfies some formula in this set, with no two elements satisfying the same formula.

Theorem 1.20 in the case of symmetric, irreflexive graphs now follows from Proposition 2.13. In particular, we can take the theory P in Proposition 2.13 to be the theory of symmetric, irreflexive graphs.

3.2 Partial Orderings

Let G be a symmetric, irreflexive, countably infinite computable graph with edge relation E.

Let G' be a presentation of G. The $\deg(G')$-computably presentable partial ordering $P_{G'} = \langle |P_{G'}|, \prec \rangle$ is defined as follows.

1. $|P_{G'}| = \{a, b\} \cup \{c_i : i \in |G'|\} \cup \{d_{ij} : i < j \in |G'|\}$.

2. The relation $\prec$ is the smallest transitive relation on $|P_{G'}|$ satisfying the following conditions.

   (a) $a \prec c_i \prec b$ for all $i \in |G'|$.

   (b) If $i < j$ and $E^{G'}(i,j)$ then $d_{ij} \prec c_i, c_j$.

(c) If $i < j \in |G'|$ and $\neg E^{G'}(i,j)$ then $c_i, c_j \prec d_{ij}$.

As an example, Figure 2 shows a portion of the partial ordering P_G in the case in which $E(0,1)$, $E(1,2)$, and $\neg E(0,2)$. An arrow from x to y represents the fact that $x \prec y$.

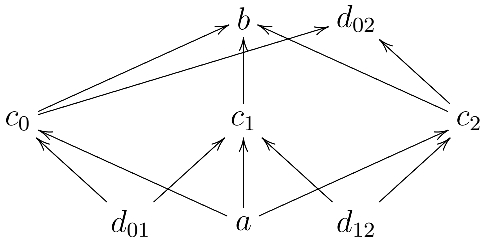

Figure 2: P_G

Fix a $\deg(G')$-computable presentation of $P_{G'}$ for which the map $g_{G'} : c_i \mapsto i$ is $\deg(G')$-computable and identify $P_{G'}$ with this presentation.

It is easy to see that, for any presentation G' of G, $P_G \cong P_{G'}$. Now let a and b be as in the definition of P_G and define

$$D(x) = \{x \in |P_G| : a \prec x \prec b\}$$

and

$$R(x,y) = \{(x,y) : x \neq y \wedge D(x) \wedge D(y) \wedge \exists z \neq a(z \prec x, y)\}.$$

Clearly, D is relatively intrinsically computable and invariant, and so is R, since

$$\exists z \neq a(z \prec x, y) \Leftrightarrow \neg\exists z \neq b(x, y \prec z).$$

(Invariance follows from the fact that a is the only element of P_G with infinitely many elements above it and b is the only element of P_G with infinitely many elements below it.) Furthermore, for any presentation G' of G, $D(P_{G'}) = \mathrm{dom}(g_{G'})$ and

$$R^{P_{G'}}(x,y) \Leftrightarrow E^{G'}(g_{G'}(x), g_{G'}(y)).$$

If $f : D(P_G) \xrightarrow[\text{onto}]{1\text{–}1} D(P_G)$ is such that $R(x,y) \Leftrightarrow R(f(x), f(y))$ then we can extend f as follows. Let a, b, and d_{ij} be as in the definition of P_G. Let $f(a) = a$, $f(b) = b$, and $f(d_{ij}) = d_{g_G \circ f(i) g_G \circ f(j)}$ if $g_G \circ f(i) < g_G \circ f(j)$, $f(d_{ij}) = d_{g_G \circ f(j) g_G \circ f(i)}$ otherwise. It can be easily verified that this extended map is an automorphism of P_G.

Finally, given a presentation G' of G, let a and b be as in the definition of $P_{G'}$ and consider the $\deg(G')$-computable set of formulas

$$\{x = a, x = b\} \cup \{x = c : c \in D(P_{G'})\} \cup \\ \{(x \prec c, c' \vee c, c' \prec x) \wedge x \neq a \wedge x \neq b : c \neq c' \in D(P_{G'})\}.$$

Clearly, every $x \in |P_{G'}|$ satisfies some formula in this set, with no two elements satisfying the same formula.

Theorem 1.20 in the case of partial orderings now follows from Proposition 2.13, with the theory P mentioned in that proposition being in this case the theory of symmetric, irreflexive graphs.

3.3 Lattices

Let G be a symmetric, irreflexive, countably infinite graph with edge relation E. We can assume that G has at least one node that is not connected to any other node, since the theory of graphs with this property is clearly complete with respect to effective dimensions, expansion by constants, and degree spectra of structures and relations.

Let G' be a presentation of G. The $\deg(G')$-computably presentable lattice $L_{G'} = \langle |L_{G'}|, \curlywedge, \curlyvee \rangle$ is the unique lattice satisfying the following conditions.

1. $|L_{G'}| = \{a, b, k\} \cup \{c_i, m_i : i \in |G'|\} \cup \{d_{ij} : i < j \wedge E^{G'}(i, j)\}$.
2. For all $x \neq y \in |L_{G'}|$, $x \curlyvee y = a$ and $x \curlywedge y = b$ except as required to satisfy the following conditions.
   (a) If $i < j$ and $E^{G'}(i, j)$ then $c_i \curlyvee c_j = d_{ij}$.
   (b) If $i \in |G'|$ then $k \curlyvee c_i = m_i$.

As an example, Figure 3 shows a portion of the lattice L_G in the case in which $E(0, 1)$, $E(1, 2)$, and $\neg E(0, 2)$. To simplify the picture, we omit the top element a and the bottom element b of the lattice.

Fix a $\deg(G')$-computable presentation of $L_{G'}$ for which the map $g_{G'} : c_i \mapsto i$ is $\deg(G')$-computable and identify $L_{G'}$ with this presentation.

Remark. It is interesting to note that L_G has height 4. Clearly, any lattice of height less than 4 is relatively computably categorical.

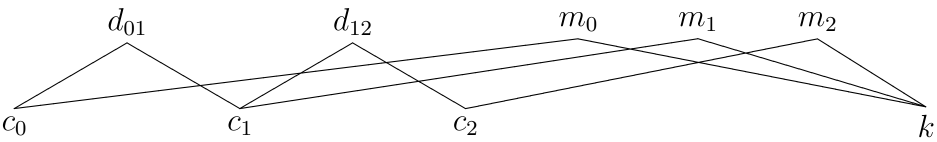

Figure 3: L_G

It is easy to see that, for any presentation G' of G, $L_G \cong L_{G'}$. Now let a, b, and k be as in the definition of L_G and define

$$D(x) = \{x \in |L_G| : k \curlyvee x \neq a \wedge k \curlyvee x \neq x\}$$

and

$$R(x, y) = \{(x, y) : x \neq y \wedge D(x) \wedge D(y) \wedge x \curlyvee y \neq a\}.$$

Clearly, D and R are relatively intrinsically computable. Furthermore, for any presentation G' of G, $D(L_{G'}) = \mathrm{dom}(g_{G'})$ and

$$R^{L_{G'}}(x, y) \Leftrightarrow E^{G'}(g_{G'}(x), g_{G'}(y)).$$

To see that D and R are invariant, it is enough to notice that, because of our assumption that G has an isolated node, k is the only element of L_G whose join with any level 2 element of L_G is not a.

If $f : D(L_G) \xrightarrow[\text{onto}]{1\text{–}1} D(L_G)$ is such that $R(x, y) \Leftrightarrow R(f(x), f(y))$ then we can extend f as follows. Let a, b, k, m_i, and d_{ij} be as in the definition of L_G. Let $f(a) = a$, $f(b) = b$, $f(k) = k$, $f(m_i) = m_{g_G \circ f(i)}$, and $f(d_{ij}) = d_{g_G \circ f(i) g_G \circ f(j)}$ if $g_G \circ f(i) < g_G \circ f(j)$, $f(d_{ij}) = d_{g_G \circ f(j) g_G \circ f(i)}$ otherwise. It can be easily verified that this extended map is an automorphism of L_G.

Finally, given a presentation G' of G, let a, b, and k be as in the definition of $L_{G'}$ and consider the $\deg(G')$-computable set of formulas

$$\{x = a, x = b, x = k\} \cup \{x = c : c \in D(L_{G'})\} \cup \\ \{c \curlyvee c' = x : (c, c') \in R^{L_{G'}}\} \cup \{k \curlyvee c = x : c \in D(L_{G'})\}.$$

Clearly, every $x \in |L_{G'}|$ satisfies some formula in this set, with no two elements satisfying the same formula.

Since, for any computable presentation L of L_G, the sublattice of L generated by any subset S of $D(L)$ has the same degree as S and is c.e. if S is c.e., Theorem 1.20 in the case of lattices now follows from Proposition 2.13, with the theory P mentioned in that proposition being in this case the theory of symmetric, irreflexive graphs with at least one isolated node.

4 A Weaker Sufficient Condition and its Application to Rings

In this section we give a strengthening of Proposition 2.13 which will be used in the next section, as well as an example of its application to rings. If Q is an equivalence relation on a set D then by a *set of Q-representatives* we mean a set of elements of D containing exactly one member of each Q-equivalence class.

4.1. Proposition. *Let T be a theory and let P be a theory of directed graphs that is complete with respect to effective dimensions, expansion by constants, and degree spectra of structures and relations. Suppose that for every countably infinite $\mathcal{G} \vDash P$ we can find an $\mathcal{A} \vDash T$; relatively intrinsically computable, invariant relations $D(x)$, $Q(x, y)$, and $R(x, y)$ on $|\mathcal{A}|$; and a map $G \mapsto A_G$ from the set of presentations of $\mathcal{G}$ to the set of presentations of $\mathcal{A}$ with the following properties.*

- *(P0) For every presentation G of $\mathcal{G}$, $\deg(A_G) = \deg(G)$.*
- *(P1′) For every presentation G of $\mathcal{G}$ there is a $\deg(G)$-computable map $g_G : D(A_G) \xrightarrow{onto} |G|$ such that, for $x, y \in D(A_G)$, $R^{A_G}(x, y) \Leftrightarrow E^G(g_G(x), g_G(y))$ and $Q^{A_G}(x, y) \Leftrightarrow g_G(x) = g_G(y)$. (Note that this implies that Q is an equivalence relation and that if $Q(x, x')$ and $Q(y, y')$ then $R(x, y) \Leftrightarrow R(x', y')$.)*
- *(P2′) For every pair S, S' of sets of Q-representatives, if $f : S \xrightarrow[onto]{1\text{–}1} S'$ is such that for every $x, y \in S$, $R(x, y) \Leftrightarrow R(f(x), f(y))$, then f can be extended to an automorphism of $\mathcal{A}$.*
- *(P3′) If G is a presentation of $\mathcal{G}$ and S is a $\deg(G)$-computable set of Q^{A_G}-representatives then there exists a $\deg(G)$-computable defining family for $\langle A_G, a\rangle_{a\in S}$.*

Then T is complete with respect to effective dimensions, expansion by constants, and degree spectra of structures and relations. Furthermore, in each of Theorems 1.8–1.11 with the extra requirement that $\mathcal{A} \vDash T$, the relation U can be chosen so that $U \subseteq D$ and $Q(x, y) \Rightarrow (U(x) \Leftrightarrow U(y))$.

Proof. It is enough to show that if the countably infinite graph $\mathcal{G} \vDash P$ and the model $\mathcal{A} \vDash T$ satisfy (P0) and (P1′)–(P3′) then Propositions 2.9–2.12 hold of $\mathcal{G}$ and $\mathcal{A}$ and, in Proposition 2.12, U can be chosen so that $Q(x, y) \Rightarrow (U(x) \Leftrightarrow U(y))$. The argument is similar to what was done in Section 2, so we present only the necessary changes. We begin with two remarks.

First, if G is a presentation of $\mathcal{G}$ and S is a set of Q^{A_G}-representatives then $g_G \restriction S$ is one-to-one. Second, if S and S' are sets of Q-representatives and $f : S \xrightarrow[\text{onto}]{1\text{–}1} S'$ is such that, for every $x \in S$, $Q(x, f(x))$, then (P1$'$) implies that, for every $x, y \in S$, $R(x, y) \Leftrightarrow R(f(x), f(y))$, so that, by (P2$'$), f can be extended to an automorphism of $\mathcal{A}$.

We now need new versions of Lemmas 2.3 and 2.4.

4.2. Lemma. *Let A and G be presentations of $\mathcal{A}$ and $\mathcal{G}$, respectively, of the same degree $\mathbf{d}$, let S be a $\mathbf{d}$-computable set of Q^A-representatives, and let $f : A \cong A_G$. Then f is $(\mathbf{d} \vee \deg(f \restriction S))$-computable.*

4.3. Lemma. *Let A and G be presentations of $\mathcal{A}$ and $\mathcal{G}$, respectively, of the same degree $\mathbf{d}$. Let S be a $\mathbf{d}$-computable set of Q^A-representatives and let S' be a $\mathbf{d}$-computable set of Q^{A_G}-representatives. Suppose that there exists a map $f : S \xrightarrow[\text{onto}]{1\text{–}1} S'$ such that, for each $x, y \in S$, $R^A(x, y) \Leftrightarrow R^{A_G}(f(x), f(y))$. Then f can be extended to a $(\mathbf{d} \vee \deg(f))$-computable isomorphism $\hat{f} : A \cong A_G$.*

The proof of Lemma 4.2 is the same as that of Lemma 2.3, using (P3$'$) in place of (P3). The proof of Lemma 4.3 is essentially the same as that of Lemma 2.4, with $D(A)$ replaced by S and $D(A_G)$ by S', and using (P2$'$) in place of (P2) and Lemma 4.2 in place of Lemma 2.3; the only other change is that the isomorphism $h : A \cong A_G$ must be such that $h(S) = S'$. The existence of such an isomorphism is an immediate consequence of our second remark above.

We now need a few definitions. Let A be a presentation of $\mathcal{A}$. Let $\widehat{D}(A) = \{x \in D(A) : y < x \Rightarrow \neg Q^A(x, y)\}$, where $<$ is the natural ordering on ω. Notice that $\widehat{D}(A)$ is a $\deg(A)$-computable set of Q^A-representatives. Let $\widetilde{G}_A$ be the graph whose domain is $\widehat{D}(A)$, with an edge between x and y if and only if $R^A(x, y)$. Clearly, there exist a $\deg(A)$-computable map h_A and a $\deg(A)$-computable graph G_A such that $h_A : \widetilde{G}_A \to G_A$ is a $\deg(A)$-computable presentation of $\widetilde{G}_A$. If A is computable then we take $G_A = \widetilde{G}_A$ and let h_A be the identity. For any presentation G of $\mathcal{G}$, let $d_G = g_G \restriction \widehat{D}(A_G)$. Note that, by our first remark above, d_G is one-to-one and hence invertible.

The following are the new versions of Lemmas 2.5–2.8.

4.4. Lemma. *If G and G' are computable presentations of $\mathcal{G}$ and $h : G \cong G'$ is an isomorphism then there exists a $\deg(h)$-computable isomorphism $\hat{f} : A_G \cong A_{G'}$ such* *that* $\hat{f} \upharpoonright \widehat{D}(A_G) \equiv d_{G'}^{-1} \circ h \circ d_G$.

4.5. Lemma. *If A and A' are computable presentations of $\mathcal{A}$ and $f : A \cong A'$ is an isomorphism then there exists a map $h : f(\widehat{D}(A)) \xrightarrow[onto]{1\text{–}1} \widehat{D}(A')$ such that $h \circ (f \upharpoonright \widehat{D}(A))$ is a* $\deg(f)$*-computable isomorphism from G_A to $G_{A'}$.*

4.6. Lemma. *If G is a computable presentation of $\mathcal{G}$ then d_G is a computable isomorphism from G_{A_G} to G.*

4.7. Lemma. *If A is a presentation of $\mathcal{A}$ then there exists a* $\deg(A)$*-computable isomorphism $f : A \cong A_{G_A}$ such that $f \upharpoonright \widehat{D}(A) \equiv d_{G_A}^{-1} \circ h_A$.*

In most cases, the proofs of these lemmas are essentially the same as those of the corresponding lemmas in Section 2, with a few obvious modifications. The only exception is Lemma 4.5, which can be proved as follows. For $x \in f(\widehat{D}(A))$, let $h(x)$ be the unique $y \in \widehat{D}(A')$ such that $Q^{A'}(x, y)$. Then $E^{G_A}(x, y) \Leftrightarrow R^A(x, y) \Leftrightarrow R^{A'}(f(x), f(y)) \Leftrightarrow R^{A'}(h \circ f(x), h \circ f(y)) \Leftrightarrow E^{G_{A'}}(h \circ f(x), h \circ f(y))$. Thus $h \circ (f \upharpoonright \widehat{D}(A))$ is a $\deg(f)$-computable isomorphism from G_A to $G_{A'}$.

We can now prove Propositions 2.9–2.12 in much the same way as before, using Lemmas 4.4–4.7 in place of Lemmas 2.5–2.8. The other necessary changes to the proofs of these propositions are described below.

No other changes to the proofs of Propositions 2.9 and 2.10 are needed.

In establishing Proposition 2.11, the proof that the computable dimension of $\langle \mathcal{A}, a \rangle$ is at least the same as that of $\langle \mathcal{G}, x \rangle$ is as before, with g_G and $g_{G'}$ replaced by d_G and $d_{G'}$, respectively.

For the other direction, if $\langle B, a^B \rangle$ and $\langle B', a^{B'} \rangle$ are computable presentations of $\langle \mathcal{A}, a \rangle$ that are not computably isomorphic then, by Lemma 4.3, there exist computable presentations $\langle A, a^A \rangle$ and $\langle A', a^{A'} \rangle$ of $\langle \mathcal{A}, a \rangle$ such that $\langle A, a^A \rangle$ is computably isomorphic to $\langle B, a^B \rangle$, $\langle A', a^{A'} \rangle$ is computably isomorphic to $\langle B', a^{B'} \rangle$, $a^A \in \widehat{D}(A)$, and $a^{A'} \in \widehat{D}(A')$. Now the proof proceeds as before, with g_{G_A} and $g_{G_{A'}}$ replaced by d_{G_A} and $d_{G_{A'}}$, respectively.

For the proof of Proposition 2.12, let f, G, and h be as in that proof and redefine $U = \{x \in D : \exists y[Q(x, y) \wedge y \in h^{-1} \circ d_G^{-1} \circ f(S)]\}$. Notice that this definition guarantees that $Q(x, y) \Rightarrow (U(x) \Leftrightarrow U(y))$. Clearly, if S is intrinsically c.e. then so is U.

Now, by Lemma 4.4, for every computable presentation $f' : \mathcal{G} \cong G'$ of $\mathcal{G}$ there exists an isomorphism $k : \mathcal{A} \cong A_{G'}$ such that $U = \{x \in D : \exists y[Q(x, y) \wedge y \in k^{-1} \circ d_{G'}^{-1} \circ$

$f'(S)]\} = \{x \in D : \neg\exists y[Q(x,y) \wedge y \in k^{-1} \circ d_{G'}^{-1} \circ f'(|\mathcal{G}| - S)]\}$, which implies that $\mathrm{DgSp}_{\mathcal{G}}(S) \subseteq \mathrm{DgSp}_{\mathcal{A}}(U)$.

On the other hand, for every computable presentation $k : \mathcal{A} \cong A$ of $\mathcal{A}$, our second remark above implies that there exists an automorphism $p : \mathcal{A} \cong \mathcal{A}$ such that $p \circ h^{-1}(\widehat{D}(A_G)) = k^{-1}(\widehat{D}(A))$. It is not hard to check that $m = k \circ p \circ h^{-1} \circ d_G^{-1} \circ f : \mathcal{G} \cong G_A$ is an isomorphism and that $S = m^{-1} \circ k(U \restriction k^{-1}(\widehat{D}(A)))$. This implies that $\mathrm{DgSp}_{\mathcal{A}}(U) \subseteq \mathrm{DgSp}_{\mathcal{G}}(S)$. $\square$

We now give a relatively simple example of the application of Proposition 4.1 to rings, via a coding based on one due to Rabin and Scott [35].

Let G be a symmetric, irreflexive, countably infinite graph with edge relation E. We can assume that there exist $x, y, z \in G$ such that $E(x,y)$, $E(x,z)$, and $E(y,z)$, since the theory of graphs with this property is clearly complete with respect to effective dimensions, expansion by constants, and degree spectra of structures and relations.

Let G' be a presentation of G. In order to simplify our notation, we will assume without loss of generality that $|G'| = \omega$. (We can do this because every infinite **d**-computable structure is computably isomorphic to a **d**-computable structure with domain ω.) The $\deg(G')$-computably presentable ring $A_{G'}$ is defined as follows.

1. $A_{G'}$ is generated by elements a, b, d, e, and c_i, $i \in \omega$.

2. Multiplication is commutative.

3. $A_{G'}$ has characteristic 0.

4. $a^2 = b^2 = ab = ad = bd = ae = be = 0$, $e^2 = a$, and $de = d^3 = b$.

5. For all $i \in \omega$, $c_i^2 = a$, $ac_i = bc_i = dc_i = 0$, and $ec_i = b$.

6. If $E^{G'}(i,j)$ then $c_i c_j = b$.

7. If $i \neq j \in \omega$ and $\neg E^{G'}(i,j)$ then $c_i c_j = 0$.

It is easy to check that $A_{G'}$ satisfies the ring axioms, using the fact that each of its elements is of the form

$$n_0 + n_1 a + n_2 b + n_3 d + n_4 d^2 + n_5 e + \sum_{i=0}^{p} n_{i+6} c_i, \tag{4.1}$$

where $p \in \omega$ and $n_0, \ldots, n_{p+6} \in \mathbb{Z}$.

Fix a $\deg(G')$-computable presentation of $A_{G'}$ for which the map $g_{G'}$ that sends $ma + nb + c_i$ to i for each $m, n \in \mathbb{Z}$ and $i \in \omega$ is $\deg(G')$-computable and identify $A_{G'}$ with this presentation.

It is easy to see that, for any presentation G' of G, $A_G \cong A_{G'}$. Now let a, b, d, and e be as in the definition of A_G and define

$$D(x) = \{x \in |A_G| : x^2 = a \wedge dx = 0 \wedge ex = b\},$$

$$R(x, y) = \{(x, y) : D(x) \wedge D(y) \wedge xy = b\},$$

and

$$Q(x, y) = \{(x, y) : D(x) \wedge D(y) \wedge xy = a\}.$$

Clearly, D, R, and Q are relatively intrinsically computable. We claim they are also invariant. To see this, fix an automorphism f of A_G. Let $P = \{x \in A_G : x^4 = 0\}$ and $P^2 = \{y \in A_G : \exists x \in P(x^2 = y)\}$.

Let x be of the form (4.1). Since x^4 is a sum of n_0^4 and terms involving a, b, d, e, or some c_i, it follows that if $x \in P$ then $n_0 = 0$. Conversely, if $n_0 = 0$ then

$$x^2 = \left(n_5^2 + \sum_{i=0}^{p} n_{i+6}^2\right)a + \left(n_3n_4 + n_3n_5 + n_5\sum_{i=0}^{p} n_{i+6} + \sum_{i=0}^{p}\sum_{j\leqslant p:E^{G'}(i,j)} n_{i+6}n_{j+6}\right)b + n_3^2d^2, \quad (4.2)$$

and hence $x^4 = 0$, so $x \in P$ if and only if $n_0 = 0$. It will be clear from this that all the elements that we consider below are in P.

We will need to consider elements whose square is a several times below. It follows easily from (4.2) that these are all of one of the forms $n_1a + n_2b + n_4d^2 \pm c_i$ or $n_1a + n_2b + n_4d^2 \pm e$. Note in particular that this means that if $x^2 = a$ then $x^3 = 0$.

Equation (4.2) also shows that every element of P^2 is of the form $ka + lb + md^2$, $k, m \in \omega$, $l \in \mathbb{Z}$. Furthermore, if $x \in P$ then $x^3 = n_3^3b$, so $f(b) = lb$ for some $l \in \mathbb{Z}$. This is only possible if $l = \pm 1$. Since every element of P^2 can be expressed as a sum of nonnegative integer multiples of a and d^2 and an integer multiple of b and P^2 is invariant, every element of P^2 can be expressed as a sum of nonnegative integer multiples of $f(a)$ and $f(d^2)$ and an integer multiple of $f(b) = \pm b$. In particular, a and d^2 can be so expressed. This means that if we write $f(a) = ka + lb + md^2$ then $k, m \leqslant 1$.

Now let x be any element of the form (4.1) such that $x^2 = f(a) = ka + lb + md^2$, $k, m \leqslant 1$. As mentioned above, for any $y \in |A_G|$, if $y^2 = a$ then $y^3 = 0$, so it must be the case that $x^3 = 0$. Now, $k = n_5^2 + \sum_{i=0}^{p} n_{i+6}^2$, $l = n_3n_4 + n_3n_5 + n_5 \sum_{i=0}^{p} n_{i+6} + \sum_{i=0}^{p} \sum_{j \leqslant p: E^{G'}(i,j)} n_{i+6}n_{j+6}$, and $m = n_3^2$. So if $k = 0$ then $n_{i+6} = 0$ for all $i \leqslant p$, which means that if at least one of l and m is nonzero then $n_3 \neq 0$, and hence $x^3 \neq 0$. If $k = 1$ then either $n_5 = \pm 1$ and $n_{i+6} = 0$ for all $i \leqslant p$ or $n_5 = 0$, $n_{i+6} = \pm 1$ for some $i \leqslant m$, and $n_{j+6} = 0$ for all $j \neq i$. Again, if at least one of l and m is nonzero then $n_3 \neq 0$, and hence $x^3 \neq 0$. Since $x^3 = 0$, this means that $l = m = 0$, which implies that $k = 1$. In other words, $f(a) = a$.

Take i_0, i_1, and i_2 such that $E^G(i_0, i_1)$, $E^G(i_0, i_2)$, and $E^G(i_1, i_2)$. For each $j \leqslant 2$, the fact that $c_{i_j}^2 = a$ implies that $f(c_{i_j})$ is of one of the forms $n_{j,1}a + n_{j,2}b + n_{j,4}d^2 + \varepsilon_j c_{i'_j}$ or $n_{j,1}a + n_{j,2}b + n_{j,4}d^2 + \varepsilon_j e$, where $\varepsilon_j = \pm 1$. The second case cannot happen for two different $j, k \leqslant 2$, since then $f(c_{i_j})f(c_{i_k}) = \pm a \neq \pm b = f(b) = f(c_{i_j}c_{i_k})$. So we can assume without loss of generality that $f(c_{i_j})$ is of the form $n_{j,1}a + n_{j,2}b + n_{j,4}d^2 + \varepsilon_j c_{i'_j}$ for $j = 0, 1$. Now, unless $\varepsilon_0 = \varepsilon_1 = \varepsilon_2$, it is not the case that $f(c_{i_0})f(c_{i_1}) = f(c_{i_0})f(c_{i_2}) = f(c_{i_1})f(c_{i_2})$, while $c_{i_0}c_{i_1} = c_{i_0}c_{i_2} = c_{i_1}c_{i_2}$. This means that $f(b) = f(c_{i_0})f(c_{i_1}) = c_{i'_0}c_{i'_1}$. Since $f(b) = \pm b$ and $c_{i'_0}c_{i'_1} \neq -b$, $f(b) = b$.

As in the case of a, if we write $f(d^2) = ka + lb + md^2$ then $k, m \leqslant 1$. But, since $f(a) = a$ and $f(b) = b$, k must equal 0, since otherwise we could not express d^2 as a sum of nonnegative integer multiples of $f(a)$ and $f(d^2)$ and an integer multiple of $f(b)$. This implies that $m = 1$. If $x^2 = d^2 + lb$ then x is of the form $n_1a + n_2b \pm d + n_4d^2$, so $f(d)$ is of this form. Since $f(d)^3 = b$, $f(d)$ is of the form $n_1a + n_2b + d + n_4d^2$. From this it follows easily that, for all $x \in |A_G|$ such that $x^2 = a$, $f(d)x = dx$. Furthermore, since $e^2 = a$ and $de = b$, $f(e)$ must be of the form $n_1a + n_2b + n_4d^2 + e$, from which it follows that, for all $x \in |A_G|$ such that $x^2 = a$, $f(e)x = ex$. This is enough to show that D, R, and Q are invariant.

We now claim that, for any presentation G' of G, $D(A_{G'}) = \mathrm{dom}(g_{G'})$, $R^{A_{G'}}(x, y) \Leftrightarrow E^{G'}(g_{G'}(x), g_{G'}(y))$, and $Q^{A_{G'}}(x, y) \Leftrightarrow g_{G'}(x) = g_{G'}(y)$.

To verify this claim, let a, b, d, e, and c_i be as in the definition of $A_{G'}$. It is easy to check that $\mathrm{dom}(g_{G'}) \subseteq D(A_{G'})$. Now let $x \in D(A_{G'})$. As mentioned above, the fact that $x^2 = a$ implies that x is of one of the forms $n_1a + n_2b + n_4d^2 \pm c_i$ or $n_1a + n_2b + n_4d^2 \pm e$. The second case cannot happen, because $e(n_1a + n_2b + n_4d^2 \pm e) = \pm e^2 = \pm a \neq b$, so $x = n_1a + n_2b + n_4d^2 \pm c_i$. Since $d(n_1a + n_2b + n_4d^2 \pm c_i) = n_4b$, $n_4 = 0$. Since $ex = b$, in fact $x = n_1a + n_2b + c_i$. This shows that $D(A_{G'}) = \mathrm{dom}(g_{G'})$.

Now suppose that $x, y \in D(A_{G'})$. Then, as we have seen, for some $i, j \in \omega$ and $m, n, m', n' \in \mathbb{Z}$, $x = ma + nb + c_i$ and $y = m'a + n' + c_j$, and hence $xy = c_ic_j$. Thus $R^{A_{G'}}(x, y) \Leftrightarrow xy = b \Leftrightarrow E^{G'}(g_{G'}(x), g_{G'}(y))$ and $Q^{A_{G'}}(x, y) \Leftrightarrow xy = a \Leftrightarrow i = j \Leftrightarrow$

$g_{G'}(x) = g_{G'}(y)$.

Let S and S' be sets of Q-representatives and let $f : S \xrightarrow[\text{onto}]{1\text{–}1} S'$ be such that $R(x,y) \Leftrightarrow R(f(x), f(y))$. We can extend f as follows. Let a, b, d, e, and c_i be as in the definition of A_G. Clearly, $S = \{m_0a+n_0b+c_0, m_1a+n_1b+c_1, \ldots\}$ for some $m_0, m_1, \ldots, n_0, n_1, \ldots \in \mathbb{Z}$, so given $x \in |A_G|$, we have $x = k_0 + k_1a + k_2b + k_3d + k_4d^2 + k_5e + \sum_{i=0}^{m} k_{i+6}s_i$ for some $m, k_0, \ldots, k_{m+6} \in \mathbb{Z}$ and $s_0, \ldots, s_m \in S$. Let $f(x) = k_0 + k_1a + k_2b + k_3d + k_4d^2 + k_5e + \sum_{i=0}^{m} k_{i+6}f(s_i)$. It can be easily verified that this extended map is an automorphism of A_G.

Finally, given a presentation G' of G and a $\deg(G')$-computable set S of $Q^{A_{G'}}$-representatives, let a, b, d, and e be as in the definition of $A_{G'}$ and let $t_0, t_1, \ldots$ be a $\deg(G')$-computable list of all terms generated by applying addition and multiplication to a, b, d, e, 1, -1, and the elements of S. Consider the $\deg(G')$-computable set of formulas $\{x = t_i : i \in \omega\}$. Every $x \in |A_{G'}|$ satisfies some formula in this set, with no two elements satisfying the same formula.

It is straightforward to check that, for any computable presentation A of A_G, if U is a subset of $D(A)$ such that $Q(x,y) \Rightarrow (U(x) \Leftrightarrow U(y))$ then the subring of A generated by U has the same degree as U and is c.e. if U is c.e.. Thus Theorem 1.20 in the case of rings of characteristic 0 follows from Proposition 4.1, with the theory P mentioned in that proposition being in this case the theory of symmetric, irreflexive graphs containing at least one triangle.

5 Integral Domains and Commutative Semigroups

In this section we present a coding of a graph into an integral domain inspired by Kudinov's coding [26] of a family of c.e. sets into an integral domain of characteristic 0 and show how this leads to a proof of Theorem 1.20 in the case of integral domains of arbitrary characteristic. Because our coding will not make use of the additive structure of the domain, we will simultaneously handle the case of commutative semigroups.

Let p be either 0 or a prime. We adopt the convention that $\mathbb{Z}_0 = \mathbb{Z}$. If $p = 0$ then let $\mathbb{F} = \mathbb{Q}$; otherwise, let $\mathbb{F} = \mathbb{Z}_p$. Let I be the set of invertible elements of $\mathbb{Z}_p$. Note that I is finite.

The graphs constructed in Subsection 3.1 have the following property: for every finite set of nodes S there exist nodes $x, y \notin S$ that are connected by an edge. Thus the theory of such graphs is complete with respect to effective dimensions, expansion by constants, and degree spectra of structures and relations.

Let G be a symmetric, irreflexive, countably infinite graph with edge relation E, having the property mentioned in the previous paragraph.

Let G' be a presentation of G. As in the previous section, we assume without loss of generality that $|G'| = \omega$. The $\deg(G')$-computably presentable integral domain $A_{G'}$ is defined to be

$$\mathbb{Z}_p\left[x_i : i \in \omega\right]\left[\frac{y}{x_i x_j} : E^{G'}(i,j)\right]\left[\frac{z}{x_i x_j} : \neg E^{G'}(i,j)\right]\left[\frac{y}{x_i^n} : i, n \in \omega\right].$$

Note that, since G is irreflexive, $\frac{z}{x_i^2}$ is included as a generator for each $i \in \omega$.

It is easy to see that $A_{G'}$ is $\deg(G')$-computably presentable. In fact, if we fix a computable presentation P of the ring $\mathbb{F}(x_i : i \in \omega)[y, z]$ then $A_{G'}$ has an obvious $\deg(G')$-computable presentation induced from that of P. (Just take as the domain of this presentation a $\deg(G')$-computable copy of the set of all elements of P that can be generated from the generators of $A_{G'}$.) In what follows, we will identify $A_{G'}$ with this presentation. We will also assume that we have chosen P so that the map $g_{G'} : ax_i \mapsto i$, $a \in I$, is $\deg(G')$-computable.

It is easy to check that if G' is a presentation of G then $A_{G'} \cong A_G$.

Let y and z be as in the definition of A_G and define

$$D(x) = \{x : x \notin I \wedge \exists r(x^2 r = z)\},$$

$$Q(x, x') = \{(x, ax) : D(x) \wedge a \in I\},$$

and

$$R(x, x') = \{(x, x') : D(x) \wedge D(x') \wedge \neg Q(x, x') \wedge \exists r(rxx' = y)\}.$$

Since A_G is a subring of $\mathbb{F}(x_i : i \in \omega)[y, z]$, it makes sense to talk of the degree in y or z (in the algebraic sense) of an element r of A_G. We will denote these by $\deg_y(r)$ and $\deg_z(r)$, respectively. Let

$$Gen = \{\pm 1\} \cup \{x_i : i \in \omega\} \cup \left\{\frac{y}{x_i x_j} : E(i,j)\right\} \cup \left\{\frac{z}{x_i x_j} : \neg E(i,j)\right\} \cup \left\{\frac{y}{x_i^n} : i, n \in \omega\right\}.$$

It will be useful to think of elements of A_G as sums of products of elements of Gen. (Of course, such a representation is not unique, but this will not matter for our purposes.)

Whenever we mention another ring B, such as $\mathbb{Z}_p[x_i, \frac{1}{x_i} : i \in \omega][y, z]$ or $\mathbb{Z}_p[x_i : i \in \omega]$, for example, we will think of A_G as a subring of B or of B as a subring of A_G, as appropriate. The relationships between such rings should be clear. For instance, if

$\deg_y(r) = \deg_z(r) = 0$ then r can be expressed as a sum of products of the generators x_i, $i \in \omega$, so that r is in the subring $\mathbb{Z}_p[x_i : i \in \omega]$ of A_G. In this case, it makes sense to talk of the degree in x_i of r, denoted by $\deg_{x_i}(r)$, for any $i \in \omega$. We will make frequent use of these and similar facts. One ring that will be mentioned often is $M = \mathbb{Z}_p[x_i, \frac{1}{x_i} : i \in \omega][y, z]$.

5.1. Lemma. *The only invertible elements of A_G are the elements of I.*

Proof. If $rs = 1$ then $\deg_y(r) = \deg_z(r) = 0$, and hence $r \in \mathbb{Z}_p[x_i : i \in \omega]$. Clearly, the only invertible elements of $\mathbb{Z}_p[x_i : i \in \omega]$ are the invertible elements of $\mathbb{Z}_p$. $\square$

5.2. Lemma. *Let $r, s \in A_G$. Suppose that $r^2 s = z$ and $r \notin I$. Then $r = ax_i$ for some $i \in \omega$ and $a \in I$.*

Proof. Clearly, $\deg_y(r) = \deg_z(r) = 0$. Since $r \notin I$, it must be the case that $r = x_i r_0 + r_1$ for some $i \in \omega$, $r_0 \in \mathbb{Z}_p[x_k : k \in \omega]$, $r_0 \neq 0$, and $r_1 \in \mathbb{Z}_p[x_k : k \neq i]$.

Now, $\deg_y(s) = 0$ and $\deg_z(s) = 1$, so that, working in M, we can write $s = \frac{z}{x_i^2}s_0 + \frac{z}{x_i}s_1 + s_2$, where $s_0 \in \mathbb{Z}_p[x_j : j \neq i]$, $s_1 \in \mathbb{Z}_p[x_j, \frac{1}{x_j} : j \neq i]$, and $s_2 \in \mathbb{Z}_p[x_j : j \in \omega][\frac{1}{x_j} : j \neq i][z]$.

Suppose that $r_1 \neq 0$. It is easy to check that

$$x_i^2 z = x_i^2 r^2 s = z r_1^2 s_0 + x_i(2z r_0 r_1 s_0 + z r_1^2 s_1) + x_i^2 t$$

for some $t \in \mathbb{Z}_p[x_j : j \in \omega][\frac{1}{x_j} : j \neq i][z]$, and hence that $z r_1^2 s_0 = x_i u$ for some $u \in \mathbb{Z}_p[x_j : j \in \omega][\frac{1}{x_j} : j \neq i][z]$. Since $\deg_{x_i}(z r_1^2 s_0) = 0$, it must be the case that $s_0 = 0$. Now $(z r_1^2 s_1)x_i = x_i^2(z - t)$. Since $\deg_{x_i}(z r_1^2 s_1) = 0$, it follows from this that $s_1 = 0$. But then $s_2 \neq 0$ and

$$x_i^2 r_0^2 s_2 = (x_i r_0 + r_1)^2 s_2 - (2x_i r_0 r_1 + r_1^2)s_2 = z - (2x_i r_0 r_1 + r_1^2)s_2.$$

Since now

$$\deg_{x_i}(x_i^2 r_0^2 s_2) = 2\deg_{x_i}(r_0) + \deg_{x_i}(s_2) + 2 >$$
$$\deg_{x_i}(r_0) + \deg_{x_i}(s_2) + 1 \geqslant \deg_{x_i}(z - (2x_i r_0 r_1 + r_1^2)s_2),$$

this is a contradiction. So in fact $r_1 = 0$, and hence $r = x_i r_0$. We need to show that $r_0 \in I$.

We have

$$x_i^2 r_0^2 s_2 = x_i^2 r_0^2 s - (r_0^2 s_0 z + x_i r_0^2 s_1 z) = z - (r_0^2 s_0 z + x_i r_0^2 s_1 z).$$

Since $s_2 \neq 0$ implies that

$$\deg_{x_i}(x_i^2 r_0^2 s_2) = 2\deg_{x_i}(r_0) + \deg_{x_i}(s_2) + 2 > \\ 2\deg_{x_i}(r_0) + 1 \geqslant \deg_{x_i}(z - (r_0^2 s_0 z + x_i r_0^2 s_1 z)),$$

it must be the case that $s_2 = 0$. Now $x_i r_0^2 s_1 z = z - r_0^2 s_0 z$. Since $s_1 \neq 0$ implies that

$$\deg_{x_i}(x_i r_0^2 s_1 z) = 2\deg_{x_i}(r_0) + 1 > 2\deg_{x_i}(r_0) \geqslant \deg_{x_i}(z - r_0^2 s_0 z),$$

it must be the case that $s_1 = 0$.

So $z = x_i^2 r_0^2 \frac{z}{x_i^2} s_0 = r_0^2 s_0 z$, and hence $r_0 \in I$. □

5.3. Corollary. *For any presentation G' of G, $D(A_{G'}) = \{ax_i : i \in \omega,\ a \in I\}$. Furthermore, D and Q are relatively intrinsically computable.*

Proof. The first statement follows immediately from Lemma 5.2; we prove the second. It is enough to show that D is relatively intrinsically computable.

Let A be a presentation of A_G. We want to show that $D(A)$ is $\deg(A)$-computable. Abusing notation, we refer to the images of y and z in A as y and z, respectively. Let $\widehat{D}(A)$ be as in Section 4. Since I is finite and $x \in D(A) \Leftrightarrow \exists a \in I(ax \in \widehat{D}(A))$, it is enough to show that $\widehat{D}(A)$ is $\deg(A)$-computable.

Clearly, $\widehat{D}(A)$ is $\deg(A)$-c.e., and hence so is the set

$$Gen_A = \widehat{D}(A) \cup \Big\{ r \in A : \exists x, x' \in \widehat{D}(A), n \in \omega(xx'r = y \vee xx'r = z \vee x^n r = y) \Big\}.$$

Given $x \in |A|$, we can write x as a sum of products of elements of Gen_A and hence $\deg(A)$-computably determine $\deg_y(x)$ and $\deg_z(x)$. If it is not the case that $\deg_y(x) = \deg_z(x) = 0$ then $x \notin \widehat{D}(A)$. Otherwise, x is a polynomial over the elements of $\widehat{D}(A)$ with coefficients in $\mathbb{Z}_p$, and checking whether a polynomial over a linearly independent $\deg(A)$-c.e. set is an element of that set can be done $\deg(A)$-computably. □

5.4. Lemma. *If $i \neq j$ and $\neg E(i,j)$ then there is no $r \in A_G$ such that $rx_ix_j = y$. Similarly, if $E(i,j)$ then there is no $r \in A_G$ such that $rx_ix_j = z$.*

Proof. The proofs of both statements are similar; we prove the first.

Assume for a contradiction that, for some $i \neq j \in \omega$ and $r \in A_G$, $\neg E(i,j)$ and $x_ix_jr = y$. We work in the ring M. Since $\deg_y(r) = 1$ and $\deg_z(r) = 0$, thinking of r as a sum of products of elements of Gen, we see that we can write $r = \frac{y}{x_i} r_0 + \frac{y}{x_j} r_1 + r_2$, where $r_0 \in \mathbb{Z}_p[x_k : k \neq i][\frac{1}{x_k} : k \neq j]$, $r_1 \in \mathbb{Z}_p[x_k : k \neq j][\frac{1}{x_k} : k \neq i]$, and $r_2 \in \mathbb{Z}_p[x_k : k \in \omega][\frac{1}{x_k} : k \neq i, j][y]$.

Let $n \in \omega$ be such that $x_i^n r_0, x_j^n r_2 \in \mathbb{Z}_p[x_k : k \in \omega][\frac{1}{x_k} : k \neq i, j]$. Then

$$(x_i x_j)^{n+1} r_2 = (x_i x_j)^n y - (x_i^n x_j^{n+1} r_0 y + x_i^{n+1} x_j^n r_1 y).$$

Since $\deg_{x_i}(x_i^n x_j^{n+1} r_0 y)$, $\deg_{x_j}(x_i^{n+1} x_j^n r_1 y)$, and $\deg_{x_i}((x_i x_j)^n y)$ are all less than or equal to n and $r_2 \in \mathbb{Z}_p[x_k : k \in \omega][\frac{1}{x_k} : k \neq i, j][y]$, it must be the case that $r_2 = 0$.

Now

$$(x_i x_j)^n y = x_i^n x_j^{n+1} r_0 y + x_i^{n+1} x_j^n r_1 y.$$

But

$$r_0 \neq 0 \Rightarrow \deg_{x_i}(x_i^n x_j^{n+1} r_0 y) \leqslant n \wedge \deg_{x_j}(x_i^n x_j^{n+1} r_0 y) > n$$

and

$$r_1 \neq 0 \Rightarrow \deg_{x_i}(x_i^{n+1} x_j^n r_1 y) > n \wedge \deg_{x_j}(x_i^{n+1} x_j^n r_1 y) \leqslant n.$$

Since we cannot have $r_0 = r_1 = 0$, this means that at least one of $\deg_{x_i}(x_i^n x_j^{n+1} r_0 y + x_i^{n+1} x_j^n r_1 y)$ and $\deg_{x_j}(x_i^n x_j^{n+1} r_0 y + x_i^{n+1} x_j^n r_1 y)$ is greater than n. But $\deg_{x_i}((x_i x_j)^n y) = \deg_{x_j}((x_i x_j)^n y) = n$, so this is a contradiction. $\square$

5.5. Corollary. *$R(A_G) = \{(x, x') : D(x) \wedge D(x') \wedge \neg Q(x, x') \wedge \exists r(rxx' = y)\} = \{(x, x') : D(x) \wedge D(x') \wedge \neg \exists r(rxx' = z)\}$, and hence R is relatively intrinsically computable. Furthermore, for any presentation G' of G, $R(A_{G'}) = \{(ax_i, bx_j) : E^{G'}(i, j) \wedge a, b \in I\}$.*

We now need to show that D, Q, and R are invariant. Fix an automorphism $f : A_G \cong A_G$. We will show that $f(D) = D$, $f(Q) = Q$, and $f(R) = R$.

5.6. Lemma. *Suppose that $i \in \omega$ and $f(x_i) = rs$ for some $r, s \in A_G$. Then either $r \in I$ or $s \in I$.*

Proof. Since $f(I) = I$ and $x_i = f^{-1}(r) f^{-1}(s)$, it is enough to show that if $x_i = r's'$ for some $r', s' \in A_G$ then either $r' \in I$ or $s' \in I$. But this follows easily from the fact that if $x_i = r's'$ then $\deg_y(r') = \deg_z(r') = \deg_y(s') = \deg_z(s') = 0$, so that $r', s' \in \mathbb{Z}_p[x_j : j \in \omega]$. $\square$

5.7. Lemma. *$f(D) = D$, which implies that $f(Q) = Q$.*

Proof. It is enough to show that $f(D) \subseteq D$. Since f is an arbitrary automorphism of A_G, the same proof will show that $f^{-1}(D) \subseteq D$, and hence that $D \subseteq f(D)$.

Let $i \in \omega$. Let $n = \deg_y(f(y))$ and let $r = f(\frac{y}{x_i^{n+1}})$. Then $f(x_i)^{n+1} r = f(y)$, and hence $n = \deg_y(f(y)) \geqslant \deg_y(f(x_i)^{n+1}) = (n+1)\deg_y(f(x_i))$. Thus it must be the case that $\deg_y(f(x_i)) = 0$. A similar argument shows that $\deg_z(f(x_i)) = 0$. Since $f(x_i) \notin I$, this means that $f(x_i) = x_j s_0 + s_1$ for some $j \in \omega$, $s_0 \in \mathbb{Z}_p[x_l : l \in \omega]$, $s_0 \neq 0$, and $s_1 \in \mathbb{Z}_p[x_l : l \neq j]$.

Let k be such that $x_j^k f(y) \in \mathbb{Z}_p[x_l : l \in \omega][\frac{1}{x_l} : l \neq j][y,z]$ and let $n = \deg_{x_j}(x_j^k f(y)) + 1$. For some $r \in A_G$, $x_j^k f(x_i)^n r = x_j^k f(y)$. Working in M, we can write

$$r = \frac{1}{x_j^{k+1}} r_0 + \frac{1}{x_j^k} r_1 + \cdots + r_{k+1},$$

where $r_0 \in \mathbb{Z}_p[x_l : l \neq j][\frac{1}{x_l} : l \in \omega][y,z]$, $r_1, \ldots, r_k \in \mathbb{Z}_p[x_l, \frac{1}{x_l} : l \neq j][y,z]$, and $r_{k+1} \in \mathbb{Z}_p[x_l : l \in \omega][\frac{1}{x_l} : l \neq j][y,z]$.

Now

$$x_j^k (x_j s_0 + s_1)^n r_{k+1} = x_j^k (x_j s_0 + s_1)^n r - x_j^k (x_j s_0 + s_1)^n (r - r_{k+1}) =$$
$$x_j^k f(y) - \left(x_j^k (x_j s_0 + s_1)^n \left(\frac{1}{x_j^{k+1}} r_0 + \frac{1}{x_j^k} r_1 + \cdots + \frac{1}{x_j} r_k \right) \right).$$

But it is easy to check that if $r_{k+1} \neq 0$ then

$$\deg_{x_j}(x_j^k (x_j s_0 + s_1)^n r_{k+1}) = n \deg_{x_j}(s_0) + \deg_{x_j}(r_{k+1}) + k + n > n \deg_{x_j}(s_0) + k + n - 1 \geqslant$$
$$\deg_{x_j} \left(x_j^k f(y) - \left(x_j^k (x_j s_0 + s_1)^n \left(\frac{1}{x_j^{k+1}} r_0 + \frac{1}{x_j^k} r_1 + \cdots + \frac{1}{x_j} r_k \right) \right) \right).$$

It follows that $r_{k+1} = 0$.

It is not hard to see that we can now repeat the above argument with k in place of $k+1$ (assuming $k > 0$). Proceeding in this fashion, we see that $r_1 = \cdots = r_{k+1} = 0$.

So

$$\frac{s_1^n r_0}{x_j} = x_j^k (x_j s_0 + s_1)^n \frac{1}{x_j^{k+1}} r_0 - x_j^k ((x_j s_0 + s_1)^n - s_1^n) \frac{1}{x_j^{k+1}} r_0 =$$
$$x_j^k f(y) - ((x_j s_0 + s_1)^n - s_1^n) \frac{1}{x_j} r_0.$$

But $s_1^n r_0 \in \mathbb{Z}_p[x_l : l \neq j][\frac{1}{x_l} : l \in \omega][y,z]$, which implies that either $s_1^n r_0 = 0$ or $\frac{s_1^n r_0}{x_j} \notin \mathbb{Z}_p[x_l : l \in \omega][\frac{1}{x_l} : l \neq j][y,z]$. Since

$$x_j^k f(y) - ((x_j s_0 + s_1)^n - s_1^n) \frac{1}{x_j} r_0 \in \mathbb{Z}_p[x_l : l \in \omega] \left[\frac{1}{x_l} : l \neq j \right] [y,z],$$

it must be the case that $s_1^n r_0 = 0$. Since $r \neq 0$, we conclude that $s_1 = 0$.

Thus $f(x_i) = s_0 x_j$. By Lemma 5.6, $s_0 \in I$. $\square$

5.8. Corollary. $f(\mathbb{Z}_p[x_i : i \in \omega]) = \mathbb{Z}_p[x_i : i \in \omega]$.

5.9. Lemma. *Let* $r \in A_G$ *be such that* $r \neq 0$, $\deg_y(r) = 0$, *and* $\deg_z(r) \leqslant n$. *For all* $i \in \omega$, $x_i^{2n+1} r \notin \mathbb{Z}_p[x_j, \frac{1}{x_j} : j \neq i][y, z]$.

Proof. We work in the ring M. Let $i \in \omega$. Thinking of r as a sum of products of elements of *Gen*, each term t in this sum can be written as $\frac{z^m}{x_i^{2m}} s$, where $m \leqslant n$ and $s \in \mathbb{Z}_p[x_j : j \in \omega][\frac{1}{x_j} : j \neq i]$. So $x_i^{2n+1} t = x_i u$ for some $u \in \mathbb{Z}_p[x_j : j \in \omega][\frac{1}{x_j} : j \neq i][z]$. Thus $x_i^{2n+1} r = x_i v$ for some $v \in \mathbb{Z}_p[x_j : j \in \omega][\frac{1}{x_j} : j \neq i][z]$, and hence $x_i^{2n+1} r \notin \mathbb{Z}_p[x_j, \frac{1}{x_j} : j \neq i][y, z]$. $\square$

5.10. Lemma. $\deg_y(f(y)) = 1$ *and* $\deg_z(f(y)) = 0$.

Proof. Let $i \in \omega$ be such that $f(y) \in \mathbb{Z}_p[x_j, \frac{1}{x_j} : j \neq i][y, z]$. Working in M, we can write $f(y) = y s_0 + s_1$, where $s_0 \in M$, $s_1 \in A_G$, and $\deg_y(s_1) = 0$. Let $n = \deg_z(s_1)$.

By Lemma 5.7, there exists an $r \in A_G$ such that $x_i^{2n+1} r = f(y) = y s_0 + s_1$. We can write $r = y r_0 + r_1$, where $r_0 \in M$, $r_1 \in A_G$, and $\deg_y(r_1) = 0$. Now $x_i^{2n+1} r_1 = s_1$. Since $\deg_z(r_1) = \deg_z(s_1) = n$, it follows from Lemma 5.9 that either $r_1 = 0$ or $s_1 \notin \mathbb{Z}_p[x_j, \frac{1}{x_j} : j \neq i][y, z]$. But the latter possibility would imply that $f(y) \notin \mathbb{Z}_p[x_j, \frac{1}{x_j} : j \neq i][y, z]$, contradicting our choice of i. So $r_1 = 0$, and hence $s_1 = 0$.

We now have $f(y) = y s_0$. A similar argument shows that $\deg_y(f^{-1}(y)) \geqslant 1$. We now need to show that $\deg_y(s_0) = \deg_z(s_0) = 0$.

Let $t \in \mathbb{Z}_p[x_j : j \in \omega]$ be such that $t s_0 \in \mathbb{Z}_p[x_j : j \in \omega][y, z]$. Then

$$f^{-1}(t) y = f^{-1}(t f(y)) = f^{-1}(t s_0 y) = f^{-1}(t s_0) f^{-1}(y).$$

By Corollary 5.8, $f^{-1}(t) \in \mathbb{Z}_p[x_j : j \in \omega]$, which means that $\deg_y(f^{-1}(t) y) = 1$ and $\deg_z(f^{-1}(t) y) = 0$. Since $\deg_y(f^{-1}(y)) \geqslant 1$, this means that $f^{-1}(t s_0) \in \mathbb{Z}_p[x_j : j \in \omega]$. By Corollary 5.8, $t s_0 \in \mathbb{Z}_p[x_j : j \in \omega]$. So $\deg_y(s_0) = \deg_z(s_0) = 0$. $\square$

5.11. Lemma. $f(y) = ty$ *for some* $t \in A_G$.

Proof. Let $i, j, i', j' \in \omega$ be such that $i \neq j$, $f(x_{i'}) = a x_i$ and $f(x_{j'}) = b x_j$ for some $a, b \in I$, $f(y) \in \mathbb{Z}_p[x_k, \frac{1}{x_k} : k \neq i, j][y, z]$, and $E(i', j')$. Such numbers exist by Lemma 5.7 and the assumption about G that we made at the beginning of this section.

Let $r = f(\frac{aby}{x_i'x_j'})$. Then $x_ix_jr = f(y)$. By Lemma 5.10, $\deg_y(f(y)) = 1$ and $\deg_z(f(y)) = 0$, and hence $\deg_y(r) = 1$ and $\deg_z(r) = 0$. Working in M and thinking of r as a sum of products of elements of Gen, we see that we can write

$$r = yr_0 + \frac{y}{x_i}r_1 + \frac{y}{x_j}r_2 + \frac{y}{x_ix_j}r_3 + r_4,$$

where $r_0 \in \mathbb{Z}_p[x_k : k \in \omega][\frac{1}{x_k} : k \neq i,j]$, $r_1 \in \mathbb{Z}_p[x_k : k \neq i][\frac{1}{x_k} : k \neq j]$, $r_2 \in \mathbb{Z}_p[x_k : k \neq j][\frac{1}{x_k} : k \neq i]$, and $r_3, r_4 \in \mathbb{Z}_p[x_k : k \neq i,j]$.

Let $n \in \omega$ be such that $x_i^n r_1, x_j^n r_2 \in \mathbb{Z}_p[x_k : k \in \omega][\frac{1}{x_k} : k \neq i,j]$. Then

$$(x_ix_j)^{n+1}r_0y + (x_ix_j)^{n+1}r_4 = (x_ix_j)^n f(y) - (x_i^n x_j^{n+1} r_1 y + x_i^{n+1}x_j^n r_2 y + (x_ix_j)^n r_3 y).$$

Since $\deg_{x_i}(x_i^n x_j^{n+1} r_1 y)$, $\deg_{x_j}(x_i^{n+1}x_j^n r_2 y)$, $\deg_{x_i}((x_ix_j)^n r_3 y)$, and $\deg_{x_i}((x_ix_j)^n f(y))$ are all less than or equal to n, $r_0, r_4 \in \mathbb{Z}_p[x_k : k \in \omega][\frac{1}{x_k} : k \neq i,j]$, and $\deg_y((x_ix_j)^{n+1}r_4) = 0$, it must be the case that $r_0 = r_4 = 0$.

Now

$$(x_ix_j)^n f(y) - (x_ix_j)^n r_3 y = x_i^n x_j^{n+1} r_1 y + x_i^{n+1} x_j^n r_2 y.$$

But

$$r_1 \neq 0 \Rightarrow \deg_{x_i}(x_i^n x_j^{n+1} r_1 y) \leqslant n \wedge \deg_{x_j}(x_i^n x_j^{n+1} r_1 y) > n$$

and

$$r_2 \neq 0 \Rightarrow \deg_{x_i}(x_i^{n+1} x_j^n r_2 y) > n \wedge \deg_{x_j}(x_i^{n+1} x_j^n r_2 y) \leqslant n,$$

which means that either $r_1 = r_2 = 0$ or at least one of $\deg_{x_i}(x_i^n x_j^{n+1} r_1 y + x_i^{n+1} x_j^n r_2 y)$ and $\deg_{x_j}(x_i^n x_j^{n+1} r_1 y + x_i^{n+1} x_j^n r_2 y)$ is greater than n. Since

$$\deg_{x_i}((x_ix_j)^n f(y) - (x_ix_j)^n r_3 y), \deg_{x_j}((x_ix_j)^n f(y) - (x_ix_j)^n r_3 y) \leqslant n,$$

it must be the case that $r_1 = r_2 = 0$. Thus $f(y) = x_ix_j \frac{y}{x_ix_j} r_3 = yr_3$. Since $r_3 \in A_G$, we are done. □

5.12. Corollary. *If* $\exists r(x_ix_jr = y)$ *then* $\exists r(x_ix_jr = f(y))$.

5.13. Lemma. $f(R) = R$.

Proof. It is enough to show that $R \subseteq f(R)$. Since f is an arbitrary automorphism of A_G, the same proof will show that $R \subseteq f^{-1}(R)$, and hence that $f(R) \subseteq R$.

By Corollaries 5.5 and 5.12, $R(x_i, x_j) \Rightarrow \exists r(x_ix_jr = y) \Rightarrow \exists r(x_ix_jr = f(y)) \Rightarrow R(f(x_i), f(x_j))$. □

Since f is an arbitrary automorphism of A_G, Lemmas 5.7 and 5.13 imply the following result.

5.14. Lemma. *D, Q, and R are invariant.*

In order to apply Proposition 4.1, we are left with showing that properties (P2′) and (P3′) in the statement of that proposition are satisfied.

5.15. Lemma. *For every pair S, S' of sets of Q-representatives, if $f : S \xrightarrow[\text{onto}]{1\text{–}1} S'$ is such that, for every $x, y \in S$, $R(x,y) \Leftrightarrow R(f(x), f(y))$, then f can be extended to an automorphism of A_G.*

Proof. Let y, z, and x_i be as in the definition of A_G. A set of Q-representatives contains one element of the form ax_i, $a \in I$, for each $i \in \omega$, and it contains no other elements. So there exist sequences $a_0, a_1, \ldots \in I$ and $b_0, b_1, \ldots \in I$ such that $S = \{a_0x_0, a_1x_1, \ldots\}$ and $S' = \{b_0x_0, b_1x_1, \ldots\}$. Thus, for some permutation π of ω, $f : a_ix_i \mapsto b_{\pi(i)}x_{\pi(i)}$.

Now, for all $i, j \in \omega$, $R(x_i, x_j) \Leftrightarrow R(x_{\pi(i)}, x_{\pi(j)})$. So it is clear from what we have previously done that the map $x_i \mapsto x_{\pi(i)}$ can be extended to an automorphism of A_G. Thus it is enough to show that the map $a_ix_i \mapsto b_{\pi(i)}x_i$, or, equivalently, the map $h : x_i \mapsto \frac{b_{\pi(i)}}{a_i}x_i$ can be extended to an automorphism of A_G. But h can clearly be extended to an automorphism of $\mathbb{F}(x_i : i \in \omega)[y, z]$ that fixes y and z. Since $\frac{b_{\pi(i)}}{a_i} \in I$, this automorphism restricts to an automorphism of A_G. □

5.16. Lemma. *For every presentation G' of G and every $\deg(G')$-computable set S of $Q^{A_{G'}}$-representatives, there exists a $\deg(G')$-computable defining family for $\langle A_{G'}, a\rangle_{a \in S}$.*

Proof. Let y, z, and x_i be as in the definition of $A_{G'}$. As mentioned above, $S = \{a_0x_0, a_1x_1, \ldots\}$ for some sequence $a_0, a_1, \ldots \in I$. Let $s_i = a_ix_i$ and consider the sets

$$Gen' = \{\pm 1\} \cup \{s_i : i \in \omega\} \cup \left\{\frac{y}{s_is_j} : E^{G'}(i,j)\right\} \cup \left\{\frac{z}{s_is_j} : \neg E^{G'}(i,j)\right\} \cup \left\{\frac{y}{s_i^n} : i, n \in \omega\right\}$$

and

$$Gen'_k = \{\pm 1\} \cup \{s_i : i \leqslant k\} \cup \left\{\frac{y}{s_is_j} : E^{G'}(i,j),\ i,j \leqslant k\right\} \cup \left\{\frac{z}{s_is_j} : \neg E^{G'}(i,j),\ i,j \leqslant k\right\} \cup \left\{\frac{y}{s_i^n} : i, n \leqslant k\right\}.$$

For each $i, j, n \in \omega$, let the formula $\varphi_{i,j,n}$ over the language of rings with additional constants $y, z, s_0, s_1, \dots$ be defined by

$$\varphi_{i,j,n} = \begin{cases} s_i s_j u_{i,j} = y \wedge s_i^n v_{i,n} = y & \text{if } E^{G'}(i,j), \\ s_i s_j u_{i,j} = z \wedge s_i^n v_{i,n} = y & \text{if } \neg E^{G'}(i,j). \end{cases}$$

(Here $u_{i,j}$ and $v_{i,n}$ are the free variables of $\varphi_{i,j,n}$.) For each sum t of products of elements of Gen', let t' be the result of substituting all occurrences of $\frac{y}{s_i s_j}$ or $\frac{z}{s_i s_j}$ in t by $u_{i,j}$, and all occurrences of $\frac{y}{s_i^n}$ by $v_{i,n}$. Let k be the least number such that t is a sum of products of elements of Gen'_k and let $\hat{t}$ be the formula

$$\exists u_{0,0}, v_{0,0}, \dots, u_{0,k}, v_{0,k}, \dots, u_{k,0}, v_{k,0}, \dots, u_{k,k}, v_{k,k} \left(t' \wedge \bigwedge_{i,j,n \leqslant k} \varphi_{i,j,n} \right).$$

Let $t_0, t_1, \dots$ be a $\deg(G')$-computable list of all sums of products of elements of Gen'. Since each s_i is a product of x_i with an element of I, each element of $A_{G'}$ is equal to t_i for some $i \in \omega$. It follows easily that $\{\hat{t}_i : i \in \omega\}$ is a defining family for $\langle A_{G'}, a \rangle_{a \in S}$. □

Lemmas 5.3, 5.14, 5.15, and 5.16 and Corollary 5.5 are enough to enable us to apply Proposition 4.1. It is straightforward to check that, for any computable presentation A of A_G, if U is a subset of $D(A)$ such that $Q(x, y) \Rightarrow (U(x) \Leftrightarrow U(y))$ then the subring of A generated by U has the same degree as U and is c.e. if U is c.e.. This establishes Theorem 1.20 in the case of integral domains of arbitrary characteristic.

Now consider the commutative semigroup generated (multiplicatively) by the elements of Gen. Let

$$D(x) = \{x : \exists r(x^2 r = z)\},$$

$$Q(x, x') = \{(x, x) : D(x)\},$$

and

$$R(x, x') = \{(x, x') : D(x) \wedge D(x') \wedge x \neq x' \wedge \exists r(r x x' = y)\}.$$

It is not hard to check that Proposition 4.1 applies in this case, with essentially the same proof as above. (Though, of course, many of the details could be simplified in this case.) This establishes Theorem 1.20 in the case of commutative semigroups.

6 Nilpotent Groups

In this section, we prove Theorem 1.20 in the case of 2-step nilpotent groups. Much of the proof consists of verifying the effectiveness of a coding of rings into groups due to Mal'cev [29]. Combined with results of Section 5, this will enable us to provide analogs of Lemmas 2.5–2.8, which can then be used to establish analogs of Propositions 2.9–2.12.

Let $R = \langle |R|, +, \cdot, 0, 1\rangle$ be a countably infinite ring with unit of characteristic $p > 2$.

Let R' be a presentation of R. The $\deg(R')$-computably presentable group $G_{R'}$ is defined to be the set of all triples (a, b, c), $a, b, c \in R'$, with multiplication given by the formula

$$(a, b, c)(x, y, z) = (a + x, b + y, b \cdot x + c + z).$$

It is easy to check that this multiplication is associative, that the triple $e = (0, 0, 0)$ is the identity element for it, and that $(a, b, c)^{-1} = (-a, -b, b \cdot a - c)$. Note that the center of $G_{R'}$ consists of all elements of the form $(0, 0, c)$.

Fix a $\deg(R')$-computable presentation of $G_{R'}$ for which the map $g_{R'} : (0, 0, c) \mapsto c$ is $\deg(R')$-computable and identify $G_{R'}$ with this presentation.

Remark. The above definition also works for nonassociative rings. When R is associative, G_R can be represented as the group of upper triangular 3×3 matrices via the isomorphism

$$(a, b, c) \mapsto \begin{pmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix},$$

$a, b, c \in R$, in which form G_R is known as the Heisenberg group of R.

We begin by establishing an analog of Lemma 2.5. Let the relation D on $|G_R|$ be defined by

$$D(x) \Leftrightarrow x \text{ is in the center of } G_R.$$

6.1. Lemma. *If $h : R \cong R'$ is an isomorphism then there exists a $\deg(h)$-computable isomorphism $f : G_R \cong G_{R'}$ such that $f \restriction D(G_R) \equiv g_{R'}^{-1} \circ h \circ g_R$.*

Proof. It is easy to check that $f((a, b, c)) = (h(a), h(b), h(c))$ is the desired isomorphism. □

Let us now consider the following properties of an expanded group $\langle \mathcal{G}, a_1, a_2\rangle$ introduced in [29]. We will denote the commutator $xyx^{-1}y^{-1}$ of group elements x and y by $[x, y]$.

(G1) $\mathcal{G}$ is 2-step nilpotent.

(G2) The subsets

$$\mathcal{C}_i = \{x \in |\mathcal{G}| : xa_i = a_i x\}, \quad i = 1, 2,$$

are Abelian subgroups of $\mathcal{G}$.

(G3) The intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$ is exactly the center $\mathcal{Z}$ of $\mathcal{G}$.

(G4) For each pair $z_1, z_2 \in \mathcal{Z}$ there exists an $h(z_1, z_2) \in |\mathcal{G}|$ such that

$$[a_1, h(z_1, z_2)] = z_1 \quad \text{and} \quad [a_2, h(z_1, z_2)] = z_2.$$

(G5) There exist isomorphisms $f_i \colon \mathcal{Z} \cong \mathcal{C}_i$, $i = 1, 2$, such that $f_1([a_2, a_1]) = a_1$, $f_2([a_2, a_1]) = a_2^{-1}$, and $[a_2, f_1(z)] = [a_1, f_2(z)] = z$ for all $z \in \mathcal{Z}$.

Let a_1 and a_2 be the elements $(1, 0, 0)$ and $(0, 1, 0)$ of G_R, respectively.

6.2. Lemma. *$\langle G_R, a_1, a_2 \rangle$ satisfies (G1)–(G4).*

Proof. Let $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{Z}$ be as in (G2) and (G3).

Direct computation shows that $\mathcal{C}_1$ consists of all triples of the form $(a, 0, c)$, $\mathcal{C}_2$ consists of all triples of the form $(0, b, c)$, and the center of G_R consists of all triples of the form $(0, 0, c)$, so G_R satisfies (G3).

Since

$$(a, b, c)(x, y, z)(a, b, c)^{-1}(x, y, z)^{-1} = (0, 0, b \cdot x - y \cdot a) \in \mathcal{Z},$$

G_R satisfies (G1).

It can be easily checked that, for $a, b, c, x, y, z \in R$, $(a, 0, c)(x, 0, y) = (x, 0, y)(a, 0, c)$ and $(0, b, c)(0, y, z) = (0, y, z)(0, b, c)$, so G_R satisfies (G2).

Finally, letting $h((0, 0, c), (0, 0, c')) = (c', -c, 0)$, we see that G_R satisfies (G4). $\square$

6.3. Lemma. *Let $m : G_R \to G$ be an isomorphism and let $b_1 = m(a_1)$ and $b_2 = m(a_2)$. Then $\langle G, b_1, b_2 \rangle$ satisfies (G1)–(G5). Moreover, the function f in (G5) can be chosen to be* $\deg(G)$*-computable.*

Proof. It is easy to check that, since $\langle G_R, a_1, a_2 \rangle$ satisfies (G1)–(G4), so does $\langle G, b_1, b_2 \rangle$.

Let $\mathcal{Z}$ be the center of G and let $\mathcal{C}_i = \{x \in |G| : xb_i = b_i x\}$, $i = 1, 2$. To prove that a $\deg(G)$-computable isomorphism f_1 as in (G5) exists, we first note that the mapping $\lambda \colon \mathcal{C}_1 \to \mathcal{Z}$ given by $\lambda \colon x \mapsto [b_2, x]$ is a homomorphism of $\mathcal{C}_1$ onto $\mathcal{Z}$. Indeed,

$$[b_2, xy] = b_2 xy b_2^{-1} y^{-1} x^{-1} = b_2 x b_2^{-1} b_2 y b_2^{-1} y^{-1} x^{-1} = b_2 x b_2^{-1} [b_2, y] x^{-1} = [b_2, x][b_2, y],$$

since $[b_2, y] \in \mathcal{Z}$, and λ is onto because of (G4).

Now, any element of $\mathcal{Z}$ is equal to $m((0,0,c))$ for some $c \in R$, and the fact that R has characteristic p implies that, for every $c \in R$, $(0,0,c)^p = (0,0,pc) = (0,0,0)$. So $\mathcal{Z}$ is an Abelian group satisfying the identity $x^p = e$, and can thus be thought of as a vector space over $\mathbb{Z}_p$ via $kz = z^k$, $k \in \mathbb{Z}_p$. Since $\mathbb{Z}_p$ is finite, there exists a $\deg(G)$-computable basis $\{z_i : i \in \omega\}$ for $\mathcal{Z}$ as a vector space with $z_0 = [b_2, b_1]$. Furthermore, the set $\{c_i : i \in \omega\}$ of elements of $\mathcal{C}_1$ such that $\lambda(c_i) = z_i$ is also $\deg(G)$-computable. Note that $c_0 = b_1$.

Given $z \in \mathcal{Z}$, there is a unique way to express z as $\prod_{i=1}^n z_i^{k_i}$ with $0 < k_1, \ldots, k_n < p$. Define $f_1(z) = \prod_{i=1}^n c_i^{k_i}$. Then

$$b_2 f_1(z) b_2^{-1} f_1(z)^{-1} = [b_2, f_1(z)] = \lambda(f_1(z)) = \lambda\left(\prod_{i=1}^n c_i^{k_i}\right) = \prod_{i=1}^n z_i^{k_i} = z.$$

Furthermore, $f_1([b_2, b_1]) = f_1(z_0) = c_0 = b_1$.

A $\deg(G)$-computable isomorphism f_2 as in (G5) can be constructed in a similar way. □

Let the relations P and M on $|G_R|$ be defined by

$$P(x, y, z) \Leftrightarrow xy = z$$

and

$$M(x, y, z) \Leftrightarrow \exists w, w'([a_1, w] = x \wedge [a_2, w'] = y \wedge [a_2, w] = [a_1, w'] = e \wedge [w', w] = z).$$

6.4. Lemma. *Let G be computable, let $m : G_R \to G$ be an isomorphism, and let $b_1 = m(a_1)$ and $b_2 = m(a_2)$. Suppose that $x_1, x_2, y_1, y_2 \in G$ are such that $[b_i, x_j] = [b_i, y_j]$, $i, j \in \{1, 2\}$. Then $[x_1, x_2] = [y_1, y_2]$.*

Proof. Let $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{Z}$ be as in the proof of Lemma 6.3.

For $i, j \in \{1, 2\}$, the fact that $[b_i, x_j] = [b_i, y_j]$ implies that $b_i x_j^{-1} y_j = x_j^{-1} y_j b_i$. So, for $j = 1, 2$, $x_j^{-1} y_j \in \mathcal{C}_1 \cap \mathcal{C}_2 = \mathcal{Z}$. Thus we have

$$x_1 x_2 x_1^{-1} x_2^{-1} = y_1 y_1^{-1} x_1 y_2 y_2^{-1} x_2 x_1^{-1} x_2^{-1} = y_1 y_2 y_1^{-1} x_1 x_1^{-1} y_2^{-1} x_2 x_2^{-1} = y_1 y_2 y_1^{-1} y_2^{-1}.$$

□

6.5. Lemma. *D, M, and P are relatively intrinsically computable. D and P are invariant, while M is mapped to itself by any automorphism of G_R that fixes a_1 and a_2. Let R' be a presentation of R. Then $D(G_{R'}) = \mathrm{dom}(g_{R'})$, and for $x, y, z \in D(G_{R'})$,*

$$P^{G_{R'}}(x, y, z) \Leftrightarrow g_{R'}(x) + g_{R'}(y) = g_{R'}(z)$$

and

$$M^{G_{R'}}(x, y, z) \Leftrightarrow g_{R'}(x) \cdot g_{R'}(y) = g_{R'}(z).$$

Proof. D and P are algebraic, and hence invariant. M is algebraic over $\{a_1, a_2\}$, and hence is mapped to itself by any automorphism of G_R that fixes a_1 and a_2.

P is obviously relatively intrinsically computable, and the fact that every presentation of G_R satisfies (G3) implies that so is D. It is also obvious that for each $x, y \in D(G_{R'})$ there is at most one z such that $P^{G_{R'}}(x, y, z)$.

Let $b_1, b_2 \in G_{R'}$ be defined by $b_1 = (1, 0, 0)$, $b_2 = (0, 1, 0)$. By Lemma 6.4,

$$\exists w, w'([b_1, w] = x \wedge [b_2, w'] = y \wedge [b_2, w] = [b_1, w'] = e \wedge [w', w] = z) \Leftrightarrow$$
$$\forall w, w'(([b_1, w] = x \wedge [b_2, w'] = y \wedge [b_2, w] = [b_1, w'] = e) \rightarrow [w', w] = z).$$

This implies that M is relatively intrinsically computable. It also implies that for each $x, y \in D(G_{R'})$ there is at most one z such that $M^{G_{R'}}(x, y, z)$.

Now let $r, s \in R'$. If we can show that $P^{G_{R'}}((0, 0, r), (0, 0, s), (0, 0, r + s))$ and $M^{G_{R'}}((0, 0, r), (0, 0, s), (0, 0, r \cdot s))$ then we will be done.

By the definition of multiplication in $G_{R'}$, $(0, 0, r)(0, 0, s) = (0, 0, r + s)$, so indeed $P^{G_{R'}}((0, 0, r), (0, 0, s), (0, 0, r + s))$.

Let $w = (0, -r, 0)$ and $w' = (s, 0, 0)$. Direct computation shows that $[b_1, w] = (0, 0, r)$, $[b_2, w'] = (0, 0, s)$, $[b_2, w] = [b_1, w'] = e$, and $[w', w] = (0, 0, r \cdot s)$, which implies that $M^{G_{R'}}((0, 0, r), (0, 0, s), (0, 0, r \cdot s))$. $\square$

For any presentation G of G_R, let $\widetilde{R}_G$ be the ring whose domain is $D(G)$, with addition defined by $x + y = z \Leftrightarrow P^G(x, y, z)$ and multiplication defined by $x \cdot y = z \Leftrightarrow M^G(x, y, z)$. Clearly, there exist a $\deg(G)$-computable map h_A and a $\deg(G)$-computable graph R_G such that $h_G : \widetilde{R}_G \to R_G$ is a $\deg(A)$-computable presentation of $\widetilde{R}_G$. If G is computable then we take $R_G = \widetilde{R}_G$ and let h_G be the identity. In any case, Lemma 6.5 implies that R_G is a $\deg(G)$-computable presentation of R.

The following lemma, which is an analog of Lemma 2.7, can be easily checked.

6.6. Lemma. *Let R' be a computable presentation of R. Then $g_{R'}$ is a computable isomorphism from $R_{G_{R'}}$ to R'.*

We need analogs of Lemmas 2.6 and 2.8. We begin with the latter.

6.7. Lemma. *Let G be a presentation of G_R. Then G is* $\deg(G)$*-computably isomorphic to G_{R_G} via a map whose restriction to $D(G)$ is equal to $g_{R_G}^{-1} \circ h_G$.*

Proof. Let f_1 and f_2 be $\deg(G)$-computable functions as in (G5). On pages 225–226 of [29], property (G5) is used to show that the mapping

$$z = (z_1, z_2, z_3) \mapsto \tau(z) = f_1(h_G^{-1}(z_1))f_2(h_G^{-1}(z_2))^{-1}h_G^{-1}(z_3)$$

is an isomorphism from G_{R_G} onto G. (Of course, in [29] h_G is not present, since there is no need there for R_G to have computable domain.) Since f_1, f_2, and h_G^{-1} are $\deg(G)$-computable, so is τ. Finally, if $x \in D(G)$ then $\tau^{-1}(x) = (0, 0, h_G(x)) = g_{R_G}^{-1} \circ h_G(x)$. $\square$

We now come to the analog of Lemma 2.6. The fact that M is not invariant creates a difficulty, but this can be remedied by showing that if G is a computable presentation of G_R, $b_1, b_2 \in G$, and there exists an automorphism h of G such that $h(a_i^G) = b_i$, $i = 1, 2$, then there exists a computable automorphism g of G such that $g(a_i^G) = b_i$, $i = 1, 2$. (See Lemma 6.13. The situation is similar to what we encountered in Section 4 in connection with Lemma 4.5.)

All the results obtained so far in this section are true for any ring R with unit of characteristic p. In order to prove the statement in the previous paragraph, however, we need to impose additional conditions on R, namely that R be an integral domain and that the only invertible elements of R be $1, \ldots, p-1$. Note that, by Lemma 5.1, the integral domains of characteristic p constructed in Section 5 have this property. Since the result we wish to prove is of interest only in the case in which R is computably presentable, we will assume for the remainder of this argument that R is computable.

Let $b_1 = (a, b, c), b_2 = (a', b', c') \in G_R$ and suppose there exists an automorphism $h : G_R \cong G_R$ such that $h(a_i) = b_i$, $i = 1, 2$.

Let $r = a \cdot b' - b \cdot a'$. Note that, since $h((0, 0, 1)) = h([a_1, a_2]) = [b_1, b_2] = (0, 0, r)$, $r \neq 0$. Let $\widetilde{R} = R\left[\frac{1}{r}\right]$. Since we are assuming that R is computable, we can take $\widetilde{R}$ to be a computable ring. We think of G_R as a subgroup of $G_{\widetilde{R}}$.

Let $g : G_{\widetilde{R}} \to G_{\widetilde{R}}$ be defined by

$$g((x,y,z)) = \Big(a\cdot x + a'\cdot y \, , \, b\cdot x + b'\cdot y \, , \, b\cdot a'x\cdot y + \frac{p+1}{2}\cdot a\cdot b\cdot (x^2 - x) + \frac{p+1}{2}\cdot a'\cdot b'\cdot (y^2-y) + c\cdot x + c'\cdot y + (a\cdot b' - b\cdot a')\cdot z \Big).$$

6.8. Lemma. *g is a computable automorphism of $G_{\widetilde{R}}$ such that $g(a_i) = b_i$, $i = 1,2$.*

Proof. It is clear that g is computable and that $g((1,0,0)) = (a,b,c)$ and $g((0,1,0)) = (a',b',c')$.

Let $(x,y,z),(x',y',z') \in G_{\widetilde{R}}$. Straightforward but tedious expansion and matching of terms shows that $g((x,y,z)(x',y',z')) = g((x,y,z))g((x',y',z'))$. (Recall that $p+1 = 1$ in $\widetilde{R}$.)

We now need to show that g is surjective and injective. To show that g is surjective, pick an arbitrary element (x,y,z) of $G_{\widetilde{R}}$. Let $d_0 = (\frac{1}{r}\cdot b'\cdot x, -\frac{1}{r}\cdot b\cdot x, 0)$ and $d_1 = (-\frac{1}{r}\cdot a'\cdot y, \frac{1}{r}\cdot a\cdot y, 0)$. It is straightforward to check that for some $z_0, z_1 \in \widetilde{R}$, $g(d_0) = (x,0,z_0)$ and $g(d_1) = (0,y,z_1)$. Let $d_2 = (0,0,\frac{1}{r}\cdot(z - z_0 - z_1))$. Then $g(d_2) = (0,0,z-z_0-z_1)$, and hence $g(d_0d_1d_2) = (x,0,z_0)(0,y,z_1)(0,0,z-z_0-z_1) = (x,y,z)$.

To see that g is injective, suppose that $g((x,y,z)) = (0,0,0)$. Then $a\cdot x + a'\cdot y = b\cdot x + b'\cdot y = 0$. Working in the field of fractions of $\widetilde{R}$, this implies that $\frac{a'}{a}\cdot y = \frac{b'}{b}\cdot y$, so that, unless $y = 0$, $r = a\cdot b' - b\cdot a' = 0$, which is a contradiction. So $y = 0$, which implies that $x = 0$. Now $g((x,y,z)) = (0,0,r\cdot z)$, so that $r\cdot z = 0$, which implies that $z = 0$. $\square$

6.9. Lemma. *Let $f : G_R \to G_{\widetilde{R}}$ be a group homomorphism such that $f(a_i) = a_i$, $i = 1,2$. Let $u,v,w \in D$ be such that $M^{G_{\widetilde{R}}}(f(u),f(v),f(w))$. Then $M^{G_R}(u,v,w)$.*

Proof. There exists a $w' \in D$ such that $M^{G_R}(u,v,w')$, and it is clear from the definition of M that $M^{G_{\widetilde{R}}}(f(u),f(v),f(w'))$. But this means that $f(w') = f(w)$, so $w' = w$. $\square$

6.10. Lemma. *$R = \widetilde{R}$.*

Proof. It is enough to show that r is invertible in R. Let s and t be such that $h((0,0,s)) = (0,0,1)$ and $h((0,0,t)) = (0,0,r^2)$. Recall that $h((0,0,1)) = (0,0,r)$. Let $i : G_R \to G_{\widetilde{R}}$ be the inclusion map and define $f \equiv g^{-1}\circ i\circ h$. Now f is a group homomorphism from G_R into $G_{\widetilde{R}}$.

It is easy to check that $f(a_1) = a_1$, $f(a_2) = a_2$, $f((0,0,1)) = (0,0,1)$, $f((0,0,s)) = (0,0,\frac{1}{r})$, and $f((0,0,t)) = (0,0,r)$. Since $M^{G_{\tilde{R}}}((0,0,\frac{1}{r}),(0,0,r),(0,0,1))$, it follows from Lemma 6.9 that $M^{G_R}((0,0,s),(0,0,t),(0,0,1))$, so that $s \cdot t = 1$.

Thus s is invertible in R. By our assumption on R, this means that $s = k$ for some $1 \leqslant k \leqslant p-1$. Let $n < p$ be such that $kn = 1 \bmod p$. It follows that $(0,0,r) = h((0,0,1)) = h((0,0,k)^n) = (h((0,0,k)))^n = (0,0,1)^n = (0,0,n)$, so r is invertible in R. $\square$

The above argument obviously holds with any computable presentation R' of R in place of R. Letting $R' = R_G$, the following lemma follows from Lemma 6.7.

6.11. Lemma. *Let G be a computable presentation of G_R and let $b_1, b_2 \in G$. If there exists an automorphism h of G such that $h(a_i^G) = b_i$, $i = 1,2$, then there exists a computable automorphism g of G such that $g(a_i^G) = b_i$, $i = 1,2$.*

6.12. Corollary. *If G and G' are computable presentations of G_R and $f : G \cong G'$ is an isomorphism then there exists an automorphism g of G' such that $g \circ f$ is a* $\deg(f)$-*computable isomorphism and $g \circ f(M^G) = M^G$.*

The following analog of Lemma 2.6 follows easily from Corollary 6.12.

6.13. Lemma. *If G and G' are computable presentations of G_R and $f : G \cong G'$ is an isomorphism then there exists an automorphism g of G' such that $(g \circ f) \upharpoonright D(G)$ is a* $\deg(f)$*-computable isomorphism from R_G to $R_{G'}$.*

Using Lemmas 6.1, 6.13, 6.6, and 6.7 in place of Lemmas 2.5, 2.6, 2.7, and 2.8, respectively, we can establish the following result by essentially the same arguments as were used in the proofs of Propositions 2.9–2.12.

6.14. Proposition. $\mathrm{DgSp}(G_R) = \mathrm{DgSp}(R)$*, and if R is computably presentable then the following hold.*

1. *For any degree $\mathbf{d}$, G_R has the same $\mathbf{d}$-computable dimension as R.*

2. *Let $x \in |R|$. There exists an $a \in D(G_R)$ such that $\langle G_R, a\rangle$ has the same computable dimension as $\langle R, x\rangle$.*

3. *Let S be a subring of R. There exists a subgroup U of G_R such that* $\mathrm{DgSp}_{G_R}(U) = \mathrm{DgSp}_R(S)$ *and if S is intrinsically c.e. then so is U.*

(The fact that U can be taken to be a subgroup in part 3 of Proposition 6.14 follows from the fact that if S is a subring of R then $g_R^{-1}(S)$ is a subgroup of G_R.)

It follows from Proposition 6.14 and the results of Section 5 that Theorem 1.20 holds in the case of 2-step nilpotent groups.

A The Universality of Directed Graphs

In this appendix, we justify the terminology adopted in Definition 1.19 by giving a sufficiently effective coding of a given countable structure into a countable graph, thus showing that if a theory satisfies this definition then it still satisfies it if "every countable graph $\mathcal{G}$" is replaced by "every countable structure $\mathcal{G}$".

Let $\mathcal{A}$ be a countable structure in the computable language L with (possibly finitely many) constants $c_0, c_1, \ldots$, functions $f_0, f_1, \ldots$, and relations $R_0, R_1, \ldots$. Let k_i be the arity of f_i and let l_i be the arity of R_i.

The directed graph $\mathcal{G}$ consists of the following nodes and edges.

1. A node x with an edge from x to itself.

2. A node x_i for each $i \in |\mathcal{A}|$, with an edge from x to each x_i.

3. For each constant c_i, a cycle of length $4i+2$ with an edge from x_j to one of the elements of this cycle, where $j = c_i^{\mathcal{A}}$.

4. For each function f_i and each tuple $(j_0, \ldots, j_{k_i-1}) \in |\mathcal{A}|$, a cycle C of length $4i+3$; a chain of elements $y_0, \ldots, y_{k_i}$, where y_0 is an element of C, with an edge from y_n to y_{n+1} for each $n < k_i$; an edge from each x_{j_n} to y_n, $n < k_i$; and an edge from y_{k_i} to x_j, where $j = f_i^{\mathcal{A}}(j_0, \ldots, j_{k_i-1})$.

5. For each relation R_i and each tuple $(j_0, \ldots, j_{l_i-1}) \in |\mathcal{A}|$ such that $R_i^{\mathcal{A}}(j_0, \ldots, j_{l_i-1})$ holds, a cycle C of length $4i+4$; a chain of elements $y_0, \ldots, y_{l_i-1}$, where y_0 is an element of C, with an edge from y_n to y_{n+1} for each $n < k_i - 1$; and an edge from each x_{j_n} to y_n, $n < l_i$.

6. For each relation R_i and each tuple $(j_0, \ldots, j_{l_i-1}) \in |\mathcal{A}|$ such that $R_i^{\mathcal{A}}(j_0, \ldots, j_{l_i-1})$ does not hold, a cycle C of length $4i+5$; a chain of elements $y_0, \ldots, y_{l_i-1}$, where y_0 is an element of C, with an edge from y_n to y_{n+1} for each $n < k_i - 1$; and an edge from each x_{j_n} to y_n, $n < l_i$.

As an example, Figure 4 shows a portion of $\mathcal{G}$ in the case in which L has one constant c_0, one unary function f_0, and one binary relation R_0, $c_0^{\mathcal{A}} = 0$, $f_0^{\mathcal{A}}(0) = 0$, $f_0^{\mathcal{A}}(1) = 0$, and the only ordered pairs of numbers less than 2 of which $R_0^{\mathcal{A}}$ holds are $(0,0)$ and $(1,0)$. The expressions under each cycle show which fact is being coded by that cycle and its connections.

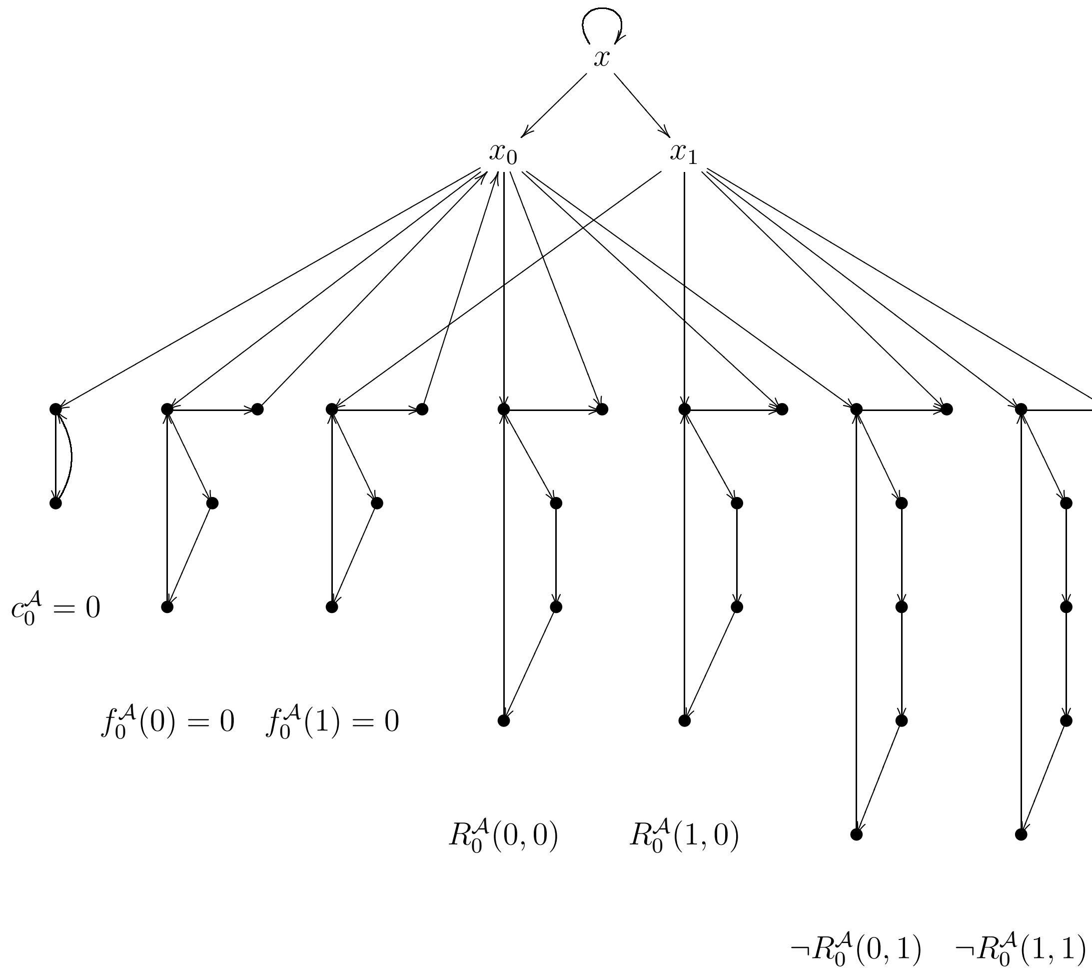

Figure 4: $\mathcal{G}$

It is not hard to check, using methods similar to those of Section 2, that $\mathcal{G}$ has the following properties.

1. $\mathrm{DgSp}(\mathcal{G}) = \mathrm{DgSp}(\mathcal{A})$.

2. If $\mathcal{A}$ is computably presentable then the following hold.

(a) For any degree $\mathbf{d}$, $\mathcal{G}$ has the same $\mathbf{d}$-computable dimension as $\mathcal{A}$.

(b) if $a \in |\mathcal{A}|$ then there exists an $x \in |\mathcal{G}|$ such that $\langle \mathcal{G}, x \rangle$ has the same computable dimension as $\langle \mathcal{A}, a \rangle$.

(c) if $S \subseteq |\mathcal{A}|$ then there exists a $U \subset \mathcal{G}$ such that $\mathrm{DgSp}_{\mathcal{G}}(U) = \mathrm{DgSp}_{\mathcal{A}}(S)$ and if S is intrinsically c.e. then so is U.

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