PRIORITY ARGUMENTS IN α-RECURSION THEORY

by

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ABSTRACT

The priority method as applied to recursion on admissible ordinals is studied and developed by generalizing to α-recursion theory several classic theorems of ordinary recursion theory whose proofs require different types of priority arguments.

The simplest form of the priority method is the Friedburg-Muchnik type of finite injury argument. This technique has been generalized to α-recursion theory by G. Sacks and S. Simpson. In chapters I and II we show how the assumption of Σ_n-admissibility can be used to extend their methods to the construction of Σ_n sets with special properties. In particular, the main result of chapter I is the following:

<u>Theorem</u>: If α is Σ_2-admissible there is a minimal α-degree α-recursive in the complete α-r.e. set. Other conditions for the existence of minimal α-degrees are also established but without the use of a priority argument. In chapter II we mix a priority argument with a forcing construction to prove the

<u>Theorem</u>: If α is Σ_n-admissible then there are Δ_n incomparable Σ_n subsets of α. Indeed, we prove that such sets exist uniformly for each α.

In chapter III we prove the <u>Splitting Theorem</u> for every admissible α: Let C be a regular α-r.e. set and D a non-α-recursive α-r.e. set. Then there are regular α-r.e. sets A and B such that $A \cup B = C$, $A \cap B = \emptyset$, $A, B \leq_\alpha C$ and such that D is not α-recursive in A or B. To prove this theorem we develop a method to handle finite injury priority requirements with unbounded preservations.

In chapter IV we sketch a proof of the <u>Density Theorem</u> for all admissible α: If $b <_\alpha c$ are α-r.e. degrees then there is an α-r.e. degree a such that $b <_\alpha a <_\alpha c$. The proof of this theorem, of course, required developing an infinite injury priority argument for all admissible ordinals.

Thesis Supervisor: Gerald E. Sacks

Title: Professor of Mathematics

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Introduction

In generalizing or axiomatizing a field of mathematics one tries to capture in a more abstract form the basic concepts of the subject. Of course, one keeps in mind the essential properties and fundamental theorems that one would like to preserve. The first major test for the correctness or even appropriateness of the attempt is whether one can then also generalize the deeper theorems and important methods of proof that are characteristic of the original field. Thus for recursion theory one attempts to reformulate in a new setting the basic notions of computation, recursive and recursively enumerable. In formulating such an abstraction one is guided, of course, by one's intuition about computation but one also has several basic facts and relationships that must be preserved. Thus one makes sure that basic theorems such as the recursion theorem and enumeration theorem are carried over.

When it comes time to apply the above test to an attempted generalization of recursion theory we feel that the key technique to be handled is the priority method. Correspondingly, the crucial theorems are the ones that depend on the priority method in ordinary recursion theory. Our reason is that the priority method is the most distinctive and characteristic technique of recursion theory, while the theorems for which it is needed are in general the deepest and most difficult in the field.

In this thesis we concern ourselves with the generalization of recursion theory to recursion on admissible ordinals. For the history of this subject and general background we refer the reader to [2], [7], [8] and especially to [13] and its bibliography. As far as the priority method is concerned the first success in this field comes in [10], where G. Sacks and S. Simpson generalize the Friedburg-Muchnik solution to Post's problem to every Σ_1-admissible α. They do much more, however, than just solve this one problem. They develop techniques that will handle any theorem about r.e. sets which in ordinary recursion theory calls for a simple finite injury priority argument. Later, in [3], M. Lerman developed an alternative approach to the finite injury method. While Sacks and Simpson draw heavily on model theoretic notions Lerman's approach is more purely recursion theoretic. Thus crucial cases in [10] depend on sequences of Σ_1-submodels while in [3] the tame Σ_2-projection takes over instead. Both of these papers significantly vindicate α-recursion theory as a generalization of ordinary recursion theory, particularly insofar as recursive enumerability is concerned.

In chapters I and II of this thesis we use the assumption of Σ_n-admissibility to extend the methods of [10] to handle constructions of Σ_n sets rather than just Σ_1 sets. In particular, assuming Σ_2-admissibility in chapter I, we use a finite injury type priority argument to construct a minimal α-degree which is α-recursive in the complete α-r.e. set. We also indicate how to construct without priorities a minimal α-degree (below 0") for each Σ_3-admissible α. Thus the improvement from Σ_3 to Σ_2 corresponds to Sacks' use of a priority argument to construct a minimal degree recursive in 0' rather than 0" in ordinary recursion theory [6]. Of course, what one would ultimately hope for is a further improvement to produce a minimal α-degree for every Σ_1-admissible. Perhaps a more complex type of priority argument is called for.

In chapter II we give another example of this method but with two additional twists. First, we present an argument which is uniform in α rather than one which splits into cases according to the internal nature of α. Second, we mix a forcing construction into the priority argument. We then prove that for any Σ_n-admissible α there is (uniformly) a pair of Σ_n subsets of α which are Δ_n-incomparable.

Clearly the next step in the attack on priority methods in α-recursion theory was to generalize the more complicated types of priority arguments. The major other form of priority argument is, of course, the infinite injury method of Sacks [6]. In [16] Lerman and Sacks use an infinite injury argument to construct minimal pairs of α-r.e. degrees for many but not all Σ_1-admissible α. We attack the infinite injury method by beginning with an intermediate type of argument -- finite injuries but unbounded preservations. In particular we prove (chapter III) the Splitting Theorem for all Σ_1-admissible α. Our method seems to be sufficiently general to handle other examples of this type of argument such as embedding partial ordering into the α-r.e. degrees below any fixed α-r.e. degree. Finally in chapter IV we sketch a proof of the Density Theorem for all Σ_1-admissibles. In addition to the techniques of chapter III we develop the machinery needed to handle a full infinite injury priority argument. Though we, of course, view this as an important success for α-recursion theory we feel that further examples of the infinite injury method must be investigated before we will be in a position to claim a general method comparable to the ones provided in [10] and chapter III for the two types of finite injury arguments. We hope to pursue this course in later work.

In closing, we would like to make a few remarks about other important tests for α-recursion theory. In addition to carrying over basic ideas a good generalization must offer something new as well. It should offer new areas of investigation within the subject itself and should promote new interactions with other fields of mathematics. We feel that α-recursion theory has had, and will continue to have, success in both of these areas. As an example of the first we cite the new problems presented by non-regularity and non-hyperregularity and refer to [13] for the beginnings of important work on these questions. We also feel that our lemma IV.2.1 will allow further progress in study of the non-regular and non-hyperregular α-r.e. degrees. As far as the second criterion is concerned, we would cite on the one hand the important contributions of model and set-theoretic methods to all work on priority arguments in α-recursion theory. Indeed, one could well claim that the subject is becoming part of set theory rather than recursion theory. On the other hand, we have the hope that the methods of α-recursion theory will prove useful in set theory and perhaps also in model theory. As a minor example of this we would cite our chapter II whose results seem to be a part of set theory or definability theory rather than recursion theory, although a priority argument is crucial to the construction. A more important example is furnished by Simpson's report [13] of work in progress on using the methods of α-recursion theory to facilitate forcing constructions over uncountable admissible initial segments of L. Finally, we expect a continued interaction between α-recursion theory and Jensen's set-theoretic work [1] on the fine structure of the constructible universe.

Chapter I

Minimal α-Degrees

In this chapter we consider the problem of constructing minimal α-degrees for admissible ordinals α. Our main result will consist of constructing, for every Σ_2-admissible α, a set recursive in the complete α-r.e. set which is of minimal α-degree though some other cases will also be covered. We assume here and throughout this thesis an acquaintence with the basic concepts of α-recursion theory such as the various notions of α-reducibility and regularity. All the necessary information can be found, for example, in the introductory sections (1,2 and the beginning of 6) of [10]. For more general information on α-recursion theory we refer the reader to [2], [8] and [9].

The first results on minimal degrees is that of Spector [14]. He proved the existence of a minimal degree recursive in $0''$ for ordinary recursion theory. Unfortunately, his proof is not suited to the needs of α-recursion theory [5]. Instead we must turn to the later improvements introduced by Sacks [6] to construct a minimal degree recursive in $0'$. Though the most important change introduced by Sacks was the use of a priority argument, his basic construction was also less restrictive than that of Spector. It was this extra freedom that MacIntyre [4] exploited to construct minimal α-degrees for all countable admissibles α. His construction, however, proceeded in ω-many steps by using a counting of α and so produced a set of minimal degree which in general was not even constructible. On the other hand a minor combinatorial twist (see section 1.7) allows one to carry out the argument for all Σ_3-admissible ordinals (we do this in section 2). The set constructed is, as in Spector's argument, recursive in $0''$. In section 4 we reintroduce the priority technique to make our construction recursive in $0'$. Now if α is Σ_2-admissible we can essentially work recursively in $0'$ without too much difficulty (section 3). Thus we can adopt the techniques of [10] to prove that the priority argument succeeds (section 5) and so our set is indeed a minimal degree as well as recursive in $0'$.

1. Definitions and the basic lemmas

A set B is of minimal α-degree iff B is not α-recursive and every C α-recursive in B is either α-recursive or has B α-recursive in it. Our general plan is to construct B in stages and to see to it that the second condition is taken care of in one of two ways corresponding to the two possible values for the degree of C . Steps will, of course, have to be taken to assure ourselves that B is not recursive but the associated requirements will be much less troublesome. This section contains some definitions and the basic lemmas that will be used to handle the more crucial minimality requirements.

1.1 Sequences: An α-finite sequence σ is a pair $\langle\sigma_0,\sigma_1\rangle$ of disjoint α-finite sets whose union is an initial segment of α called the <u>length of σ</u> (lth σ) . One should think of an α-finite sequences σ as being an initial segment of a representing function with σ_0 as the zeros and σ_1 as the ones of the function. We define concatenation of two α-finite sequences σ and τ , written $\sigma*\tau$, in the obvious way:
$\underline{\sigma*\tau} = \langle\sigma_0 \cup \{\mathrm{lth}\ \sigma + \beta \mid \beta \in \tau_0\}, \sigma_1 \cup \{\mathrm{lth}\ \sigma + \beta \mid \beta \in \tau_1\}\rangle$.
In particular $\sigma*0$ and $\sigma*1$ correspond to extending the representing function associated with σ by assigning the values 0 and 1 respectively to the ordinal lth σ . In general we say that σ <u>extends</u> τ ($\sigma \supseteq \tau$ or $\tau \subseteq \sigma$) if $\sigma_i \subseteq \tau_i$, i = 0,1 and we call σ and τ <u>incompatible</u> if $\sigma \not\subseteq \tau$ and $\tau \not\subseteq \sigma$.
In this section only ρ , σ and τ will be used to denote α-finite sequences exclusively.

1.2 Trees: A <u>tree</u> is a partial α-recursive function T from α-finite sequences to α-finite sequences such that

1) If one of $T(\sigma * 0)$, $T(\sigma * 1)$ is defined then so are all of $T(\sigma)$, $T(\sigma * 0)$ and $T(\sigma * 1)$. In this case $T(\sigma * 0)$ and $T(\sigma * 1)$ are proper incompatible extensions of $T(\sigma)$ which we will also denote by $T(\sigma)^0$, $T(\sigma)^1$.

2) If lth σ is a limit ordinal then $T(\sigma)$ is the limit (or union) of $T(\tau)$ for $\tau \subseteq \sigma$, more precisely $T(\sigma)_i = \bigcup_{\tau \subset \sigma} T(\tau)_i$ for $i = 0,1$ and $T(\sigma)$ is defined iff $T(\tau)$ is defined for every $\tau \subseteq \sigma$.

As examples of the many obvious but useful properties of trees we note the following:

3) If $\sigma \subsetneq \tau$ and $T(\tau)$ is defined then so is $T(\sigma)$ and $T(\sigma) \subsetneq T(\tau)$.

4) If σ and τ are incompatible and $T(\sigma)$, $T(\tau)$ are defined then they are incompatible.

We say that an α-finite sequence <u>τ is on T</u> if there is a σ (necessarily unique) such that $T(\sigma) = \tau$. A set $B \subseteq \alpha$ is said to be on T if there are unboundedly many initial segments of its representing function on T i.e.

$$(\forall\gamma)(\exists\beta > \gamma)(\langle B \cap \beta, \overline{B} \cap \beta\rangle \text{ is on } T) .$$

We call $\langle B \cap \beta, \overline{B} \cap \beta\rangle$ a beginning of B and denote it by B_β . Note that any set that is on some tree is regular.

1.3 Reduction Procedures: In order to handle the minimality requirements we define an α-r.e. approximation to $[\epsilon]^B$:

$$[\epsilon]^{\tau}(x) = y \quad \text{if}$$

$$(\exists M,N)(\langle M,N,x,y\rangle \in R_\epsilon \ \& \ M \subseteq \tau_0 \ , \ N \subseteq \tau_1) \ .$$

(Recall that R_ϵ is ϵ-th α-r.e. set) .

Clearly if τ is a beginning of B and $[\epsilon]^{\tau}(x) = y$ then $[\epsilon]^{B}(x) = y$. Conversely if B is regular and $[\epsilon]^{B}(x) = y$ there is a beginning of B , τ , such that $[\epsilon]^{\tau}(x) = y$, indeed if $\sigma \supseteq \tau$ then $[\epsilon]^{\sigma}(x) = y$ as well. Using this approximation to $[\epsilon]^{B}$ we define the key notion of splitting and prove the lemma that is associated with satisfying a minimality requirement by making $[\epsilon]^{B}$ recursive:

<u>σ and τ split ρ for ϵ</u> iff $\sigma, \tau \supseteq \rho$ and there are distinct y_1 and y_2 such that for some x $[\epsilon]^{\sigma}(x) = y_1$ and $[\epsilon]^{\tau}(x) = y_2$. This is clearly an α-r.e. relation of σ and τ .

Lemma 1.4: Suppose $B \subseteq \alpha$ is on a tree T and for some beginning B_β of B no pair of sequences on T split B_β for ϵ . Then if $[\epsilon]^{B}$ is a representing function it is α-recursive.

Proof: To compute $[\epsilon]^{B}(x)$ look for any τ on T which extends B_β such that $[\epsilon]^{\tau}(x)$ is defined. By our remarks above there is beginning of B , τ , such that $[\epsilon]^{\tau}(x) = [\epsilon]^{B}(x)$ and so there is one extending B_β . On the other hand, since there is no splitting, any τ extending B_β for which $[\epsilon]^{\tau}(x)$ is defined must give this same answer. Since this search is α-recursive so is $[\epsilon]^{B}$. $\square$

We now turn to the second way of satisfying the minimality requirement - making sure that B is α-recursive in $[\epsilon]^B$.

1.5 T is a <u>splitting tree for ϵ</u> if whenever $T(\sigma*0)$ and $T(\sigma*1)$ are defined they split $T(\sigma)$ for ϵ .

Lemma 1.6: If a hyperregular set B is on T , a splitting tree for ϵ, and $[\epsilon]^B$ is a representing function then $B \leq_\alpha [\epsilon]^B$.

Proof: Since B is regular (it is on T) and hyperregular it suffices [7] to prove that $B \leq_{c\alpha} [\epsilon]^B$ and so to show how to calculate initial segments of B from $[\epsilon]^B$ step by step: say $T(\sigma)$ has been calculated to be a beginning of B . To advance one more step all we have to do is decide which of $T(\sigma*0)$ and $T(\sigma*1)$ is a beginning of B . Since T is a splitting tree for ϵ we can α-recursively find an x and distinct y_1 and y_2 such that $[\epsilon]^{T(\sigma*0)}(x) = y_1$ and $[\epsilon]^{T(\sigma*1)}(x) = y_2$. Since $[\epsilon]^B$ is a representing function there is exactly one value for $[\epsilon]^\tau(x)$ for any τ a beginning of B for which $[\epsilon]^\tau(x)$ is defined. Thus $[\epsilon]^B(x) = y_i$ for i = 0 or 1 and $T(\sigma*i)$ is then beginning of B . □

This lemma can in fact be proven without the assumption that B is hyperregular [4] but the proof is more difficult and less intuitive. In any case our final set B will be hyperregular.

Now for some operations on trees that will indicate how we intend to exploit lemmas 1.4 and 1.6.

1.7 The Splitting Trees: For τ on T the tree $Sp(T,\epsilon,\tau)$ is computed as follows: $Sp(T,\epsilon,\tau)(\emptyset) = \tau$. If $Sp(T,\epsilon,\tau)(\sigma)$ has been computed as $T(\rho)$, look for the least computation showing that some pair of α-finite sequences η_0, η_1 on T with $\eta_i \supseteq T(\rho * i)$ split $T(\rho)$ for ϵ. When this pair is found set $Sp(T,\epsilon,\tau)(\sigma * i) = \eta_i$. (Of course if there is no such pair they are undefined). Finally if lth σ is a limit ordinal we set $Sp(T,\epsilon,\tau)(\sigma) = \bigcup_{\rho \subseteq \sigma} Sp(T,\epsilon,\tau)(\rho)$.

This definition has one oddity that should be remarked on — the requirement that $\eta_i \supseteq T(\rho * i)$. Though not needed in ordinary minimal degree arguments it is included here to make sure that when the operation is iterated into the transfinite the resulting trees do not shrink down to a single path (see 1.9 and 1.10 for further details). This twist was introduced independently by MacIntyre [5] to handle the case of α a regular cardinal of L.

1.8 The Full Tree: Clearly $Sp(T,\epsilon,\tau)$ is a splitting tree for ϵ. Thus, if for every ϵ we could arrange for our final set B to be on some $Sp(T,\epsilon,\tau)$ we would satisfy every minimality requirement by lemma 1.6. This however may not be possible — some beginning τ of B which is on some given T may have no extension on $Sp(T,\epsilon,\tau)$ and so we would seem to be stuck. What we do in this case is fall back on lemma 1.4. If τ has no proper extension on $Sp(T,\epsilon,\tau)$ then there are no η_0, η_1 on T extending τ^0 and τ^1 respectively which split τ for ϵ (recall that τ^0 and τ^1 are the immediate extensions of τ on T). There may, however, be ρ_1, ρ_2 on T which split τ for

ϵ and so lemma 1.4 would not apply directly. There are two possibilities. If there is a split ρ_1, ρ_2 extending τ^1 and $[\epsilon]^{\rho_1}(x) \neq [\epsilon]^{\rho_2}(x)$ then $[\epsilon]^{\rho}(x)$ is undefined for all $\rho \supseteq \tau^0$. In this case we try to assure ourselves that τ^0 is a beginning of B and that B lies on T for then by our remarks in 1.3 $[\epsilon]^B$ is not total and so does not concern us. On the other hand if there is no such split and we make τ^1 a beginning of B then lemma 1.4 will tell us that (modulo the hyperregularity of B) we have successfully handled the minimality requirement for ϵ. In either case we will take $\rho = \tau^0$ or τ^1 as indicated and form a new tree on which B will hopefully lie. The desired tree is Fu (T,ρ), the full subtree of T above ρ, which is defined by Fu $(T,\rho)(\sigma) = T(\rho_1 * \sigma)$ where $T(\rho_1) = \rho$. (T' is a subtree of T if every sequence on T' is on T; it lies above ρ if every sequence on T' extends ρ).

1.9 Intersecting Trees: Since our construction will take place in infinitely many steps we must provide a method to carry us through the limit stages (at successor stages we will essentially just use Sp or Fu). To that end we define an α-recursive function RI (recursive intersection) which operates on α-finite sequences of trees $\{T_\nu\}_{\nu<\lambda}$.

$$\text{RI } \{T_\nu\} \ (\emptyset) = \bigcup_{\nu<\lambda} T_\nu \ (\emptyset) \quad \text{if they are compatible.}$$

If RI $\{T_\nu\}$ (σ) has been calculated and is on every T_ν (say $T_\nu(\tau_\nu) = \sigma$) then we set RI $\{T_\nu\}(\sigma * i) = \bigcup T_\nu(\tau_\nu * i)$

assuming they are compatible. Of course if $lth\ \sigma$ is a limit ordinal $RI\ \{T_\nu\}\ (\sigma) = \bigcup_{\tau \subseteq \sigma} RI\ \{T_\nu\}\ (\tau)$. Since the sequence $\{T_\nu\}_{\nu<\lambda}$ is α-finite we can α-recursively compute all of $T_\nu(\tau_\nu)$ if they are in fact defined and if they are compatible this union must be of length less than α by the admissiblity of α . Thus $RI\ \{T_\nu\}$ is a partial α-recursive function. It may however be empty and even if not it may fail to be a tree since $RI\ \{T_\nu\}\ (\sigma*0)$ can be defined without $RI\ \{T_\nu\}\ (\sigma*1)$ being defined. The following important lemma obviates these difficulties in our construction.

Lemma 1.10: If $\{T_\nu\}_{\nu<\lambda}$ is an α-finite sequence of non empty trees such that for every $\nu < \lambda$, $T_{\nu+1}$ is gotten from T_ν by an application of Sp or Fu and every T_γ for limit $\gamma < \lambda$ is $RI\ \{T_\nu\}_{\nu<\lambda}$, then $RI\ \{T_\nu\}_{\nu<\lambda}$ is a non empty tree and indeed if σ is on $RI\ \{T_\nu\}$ and has a proper extension on every T_ν then it has one on $RI\ \{T_\nu\}$.

<u>Proof</u>: Assume the lemma for all limit ordinals less than λ . It is then clear that the T_ν are nested, i.e. if $\nu < \nu'$ then T_ν is a subtree of $T_{\nu'}$. Thus $T_\nu(\emptyset)$ is a nested sequence and $RI\ \{T_\nu\}(\emptyset) = \bigcup T_\nu(\emptyset)$. If then $\sigma = T_\nu(\tau_\nu)$ is on $RI\ \{T_\nu\}$ and $T_\nu\ (\tau_\nu*i)\ i = 0,1$ are defined for every ν , we claim $T_\nu\ (\tau_\nu*i)$ are nested sequences. By induction it suffices to show that for any $\nu < \lambda$ $T_\nu(\tau_\nu*i) \subseteq T_{\nu+1}(\tau_{\nu+1}*i)$. We have two possibilities $T_{\nu+1} = Sp(T_\nu,\epsilon,\rho)$ or $T_{\nu+1} = Fu(T_\nu,\rho)$ for some ρ on T_ν . In either case it is clear that

$\rho \subseteq RI\ \{T_\nu\}\ (\emptyset)$ and so σ is on T_ν above ρ . In the first case we recall from the definition of Sp that since $T_{\nu+1}(\tau_{\nu+1}) = T_\nu(\tau_\nu)$ we have that $T_{\nu+1}(\tau_{\nu+1}*i) \supseteq T_\nu(\tau_\nu*i)$. In the second case it is immediate from the definition of Fu that $T_{\nu+1}(\tau_{\nu+1}*i) = T_\nu(\tau_\nu*i)$. Thus the $T_\nu(\tau_\nu*i)$ are compatible and so $RI\ \{T_\nu\}\ (\sigma*i) = \cup\ T_\nu(\tau_\nu*i)$. $\square$

We remark that Sp , Fu and RI not only give recursive trees but that Gödel numbers for these trees can be found uniformly from the arguments.

2. Minimal degrees without priorities

In this section we give the basic minimal degree argument without priorities as in [4,5] where MacIntyre proved that every countable admissible and every regular cardinal of L have minimal degrees. The method actually can be seen to give the result for every Σ_3-admissible as well as for a few ordinals which are not even Σ_2-admissible. We also include some relevant examples.

2.1. The construction: Say we have a list of the α reduction procedures of length $\gamma \leq \alpha$ given by $k: \gamma \xrightarrow{\text{onto}} \alpha$. We construct our set B in γ many stages. At each stage σ we will have an initial segment β_σ of B of length $\geq \bigcup_{\delta<\sigma} k(\delta)$ and a total tree T_σ on which β_σ lies:

Stage 0: Let T_0 be the identity map and let β_0 be $\emptyset$.

Stage $\sigma = \delta+1$: Since T_δ is total we can choose an extension β of β_δ of length at least $\bigcup_{\eta\leq\delta} k(\eta)$ which is incompatible with $R_{k(\delta)}$. If $Sp(T_\delta, k(\delta), \beta)$ is total let it be $T_{\delta+1}$ and let $\beta_\sigma = \beta$. Otherwise let $T_{\delta+1} = Fu(T_\sigma, \beta^i)$ where $i = 0$ or 1 is chosen as indicated by the discussion of 1.8 and let $\beta_\sigma = \beta^i$. In either case $T_{\delta+1}$ is total.

Stage λ: Let $\beta_\lambda = \bigcup_{\sigma<\lambda} \beta_\sigma$ and $T_\lambda = RI\ \{T_\sigma\}_{\sigma<\lambda}$. If $\{T_\sigma\}_{\sigma<\lambda}$ is α-finite then by lemma 1.10 T_λ is a total subtree of every T_σ and $\beta_\lambda = \bigcup_{\sigma<\lambda} T_\sigma(\emptyset) = T_\lambda(\emptyset)$.

The only difficulty arising in the construction is the possibility that at some limit stage the sequence $\{T_\sigma\}_{\sigma<\lambda}$ might not be α-finite. A reasonably straightforward calculation, however, will show that if k is the identity and α is Σ_3-admissible then $\{T_\sigma\}_{\sigma<\lambda}$ is always α-finite. Alternatively we can in a brute force way require that (*) γ be ω or a regular cardinal of some model of $V = L$ and that every subset of α of size less than γ in the model is in fact a member of L_α. Examples of ordinals with this second property include all countable admissibles as well as all admissibles α such that α^* is a regular cardinal of some model of $V = L$ and more generally all α such that the last cardinal of L_α is a regular cardinal of some model of $V = L$ in which it is also the cofinality of α. To see that this last example is correct consider any $C \subseteq \alpha$ whose cardinality in the model is less than $\bar{\bar{\alpha}}$. Since $cf(\alpha) = \bar{\bar{\alpha}}$, C is bounded below α, say by δ. As $\bar{\bar{\alpha}}$ is the last cardinal of L_α there is an α-finite map $f\colon \delta \xrightarrow{1-1} \bar{\bar{\alpha}}$. Now $\bar{\bar{C}} < \bar{\bar{\alpha}}$ assures us that $f[C]$ is a bounded subset of $\bar{\bar{\alpha}}$ and so in fact a member of $L_{\bar{\bar{\alpha}}} \subseteq L_\alpha$. We can thus invert f on the α-finite set $f[C]$ to obtain C itself as an α-finite set.

2.2 The minimality of B: Assuming the construction of 2.1 can be carried out we

Claim: B is of minimal degree.

Proof: B is not α-recursive nor even α-r.e. since we made sure at step δ that $B \neq R_{k(\delta)}$ for each $\delta < \gamma$ (of course $k[\gamma] = \alpha$). Consider now any α-reduction procedure $t = k(\delta)$. B is clearly on every T_ν, $\nu < k$ and so if $T_{\delta+1} = Sp(T_\delta, k(\delta), \beta_{\delta+1})$ and $[\epsilon]^B$ is a representing function then lemma 1.6 tells us that $B \leq_\alpha [\epsilon]^B$. If on the other hand $T_{\delta+1}$ is $Fu\ (T_\delta, \beta_\delta)$, then the discussion of 1.8 assures us that (modulo the hyperregularity of B) if $[\epsilon]^B$ is a representing function it is recursive. Though we have not taken any steps to make B hyperregular we could easily do so as is done in [5]. Alternatively we can again refer to a proof of lemma 1.4 that does not use hyperregularity [4]. □

2.3 Some counterexamples: Our first example shows that the construction of 2.1 does not handle as many ordinals as one might at first believe. More specifically we indicate how to construct examples of admissible ordinals α such that $\bar{\bar{\alpha}}$ is a regular cardinal of L and $cf(\alpha) = \bar{\bar{\alpha}}$ but α does not have the property (*). Working in L, we let β be the first admissible after ω_2 which is of cofinality ω_1 and let S be a Skolem-function for L_{ω_3}. Define sets as follows $A_0 = \omega_1 \cup \{\omega_1\} \cup \{\omega_2\} \cup \{\beta\}$, $A_{i+1} = S[(\cup A_i \cap \omega_2) + 1 \cup A_i]$ $A = \bigcup_{i<\omega} A_i$. A is clearly an elementary submodel of L_{ω_3} of cardinality ω_1 and $A \cap \omega_2$ is an initial segment of ω_2. If $\bar{A} = L_\delta$ $(\omega_1 < \delta < \omega_2)$ is the transitive collapse of A then $\bar{\omega}_2 = \cup A \cap \omega_2$ has cofinality ω since $\bar{\omega}_2 = \bigcup_{i<\omega} A_i \cap \omega_2$. Since $\bar{\omega}_2$ is a regular cardinal of L_δ, however, the sequence $\cup (A_i \cap \omega_2)$ is not a member of L_δ and so of course not in $L_{\bar{\beta}}$. $\bar{\beta}$ on the other hand is admissible and of cofinality ω_1

by the elementariness of A. It is our desired example.

Our second example touches on the question of hyperregularity and exhibits various ordinals which do not have non-hyperregular minimal degrees.

<u>Claim</u>: (Assume $V = L$). If ω_1 is the last cardinal of L_α and $cf(\alpha) = \omega$ then any non-hyperregular set B is complete i.e. every α-r.e. set is α-recursive in B.

Proof: Since B is non-hyperregular, there is a function $f \leq_{w\alpha} B$ whose domain is an initial segment of α and whose range is unbounded in α. As ω_1 is the last cardinal of L_α, we can clearly take f such that $\mathrm{dom}\, f \leq \omega_1$. On the other hand, $cf(\alpha) = \omega$ implies that some countable subset of $\mathrm{dom}\, f$ is sent unboundedly into α. Finally, since $\alpha > \omega_1$ and $V = L$, this subset is constructible in L_α and so we may take $\mathrm{dom}\, f = \omega$. Let C be any α-r.e. set, which we can assume to be regular without any loss of generality [9]. For some limited formula $\varphi(x,y)$, $y \in C \iff L_\alpha \models \exists x\, \varphi(x,y)$. We define a function $j\colon \omega \to \omega$ as follows: $j(n)$ is the least m such that $\forall y < f(n)\ [\exists x \in L_\alpha\, \varphi(x,y) \iff \exists x \in L_{f(m)}\, \varphi(x,y)]$. j exists since C is regular, while $V = L$ assures us that $j \in L_{\omega_1} \subseteq L_\alpha$. Using j as a parameter, it is easy to see that C is α-recursive in B. Given a set m and the task of deciding if $m \cap C = \emptyset$, just compute finitely many values of f until an n is reached such that $f(n) > \cup M$. Now we just check α-recursively if there is a y in M and an $x < j(n)$ such that $\varphi(x,y)$. If yes then $y \in M \cap C$ and if not $M \cap C = \emptyset$. $\square$

This argument is essentially just a simplification of one in [13] where Simpson proves that any non-hyperregular α-r.e. degree is complete for various α's . Though he uses a special representation theorem for α-r.e. sets which is not true for arbitrary subsets of α, it is not really needed. We note, for example, that the argument given here also works for $\aleph_\omega$ and α's such that $\aleph_\omega$ is the last cardinal of L_α and such that $cf(\alpha) = \omega$.

Finally we would like to point out that by Cor III.3.4 no α-r.e. degree can be minimal.

3. Σ_2-Admissible Ordinals

An ordinal α is Σ_2-admissible if there is no function Σ_2 over L_α which maps a proper initial segment of α onto an unbounded subset of α. Though we will be using Σ_2 functions and so Σ_2-admisibility rather heavily, we would like to disguise this fact and make things look as much like the Σ_1 case as possible. We will then be able to carry out most of our arguments formally as in the case of Σ_1-functions and admissibility. Thus we follow Jensen [1] in making a reduction from Σ_2 to Σ_1 by adding on an extra predicate A to the structure L_α and working in the expanded $\langle L_\alpha, A\rangle$. To be specific we let A be a complete Σ_1-set, e.g. $A = \{\langle\eta,\nu\rangle \mid \eta \in R_\nu\}$ or for the more model-theoretically oriented we could let A code Σ_1 satisfaction for L_α in a more obvious way. Now note the following facts:

Prop. 3.1: 1) α is Σ_2 admissible $\Leftrightarrow$ $\langle L_\alpha, A\rangle$ is admissible $\Leftrightarrow$ A is regular and hyperregular.

For such α we also have

2) $B \subseteq \alpha$ is $\alpha - A$ recursively enumerable, i.e. Σ_1 over $\langle L_\alpha, A\rangle$, iff B is recursively enumerable in A iff B is Σ_2 over L_α. Similarly B is α-A recursive, i.e. Δ_1 over $\langle L_\alpha, A\rangle$, iff B is α-recursive in A iff B is Δ_2 over L_α.

3) On the other hand $B \subseteq \alpha$ is α-A finite, i.e. α-A recursive and bounded, iff B is α-finite. Thus for example the notion of α-A cardinal coincides with that of α-cardinal.

Except for the second equivalence of (1), which can be found in [7,8], the proofs of all these facts are by straightforward quantifier manipulations using the Σ_2 admissibility of α and so are omitted. A remark on the roles these facts will play is, however, useful. (2) tells us that adjoining A really does reduce Σ_2 to Σ_1, while (1) assures us that we can still carry out in $\langle L_\alpha, A\rangle$ most of the familiar operations of recursion theory on Σ_1-admissibles. Finally (3), though essientially a restatement of the Σ_2-admissibility of α, embodies the form in which this property will be most heavily used.

The basic recursion theoretic facts such as Gödel numbering, the enumeration theorem and the recursion theorem are of course true in any admissible structure, but $\langle L_\alpha, A\rangle$ is much closer to an admissible ordinal than an arbitrary admissible set. We give two examples of the analogy and the reduction of Σ_2 to Σ_1 that is introduced. First consider the basic lemma of the Sacks-Simpson priority argument [10] in this generalized setting.

Lemma 3.2: Let $\aleph$ be a regular α-A cardinal and let $\{I_\nu \mid \nu < \rho\}$ for some $\rho < \aleph$ be a uniformly α-A-r.e. sequence of α-A-finite sets each of α-A-cardinality less than $\aleph$. Then $\bigcup_{\nu<\rho} I_\nu$ is α-A finite and of α-A cardinality $< \aleph$.

Proof: Since we will later need the proof of this lemma as well as the statement we repeat the proof from [10]. Let g be a 1-1 α-A recursive enumeration of $\bigcup_{\nu<\rho} I_\nu \times \{\nu\}$. If the domain of g is less than $\aleph$ we are of course done (by admissibility). So say $\aleph \subset \mathrm{dom}\ g$ and consider $g[\aleph]$. Since g is α-A recursive and $\langle L_\alpha, A\rangle$ is admissible, $g[\aleph]$ is α-A finite and of α-A cardinality $\aleph$. Clearly $g[\aleph] = \bigcup_{\nu<\rho} J_\nu$, where J_ν is $g[\aleph] \cap I_\nu \times \{\nu\}$, and so is of α-A cardinality $< \aleph$. On the other hand, $\bigcup J_\nu$ is an α-A-recursive union and so we have contradicted the regularity of $\aleph$. □

To illustrate the reduction of Σ_2 to Σ_1 we translate this lemma into the language of L_α.

Lemma 3.2': Let $\aleph$ be a regular α-cardinal and let $\{I_\nu \mid \nu < \rho\}$ for some $\rho < \aleph$ be a sequence of α-finite sets of α-cardinality $< \aleph$ which is uniformly Σ_2 over L_α. Then $\bigcup I_\nu$ is α-finite and of α-cardinality less than $\aleph$.

3.3 The α-A projection: Following Jensen we define $\rho^1_{\alpha,A}$ to be the least ordinal ρ such that there is a partial function $f\colon \rho \xrightarrow{\text{onto}} \alpha$ which is Σ_1 (L_α, A). For Σ_2-admissible α, $\rho^1_{\alpha,A}$ enjoys most of the properties generally associated with α^* and will act as its analog. Thus for example, if $\rho^1_{\alpha,A} < \alpha$ it is the last α-A (and so α) cardinal, any α-A r.e. set bounded below $\rho^1_{\alpha,A}$ is α-A finite, and we can index the reduction procedures of $\langle L_\alpha, A\rangle$ by a $\rho^1_{\alpha,A}$ list recursively in $\langle L_\alpha, A\rangle$. Of course $\rho^1_{\alpha,A}$ is just the usual Σ_2 projectum of α.

All such facts as 3.2' and those about $\rho^1_{\alpha,A}$ could of course be proven directly in L_α, though the manipulations become more involved. The advantage of working in $\langle L_\alpha, A\rangle$ is that there is no need to reprove them — they all become obvious. For example we can essentially mimic the Sacks-Simpson solution of Posts problem in α-recursion theory [10] to get an analogous incomparability result for Σ_2 sets (ch. II).

One word of warning is in order, however. Though recursion theoretic and combinatorial arguments tend to carry over to $\langle L_\alpha, A\rangle$ without any changes, model theoretic ones do not. The main obstacles arise because a Σ_1 (L_α, A)-Skolem hull of a transitive set is not necessarily transitive, while one taken relative to L_α is always transitive. In particular, the notion of an α-stable ordinal (i.e. one which is a Σ_1 submodel of L_α) does not prove useful in this setting and will be replaced.

Remark 3.4: Nothing in this section is specific to Σ_2. Indeed everything remains correct when 2 is replaced by n throughout for any $n > 2$. The only real break occurs when one goes from Σ_1 to Σ_2.

4. The Construction

4.1: We assume α is Σ_2 admissible and work in $\langle L_\alpha, A\rangle$ in the sense that our construction will be α-A recursive and our priority arguments will be done accordingly in $\langle L_\alpha, A\rangle$. On the other hand, we are trying to build a set which is of minimal degree in L_α and so we are basically interested in reduction procedures in L_α rather than $\langle L_\alpha, A\rangle$ and so we compromise on our notation: all reduction procedures $[\epsilon]$ and r.e. sets R_ϵ will be assumed to be ones from L_α and all trees are α-recursive functions.

At the end of each stage σ of our construction we will have an α-A-finite sequence of trees $\{T_i^\sigma\}_{i\le f_\sigma}$ for some $f_\sigma \le \sigma$ such that each T_{i+1}^σ is obtained from T_i^σ by an application of Sp or Fu and each T_λ^σ is RI $\{T_i^\sigma\}_{i<\lambda}$. We will also have an α-finite sequence β_σ which is on every T_i^σ such that $\beta_{\sigma'} \subsetneq \beta_\sigma$ for all $\sigma' < \sigma$. $B = \bigcup_{\sigma<\alpha} \beta_\sigma$ will of course be our desired set of minimal degree. The construction will differ from the straightforward one of §2 in that the trees will no longer be required to be total functions since such decisions are not α-A recursive. As a result f_σ can no longer be a strictly increasing function. The impact of the priority argument in the next section will be that though f_σ is not monotone it eventually stays above each point of interest, and so for each i there is eventually a T_i which takes care of the requirement of priority i. Unfortunately, we cannot in all cases recursively specify the priorities in advance, but must content ourselves, as in section 5 of [10], with an approximation procedure which eventually orders the priorities correctly. This approximation will be given by an α-A recursive function $k(\sigma,\epsilon)$ which tells us what requirement has priority ϵ at stage σ. The precise definition of $k(\sigma,\epsilon)$ depends on the nature of α and will be given in §5. For the purposes of understanding the construction it will suffice to note that $k(\sigma,\epsilon)$ will have the following properties:

1) There is a $\delta \leq \alpha$ called the domain of k such that $k(\sigma,\epsilon)$, as a function of ϵ, is eventually constant on every initial segment of δ. (δ should be thought of as the final order type of the priorities). We let $k(\epsilon) = \lim_{\sigma\to\alpha} k(\sigma,\epsilon)$ for $\epsilon < \delta$.

2) For every $\gamma < \alpha$ there is an $\epsilon < \delta$ such that $k(\epsilon) = \delta$.

3) If τ is the least θ such that $(\forall\sigma \geq \theta)(k(\sigma,\epsilon) = k(\epsilon))$ then τ is zero or a successor.

The first two properties are to make the priority argument work, while the last is just technically convenient for the construction. At first reading it is probably best to think of k as the identity function as it will in fact be when $\rho^1_{\alpha,A} = \alpha$, and to ignore the complications introduced by changes in the value of $k(\sigma,\epsilon)$.

4.2 We begin the construction at stage 0 by setting $f_0 = 0$, T^0_0 = identity and $\beta_0 = \emptyset$. We continue as follows:

Stage $\sigma = \gamma + 1$. Keeping in mind the convention that when one sets x equal to the least $y < t$ with some property and there is no such $y < t$ then $x = t$, we let η be the least $\epsilon < f_\gamma$ such that $k(\sigma,\epsilon) \neq k(\gamma,\epsilon)$ and set f_σ equal to the least $i < \eta + 1$ such that there is no proper extension of β_γ on T_i^γ.

Case 1: $f_\sigma = \eta + 1$. By the minimalify of f_σ, β_γ has a proper extension on T_η^γ. By 1.2.1 it has two incompatible proper extensions on T_η^γ one of which, say β_γ^0, is necessarily incompatible with $R_{k(\sigma,\eta)}$. We let $\beta_\sigma = \beta_\gamma^0$ and set

$$T_i^\sigma = T_i^\gamma \quad \text{for} \quad i < f_\sigma = \eta + 1$$

$$\text{and} \qquad T_{f_\sigma}^\sigma = Sp(T_\eta^\sigma, k(\sigma,\eta), \beta_\sigma) .$$

Case 2: $f_\sigma < \eta$. By lemma 1.10 f_σ is again a successor, say $\nu + 1$. Now if $T_{\nu+1}^\gamma$ were of the form $In(T_\nu^\gamma, \beta)$ then any sequence such as β_γ which is on both and has a proper extension on T_ν^γ has one on $T_{\nu+1}^\gamma$. Since this would contradict the definition of f_σ, we must have $T_{\nu+1}^\gamma = Sp(T_\nu^\gamma, k(\gamma,\nu), \beta)$ for some $\beta \subseteq \beta_\gamma$, as these are the only trees we ever put in at successor places. We now let β_σ be the immediate successor of β_γ on T_ν^γ dictated by 1.8 and set

$$T_i^\sigma = T_i^\gamma \qquad i < f_\sigma = \nu + 1$$

$$T_{f_\sigma}^\sigma = In(T_\nu^\gamma, \beta_\sigma) .$$

Note that all the information needed to carry out these steps is α-A-recursive. Essentially we ask only if certain partial α-recursive functions are defined at specified points, and except for these questions which are trivially recursive in A and computing the α-A recursive function k we proceed α-recursively.

Stage λ: Let $\eta = \mu\theta(\{\gamma \mid f_\gamma \leq \theta\}$ is unbounded in $\lambda)$. Since T_i^σ can change at stage σ only if $f_\sigma \leq i$, the definition of η assures us that for $i < \eta$, T_i^σ is eventually constant as $\sigma \to \lambda$. Similarly $\lim_{\sigma\to\lambda} k(\sigma,i)$ exists for every $i < \eta$. We now set $f_\lambda = (\mu\theta < \eta)((\exists\nu < \theta)(\lim_{\sigma\to\lambda} k(\sigma,\nu) \neq k(\lambda,\nu))$. By induction the β_σ $(\sigma<\lambda)$ are nested and so we can set $\beta_\lambda = \bigcup_{\sigma<\lambda} \beta_\sigma$. $T_i^\lambda = \lim_{\sigma\to\lambda} T_i^\sigma$ for $i < f_\lambda$, of course, but there are two possibilities for $T_{f_\lambda}^\lambda$. If $f_\lambda = \nu+1$ (which can happen when the changes in $k(\sigma,\nu)$ are unbounded in λ or when $f_\lambda < \eta$), we set $T_{f_\lambda}^\lambda = Sp(T_\nu^\lambda, k(\lambda,\nu), \beta_\lambda)$. On the other hand if f_λ is a limit ordinal we set $T_{f_\lambda}^\lambda = RI\ \{T_i^\lambda\}_{i<f_\lambda}$. Note that since our construction is α-A recursive everything generated by some stage is α-A finite and so α-finite (Prop. 3.1.3). In particular $\{T_i^\lambda\}_{i<f_\lambda}$ is α-finite and so $T_{f_\lambda}^\lambda$ is a tree. This is the crucial use of the Σ_2-admissibility of α in the construction.

The construction is now complete. In the next section we will make the choice of k explicit and show that the priority argument succeeds i.e. for every $\epsilon <$ dom k $\{\sigma \mid f_\sigma < \epsilon\}$ is bounded. Before we do this however we will show that this suffices to establish the minimality of B .

Lemma 4.3: For every $\epsilon < \mathrm{dom}\, k$ there is a last stage σ_ϵ for which $f_{\sigma_\epsilon} = \epsilon$. Moreover if $\epsilon < \epsilon'$ then $\sigma_\epsilon < \sigma_{\epsilon'}$

Proof: Since $\{\sigma \mid f_\sigma < \epsilon + 1\}$ is bounded we can let $\gamma = \bigcup \{\sigma \mid f_\sigma < \epsilon + 1\}$ and then look at f_γ . To prove the lemma it clearly suffices to show that $f_\gamma = \epsilon$. An inspection of the successor stage of the construction shows us that $f_{\gamma+1} \leq f_\gamma + 1$ and so $f_\gamma < \epsilon$ would contradict the definition of γ . On the other hand if $f_\gamma > \epsilon$ and $\gamma = \nu + 1$ then $\bigcup \{\sigma \mid f_\sigma < \epsilon + 1\} \leq \nu < \gamma$ — a contradiction. Finally we have the possibility that γ is a limit ordinal and $f_\gamma > \epsilon$. In this case $\{\sigma \mid f_\sigma < \epsilon + 1\}$ must be unbounded in γ and so the limit stage part of the construction tells us that $f_\gamma \leq \epsilon$ — again a contradiction and so $f_\gamma = \epsilon$. □

Lemma 4.4: B is not α-recursive.

Proof: We in fact show that B is not α-r.e. Consider any R_ϵ . By 4.2.1 and 4.2.2 we can find an η such that $k(\eta) = \epsilon$ and also the least τ such that $(\forall \sigma \geq \tau)(k(\sigma, \eta) = k(\tau, \eta) = k(\eta))$. Since τ is the least such ordinal, $f_\tau \leq \eta + 1$ and so by lemma 4.3 there is a first $\sigma \geq \tau$ such that $f_\sigma = \eta + 1$. We claim that σ is a successor: If σ is a limit ordinal greater than τ then $f_\sigma = \eta + 1$ implies, by our choice of τ , that $\{\gamma \mid f_\gamma = \eta + 1\}$ is unbounded in σ and so σ cannot be least. On the other hand 4.2.3 assures us that τ is a successor. Now since $f_\tau \leq \eta + 1$ and

σ is the first stage $\geq \tau$ in which $f_\sigma = \eta + 1$ we must have been in case 1 of the construction at stage σ and so we made sure that β_σ was incompatible with R_ϵ and so B, which extends β_σ, cannot equal R_ϵ . □

Note however that since our construction is α-A-recursive so is B . Thus by Proposition 3.1 B is α-recursive in A , which is hyperregular, and so B is hyperregular. In particular we can employ Lemma 1.6 once we choose our trees. The trees we need are obviously the limits of the T_i^σ's as $\sigma \to \alpha$. The limits exist because of the priority argument and the fact that if $f_\sigma > i$ for all $\sigma > \tau$ then $T_i^\sigma = T_i^\tau$ for all $\sigma > \tau$. We set $T_i = \lim T_i^\sigma$ and note that B is on every T_i since β_σ is on $T_i^\sigma = T_i$ for every $\sigma > \tau$. Now for the final lemma.

Lemma 4.5: If $[\epsilon]^B$ is a representing function then it is either recursive or B is recursive in it.

Proof: Let η and τ be as in lemma 4.2. By lemma 4.3 we can take $\sigma \geq \tau$ to be the last stage at which $f_\sigma = \eta + 1$ and so $T_{\eta+1}^\sigma = T_{\eta+1}$ and $T_\eta^\sigma = T_\eta$ (again by 4.3). There are two possibilities for $T_\eta^\sigma = T_\eta$ — it can be $Sp(T_\eta^\sigma, k(\sigma,\eta), \beta_\sigma) = Sp(T_\eta, \epsilon, \beta_\sigma)$ or $In(T_\eta^\sigma, \beta_\sigma) = In(T_\eta, \beta_\sigma)$. In the first case (which can occur either if σ is a limit ordinal or if it is a successor and we were on case 1 at stage σ), B is recursive in $[\epsilon]^B$ by lemma 1.6. In the second case σ must be a successor and we must have been in case 2 at stage σ and so 1.8 tells us that $[\epsilon]^B$ is recursive. □

Thus modulo the proof of the priority argument in §5 we have proven our main result.

Theorem 4.6: If α is Σ_2 admissible there is a minimal α-degree recursive in the complete α-r.e. set. $\square$

5. The priority argument

In this section we prove

Theorem 5.1: For each $\epsilon < \mathrm{dom}\, k$ the set $I_\epsilon = \{\sigma \mid f_\sigma < \epsilon\}$ is bounded.

The proof splits into three cases as in [10]. For each case we will use a different k to assign priorities and then prove that we stop returning to each initial segment of priorities. Before we split into cases, however, there is one basic fact about the construction which we need in every case.

Lemma 5.2: Let $\eta < \nu \leq \alpha$ and suppose that for all $\sigma \in [\eta,\nu)$, $f_\sigma \geq \epsilon$, and that $k(\sigma,\gamma) = k(\eta,\gamma)$ for all $\gamma < \epsilon$, then $f_\sigma = \epsilon$ for at most two σ's in $[\eta,\nu)$.

Proof: It is clear that f_σ can be a limit ordinal λ only if $\{\gamma \mid f_\gamma < \lambda + 1\}$ is unbounded on σ . This would imply however that $\{\gamma \mid f_\gamma < \lambda\}$ is also unbounded in σ . Thus our assumption that $f_\sigma \geq \epsilon$ for $\sigma \in [\eta,\nu)$ assures us that $f_\sigma \neq \epsilon$ for $\sigma \in (\eta,\nu)$ if ϵ is a limit ordinal. If $\epsilon = \gamma + 1$ let σ be the least $\sigma \in [\eta,\nu)$ such that $f_\sigma = \epsilon$. There are two possibilities for T^σ_ϵ : $\mathrm{In}(T^\sigma_\gamma, \beta_\sigma)$ and $\mathrm{Sp}(T^\sigma_\gamma, k(\sigma,\gamma), \beta_\sigma)$. Since $f_\delta \geq \epsilon$ for $\delta \in [\eta,\nu)$, T^δ_γ does not change in this interval. Thus in the first case a return to ϵ after σ and before ν would require some $\beta_{\delta+1}$ $(\sigma < \delta < \nu)$ to have a proper extension on $T^\delta_\gamma = T^\sigma_\gamma$ but not on $T^\delta_{\gamma+1} = \mathrm{In}(T^\sigma_\gamma, \beta_\sigma)$. Since $\beta_{\delta+1} \supseteq \beta_\sigma$ this is impossible and we do not return to ϵ before ν. Alternatively assume $T^{\sigma}_{\epsilon} = Sp(T^{\sigma}_{\gamma}, k(\sigma,\gamma), \beta_{\sigma})$ and that we first return to ϵ at a later stage δ. In this case $f_{\theta} > \epsilon$ for $\sigma < \theta < \delta$ and so we must be in case 2 of the construction at stage δ and so $T^{\delta}_{\epsilon} = In(T^{\delta}_{\gamma+1}, \beta_{\delta})$. Now just as above we see that we can not return to ϵ before stage ν. □

Proof of 5.1: case 1. $\rho^{1}_{\alpha,A} < \alpha$. Let f be a partial α-A recursive map from $\rho^{1}_{\alpha,A}$ onto α. Let $k(\sigma,\epsilon) = f(\epsilon)$ if there is a computation with Gödel number less than σ which shows that $f(\epsilon)$ is defined and a Gödel number for the empty function otherwise. If we take domain $k = \rho^{1}_{\alpha,A}$, k has all the properties required of it in section 4.2. The only one which is perhaps not immediately obvious is the first. It follows, however, directly from the fact that any α-A r.e. subset of $\rho^{1}_{\alpha,A}$ is α-A finite (3.3). Note that in this case k has the additional property $(*)$ $k(\sigma,\epsilon)$ changes at most once for each $\epsilon < \rho^{1}_{\alpha,A}$.

The theorem is now an immediate consequence of an induction on ϵ. Let ϵ^{+} be the first infinite α-A cardinal after ϵ (it is, of course, regular) and inductively assume that $\overline{\overline{I}}_{\epsilon} < \epsilon^{+}$. At a successor stage we consider the map which takes each σ such that $f_{\sigma} = \epsilon$ to the least element of I_{ϵ} above it. By 5.2 and $(*)$ this map is at worst three-to-one on its domain and there are at most three σ's with $f_{\sigma} = \epsilon$ which are above all elements of I_{ϵ} and so not in its domain. Thus $\overline{\overline{I}}_{\epsilon+1} \leq \overline{\overline{I}}_{\epsilon} \cdot 3 + 3 < \epsilon^{+}$. Finally, at limit levels, lemma 3.2 carries the induction along for us. □

Case 2: $\rho^1_{\alpha,A} = \alpha$ and there is no last α-A cardinal. This is the simplest case. We let $k(\sigma,\epsilon) = \epsilon$ for all σ, ϵ. The domain of k is now α and it trivially has the properties required in 4.2 as well as (*). Since there is no last cardinal we can argue exactly as in case 1. □

Case 3: $\rho^1_{\alpha,A} = \alpha$ and there is a last α-A cardinal $\aleph$. As usual this is the most interesting case. Indeed it is the only one in which we deviate from [10].

Since $\langle L_\alpha, A\rangle$ is admissible it is easy to define a Σ_1 partial function h which is a Skolem function for all Σ_1 formulas of $\langle L_\alpha, A\rangle$ [1]. We use h to define a sequence $\{\delta_\zeta\}$: $\delta_0 = \cup\, h[\aleph]$, $\delta_{\zeta+1} = \cup\, h[\delta_\zeta + 1]$ and $\delta_\lambda = \bigcup_{\zeta<\lambda} \delta_\zeta$. Since $\rho^1_{\alpha,A} = \alpha$ this sequence is unbounded in α (an easy consequence of 3.3). We now think of the priorities as being arranged in blocks $[0, \delta_{\zeta+1})$ and within each block as arranged in an $\aleph$-list by using canonically chosen maps $f_\zeta : \aleph \xrightarrow{\text{onto}} \delta_{\zeta+1}$.

Let $h^\sigma(x) = h(x)$ if there is a computation $< \sigma$ showing that $h(x)$ is defined and $h^\sigma(x) = 0$ otherwise. We approximate $\{\delta_\zeta\}$ by sequences $\{\delta^\sigma_\zeta\}$: $\delta^\sigma_0 = h^\sigma[\aleph]$, $\delta^\sigma_{\zeta+1} = \cup\, h^\sigma[\delta^\sigma_\zeta + 1]$ and $\delta^\sigma_\lambda = \bigcup_{\zeta<\lambda} \delta^\sigma_\zeta$. Finally let f^σ_ζ be the least map $f: \aleph \to \delta^\sigma_{\zeta+1}$.

We would now naturally define our approximation to the priority listing by $k(\sigma,\epsilon) = f_\nu^\sigma(\eta)$ where $\epsilon = \aleph\cdot\nu+\eta$, $\eta < \aleph$, except that this would not satisfy 4.2.3. To accomodate this requirement we perturb k slightly and set $k(\sigma+1, \aleph\cdot\nu+\eta) = f_\nu^{\sigma+1}(\eta)$ and $k(\lambda, \aleph\cdot\nu+\eta) = f_\nu^\lambda(\eta)$ if this value is also $\lim_{\sigma\to\lambda} f_\nu^\sigma(\eta)$ and $f_\nu^{\sigma+1}(\eta)+1$ otherwise. This alteration clearly assures us that k satisfies 4.2.3 . We must now check the other requirements.

1) The computation of k is entirely determined by the values of $\{\delta_\zeta^\sigma\}$ except for the above perturbation which does not affect the requirements other than 4.2.3 . Thus if $\{\delta_\zeta^\sigma\}_{\zeta\le\gamma}$ reaches a constant value so does $k(\sigma,\epsilon)$ for $\epsilon < \aleph\cdot\gamma$. Once $\delta_\zeta^\sigma = \delta_\zeta$ for $\zeta \le \gamma$ all we need to correctly compute $\delta_{\gamma+1}$ are the computations of $h(x)$ for $x \in \delta_\gamma+1$. They are of course values of the Σ_1-Skolem function h and so by definition occur by $\delta_{\gamma+1}$. Thus at stage $\sigma = \delta_{\gamma+1}+1$ we compute $\delta_{\gamma+1}^\sigma$ correctly and of course never change again. Finally if $\delta_\zeta^{\sigma+1}$ are all correct for $\zeta < \lambda$ then by definition alone we see that $\delta_\lambda^{\sigma+1}$ is also correct. □

2) As $\{\delta_\zeta\}$ is unbounded in α for each ϵ there is a ζ such that $\delta_\zeta \le \epsilon < \delta_{\zeta+1}$ and so for some $\nu < \aleph$ $f_\zeta(\nu) = \epsilon$. Now once we have computed $\delta_{\zeta+1}$ correctly we will have $k(\sigma, \aleph\cdot\zeta+\nu) = \epsilon$. Indeed we can say more, for every $\sigma \ge \delta_{\zeta+1}+1$, $k(\sigma, \aleph\cdot\zeta+\nu) = \epsilon$. □

To finish this final case we induct on $\aleph\cdot\nu+\eta$. Setting $J_{\aleph\cdot\nu+\eta} = I_{\aleph\cdot\nu+\eta} - \delta_{\nu+1}$ we prove that a) $J_{\aleph\cdot\nu+\eta}$ is bounded below

$\delta_{\nu+2}$ and b) $J_{\aleph\cdot\nu+\eta} < \eta^+$. Assume we have succeeded for $\aleph\cdot\nu+\eta$. By induction $I_{\aleph\cdot\nu} \subseteq \delta_{\nu+1}$ and so using our calculation for k above we can employ lemma 5.2 to show that $J_{\aleph\cdot\nu+\eta+1} < \eta^+$. Moreover as $J_{\aleph\cdot\nu+\eta}$ is bounded in $\delta_{\nu+2}$ there is an $x \leq \delta_{\nu+1}$ such that $h(x) \geq J_{\aleph\cdot\nu+\eta}$. By lemma 5.2 for some $n \leq 2$ there are exactly n σ's above $h(x)$ such that $f_\sigma = \aleph\cdot\nu+\eta+1$. This can be written as a Σ_1 sentence with parameters $\leq \delta_{\nu+1}$. Thus since h is a Σ_1 Skolem function they must all occur below the limit ordinal $\delta_{\nu+2} = h[\delta_{\nu+1}+1]$. For η a limit ordinal less than $\aleph$ we can again argue as in case 1 for the cardinality of $J_{\aleph\cdot\nu+\eta}$. As for the rest we must look more closely. An examination of the proof of lemma 3.2 shows that when applied to $J_{\aleph\cdot\nu+\eta}$ the function g which enumerates $J_{\aleph\cdot\nu+\eta}$ is defined from parameters $\leq \delta_{\nu+1}$. The domain of g is of course some $\gamma < \aleph$ and so $g[\gamma] \subseteq \delta_{\nu+2}$. Moreover the sentence $(\exists x)(\forall y \in \gamma)(g(y) < x)$ is certainly true in $\langle L_\alpha, A\rangle$ as g is α-A recursive. Thus a bound on $g[\gamma]$ is given as the value of the Σ_1 Skolem function h on some ordinal $\leq \delta_{\nu+1}$ and $J_{\aleph\cdot\nu+\eta} = g[\gamma]$ is bounded below $\delta_{\nu+2} = h[\delta_{\nu+1}+1]$. At stages $\aleph\cdot\gamma$ of the induction everything proceeds trivially: $J_{\aleph\cdot\nu+\eta} \subseteq \delta_{\nu+2}$ for every $\nu < \gamma$, $\nu < \aleph$ clearly implies $J_{\aleph\cdot\gamma} \subseteq \delta_{\gamma+1}$ and so $J_{\aleph\cdot\gamma}$ is in fact empty. □

6. Open Questions

1. The basic problem is of course to construct a minimal degree for arbitrary Σ_1-admissibles. In particular we view $\aleph_\omega$ (of L) as a good test case for new approaches. It is not hard to convince oneself that something different is actually needed: consider the functions $[n]^B$ given by

$$[n]^B(x) = \begin{cases} B(x) & \text{if } Lx \models \exists < n \text{ cardinals} \\ 0 & \text{otherwise} . \end{cases}$$

In any procedure like that of section 4 T_n will be a splitting tree until one reaches stage $\aleph_n$ at which point we must return to it and make it the full subtree of T_{n-1}. Thus there will never be a stage at which all of T_i $i < \omega$ will have reached their limits. Thus there will never be a chance to properly attack requirements after ω. On the other hand counterexamples seem even less likely.

2. In connection with the second example of 2.3 we ask: For what admissible ordinals α are there non-hyperregular minimal α-degrees? Indeed doing much of anything with non-hyperregular sets is something of a challenge. See [13] where some important first steps have been taken.

3. When can we embed arbitrary finite lattices (or even distributive ones) as initial segments of the α-degrees of unsolvability?

Chapter II

Σ_n Sets Which Are Δ_n-Incomparable (Uniformly)

G. Sacks and S. Simpson have asked [10, §7 question 5] if it is obvious that there are, for each Σ_n-admissible α, Σ_n (over L_α) subsets of α which are Δ_n-incomparable. If one understands "B is Δ_n in C" to mean that there are Σ_n/L_α reduction procedures which put out B and $\overline{B}$ when one feeds in C, then the answer is an unqualified " yes". In this sense "Δ_n in" is a direct generalization of "α-recursive in" (replace Σ_1 by Σ_n in the definition) and so amenable to the methods of I.3 and I.5. Indeed one simply chooses a complete Σ_{n-1} set A and mimics the construction of [10] to produce two α-A-r.e. sets B and C neither of which is α-A-recursive in the other. By the remarks on translation (I.3) this will immediately give the desired result for this definition of "Δ_n in".

There is, however, the more obvious and natural notion of "Δ_n in" to be considered: B is Δ_n in C iff there are Σ_n and Π_n formulas of $\langle L_\alpha, C\rangle$ which define B. The answer in this case is still "yes" but the proof is no longer quite so obvious. First of all, this later definition is easily seen to give a much stronger reducibility than the former for $n \geq 2$. Consider, for example, the empty set $\emptyset$ along with complete Σ_{n-1} and Σ_n sets

A and B. Clearly $B \in \Delta_n(A)$ and $A \in \Delta_n(\emptyset)$ according to the second definition. With the first definition, however, anything Δ_n in A is Δ_n and so $B \notin \Delta_n(A)$. (One should also note that as $B \notin \Delta_n(\emptyset)$, the second notion of "$\Delta_n$ in" is not transitive while the first clearly is).

The second definition, moreover, is really model-theoretic in nature and one can gain no direct hold on it via purely recursion-theoretic constructions. The reason for this is that individual facts about B derived via the second definition of "B is Δ_n in C" depend, in general, on unboundedly much of C. Of course by its very nature the first definition uses only α-finitely much about C for any single decision. Since at any stage of a construction inside L_α we can only handle α-finitely much information, we would seem to be at a loss as far as the strong notion of "Δ_n in" is concerned. It would seem that all we could hope to handle would be reduction procedures which were Σ_n/L_α.

The solution to this difficulty, which was suggested by Prof. Sacks, is to mix a forcing construction into the priority argument to produce sets which are sufficiently generic so that the two notions of reducibility coincide. More specifically, one hopes to make the truth of Σ_n sentences about C depend on only α-finitely much of C via a forcing relation which is itself Σ_n. This would then give us a method of handling the strong notion of "Δ_n in" directly in a construction inside L_α. We carry out this suggestion below to prove the theorem.

We also answer question 3 of [10] in this chapter by carrying out our construction of Δ_n-incomparable Σ_n sets uniformly. Thus for each n we show that there are integers k and ℓ which are, for every Σ_n-admissible α, Gödel numbers for Σ_n subsets of α which are Δ_n-incomparable. (M. Leman and S. Simpson have informed us that they have also found a uniform solution for the case $n = 1$.)

1. The Forcing Relation

We assume that the reader has a general familiarity with various standard forcing procedures. We therefore describe the particular system with which we are concerned but give no proofs for the standard theorems. The basic techniques and lemmas for the subject in various recursion-theoretic settings can be found for example in [8].

1.1 Definitions: The general framework within which we will be working is that of Cohen style class forcing over models of weak set theories. More specifically we will be forcing over L_α's for α Σ_n-admissible. Our forcing conditions will be α-finite sequences ordered by inclusions (see I.1). The intended interpretation, of course, is that the α-finite sequence is an initial segment of the representing function for our generic predicate. As far as language is concerned ours will contain, in addition to the usual logical symbols, bounded quantifiers, $\in$, $=$, constant symbols $\underline{a}$ for each $a \in L_\alpha$ and a unary predicate symbol $\underline{G}$ for the generic predicate. We will, however, in our construction only consider formulas in <u>strict</u> prenex normal form, i.e. ones in which all bounded quantifiers follow all unbounded ones. Thus by $\neg\phi$ we will mean the formula gotten by changing all quantifiers in ϕ and negating the matrix.

The forcing relation $\sigma \Vdash \phi$ is defined by induction on the number of unrestricted quantifiers in ϕ. For Δ_0 formulas ϕ, forcing equals truth. More precisely, $\sigma \Vdash \phi$ iff every constant symbol in ϕ belongs to $L_{lth\ \sigma}$ and $L_{lth\ \sigma} \models \phi$ when G is interpreted as having σ as an initial segment of its representing function. The inductive step is as usual: $\sigma \Vdash \exists x\phi(x)$ iff $(\exists a \in L_\alpha)(\sigma \Vdash \phi(\underline{a}))$ and $\sigma \Vdash \forall x\phi(x)$ iff $(\forall\tau \supseteq \sigma)(\tau \nVdash \exists x \neg\phi(x))$. With this definition it is easy to prove all the standard facts about forcing in the usual inductive way. One should also note that for Σ_n formulas ϕ $(n \geq 1)$ $\sigma \Vdash \phi$ is a Σ_n relation while for $n = 0$ it is Δ_1. Thus in every case, for formulas in $\Sigma_{n-1} \cup \Pi_{n-1}$, $\sigma \Vdash \phi$ is Δ_n. These basic facts will enable us to use the Σ_n-admissibility of α to carry out the forcing part of our construction.

1.2 Σ_n-genericity: We call a subset G of α Σ_n-generic iff for every Σ_n sentence ϕ of our language $\langle L_\alpha, G\rangle \models \phi \iff (\exists\sigma \subseteq G)(\sigma \Vdash \phi)$. By making our Σ_n sets B and C Σ_n-generic we will be able to handle the strong notion of "Δ_n in" via α-finite facts, i.e. forcing conditions. As an example of the power this gives us we note that if G is Σ_n-generic then B is Δ_n in G in the strong sense iff it is also true in the weak sense. We must prove the if part: Say there are Σ_n formulas ϕ and ψ such that $(x)(x \in B \Leftrightarrow \langle L_\alpha, G\rangle \models \phi(x)$ & $x \notin B \Leftrightarrow \langle L_\alpha, G\rangle \models \psi(x))$. Then by the Σ_n-genericity of G we have that for all x, $x \in B \Leftrightarrow (\exists\sigma \subseteq G)(\sigma \Vdash \phi(x))$ and $x \notin B \Leftrightarrow (\exists\sigma \subseteq G)(\sigma \Vdash \psi(x))$. Since forcing for Σ_n sentences is Σ_n this is precisely the statement that B is Δ_n in G in the weak sense.

Finally we remark that by the usual lemmas about forcing (in particular the "forcing equals truth" lemma) being Σ_n-generic is equivalent to there being, for each Σ_{n-1} formula ϕ, a $\sigma \subseteq G$ such that $\sigma \Vdash \phi$ or $\sigma \Vdash \neg\phi$. Indeed these are the requirements that we will try to satisfy in our construction.

1.3 Before proceeding we should comment on the restriction to strict prenex formulas. This corresponds to yet another variation in the meaning of "Σ_n in" (or "Δ_n in"). Though we will for the sake of simplicity primarily concern ourselves with strict Σ_n formulas, we will indicate in section 4 how to add on additional requirements to the construction and handle Σ_n formulas in which there are no restrictions on the positions of the bounded quantifiers.

2. The Construction

We first give an intuitive picture of what we are trying to do and then give the actual details of the construction. In both sections, B and C will play virtually symmetric roles and we will generally confine our attentions to the B side of the construction. When necessary we shall distinguish between B and C notations by subscripting. Throughout the rest of this chapter we fix n and assume α is Σ_n-admissible.

2.1 The idea: The construction will have two goals. The first will be to decide by forcing conditions contained in B each Σ_{n-1} sentence. The second will be to make sure that no Σ_n sentence defines the complement of C when G is interpreted as B. To begin, we will order the Σ_n sentences in some way and so assign priorities to these requirements. Once this has been done we proceed towards our first goal by, at each stage η, extending our current guess at an initial segment of B to force the Σ_{n-1} formula of highest priority not yet decided or its negation. By the usual lemmas about forcing, such an extension always exists. Moreover as forcing for $\Sigma_{n-1} \cup \Pi_{n-1}$ sentences is Δ_n there is a Δ_n function giving such an extension.

As far as our second goal is concerned we will proceed in the usual fashion. For each Σ_n formula ϕ_ε we will have a potential witness x_ε which is not yet in C. If our current approximation to B forces $(\underline{x}_\phi, \underline{C})$ then we put x_ε into C thus making sure that ϕ does not define $\bar{C}$. Of course in both parts we will protect the necessary negative information about B only against later encroachments by requirements of lower priority.

The problem with making this construction independent of α arises, of course, with the specification of the priority ordering. In chapter I as in [10] various orderings depending on internal characteristics of α are specified in advance. We will here uniformly generate (via approximations, of course) a priority listing that, for each α, turns out to be essentially the same as the one given in I.5.1. The idea is to mimic the proof for case 3 in I.5 (there is a last α-cordinal and $\rho_\alpha^n = \alpha$). We first use a parameterless Σ_n Skolem function h to generate blocks of requirements. Then within each block we act as if we were indeed in the situation of case 3, that is, if the block is δ we consider the requirements for formulas in $h[\delta]$ ordered in a $\bar{\bar{\delta}}$ type list. The proofs will then follow I.5 and [10].

2.2 The priority listing: Since α is Σ_n-admissible one can easily get a parameterless Σ_n-Skolem function

h [1]. We generate blocks of requirements as in I.5.1 (3): $\delta_0 = \omega$, $\delta_{\zeta+1} = \bigcup h[\delta_\zeta+1]$, $\delta_\lambda = \bigcup_{\zeta<\lambda} \delta_\zeta$.

If there is a last α-cardinal and $\rho_\alpha^n = \alpha$ then these blocks will eventually coincide with those of I.5.3. If there is no last cardinal then the δ_ζ will be unbounded in α and will include each α-cardinal thus mimicing I.5.1 (2). Finally if $\rho_\alpha^n < \alpha$ then there will be a last δ_ζ which projects onto α (via h) and so the situation will be as in I.5.1 (1). Of course in the construction we can only approximate the δ_ζ but this is what will happen "in the limit".

Using the notation of I.5.1 (3) we let δ_ζ^η be the approximation to δ_ζ at stage η. We also let f_ζ^η be the least 1-1 map from θ_ζ^η onto $\bigcup_{n<\omega} (\delta_\zeta^\eta+1)^n \times \omega$ where θ_ζ^η is the least ordinal for which there is such a map constructed by stage η (i.e. θ_ζ^η approximates the α-cardinality of δ_ζ^η). We now define R_β (requirement β) to have higher priority than R_γ at stage η if $\beta = h^\eta f_\zeta^\eta(\varepsilon)$ for some $\varepsilon < \theta_\zeta^\eta$ and there is no pair ζ',ε' with either $\zeta' < \zeta$ and $\varepsilon' < \theta_\zeta^\eta$, or $\zeta' = \zeta$ and $\varepsilon' < \varepsilon$ such that $\gamma = h^\eta f_\zeta^\eta(\varepsilon')$. Moreover for fixed ζ and $\varepsilon < \theta_\zeta^\eta$ a requirement $R_{h^\eta f^\eta(\varepsilon)}$ associated with B has higher priority than the $R_{h^\eta f_\zeta^\eta(\varepsilon)}$ associated with C.

2.3 The requirements: We begin by making an α-recursive list $\{\phi_\varepsilon\}_{\varepsilon<\alpha}$ of the Σ_n formulas of our language. There will be two types of requirements R_ε (for both B and C). If ϕ_ε is a Σ_{n-1} sentence, then at stage η R_ε requires that σ^η, our current initial segment of B, forces ϕ_ε or $\neg\phi_\varepsilon$. If ϕ_ε is a Σ_n-formula with one free variable then R_ε requires that $\phi_\varepsilon(x,B)$ does not define $\overline{C}$. We say that any requirement R_ε of the first type is satisfied at stage η if $\sigma^\eta \Vdash \phi_\varepsilon$ or $\sigma^\eta \Vdash \neg\phi_\varepsilon$. It is a standard fact about forcing that at any stage $\eta + 1$ we can satisfy any unsatisfied requirement of this type by choosing a proper extension of σ^η. Moreover, by the remarks at the end of 1.1, this operation is Δ_n (and so relatively harmless).

Satisfying the second type of requirement is more difficult but follows the usual pattern of the Friedburg-Muchnik theorem.

We first split α into α many unbounded, pairwise disjoint recursive subsets Z_ε, $\varepsilon < \alpha$. The idea is that potential witnesses to the fact that ϕ_ε does not define $\overline{C}$ will come from Z_ε. At each stage η we will have a specific potential witness x^η_ε from Z_ε for every $\varepsilon < \alpha$. We can satisfy a requirement R_ε of the second type at stage $\eta + 1$, if $\sigma^\eta_B \Vdash \phi_\varepsilon(\underline{x}^\eta_\varepsilon, \underline{G})$ via a witness for the first quantifier which is in L_η, by putting x^η_ε into C. Of course if x^η_ε is in C and we know that $\sigma^\eta_B \Vdash \phi_\varepsilon(\underline{x}^\eta_\varepsilon, \underline{G})$ then R_ε is already satisfied. Note that this procedure is also Δ_n since we have put a bound on the witness for the initial existential quantifier.

Of course if ϕ_ε is not of either of these forms then R_ε is null and we never concern ourselves with it.

2.4 The construction: At each stage η we will have an α-finite sequences σ_B^η and σ_C^η which are intended to approximate initial segments of the representing functions for B and C. These sequences will be changed from step to step only by adding new elements to B and C and/or by increasing their length. Since the process of going from one step to the next will be Δ_n the sets we enumerate will be Σ_n.

We begin at stage 0 by setting, for both B and C, $\sigma^\eta = \emptyset$ and x_ε^0 equal to the least elements of Z_ε for each $\varepsilon < \alpha$. At stage $\eta + 1$ we let R_β be the unsatisfied requriement of highest priority which we can now satisfy. To be specific let us assume that R_β is associated with B. We now satisfy R_β as dictated by 2.3 and make some changes in the x_ε^η to reflect the injuries this causes. More precisely, if R_β is a requirement of the first type we let $\sigma_B^{\eta+1}$ be an appropriate extension of σ_B^η. We now set $(x_\varepsilon^{\eta+1})_C = (x_\varepsilon^\eta)_C$ for all ε of higher priority than β. For ε of lower priority than β we set $(x_\varepsilon^{\eta+1})_C$ equal to the least element of Z_ε above

$(\sigma_B^{\eta+1})_0$. Finally $\sigma_C^{\eta+1} = \sigma_C^\eta$ and $(x_\varepsilon^{\eta+1})_B = (x_\varepsilon^\eta)_B$ for all $\varepsilon < \alpha$. If R_β is of the second type then we add $(x_\varepsilon^\eta)_B$ to C. Formally we set

$$\sigma_C^{\eta+1} = \langle (\sigma_C^\eta)_0 \cup \{(x_\varepsilon^\eta)_B\}, (\sigma_C^\eta)_1 \cup (x_\varepsilon^\eta)_B \rangle.$$

Everything else remains unchanged. Had R_β been a C requirement we would of course have proceeded dually.

At limit stages λ we simply set, for both B and C, $\sigma^\lambda = \langle \bigcup_{\eta<\lambda} (\sigma^\eta)_0, \bigcup_{\eta<\lambda} (\sigma^\eta)_1 - \bigcup_{\eta<\lambda} (\sigma^\eta)_0 \rangle$ and x_ε^λ = the least element of $Z_\varepsilon \geq \bigcup_{\eta<\lambda} x_\varepsilon^\eta$.

3. The Priority Argument

In this section we will show that, for each β, we act to satisfy requirements with higher priority than R_β only boundedly often. We can then easily conclude that the sets B and C constructed above have the desired properties.

3.1 The first point to consider is our approximation for the δ_ζ. Let γ be the least ordinal such that $\bigcup_{\zeta\leq\gamma} \delta_\zeta = \alpha$. It can be either a limit or successor ordinal. In either case the Σ_n-admissibility of α straightforwardly implies that, for each $\nu < \gamma$, there is a stage $\eta'_\nu < \delta_{\nu+1}$ such that $\delta^\eta_\zeta = \delta_\zeta$ for all $\zeta \leq \nu$ and $\eta > \eta'_\nu$. Once such a stage has been reached we can, of course, find a later stage $\eta_\nu < \delta_{\nu+1}$, by which $\theta^\eta_\zeta = \bar{\bar{\delta}}_\zeta$ for all $\zeta \leq \nu$. From such a stage onward all requirements whose priorities lie in any block $\leq \nu$ are fixed. Finally we note that for every $\nu < \gamma$ and $\varepsilon < \theta_\nu$ there is a stage $\eta_{\nu,\varepsilon} < \delta_{\nu+1}$ such that $h^\eta f^\eta_\nu(x) = hf_\nu(x)$ for every $x < \varepsilon$ and $\eta \geq \eta_\varepsilon$. Thus we can always assume any initial segment of the priority listing to be fixed from some stage onward. We can now prove the main lemma.

<u>Lemma 3.2</u>: For each $\nu < \gamma$, $\varepsilon < \theta_\nu$ there is a bound

on the stages at which we act to satisfy requirements of higher priority than $R_{hf_\nu(\varepsilon)}$.

Proof: We proceed by induction to prove an even stronger result: 1) $J_{\nu,\varepsilon}$, the set of stages after $\eta_{\nu,\varepsilon}$ at which we act to satisfy requirements of higher priority than $R_{hf_\nu(\varepsilon)}$, is bounded below $\delta_{\nu+1}$ and 2) $\bar{\bar{J}}_{\nu,\varepsilon} < \varepsilon^+$.

The proof is similar to I.5.1. Corresponding to lemma I.5.2 we have the fact that once we have stopped acting to satisfy all requirements of higher priority than R_β we act at most once more to satisfy R_β. Thus for a successor (2) is immediate by induction. Moreover our acting to satisfy an R_β is a Σ_n fact and so the stage at which it occurs lies inside the appropriate Σ_n Skolem hull. As h is a Σ_n Skolem function, we have that (1) also continues to hold by the definition of $\delta_{\nu+1}$.

For ε a limit ordinal lemma I.3.2' (for Σ_n-admissibles) carries along the induction for (2) since our construction is Δ_n. To see that (1) remains true we argue more closely from the proof of I.3.2' as we did from that of I.3.2 in the last paragraph of I.5. Of course, at stages of the induction corresponding to $\varepsilon = 0$ we proceed trivially as in I.5: $J_{\nu,\varepsilon} \subseteq \delta_{\nu+1}$ for every $\varepsilon < \theta_\nu$ implies that $J_{\nu+1,0} \subseteq \delta_{\nu+1} < \delta_{\nu+2}$ and $\bar{\bar{J}}_{\nu+1,0} = 0$. The argument for limit ν is equally trivial.

We now proceed to our goal via some simple lemmas.

Lemma 3.3: For every $\beta < \alpha$ there is a $\nu < \gamma$ and $\varepsilon < \theta_\nu$ such that $\beta = hf_\nu(\varepsilon)$.

Proof: If γ is a limit ordinal the result is obvious, so assume $\gamma = \nu + 1$. By the definition of γ, $\delta_\gamma = \bigcup h[\delta_\nu+1] = \alpha$. Thus the Σ_n-admissibility of α tells us that $D = \text{dom } h \cap \delta_\nu + 1$ is not α-finite. It is, however, definable over $h[\delta_\nu+1]$ and so over its transitive collapse M. Thus $M = L_\alpha$. Of course, $h[\delta_\nu+1] \models (\forall x)(\ y \leq \delta_\nu)(h(y) = x)$, but since h has no parameters we immediately have that $M = L_\alpha \models (\forall x)(\ y \leq \delta_\nu)(h(y) = x)$.

Lemma 3.4: B and C are Σ_n-generic.

Proof: Consider any Σ_{n-1} sentence ϕ_β. By lemma 3.3 there are $\nu < \gamma$ and $\varepsilon < \theta_\nu$ such that $hf_\nu(\varepsilon) = \beta$. By lemma 3.2 we can go to a stage after which we never act to satisfy a requirement of higher priority than R_β. At such a stage we must satisfy R_β if it is not yet satisfied. Moreover, the procedure for moving the x_ε, and so protecting this requirement assures us that R_β will remain satisfied. Thus each ϕ_ε is in fact decided by an initial segment of each set constructed.

We can now conclude the proof.

<u>Theorem 3.5</u>: B and C are Δ_n-incomparable.

<u>Proof</u>: Let ϕ_β be any Σ_n formula with one free variable and let ν and ε be as given by lemma 3.3. For the sake of definiteness we show that ϕ_β does not define $\bar{C}$ when $\underline{G}$ is interpreted as B. Let η' be the stage given for ν and ε by lemma 3.2. We see from the construction that x_β^η is constant for $\eta > \eta'$. Let its value be x_β. If $x_\beta \in \bar{C}$ then by assumption $\langle L_\alpha, B\rangle \models \phi(\underline{x}_\beta, \underline{G})$. Say $\phi(x, \underline{G})$ is $\exists y \psi(y, x, \underline{G})$ where ψ is Σ_{n-1} and let a be a witness for the initial quantifier. Thus $\langle L_\alpha, B\rangle \models \psi(\underline{a}, \underline{x}_\beta, \underline{G})$ and so by 3.4 there is a stage $\eta \geq \eta'$, rk(a) such that $\sigma^\eta \Vdash \psi(\underline{a}, \underline{x}_\beta, \underline{G})$ and σ^η represent an initial segment of B. We now see that if R_β is not satisfied at stage $\eta + 1$ (i.e. x_β is not yet in C) we can and indeed must then satisfy it. To do this we put x_β into C -- a contradiction.

Finally we note that our construction does not depend on α nor does it mention any infinite parameters. Thus we have in fact established the theorem with the uniformity promised at the beginning of the chapter.

4. Preserving Σ_n-admissibility

As promised in 1.3 we now indicate how our theorem can be strengthened to handle definitions which are Σ_n but not necessarily in strict prenex form.

4.1 The idea: If the structure L_α remains Σ_n-admissible when B is added on as an additional predicate, then every $\Sigma_n(B)$ formula is equivalent to one in strict prenex form. Thus it suffices to insure that the addition of B or C as an additional predicate does not destroy the Σ_n-admissibility of L_α. We achieve this result by adding requirements to our construction similar to those used in [10,§6] and III.4 to make sure that various sets are hyperregular. We treat each strict prenex $\Sigma_n(B)$ formula with two free variables as a function and try to preserve computations on certain initial segments on which it is total. Of course, this is handled via the forcing relation and so the construction remains Δ_n. By the priority argument we eventually keep preserving the first computation (actually a forcing condition in this context) for each value. Thus the function itself becomes Δ_n and its range bounded. This obviously suffices to make $\langle L_\alpha, B\rangle$ Σ_n-admissible.

4.2 The new requirements: Trivial manipulations show that it suffices to consider, for a function ϕ_β, preservations on initial segments up to the cofinality (in L_α) of θ_ν (where $\beta = hf_\nu(\varepsilon)$ for some $\varepsilon < \theta_\nu$). To do this we add on requirements R_β for β's such that ϕ_β is a (strict prenex) Σ_n formula with two free variables. Let $cf^\eta(\theta)$ be the L_α-cofinality of θ as computed by stage η. Also let $y^\eta_{\nu,\varepsilon}$ be the least $y < cf^\eta(\theta^\eta_\nu)$ for which σ^η does not force $\phi_{h^\eta f^\eta_\nu(\varepsilon)}$ to have a specific value. If $\beta = h^\eta f^\eta_\nu(\varepsilon)$ we say that we can satisfy an R_β of our new kind if $y^\eta_{\nu,\varepsilon} < cf^\eta(\theta^\eta_\nu)$ and we can find at this stage an extension of σ^η which forces $\phi_\beta(y^\eta_{\nu,\varepsilon})$ to have a specific value. These new requirements fit naturally into the priority ordering. The construction proceeds as before by always trying to satisfy the requirement of highest priority possible and then appropriately moving the x^η_ε corresponding to requirements of lower priority.

4.3 The proof: We have only to indicate how the new requirements are handled in the proof of the priority lemma. Everything else in §3 remains the same. Moreover, once the priority lemma is established the Σ_n-admissibility of $\langle L_\alpha, B\rangle$ follows as indicated in 4.1. We therefore conclude with the required lemma.

Lemma 4.4: For each $\nu < \gamma$, $\varepsilon < \theta_\nu$ there is a bound on the stages at which we act to satisfy requirements of higher priority than $R_{hf_\nu(\varepsilon)}$.

Proof: First note that we can adjust $\eta_{\nu,\varepsilon}$ if necessary to make sure that $cf^{\eta_{\nu,\varepsilon}}(\theta_\nu) = cf(\theta_\nu)$. Keeping in mind the proof of 3.2, we see that it suffices to prove that for each $\theta < \gamma$, $\varepsilon < \theta_\nu$
$K_\varepsilon = \bigcup\{J_{\nu,\varepsilon'} | \varepsilon' < \varepsilon \;\&\; R_{hf_\nu(\varepsilon')}$ is of the new type$\}$ is bounded below $\delta_{\nu+1}$. For ε a successor this fact is immediate. If ε is a limit ordinal then by the arguments of 3.2 we have a bound $\eta < \delta_{\nu+1}$ on
$\bigcup\{J_{\nu,\varepsilon'} | \varepsilon' < \varepsilon \;\&\; R_{hf_\nu(\varepsilon)}$ is of an old type$\}$. We consider the set $K_\varepsilon - \eta$. Any σ^τ for τ a member of $K_\varepsilon - \eta$ is necessarily permanent by the definition of η. Thus for each $\varepsilon' < \varepsilon$ and $y < cf\ \theta_\nu$ there can be at most one corresponding element, τ, of $K_\varepsilon - \eta$ at which we acted to force $\phi_{hf_\nu(\varepsilon')}(y^\tau_{\nu,\varepsilon'})$ to have a value.

We first handle the set W of $\varepsilon' < \varepsilon$ for which there are $y^\tau_{\nu,\varepsilon'}$ unbounded in $cf(\theta_\nu)$ with $\tau \in K_\varepsilon - \eta$. Since W is a Σ_n subset of $\varepsilon < \theta_\nu \leq \rho^n_\alpha$ it is α-finite. We now have a total 1-1 Σ_n map from $W \times cf\ \theta_\nu$ onto the elements of $K_\varepsilon - \eta$ corresponding to them. Thus by the Σ_n-admissibility of α its range is bounded. Moreover, the existence of a bound is itself a Σ_n fact in parameters $\leq \delta_\nu$, and so the bound is a member of the Σ_n Skolem hull of $\delta_\nu + 1$. In particular, this part of $K_\varepsilon - \eta$ is bounded below $\delta_{\nu+1}$.

Now we deal with the elements of $K_\varepsilon - \eta$ not in W. We claim that the associated $y^\tau_{\nu,\varepsilon'}$'s are bounded below θ_ν. If $cf(\theta_\nu) < \theta_\nu$ then this is immediate. On the other hand, suppose θ_ν is a regular α-cardinal and the $y^\tau_{\nu,\varepsilon'}$'s are unbounded in θ_ν. We see then that the $y^\tau_{\nu,\varepsilon'}$'s involved must cover a final segment of θ_ν. Thus we can define a Δ_n map from this final segment of θ_ν into ε by sending $y \mapsto \varepsilon'$ where $y = y^\tau_{\nu,\varepsilon'}$ and τ is the least such member of $K_\varepsilon - \eta$. Since this map is Δ_n and α is Σ_n-admissible we have contradicted the regularity of θ_ν.

Let $S < \theta_\nu$ be the bound just established. The elements of K_ε are in a natural 1-1 correspondence with a subset of $\varepsilon \times S$. The correspondence is Δ_n and the subset is Σ_n. As $\varepsilon, S < \theta_\nu \leq \rho^\alpha_n$, however, the subset is in fact α-finite. Thus the Σ_n-admissibility of α tells us that $K_\varepsilon - \eta$ is bounded. As before the existence of this bound is a Σ_n fact depending on parameters below δ_ν and so is indeed attained below $\delta_{\nu+1}$.

Chapter III

The Splitting Theorem

In ordinary recursion theory one can distinguish various types of priority arguments. The major split is between finite and infinite injury constructions but finer distinctions can and should be drawn. Thus for example one should mark the difference between the arguments for the Friedburg-Muchnik solution of Post's problem [10] and Sacks' splitting theorem [6,12]. In the former there is an *a priori* recursive bound on the preservations initiated for a given requirement and so on the injuries it inflicts on lesser requirements. There are, however, no such bounds available in the proof of the latter theorem.

As long as one remains within the confines of ordinary recursion theory these distinctions are relatively unimportant. All that one ever needs in the proofs is that the injuries to each initial segment of requirements are bounded. Since the union over an initial segment of ω of finite sets is finite, the proofs go through without regard to the more subtle questions of the existence of recursive bounds on the injury sets. The situation changes dramatically when one enters the realm of recursion theory generalized to all admissible ordinals α. The problem is that the union over an initial segment of α of sets each of which is α-finite need not be α-finite. In general the collection itself must be given as an α-finite union of α-finite sets for one to be sure that the union is α-finite. Even in the proof of the Friedburg-Muchnik theorem, however, one is not so lucky as to have everything presented on a silver platter. Indeed, a crucial point in the Sacks-Simpson proof of the theorem in α-recursion theory [10] is their lemma 2.3 (see I.3) which enables them to handle unions which are only α-r.e. Unfortunately, their lemma only applies to collections with a uniform a priori bound on the size of the members. It is therefore admirably suited to priority arguments of the Friedburg-Muchnik type but does not seem to suffice for ones like the splitting theorem which lack appropriate bounds. Indeed to date the only priority arguments that have been carried out for all admissibles have been of the Friedburg-Muchnik type. In this chapter we exhibit a construction that enables one to handle priority arguments of the second type. In particular we prove a strong form of the splitting theorem for all admissible ordinals.

Theorem: Let C be a regular α-r.e. set and D be a non-recursive α-r.e. set. Then there are regular α-r.e. sets A and B such that $A \cup B = C$, $A \cap B = \phi$, $A, B \leq_{\alpha} C$ and such that D is not α-recursive in A or B.

Note: We have been informed by M. Lerman that C. T. Chong has independently proven that there are incomparable α-r.e. degrees below every α-r.e. degree. Though an even stronger result is a simple corollary of our theorem (Cor. 3.3) the same type of priority argument is needed even for the weaker result. We have not seen his proof but have been given to understand that his methods are rather different from ours.

1. The Construction

1.1 The intuitive picture: We begin by noting that since every α-r.e. degree contains a regular set we can assume without loss of generality that D is regular. We let c and d be one-one α-recursive functions which enumerate C and D respectively. As usual, we denote the associated approximations to C and D by $C^{\sigma} = \bigcup_{i<\sigma} c(i)$ and $D^{\sigma} = \bigcup_{i<\sigma} d(i)$.

The general plan of the construction calls for putting $c(\sigma)$ into exactly one of A and B at stage σ. This will assure us that $A \cup B = C$, $A \cap B = \phi$ and that $A, B \leq_{\alpha} C$. We will also have various elements that we wish to keep out of A and B for the sake of requirements associated with the condition that D not be recursive in A or B. These requirements will be ordered by a priority system that will be developed along with the construction. At any stage σ we will have an approximation to the final priority listing which we will use to determine whether $c(\sigma)$ is put into A or B. Essentially we will choose whichever will preserve the requirements of the highest possible priority.

In line with Sacks' original approach to this theorem [6] we try to insure that $D \not\leq_{\alpha} A$ by the roundabout method of preserving (for each ε) computations of $[\varepsilon]^{A}$

on initial segments as long as they seem to agree with D. The idea is that if for some ε, $[\varepsilon]^A = D$, then we would eventually be preserving the first available computation of $[\varepsilon]^A(x)$ for each x. We would then be able to compute $[\varepsilon]^A$ and so D α-recursively -- a contradiction. On the basis of such considerations, however, we can only argue that there is some bound on the x's for which we preserve computations of $[\varepsilon]^A(x)$. We cannot assign any uniform a priori value to this bound and so find ourselves heir to the problems discussed in the introduction to this chapter.

Our strategy for handling these difficulties is to first arrange the requirements $[\varepsilon]^A \neq D$ in blocks P and then to consider $[\varepsilon]^A \neq D$ for all ε in P as a single requirement. Since the preservations for such requirements only involve keeping elements out of A, there can be no conflict within a given block. Moreover, considering the whole block P as a single requirement will not interfere with our ability to recover D should the preservations associated with P be unbounded. Finally we will have the determination of the size and number of these blocks interwoven with the construction in such a way that the blocks progress through the list of reduction procedures cofinally with the progression of preservations and injuries through α.

1.2 The actual construction: Since A and B play entirely analogous roles in the construction we will describe explicitly only the A part. It is of course understood that similar steps are to be taken on behalf of B. Before beginning the construction we let $f:\alpha \xrightarrow{1-1} \alpha^*$ be an α-recursive projection. The rest of our terminology will be defined simultaneously with the construction.

At stage σ we will have blocks P^σ_γ each of which will be an initial segment of α^*. For each γ we find the least x for which there is no $\gamma - A$ requirement with argument x associated with a $\gamma - A$ active reduction procedure. If there is a $\gamma - A$ active reduction procedure ε in P^σ_γ for which $[f^{-1}\varepsilon]^{A^\sigma}_\sigma(x) = D^\sigma(x)$, we take the least such computation and <u>create a $\gamma - A$ requirement with argument x associated with ε.</u> This requirement consists of the elements required to be out of A by the chosen computation.

If at any stage we put an element of a $\gamma - A$ requirement into A we <u>destroy</u> the requirement. A reduction procedure $\varepsilon < \alpha^*$ is <u>$\gamma - A$ active</u> at stage σ unless there is a $\gamma - A$ requirement (as yet undestroyed) with argument x associated with ε such that $[f^{-1}\varepsilon]^{A^\sigma}_\sigma(x) \neq D^\sigma(x)$ --- i.e. x has been enumerated in D since the requirement was created. The idea is that as long as we seem to have a computation showing that $[f^{-1}\varepsilon]^A \neq D$ we need pay no further attention to ε.

Now for the definition of the blocks P :

$$P_0^\sigma = \{0\}, \forall \sigma.$$

$$P_\gamma^\sigma = \bigcup_{\delta<\gamma} P_\delta^\sigma \cup f(y_\gamma^\sigma) + 1$$

where $y_\gamma^\sigma = \bigcup\{\tau<\gamma \mid (\exists\delta<\gamma)$ (a δ-requirement is created at stage $\tau \vee$ an element of a δ-requirement is enumerated in C at stage $\tau\}$.

The idea is that (for some λ) each block P_γ^σ, $\gamma < \lambda$, will eventually reach a constant value which will reflect (via f) a bound on all injuries caused by our having to preserve δ-requirements for $\delta < \lambda$. Once we have such a bound we can show that $P_{\gamma+1}$ is bounded via the argument about recovering D alluded to in 1.1. Moreover if λ is the least ordinal such that $\bigcup_{\gamma<\lambda} P_\gamma = \alpha^*$ then every P_γ will be so handled and we will succeed in getting $D \nleq A, B$.

Finally we take $c(\sigma)$ and put it into A or B so as to preserve as much as possible. More precisely, we consider the sets $I_A(I_B)$ of $A(B)$ requirements which would be destroyed by putting $c(\sigma)$ into $A(B)$. Let $\delta_A(\delta_B)$ be the least ordinal $\gamma < \lambda$ such that $I_A(I_B)$ contains a γ-requirement. If $\delta_A \leq \delta_B$ we put $c(\sigma)$ into B; otherwise it goes into A. Thus we have given a $\delta - A$

requirement priority over a $\delta - B$ requirement iff $\gamma \leq \delta$. This then completes our description of the construction.

2. The Priority Argument

Our primary concern is to show that enough blocks eventually settle down so that we can argue that $[f^{-1}\varepsilon]^A \neq D$ for every $\varepsilon < \alpha^*$. We proceed inductively.

<u>Lemma 2.1</u>: If by some stage τ, p_δ^σ and y_δ^σ have stabilized at values p_δ and y_δ respectively for $\delta < \gamma$ and $\bigcup_{\delta<\gamma} p_\delta < \alpha^*$ then p_γ^σ and y_γ^σ eventually stabilize (of course at values less than α^* and α respectively).

<u>Proof</u>: Note that $\bigcup_{\delta<\gamma} p_\delta < \alpha^*$ implies that $\bigcup_{\delta<\gamma} y_\delta < \alpha$ since no sequence unbounded in α can project down to one bounded in α^*. To establish the lemma it clearly suffices to show that y_γ^σ eventually stabilizes. For γ a limit ordinal this is immediate from our assumptions and the definition of y_γ^σ. We therefore consider the case $\gamma = \nu + 1$ and show that there is a bound on the stages at which ν-requirements are created and at which elements of such requirements are enumerated in C.

Let $y = \bigcup_{\delta<\gamma} y_\delta$ and look at the construction from stage y onward. Everything connected with δ-requirements for $\delta < \nu$ has stopped acting up by stage y. In particular we never have to worry about preserving any δ-requirement for $\delta < \nu$ after stage y. Thus any $\nu - A$ requirement existing at stage y or created thereafter is never destroyed. The computation associated with such a requirement is, therefore, correct, i.e. $[f^{-1}\varepsilon]_{\sigma}^{A^{\sigma}}(x) = [f^{-1}\varepsilon]^{A}(x)$. We must show that there are not too many of them.

Consider the set W of ε in P_ν such that ε is ν-inactive at some stage after y. This is a Σ_1 subset of $P_\nu < \alpha^*$ and so α-finite. Moreover by the above remarks once any ε becomes ν-inactive after stage y it remains so forever. Of course the stages at which each ε in W becomes ν-inactive are contained in the Σ_1-hull of W. Since W is an α-finite subset of $P_\nu < \alpha^*$ this hull is bounded, say by τ. After stage τ no $\nu - A$ requirement which is associated with any ε in W can be created. Moreover any $\nu - A$ requirement created after stage τ gives the correct value of D. The point is that the computation from A is never changed while the only change in D that can occur is that a new element of D is enumerated. This, however, would put the reduction procedure involved into W by definition contradicting our choice of τ.

Finally it is clear from the construction that the only way α-infinitely many $\nu - A$ requirements could be created would be to eventually have one associated with a ν-active requirement for each $x < \alpha$. Were this to occur we could calculate D recursively as follows: to decide if $K \quad D = \emptyset$, begin at stage τ and proceed until a $\nu - A$

requirement with argument x associated with an ε not in W has been created for every $x \varepsilon K$. (Since such a stage exists for each $x \varepsilon K$ and the map from x to that stage is recursive, there is one stage by which it has all happened.) Now simply check the values of the computations associated with each argument to get the true value of $D(x)$. Since D is not α-recursive, there is a bound on the stages at which $\nu - A$ requirements are created. Moreover the collection of $\nu - A$ requirements is an α-finite set and since C is regular there is a bound on the stages at which elements of these requirements are enumerated in C. Thus the contribution to y_γ^σ from $\gamma - A$ requirements eventually stabilizes. Beginning at such a stage the same argument now shows that the contributions from $\gamma - B$ requirements also stabilizes. Thus y_γ^σ and hence P_γ^σ attain constant values below α and α^* respectively. $\square$

In view of this lemma we can let λ be the least ordinal such that $\bigcup_{\delta<\lambda} P_\delta = \alpha^*$ and be assured that λ is a limit ordinal and that P_δ does in fact exist for each $\delta < \lambda$. We are now in a position to prove that our construction has succeeded. As noted before $A \cup B = C$ and $A \cap B = \phi$ are immediate while $A, B \leq_\alpha C$ is only slightly less obvious. To check if $K \cap A = \phi$ just find a stage σ such that $C^\sigma \cap K = C \cap K$ and ask if $A^\sigma \cap K = \phi$. This clearly represents a reduction procedure for A from C as well as a proof that A and B are regular. As A and

B are α-r.e. by construction we only have to prove the following:

<u>Lemma 2.2</u>: D is not α-recursive in A or B.

<u>Proof</u>: Assume not. For the sake of definiteness say $[f^{-1}\varepsilon]^A = D$. Let ν be the least ordinal $< \lambda$ such that $\varepsilon < P_\nu$. By lemma 2.1 there is a least x which is not the argument of a $\nu - A$ requirement associated with a $\nu - A$ active ε' at any stage after y_ν. As $[f^{-1}\varepsilon]^A = D$, ε cannot be $\nu - A$ inactive at any stage after y_ν. Moreover since A is regular there is a stage $\sigma \geq y_\nu$ by which $[f^{-1}\varepsilon]_\sigma^{A^\sigma}(x) = [f^{-1}\varepsilon]^A(x)$. We may of course also assume that $D^\sigma(x) = D(x)$. Since $D(x) = [f^{-1}\varepsilon]^A(x)$ we create, at stage σ, a $\nu - A$ requirement with argument x associated with the $\nu - A$ active reduction procedure ε. This of course contradicts our choice of x. $\square$

3. Some Corollaries about α-degrees

In order to derive some interesting consequences about α-degrees from the splitting theorem we need a simple lemma.

Lemma 3.1: If A and B are disjoint regular α-r.e. sets then $\deg(A \cup B) = \deg(A) \vee \deg(B)$.

Proof: That $A \cup B$ is α-recursive in $\deg(A) \vee \deg(B)$ is obvious. To prove the other direction it suffices to show that A and B are α-recursive in $A \cup B$, i.e. that we can get A^c (and B^c) from $A \cup B$. SInce A and B are disjoint this is immediate from the observation that $A^c = (A \cup B)^c \cup B$ and the fact that B is α-r.e. and regular. □

We can now prove the usual corollaries of the splitting theorem.

Corollary 3.2: Let c and d be α-r.e. degrees such that d is not α-recursive. Then there are α-r.e. degrees a and b such that $c = a \vee b$, $d \nleq_\alpha a$ and $d \nleq_\alpha b$. □

Corollary 3.3: If c is a non-zero α-r.e. degree, then there are α-r.e. degrees a and b such that

$c = a \quad b$, $0 <_{\alpha} a <_{\alpha} c$, $0 <_{\alpha} b <_{\alpha} c$ and a is incomparable with b.

<u>Proof</u>: Let C be regular set of degree c and let $D = C$ in the theorem. Let a and b be the degrees of the sets A and B guaranteed by the theorem. Then if a and b are comparable (e.g. one is α-recursive) then $a \vee b$ is a or b. But by the lemma $a \vee b = c$ and the theorem assures us that $c \nleq a$ and $c \nleq b$. □

<u>Corollary 3.4</u>: No α-r.e. degree is minimal. □

<u>Corollary 3.5</u>: If d is an incomplete, non-α-recursive α-r.e. degree then there is an α-r.e. degree incomparable with d.

<u>Proof</u>: Let the C of the theorem be a regular complete α-r.e. set and let D be a regular set of degree d. Let a and b be the degrees of the sets given by the theorem. If both a and b are comparable with d then both are recursive in it and so $a \vee b \leq_{\alpha} d$ but $a \vee b = c$ and $d \lneq_{\alpha} c$ -- a contradiction. □

4. A Strengthening for α-calculability

4.1 In this section we will show how our main theorem can be strengthened to make sure that D is not α-calculable from A or B. Since a set is α-calculable from a regular hyperregular set E iff it is α-recursive in E, it will suffice to make the A and B of our theorem hyperregular. We will do this by adding on as new requirements the preservations usually invoked to make sets hyperregular [10]. More specifically, we will preserve computations for $[\varepsilon]^A(x)$ roughly on the longest initial segment on which it is total. This will make $[\varepsilon]^A$ essentially α-recursive on this segment and so its range will be bounded. Thus A will be hyperregular by definition.

4.2 The construction: We begin by noting that we can place an a priori bound on the length of an initial segment about which we must concern ourselves. Trivial manipulations show that there is no need to preserve $[\varepsilon]^A$ on any segment longer than ε. Moreover if there is a last α-cardinal whose cofinality in L_α is γ we can get by with preserving $[\varepsilon]^A$ on γ. Keeping this in mind we make the following additions to our construction:

At stage σ we find, for each ν and each $\varepsilon < p_\nu$, the least $x < \varepsilon(\gamma)$ such that there is no $\nu' - A$ requirement with argument x associated with ε. If there is such an x and $[f^{-1}\varepsilon]_{\sigma}^{A^{\sigma}}(x)$ is convergent we create a $\nu' - A$ requirement with argument x associated with ε. The requirement consists of the negative facts about A^{σ} used in this computation. We adjust the definition of P_{ν}^{σ} to bound the creation of γ'-requirements and the enumeration by C of elements contained in γ'-requirements for $\gamma < \nu$. Of course a $\gamma' - A$ requirement is destroyed if any of its elements is put into A. Finally we insert these requirements into the priority listing by giving ν'-requirements (for A and B) the same priority as the appropriate requirements. (Since $\nu - A$ and $\nu' - A$ requirements do not conflict this presents no problems.) We now determine whether to put $c(\sigma)$ into A or B as before.

4.3 The priority argument: Our main goal is again to show that the P_{ν}^{σ} and y_{ν}^{σ} stabilize. Once that has been done we can argue for the hyperregularity of A as follows: Consider any $\varepsilon < \alpha^*$. Let ν be the least ordinal such that $\varepsilon < P_{\nu}$. Once we are beyond stage y_{ν} any computations of $[f^{-1}\varepsilon]^{A}(x)$ for $x < \varepsilon(\gamma)$ covering an initial segment of $\varepsilon(\gamma)$ are made into requirements which are never destroyed. Moreover if $[f^{-1}\varepsilon]^{A}(x)$ is defined at all we eventually get such a computation (A is still regular). Thus we can compute $[f^{-1}\varepsilon]^{A}$ on the longest initial segment $< \varepsilon(\gamma)$ on which it is defined by just going through the construction from stage y onward. Since this is an α-recursive procedure $[f^{-1}\varepsilon]^A$ is bounded on this segment and A is hyperregular. We therefore conclude our proof by establishing lemma 2.1 in this context.

Lemma 4.4: If by some stage τ, P_δ^σ and y_δ^σ have stabilized at values P_δ and y_δ respectively and $\bigcup_{\delta<\gamma} P_\delta < \alpha^*$ then P_γ^σ and y_γ^σ eventually stabilize.

Proof: As before, we need only consider the case $\gamma = \nu + 1$ and we know that the contributions for ν-requirements are bounded. We have only to show that there is a bound on the stages at which ν'-requirements are created. (The regularity of C takes care of the other component of y_γ.) If there is no last α-cardinal or the last α-cardinal is singular we can argue directly for this bound. The ν'-requirements created correspond to an α-r.e. subset of $P_\nu \times P_\nu$ or $P_\nu \times \gamma$ ($\langle$the associated requirement, the argument of the requirement$\rangle$) each of which is strictly less than α^* by assumption. Thus the set is α-finite and so enumerated in bounded time.

If the last α-cardinal γ is regular we first consider the set W of $\varepsilon < P_\nu$ such that for every $x < \gamma$ there are ν'-requirements created after stage y_ν which are associated with ε. This subset of P_ν is α-r.e. and so α-finite. Similarly all these requirements associated with ε's in W form an α-finite set and so are bounded. Were there some bound strictly less than γ on the arguments for which we create ν'-requirements, we could argue as above, since $\gamma \leq \alpha^*$. On the other hand if there were no such bound, we could define an α-recursive map $\gamma \to p_\nu$ by sending x to the ε associated with the first ν'-requirement with argument x created after y_ν. Since this would exhibit γ as an α-finite union over $p_\nu < \gamma$ of α-finite sets bounded below γ it would contradict the regularity of γ in L_α. Thus we can indeed conclude as in the previous cases. □

Of course there is now no difficulty in deriving all the corollaries of section 3 for α-calculability degrees as well.

Chapter IV

The Density Theorem

Of all the types of priority arguments in ordinary recursion theory surely the most powerful (and difficult) is the infinite injury method developed by Sacks [6]. As a prime example of this method we cite Sack's theorem that the r.e. degrees are dense [17]. This theorem has, indeed, been seen as a touchstone for the infinite injury method in α-recursion theory [10, Q. 4], [16, Q. 2]. The best result to date has been Driscoll's theorem [15] that the α-r.e. degrees are dense for $\alpha^* = \omega$. In this chapter we sketch a proof of the density theorem for every admissible α. Hopefully this beginning will also give some insight into the general method of infinite injury arguments in α-recursion theory. A full proof with more detailed discussion will appear elsewhere [18].

1. The General Plan

Let two regular α-r.e. sets $B <_\alpha C$ be given. We wish to construct an α-r.e. set A such that $B <_\alpha A <_\alpha C$. Our whole construction will be controlled by C and so we will have that $A \leq_\alpha C$. To insure that A has all the other required properties we will have three types of requirements. First we will require that $A^{(0)} \equiv \{x \mid \langle 0,x \rangle \in A\} = B$ to make sure that $B \leq_\alpha A$. Secondly, we will have negative requirements resembling those of chapter III to guarantee that $C \not\leq_\alpha A$. Finally we will have positive requirements that feed parts of C into certain areas of A as long as it appears that $A \leq_\alpha B$. These requirements will also be formed along the lines of chapter III. The idea is that, should A turn out to be α-recursive in B, we would then have put all of C into A in a simple way. This would imply that $C \leq_\alpha A \leq_\alpha B$ -- a contradiction and so $A \not\leq_\alpha B$ as required.

As in chapter III, the main overall problem is to arrange the requirements in a sufficiently short sequence. Because of the complexity of the construction we are forced to use the Σ_2-cofinality of $\langle L_\alpha, B \rangle$, written $\Sigma_2(B) - cf(2)$, as our basic ordering of priorities. Though this is a natural choice it is difficult to implement since it is given by a Σ_3 function while our construction must be Δ_1. This restriction requires an approximation procedure for the priority listing which is quite weak. It converges only in the sense that the correct value is the least one returned to unboundedly often. This phenomena is typical of the infinite injury method in ordinary recursion theory and it was only to be expected that it be manifested in the priority listing itself in α-recursion theory.

Of course, we also have the usual problem of infinite injury arguments. We must not allow positive requirements to be thwarted by an unbounded succession of temporary negative requirements. This problem is solved roughly as in [12] by not allowing elements associated with positive requirements into negative requirements. Instead we put in a collection of other elements which are also being kept out by related negative requirements. (Admittedly this is vague.) The machinations necessary to carry this out will cause a split in the proof (though not in the construction) according to whether or not the Σ_2 - cf(2) equals ω or not. The idea is that we will have to be able to close under the operations involved in forming negative requirements. This closure is done naturally in ω many steps of a Σ_2 process. Thus if $\Sigma_2 - cf(\alpha) > \omega$ all will be well. If, on the other hand, $\Sigma_2 - cf(\alpha) = \omega$, we will prove that only finitely many steps are needed after all.

Finally there will be a whole new set of problems introduced if B is not hyperregular. Since we will essentially be working relative to B, its non-hyperregularity raises the obvious but serious difficulty of our not being able to iterate simple procedures. That is, as $\langle L_\alpha, B\rangle$ is not admissible we cannot perform normal recursions. We handle this difficulty by restricting ourselves to recursive cofinality-(B) many steps for several parts of the construction. Even this drastic restriction would not be enough were it not for the crucial lemma 2.1 which allows us to work below $\rho^1_{\alpha,B}$ as well.

that $k_x = \{y | k(x,y)$ is convergent$\}$ is a proper initial segment of α for every $x < \theta$ but $\bigcup_{x<\theta} k_x = \alpha$.

b) γ = the least θ such that there is a partial function, $k(x,y)$, α-recursive in B such that k_x is a proper initial segment of β for each $x < \gamma$ but $\bigcup_{x<\theta} k_x = \beta$.

Lemma 2.3: $f[g[\gamma]]$ is unbounded in ρ and for every $\delta < \gamma$, $f[g[\delta]]$ is bounded in ρ.

2.4: As for approximating f, g, and h, we begin by approximating B as usual via an α-recursive enumeration that gives us B^σ at stage σ. As f, f^{-1} and h are α-recursive in B, there are e_f, $e_{f^{-1}}$ and e_h such that $[e_f]^B = f$, $[e_{f^{-1}}]^B = f^{-1}$, and $[e_h]^B = h$. We can therefore approximate them quite easily by setting $f^\sigma(x) = [e_f]_\sigma^{B^\sigma}(x)$ for every x and acting similarly for f^{-1} and h. (Note that we always take the least available computation giving a value for $[e]_\sigma^{B^\sigma}(x)$ to determine a value.) We approximate g via the k of lemma 2.2 (a). As k is α-recursive in B, $k = [e_k]^B$ and we set $g^\sigma(x)$ = least y such that $[e_k]_\sigma^{B^\sigma}(x,y)$ is divergent. We add the proviso, however, that if at a later stage σ' we discover that some incorrect information about B was used to get this value, $g^{\sigma'}(x)$

reverts to the least y for which incorrect information about B was used in the computation for $[e_k]^B(x,y)$.

The approximation to f and h are eventually correct on each initial segment of α and β respectively. As for f^{-1}, the approximation is eventually correct on $\mathrm{dom}\, f^{-1} \cap \delta$ for each $\delta < \sigma$. For $x \notin \mathrm{dom}\, f^{-1}$ all we can say is that any proposed value is eventually seen to be incorrect. Finally for $x < \alpha$, $g(x)$ is the least y such that $g^\sigma(x) = y$ for unboundedly many σ.

3. The Requirements and Construction

We will try to give an intuitive picture of the requirements and leave the formal details to [18]. To facilitate this we supress the approximation procedures and write simply f, f^{-1}, g and h. Before beginning in earnest we insist that when any x is enumerated in B we put $\langle 0,x\rangle$ into A. This takes precedence over all other requirements and insures that $B \leq_\alpha A$.

3.1 The positive requirements: For each $x < \gamma$ we consider $A^{g(x)}$ as a square array. At stage σ, we first find the least $y < \beta$ such that we cannot compute $A\restriction h(y)$ from B via some p-active reduction procedure $< fg(x)$. ("ρ-active" is defined analogously to "active" in III 1.2.) We then initiate positive requirements with priority x that try to copy $C\restriction h(y)$ into the σ^{th} row of $A^{g(x)}$. Of course, if at any later stage we discover that incorrect information about B was used to compute some $A\restriction h(z)$ we then give up trying to copy $C\restriction h(y) - C\restriction h(z)$ into $(A^{g(x)})^\sigma$.

3.2 The negative requirements: For each $x < \gamma$ we find the least $y < \beta$ such that we cannot compute $C\restriction h(y)$ from A via some n-active reduction procedure $< fg(x)$.

For each $z < y$ we have a set $N_{\sigma,x,z}$ of negative facts about A used in the computation of $C\restriction h(z)$. We would like to make these into negative requirements but that would allow elements to be kept out of A by a succession of temporary requirements which we cannot bound. We instead associate a negative requirement $R_{\sigma,x,z}$ with priority x with each $N_{\sigma,x,z}$. The main restriction is that no element with an associated positive requirement gets put into $R_{\sigma,x,z}$. Rather than putting any such elements into $R_{\sigma,x,z}$ we put in all other elements of negative requirements of high priority which are currently keeping them out. Indeed, we basically close $N_{\sigma,x,z}$ under such an operation to get $R_{\sigma,x,z}$. If any element of $R_{\sigma,x,z}$ is later put into A or any information about B used to compute $C\restriction h(z)$ is found to be incorrect, we destroy $R_{\sigma,x,w}$ for $w \geq z$.

3.3 The construction: We simply act as dictated by the requirements described above. At each stage we put in any element for which we have a positive requirement with priority x unless it belongs to a negative requirement with priority $\leq x$.

4. The Proof -- An Outline

The priority argument and proof that our construction succeeds are quite long and depend somewhat on details we have neglected. Thus in this section we merely give the sequence of lemmas via which the proof proceeds with only a general idea as to how one proves the lemmas.

Lemma 4.1: One can tell α-recursively if one ever begins to copy C into a given part of A. Moreover, α-recursively in B one can tell whether one ever later stops trying to copy C into this part of A. Finally, if one never does give this up the associated computation relative to B is correct. This lemma is immediate.

Lemma 4.2: If a negative requirement $R_{\sigma,x,z}$ is permanent the associated computation relative to A is correct.

The proof depends on the well-foundedness (derived from the priority ordering) of the procedure used to form $R_{\sigma,x,z}$ from $N_{\sigma,x,z}$.

Now for the heart of the proof one proceeds by a simultaneous induction on $x < \gamma$ to establish the following:

Lemma 4.3: 1) We can tell α-recursively in B if a given negative requirement of priority $x' < x$ is permanent.

2) There are only α-finitely many permanent negative requriements of priority $x' < x$.

3) A permanent positive requirement of priority $x' < x$ gets put into A unless it belongs to a permanent negative requirement of priority $x'' \leq x'$.

4) There are only α-finitely many elements at which we permanently try to copy C in A with priority $x' < x$. Moreover, there are only α-finitely many elements for which we actually create permanent positive requirements with priority $x' < x$.

At successor stages, (1) depends on (3) and (4) plus the fact that we can then α-recursively check if a negative requirement is destroyed by a temporary positive requirement. Of course, in every part we can tell immediately (from B) if any incorrect information about B was used. The proofs of (2) and (4) are similar to III.2.1. If (2) failed we could show that $C \leq_\alpha B$, while a failure of (4) would give first that $A \leq_\alpha B$ and then $C \leq_\alpha A \quad B$. In either case we contradict our basic assumption that $B <_\alpha C$. Finally (3) follows from the condition that no element with a positive requirement is put into a negative requirement thereafter. Thus all the negative ones to which it belongs are eventually destroyed.

At limit levels we note that the whole procedure was α-recursive in B except that certain initial segments of C were also needed. It turns out that the amount of C needed is given by a $\Sigma_2(B)$ function. Thus the definition of γ assures us that one needed only a proper initial segment of C. Since C is regular this set is α-finite and one can now go through the proof up to this stage uniformly to get the required bounds.

We are now in a position to show that the set A has all the desired properties.

Lemma 4.4: A is regular and $A \leq_\alpha C$. Basically the idea is to use 4.3 (3). One must, however, search through a ramifying tree of requirements. The tree is, of course, well founded and the only problem is to make sure the tree generated by any α-finite set is bounded below α. In general the operations that one must close under are Σ_2 and so if $\Sigma_2 - cf(\alpha) > \omega$ all is well. If $\Sigma_2 - cf(\alpha) = \omega$ then it turns out that finitely many steps suffice to close off, since in this case all priorities are integers and each step insures a strict decrease in priority.

Lemma 4.5: $C \not\leq_\alpha A$.

<u>Lemma 4.6</u>: $A \not\leq_\alpha B$.

In both of these lemmas the idea is like that of III.2.2. The main new obstacle for 4.5 is to show that for certain correct computations of C from A one eventually gets an associated negative requirement which is permanent. In addition to 4.3 one must analyze the procedure for constructing negative requirements. We are again faced with a closure operation problem and again the proof splits according to whether $\Sigma_2 - cf(\alpha) = \omega$ or not. Since it is easy to see that the relevant correct computations of A from B eventually produce permanent positive requirements, there are no extra difficulties in the proof of 4.6.

As we insured that $B \leq_\alpha A$ in 3, we now have all the required properties for A: $B <_\alpha A <_\alpha C$.

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BIOGRAPHY

Richard A. Shore was born in Boston, Massachusetts on August 18, 1946. He was graduated from the Boston Latin School in June, 1963, and entered Harvard University the following September. He spent the academic year 1964-1965 on leave from Harvard at the Hebrew University in Jerusalem, Israel. His study there constituted his Junior Year at the Hebrew Teachers' College, Brookline, Massachusetts, from which he received a B.J.Ed. in June, 1966. In June, 1968 he received an A.B. summa cum laude in mathematics from Harvard University and was elected to $\phi\beta\kappa$. The following September he began four years of graduate work in mathematics at the Massachusetts Institute of Technology. During his first two years at M.I.T. he held a full-time teaching assistantship and a staff scholarship. In his third and fourth years he was supported by a National Science Foundation Traineeship and Fellowship, respectively. With Prof. E. M. Kleinberg he has published two papers in the Journal of Symbolic Logic: "On Large Cardinals and Partition Relations" and "Weak Compactness and Square Bracket Partition Relations". He has also been elected to full membership in $\Sigma\chi$ and is a member of the American Mathematical Society and the Association for Symbolic Logic. In September, 1971 he bought a Persian kitten, named her Theorem, and began work on this thesis.