ON THE UNIFORM COMPUTATIONAL CONTENT OF RAMSEY'S THEOREM

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Abstract. We study the uniform computational content of Ramsey's Theorem in the Weihrauch lattice. Our central results provide information on how Ramsey's Theorem behaves under product, parallelization and jumps. From these results we can derive a number of important properties of Ramsey's Theorem. For one, the parallelization of Ramsey's Theorem for size $n \geq 1$ and an arbitrary finite number of colors $k \geq 2$ is equivalent to the n–th jump of Weak Kőnig's Lemma. In particular, Ramsey's Theorem for size $n \geq 1$ is $\mathbf{\Sigma}^0_{n+2}$–measurable in the effective Borel hierarchy, but not $\mathbf{\Sigma}^0_{n+1}$–measurable. Secondly, we obtain interesting lower bounds, for instance the n–th jump of Weak Kőnig's lemma is reducible to (the stable version of) Ramsey's Theorem of size $n+2$ for $n \geq 2$. Another result that can be derived from our study of products is the fact that with strictly increasing numbers of colors Ramsey's Theorem forms a strictly increasing chain in the Weihrauch lattice. Our study of jumps also shows that certain uniform variants of Ramsey's Theorem that are indistinguishable from a non-uniform perspective play an important role. For instance, the colored version of Ramsey's Theorem explicitly includes the color of the homogeneous set as output information and the jump of this problem (but not the uncolored variant) is equivalent to the stable version of Ramsey's Theorem of the next larger size. Finally, we briefly discuss the particular case of Ramsey's Theorem for pairs and we provide some new separation techniques for problems that involve jumps in this context. In particular, we study uniform results regarding the relation of boundedness and induction problems to Ramsey's Theorem and we show that there are some significant differences to the non-uniform situation in reverse mathematics.

Keywords: Computable analysis, Weihrauch lattice, Ramsey's Theorem, reverse mathematics.

Contents

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1. Introduction

In this paper we study uniform computational properties of Ramsey's Theorem for size n and k colors. We briefly recall some basic definitions.

By $[M]^n := \{A \subseteq M : |A| = n\}$ we denote the set of subsets of M with exactly n elements. We identify k with the set $\{0, ..., k-1\}$ for every $k \in \mathbb{N}$. We also allow the case $k = \mathbb{N}$. Any map $c : [\mathbb{N}]^n \to k$ with finite range is called a *coloring* (*of* $[\mathbb{N}]^n$). A subset $M \subseteq \mathbb{N}$ is called *homogeneous* (*for* c) if there is some $i \in k$ such that $c(A) = i$ for every $A \in [M]^n$. In this situation we write $c(M) = i$, which is understood to imply that M is homogeneous. Frank P. Ramsey proved the following theorem [30].

Theorem 1.1 (Ramsey's Theorem 1930)**.** *For every coloring $c : [\mathbb{N}]^n \to k$ with $n, k \geq 1$ there exists an infinite homogeneous set $M \subseteq \mathbb{N}$.*

We will abbreviate Ramsey's Theorem for cardinality n and k colors by $\mathsf{RT}_{n,k}$.[1] The computability theoretic study of Ramsey's Theorem started in the same year when Specker proved that there exists a computable counterexample for Ramsey's Theorem for pairs [33], which shows that Ramsey's Theorem cannot be proved constructively.

Theorem 1.2 (Specker 1969)**.** *There exists a computable coloring $c : [\mathbb{N}]^2 \to k$ without a computable infinite homogeneous set $M \subseteq \mathbb{N}$.*

Jockusch provided a very simple proof of Specker's theorem and he improved Specker's result by showing the following [21].

Theorem 1.3 (Jockusch 1972)**.** *For every computable coloring $c : [\mathbb{N}]^n \to 2$ with $n \geq 1$ there exists an infinite homogeneous set $M \subseteq \mathbb{N}$ such that $M' \leq_{\mathrm{T}} \emptyset^{(n)}$. However, there exists a computable coloring $c : [\mathbb{N}]^n \to 2$ for each $n \geq 1$ without an infinite homogeneous set $M \subseteq \mathbb{N}$ that is computable in $\emptyset^{(n-1)}$.*

Another cornerstone in the study of Ramsey's Theorem was the Cone Avoidance Theorem [31, Theorem 2.1] that was originally proved by Seetapun.

Theorem 1.4 (Seetapun 1995)**.** *Let $c : [\mathbb{N}]^2 \to 2$ be a coloring that is computable in $B \subseteq \mathbb{N}$ and let $(C_i)_i$ be a sequence of sets $C_i \subseteq \mathbb{N}$ such that $C_i \not\leq_{\mathrm{T}} B$ for all $i \in \mathbb{N}$. Then there exists an infinite homogeneous set M for c such that $C_i \not\leq_{\mathrm{T}} M$ for all $i \in \mathbb{N}$.*

This theorem was generalized by Cholak, Jockusch and Slaman who proved in particular the following version [9, Theorem 12.2].

Theorem 1.5 (Cholak, Jockusch and Slaman 2001)**.** *For every computable coloring $c : [\mathbb{N}]^n \to k$ there exists an infinite homogeneous set $M \subseteq \mathbb{N}$ such that $\emptyset^{(n)} \not\leq_{\mathrm{T}} M'$.*

Cholak, Jockusch and Slaman also improved Jockusch's Theorem 1.3 for the case of Ramsey's Theorem for pairs [9, 10].

Theorem 1.6 (Cholak, Jockusch and Slaman 2001)**.** *For every computable coloring $c : [\mathbb{N}]^2 \to 2$ there exists an infinite homogeneous set $M \subseteq \mathbb{N}$, which is low$_2$, i.e., such that $M'' \leq_{\mathrm{T}} \emptyset''$.*

We will make use of these and other earlier results in our uniform study of Ramsey's Theorem. Firstly, a substantial number of results in this article are based on the second author's PhD thesis [29]. The first uniform results on Ramsey's Theorem were published by Dorais, Dzhafarov, Hirst, Mileti and Shafer [13] and by Dzhafarov [14, 15]. Among other things they proved the following Squashing Theorem [13,

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[1] We do not use the more common abbreviation RT^n_k since we will use upper indices to indicate the number of jumps or products.

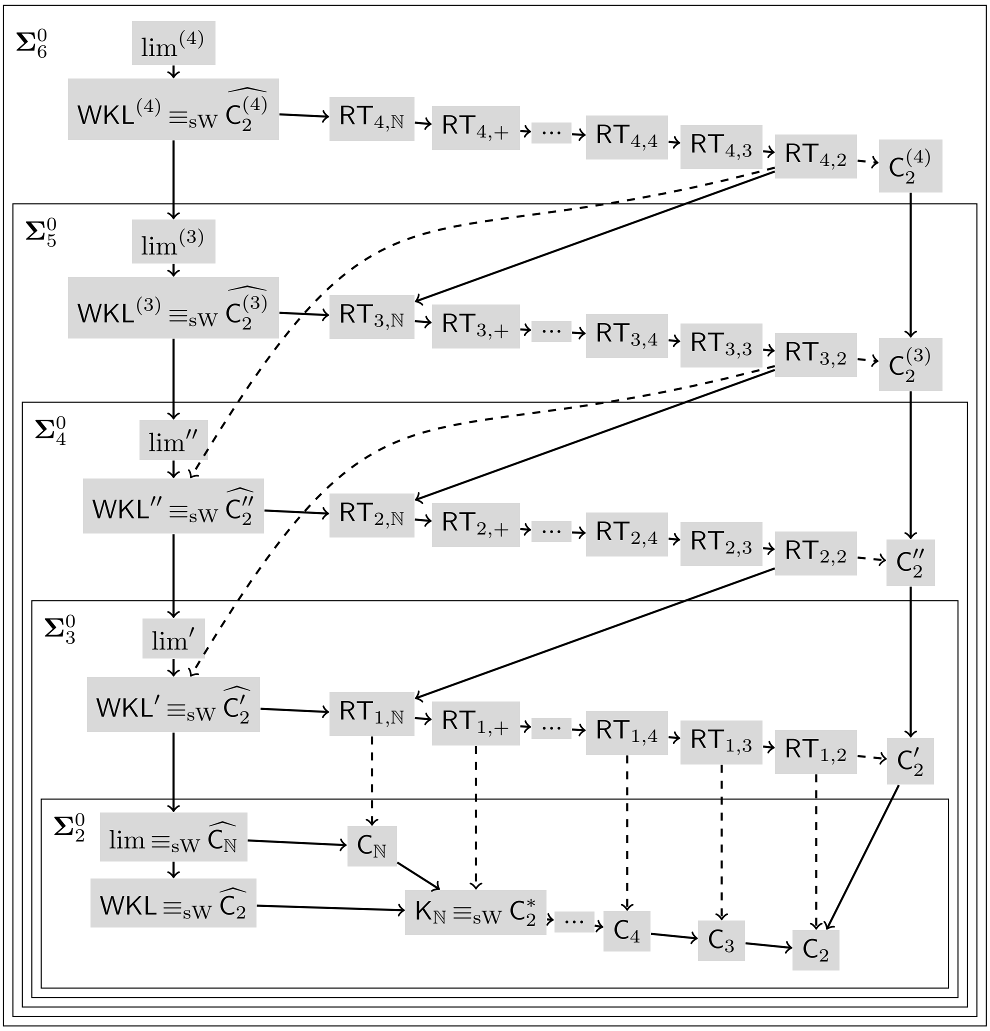

FIGURE 1. Ramsey's Theorem for different sizes and colors in the Weihrauch lattice: all solid arrows indicate strong Weihrauch reductions against the direction of the arrow, all dashed arrows indicate ordinary Weihrauch reductions.

Theorem 2.5] that establishes a relation between products and parallelization for problems such as Ramsey's Theorem. Here a problem $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ is called *finitely tolerant* if there is a computable $T :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that for all $p, q \in \mathrm{dom}(f)$ and $k \in \mathbb{N}$ with $(\forall n \geq k)\ p(n) = q(n)$ it follows that $r \in f(q)$ implies $T\langle r, k\rangle \in f(p)$. Intuitively, it means that for two almost identical inputs and a solution for one of these inputs we can compute a solution that fits to the other one.

Theorem 1.7 (Squashing Theorem [13]). *If $f, g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ are such that f is total and finitely tolerant, then $g \times f \leq_{\mathrm{W}} f \Longrightarrow \widehat{g} \leq_{\mathrm{W}} f$.*

There is an analogous version of the Squashing Theorem for strong Weihrauch reducibility, which additionally requires that $\mathrm{dom}(f) = \mathrm{dom}(g) = 2^{\mathbb{N}}$. This theorem allowed the authors of [13] to prove that Ramsey's Theorem for strictly increasing numbers of colors forms a strictly increasing chain with respect to *strong* Weihrauch reducibility [13, Theorem 3.1]:

$$\mathsf{RT}_{n,2} \leq_{\mathrm{sW}} \mathsf{RT}_{n,3} \leq_{\mathrm{sW}} \mathsf{RT}_{n,3} \leq_{\mathrm{sW}} \dots.$$

However, they left it an open question [13, Question 7.1] whether an analogous statement also holds for ordinary Weihrauch reducibility. We will be able to use our Theorem 3.24 on products to answer this question in the affirmative (see Theorem 4.21). Independently, similar results were obtained by Hirschfeldt and Jockusch [18] and Patey [27]. Altogether, the diagram in Figure 1 displays how Ramsey's Theorem for different sizes and colors is situated in the Weihrauch lattice.

We briefly describe the further organization of the paper. In the following Section 2 we point the reader to sources where the basic concepts and definitions on the Weihrauch lattice can be found and we collect a number of facts that are useful for our study. In Section 3 we introduce the uniform versions of Ramsey's Theorem that we are going to consider in this study, we establish a core lower bound in Theorem 3.5 that will be used to derive almost all other lower bound results. We also prove key results on products in Theorem 3.24 and on parallelization in Theorem 3.28 that also lead to our main lower bound results, which are formulated in Corollary 3.27 and Corollary 3.30. In Section 4 we discuss jumps of Ramsey's Theorem together with upper bound results. In particular, we prove Theorem 4.3, which is our main result on jumps and shows that the stable version of Ramsey's Theorem can be seen as the jump of the colored version of the size below. These results lead to Corollary 4.14, which shows that a composition with one jump of Weak Kőnig's Lemma is sufficient to bring Ramsey's Theorem from one size to the next size. From this result we can conclude our main upper bound result in Corollary 4.15, which shows that the n–th jump of Weak Kőnig's Lemma is an upper bound of Ramsey's Theorem of size n. Together with our lower bound results this finally leads to the classification of the parallelization of Ramsey's Theorem of size n as the n–th jump of Weak Kőnig's Lemma in Corollary 4.18 and to the classification of the exact Borel degree of Ramsey's Theorem in Corollary 4.19. We also use these results to obtain the above mentioned result on increasing numbers of colors. In Section 5 we briefly discuss the special case of Ramsey's Theorem for pairs, we summarize some known results, provide some new insights and formulate some open questions. In particular, we separate some uniform versions of Ramsey's Theorem with the help of the separation techniques for jumps that are developed in Section 6. Finally, in Section 7 we discuss the relation of closed and compact choice problems to Ramsey's Theorem, which corresponds to the discussion of boundedness and induction principles in reverse mathematics.

2. Preliminaries

We use the theory of the Weihrauch lattice as a framework for the uniform classification of the computational content of mathematical problems as it has been developed in [5, 4, 3, 7, 6, 8]. We refrain from giving another introduction into this theory and the main ingredients, as the reader can find all the necessary definitions and references in [6] or [8]. In particular, the concepts of *Weihrauch reducibility* $f \leq_{\mathrm{W}} g$, *strong Weihrauch reducibility* $f \leq_{\mathrm{sW}} g$, of algebraic operations such as *product* $f \times g$, *coproduct* $f \sqcup g$, *compositional product* $f * g$, *finite parallelization* f^*, *parallelization* $\widehat{f}$ and *jump* f' are precisely defined in these references. Moreover, problems such as the choice problem C_X and the Bolzano-Weierstraß Theorem BWT_X for a space X are defined and discussed in these references too.

We just summarize a number of facts that are of particular importance. The following facts were established in [7, Fact 3.4, Proposition 5.6(2), Proposition 5.7(3), Theorem 11.2, Corollary 13.8] or are easy to see.

Fact 2.1. *We obtain*

(1) $\mathsf{BWT}_k^{(n-1)} \equiv_{\mathrm{sW}} \mathsf{C}_k^{(n)}$ *for all* $k \in \mathbb{N}$, $n \geq 1$,
(2) $\mathsf{WKL}^{(n)} \equiv_{\mathrm{sW}} \widehat{\mathsf{C}_k^{(n)}}$ *for all* $k \geq 2$ *and* $n \in \mathbb{N}$,
(3) $\lim^{(n)} \equiv_{\mathrm{sW}} \widehat{\lim_k^{(n)}}$ *for all* $k \geq 2$, $k = \mathbb{N}$ *and* $n \in \mathbb{N}$,
(4) $\lim_k^{(n)} <_{\mathrm{sW}} \mathsf{BWT}_k^{(n)}$ *and* $\lim_k^{(n)} <_{\mathrm{W}} \mathsf{BWT}_k^{(n)}$ *for all* $k \geq 2$, $k = \mathbb{N}$ *and* $n \in \mathbb{N}$,
(5) $\mathsf{WKL}^{(n)} <_{\mathrm{sW}} \lim^{(n)}$ *and* $\mathsf{WKL}^{(n)} <_{\mathrm{W}} \lim^{(n)}$ *for all* $n \in \mathbb{N}$,
(6) $(\mathsf{WKL}')^{[n]} \equiv_{\mathrm{W}} \mathsf{WKL}^{(n)}$ *for all* $n \geq 1$.

We recall that $\langle n, k\rangle := \frac{1}{2}(n+k+1)(n+k)+k$ and inductively this can be extended to finite tuples by $\langle i_{n+1}, i_n, ..., i_0\rangle := \langle i_{n+1}, \langle i_n, ..., i_0\rangle\rangle$ for all $n \geq 1$ and $i_0, ..., i_{n+1} \in \mathbb{N}$. Likewise, we define $\langle p_0, p_1, p_2, ...\rangle \in \mathbb{N}^{\mathbb{N}}$ for $p_i \in \mathbb{N}^{\mathbb{N}}$ by $\langle p_0, p_1, p_2, ...\rangle\langle n, k\rangle := p_n(k)$. While $\lim_X :\subseteq X^{\mathbb{N}} \to X$ denotes the ordinary limit operation for $X \subseteq \mathbb{N}$, we denote by lim the map $\lim :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, \langle p_0, p_1, p_2, ...\rangle \mapsto \lim_{i\to\infty} p_i$, which is essentially the ordinary limit map of Baire space (with respect to the product topology on $\mathbb{N}^{\mathbb{N}}$), but for convenience the input sequence is tupled into one point in Baire space. Correspondingly, we define the limit $\lim_{2^{\mathbb{N}}} :\subseteq 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ on Cantor space $2^{\mathbb{N}}$. By $f^{(n)}$ we denote the *n–fold jump* of f and we denote the *n–fold composition* of f with itself by $f^{[n]}$, i.e., $f^{[0]} = \mathrm{id}$, $f^{[n+1]} := f^{[n]} \circ f$.

3. The Uniform Scenario, Lower Bounds and Products

In this section we plan to introduce the uniform versions of Ramsey's Theorem that we are going to study and we will prove some basic facts about them. While in non-uniform settings such as reverse mathematics [32] certain information on infinite homogeneous sets can just be assumed to be available, we need to make this more explicit. For instance, it turned out to be useful to consider an enriched version $\mathsf{CRT}_{n,k}$ of Ramsey's Theorem that provides additional information on the color of the produced infinite homogeneous set. In [9] the stable version of Ramsey's Theorem $\mathsf{SRT}_{n,k}$ was introduced, which is a restriction of Ramsey's Theorem that we consider too.

We need to introduce some notation first. For every $n \geq 1$ we assume that we use some total standard numbering $\vartheta_n : \mathbb{N} \to [\mathbb{N}]^n$ that can be defined, for instance, by $\vartheta_n\langle i_0, i_1, ..., i_{n-1}\rangle := \left\{k + \sum_{j=0}^k i_j : k = 0, ..., n-1\right\}$ for all $i_0, ..., i_{n-1} \in \mathbb{N}$, i.e., the set $\vartheta_n\langle i_0, i_1, ..., i_{n-1}\rangle$ contains the numbers $i_0 < i_0 + i_1 + 1 < i_0 + i_1 + i_2 + 2 < ... < i_0 + i_1 + i_{n-1} + n - 1$. However, we will not make any technical use of this specific definition of ϑ_n. Occasionally, we use the notation $\{i_0 < i_1 < ... < i_{n-1}\}$ for a set $\{i_0, i_1, ..., i_{n-1}\} \in [\mathbb{N}]^n$ with the additional property that $i_0 < i_1 < ... < i_{n-1}$. By $\mathcal{C}_{n,k}$ we denote the set of colorings $c : [\mathbb{N}]^n \to k$ which is represented via the natural function space representation $[\vartheta_n \to \mathrm{id}_k]$ (or equivalently, we can say that p is a name for c, if $p(i) = c(\vartheta_n(i))$). By $\mathcal{C}_{n,\mathbb{N}}$ we denote the set of all colorings $c : [\mathbb{N}]^n \to \mathbb{N}$ with a finite range, represented by $[\vartheta_n \to \mathrm{id}_{\mathbb{N}}]$. By $\mathcal{H}_c$ we denote the set of infinite homogeneous sets $M \subseteq \mathbb{N}$ for the coloring c. A coloring $c : [\mathbb{N}]^n \to k$ is called *stable*, if $\lim_{i\to\infty} c(A \cup \{i\})$ exists for all $A \in [\mathbb{N}]^{n-1}$. The expression $k \geq a, \mathbb{N}$ with $a \in \mathbb{N}$ is supposed to mean $k \in \{x \in \mathbb{N} : x \geq a\} \cup \{\mathbb{N}\}$.

Definition 3.1 (Uniform variants of Ramsey's Theorem)**.** For all $n \geq 1$ and $k \geq 1, \mathbb{N}$ we define

(1) $\mathsf{RT}_{n,k} : \mathcal{C}_{n,k} \rightrightarrows 2^{\mathbb{N}}, \mathsf{RT}_{n,k}(c) := \mathcal{H}_c$,
(2) $\mathsf{CRT}_{n,k} : \mathcal{C}_{n,k} \rightrightarrows k \times 2^{\mathbb{N}}, \mathsf{CRT}_{n,k}(c) := \{(c(M), M) : M \in \mathcal{H}_c\}$,
(3) $\mathsf{SRT}_{n,k} :\subseteq \mathcal{C}_{n,k} \rightrightarrows 2^{\mathbb{N}}, \mathsf{SRT}_{n,k}(c) := \mathsf{RT}_{n,k}(c)$,
where $\mathrm{dom}(\mathsf{SRT}_{n,k}) := \{c \in \mathcal{C}_{n,k} : c \text{ stable}\}$,
(4) $\mathsf{CSRT}_{n,k} :\subseteq \mathcal{C}_{n,k} \rightrightarrows k \times 2^{\mathbb{N}}, \mathsf{CSRT}_{n,k}(c) := \{(c(M), M) : M \in \mathcal{H}_c\}$,
where $\mathrm{dom}(\mathsf{CSRT}_{n,k}) := \{c \in \mathcal{C}_{n,k} : c \text{ stable}\}$,

(5) $\mathsf{RT}_{n,+} := \bigsqcup_{k\geq 1}\mathsf{RT}_{n,k}$, $\mathsf{RT} := \bigsqcup_{n\geq 1}\mathsf{RT}_{n,+}$.

All formalized versions of Ramsey's Theorem mentioned here are well-defined by Ramsey's Theorem 1.1. We call n the *size* of the respective version of Ramsey's Theorem. Here $\mathsf{CRT}_{n,k}$ enriches $\mathsf{RT}_{n,k}$ by the information on the color of the resulting infinite homogeneous set and $\mathsf{SRT}_{n,k}$ is a restriction of $\mathsf{RT}_{n,k}$ defined only for stable colorings. The coproduct $\mathsf{RT}_{n,+}$ as well as $\mathsf{RT}_{n,\mathbb{N}}$ can both be seen as different uniform versions of the principle $(\forall k)\ \mathsf{RT}_{n,k}$ that is usually denoted by $\mathsf{RT}^n_{<\infty}$ in reverse mathematics (see, e.g., [17]). In case of $\mathsf{RT}_{n,\mathbb{N}}$ the finite number of colors is left unspecified, whereas in case of $\mathsf{RT}_{n,+}$, the number of colors is an input parameter.

We emphasize that we use Cantor space $2^{\mathbb{N}}$, i.e., the infinite homogeneous sets $M\in\mathcal{H}_c$ are represented via their characteristic functions $\chi_M:\mathbb{N}\to\{0,1\}$. However, by definition any infinite subset $A\subseteq M$ of an infinite homogeneous set $M\in\mathcal{H}_c$ is in $\mathcal{H}_c$ too and given an enumeration of an infinite set M, we can find a characteristic function χ_A of an infinite subset $A\subseteq M$. This means that we can equivalently think about sets in $2^{\mathbb{N}}$ as being represented via enumerations.[2]

We obtain the following obvious strong reductions between the different versions of Ramsey's Theorem.

Lemma 3.2 (Basic reductions)**.** $\mathsf{SRT}_{n,k}\leq_{\mathrm{sW}}\mathsf{RT}_{n,k}\leq_{\mathrm{sW}}\mathsf{CRT}_{n,k}$ *and* $\mathsf{SRT}_{n,k}\leq_{\mathrm{sW}}\mathsf{CSRT}_{n,k}\leq_{\mathrm{sW}}\mathsf{CRT}_{n,k}$ *for all $n\geq 1$ and $k\geq 1,\mathbb{N}$.*

We note that the colored versions of Ramsey's Theorem are Weihrauch equivalent to the corresponding uncolored versions. This is because given a coloring c and an infinite homogeneous set $M\in\mathcal{H}_c$, we can easily compute $c(M)$ by choosing some points $i_0<i_1<...<i_{n-1}$ in M and by computing $c\{i_0,i_1,...,i_{n-1}\}$. Hence, we obtain the following corollary.

Corollary 3.3. $\mathsf{RT}_{n,k}\equiv_{\mathrm{W}}\mathsf{CRT}_{n,k}$ *and* $\mathsf{SRT}_{n,k}\equiv_{\mathrm{W}}\mathsf{CSRT}_{n,k}$ *for all $n\geq 1$, $k\geq 1,\mathbb{N}$.*

The diagram in Figure 2 illustrates the situation and it displays further information on lower and upper bounds that is justified by proofs that we will only provide step by step. The diagram illustrates the non-trivial case $n\geq 2$, $k\geq 2,\mathbb{N}$ whereas the bottom cases where $n=1$ or $k=1$ are described in the following result.

Proposition 3.4 (The bottom cases)**.** *Let $n\geq 1$ and $k\geq 2,\mathbb{N}$. Then we obtain*

(1) $\mathsf{CSRT}_{n,1}\equiv_{\mathrm{sW}}\mathsf{SRT}_{n,1}\equiv_{\mathrm{sW}}\mathsf{RT}_{n,1}\equiv_{\mathrm{sW}}\mathsf{CRT}_{n,1}$ *are computable and not cylinders,*

(2) $\lim_k\equiv_{\mathrm{W}}\mathsf{SRT}_{1,k}<_{\mathrm{W}}\mathsf{BWT}_k\equiv_{\mathrm{W}}\mathsf{RT}_{1,k}$,

(3) $\lim_k\leq_{\mathrm{sW}}\mathsf{CSRT}_{1,k}$ *and* $\mathsf{BWT}_k\leq_{\mathrm{sW}}\mathsf{CRT}_{1,k}$.

Proof. We note that $\mathsf{CSRT}_{n,1},\mathsf{SRT}_{n,1},\mathsf{RT}_{n,1}$ and $\mathsf{CRT}_{n,1}$ can yield any infinite subset of $\mathbb{N}$ (together with the color 1 in case of $\mathsf{CSRT}_{n,1},\mathsf{CRT}_{n,1}$) and hence they are all constant as multi-valued functions, hence computable and strongly equivalent to each other. Since the identity cannot be strongly reduced to constant multi-valued problems, it follows that all the aforementioned problems are not cylinders. It is easy to see that $\lim_k\equiv_{\mathrm{W}}\mathsf{SRT}_{1,k}$ and $\mathsf{BWT}_k\equiv_{\mathrm{W}}\mathsf{RT}_{1,k}$, whereas $\lim_k<_{\mathrm{W}}\mathsf{BWT}_k$ is given by Fact 2.1(4). The strong reductions $\lim_k\leq_{\mathrm{sW}}\mathsf{CSRT}_{1,k}$ and $\mathsf{BWT}_k\leq_{\mathrm{sW}}\mathsf{CRT}_{1,k}$ are also clear. □

This result demonstrates that the Ramsey Theorem for size 1 and k colors $\mathsf{RT}_{1,k}$ is equivalent to the Bolzano-Weierstraß Theorem for k colors BWT_k and hence the general Ramsey Theorem can be seen as a generalization of the Bolzano-Weierstraß

[2]More formally, we could equivalently consider the output space of $\mathsf{RT}_{n,k}$ and its variants as $\mathcal{A}_+(\mathbb{N})$ or $\mathcal{A}(\mathbb{N})$, i.e., as space of subsets of $\mathbb{N}$ equipped with positive or full information, respectively, which corresponds topologically to the lower Fell topology and the Fell topology, respectively.

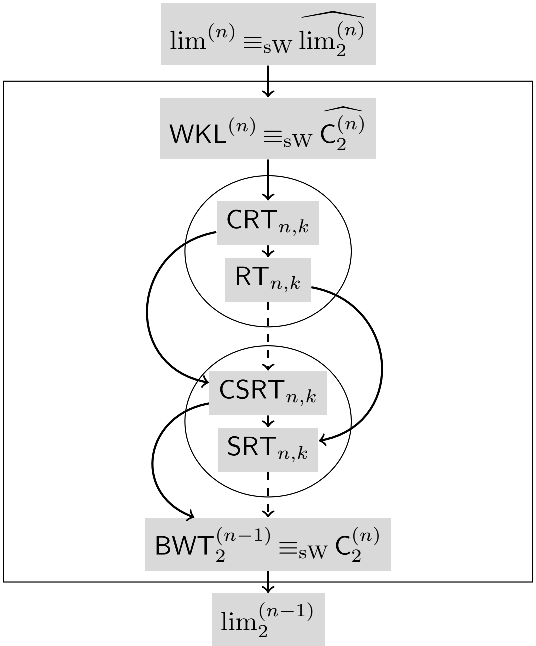

FIGURE 2. The degree of Ramsey's Theorem for fixed $n \geq 2$ and $k \geq 2, \mathbb{N}$ in the Weihrauch lattice: all solid arrows indicate strong Weihrauch reductions against the direction of the arrow, all dashed arrows indicate ordinary Weihrauch reductions; the circle indicates what falls into a single Weihrauch degree and the square box indicates what falls into a single parallelized Weihrauch degree.

Theorem. Next we want to prove the following interesting lower bound on the complexity of Ramsey's Theorem.

Theorem 3.5 (Lower bound)**.** $\mathsf{C}_2^{(n)} \leq_{\mathrm{sW}} \mathsf{CSRT}_{n,2}$ *for all* $n \geq 2$.

Proof. We note that by Fact 2.1 $\mathsf{C}_2^{(n)} \equiv_{\mathrm{sW}} \mathsf{BWT}_2^{(n-1)} \equiv_{\mathrm{sW}} \mathsf{BWT}_2 \circ \lim_{2^{\mathbb{N}}}^{[n-1]}$. Let $p \in \mathrm{dom}(\mathsf{BWT}_2 \circ \lim_{2^{\mathbb{N}}}^{[n-1]})$ and $q := \lim_{2^{\mathbb{N}}}^{[n-1]}(p)$. Then

$$q(i_0) = \lim_{i_1 \to \infty} \lim_{i_2 \to \infty} ... \lim_{i_{n-1} \to \infty} p\langle i_{n-1}, ..., i_0 \rangle$$

for all $i_0 \in \mathbb{N}$. We compute the coloring $c : [\mathbb{N}]^n \to 2$ with

$$c\{i_0 < i_1 < .... < i_{n-1}\} := p\langle i_{n-1}, i_{n-2}, ..., i_1, i_0 \rangle.$$

It is clear that c is a stable coloring and with the help of $\mathsf{CSRT}_{n,2}$ we can compute $(c(M), M) \in \mathsf{CSRT}_{n,2}(c)$. We claim that $c(M) \in \mathsf{BWT}_2(q)$. This proves $\mathsf{BWT}_2^{(n-1)} \leq_{\mathrm{sW}} \mathsf{CSRT}_{n,2}$. We still need to prove the claim.
<u>1. Case:</u> q is a convergent binary sequence, i.e., $x := \lim_2(q) \in \{0, 1\}$ exists.
In this case $q(i_0) = x$ for all sufficiently large i_0 and $\mathsf{BWT}_2(q) = \{x\}$. Since M is infinite, there will be such a sufficiently large $i_0 \in M$. Since $q = \lim_{2^{\mathbb{N}}}^{[n-1]}(p)$ it follows that there will be sufficiently large $i_{n-1} > ... > i_1 > i_0$ in M such that $p\langle i_{n-1}, i_{n-2}, ..., i_1, i_0 \rangle = x$ and hence $c(M) = x \in \{x\} = \mathsf{BWT}_2(q)$.
<u>2. Case:</u> q is not a convergent binary sequence.
In this case $\mathsf{BWT}_2(q) = \{0, 1\}$ and $c(M) \in \mathsf{BWT}_2(q)$ holds automatically. $\square$

It is clear that Ramsey's Theorem for any number of colors k can be reduced to Ramsey's Theorem for any higher number of colors. This is because any coloring

$c : [\mathbb{N}]^n \to k$ can be seen as a coloring for any number $m \geq k$ of colors and this idea applies to all versions of Ramsey's Theorem that we have considered.

Lemma 3.6 (Increasing colors)**.** *For all $n, k \geq 1$ we obtain:*

(1) $\mathsf{SRT}_{n,k} \leq_{\mathrm{sW}} \mathsf{SRT}_{n,k+1} \leq_{\mathrm{sW}} \mathsf{SRT}_{n,\mathbb{N}}$,
(2) $\mathsf{CSRT}_{n,k} \leq_{\mathrm{sW}} \mathsf{CSRT}_{n,k+1} \leq_{\mathrm{sW}} \mathsf{CSRT}_{n,\mathbb{N}}$,
(3) $\mathsf{RT}_{n,k} \leq_{\mathrm{sW}} \mathsf{RT}_{n,k+1} \leq_{\mathrm{sW}} \mathsf{RT}_{n,\mathbb{N}}$,
(4) $\mathsf{CRT}_{n,k} \leq_{\mathrm{sW}} \mathsf{CRT}_{n,k+1} \leq_{\mathrm{sW}} \mathsf{CRT}_{n,\mathbb{N}}$.

We will later come back to the question whether these reductions are strict. In particular, with Theorem 3.5 and Lemma 3.6 we have now established the lower bound that is indicated in the diagram in Figure 2.

Corollary 3.7 (Lower bound)**.** $\mathsf{BWT}_2^{(n-1)} \leq_{\mathrm{sW}} \mathsf{CSRT}_{n,k}$ *for all $n \geq 2$, $k \geq 2, \mathbb{N}$ and* $\lim_2^{(n-1)} \leq_{\mathrm{sW}} \mathsf{CSRT}_{n,k}$ *for all $n \geq 1$ and $k \geq 2, \mathbb{N}$.*

The second statement follows from the first one by Fact 2.1 in case of $n \geq 2$ and follows from Proposition 3.4 in case of $n = 1$. We note that by Corollary 3.15 below $\mathsf{CSRT}_{n,k}$ cannot be replaced by $\mathsf{SRT}_{n,k}$ in this result. It follows from the Theorem of Cholak, Jockusch and Slaman 1.5 that the binary limit $\lim_2$ cannot be replaced by the limit on Baire space in the previous result.

Theorem 3.8 (Limit avoidance)**.** $\lim^{(n-1)} \not\leq_{\mathrm{W}} \lim * \mathsf{RT}_{n,\mathbb{N}}$ *for all $n \geq 2$.*

Proof. Let us assume that $\lim^{(n-1)} \leq_{\mathrm{W}} \lim * \mathsf{RT}_{n,\mathbb{N}}$ holds for some $n \geq 2$. Then $\lim^{(n-1)}$ maps some computable $p \in \mathbb{N}^{\mathbb{N}}$ to $\emptyset^{(n)}$ and the reduction maps p to a computable coloring $c : [\mathbb{N}]^n \to \mathbb{N}$ with finite range(c). By the Theorem of Cholak, Jockusch and Slaman 1.5 there exists an infinite homogeneous set $M \subseteq \mathbb{N}$ for c such that $\emptyset^{(n)} \not\leq_{\mathrm{T}} M'$. Now the assumption is that there is a limit computation performed on p and M that produces $\emptyset^{(n)}$. But any result produced by such a limit computation can also be computed from M' since p is computable (and for all computable functions $F, G :\subseteq \mathbb{N} \to \mathbb{N}$ there is a computable function $H :\subseteq \mathbb{N} \to \mathbb{N}$ such that $F \circ \lim \circ G = H \circ \mathsf{J}$). Hence, $\emptyset^{(n)} \leq_{\mathrm{T}} M'$, which is a contradiction. $\square$

We obtain the following corollary since $\lim * \lim^{(n-2)} \equiv_{\mathrm{W}} \lim^{(n-1)}$.

Corollary 3.9 (Limit avoidance)**.** $\lim^{(n-2)} \not\leq_{\mathrm{W}} \mathsf{RT}_{n,\mathbb{N}}$ *for all $n \geq 2$.*

Since $\widehat{\lim_2} \equiv_{\mathrm{sW}} \lim$ by Fact 2.1, we can conclude from Corollaries 3.9 and 3.7 that Ramsey's Theorem is not parallelizable.

Corollary 3.10 (Parallelizability)**.** $\mathsf{RT}_{n,k}$ *and* $\mathsf{SRT}_{n,k}$ *are not parallelizable for all $n \geq 1$ and $k \geq 2, \mathbb{N}$.*

With the help of the Squashing Theorem 1.7 we can draw the following conclusion [13, Lemma 3.3].

Corollary 3.11 (Idempotency)**.** $\mathsf{RT}_{n,k}$ *is not idempotent for all $n \geq 1$ and $k \geq 2$.*

Another noticeable consequence of Corollary 3.7 is the following.

Corollary 3.12 (Cylinders)**.** $\widehat{\mathsf{CSRT}_{n,k}}$ *and* $\widehat{\mathsf{CRT}_{n,k}}$ *are cylinders for all $n \geq 1$ and $k \geq 2, \mathbb{N}$.*

The claim follows since $\mathrm{id} \leq_{\mathrm{sW}} \lim \equiv_{\mathrm{sW}} \widehat{\lim_2} \leq_{\mathrm{sW}} \widehat{\mathsf{CSRT}_{n,k}}$ holds by Corollary 3.7 and Fact 2.1. The next lemma now formulates a simple finiteness condition that Ramsey's Theorem satisfies as a consequence of which Ramsey's Theorem has very little uniform computational power.

Lemma 3.13 (Finite Intersection Lemma)**.** *Let $c_i : [\mathbb{N}]^n \to k$ be colorings for $i = 1, ..., m$ with $m, n \geq 1$, $k \geq 1, \mathbb{N}$. Then we obtain $\bigcap_{i=1}^m \mathsf{RT}_{n,k}(c_i) \neq \emptyset$.*

Proof. We first consider $k \geq 1$. We use a bijection $\alpha : \{0, 1, ..., k-1\}^m \to \{0, 1, ..., k^m - 1\}$ in order to construct a map $f : (\mathcal{C}_{n,k})^m \to \mathcal{C}_{n,k^m}$ by

$$f(c_1, ..., c_m)(A) := \alpha(c_1(A), ..., c_m(A))$$

for all colorings $c_1, ..., c_m \in \mathcal{C}_{n,k}$ and $A \in [\mathbb{N}]^n$. Given $c_1, ..., c_m \in \mathcal{C}_{n,k}$ we consider $c := f(c_1, ..., c_m)$ and we claim that $\mathsf{RT}_{n,k^m}(c) \subseteq \bigcap_{i=1}^m \mathsf{RT}_{n,k}(c_i)$. To this end, let $M \in \mathsf{RT}_{n,k^m}(c)$ and $x := c(M)$. If $(x_1, ..., x_m) := \alpha^{-1}(x)$, then we obtain $c_i(A) = x_i$ for all $i = 1, ..., m$ and $A \in [M]^n$ and hence M is homogeneous for all $c_1, ..., c_m$. This proves the claim. It follows by Ramsey's Theorem 1.1 that $\mathsf{RT}_{n,k^m}(c) \neq \emptyset$.

We now consider the case $k = \mathbb{N}$. In this case we use Cantor's tupling function $\alpha : \mathbb{N}^m \to \mathbb{N}$ in order to construct a map $f : (\mathcal{C}_{n,\mathbb{N}})^m \to \mathcal{C}_{n,\mathbb{N}}$ analogously as above. We obtain $\mathsf{RT}_{n,\mathbb{N}}(c) \subseteq \bigcap_{i=1}^m \mathsf{RT}_{n,\mathbb{N}}(c_i)$. It follows by Ramsey's Theorem 1.1 that $\mathsf{RT}_{n,\mathbb{N}}(c) \neq \emptyset$.

In any case this proves the claim of the lemma. ☐

We recall that for a multi-valued function $f :\subseteq X \rightrightarrows Y$ we have defined $\#f$, the *cardinality* of f, in [6] as the supremum of all cardinalities of sets $M \subseteq \mathrm{dom}(f)$ such that $\{f(x) : x \in M\}$ contains only pairwise disjoint sets. The cardinality yields an invariant for strong Weihrauch reducibility, i.e., $f \leq_{\mathrm{sW}} g$ implies $\#f \leq \#g$ [6, Proposition 3.6]. Lemma 3.13 now implies the following perhaps surprising fact.

Corollary 3.14 (Cardinality)**.** $\#\mathsf{RT}_{n,k}^{(m)} = \#\widehat{\mathsf{RT}_{n,k}^{(m)}} = 1$ *for all $n \geq 1$, $k \geq 1, \mathbb{N}$ and $m \geq 0$.*

In case of the parallelization we only need to apply Lemma 3.13 pairwise to any component of the sequence, which requires the Axiom of Countable Choice. We recall that a multi-valued function f was called *discriminative* in [8] if $\mathsf{LLPO} \leq_{\mathrm{W}} f$ and *indiscriminative* otherwise. Likewise, we could call f *strongly discriminative* if $\mathsf{LLPO} \leq_{\mathrm{sW}} f$. Since $\#\mathsf{LLPO} = 2$ it follows that Corollary 3.14 implies in particular that Ramsey's Theorem is not strongly discriminative, not even in its parallelized form.

Corollary 3.15 (Strong discrimination)**.** $\mathsf{LLPO} \equiv_{\mathrm{sW}} \mathsf{C}_2 \not\leq_{\mathrm{sW}} \widehat{\mathsf{RT}_{n,k}^{(m)}}$ *for all $n \geq 1$, $k \geq 1, \mathbb{N}$ and $m \geq 0$.*

Since $\mathsf{C}_2' \equiv_{\mathrm{sW}} \mathsf{BWT}_2$ by Fact 2.1, we can conclude that CSRT cannot be replaced by SRT in Theorem 3.5 and Corollary 3.7 (without simultaneously replacing the strong reduction by an ordinary one).[3] We can also conclude from Corollary 3.14 that the parallelized uncolored versions of Ramsey's Theorem are not cylinders since $\#\mathrm{id} = |\mathbb{N}^\mathbb{N}|$.

Corollary 3.16 (Cylinders)**.** $\widehat{\mathsf{RT}_{n,k}^{(m)}}$ *and* $\widehat{\mathsf{SRT}_{n,k}^{(m)}}$ *are not cylinders for all $n \geq 1$, $k \geq 1, \mathbb{N}$ and $m \geq 0$.*

Since $\#\mathsf{CSRT}_{n,k} \geq k$ holds (because there are monochromatic colorings for each color), we can also conclude from Corollary 3.14 that the colored versions of Ramsey's Theorem are not strongly equivalent to the uncolored ones (in case of at least two colors).

Corollary 3.17. $\mathsf{CSRT}_{n,k} \not\leq_{\mathrm{sW}} \widehat{\mathsf{RT}_{n,k}^{(m)}}$ *for all $n \geq 1$, $k \geq 2, \mathbb{N}$ and $m \geq 0$.*

[3] It follows from Corollary 5.25 below that in Corollary 3.15 LLPO cannot be replaced by $\mathsf{ACC}_\mathbb{N}$, as defined in [8].

The following result is a consequence of Lemma 3.13 and its proof. For a sequence $(f_i)_i$ of multi-valued functions $f_i :\subseteq X \rightrightarrows Y$ we denote the *intersection* by $\bigcap_{i=1}^m f_i :\subseteq X \rightrightarrows Y, x \mapsto \bigcap_{i=1}^m f_i(x)$, where $\mathrm{dom}(\bigcap_{i=1}^m f_i)$ contains all points $x \in X$ such that $\bigcap_{i=1}^m f_i(x) \neq \emptyset$.

Corollary 3.18 (Products)**.** *For all $n, m, k \geq 1$ we obtain*

(1) $\mathsf{RT}_{n,k}^m \leq_{\mathrm{sW}} \bigcap_{i=1}^m \mathsf{RT}_{n,k} \leq_{\mathrm{sW}} \mathsf{RT}_{n,k^m}$,
(2) $\mathsf{SRT}_{n,k}^m \leq_{\mathrm{sW}} \bigcap_{i=1}^m \mathsf{SRT}_{n,k} \leq_{\mathrm{sW}} \mathsf{SRT}_{n,k^m}$,
(3) $\mathsf{RT}_{n,\mathbb{N}}^m \leq_{\mathrm{sW}} \bigcap_{i=1}^m \mathsf{RT}_{n,\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{RT}_{n,\mathbb{N}}$,
(4) $\mathsf{SRT}_{n,\mathbb{N}}^m \leq_{\mathrm{sW}} \bigcap_{i=1}^m \mathsf{SRT}_{n,\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{SRT}_{n,\mathbb{N}}$.

Proof. The functions $f : (\mathcal{C}_{n,k})^m \to \mathcal{C}_{n,k^m}$ and $f : (\mathcal{C}_{n,\mathbb{N}})^m \to \mathcal{C}_{n,\mathbb{N}}$ constructed in the proof of Lemma 3.13 are computable and hence they yield the reductions $\bigcap_{i=1}^m \mathsf{RT}_{n,k} \leq_{\mathrm{sW}} \mathsf{RT}_{n,k^m}$ and $\bigcap_{i=1}^m \mathsf{RT}_{n,\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{RT}_{n,\mathbb{N}}$, respectively. Additionally, both maps f have the property that they map stable colorings $c_1, ..., c_m$ to stable colorings $c := f(c_1, ..., c_m)$, hence they also yield the corresponding reductions in the stable case. The other reductions follow from Lemma 3.13. $\square$

We note that by [13, Proposition 2.1] we also have $\mathsf{RT}_{n,k} \times \mathsf{RT}_{n,l} \leq_{\mathrm{sW}} \mathsf{RT}_{n,kl}$ for all $n, k, l \geq 1$. We also note that the following result is implicitly included in [13, Section 1] (the proof given there is for the case $l = 1$ and $k = 2$ and can be generalized straightforwardly).

Proposition 3.19 (Compositional products)**.** $\mathsf{RT}_{n,k+l} \leq_{\mathrm{W}} \mathsf{RT}_{n,k} * \mathsf{RT}_{n,l+1}$ *for all $n, k, l \geq 1$.*

From Corollary 3.18 we can directly conclude that Ramsey's Theorem for an unspecified finite number of colors is idempotent.

Corollary 3.20 (Idempotency)**.** $\mathsf{RT}_{n,\mathbb{N}}$ *and* $\mathsf{SRT}_{n,\mathbb{N}}$ *are idempotent for all $n \geq 1$.*

Since the sequence $(f_m)_m$ of maps $f_m : (\mathcal{C}_{n,\mathbb{N}})^m \to \mathcal{C}_{n,\mathbb{N}}$ from the proof of Corollary 3.18 is uniformly computable, we obtain the following corollary.

Corollary 3.21 (Finite parallelization)**.** $\mathsf{RT}_{n,k}^* \leq_{\mathrm{W}} \mathsf{RT}_{n,+}$ *and* $\mathsf{SRT}_{n,k}^* \leq_{\mathrm{W}} \mathsf{SRT}_{n,+}$ *for all $n \geq 1$ and $k \geq 1, \mathbb{N}$.*

We note that $\mathsf{RT}_{n,k}^* = \bigsqcup_{m \geq 0} \mathsf{RT}_{n,k}^m$, where $\mathsf{RT}_{n,k}^0 = \mathrm{id}$ and hence we obtain only an ordinary Weihrauch reduction in the previous result. Corollary 3.21 leads to the obvious question, whether additional factors can make up for color increases.

Question 3.22 (Colors and factors)**.** Does $\mathsf{RT}_{n,k}^* \equiv_{\mathrm{W}} \mathsf{RT}_{n,+}$ hold for $n, k \geq 2$?

We note that the equivalence is known to hold in case of $n = 1$. In [28, Theorem 32] it was proved that $\mathsf{C}_{n+1} \leq_{\mathrm{sW}} \mathsf{C}_2^n$ holds for all $n \geq 1$ (only "$\leq_{\mathrm{W}}$" was claimed but the proof shows "$\leq_{\mathrm{sW}}$"). This implies $\mathsf{BWT}_{n+1} \leq_{\mathrm{sW}} \mathsf{BWT}_2^n$ and we obtain the following.

Proposition 3.23 (Colors and factors)**.** $\mathsf{RT}_{1,n+1} \leq_{\mathrm{W}} \mathsf{RT}_{1,2}^n$ *for all $n \geq 1$ and, in particular,* $\mathsf{RT}_{1,2}^* \equiv_{\mathrm{W}} \mathsf{RT}_{1,+}$.

As a consequence of the next result we obtain that any unspecified finite number of colors can be reduced to two colors for the price of an increase of the size. Simultaneously, the following theorem also gives us a handle to show that the complexity of Ramsey's Theorem increases with increasing numbers of colors.

Theorem 3.24 (Products)**.** $\mathsf{RT}_{n,\mathbb{N}} \times \mathsf{RT}_{n+1,k} \leq_{\mathrm{sW}} \mathsf{RT}_{n+1,k+1}$ *and* $\mathsf{SRT}_{n,\mathbb{N}} \times \mathsf{SRT}_{n+1,k} \leq_{\mathrm{sW}} \mathsf{SRT}_{n+1,k+1}$ *for all $n \geq 1$ and $k \geq 1, \mathbb{N}$.*

Proof. We start with the first reduction. Given a coloring $c_1 : [\mathbb{N}]^n \to \mathbb{N}$ with finite range and a coloring $c_2 : [\mathbb{N}]^{n+1} \to k$ we construct a coloring $c^+ : [\mathbb{N}]^{n+1} \to k+1$ as follows:

$$c^+(A) := \begin{cases} c_2(A) & \text{if } A \text{ is homogeneous for } c_1 \\ k & \text{otherwise} \end{cases}$$

for all $A \in [\mathbb{N}]^{n+1}$. Let $M \in \mathsf{RT}_{n+1,k+1}(c^+)$ and let $p : \mathbb{N} \to \mathbb{N}$ be the principal function of M. Then we define a coloring $c_p : [\mathbb{N}]^n \to \mathbb{N}$ by $c_p(A) := c_1(p(A))$ for all $A \in [\mathbb{N}]^n$. By construction, c_p has finite range too. Let $B \in \mathsf{RT}_{n,\mathbb{N}}(c_p)$. Then $p(B)$ is homogeneous for c_1 and $p(B) \subseteq M$. Hence any $A \in [p(B)]^{n+1}$ is also homogeneous for c_1, which implies $c^+(A) = c_2(A)$ and hence $c^+(M) = c_2(A) < k$. This implies $M \in \mathsf{RT}_{n+1,k}(c_2)$ and all $A \in [M]^{n+1}$ are homogeneous for c_1. We claim that this implies $M \in \mathsf{RT}_{n,\mathbb{N}}(c_1)$. The claim yields $\mathsf{RT}_{n+1,2}(c^+) \subseteq \mathsf{RT}_{n,\mathbb{N}}(c_1) \cap \mathsf{RT}_{n+1,k}(c_2)$ and hence the desired reduction follows.

We still need to prove the claim. To this end we show that all $A \in [M]^{n+1}$ are homogeneous for c_1 with respect to one fixed color. Firstly, we note that for every two $A, B \in [M]^{n+1}$ there is a finite sequence $A_1, ..., A_l \in [M]^{n+1}$ such that $A_1 = A$, $A_l = B$ and $|A_i \cap A_{i+1}| \geq n$ for all $i = 1, ..., l-1$. This is because each element of A can be replaced step by step by one element of B. Now we fix some $i \in \{1, ..., l-1\}$. Since A_i and A_{i+1} are homogeneous for c_1 by assumption and they share an n–element subset, it is clear that $c_1(A_i) = c_1(A_{i+1})$. Since this holds for all $i \in \{1, ..., l-1\}$, we obtain $c_1(A) = c_1(B)$. This means that all $A \in [M]^{n+1}$ are homogeneous for c_1 with respect to the same fixed color and hence, in particular, all $A \in [M]^n$ share the same color with respect to c_1, i.e., $M \in \mathsf{RT}_{n,\mathbb{N}}(c_1)$.

Now we still show that the same construction also proves the second reduction regarding stable colorings. For this it suffices to show that $c^+ : [\mathbb{N}]^{n+1} \to k+1$ is stable for all stable colorings $c_1 : [\mathbb{N}]^n \to \mathbb{N}$ and $c_2 : [\mathbb{N}]^{n+1} \to k$. For this purpose let c_1, c_2 be stable and let $A \in [\mathbb{N}]^n$. Then $[A]^{n-1} = \{B_0, B_1, ..., B_{n-1}\}$ and for each $i = 0, ..., n-1$ there is some $l_i \geq \max(B_i)$ and some $x_i \in \mathbb{N}$ such that $c_1(B_i \cup \{j\}) = x_i$ for $j > l_i$, since c_1 is stable. There is also some $l_n \geq \max(A)$ and some $x_n \in k$ such that $c_2(A \cup \{j\}) = x_n$ for $j > l_n$, since c_2 is stable. Let $l := \max\{l_0, ..., l_{n-1}, l_n\}$. Then

$$c^+(A \cup \{j\}) = \begin{cases} c_2(A \cup \{j\}) & \text{if } c_1(A) = x_0 = x_1 = ... = x_{n-1} \\ k & \text{otherwise} \end{cases}$$

for all $j > l$. Hence c^+ is stable. □

We note that in case of $k = 1$ we get the following corollary.

Corollary 3.25 (Color reduction)**.** $\mathsf{RT}_{n,\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{RT}_{n+1,2}$ *and* $\mathsf{SRT}_{n,\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{SRT}_{n+1,2}$ *for all* $n \geq 1$.

We note that the increase of the size from n to $n+1$ in this corollary is necessary in case of RT by Corollaries 3.11 and 3.21.

We mention that the proof of Theorem 3.24 also shows that the coloring c^+ constructed therein has only infinite homogeneous sets of colors other than k. In case of $k = 1$ this means that only infinite homogeneous sets of color 0 occur. We denote by $0\text{-}\mathsf{SRT}_{n+1,2}$ the stable version $\mathsf{SRT}_{n+1,2}$ of Ramsey's Theorem restricted to colorings that only admit infinite homogeneous sets of color 0. We obtain the following corollary.

Corollary 3.26 (Color reduction)**.** $\mathsf{SRT}_{n,\mathbb{N}} \leq_{\mathrm{sW}} 0\text{-}\mathsf{SRT}_{n+1,2}$ *for all* $n \geq 1$.

With the help of Proposition 3.4(2) in the case $k = \mathbb{N}$ we obtain the following corollary of Corollary 3.25.

Corollary 3.27 (Discrete lower bounds)**.** $\lim_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{SRT}_{2,2}$ *and* $\mathsf{BWT}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{RT}_{2,2}$.

In order to accommodate the full parallelization of Ramsey's Theorem, we even need to increase the size by 2. The next result shows that this is sufficient and it uses the ideas that have already been applied in the proofs of Corollary 3.18 and Theorem 3.24.

Theorem 3.28 (Delayed Parallelization)**.** $\widehat{\mathsf{RT}_{n,k}} \leq_{\mathrm{sW}} \mathsf{RT}_{n+2,2}$ *and* $\widehat{\mathsf{SRT}_{n,k}} \leq_{\mathrm{sW}} \mathsf{SRT}_{n+2,2}$ *for all* $n \geq 1$ *and* $k \geq 1, \mathbb{N}$.

Proof. We start with the reduction $\widehat{\mathsf{RT}_{n,k}} \leq_{\mathrm{sW}} \mathsf{RT}_{n+2,2}$ for $k \geq 1$. Given a sequence $(c_i)_i$ of colorings $c_i : [\mathbb{N}]^n \to k$, we want to determine infinite homogeneous sets M_i for all of them in parallel, using $\mathsf{RT}_{n+2,2}$. The sequence $(f_m)_m$ of functions $f_m : (\mathcal{C}_{n,k})^m \to \mathcal{C}_{n,k^m}$, defined as in the proof of Lemma 3.13, is computable and we use it to compute a sequence $(d_m)_m$ of colorings $d_m \in \mathcal{C}_{n,k^m}$ by $d_m := f_m(c_0, ..., c_{m-1})$. Given the sequence $(d_m)_m$, we can compute a sequence $(d_m^+)_m$ of colorings $d_m^+ : [\mathbb{N}]^{n+1} \to 2$ by

$$d_m^+(A) := \begin{cases} 0 & \text{if } A \text{ is homogeneous for } d_m \\ 1 & \text{otherwise} \end{cases}$$

for all $A \in [\mathbb{N}]^{n+1}$. Now, in a final step we compute a coloring $c : [\mathbb{N}]^{n+2} \to 2$ with

$$c(\{m\} \cup A) := d_m^+(A)$$

for all $A \in [\mathbb{N}]^{n+1}$ and $m < \min(A)$. Given an infinite homogeneous set $M \in \mathsf{RT}_{n+2,2}(c)$ we determine a sequence $(M_i)_i$ as follows: for each fixed $i \in \mathbb{N}$ we first search for a number $m > i$ in M and then we let $M_i := \{x \in M : x > m\}$. It follows from the definition of c that M_i is homogeneous for d_m^+ and following the reasoning in the proof of Theorem 3.24, we obtain that M_i is also homogeneous for d_m. Following the reasoning in the proof of Lemma 3.13, we finally conclude that $M_i \in \bigcap_{j=0}^{m-1} \mathsf{RT}_{n,k}(c_j)$, hence, in particular, $M_i \in \mathsf{RT}_{n,k}(c_i)$, which was to be proved.

We note that the entire construction preserves stability. As shown in the proof of Corollary 3.18, the function f preserves stability. Hence, given a sequence $(c_i)_i$ of stable colorings, also the sequence $(d_m)_m$ consists of stable colorings. Likewise, it was shown in the proof of Theorem 3.24 that in this case also the sequence $(d_m^+)_m$ consists of stable colorings. It follows immediately from the construction of c that also c is stable, since

$$\lim_{j\to\infty} c(\{m\} \cup A \cup \{j\}) = \lim_{j\to\infty} d_m(A \cup \{j\})$$

for all $A \in [\mathbb{N}]^n$ and $m < \min(A)$. Altogether, this proves $\widehat{\mathsf{SRT}_{n,k}} \leq_{\mathrm{sW}} \mathsf{SRT}_{n+2,2}$. The case $k = \mathbb{N}$ can be handled analogously. □

Again the observation made after Corollary 3.25 applies: the colorings d_m^+ can only have infinite homogeneous sets of color 0 and hence also c can only have infinite homogeneous sets of color 0. This yields the following corollary.

Corollary 3.29 (Delayed Parallelization)**.** $\widehat{\mathsf{SRT}_{n,k}} \leq_{\mathrm{sW}} 0\text{-}\mathsf{SRT}_{n+2,2}$ *for all* $n \geq 1$ *and* $k \geq 1, \mathbb{N}$.

Since Ramsey's Theorem is not parallelizable by Corollary 3.10, it is clear that some increase in the size is necessary in order to accommodate the parallelization. The following corollary is a consequence of Theorem 3.28, Theorem 3.5, Corollary 3.3, Proposition 3.4 and Fact 2.1.

Corollary 3.30 (Lower bounds)**.** $\lim \leq_{\mathrm{W}} \mathsf{SRT}_{3,2}$, $\mathsf{WKL}' \leq_{\mathrm{W}} \mathsf{RT}_{3,2}$ *and* $\mathsf{WKL}^{(n)} \leq_{\mathrm{W}} \mathsf{SRT}_{n+2,2}$ *for all* $n \geq 2$.

Corollary 3.30 generalizes [18, Corollary 2.3] (see Corollary 5.14). Corollary 3.9 together with Corollary 3.30 show that both corollaries are optimal in the sense that $\lim^{(n-2)}$ cannot be replaced by $\lim^{(n-3)}$ in the statement of the Corollary 3.9 and

$\mathsf{WKL}^{(n)}$ cannot be replaced by $\lim^{(n)}$ in the statement of Corollary 3.30. In particular, $n+2$ in Theorem 3.28 is also optimal and cannot be replaced by $n+1$. However, the following question remains open.

Question 3.31. Does $\mathsf{WKL}' \leq_{\mathrm{W}} \mathsf{SRT}_{3,2}$ hold?

As a combination of Corollaries 3.27 and 3.30 we obtain the following result on Bolzano-Weierstraß cones and limit cones.

Corollary 3.32 (Cones)**.** *$\lim_2 \leq_{\mathrm{sW}} \mathsf{BWT}_2 \leq_{\mathrm{W}} \mathsf{RT}_{1,2}$, $\lim_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{RT}_{2,2}$ and $\lim \leq_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{N}^{\mathbb{N}}} \leq_{\mathrm{W}} \mathsf{RT}_{3,2}$.*

4. Jumps, Increasing Size and Color and Upper Bounds

The purpose of this section is to provide a useful upper bound on Ramsey's Theorem. Simultaneously, we will demonstrate how the complexity of Ramsey's Theorem increases with increasing size. The proof is subdivided into several steps. The first and crucial step made by Theorem 4.3 is interesting by itself and connects the jump of the colored version of Ramsey's Theorem with the stable version of the next higher size. This is one result where the usage of the colored version of Ramsey's Theorem is essential. We start with a result that prepares the first direction of this theorem.

Proposition 4.1 (Jumps)**.** *$\mathsf{CRT}'_{n,k} \leq_{\mathrm{sW}} \mathsf{CSRT}_{n+1,k}$ for all $n \geq 1$ and $k \geq 1, \mathbb{N}$.*

Proof. Let $(c_i)_i$ be a converging sequence of colorings $c_i : [\mathbb{N}]^n \to k$ and let the coloring $c_\infty : [\mathbb{N}]^n \to k$ be its limit. We compute the coloring $c : [\mathbb{N}]^{n+1} \to k$ with

$$c(A \cup \{i\}) := c_i(A)$$

for all $A \in [\mathbb{N}]^n$ and $i > \max(A)$. Then c is stable and we claim that $\mathsf{RT}_{n+1,k}(c) \subseteq \mathsf{RT}_{n,k}(c_\infty)$. To this end, let $M \in \mathsf{RT}_{n+1,k}(c)$ and let $A \in [M]^n$. Since M is infinite and $\lim_{i\to\infty} c(A \cup \{i\}) = c_\infty(A)$, we obtain $c(M) = c_\infty(A)$. Since this holds for all $A \in [M]^n$, we obtain that M is homogeneous for c_∞, i.e., $M \in \mathsf{RT}_{n,k}(c_\infty)$. We note that we also obtain $c_\infty(M) = c(M)$ and hence this proves $\mathsf{CRT}'_{n,k} \leq_{\mathrm{sW}} \mathsf{CSRT}_{n+1,k}$. □

We obtain the following corollary, since the jump operator is monotone with respect to strong reductions.

Corollary 4.2 (Jumps)**.** *$\mathsf{RT}^{(m)}_{n,k} \leq_{\mathrm{W}} \mathsf{SRT}_{n+m,k}$ for all $m, n \geq 1$, $k \geq 1, \mathbb{N}$.*

Now we are prepared to formulate our main result on jumps of Ramsey's Theorem.

Theorem 4.3 (Jumps)**.** *$\mathsf{CRT}'_{n,k} \equiv_{\mathrm{W}} \mathsf{SRT}_{n+1,k}$ for all $n \geq 1$, $k \geq 1, \mathbb{N}$.*

Proof. By Proposition 4.1 and Corollary 3.3 we obtain $\mathsf{CRT}'_{n,k} \leq_{\mathrm{W}} \mathsf{SRT}_{n+1,k}$ and it remains to prove $\mathsf{SRT}_{n+1,k} \leq_{\mathrm{W}} \mathsf{CRT}'_{n,k}$. Let $c : [\mathbb{N}]^{n+1} \to k$ be a stable coloring. Then we can define a sequence $(c_i)_i$ of colorings $c_i : [\mathbb{N}]^n \to k$ by

$$c_i(A) := \begin{cases} c(A \cup \{i\}) & \text{if } i > \max(A) \\ 0 & \text{otherwise} \end{cases}$$

for all $A \in [\mathbb{N}]^n$. Then $(c_i)_i$ is a converging sequence of colorings and let the limit be denoted by $c_\infty : [\mathbb{N}]^n \to k$. With the help of $\mathsf{CRT}_{n,k}$ we can compute $(c_\infty(M_\infty), M_\infty) \in \mathsf{CRT}_{n,k}(c_\infty)$. We will now describe how we can use this set M_∞ together with c_∞ and c in order to computably enumerate an infinite homogeneous set $M \in \mathsf{SRT}_{n+1,k}(c)$. The set $M = \bigcup_{i=0}^\infty M_i$ will be defined inductively using sets $M_i \in [M_\infty]^{n+i}$. We start with choosing some $M_0 \in [M_\infty]^n$. Then we continue in steps $i = 0, 1, 2...$ as follows:

Let us assume that we have $M_i \in [M_\infty]^{n+i}$. For all $A \in [M_i]^n$ we obtain

$$\lim_{j\to\infty} c(A \cup \{j\}) = c_\infty(A) = c_\infty(M_\infty).$$

Hence, we can effectively find an $m > \max(M_i)$ such that $M_{i+1} := M_i \cup \{m\}$ satisfies $c(M_{i+1}) = c_\infty(M_\infty)$.

Since all the sets M_i with $i \geq 1$ are homogeneous sets for c with the same color $c_\infty(M_\infty)$, the set $M := \bigcup_{i=0}^\infty M_i$ is an infinite homogeneous set for c with $c(M) = c_\infty(M_\infty)$. This proves the desired reduction $\mathsf{SRT}_{n+1,k} \leq_{\mathrm{W}} \mathsf{CRT}'_{n,k}$. □

We note that the color provided by $\mathsf{CRT}'_{n,k}$ is not needed if the color of the infinite homogeneous set that is to be constructed is known in advance. Hence, the proof yields also the following result.

Corollary 4.4 (Jumps in case of known color). *$0\text{-}\mathsf{SRT}_{n+1,2} \leq_{\mathrm{W}} \mathsf{RT}'_{n,2}$ for all $n \geq 1$.*

We note that Corollaries 4.4 and 3.26 allow us to improve the bound given in Corollary 3.25 somewhat.

Corollary 4.5 (Color reduction with jumps). *$\mathsf{SRT}_{n,\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{RT}'_{n,2}$ for all $n \geq 1$.*

In particular, this implies $\lim_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{RT}'_{1,2}$ by Proposition 3.4. Likewise the lower bounds established in Corollary 3.30 can be improved with the help of Corollaries 4.4 and 3.29. We collect all these lower bound results in the following corollary.

Corollary 4.6 (Lower bounds with jumps). *$\lim_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{RT}'_{1,2}$, $\lim \leq_{\mathrm{W}} \mathsf{RT}'_{2,2}$ and $\mathsf{WKL}^{(n)} \leq_{\mathrm{W}} \mathsf{RT}'_{n+1,2}$ for all $n \geq 2$.*

Analogously to Question 3.31 the following question remains open.

Question 4.7. Does $\mathsf{WKL}' \leq_{\mathrm{W}} \mathsf{RT}'_{2,2}$ hold?

Roughly speaking, Theorem 4.3 indicates that any increase in the size of Ramsey's Theorem corresponds to a jump. We can also conclude from Corollary 4.2 that Ramsey's Theorem is increasing with respect to increasing size.

Lemma 4.8 (Increasing size). *$\mathsf{SRT}_{n,k} \leq_{\mathrm{W}} \mathsf{RT}_{n,k} \leq_{\mathrm{W}} \mathsf{SRT}_{n+1,k} \leq_{\mathrm{W}} \mathsf{RT}_{n+1,k}$ for all $n \geq 1$, $k \geq 1, \mathbb{N}$.*

Proof. It follows from Lemma 3.2 and Corollaries 4.2 and 3.3 that $\mathsf{SRT}_{n,k} \leq_{\mathrm{W}} \mathsf{RT}_{n,k} \leq_{\mathrm{W}} \mathsf{RT}'_{n,k} \leq_{\mathrm{sW}} \mathsf{CRT}'_{n,k} \leq_{\mathrm{W}} \mathsf{SRT}_{n+1,k} \leq_{\mathrm{W}} \mathsf{RT}_{n+1,k}$. □

We will soon see in Corollary 4.20 and 5.3 that the reductions in this lemma are all strict in certain cases. We note that $\mathsf{CRT}'_{n,k} \leq_{\mathrm{W}} \mathsf{CRT}_{n,k} * \lim$ and the latter problem is very stable and has several useful descriptions.

Lemma 4.9 (Jump of the cylindrification). *For all $n \geq 1$, $k \geq 1, \mathbb{N}$ we obtain $\mathsf{CRT}_{n,k} * \lim \equiv_{\mathrm{W}} \mathsf{RT}_{n,k} * \lim \equiv_{\mathrm{W}} (\mathsf{RT}_{n,k} \times \mathrm{id})' \equiv_{\mathrm{W}} \mathsf{RT}'_{n,k} \times \lim$.*

Since $\mathsf{RT}_{n,k} \times \mathrm{id}$ is the cylindrification of Ramsey's Theorem, this lemma characterizes the jump of the cylindrification of Ramsey's Theorem up to Weihrauch equivalence. In particular, we obtain the following consequence of Theorem 4.3.

Corollary 4.10. *$\mathsf{SRT}_{n+1,k} \leq_{\mathrm{W}} \mathsf{RT}_{n,k} * \lim$ for all $n \geq 1$, $k \geq 1, \mathbb{N}$.*

This result is the crucial step towards our upper bound result. The next step involves a usage of the cohesiveness problem. We recall that a set $A \subseteq \mathbb{N}$ is called *cohesive* for a sequence $(R_i)_i$ of sets $R_i \subseteq \mathbb{N}$ if $A \cap R_i$ or $A \cap (\mathbb{N} \setminus R_i)$ is finite for each $i \in \mathbb{N}$. In other words, up to finitely many exceptions, A is fully included in any R_i or its complement. By $\mathsf{COH} : (2^{\mathbb{N}})^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$ we denote the cohesiveness problem, where $\mathsf{COH}(R_i)$ contains all sets $A \subseteq \mathbb{N}$ that are cohesive for $(R_i)_i$. The uniform computational content of COH has already been studied in [13, 8, 14]. The relevance of the cohesiveness problem for Ramsey's Theorem has originally been noticed by Cholak et al. who proved over RCA_0 that $\mathsf{RT}_{2,2} \iff \mathsf{SRT}_{2,2} \wedge \mathsf{COH}$ [9, Lemma 7.11] (see [10] for a correction). We use the same idea to prove the following, which was in case of $n = k = 2$ also observed by Dorais et al. [13].

Proposition 4.11. $\mathsf{RT}_{n,k} \leq_{\mathrm{W}} \mathsf{SRT}_{n,k} * \mathsf{COH}$ *for all* $n \geq 1$*,* $k \geq 1, \mathbb{N}$*.*

Proof. We fix some $n, k \geq 1$. Given a coloring $c : [\mathbb{N}]^n \to k$ we compute a sequence $(R_i)_i$ of sets $R_i \subseteq \mathbb{N}$ as follows:

$$R_{\langle i,j\rangle} := \{r \geq \max \vartheta_{n-1}(i) : c(\vartheta_{n-1}(i) \cup \{r\}) = j\}$$

for all $i, j \in \mathbb{N}$. With the help of COH we can compute an infinite cohesive set $Y \in \mathsf{COH}(R_i)_i$ for the sequence $(R_i)_i$. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be the principal function of Y. Since Y is cohesive for $(R_i)_i$ and range(c) is finite, it follows that for all $B \in [\mathbb{N}]^{n-1}$ there is some $j \in \mathbb{N}$ such that $c(B \cup \{r\}) = j$ holds for almost all $r \in Y$. Hence, the coloring $c_\sigma : [\mathbb{N}]^n \to k$, defined by

$$c_\sigma(A) := c(\sigma(A))$$

for all $A \in [\mathbb{N}]^n$ is stable. With the help of $\mathsf{SRT}_{n,k}$ we can compute an infinite homogeneous set $M_\sigma \in \mathsf{SRT}_{n,k}(c_\sigma)$. It is clear that $M := \sigma(M_\sigma)$ is an infinite homogeneous set for c. $\square$

Now we can combine Corollary 4.10 with Proposition 4.11 in both possible orders in order to obtain the following result.

Corollary 4.12. *For all* $n \geq 1$*,* $k \geq 1, \mathbb{N}$ *we obtain*

(1) $\mathsf{RT}_{n+1,k} \leq_{\mathrm{W}} \mathsf{RT}_{n,k} * \lim * \mathsf{COH}$*,*
(2) $\mathsf{SRT}_{n+1,k} \leq_{\mathrm{W}} \mathsf{SRT}_{n,k} * \mathsf{COH} * \lim$*.*

The diagram in Figure 3 illustrates the situation. The first bound given by Corol-

$$\begin{tikzcd}
\mathsf{SRT}_{n,k} \arrow[r, bend right, dashed, "\mathsf{COH}"'] & \mathsf{RT}_{n,k} \arrow[l] \arrow[r, bend right, dashed, "\lim"'] & \mathsf{SRT}_{n+1,k} \arrow[l] \arrow[r, bend right, dashed, "\mathsf{COH}"'] & \mathsf{RT}_{n+1,k} \arrow[l]
\end{tikzcd}$$

FIGURE 3. Upper bounds for Ramsey's Theorem for $n \geq 2$, $k \geq 2, \mathbb{N}$: all solid arrows indicated strong Weihrauch reductions against the direction of the arrow, all circled dashed arrows are labeled with an upper bound of the inverse implication.

lary 4.12 is particularly useful, since the following result was proved in [8, Corollary 14.14].

Fact 4.13. $\mathsf{WKL}' \equiv_{\mathrm{W}} \lim * \mathsf{COH}$*.*

Fact 4.13 together with Corollary 4.12 yields the following.

Corollary 4.14 (Induction)**.** $\mathsf{RT}_{n+1,k} \leq_{\mathrm{W}} \mathsf{RT}_{n,k} * \mathsf{WKL}'$ *for all* $n \geq 1$*,* $k \geq 1, \mathbb{N}$*.*

This corollary can be interpreted such that WKL' is sufficient to transfer $\mathsf{RT}_{n,k}$ into $\mathsf{RT}_{n+1,k}$. An obvious question is whether WKL' is minimal with this property.

Corollary 4.14 could also be proved directly using so-called Erdös-Rado trees.[4] It reflects the fact that Ramsey's Theorem can be proved inductively, where the induction base follows from the Bolzano-Weierstraß Theorem for the k–point space by $\mathsf{RT}_{1,k} \equiv_{\mathrm{W}} \mathsf{BWT}_k \leq_{\mathrm{W}} \mathsf{WKL}'$, which holds by Proposition 3.4 and Fact 2.1. Altogether, this means that we can derive the following result from Corollary 4.14 and Fact 2.1(6).

Corollary 4.15 (Upper bound)**.** $\mathsf{RT}_{n,k} \leq_{\mathrm{sW}} \mathsf{WKL}^{(n)}$ *for all* $n \geq 1$*,* $k \geq 1, \mathbb{N}$*.*

[4] This approach has been used in [29, Proposition 6.6.1]; for a proof theoretic analysis of Ramsey's Theorem this method has been applied in [23, 24, 1] and in reverse mathematics it has been used to prove that Ramsey's Theorem is provable over ACA_0; see [16] for the Erdös-Rado method in general.

We note that we get a strong reduction here, since $\mathsf{WKL}^{(n)}$ is a cylinder. The upper bound is tight in terms of the number of jumps and also up to parallelization. Using the upper bound from Corollary 4.15 together with the lower bound from Corollary 3.30 yields the following characterization of Ramsey's Theorem RT.

Corollary 4.16 (Ramsey's Theorem). $\mathsf{RT}\equiv_{\mathrm{W}}\bigsqcup_{n=0}^\infty\lim^{(n)}\equiv_{\mathrm{W}}\bigsqcup_{n=0}^\infty\mathsf{WKL}^{(n)}$.

This degree corresponds to the class ACA_0' in reverse mathematics (see [17, Theorem 6.27]). Since it is clear that $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ is join-irreducible [3, Corollary 5.6] and it is easy to see that $\bigsqcup_{n=0}^\infty\lim^{(n)}\leq_{\mathrm{W}}\mathsf{C}_{\mathbb{N}^\mathbb{N}}$, we obtain the following corollary.

Corollary 4.17. $\mathsf{RT}<_{\mathrm{W}}\mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

Here $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ can be seen as a possible counterpart of ATR_0 in reverse mathematics. Using again Fact 2.1, Corollary 3.3, the lower bound from Corollary 3.7 and the upper bound from Corollary 4.15 we obtain the following characterization of the parallelization of Ramsey's Theorem.

Corollary 4.18 (Parallelization). *$\widehat{\mathsf{RT}_{n,k}}\equiv_{\mathrm{W}}\mathsf{WKL}^{(n)}\equiv_{\mathrm{sW}}\widehat{\mathsf{CRT}_{n,k}}$ for all $n\geq 1$, $k\geq 2,\mathbb{N}$ and $\widehat{\mathsf{SRT}_{n,k}}\equiv_{\mathrm{W}}\mathsf{WKL}^{(n)}\equiv_{\mathrm{sW}}\widehat{\mathsf{CSRT}_{n,k}}$ for all $n\geq 2$, $k\geq 2,\mathbb{N}$.*

We mention that $\widehat{\mathsf{SRT}_{1,k}}\equiv_{\mathrm{W}}\lim\equiv_{\mathrm{sW}}\widehat{\mathsf{CSRT}_{1,k}}$ holds for $k\geq 2,\mathbb{N}$ by Corollary 3.7. We note that in case of the colored versions of Ramsey's Theorem we get strong equivalences since all involved problems are cylinders. By Corollary 3.16 the parallelized uncolored versions of Ramsey's Theorem are not cylinders and hence the Weihrauch equivalences $\equiv_{\mathrm{W}}$ in the previous corollary cannot be replaced by strong ones $\equiv_{\mathrm{sW}}$.

Since $\mathsf{WKL}^{(n)}\leq_{\mathrm{W}}\lim^{(n)}$ and $\mathsf{WKL}^{(n)}\not\leq_{\mathrm{W}}\lim^{(n-1)}$ for all $n\geq 1$ by Fact 2.1 and $\lim^{(n)}$ is $\mathbf{\Sigma}^0_{n+2}$–complete in the (effective) Borel hierarchy by [2, Proposition 9.1], we obtain the following corollary that characterizes the Borel complexity of Ramsey's Theorem.

Corollary 4.19 (Borel complexity). *$\mathsf{SRT}_{n,k}$ and $\mathsf{RT}_{n,k}$ are both effectively $\mathbf{\Sigma}^0_{n+2}$–measurable, but not $\mathbf{\Sigma}^0_{n+1}$–measurable for all $n\geq 2$, $k\geq 2,\mathbb{N}$.*

Both positive statements and the negative statement on $\mathsf{RT}_{n,k}$ also hold for $n=1$. Since $\mathbf{\Sigma}^0_n$–measurability is preserved downwards by Weihrauch reducibility (see [2, Proposition 5.2]), this in turn implies that Ramsey's Theorem actually forms a strictly increasing chain with increasing size.

Corollary 4.20 (Increasing size). *$\mathsf{RT}_{n,k}<_{\mathrm{W}}\mathsf{RT}_{n+1,2}$, $\mathsf{RT}_{n,k}<_{\mathrm{sW}}\mathsf{RT}_{n+1,2}$, $\mathsf{SRT}_{n,k}<_{\mathrm{W}}\mathsf{SRT}_{n+1,2}$ and $\mathsf{SRT}_{n,k}<_{\mathrm{sW}}\mathsf{SRT}_{n+1,2}$ for all $n\geq 1$ and $k\geq 2,\mathbb{N}$.*

Here the positive parts of the reduction hold by Corollary 3.25. We can also draw some conclusions on increasing numbers of colors. In [13, Theorem 3.1] Dorais et al. have proved that $\mathsf{RT}_{n,k}<_{\mathrm{sW}}\mathsf{RT}_{n,k+1}$ holds for all $n,k\geq 1$. Their main tool was a version of the Squashing Theorem 1.7 for strong Weihrauch reducibility. With the help of Theorem 3.24 we can strengthen this result to ordinary Weihrauch reducibility, which answers [13, Question 7.1]. This result was independently obtained by Hirschfeldt and Jockusch [18, Theorem 3.3] and Patey [27, Corollary 3.15].

Theorem 4.21 (Increasing numbers of colors). *$\mathsf{RT}_{n,k}<_{\mathrm{W}}\mathsf{RT}_{n,k+1}$ for all $n,k\geq 1$.*

Proof. Let us assume that $\mathsf{RT}_{n,2}\times\mathsf{RT}_{n+1,k}\leq_{\mathrm{W}}\mathsf{RT}_{n+1,k}$ holds for some $n,k\geq 1$. Then by the Squashing Theorem 1.7 we obtain $\widehat{\mathsf{RT}_{n,2}}\leq_{\mathrm{W}}\mathsf{RT}_{n+1,k}$ and hence by Corollary 4.18

$$\lim{}^{(n-1)}\leq_{\mathrm{W}}\mathsf{WKL}^{(n)}\equiv_{\mathrm{W}}\widehat{\mathsf{RT}_{n,2}}\leq_{\mathrm{W}}\mathsf{RT}_{n+1,k}$$

in contradiction to Corollary 3.9. Hence $\mathsf{RT}_{n,2}\times\mathsf{RT}_{n+1,k}\not\leq_{\mathrm{W}}\mathsf{RT}_{n+1,k}$ for all $n,k\geq 1$. On the other hand, we have $\mathsf{RT}_{n,2}\times\mathsf{RT}_{n+1,k}\leq_{\mathrm{W}}\mathsf{RT}_{n+1,k+1}$ by Theorem 3.24. This implies $\mathsf{RT}_{n+1,k} <_{\mathrm{W}} \mathsf{RT}_{n+1,k+1}$ for all $n, k \geq 1$. The claim $\mathsf{RT}_{1,k} <_{\mathrm{W}} \mathsf{RT}_{1,k+1}$ for all $k \geq 1$ was already known [7, Theorem 13.4] via Proposition 3.4. □

From this result we can also conclude that the two uniform versions of $\mathsf{RT}^n_{<\infty}$ are not equivalent. This is because $\mathsf{RT}_{n,\mathbb{N}}$ and $\mathsf{SRT}_{n,\mathbb{N}}$ are fractals and hence join irreducible by [7, Proposition 2.6] and hence the strictness of the reductions given in the following corollary follows from Theorem 4.21.

Corollary 4.22 (Arbitrary numbers of colors)**.** *$\mathsf{RT}_{n,+} <_{\mathrm{W}} \mathsf{RT}_{n,\mathbb{N}}$ and $\mathsf{SRT}_{n,+} <_{\mathrm{W}} \mathsf{SRT}_{n,\mathbb{N}}$ for all $n \geq 1$.*

Another consequence of the Squashing Theorem 1.7 is the following result that shows that the complexity of Ramsey's Theorem also grows with an increasing number of factors. This generalizes Corollary 3.11.

Proposition 4.23 (Increasing number of factors)**.** *$\mathsf{RT}^m_{n,k} <_{\mathrm{W}} \mathsf{RT}^{m+1}_{n,k}$ for all $n, m \geq 1$ and $k \geq 2$.*

Proof. Let us assume that $\mathsf{RT}^{m+1}_{n,k} \equiv_{\mathrm{W}} \mathsf{RT}_{n,k} \times \mathsf{RT}^m_{n,k} \leq_{\mathrm{W}} \mathsf{RT}^m_{n,k}$ holds for some $n, m \geq 1$ and $k \geq 2$. Then by the Squashing Theorem 1.7 we obtain $\widehat{\mathsf{RT}_{n,k}} \leq_{\mathrm{W}} \mathsf{RT}^m_{n,k}$ and hence by Corollaries 3.21, 3.25 and 4.18 we can conclude

$$\lim^{(n-1)} \leq_{\mathrm{W}} \mathsf{WKL}^{(n)} \equiv_{\mathrm{W}} \widehat{\mathsf{RT}_{n,k}} \leq_{\mathrm{W}} \mathsf{RT}^m_{n,k} \leq_{\mathrm{W}} \mathsf{RT}^*_{n,k} \leq_{\mathrm{W}} \mathsf{RT}_{n+1,2},$$

in contradiction to Corollary 3.9. Hence, $\mathsf{RT}^{m+1}_{n,k} \not\leq_{\mathrm{W}} \mathsf{RT}^m_{n,k}$. □

We note that Question 3.22 remains, whether additional factors can make up for color increases. The upper bound given in Corollary 4.15 also implies the following result on the arithmetic complexity of homogeneous sets with the help of the Uniform Lower Basis Theorem [3, Theorem 8.3].

Corollary 4.24 (Arithmetic complexity)**.** *Every computable sequence $(c_i)_i$ of colorings $c_i : [\mathbb{N}]^n \to k$ for $n \geq 1$ and $k \geq 2$ admits a sequence $(M_i)_i$ such that $\langle M_0, M_1, ...\rangle' \leq_{\mathrm{T}} \emptyset^{(n+1)}$, and such that M_i is an infinite homogeneous set for c_i for each $i \in \mathbb{N}$.*

The condition $\langle M_0, M_1, ...\rangle' \leq_{\mathrm{T}} \emptyset^{(n+1)}$ cannot be replaced by "computable relative to $\emptyset^{(n)}$" by Corollary 4.18 and in this sense this result is optimal. Hence, we have a striking difference in the arithmetic complexity between the non-uniform Ramsey Theorem as witnessed by Theorems 1.3 and 1.6 and the uniform sequential version of Ramsey's Theorem for sequences as witnessed by Corollary 4.24. This yields another proof of Corollary 3.10.

Since jumps commute with parallelization by [7, Proposition 5.7(3)], Corollary 4.18 yields also the following corollary, which states that under parallelization a jump of the colored versions of Ramsey's Theorem corresponds exactly to an increase in size by one.

Corollary 4.25 (Parallelized jumps)**.** *For all $n \geq 2$, $k \geq 2, \mathbb{N}$ we obtain $\widehat{\mathsf{CSRT}'_{n,k}} \equiv_{\mathrm{sW}} \widehat{\mathsf{CSRT}_{n+1,k}} \equiv_{\mathrm{sW}} \widehat{\mathsf{CRT}'_{n,k}} \equiv_{\mathrm{sW}} \widehat{\mathsf{CRT}_{n+1,k}}$.*

A similar property also holds for the uncolored version of Ramsey's Theorem $\mathsf{RT}_{n,k}$, but in this case it cannot be so easily concluded since the parallelized uncolored theorem is not a cylinder by Corollary 3.16. Hence, we need the following reformulation of Corollary 4.10 that is justified by Lemma 4.9.

Corollary 4.26. *$\mathsf{SRT}_{n+1,k} \leq_{\mathrm{W}} \mathsf{RT}'_{n,k} \times \lim$ for all $n \geq 1$, $k \geq 1, \mathbb{N}$.*

Now we are prepared to prove the following result on parallelized jumps of Ramsey's Theorem.

Corollary 4.27 (Parallelized jumps). $\widehat{\mathsf{RT}'_{n,k}} \equiv_{\mathrm{W}} \widehat{\mathsf{RT}_{n+1,k}}$ *for all* $n \geq 1$ *and* $k \geq 2, \mathbb{N}$.

Proof. It follows from Corollary 4.2 that $\mathsf{RT}'_{n,k} \leq_{\mathrm{W}} \mathsf{RT}_{n+1,k}$ and hence one direction of the claims follows since parallelization is a closure operator. By Corollaries 4.18, 4.26 and 4.6 it follows that

$$\widehat{\mathsf{RT}_{n+1,k}} \equiv_{\mathrm{W}} \widehat{\mathsf{SRT}_{n+1,k}} \leq_{\mathrm{W}} \widehat{\mathsf{RT}'_{n,k}} \times \lim \equiv_{\mathrm{W}} \widehat{\mathsf{RT}'_{n,k}} \times \widehat{\lim_{\mathbb{N}}} \leq_{\mathrm{W}} \widehat{\mathsf{RT}'_{n,k}},$$

which completes the proof. $\square$

We leave it open whether a corresponding fact can be established for the stable version of Ramsey's Theorem (which is not the case for $n = 1$). We have now completely provided all the positive information that is displayed in the diagram in Figure 2. What still remains to be done are some of the separations.

5. Ramsey's Theorem for Pairs

In this section we want to discuss Ramsey's Theorem for pairs $\mathsf{RT}_{2,2}$, which is of particular interest. For one, we derive some conclusions from general results that we proved before, we mention some results that have been proved by other authors and we raise some open questions. The neighborhood of Ramsey's Theorem in the Weihrauch lattice is illustrated in the diagram in Figure 4.

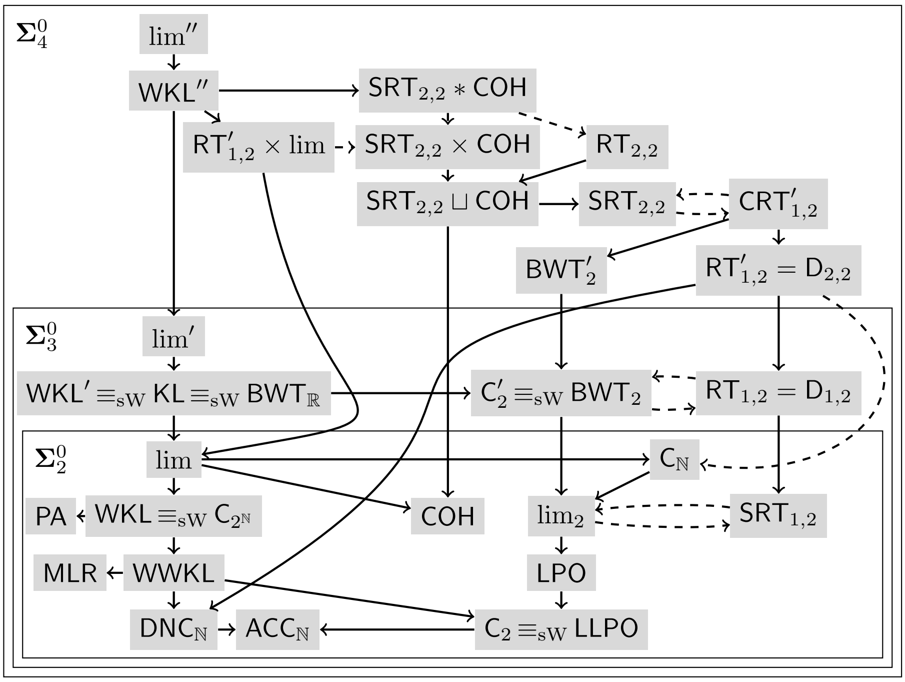

Figure 4. Ramsey's Theorem for pairs and two colors in the Weihrauch lattice: all solid arrows indicated strong Weihrauch reductions against the direction of the arrow, all dashed arrows indicate ordinary Weihrauch reductions, the boxes indicate the given levels of the effective Borel hierarchy.

We start with two straightforward separation results that shows that the jump of the cylindrification of $\mathsf{RT}_{1,k}$ is incomparable with $\mathsf{RT}_{2,2}$.

Proposition 5.1. $\mathsf{RT}_{2,2} \not\leq_{\mathrm{W}} \mathsf{RT}'_{1,\mathbb{N}} \times \lim$ *and* $\mathsf{RT}'_{1,2} \times \lim \not\leq_{\mathrm{W}} \mathsf{RT}_{2,\mathbb{N}}$.

Proof. For every computable input $\mathsf{RT}'_{1,\mathbb{N}} \times \lim$ clearly has a limit computable output, since for a limit computable coloring $c : [\mathbb{N}]^1 \to \mathbb{N}$ we can always choose the maximal set of all points A of one color as an output $A \in \mathsf{RT}_{1,\mathbb{N}}(c)$. However, $\mathsf{RT}_{2,2}$ has computable inputs $c : [\mathbb{N}]^2 \to 2$, which have no limit computable outputs $A \in \mathsf{RT}_{2,2}(c)$ by Theorem 1.3. Hence, $\mathsf{RT}_{2,2} \not\leq_{\mathrm{W}} \mathsf{RT}'_{1,\mathbb{N}} \times \lim$.

While $\lim \leq_{\mathrm{W}} \mathsf{RT}'_{1,2} \times \lim$ holds obviously, we obtain $\lim \not\leq_{\mathrm{W}} \mathsf{RT}_{2,\mathbb{N}}$ by Theorem 3.8. Hence, $\mathsf{RT}'_{1,2} \times \lim \not\leq_{\mathrm{W}} \mathsf{RT}_{2,\mathbb{N}}$. □

This result yields the following corollary.

Corollary 5.2. $\mathsf{RT}'_{1,2} \times \lim |_{\mathrm{W}} \mathsf{RT}_{2,2}$.

Since we have $\mathsf{SRT}_{2,\mathbb{N}} \leq_{\mathrm{W}} \mathsf{RT}_{1,\mathbb{N}} * \lim \equiv_{\mathrm{W}} \mathsf{RT}'_{1,\mathbb{N}} \times \lim$ by Corollary 4.10 and Lemma 4.9, we also obtain the following corollary of Proposition 5.1

Corollary 5.3. $\mathsf{RT}_{2,2} \not\leq_{\mathrm{W}} \mathsf{SRT}_{2,\mathbb{N}}$ *and hence, in particular,* $\mathsf{RT}_{2,2} \not\leq_{\mathrm{W}} \mathsf{SRT}_{2,2}$.

The corresponding non-uniform result in reverse mathematics was much harder to obtain and it solved a longstanding open question [12] and it does so using a non-standard model, whereas our result would follow from a separation with a natural number model.

Among other things, we are going to discuss the relation of $\mathsf{RT}_{2,2}$ to the cohesiveness problem. The positive part of the following proof is essentially the proof of Cholak, Jockusch and Slaman [9, Theorem 12.5].[5]

Proposition 5.4 (Cohesiveness)**.** $\mathsf{COH} \leq_{\mathrm{sW}} \mathsf{RT}_{2,2}$.

Proof. Let $(R_i)_i$ be a sequence of sets $R_i \subseteq \mathbb{N}$. We compute a sequence $(S_i)_i$ by $S_{2i} := R_i$ and $S_{2i+1} := \{i\}$ for all $i \in \mathbb{N}$. We let $d(i,j) := \min\{k : \chi_{S_k}(i) \neq \chi_{S_k}(j)\}$. The definition of $(S_i)_i$ ensures that d is well-defined for all $i < j$ and it can be computed from $(S_i)_i$. We compute a coloring $c : [\mathbb{N}]^2 \to 2$ with

$$c\{i<j\} := \begin{cases} 0 & \text{if } i \in S_{d(i,j)} \\ 1 & \text{otherwise} \end{cases}$$

and we consider an infinite homogeneous set $M \in \mathsf{RT}_{2,2}(c)$ and $i \in \mathbb{N}$. We claim that M is cohesive for $(S_i)_i$ and hence for $(R_i)_i$. Let us assume for a contradiction that k is the smallest number such that $M \cap S_k$ and $M \cap (\mathbb{N} \setminus S_k)$ are infinite. Since k is minimal, there is a number $m \in \mathbb{N}$ such that $d(i,j) \geq k$ for all $i,j \geq m$ in M. There are also sufficiently large $i_0, i_1, i_2 \geq m$ in M such that $i_0 < i_1 < i_2$, $\chi_{S_k}(i_0) \neq \chi_{S_k}(i_1)$ and $\chi_{S_k}(i_1) \neq \chi_{S_k}(i_2)$. This implies $d(i_0,i_1) = k = d(i_1,i_2)$, which in turn yields $c\{i_0,i_1\} \neq c\{i_1,i_2\}$, in contradiction to the homogeneity of M. Hence, M is cohesive for $(R_i)_i$ and we obtain $\mathsf{COH} \leq_{\mathrm{sW}} \mathsf{RT}_{2,2}$. □

We obtain $\mathsf{RT}_{2,2} \not\leq_{\mathrm{W}} \mathsf{COH}$ since $\mathsf{COH} \leq_{\mathrm{W}} \mathsf{WKL}'$ (i.e., COH is $\mathbf{\Sigma}^0_3$–measurable) by Fact 4.13, while $\mathsf{RT}_{2,2}$ is $\mathbf{\Sigma}^0_4$–measurable by Corollary 4.19. Together with Proposition 4.11, Lemma 3.2, Corollary 4.15 and Facts 4.13 and 2.1 we obtain the following corollary.

Corollary 5.5. $\mathsf{SRT}_{2,2} \sqcup \mathsf{COH} \leq_{\mathrm{sW}} \mathsf{RT}_{2,2} \leq_{\mathrm{W}} \mathsf{SRT}_{2,2} * \mathsf{COH} \leq_{\mathrm{sW}} \mathsf{WKL}''$.

Proof. We note that the last mentioned reduction follows from Corollary 4.15 and Facts 4.13 and 2.1 since

$$\mathsf{SRT}_{2,2} * \mathsf{COH} \leq_{\mathrm{W}} \mathsf{WKL}'' * \mathsf{COH} \equiv_{\mathrm{W}} \mathsf{WKL}' * \lim * \mathsf{COH} \equiv_{\mathrm{W}} \mathsf{WKL}' * \mathsf{WKL}' \equiv_{\mathrm{W}} \mathsf{WKL}''.$$

□

[5]We mention that it has been noted in [10] that the proof of [9, Lemma 7.11] is flawed and likewise it is also not sufficient for our purposes.

Given the fact that $\mathsf{SRT}_{2,2} \sqcup \mathsf{COH} \leq_{\mathrm{sW}} \mathsf{SRT}_{2,2} \times \mathsf{COH} \leq_{\mathrm{W}} \mathsf{SRT}_{2,2} * \mathsf{COH}$, it would be desirable to clarify the relation of $\mathsf{RT}_{2,2}$ to all the three mentioned problems.

Question 5.6. How does $\mathsf{RT}_{2,2}$ exactly relate to $\mathsf{SRT}_{2,2} \sqcup \mathsf{COH}$, $\mathsf{SRT}_{2,2} \times \mathsf{COH}$ and $\mathsf{SRT}_{2,2} * \mathsf{COH}$?

We can at least say something. We have $\mathsf{COH} \leq_{\mathrm{W}} \lim$ by [8, Proposition 12.10] and by Corollary 4.10 and Lemma 4.9 we know that $\mathsf{SRT}_{2,2} \leq_{\mathrm{W}} \mathsf{RT}_{1,2}' \times \lim$. Since lim is idempotent we obtain the following corollary.

Corollary 5.7. $\mathsf{SRT}_{2,2} \times \mathsf{COH} \leq_{\mathrm{W}} \mathsf{RT}_{1,2}' \times \lim$.

With Proposition 5.1 we arrive at the following conclusion.

Corollary 5.8. $\mathsf{RT}_{2,2} \not\leq_{\mathrm{W}} \mathsf{SRT}_{2,2} \times \mathsf{COH}$.

We also mention that Propositions 3.18 and 5.4 imply the following.

Corollary 5.9. $\mathsf{SRT}_{2,2} \times \mathsf{COH} \leq_{\mathrm{W}} \mathsf{RT}_{2,4}$.

We recall that the problem $\mathsf{RT}_{1,2}'$ is also studied under the name D_2^2 (see for instance [9, 11]). We introduce the following notation, where we use lower indices again.

Definition 5.10. We define $\mathsf{D}_{n,k} := \mathsf{RT}_{1,k}^{(n-1)}$ for all $n, k \geq 1$.

Dzhafarov proved that $\mathsf{COH} \not\leq_{\mathrm{sW}} \mathsf{D}_{2,k}$ holds for all $k \in \mathbb{N}$ [14, Corollary 1.10]. In fact, his key theorem [14, Theorem 1.5] yields even the following stronger result.

Corollary 5.11. $\mathsf{COH} \not\leq_{\mathrm{sW}} \mathsf{CRT}_{1,k}^{(m)}$ *for all* $k, m \in \mathbb{N}$.

In a subsequent paper [15] Dzhafarov proved the following result.

Theorem 5.12. $\mathsf{COH} \not\leq_{\mathrm{W}} \mathsf{SRT}_{2,+}$ *and hence, in particular,* $\mathsf{COH} \not\leq_{\mathrm{W}} \mathsf{SRT}_{2,2}$.

Since Kőnig's Lemma KL is often discussed in the context of Ramsey's Theorem for pairs, we mention it briefly in passing. By $\mathsf{KL} :\subseteq \mathrm{Tr}_{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ we denote the multi-valued function that is defined on all infinite finitely branching trees $T \subseteq \mathbb{N}^*$ and such that $\mathsf{KL}(T) = [T]$ is the set of infinite paths of T. The following result is essentially based on results from [7].

Theorem 5.13 (Kőnig's Lemma)**.** $\mathsf{KL} \equiv_{\mathrm{sW}} \mathsf{WKL}'$.

Proof. By [7, Corollaries 11.6 and 11.7] we have $\mathsf{WKL}' \equiv_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{N}^{\mathbb{N}}}$ and it is easy to see that $\mathsf{KL} \leq_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{N}^{\mathbb{N}}}$: Given a finitely bounded tree $T \subseteq \mathbb{N}^*$ we just compute a sequence $(x_n)_n$ in $\mathbb{N}^{\mathbb{N}}$ with $x_n := w_n\widehat{0}$, where w_n is the n–th word in T. Since T is finitely bounded, the set $\overline{\{x_n : n \in \mathbb{N}\}}$ is compact and hence $(x_n)_n$ has cluster points. It is clear that all cluster points of $(x_n)_n$ are infinite paths of T. This proves $\mathsf{KL} \leq_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{N}^{\mathbb{N}}}$.

We prove $\mathsf{WKL}' \leq_{\mathrm{sW}} \mathsf{KL}$ directly. Given a sequence $(T_n)_n$ of binary trees with an infinite limit T_∞, we compute a finitely branching tree $T \subseteq \mathbb{N}^*$ such that every infinite path $p \in [T]$ modulo 2 is an infinite path of T_∞, i.e., if $q(i) := p(i) \bmod 2$ for all $i \in \mathbb{N}$, then $q \in [T_\infty]$. This clearly shows $\mathsf{WKL}' \leq_{\mathrm{sW}} \mathsf{KL}$. The tree T is constructed in stages $n = 1, 2, ...$ In stage n the tree T_n is inspected up to level n and copied to the output, possibly with changes modulo 2. We describe this construction that involved a case distinction more precisely.

1. Case: If T_n and T_{n-1} coincide up to level n, then the tree on the output is just extended by the corresponding nodes from T_n on level n.

2. Case: If there is some level $i \leq n$ on which T_{n-1} and T_n differ and i is the first such level, then the changes modulo 2 happen on level i. Basically T_n is copied to the output up to level n, but on level i the values 0 and 1 are replaced by $2n$ and $2n + 1$, respectively.

In both cases the output nodes with values less or equal to $2n+1$ and up to level n, which are currently not in use, are marked as not belonging to the tree.

This construction leads to a finitely branching tree T since $(T_n)_n$ converges. That is, each level i of T stabilizes eventually with a finite number of branches. The constructed tree clearly satisfies the required condition, i.e., each branch $q \in [T]$ modulo 2 yields an infinite branch of T_∞. □

Now Theorem 5.13 and Corollary 3.30 yield the following result, which was independently proved in [18, Corollary 2.3] by a very different method.

Corollary 5.14 (Kőnig's Lemma)**.** $\mathsf{KL} \leq_{\mathrm{W}} \mathsf{RT}_{3,2}$.

We note that by Corollary 3.15 the reduction cannot be replaced by a strong one. Likewise, one obtains $\mathsf{WWKL} \not\leq_{\mathrm{sW}} \mathsf{RT}_{n,k}$ for all $n, k \geq 1$, which answers [18, Question 5.7] in the su-case and removes the corresponding question marks in [18, Figure 5.5]. We note that by Corollary 3.9 we obtain that Corollary 5.14 is optimal in the following sense.

Corollary 5.15. $\mathsf{KL} \not\leq_{\mathrm{W}} \mathsf{RT}_{2,\mathbb{N}}$.

However, we note the related Questions 3.31 and 4.7. Liu proved the following theorem and lemma [25, Theorem 1.5 and proof of Corollary 1.6].

Theorem 5.16 (Liu 2012)**.** *For any set C not of PA–degree and any $A \subseteq \mathbb{N}$ there exists an infinite subset G of A or $\mathbb{N} \setminus A$ such that $G \oplus C$ is also not of PA–degree.*

Lemma 5.17 (Liu 2012)**.** *For any set C not of PA–degree and any uniformly C–computable sequence $(C_i)_i$ there exists a set G which is cohesive for $(C_i)_i$ and such that $G \oplus C$ is also not of PA–degree.*

From Liu's Theorem 5.16 we can directly derive $\mathsf{PA} \not\leq_{\mathrm{W}} \mathsf{D}_{n,2}$ for all $n \geq 1$ and we also obtain the following corollary.

Corollary 5.18 (Peano arithmetic)**.** $\mathsf{PA} \not\leq_{\mathrm{W}} \mathsf{CRT}_{1,2}^{(n-1)}$ *for all $n \geq 1$.*

In particular, this implies $\mathsf{PA} \not\leq_{\mathrm{W}} \mathsf{SRT}_{2,2}$, since $\mathsf{SRT}_{2,2} \equiv_{\mathrm{W}} \mathsf{CRT}'_{1,2}$ by Theorem 4.3. Using Liu's Lemma 5.17 this can be strengthened as follows.

Corollary 5.19 (Peano arithmetic)**.** $\mathsf{PA} \not\leq_{\mathrm{W}} \mathsf{SRT}_{2,2} * \mathsf{COH}$.

Since $\mathsf{RT}_{2,2} \leq_{\mathrm{W}} \mathsf{SRT}_{2,2} * \mathsf{COH}$ by Proposition 4.11, this implies in particular $\mathsf{PA} \not\leq_{\mathrm{W}} \mathsf{RT}_{2,2}$. Since $\mathsf{PA} <_{\mathrm{W}} \mathsf{WKL}$ [8, Theorem 5.5, Corollary 6.4] and we get also the following conclusion.

Corollary 5.20 (Weak Kőnig's Lemma)**.** $\mathsf{WKL} \not\leq_{\mathrm{W}} \mathsf{RT}_{2,2}$.

Likewise, one should be able to use the methods provided by Liu [26] to show that $\mathsf{MLR} \not\leq_{\mathrm{W}} \mathsf{RT}_{2,2}$ and hence $\mathsf{WWKL} \not\leq_{\mathrm{W}} \mathsf{RT}_{2,2}$. It would also be interesting to find out whether a variant of the above ideas can be used to answer Question 4.7. A one-dimensional version of Weak-Kőnig's Lemma is the Intermediate Value Theorem IVT. Hence, from the uniform perspective it is an obvious question to ask how IVT is related to $\mathsf{RT}_{2,2}$. In order to approach this question we provide a new upper bound on IVT which is of independent interest.

Proposition 5.21 (Intermediate Value Theorem)**.** $\mathsf{IVT} \leq_{\mathrm{W}} \mathsf{BWT}_{\mathbb{N}}$.

Proof. It is known that IVT is equivalent to connected choice $\mathsf{CC}_{[0,1]}$ [4, Theorem 6.2]. Hence, we can assume that we have two monotone sequences $(a_n)_n$ and $(b_n)_n$ of rational numbers in $[0,1]$ with $a_n \leq a_{n+1} < b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$ and the goal is to find some $x \in [0,1]$ with $a := \sup_{n\in\mathbb{N}} a_n \leq x \leq \inf_{n\in\mathbb{N}} b_n =: b$. Now we produce a sequence $(c_n)_n$ of natural numbers in stages $n = 0, 1, 2, ...$ as follows, where we use a variable $k \in \mathbb{N}$ that is initially $k = 0$. If at stage n we find that $|b_n - a_n| < 2^{-k-1}$, then we let $k := k + 1$ and we choose $c_n := 0$. Otherwise, we choose some value $c_n \neq 0$ such that $a_n < \overline{c_n} < b_n$, where $\overline{c_n}$ is the rational number with code c_n. If possible, we choose c_n such that it is identical to the previous such number chosen and if that is impossible, then we chose c_n with $\overline{c_n} := a_n + \frac{b_n - a_n}{2}$. Now there are two possible cases. If $a = b$, then the sequence $(c_n)_n$ contains infinitely many zeros and at most one other number c occurs infinitely often in $(c_n)_n$ and it must be such that $a = b = \overline{c}$. Otherwise, if $a \neq b$, then the sequence $(c_n)_n$ contains only finitely many zeros and only exactly one number c different from zero occurs infinitely often. This number satisfies $a \leq \overline{c} \leq b$. The result of $\mathsf{BWT}_{\mathbb{N}}(c_n)_n$ is one number c that occurs infinitely often in $(c_n)_n$. If $c = 0$, then we can compute $x := a = b$ with the help of the input $(a_n)_n$, $(b_n)_n$. If $c \neq 0$, then $x := \overline{c}$ is a suitable result. □

With the help of Propositions 3.4 we obtain the following conclusion.

Corollary 5.22 (Intermediate Value Theorem)**.** $\mathsf{IVT} \leq_{\mathrm{W}} \mathsf{RT}_{1,\mathbb{N}}$.

Together with Corollary 3.32 we obtain the following result.[6]

Corollary 5.23 (Intermediate Value Theorem)**.** $\mathsf{IVT} \leq_{\mathrm{W}} \mathsf{RT}_{2,2}$.

Ramsey's Theorem for pairs $\mathsf{RT}_{2,2}$ and $\mathsf{SRT}_{2,2}$ cannot be reducible to any of $\mathsf{PA}, \mathsf{MLR}$ or COH for mere reasons of Borel measurability. However, we can even say more. Since it is easy to see that $\mathsf{PA}, \mathsf{MLR}$ and COH are all densely realized, it follows by [8, Proposition 4.3] that $\mathsf{PA}^{(m)}, \mathsf{MLR}^{(m)}$ and $\mathsf{COH}^{(m)}$ are all indiscriminative. In particular we obtain the following corollary.

Corollary 5.24. *$\mathsf{SRT}_{1,2} \not\leq_{\mathrm{W}} \mathsf{PA}^{(m)}$, $\mathsf{SRT}_{1,2} \not\leq_{\mathrm{W}} \mathsf{MLR}^{(m)}$ and $\mathsf{SRT}_{1,2} \not\leq_{\mathrm{W}} \mathsf{COH}^{(m)}$ for all $m \in \mathbb{N}$.*

We mention that it was noted by Hirschfeldt and Jockusch [18, Figure 5.5] that the following corollary follows from [19, Theorem 2.3].

Corollary 5.25 (Diagonally non-computable functions)**.** $\mathsf{DNC}_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{RT}_{1,2}' = \mathsf{D}_{2,2}$.

More information on the uniform content and relations between the problems $\mathsf{DNC}, \mathsf{PA}, \mathsf{WKL}, \mathsf{WWKL}, \mathsf{MLR}, \mathsf{COH}$ and other problems can be found in [8].

6. Separation Techniques for Jumps

In this section we plan to prove some further separation results related to Ramsey's Theorem for pairs $\mathsf{RT}_{2,2}$ (see the diagram in Figure 4). For this purpose we collect some classical results that yield separation techniques for jumps. Often continuity arguments can be used for separation results. In case of reductions to jumps of the form $f \leq_{\mathrm{W}} g'$, such continuity arguments are not always applicable, since jumps g' implicitly involve limits. However, continuity arguments can occasionally be replaced by arguments based on the existence of continuity points, which are still available for limit computable functions. The classical techniques that we recall here concern $\mathbf{\Sigma}_2^0$–measurable functions, which form the topological counterpart of limit computable functions.

We recall that a function $f : X \to Y$ on metric spaces X and Y is called $\mathbf{\Sigma}_2^0$–*measurable* (or of *Baire class* 1) if $f^{-1}(U) \in \mathbf{\Sigma}_2^0$ for every open set $U \subseteq Y$, where $\mathbf{\Sigma}_2^0$ denotes the second class of the Borel hierarchy (i.e. the class of F_σ–sets). Baire proved that the set of points of continuity of every $\mathbf{\Sigma}_2^0$–measurable function is a comeager G_δ–set under certain conditions [22, Theorem 24.14]. We recall that a set is called *comeager* if it contains a countable intersection of dense open sets.

[6]This result was an open question in an earlier version of this article and it was first proved by Ludovic Patey (personal communication).

Proposition 6.1 (Baire)**.** *Let X, Y be metric spaces and let Y additionally be separable. If $f : X \to Y$ is $\mathbf{\Sigma}^0_2$–measurable, then the set of points of continuity of f is a comeager G_δ–set in X.*

We recall that a topological space X is called a *Baire space*[7], if every comeager set $C \subseteq X$ is dense in the space X. Hence we get the following corollary.

Corollary 6.2. *Let X and Y be metric spaces, where X is additionally a Baire space and Y is separable. If $f : X \to Y$ is $\mathbf{\Sigma}^0_2$–measurable, then $f|_U$ has a point of continuity for every non-empty open set $U \subseteq X$.*

By the Baire Category Theorem [22, Theorem 8.4], every complete separable metric space is a Baire space. Moreover, closed subspaces of complete separable metric spaces are complete again. Hence Corollary 6.2 easily implies the direction "(1) $\Longrightarrow$ (2)" of the following characterization of $\mathbf{\Sigma}^0_2$–measurable functions [22, Theorem 24.15].

Theorem 6.3 (Baire Characterization Theorem)**.** *Let X, Y be separable metric spaces and let X additionally be complete. For every function $f : X \to Y$ the following conditions are equivalent to each other:*

(1) *f is $\mathbf{\Sigma}^0_2$–measurable,*
(2) *$f|_A$ has a point of continuity for every non-empty closed set $A \subseteq X$.*

We note that while Theorem 6.3 yields a stronger conclusion than Corollary 6.2, it also requires stronger conditions on the domain X. In certain situations, we will be dealing with Baire spaces $X^{\mathbb{N}}$, which are not Polish spaces and hence we will have to take recourse to Corollary 6.2.

We recall [2, Proposition 9.1] that the limit map lim is a prototype of a $\mathbf{\Sigma}^0_2$–measurable function (relatively to its domain of convergent sequences) and hence the Characterization Theorem of Baire 6.3 can be used as a separation tool for certain reductions to jumps, since limit computable functions are closed under composition with continuous functions.

Fact 6.4. $\lim :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ *is $\mathbf{\Sigma}^0_2$–measurable relatively to its domain.*

For the following result we use the Baire Characterization Theorem 6.3 as a separation technique.

Theorem 6.5. $\mathsf{BWT}_{k+1} \not\leq_{\mathrm{W}} \mathsf{RT}'_{1,k} \times \lim$ *for all $k \geq 1$.*

Proof. Since $\mathsf{RT}'_{1,k} \times \lim$ is a cylinder, it suffices to prove $\mathsf{BWT}_{k+1} \not\leq_{\mathrm{sW}} \mathsf{RT}'_{1,k} \times \lim$. Let us assume for a contradiction that $\mathsf{BWT}_{k+1} \leq_{\mathrm{sW}} \mathsf{RT}'_{1,k} \times \lim$, i.e., there are computable functions H, K such that HGK is a realizer of BWT_{k+1} whenever G is a realizer of $\mathsf{RT}'_{1,k} \times \lim = (\mathsf{RT}_{1,k} \times \mathrm{id}) \circ (\lim \times \lim)$. Since BWT_{k+1} is total, it follows that $\langle K_1, K_2 \rangle := \langle \lim \times \lim \rangle \circ K$ is total too and this function is $\mathbf{\Sigma}^0_2$–measurable. We construct a number of points $p_0, ..., p_k \in \mathbb{N}^{\mathbb{N}}$ inductively. By the Theorem 6.3 of Baire there is a point $p_0 \in \mathbb{N}^{\mathbb{N}}$ of continuity of $\langle K_1, K_2 \rangle$. Let $\langle s_0, r_0 \rangle := \langle K_1(p_0), K_2(p_0) \rangle$, let $q_0 \in \mathsf{RT}_{1,k}(s_0)$ be a maximal homogeneous set and let $c_0 \leq k-1$ be the corresponding color. Let $k_0 \leq k$ be such that $H\langle q_0, r_0 \rangle = k_0$. By continuity of H there exists a prefix $w_0 \sqsubseteq \langle q_0, r_0 \rangle$ such that $H(w_0\mathbb{N}^{\mathbb{N}}) = \{k_0\}$. We can assume that $w_0 = \langle x_0, y_0 \rangle$ is of even length $2n_0$. Since $\langle K_1, K_2 \rangle$ is continuous at p_0, there is a $v_0 \sqsubseteq p_0$ such that $K_1(v_0\mathbb{N}^{\mathbb{N}}) \subseteq s_0|_{n_0}\mathbb{N}^{\mathbb{N}}$ and $K_2(v_0\mathbb{N}^{\mathbb{N}}) \subseteq y_0\mathbb{N}^{\mathbb{N}}$. Now we consider $N_0 := \{0, ..., k\} \setminus \{k_0\}$ and the closed set $A_0 := v_0 N_0^{\mathbb{N}}$. Again by the Theorem 6.3 of Baire there exists a point $p_1 \in A_0$ of continuity of $\langle K_1, K_2 \rangle|_{A_0}$. Let $\langle s_1, r_1 \rangle := \langle K_1(p_1), K_2(p_1) \rangle$, let $q_1 \in \mathsf{RT}_{1,k}(s_1)$ be a maximal homogeneous set and let $c_1 \leq k-1$ be the corresponding color. Let $k_1 \leq k$ be such that $H\langle q_1, r_1 \rangle = k_1$. Since $p_1 \in A_0$ contains k_0 at most

[7]We distinguish between *a* Baire space X in general and *the* Baire space $\mathbb{N}^{\mathbb{N}}$, in particular, which is also an instance of a Baire space.

finitely many times, it follows that $k_1 \neq k_0$. By continuity of H there exists a prefix $w_1 \sqsubseteq \langle q_1, r_1\rangle$ such that $H(w_1\mathbb{N}^\mathbb{N}) = \{k_1\}$. Again we can assume that $w_1 = \langle x_1, y_1\rangle$ is of even length $2n_1$ and additionally we can assume $n_1 > n_0$. Since $v_0 \sqsubseteq p_1$ we have $y_0 \sqsubseteq K_2(p_1) = r_1$ and we obtain

$$s_1|_{n_0} = K_1(p_1)|_{n_0} = K_1(p_0)|_{n_0} = s_0|_{n_0}.$$

If $c_1 = c_0$, i.e., if the maximal homogeneous sets q_0 and q_1 of s_0 and s_1, respectively, have the same color, then $x_0 = q_0|_{n_0} = q_1|_{n_0}$ follows and hence $w_0 \sqsubseteq \langle q_1, r_1\rangle$, which implies $H\langle q_1, r_1\rangle = k_0$ and hence $k_0 = k_1$. Since $k_0 \neq k_1$, we can conclude that $c_0 \neq c_1$. Now we continue as before: since $\langle K_1, K_2\rangle|_{A_0}$ is continuous at p_1, there is a $v_1 \sqsubseteq p_1$ such that $K_1(v_1\mathbb{N}^\mathbb{N}) \subseteq s_1|_{n_1}\mathbb{N}^\mathbb{N}$ and $K_2(v_1\mathbb{N}^\mathbb{N}) \subseteq y_1\mathbb{N}^\mathbb{N}$. Now we consider $N_1 := \{0, ..., k\} \setminus \{k_0, k_1\}$ and the closed set $A_1 := v_1 N_1^\mathbb{N}$. Again by the Theorem 6.3 of Baire there exists a point $p_2 \in A_1$ of continuity of $\langle K_1, K_2\rangle_{A_1}$. In this case we eventually obtain colors $k_2 \leq k, c_2 \leq k-1$ such that $k_2 \notin \{k_0, k_1\}$ and $c_2 \notin \{c_0, c_1\}$. This construction can be repeated k times since there are $k+1$ colors in $\{0, ..., k\}$ until $p_0, ..., p_k$ and $c_0, ..., c_k$ are determined. However, the construction yields that the $c_0, ..., c_k$ have to be pairwise different, which is a contradiction since there are only k colors $c \in \{0, ..., k-1\}$ available. ☐

We get the following immediate corollary with the help of Lemma 4.9 and Corollary 4.10.

Corollary 6.6. $\mathsf{BWT}_{k+1} \not\leq_{\mathrm{W}} \mathsf{SRT}_{2,k}$ *for all* $k \geq 1$.

Since $\mathsf{BWT}_{k+1} \leq_{\mathrm{W}} \mathsf{RT}_{1,k+1} \leq_{\mathrm{W}} \mathsf{SRT}_{2,k+1}$ by Proposition 3.4 and Lemma 4.8, we obtain also the following separation result as a consequence of Corollary 6.6.

Corollary 6.7. $\mathsf{SRT}_{2,k+1} \not\leq_{\mathrm{W}} \mathsf{SRT}_{2,k}$ *for all* $k \geq 1$.

As next separation result we want to prove $\mathsf{BWT}_2' \not\leq_{\mathrm{W}} \mathsf{RT}_{1,2}'$. To this end we are going to use Corollary 6.2 as a separation tool. It is not too difficult to see that for every topological space X that is endowed with the discrete topology, the product space $X^\mathbb{N}$ is a Baire space. We will apply this observation to the particular case of the set

$$X_2 := \{p \in 2^\mathbb{N} : (\exists k)(\forall n \geq k)\; p(n) = p(k)\}$$

of convergent sequences of zeros and ones. On the other hand, X_2 is also endowed with the subspace topology that it inherits from the ordinary product topology on $\mathbb{N}^\mathbb{N}$. In both cases, we assume that $X_2^\mathbb{N}$ is equipped with the respective product topology. In order to distinguish both topologies, we speak about "discrete continuity" of a map $f : X_2^\mathbb{N} \to Y$, if X_2 is endowed with the discrete topology and about continuity if the usual subspace topology of $\mathbb{N}^\mathbb{N}$ is used for X_2. In the latter case, the map

$$h_2 : X_2^\mathbb{N} \to \mathbb{N}^\mathbb{N}, ((x_{i,j})_i)_j \mapsto \langle (x_{0,j})_j, (x_{1,j})_j, (x_{2,j})_j, ...\rangle,$$

is a computable embedding, i.e., h_2 as well as the partial inverses h_2^{-1} are both computable and continuous. Intuitively, h_2 swaps rows and columns and applies the Cantor encoding $\langle ...\rangle$, both being purely technical changes. Hence we will identify $X_2^\mathbb{N}$ with (a subspace of) $\mathbb{N}^\mathbb{N}$ via h_2 in the following proof.

Theorem 6.8. $\mathsf{BWT}_2' \not\leq_{\mathrm{W}} \mathsf{RT}_{1,2}'$.

Proof. Let us assume for a contradiction that $\mathsf{BWT}_2' \leq_{\mathrm{W}} \mathsf{RT}_{1,2}'$. Then there are computable functions H, K such that $H\langle \mathrm{id}, GK\rangle$ is a realizer of BWT_2' whenever G is a realizer of $\mathsf{RT}_{1,2}'$. We tacitly identify $X_2^\mathbb{N}$ with a subspace of $\mathbb{N}^\mathbb{N}$ via h_2. Since $\mathsf{BWT}_2' : 2^\mathbb{N} \rightrightarrows \{0,1\}$ is total, it follows that $\lim K : X_2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is also total and $\mathbf{\Sigma}_2^0$–measurable. If we endow X_2 with the discrete topology, then $X_2^\mathbb{N}$ is a Baire space and by Corollary 6.2 there exists a point of discrete continuity $x \in X_2^\mathbb{N}$ of $\lim K$. Let

$c := \lim K(x)$ be the corresponding coloring. Then there is some maximal infinite homogeneous set $q \in \mathsf{RT}'_{1,2}(c)$ and there is a realizer G of $\mathsf{RT}'_{1,2}$ that maps $K(x)$ to q. By continuity of H there is some $m \in \mathbb{N}$ such that $H\langle x|_m X_2^{\mathbb{N}}, q|_m \mathbb{N}^{\mathbb{N}}\rangle = \{e\}$ for some $e \in \{0,1\}$. Without loss of generality, we assume $e = 0$. Hence, by discrete continuity of $\lim K$ at x, there is some $k > m$ such that $\lim K(x|_k X_2^{\mathbb{N}}) \subseteq c|_m 2^{\mathbb{N}}$.

We construct some $y = (x|_k, z_0, z_1, z_2...) \in X_2^{\mathbb{N}}$ such that all $z_i \in X_2$ converge to 1 and the coloring $c_y := \lim K(y)$ has maximal homogeneous sets $q_0, q_1 \in \mathsf{RT}_{1,2}(c_y)$ of different colors. This yields a contradiction: since $x|_k \sqsubseteq y$, we obtain $c|_m \sqsubseteq c_y$ and hence $q|_m$ must be prefix of either q_0 or q_1, say q_0, and this implies $H\langle y, q_0\rangle = e = 0$, whereas the correct result would have to be 1, since all the z_i converge to 1. We describe the inductive construction of y in even and odd stages $i \in \mathbb{N}$.

Stage 0: Let $y_0 := x|_k\widehat{0}\widehat{0}...$, $c_0 := \lim K(y_0)$ and let $r_0 \in \mathsf{RT}_{1,2}(c_0)$. Then by continuity of H there is some $m_0 > k$ such that $H\langle x|_k w_{m_0} X_2^{\mathbb{N}}, r_0|_{m_0}\mathbb{N}^{\mathbb{N}}\rangle = \{0\}$ with $w_m := (0^m\widehat{1})^m = (0^m\widehat{1}, 0^m\widehat{1}, ..., 0^m\widehat{1}) \in X_2^m$ for all $m \in \mathbb{N}$.

Stage 1: Let $y_1 := x|_k w_{m_0}\widehat{1}\widehat{1}...$, $c_1 := \lim K(y_1)$ and let $r_1 \in \mathsf{RT}_{1,2}(c_1)$. Then also $0^{m_0}\frown r_1 \in \mathsf{RT}_{1,2}(c_1)$ and by continuity of H there is some $m_1 > k + m_0$ such that $H\langle y_1|_{m_1} X_2^{\mathbb{N}}, 0^{m_0}\frown r_1|_{m_1}\mathbb{N}^{\mathbb{N}}\rangle = \{1\}$.

We know assume that $i \geq 1$ and that $y_0, ..., y_{2i-1} \in X_2^{\mathbb{N}}$ and $m_0, ..., m_{2i+1} \in \mathbb{N}$ have already been constructed.

Stage $2i$: Let $y_{2i} := y_{2i-1}|_{m_{2i-1}}\widehat{0}\widehat{0}...$, $c_{2i} := \lim K(y_{2i})$ and let $r_{2i} \in \mathsf{RT}_{1,2}(c_{2i})$. Then by continuity of H there is some $m_{2i} > m_{2i-1}$ such that

$$H\langle y_{2i-1}|_{m_{2i-1}} w_{m_{2i}} X_2^{\mathbb{N}}, 0^{m_{2i-1}}\frown r_{2i}|_{m_{2i}}\mathbb{N}^{\mathbb{N}}\rangle = \{0\}.$$

Stage $2i+1$: Let $y_{2i+1} := y_{2i-1}|_{m_{2i-1}} w_{m_{2i}}\widehat{1}\widehat{1}...$, $c_{2i+1} := \lim K(y_{2i+1})$ and let $r_{2i+1} \in \mathsf{RT}_{1,2}(c_{2i+1})$. Then also $0^{m_{2i}}\frown r_{2i+1} \in \mathsf{RT}_{1,2}(c_{2i+1})$ and by continuity of H there is some $m_{2i+1} > m_{2i-1} + m_{2i}$ such that

$$H\langle y_{2i+1}|_{m_{2i+1}} X_2^{\mathbb{N}}, 0^{m_{2i}}\frown r_{2i+1}|_{m_{2i+1}}\mathbb{N}^{\mathbb{N}}\rangle = \{1\}.$$

This construction yields a sequence $(y_i)_i$ in $X_2^{\mathbb{N}}$ and a strictly increasing sequence $(m_i)_i$ in $\mathbb{N}$ such that $y_{2i-1}|_{m_{2i-1}} \sqsubseteq y_{2i+1}$ for all i. Since all the blocks w_m that are added to the initial $x|_k$ in the odd stages are of the form $w_m = (z_1, ..., z_m)$, with $z_i \in X_2$ that all converge to 1, it follows that $(y_{2i+1})_i$ converges to a $y = (x|_k, z_0, z_1, z_2...) \in X_2^{\mathbb{N}}$, with z_i that all converge to 1. Let $c_y := \lim K(y)$ be the corresponding coloring. We now prove that there are homogeneous sets in $\mathsf{RT}_{1,2}(c_y)$ of different color. We fix some maximal homogeneous set $r \in \mathsf{RT}_{1,2}(c_y)$. It suffices to prove that there are infinitely many i with $r(i) = 0$. The inductive proof follows the stages of the construction above and uses the corresponding objects constructed therein.

Stage 0, 1: We obtain $r_0|_{m_0} \not\sqsubseteq r$, since otherwise there is some $s \in \mathsf{RT}_{1,2}(c_y)$ such that $r_0|_{m_0} \sqsubseteq s$ and $H\langle y, s\rangle = 1$. Since $x|_k w_{m_0} \sqsubseteq y$ this contradicts the choice of m_0 in Stage 0. In particular, we obtain that there is some i with $0 \leq i < m_0$ and $r(i) = 0$.

Stages $2i, 2i+1$: Likewise, $0^{m_{2i-1}}\frown r_{2i}|_{m_{2i}} \not\sqsubseteq r$, since otherwise there is some $s \in \mathsf{RT}_{1,2}(c_y)$ such that $0^{m_{2i-1}}\frown r_{2i}|_{m_{2i}} \sqsubseteq s$ and $H\langle y, s\rangle = 1$. Since $y_{2i-1}|_{m_{2i-1}} w_{m_{2i}} \sqsubseteq y$ this contradicts the choice of m_{2i} in Stage $2i$. In particular, there is some i with $m_{2i-1} \leq i < m_{2i}$ and $r(i) = 0$.

Altogether, this proves that there are infinitely many i with $r(i) = 0$, since $(m_i)_i$ is strictly increasing. □

By Corollary 4.6 we have $\lim_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{RT}'_{1,2}$. On the other hand, the following corollary is a consequence of [7, Facts 3.2(1) and 3.6, Theorem 13.3].

Fact 6.9. $\lim_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{BWT}_k^{(m)}$ *for all* $k, m \geq 1$.

Now it follows with Fact 6.9 that $\mathsf{RT}'_{1,2} \not\leq_{\mathrm{W}} \mathsf{BWT}'_2$. Hence, together with Theorem 6.8 we obtain the following corollary.

Corollary 6.10. $\mathsf{BWT}'_2 \,|_{\mathrm{W}}\, \mathsf{RT}'_{1,2}$.

By Proposition 3.4 we have $\mathsf{BWT}_2 \equiv_{\mathrm{W}} \mathsf{RT}_{1,2}$, but Corollary 6.10 shows that the situation is very different for strong Weihrauch reducibility.

Corollary 6.11. $\mathsf{BWT}_2 \,|_{\mathrm{sW}}\, \mathsf{RT}_{1,2}$.

It is clear that $\mathsf{D}_{2,2} = \mathsf{RT}'_{1,2} \leq_{\mathrm{sW}} \mathsf{CRT}'_{1,2} \equiv_{\mathrm{W}} \mathsf{SRT}_{2,2}$ by Theorem 4.3 and by Theorem 3.5 we also obtain $\mathsf{BWT}'_2 \leq_{\mathrm{sW}} \mathsf{CRT}'_{1,2}$. Now Corollary 6.10 also leads to the following separation result.

Corollary 6.12. $\mathsf{SRT}_{2,2} \not\leq_{\mathrm{W}} \mathsf{D}_{2,2}$.

Altogether, we have now separated several of the degrees illustrated in the diagram of Figure 4.

7. Boundedness and Induction

The purpose of this section is to collect some results on how Ramsey's Theorem is related to boundedness and induction. In reverse mathematics there is a well-known strict hierarchy of induction principles $\mathsf{I}\Sigma^0_n$ and boundedness principles $\mathsf{B}\Sigma^0_n$ (see [17, Theorem 4.32]):

$$\mathsf{B}\Sigma^0_1 \leftarrow \mathsf{I}\Sigma^0_1 \leftarrow \mathsf{B}\Sigma^0_2 \leftarrow \mathsf{I}\Sigma^0_2 \leftarrow ...$$

The n–th jump $\mathsf{C}_{\mathbb{N}}^{(n)}$ of closed choice on the natural numbers can be seen as a uniform analogue of $\mathsf{I}\Sigma^0_{n+1}$ and the n–th jump $\mathsf{K}_{\mathbb{N}}^{(n)}$ of compact choice[8] can be seen as a uniform analogue of $\mathsf{B}\Sigma^0_{n+1}$. Analogously to the above inclusion chain we obtain

$$\mathsf{K}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{K}'_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{C}'_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{K}''_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{C}''_{\mathbb{N}} \leq_{\mathrm{W}} ...$$

and at least the first four reductions are known to be strict (see [7, Corollary 10.10 and Theorems 11.2 and 12.10]). We note that $\mathsf{K}'_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{N}}$ is the Bolzano-Weierstraß Theorem on $\mathbb{N}$ [7, Theorem 11.12] and $\mathsf{C}'_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{CL}_{\mathbb{N}}$ the cluster point problem on $\mathbb{N}$ [7, Theorem 9.4]. The above reduction chain is based on the following observation.

Lemma 7.1. $\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{K}'_{\mathbb{N}}$.

Proof. We obtain $\mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{sW}} \lim_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{K}'_{\mathbb{N}}$. □

In reverse mathematics a considerably amount of work has been spent in order to calibrate Ramsey's Theorem according to the above boundedness and induction principles. Hence, it is natural to ask how it compares uniformly to choice principles. We start with some easy observations and as a preparation we prove the following result. The proof is a simple variation of the proof of [7, Theorem 13.3].

Proposition 7.2 (Choice and cardinality)**.** *Let $f :\subseteq X \rightrightarrows \mathbb{N}$ and $X \subseteq \mathbb{N}$ be such that $\mathsf{C}_X \leq_{\mathrm{W}} f$. Then $|X| \leq |\mathrm{range}(f)|$.*

Proof. Let us assume that $\mathsf{C}_X \leq_{\mathrm{W}} f$. We use the representation ψ_- of closed subsets of X. Then there are computable H, K such that $H\langle \mathrm{id}, GK\rangle \vdash \mathsf{C}_X$ for all $G \vdash f$. We fix some realizer $G \vdash f$. For simplicity and without loss of generality we assume that G and H have target space $\mathbb{N}$. We consider the following claim: for each $i \in \mathbb{N}$ with $|X| > i$ there exists

(1) $k_i \in X \setminus \{k_0, ..., k_{i-1}\}$,
(2) $n_i \in \mathrm{range}(f) \setminus \{n_0, ..., n_{i-1}\}$,

[8]Compact choice $\mathsf{K}_{\mathbb{N}}$ is defined as closed choice $\mathsf{C}_{\mathbb{N}}$, except that the input set is not just described negatively, but additionally a bound is provided, see [7] for precise definitions.

(3) p_i is a name of a closed subset $A_i \subseteq X$,
(4) $w_i \sqsubseteq p_i$,

such that $w_{i-1} \sqsubseteq w_i$, $GK(p_i) = n_i$ and $H\langle w_i\mathbb{N}^\mathbb{N}, n_i\rangle = k_i$. Let w_{-1} be the empty word. We prove this claim by induction on i. Let us assume that $|X| > 0$ and let p_0 be a name of $A_0 := X$. Then $k_0 := H\langle p_0, GK(p_0)\rangle \in A_0$ and $n_0 := GK(p_0)$. By continuity of H there is some $w_0 \sqsubseteq p_0$ such that such that $H\langle w_0\mathbb{N}^\mathbb{N}, n_0\rangle = k_0$. Let us now assume that $|X| > 1$ and let p_1 be a name of $A_1 := X \setminus \{k_0\}$. Then $k_1 := H\langle p_1, GK(p_1)\rangle \in A_1$ and $n_1 := GK(p_1)$. Since $k_1 \neq k_0$, we obtain $n_0 \neq n_1$ since $H\langle w_0\mathbb{N}^\mathbb{N}, n_0\rangle = k_0$. By continuity of H there is some $w_1 \sqsubseteq p_1$ with $w_0 \sqsubseteq w_1$ such that $H\langle w_1\mathbb{N}^\mathbb{N}, n_1\rangle = k_1$. The proof can now continue inductively as above which proves the claim. The claim implies $|X| \leq |\mathrm{range}(f)|$. $\square$

From this result we can conclude the following observation.

Proposition 7.3 (Finite Choice). $\mathsf{C}_k \leq_\mathrm{W} \mathsf{SRT}_{1,k}$ *and* $\mathsf{C}_{k+1} \not\leq_\mathrm{W} \mathsf{RT}_{1,k}$ *for all* $k \geq 1$.

Proof. By Lemma 3.4 we have $\lim_k \equiv_\mathrm{W} \mathsf{SRT}_{1,k}$, hence the first reduction follows from [7, Corollary 13.8] and can easily been proved directly. By Lemma 3.4 we have $\mathsf{BWT}_k \equiv_\mathrm{W} \mathsf{RT}_{1,k}$ and hence the second claim follows from Proposition 7.2. $\square$

Since $\mathsf{K}_\mathbb{N} \equiv_\mathrm{W} \mathsf{C}_2^*$ by [7, Proposition 10.9] we get the following conclusion.

Proposition 7.4 (Compact choice). $\mathsf{K}_\mathbb{N} \leq_\mathrm{W} \mathsf{SRT}_{1,+}$ *and* $\mathsf{K}_\mathbb{N} \not\leq_\mathrm{W} \mathsf{RT}_{1,k}$ *for all* $k \geq 1$.

Proof. Since $\mathsf{C}_2 \leq_\mathrm{W} \mathsf{SRT}_{1,2}$, we obtain $\mathsf{K}_\mathbb{N} \equiv_\mathrm{W} \mathsf{C}_2^* \leq_\mathrm{W} \mathsf{SRT}_{1,2}^* \leq_\mathrm{W} \mathsf{SRT}_{1,+}$ by Corollary 3.21. On the other hand, $\mathsf{C}_{k+1} \not\leq_\mathrm{W} \mathsf{RT}_{1,k}$ and $\mathsf{C}_{k+1} \leq_\mathrm{W} \mathsf{K}_\mathbb{N}$ implies $\mathsf{K}_\mathbb{N} \not\leq_\mathrm{W} \mathsf{RT}_{1,k}$ for all $k \geq 1$. $\square$

Since $\mathsf{K}_\mathbb{N}' \equiv_\mathrm{sW} \mathsf{BWT}_\mathbb{N}$ by [7, Theorem 11.12] we get the following reformulation of Proposition 3.4 as a first statement. Corollary 6.6 implies $\mathsf{K}_\mathbb{N}' \not\leq_\mathrm{W} \mathsf{SRT}_{2,k}$ for all $k \geq 1$ and since $\mathsf{K}_\mathbb{N}'$ is join irreducible (as any jump is by [7, Proposition 5.8]) we obtain the following conclusion as a second statement.

Corollary 7.5 (Jump of compact choice). $\mathsf{K}_\mathbb{N}' \equiv_\mathrm{W} \mathsf{RT}_{1,\mathbb{N}}$ *and* $\mathsf{K}_\mathbb{N}' \not\leq_\mathrm{W} \mathsf{SRT}_{2,+}$.

The first statement corresponds to a well-known theorem of Hirst [20], which says that $\mathsf{RT}^1_{<\infty}$ is equivalent to $\mathsf{B}\Sigma^0_2$ over RCA_0 (see also [17, Theorem 6.81]), whereas the second claim shows that the reverse mathematics result [9] (see also [17, Theorem 6.82]) that $\mathsf{SRT}_{2,2}$ proves $\mathsf{RT}^1_{<\infty}$ over RCA_0 cannot be proved uniformly. Actually, the proof of the reverse mathematics result contains a non-constructive case distinction that requires $\mathsf{LPO}' \leq_\mathrm{sW} \mathsf{LLPO}'' \leq_\mathrm{W} \mathsf{SRT}_{2,2}$. Hence, the uniform content of the proof is rather captured by the following corollary.

Corollary 7.6. $\mathsf{K}_\mathbb{N}' \leq_\mathrm{W} \mathsf{SRT}_{2,2} * \mathsf{SRT}_{2,2}$.

Corollary 7.5 yields also one direction of the following corollary and the other direction follows since $\mathsf{RT}_{1,\mathbb{N}}$ is $\mathbf{\Sigma}^0_3$–measurable, but $\mathsf{SRT}_{2,2}$ and $\mathsf{SRT}_{2,\mathbb{N}}$ are not by Corollary 4.19.

Corollary 7.7. $\mathsf{RT}_{1,\mathbb{N}} \,|_\mathrm{W}\, \mathsf{SRT}_{2,2}$ *and* $\mathsf{RT}_{1,\mathbb{N}} \,|_\mathrm{W}\, \mathsf{SRT}_{2,+}$.

We can conclude the following results on $\mathsf{C}_\mathbb{N}$ from earlier results.

Proposition 7.8 (Closed choice). $\mathsf{C}_\mathbb{N} \equiv_\mathrm{W} \mathsf{SRT}_{1,\mathbb{N}}$ *and* $\mathsf{C}_\mathbb{N} \not\leq_\mathrm{W} \mathsf{RT}_{1,+}$.

Proof. The first claim follows from Proposition 3.4 since $\mathsf{C}_\mathbb{N} \equiv_\mathrm{W} \lim_\mathbb{N}$ and the second claim follows from the fact that $\mathsf{C}_\mathbb{N}$ is join irreducible by [7, Fact 3.2], but $\mathsf{C}_\mathbb{N} \leq_\mathrm{W} \mathsf{RT}_{1,k}$ for some $k \geq 1$ is impossible by Proposition 7.4 since $\mathsf{K}_\mathbb{N} \leq_\mathrm{W} \mathsf{C}_\mathbb{N}$. $\square$

Altogether, we have justified the way choice problems are displayed in Figure 1. Finally, we provide the following reduction that can be derived from earlier results.

Theorem 7.9 (Jumps of compact choice)**.** $\mathsf{K}_{\mathbb{N}}^{(n)} \leq_{\mathrm{W}} \mathsf{SRT}_{n,\mathbb{N}}$ *for all* $n \geq 2$.

Proof. By Propositions 3.4 and 4.1 we obtain

$$\mathsf{K}_{\mathbb{N}}^{(n)} \equiv_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{N}}^{(n-1)} \leq_{\mathrm{sW}} \mathsf{CRT}_{1,\mathbb{N}}^{(n-1)} \leq_{\mathrm{sW}} \mathsf{CSRT}_{n,\mathbb{N}} \leq_{\mathrm{W}} \mathsf{SRT}_{n,\mathbb{N}}.$$

□

The special case for $n = 2$ can be seen as the uniform version of a theorem of Cholak, Jockusch and Slaman [9], see also [17, Theorem 6.89], which states that $\mathsf{SRT}^2_{<\infty}$ proves $\mathsf{B}\Sigma^0_3$ over RCA_0. In light of Corollary 7.5 and Theorem 7.9 there is quite some gap in between SRT_{2+} and $\mathsf{SRT}_{2,\mathbb{N}}$. The diagram summarizes our calibration of choice problems by Ramsey's Theorem.

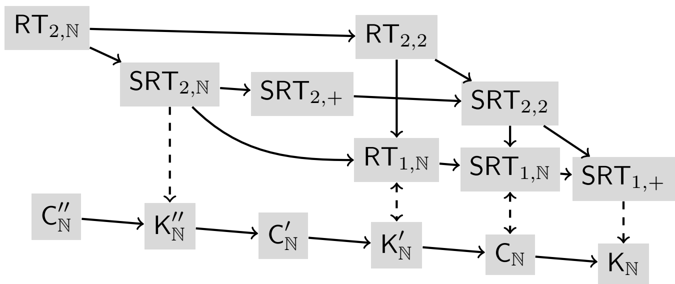

Figure 5. Closed and compact choice on natural numbers calibrated with Ramsey's Theorem in the Weihrauch lattice.

Further questions could be studied along these lines. We mention the following question. The first part of this question is related to [17, Theorem 6.85] and the second part to [17, Open Question 6.92].

Question 7.10 (Jump of closed choice)**.** Does $\mathsf{C}_{\mathbb{N}}' \leq_{\mathrm{W}} \mathsf{RT}_{2,2}$ hold or $\mathsf{C}_{\mathbb{N}}'' \leq_{\mathrm{W}} \mathsf{RT}_{2,\mathbb{N}}$?

In light of [17, Theorem 6.85] a positive answer to the first part of this question seems unlikely. Another natural question is how cluster point problems are related to Ramsey's Theorem. For instance, by [7, Proposition 9.15] we have $\mathsf{CL}_{\mathbb{R}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}' \times \mathsf{CL}_{2^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}' \times \mathsf{WKL}'$. Hence, as a consequence of Theorem 7.9 and Corollaries 5.14, 3.18 and 3.25 we obtain

$$\mathsf{CL}_{\mathbb{R}} \leq_{\mathrm{W}} \mathsf{K}_{\mathbb{N}}'' \times \mathsf{WKL}' \leq_{\mathrm{W}} \mathsf{RT}_{2,\mathbb{N}} \times \mathsf{RT}_{3,2} \leq_{\mathrm{W}} \mathsf{RT}_{3,3}.$$

Hence, we have the following corollary.

Corollary 7.11 (Cluster point problem)**.** $\mathsf{CL}_{\mathbb{R}} \leq_{\mathrm{W}} \mathsf{RT}_{3,3}$.

However, it is not immediately clear whether the following holds.

Question 7.12. $\mathsf{CL}_{\mathbb{R}} \leq_{\mathrm{W}} \mathsf{RT}_{3,2}$?

8. Conclusion

We have studied the uniform computational content of Ramsey's Theorem in the Weihrauch lattice and we have clarified many aspects of Ramsey's Theorem in this context. Key results are the lower bound provided in Theorem 3.5, the Theorems on Products 3.24 and Parallelization 3.28, as well as the Theorem on Jumps 4.3 and the upper bounds in Corollary 4.15 derived from it. From this tool box of key results (together with the Squashing Theorem 1.7) we were able to derive a number of interesting consequences, such as the characterization of the parallelization of Ramsey's Theorem in Corollary 4.18 and the effect of increasing numbers of colors in Theorem 4.21. The separation tools provided in Section 6 have led to some further clarity. A number of important questions regarding the uniform behavior of Ramsey's Theorem were left open. Hopefully, some future study will shed further light on this question.

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