SEPARATING COMBINATORIAL PRINCIPLES OVER WEAK KÖNIG'S LEMMA

DAMIR D. DZHAFAROV, JUN LE GOH, AND RICHARD A. SHORE

Abstract. We study the reverse mathematical content of two combinatorial principles weaker than Ramsey's theorem for pairs: the ascending-descending sequence principle (ADS), which says that every infinite linear order has an infinite ascending or descending sequence; and the chain-antichain principle (CAC), which says that every infinite partial order has an infinite chain or antichain. It is easy to show that CAC implies ADS over the base theory RCA_0. On the other hand, Lerman, Solomon, and Towsner [8] showed that ADS does not imply CAC, nor even the chain-antichain principle for stable partial orders (SCAC). Their proof begins by showing that SCAC does not computably reduce to ADS, followed by an iterated forcing construction to add solutions to ADS without adding a solution to a fixed suitably generic instance of SCAC. Using an extension of their methods, we strengthen their result to show that ADS does not imply SCAC even over Weak König's Lemma (WKL_0), rather than merely over RCA_0. Roughly, this means that even if we allow ourselves the use of compactness, we cannot prove SCAC from ADS using computable methods.

1. Introduction

Reverse mathematics is a program which aims to calibrate the proof-theoretic strength of theorems in countable and separable mathematics. This calibration is done by formalizing theorems in the setting of second-order arithmetic, and then proving implications or nonimplications between them over a weak base theory. The standard base theory is the subsystem RCA_0 of second-order arithmetic (to be defined at the end of the introduction), which roughly corresponds to computable mathematics. See [12] for background on reverse mathematics.

Several major subsystems of second-order arithmetic other than RCA_0 admit characterizations via computability-theoretic notions such as the Turing jump or hyperjump. It is therefore not surprising that

Date: November 24, 2015.

Dzhafarov's research was supported by NSF Grant DMS-1400267. Goh's and Shore's research were supported by NSF Grant DMS-1161175.

many results in reverse mathematics can be proved by methods in computability theory. Consider, for example, theorems Φ of the form $\forall A(\Theta(A) \to \exists B\, \Xi(A, B))$, where Θ and Ξ are arithmetic formulas. We call these theorems Π^1_2 *principles*. We say that A is a Φ-*instance* if $\Theta(A)$ holds, and we say that B is a *solution* to the Φ-instance A if $\Xi(A, B)$ holds. A Π^1_2 principle of particular interest is Ramsey's theorem (RT^n_k) : every k-coloring of n-tuples of natural numbers has an infinite homogeneous set.

The simplest and most prevalent interaction between reverse mathematics and computability theory is as follows. Let Φ be a Π^1_2 principle. Roughly, if there is a Φ-instance with all solutions being computationally hard, then Φ has high proof-theoretic strength, while if every Φ-instance has a computationally easy solution, then Φ has low proof-theoretic strength. That is, being "harder to compute" roughly corresponds to being "harder to prove" [11]. This interaction is often used for proving nonimplications between Π^1_2 principles. For example, it figures strongly in the study of combinatorial principles, such as Ramsey's theorem, from the point of view of mathematical logic (see, e.g., [1], [3]).

We are particularly interested in results in reverse mathematics which are derived from more sophisticated interactions with computability theory. In this paper, we consider one such result due to Lerman, Solomon, and Towsner (LST) [8]. Consider the *ascending-descending sequence principle*

> ADS: every infinite linear order has an infinite ascending or descending sequence,

and the *chain-antichain principle*

> CAC: every infinite partial order has an infinite chain or antichain.

In the setting of second-order arithmetic, we only consider countably infinite linear orders and partial orders, and their domain will be the first-order universe $\mathbb{N}$.

It is easy to see that CAC implies ADS (over RCA_0; proof below). LST [8] showed that this implication is strict, i.e., ADS does not imply CAC over RCA_0. We extend their work to show that ADS does not imply CAC even over the base theory WKL_0 (to be defined at the end of the introduction). WKL_0 roughly corresponds to computable mathematics enhanced with the ability to make compactness arguments.

We proceed to give some background on ADS and CAC (see also [5]). We will also define stable versions of ADS and CAC, so that we may state the results of [8] and our result more accurately.

Firstly, we may computably transform instances of ADS into instances of CAC as follows. Given a linear order $\prec$, define a partial order $<_M$ by $a <_M b$ if and only if $a \prec b$ and $a <_{\mathbb{N}} b$. Every $<_M$-chain is a $\prec$-ascending sequence, and every $<_M$-antichain (listed in $<_{\mathbb{N}}$-order) is a $\prec$-descending sequence. By formalizing the above construction in RCA_0, we may prove that CAC implies ADS over RCA_0 [5].

We may also computably transform instances of CAC into instances of Ramsey's theorem for pairs (RT^2_2). Given a partial order, define a 2-coloring of pairs by coloring (a, b) red if and only if a is comparable to b in the partial order. Every homogeneous set for this coloring is either a chain or antichain for the partial order. By formalizing the above construction in RCA_0, we may prove that RT^2_2 implies CAC over RCA_0 [5]. Hence CAC and ADS are both consequences of RT^2_2.

In [1], Cholak, Jockusch, and Slaman (CJS) introduced an extremely fruitful method of studying RT^2_2. They say that a 2-coloring of pairs $c : [\mathbb{N}]^2 \to 2$ is *stable* if for every n,

$$\exists x(\forall y > x)[c(n, x) = c(n, y)].$$

CJS showed that RT^2_2 is equivalent to the conjunction of its stable version SRT^2_2 and the cohesive principle COH (see [1]; we will not analyse SRT^2_2 or COH in this paper). Inspired by CJS, Hirschfeldt and Shore (HS) [5] initiated the study of the stable versions of CAC and ADS. They say that an infinite linear order $\prec$ is *stable* if for every n, either

$$\exists x(\forall y > x)[n \prec y] \quad \text{or} \quad \exists x(\forall y > x)[y \prec n].$$

There is an analogous definition (to be given at the end of the introduction) for stability of partial orders. The *stable chain-antichain principle* (SCAC) is the chain-antichain principle for stable partial orders, and the *stable ascending-descending sequence principle* (SADS) is the ascending-descending sequence principle for stable linear orders. The aforementioned transformation of instances of ADS into instances of CAC also transforms instances of SADS into instances of SCAC, so we may show that SADS follows from SCAC over RCA_0 [5].

It turns out that the only implications between CAC, ADS, SCAC, and SADS over RCA_0 are the ones mentioned thus far: HS [5] showed that $\mathsf{SCAC} \nvdash \mathsf{ADS}$ and LST [8] showed that $\mathsf{ADS} \nvdash \mathsf{SCAC}$[1]. These relations are summarized in Figure 1.

Since compactness arguments are routine in combinatorics, it is natural to ask whether the nonimplications above hold over WKL_0. Note that dovetailing constructions in HS [5] gives us $\mathsf{WKL}_0 + \mathsf{SCAC} \nvdash \mathsf{ADS}$.

[1]It follows that $\mathsf{ADS} \nvdash \mathsf{SRT}^2_2$, and so $\mathsf{ADS} \nvdash \mathsf{SEM}$.

$$\begin{tikzcd} & \mathsf{CAC} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ \mathsf{ADS} \arrow[rr, leftrightarrow, "/" marking] \arrow[dr, Rightarrow] & & \mathsf{SCAC} \arrow[dl, Rightarrow] \\ & \mathsf{SADS} & \end{tikzcd}$$

FIGURE 1. The above relations hold over RCA_0. The double arrows indicate strict implications and the negated single arrows indicate nonimplications in both directions.

In this paper, we prove that $\mathsf{WKL}_0 + \mathsf{ADS} \nvdash \mathsf{SCAC}$[2], which implies that all relations in Figure 1 hold over WKL_0.

We conclude the introduction with some definitions which were omitted above. If $\prec$ is an infinite linear order, we say that G is a $\prec$-ascending sequence if for all a and b in G such that $a <_{\mathbb{N}} b$, we have $a \prec b$. We define $\prec$-descending sequences analogously. If M is an infinite partial order, define

$$\begin{aligned} A^*(M) &= \{n : \exists x \forall y > x (n <_M y)\} \\ B^*(M) &= \{n : \exists x \forall y > x (n \,|_M\, y)\} \\ C^*(M) &= \{n : \exists x \forall y > x (y <_M n)\}. \end{aligned}$$

M is *stable* if either $A^*(M) \cup B^*(M) =^* \mathbb{N}$ or $C^*(M) \cup B^*(M) =^* \mathbb{N}$, where $=^*$ denotes equality up to a finite set. Justification for this definition of stability can be found in [5, Proposition 5.4]. LST [8] construct a partial order M such that $A^*(M) \cup B^*(M) = \mathbb{N}$, with the additional property that any M-computable set intersects both $A^*(M)$ and $B^*(M)$ and hence is neither a $<_M$-chain nor a $<_M$-antichain.

Next, we define the theories RCA_0 and WKL_0. The language L_2 of second-order arithmetic is a two-sorted language formed by augmenting the usual first-order language of arithmetic with set variables and the membership relation $\in$. An L_2-structure is of the form

$$N = (|N|, \mathcal{S}_N, \in_N, +_N, \cdot_N, 0_N, 1_N, <_N),$$

where number parameters come from the first-order universe $|N|$ and set parameters come from the second-order universe $\mathcal{S}_N \subseteq \mathcal{P}(|N|)$.

All of the theories we consider will contain axioms for 0, 1, $+$, $\cdot$, and $<$ which say that $\mathbb{N}$ is an ordered semiring, together with the following

[2] As before, it follows from this that $\mathsf{WKL}_0 + \mathsf{ADS} \nvdash \mathsf{SRT}^2_2, \mathsf{SEM}$.

induction axiom:

$$\forall X((0 \in X \wedge \forall n(n \in X \to n+1 \in X)) \to \forall n(n \in X)).$$

In addition to the above, the system *Recursive Comprehension Axiom* (RCA_0) contains comprehension for Δ^0_1 relations, i.e., for all Σ^0_1 formulas φ and Π^0_1 formulas ψ, we have

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \to \exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

and induction for Σ^0_1 formulas, i.e., for all Σ^0_1 formulas φ, we have

$$(\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(n+1))) \to \forall n \varphi(n).$$

The system *Weak König's Lemma* (WKL_0) contains, in addition to RCA_0, the statement WKL: every infinite binary tree (i.e., subtree of $2^{<\mathbb{N}}$) has an infinite path.

2. Proof strategy

We begin by surveying some known methods for proving nonimplications between Π^1_2 principles, with attention to whether the proofs of nonimplications $\Phi_1 \nvdash \Psi$ and $\Phi_2 \nvdash \Phi$ can be combined to give $\Phi_1 + \Phi_2 \nvdash \Psi$. In this paper, $\Phi \nvdash \Psi$ always means $\mathsf{RCA}_0 + \Phi \nvdash \Psi$.

We say that Ψ *computably reduces* to Φ [2] if every Ψ-instance A computes a Φ-instance B such that any solution to B, together with A, computes a solution to A. For example, the aforementioned transformation of instances of ADS into instances of CAC is a computable reduction from ADS to CAC[3].

Strong failures of computable reducibility often lead to nonimplication proofs. For example, if (1) there is a Ψ-instance with no "simple" solution; and (2) every Φ-instance has "simple" solutions, then one could construct a model of $\Phi + \neg\Psi$ by starting from said Ψ-instance and repeatedly adding "simple" solutions to Φ-instances in the model. Since the array of primitive recursive sets has no low cohesive set (Jockusch, Stephan [7]) and WKL always admits low solutions, we have $\mathsf{WKL} \nvdash \mathsf{COH}$ (CJS [1, Lemma 9.14]). HS [5, Proposition 2.15, Theorem 3.4] showed that there is a computable ADS-instance with no low solution, and that SCAC always has low solutions, so $\mathsf{SCAC} \nvdash \mathsf{ADS}$.

In this case nonimplications combine easily by dovetailing. We may start from a computable ADS-instance with no low solution and add low solutions to WKL and SCAC alternately. This gives a model witnessing $\mathsf{WKL}_0 + \mathsf{SCAC} \nvdash \mathsf{ADS}$. Another combination of nonimplications is due to Wang [13]. We may consider a set X to be "simple" if for all sets

[3]This is in fact a *strong computable* reduction. Notions of computability theoretic reducibilities between Π^1_2 principles have gained recent interest, see [2] and [4].

Y, sets which are "complex" relative to Y remain "complex" relative to $X \oplus Y$. Taking "complex" to mean Δ^0_2, Wang [13] showed that SADS is not implied by a conjunction of five particular Π^1_2 principles[4], including WKL. Taking "complex" to mean hyperimmune, Patey [10] later proved a variation of Wang's result, among other results.

Another method for proving $\Phi \nvdash \Psi$ is to prove that (1) there is a Ψ-instance A_0 with no A_0-computable solution; and (2) for every Ψ-instance A, every $X \geq_T A$ which does not compute any solution to A, and every Φ-instance $B \leq_T X$, there is a solution G to B such that $X \oplus G$ does not compute any solution to A. Then we may construct a model of $\Phi + \neg\Psi$ as before. This method is more complicated than the previous one because the construction of a solution to the Φ-instance B depends on the Ψ-instance A.

In this way, CJS [1, Lemma 9.16] showed that $\mathsf{COH} \nvdash \mathsf{WKL}$ and HS [5, Proposition 2.24] showed that $\mathsf{WKL} \nvdash \mathsf{SADS}$. HS [5, Theorem 2.20] also strengthened CJS's result to include all "restricted Π^1_2" principles such as WKL and SADS, i.e., they showed that COH is "restricted Π^1_2"-conservative over RCA_0. Hirschfeldt, Shore, and Slaman [6, Proposition 3.14] later showed that the atomic model theorem (AMT) is "restricted Π^1_2"-conservative over RCA_0 as well. Observe that we may combine nonimplications by dovetailing, so the above results combine to give $\mathsf{WKL}_0 + \mathsf{AMT} + \mathsf{COH} \nvdash \mathsf{SADS}$[5].

More generally, in order to prove $\Phi \nvdash \Psi$, it suffices to have a single Ψ-instance A such that (1) A has no A-computable solution; and (2) for every $X \geq_T A$ which does not compute any solution to A and every Φ-instance $B \leq_T X$, there is a solution G to B such that $X \oplus G$ does not compute a solution to A. In fact when constructing a model witnessing $\Phi \nvdash \Psi$, we do not need (2) to hold for *every* $X \geq_T A$ which does not compute any solution to A, but merely X of the form $A \oplus G_0 \oplus \cdots \oplus G_i$, where the G_i are solutions to the Φ-instances which we considered previously in the construction. That is, inductively for all i, we want that

> for every Φ-instance $B_{i+1} \leq_T A \oplus G_0 \oplus \cdots \oplus G_i$, there is a solution G_{i+1} to B_{i+1} such that $A \oplus G_0 \oplus \cdots \oplus G_{i+1}$ does not compute any solution to A.

In general one cannot expect the above to hold for arbitrary choices of A and each G_i, so we will set up requirements to ensure this. We will

---

[4]To be precise, Wang [13] showed that $\mathsf{WKL}_0 + \mathsf{COH} + \mathsf{EM} + \Pi^0_1\mathsf{G} + \mathsf{RRT}^2_2 \nvdash \mathsf{SADS}, \mathsf{STS}^2$.

[5]This result was later superseded by the aforementioned result of Wang [13]. Note that $\Pi^0_1\mathsf{G} \vdash \mathsf{AMT}$ [6, pg. 5823].

add solutions to Φ-instances by forcing, and furthermore ensure that inductively for all i,

($*$) for every Φ-instance $B_{i+1} \leq_T A \oplus G_0 \oplus \cdots \oplus G_i$ and every $(A \oplus G_0 \oplus \cdots \oplus G_i)$-computable Φ-forcing for adding solutions to B_{i+1}, certain requirements in the forcing are appropriately dense.

We would then construct G_{i+1} by meeting said requirements, which would have been set up to ensure that (1) $A \oplus G_0 \oplus \cdots \oplus G_{i+1}$ does not compute any solution to A; and (2) ($*$) holds for $A \oplus G_0 \oplus \cdots \oplus G_{i+1}$.

This idea of using genericity to ensure genericity in future forcings was introduced to reverse mathematics by LST [8], who used this method to show that $\mathsf{ADS} \nvdash \mathsf{SCAC}$. Observe that this method is even more complicated than the previous one: A and each G_i have to be constructed in a way that ensures that G_{i+1} exists (which itself must be constructed in a way that ensures that G_{i+2} exists, etc.) Furthermore, it is nontrivial to combine nonimplications which are proved using this method, because ($*$) for a principle Φ_1 does not a priori imply ($*$) for another principle Φ_2. Adding solutions to Φ_1 and Φ_2 alternately would involve showing that inductively for all i,

($*_{\Phi_1}$) for every Φ_1-instance $B_{2i} \leq_T A \oplus G_0 \oplus \cdots \oplus G_{2i-1}$ and every $(A \oplus G_0 \oplus \cdots \oplus G_{2i-1})$-computable Φ_1-forcing for adding solutions to B_{2i}, certain requirements in the forcing are appropriately dense.

($*_{\Phi_2}$) for every Φ_2-instance $B_{2i+1} \leq_T A \oplus G_0 \oplus \cdots \oplus G_{2i}$ and every $(A \oplus G_0 \oplus \cdots \oplus G_{2i})$-computable Φ_2-forcing for adding solutions to B_{2i+1}, certain requirements in the forcing are appropriately dense.

In this paper, we extend the construction in LST [8] to obtain a Turing ideal witnessing that $\mathsf{WKL}_0 + \mathsf{ADS} \nvdash \mathsf{SCAC}$. They construct a stable partial order M with no M-computable chain or antichain, and show that the model topped by M satisfies the appropriate version of ($*_{\mathsf{ADS}}$). We show that

- if we start from a model satisfying ($*_{\mathsf{ADS}}$) and do the ADS forcing in [8], then the extension satisfies the appropriate version of ($*_{\mathsf{WKL}}$);
- if we start from a model satisfying ($*_{\mathsf{WKL}}$) and do WKL forcing (defined in Section 4), then the extension satisfies the appropriate version of ($*_{\mathsf{ADS}}$).

It follows that we may do iterated forcing to add solutions to WKL and ADS alternately without adding a solution to M.

Note that this construction is not modular, and that each diagonalization step in this construction has to anticipate future steps. We do not know if there is a proof of our result, or even LST's original result, which does not involve these complications. In particular, the modular methods of Wang [13] and Patey [10] do not seem to apply here. See Patey [9, pg. 14] and LST [8, pg. 43] for related discussion.

3. ADS forcing

We recall the ADS forcing used in [8], with minor modifications[6]. Firstly, when adding solutions to ADS, we need only consider *stable-ish* linear orders:

Definition 1. A linear order $\prec$ on $\mathbb{N}$ is *stable-ish* if there is a nonempty initial segment V which has no $\prec$-maximum, and furthermore $\mathbb{N}\backslash V$ is nonempty and has no $\prec$-minimum.

Lemma 1 (Lemma 2.4, [8])**.** *If $\prec$ is not stable-ish then $\prec$ computes an infinite $\prec$-ascending or $\prec$-descending sequence.*

For the remainder of this section, suppose Φ_e^X defines a stable-ish linear order $\prec_e^X$, and fix V as in Definition 1.

Definition 2. Define

$$\begin{aligned}\mathbb{A}_e^X &= \{\sigma : \sigma \text{ is a finite } \prec_e^X\text{-ascending sequence}\}\\ \mathbb{D}_e^X &= \{\tau : \tau \text{ is a finite } \prec_e^X\text{-descending sequence}\}\\ \mathbb{P}_e^X &= \{(\sigma,\tau) : \sigma \in \mathbb{A}_e^X, \tau \in \mathbb{D}_e^X, \sigma \prec_e^X \tau\},\end{aligned}$$

where $\sigma \prec \tau$ means $\sigma(|\sigma|-1) \prec \tau(|\tau|-1)$. We assume that $\prec_e^X$-ascending and $\prec_e^X$-descending sequences are also $<_{\mathbb{N}}$-ascending; note that the above sets are computable in X.

For $p \in \mathbb{P}_e^X$, we let σ_p and τ_p denote the ascending and descending parts of p respectively. For $p, q \in \mathbb{P}_e^X$, we say $q \leq p$ if $\sigma_p \subseteq \sigma_q$ and $\tau_p \subseteq \tau_q$.

$\mathbb{P}_e^X$ is our set of forcing conditions. For clarity we may denote it by ADS_e^X.

Definition 3. Define

$$\mathbb{V}_e^X = \{(\sigma,\tau) \in \mathbb{P}_e^X : \sigma \subseteq V, \tau \subseteq \mathbb{N}\backslash V\}.$$

[6]Our half requirements $\mathcal{R}^{X,A,B}$ are always closed under extensions even for finite A and B, so our definition for half requirements being essential is simpler. (Our definition for full requirements being essential is the same as that in [8].) Our definition for the general context for ADS forcing is slightly weaker.

Definition 4. For $p \in \mathbb{P}_e^X$, a *split pair below* p is a pair $q_0, q_1 \in \mathbb{P}_e^X$ such that $q_0 = (\sigma_p{}^\frown\sigma', \tau_p)$, $q_1 = (\sigma_p, \tau_p{}^\frown\tau')$, and $\sigma' \prec_e^X \tau'$.

Lemma 2 (Lemma 2.4, [8])**.** *If $p \in \mathbb{V}_e^X$ and $q_0, q_1 \in \mathbb{P}_e^X$ is a split pair below p, then either $q_0 \in \mathbb{V}_e^X$ or $q_1 \in \mathbb{V}_e^X$.*

We have two types of requirements: half requirements and full requirements. The $\mathbb{A}$-side and $\mathbb{D}$-side half requirements will be subsets of $\mathbb{A}_e^X$ and $\mathbb{D}_e^X$, respectively. In the following, A and B are subsets of $\mathbb{N}$ while A^* and B^* are infinite subsets of $\mathbb{N}$.

Definition 5. For each index i, define the $\mathbb{A}$-side half requirement

$$\mathcal{R}_i^{X,A,B} = \{\tau \in \mathbb{A}_e^X : (\exists \overline{a} \in A)(\exists \overline{b} \in B)[\Phi_i^{X\oplus\tau}(\overline{a})\downarrow = \Phi_i^{X\oplus\tau}(\overline{b})\downarrow = 1]\}.$$

When we do not need to refer to Φ_i directly, we suppress the subscript i.

We say that $\mathcal{R}^X$ is *essential* in G if for all x, there is some finite $A > x$ and n such that $G \upharpoonright n \in \mathcal{R}^{X,A,A}$.

We say that G *satisfies* $\mathcal{R}^{X,A^*,B^*}$ if either $\mathcal{R}^X$ is not essential in G, or some initial segment of G lies in $\mathcal{R}^{X,A^*,B^*}$.

Define $\mathbb{D}$-side half requirements $\mathcal{S}_i^{X,A,B}$ similarly.

Observe that $\mathcal{R}^{X,A,B}$ is closed under extensions and depends positively on A and B. Also observe that for $\prec_e^X$-ascending sequences G, $\mathcal{R}_i^X$ is essential in G if and only if $\Phi_i^{X\oplus G}(\overline{a})\downarrow = 1$ for infinitely many $\overline{a}$. Similar statements hold for $\mathcal{S}^{X,A,B}$ and $\prec_e^X$-descending sequences G.

Definition 6. For each $\mathbb{A}$-side half requirement $\mathcal{R}^{X,A,B}$ and $\mathbb{D}$-side half requirement $\mathcal{S}^{X,A,B}$, define the full requirement

$$\mathcal{J}_{\mathcal{R},\mathcal{S}}^{X,A,B} = \{(\sigma, \tau) \in \mathbb{P}_e^X : \sigma \in \mathcal{R}^{X,A,B} \vee \tau \in \mathcal{S}^{X,A,B}\}.$$

We say that $\mathcal{J}^X$ is *essential* below $p \in \mathbb{P}_e^X$ if for all x, there is some finite $A > x$ such that for all y, there is some finite $B > y$ and a split pair q_0, q_1 below p with $q_0, q_1 \in \mathcal{J}^{X,A,B}$.

If $p_0 > p_1 > \cdots$ is a sequence in $\mathbb{P}_e^X$, we say that $\{p_n\}$ *satisfies* $\mathcal{J}^{X,A^*,B^*}$ if either some $p_n \in \mathcal{J}^{X,A^*,B^*}$, or there are only finitely many p_n such that $\mathcal{J}^X$ is essential below p_n.

Observe that $\mathcal{J}^{X,A,B}$ is closed under extensions and depends positively on A and B.

Lemma 3. *Let $p_0 > p_1 > \cdots$ be an infinite sequence of ADS_e^X conditions with $p_m = (\sigma_m, \tau_m)$. Let $\sigma = \bigcup \sigma_m$ and $\tau = \bigcup \sigma_m$. If $\mathcal{R}^X$ is essential in σ and $\mathcal{S}^X$ is essential in τ, then $\mathcal{J}_{\mathcal{R},\mathcal{S}}^X$ is essential below every p_m.*

Proof. Given $p_m = (\sigma_m, \tau_m)$ and x, let $A_0, A_1 > x$ and $n_0, n_1 \geq m$ be such that $\sigma_{n_0} \in \mathcal{R}^{X,A_0,A_0}$ and $\tau_{n_1} \in \mathcal{S}^{X,A_1,A_1}$. Then, given y, let $B_0, B_1 > y$ and $n_0' \geq n_0, n_1' \geq n_1$ be such that $\sigma_{n_0'} \in \mathcal{R}^{X,B_0,B_0}$ and $\tau_{n_1'} \in \mathcal{S}^{X,B_1,B_1}$. Since $n_0' \geq n_0$, we have $\sigma_{n_0'} \in \mathcal{R}^{X,A_0,A_0}$. So

$$\sigma_{n_0'} \in \mathcal{R}^{X,A_0,A_0} \cap \mathcal{R}^{X,B_0,B_0} \subseteq \mathcal{R}^{X,A_0,B_0} \subseteq \mathcal{R}^{X,A_0 \cup A_1, B_0 \cup B_1}.$$

Similarly, $\tau_{n_1'} \in \mathcal{S}^{X,A_0 \cup A_1, B_0 \cup B_1}$. So $(\sigma_{n_0'}, \tau_m)$ and $(\sigma_m, \tau_{n_1'})$ is a split pair below p_m which lies in $\mathcal{J}^{X,A_0 \cup A_1, B_0 \cup B_1}$. □

Definition 7. A sequence $(\sigma_0, \tau_0) > (\sigma_1, \tau_1) > \cdots$ in $\mathbb{P}_e^X$ is ADS_e^X-*generic* if

(1) it satisfies every $\mathcal{J}^{X,A^*,B^*}$;
(2) every (σ_n, τ_n) lies in $\mathbb{V}_e^X$.

G is ADS_e^X-*generic* if it is $\prec_e^X$-ascending (resp. descending) and satisfies all $\mathcal{R}^{X,A^*,B^*}$ (resp. $\mathcal{S}^{X,A^*,B^*}$).

Lemma 4. *Suppose that for any full requirement $\mathcal{J}^X$ and ADS_e^X condition p, we may find a split pair q_0, q_1 below p such that any infinite descending sequence in $\mathbb{P}_e^X$ containing q_0 or q_1 satisfies $\mathcal{J}^{X,A^*,B^*}$. Then there is an ADS_e^X-generic sequence, from which we may construct an ADS_e^X-generic set.*

Proof. Starting from $(\emptyset, \emptyset) \in \mathbb{V}_e^X$, we may construct an ADS_e^X-generic sequence $(\sigma_0, \tau_0) > (\sigma_1, \tau_1) > \cdots$ by iteratively applying the assumption and using Lemma 2 to choose an element of the split pair which lies in $\mathbb{V}_e^X$.

Let $\sigma = \bigcup \sigma_n$ and $\tau = \bigcup \tau_n$. We show that either σ satisfies every $\mathcal{R}^{X,A^*,B^*}$, in which case σ is an ADS_e^X-generic, or τ satisfies every $\mathcal{S}^{X,A^*,B^*}$, in which case τ is an ADS_e^X-generic.

Suppose σ does not satisfy $\mathcal{R}^{X,A^*,B^*}$. Then $\mathcal{R}^X$ is essential in σ. Given $\mathcal{S}^{X,A^*,B^*}$, we show that τ satisfies it. Suppose τ is essential in $\mathcal{S}^X$ (otherwise we are done). Then by Lemma 3, $\mathcal{J}_{\mathcal{R},\mathcal{S}}^X$ is essential below every (σ_n, τ_n). Since $(\sigma_0, \tau_0) > (\sigma_1, \tau_1) > \cdots$ satisfies $\mathcal{J}_{\mathcal{R},\mathcal{S}}^X$, we must have that some $(\sigma_n, \tau_n) \in \mathcal{J}_{\mathcal{R},\mathcal{S}}^X$. Since σ does not satisfy $\mathcal{R}^{X,A^*,B^*}$, we cannot have $\sigma_n \in \mathcal{R}^{X,A^*,B^*}$ and so we must have $\tau_n \in \mathcal{S}^{X,A^*,B^*}$. So τ satisfies $\mathcal{S}^{X,A^*,B^*}$ as desired. □

Assume, as in [8], that we have defined a stable partial order M together with A^* $(= A^*(M))$ and B^* $(= B^*(M))$.

Lemma 5. *If $M \leq_T X$, X does not compute a solution to M, and G is ADS_e^X-generic, then $X \oplus G$ does not compute a solution to M either.*

Proof. Suppose i is such that $\Phi_i^{X\oplus G}$ is the characteristic function of a subset of $\mathbb{N}$. If G is $\prec_e^X$-ascending, consider the requirement $\mathcal{R}_i^{X,A^*,B^*}$:

$$\{\tau \in \mathbb{A}_e^X : (\exists a \in A^*)(\exists b \in B^*)[\Phi_i^{X\oplus\tau}(a)\downarrow = \Phi_i^{X\oplus\tau}(b)\downarrow = 1]\}.$$

Since G is ADS_e^X-generic, G satisfies $\mathcal{R}_i^{X,A^*,B^*}$. We now have two cases.

Case 1. If some $\tau \subseteq G$ lies in $\mathcal{R}_i^{X,A^*,B^*}$, then there is an $a \in A^*$ and $b \in B^*$ such that $\Phi_i^{X\oplus G}(a)\downarrow = \Phi_i^{X\oplus G}(b)\downarrow = 1$. Hence $\Phi_i^{X\oplus G}$ cannot be a chain or an antichain in M.

Case 2. Otherwise $\mathcal{R}_i^X$ is not essential in G, i.e., $\Phi_i^{X\oplus G}$ is finite. In particular, $\Phi_i^{X\oplus G}$ is not a solution to M.

If G is $\prec_e^X$-descending, the proof proceeds by considering $\mathcal{S}_i^{X,A^*,B^*}$ instead. □

We define the *general context for* ADS *forcing* to be a set X and an index e such that

(1) $M \leq_T X$;
(2) X does not compute a solution to M;
(3) Φ_e^X defines a stable-ish linear order on $\mathbb{N}$, denoted $\prec_e^X$;
(4) for any full requirement $\mathcal{J}^X$ and ADS_e^X condition p, we may find a split pair q_0, q_1 below p such that any infinite descending sequence in $\mathbb{P}_e^X$ containing q_0 or q_1 satisfies $\mathcal{J}^{X,A^*,B^*}$.

4. WKL forcing

In order to construct a path on an infinite X-computable binary tree T_d^X, we force with infinite X-computable binary subtrees T^X of T_d^X, ordered by inclusion. We denote this by WKL_d^X forcing. In the following, A and B are subsets of $\mathbb{N}$ while A^* and B^* are infinite subsets of $\mathbb{N}$.

Definition 8. For each index i, define the requirement $\mathcal{K}_i^{X,A,B}$ to be the set of T^X such that

$$(\exists n)(\forall \sigma \in T^X)_{|\sigma|=n}(\exists \overline{a} \in A)(\exists \overline{b} \in B)[\Phi_i^{X\oplus\sigma}(\overline{a})\downarrow = \Phi_i^{X\oplus\sigma}(\overline{b})\downarrow = 1].$$

When we do not need to refer to Φ_i directly, we suppress the subscript i.

We say that $\mathcal{K}^X$ is *essential* in G if for all x, there is some finite $A > x$ and $T^X \in \mathcal{K}^{X,A,A}$ with $G \in [T^X]$.

We say that G *satisfies* $\mathcal{K}^{X,A^*,B^*}$ if either $\mathcal{K}^X$ is not essential in G, or there is some $T^X \in \mathcal{K}^{X,A^*,B^*}$ with $G \in [T^X]$.

Observe that $\mathcal{K}^{X,A,B}$ is closed under extensions and depends positively on A and B. Also observe that for $G \in [T_d^X]$, $\mathcal{K}_i^X$ is essential in G if and only if $\Phi_i^{X\oplus G}(\overline{a})\downarrow = 1$ for infinitely many $\overline{a}$, and G satisfies

$\mathcal{K}_i^{X,A^*,B^*}$ if and only if either $\mathcal{K}_i^X$ is not essential in G, or there is some $\sigma \subseteq G$ such that

$$(\exists \overline{a} \in A^*)(\exists \overline{b} \in B^*)[\Phi_i^{X\oplus\sigma}(\overline{a})\downarrow = \Phi_i^{X\oplus\sigma}(\overline{b})\downarrow = 1].$$

Definition 9. G is WKL_d^X*-generic* if

(1) it satisfies all $\mathcal{K}^{X,A^*,B^*}$;
(2) it is arithmetically generic over the model topped by X.

Lemma 6. *Suppose that for any requirement $\mathcal{K}^X$ and any WKL_d^X condition T_e^X, we may extend T_e^X to $T_{e'}^X$ such that any $H \in [T_{e'}^X]$ satisfies $\mathcal{K}^{X,A^*,B^*}$. Then there is a WKL_d^X-generic.*

Proof. Construct a nested sequence of WKL_d^X conditions by alternately extending to force that some $\mathcal{K}^{X,A^*,B^*}$ is satisfied or to meet some arithmetic dense set of WKL_d^X conditions. The unique G which lies on all of the WKL_d^X conditions in the sequence will be a WKL_d^X-generic. $\square$

Assume, as in [8], that we have defined a stable partial order M together with A^* $(= A^*(M))$ and B^* $(= B^*(M))$.

Lemma 7. *If $M \leq_T X$, X does not compute a solution to M, and G is WKL_d^X-generic, then $X \oplus G$ does not compute a solution to M either.*

Proof. Given i such that $\Phi_i^{X\oplus G}$ is the characteristic function of a subset of $\mathbb{N}$, consider the requirement $\mathcal{K}_i^{X,A^*,B^*}$:

$$\{T^X : (\exists n)(\forall \sigma \in T^X)_{|\sigma|=n}(\exists a \in A^*)(\exists b \in B^*)[\Phi_i^{X\oplus\sigma}(a)\downarrow = \Phi_i^{X\oplus\sigma}(b)\downarrow = 1]\}.$$

Since G is WKL_d^X-generic, G satisfies $\mathcal{K}_i^{X,A^*,B^*}$. We now have two cases.

Case 1. If there is some $T^X \in \mathcal{K}_i^{X,A^*,B^*}$ such that $G \in [T^X]$, then take n and $\sigma = G \upharpoonright n$. Then there is an $a \in A^*$ and $b \in B^*$ such that $\Phi_i^{X\oplus G}(a)\downarrow = \Phi_i^{X\oplus G}(b)\downarrow = 1$. Hence $\Phi_i^{X\oplus G}$ cannot be a chain or an antichain in M.

Case 2. Otherwise $\mathcal{K}_i^X$ is not essential in G, i.e., $\Phi_i^{X\oplus G}$ is finite. In particular, $\Phi_i^{X\oplus G}$ is not a solution to M. $\square$

We define the *general context for* WKL *forcing* to be a set X and an index d such that

(1) $M \leq_T X$;
(2) X does not compute a solution to M;
(3) Φ_d^X defines an infinite binary tree, denoted T_d^X;
(4) for any requirement $\mathcal{K}^X$ and any WKL_d^X condition T_e^X, we may extend T_e^X to $T_{e'}^X$ such that any $H \in [T_{e'}^X]$ satisfies $\mathcal{K}^{X,A^*,B^*}$.

5. Iterating ADS and WKL forcings

Theorem 1. *If X and d satisfy the general context for* WKL *forcing, G is a WKL^X_d-generic, and $\prec_e^{X\oplus G}$ is a stable-ish linear order, then $X \oplus G$ and e satisfy the general context for* ADS *forcing.*

Proof. Lemma 7 ensures that $X \oplus G$ does not compute a solution to M.

It remains to show that given $\mathcal{J}^{X\oplus G}_{\mathcal{R}_i,\mathcal{S}_j}$ ($\mathcal{J}^{X\oplus G}$ for short) and $p \in \mathbb{P}^{X\oplus G}_e$, we may find a split pair q_0, q_1 below p such that any infinite descending sequence of conditions containing q_0 or q_1 satisfies $\mathcal{J}^{X\oplus G,A^*,B^*}$. Since G is arithmetically generic, fix $T^X_0 \subseteq T^X_d$ forcing that $\prec_e^{X\oplus G}$ is a linear order on $\mathbb{N}$ and $p \in \mathbb{P}^{X\oplus G}_e$. Then consider the set of $T^X \subseteq T^X_0$ such that

$$(\exists n)(\forall \rho \in T^X)_{|\rho|=n}(\exists \text{ split pair } q_0, q_1 \in \mathbb{P}^{X\oplus\rho}_e \text{ below } p)(\exists \overline{a_0}, \overline{a_1} \in A)(\exists \overline{b_0}, \overline{b_1} \in B)$$
$$[((\Phi_i^{X\oplus\rho\oplus\sigma_0}(\overline{a_0})\downarrow = \Phi_i^{X\oplus\rho\oplus\sigma_0}(\overline{b_0})\downarrow = 1) \vee (\Phi_i^{X\oplus\rho\oplus\tau_0}(\overline{a_0})\downarrow = \Phi_i^{X\oplus\rho\oplus\tau_0}(\overline{b_0})\downarrow = 1))$$
$$\wedge ((\Phi_j^{X\oplus\rho\oplus\sigma_1}(\overline{a_1})\downarrow = \Phi_j^{X\oplus\rho\oplus\sigma_1}(\overline{b_1})\downarrow = 1) \vee (\Phi_j^{X\oplus\rho\oplus\tau_1}(\overline{a_1})\downarrow = \Phi_j^{X\oplus\rho\oplus\tau_1}(\overline{b_1})\downarrow = 1))],$$

where $q_0 = (\sigma_0, \tau_0)$ and $q_1 = (\sigma_1, \tau_1)$. Observe that the above is some $\mathcal{K}^{X,A,B}_l$. Since G is WKL^X_d-generic, G satisfies $\mathcal{K}^{X,A^*,B^*}_l$. We now have two cases.

<u>Case 1.</u> If there is some $T^X \in \mathcal{K}^{X,A^*,B^*}_l$ such that $G \in [T^X]$, then take n and $\rho = G \upharpoonright n$. We get a split pair $q_0, q_1 \in \mathbb{P}^{X\oplus G}_e$ below p. We may check that $q_0, q_1 \in \mathcal{J}^{X\oplus G,A^*,B^*}$.

<u>Case 2.</u> Otherwise, fix x witnessing that $\mathcal{K}^X_l$ is not essential in G, i.e., for all finite $A > x$, T^X with $G \in [T^X]$, and n, there is some $\rho \in T^X$ of length n such that for all split pairs $q_0, q_1 \in \mathbb{P}^{X\oplus\rho}_e$ below p, we have

$$(\forall \overline{a_0}, \overline{a_1} \in A)\neg[(\Phi_i^{X\oplus\rho\oplus\sigma_0}(\overline{a_0})\downarrow = 1 \vee \Phi_i^{X\oplus\rho\oplus\tau_0}(\overline{a_0})\downarrow = 1)$$
$$\wedge\ (\Phi_j^{X\oplus\rho\oplus\sigma_1}(\overline{a_1})\downarrow = 1 \vee \Phi_j^{X\oplus\rho\oplus\tau_1}(\overline{a_1})\downarrow = 1)].$$

For each finite $A > x$ and n, we may take T^X to be the full tree (in T^X_0) with root $G \upharpoonright n$. So ρ above could be taken to be $G \upharpoonright n$.

We show that $\mathcal{J}^{X\oplus G}$ is not essential below p. (It follows that $\mathcal{J}^{X\oplus G}$ is not essential below any q extending p, so any infinite descending sequence of conditions containing p will satisfy $\mathcal{J}^{X\oplus G,A^*,B^*}$.)

Suppose for contradiction that $\mathcal{J}^{X\oplus G}$ is essential below p. Take a finite $A > x$ such that for all y, there is some finite $B > y$ and a split pair q_0, q_1 below p with $q_0, q_1 \in \mathcal{J}^{X\oplus G,A,B}$. Take $y = x$ and get some finite $B > x$ and a split pair q_0, q_1 below p with $q_0, q_1 \in \mathcal{J}^{X\oplus G,A,B}$. Setting $A' = A \cup B > x$, it follows that $q_0, q_1 \in \mathcal{J}^{X\oplus G,A',A'}$.

But now by choice of x, for all sufficiently long $\rho \subseteq G$ (such that $q_0, q_1 \in \mathbb{P}_e^{X\oplus\rho}$ is a split pair below p), we have

$$(\forall \overline{a_0}, \overline{a_1} \in A')\neg[(\Phi_i^{X\oplus\rho\oplus\sigma_0}(\overline{a_0})\downarrow = 1 \vee \Phi_i^{X\oplus\rho\oplus\tau_0}(\overline{a_0})\downarrow = 1) \\ \wedge (\Phi_j^{X\oplus\rho\oplus\sigma_1}(\overline{a_1})\downarrow = 1 \vee \Phi_j^{X\oplus\rho\oplus\tau_1}(\overline{a_1})\downarrow = 1)].$$

So

$$(\forall \overline{a_0}, \overline{a_1} \in A')\neg[(\Phi_i^{X\oplus G\oplus\sigma_0}(\overline{a_0})\downarrow = 1 \vee \Phi_i^{X\oplus G\oplus\tau_0}(\overline{a_0})\downarrow = 1) \\ \wedge (\Phi_j^{X\oplus G\oplus\sigma_1}(\overline{a_1})\downarrow = 1 \vee \Phi_j^{X\oplus G\oplus\tau_1}(\overline{a_1})\downarrow = 1)],$$

contradicting $q_0, q_1 \in \mathcal{J}^{X\oplus G, A', A'}$. □

Theorem 2. *If X and d satisfy the general context for* ADS *forcing, G is an ADS_d^X-generic, and $T_{d'}^{X\oplus G}$ is an infinite binary tree, then $X \oplus G$ and d' satisfy the general context for* WKL *forcing.*

Proof. Lemma 5 ensures that $X \oplus G$ does not compute a solution to M.

It remains to show that given $\mathcal{K}_i^{X\oplus G}$ and $T_e^{X\oplus G} \subseteq T_{d'}^{X\oplus G}$, we may extend $T_e^{X\oplus G}$ to $T_{e'}^{X\oplus G}$ such that any $H \in [T_{e'}^{X\oplus G}]$ satisfies $\mathcal{K}_i^{X\oplus G, A^*, B^*}$.

Define $k : 2^\omega \to \omega$ as follows: let $k(\tau)$ be the largest n such that $\Phi_e^{X\oplus\tau}$ is total on 2^n and defines a binary tree of height n. Observe that (1) given τ, X can compute $k(\tau)$; (2) given l, $X \oplus G$ can compute the least $\tau \subseteq G$ such that $k(\tau) \geq l$.

Without loss of generality, assume that G is a $\prec_d^X$-ascending sequence. Consider the set of $\tau \in \mathbb{A}_d^X$ such that

$$(\forall \sigma \in \Phi_e^{X\oplus\tau})_{|\sigma|=k(\tau)}(\exists \overline{a} \in A)(\exists \overline{b} \in B)[\Phi_i^{X\oplus\tau\oplus\sigma}(\overline{a})\downarrow = \Phi_i^{X\oplus\tau\oplus\sigma}(\overline{b})\downarrow = 1].$$

Observe that this set is some $\mathcal{R}_j^{X,A,B}$. Since G is ADS_d^X-generic, G satisfies $\mathcal{R}_j^{X,A^*,B^*}$. We now have two cases.

<u>Case 1.</u> If some $\tau \subseteq G$ lies in $\mathcal{R}_j^{X,A^*,B^*}$, then we may check that $T_e^{X\oplus G} \in \mathcal{K}_i^{X\oplus G,A^*,B^*}$ (take $n = k(\tau)$). So $T_e^{X\oplus G}$ is already as desired.

<u>Case 2.</u> Otherwise, fix x witnessing that $\mathcal{R}_j^X$ is not essential in G, i.e., for all finite $A > x$ and $\tau \subseteq G$, there is some $\sigma \in \Phi_e^{X\oplus\tau}$ of length $k(\tau)$ such that

$$(\forall \overline{a} \in A)\neg[\Phi_i^{X\oplus\tau\oplus\sigma}(\overline{a})\downarrow = 1].$$

We then extend $T_e^{X\oplus G}$ to

$$T_{e'}^{X\oplus G} = \{\sigma \in T_e^{X\oplus G} : |\sigma| < x \text{ or } \\ (\forall \overline{a} \in (x, |\sigma|))\neg[\Phi_i^{X\oplus\tau\oplus\sigma}(\overline{a})\downarrow = 1], \text{ where} \\ \tau \subseteq G \text{ is the least such that } k(\tau) \geq |\sigma|\}.$$

Observe that $T_{e'}^{X\oplus G}$ is an infinite binary subtree of $T_e^{X\oplus G}$.

Given some $H \in [T_{e'}^{X\oplus G}]$, we show that $\mathcal{K}_i^{X\oplus G}$ is not essential in H. Given some finite $A > x$ and $T^{X\oplus G} \subseteq T_{d'}^{X\oplus G}$ with $H \in [T^{X\oplus G}]$, we show that $T^{X\oplus G} \notin \mathcal{K}_i^{X\oplus G,A,A}$. Suppose not, i.e., there is some n such that for all $\sigma \in T^{X\oplus G}$ of length n,

$$(\exists \overline{a} \in A)[\Phi_i^{X\oplus G\oplus\sigma}(\overline{a})\downarrow = 1].$$

Now take $\tau \subseteq G$ and $\sigma \subseteq H$ such that (1) τ contains the use of the above computation; (2) $|\sigma| > \max\{n, x\}$; (3) τ is the least such that $k(\tau) \geq |\sigma|$. Then

$$(\exists \overline{a} \in A)[\Phi_i^{X\oplus\tau\oplus\sigma}(\overline{a})\downarrow = 1].$$

But then $\overline{a} \in (x, |\sigma|)$, contradicting $\sigma \in T_{e'}^{X\oplus G}$. □

6. Combining the ingredients

Finally, we combine all of the ingredients to construct a Turing ideal satisfying WKL and ADS but not SCAC.

Start from the Turing ideal of all computable sets. Let M be a partial order generic for the ground forcing in [8], with associated A^* and B^*. By (C1)–(C3) in Section 2.3 of [8], M is stable, M does not compute a solution to itself, and if $\prec_e^M$ is a stable-ish linear order, $\mathcal{J}^M$ is a full requirement in $\mathbb{P}_e^M$, and p is a condition in $\mathbb{P}_e^M$, then we may find a split pair q_0, q_1 below p such that any infinite descending sequence in $\mathbb{P}_e^M$ containing q_0 or q_1 satisfies $\mathcal{J}^{M,A^*,B^*}$. Thus, we are ready to do ADS_e^M forcing for any e for which $\prec_e^M$ is a stable-ish linear order.

Henceforth do ADS forcing and WKL forcing alternately (dovetailing appropriately such that all stable-ish linear orders and all infinite binary trees are taken care of). We may do so by Theorems 1 and 2.

The union of the Turing ideals we construct, which is itself a Turing ideal, satisfies WKL and ADS but not SCAC.

References

[1] P. A. Cholak, C. G. Jockusch, T. A. Slaman. On the Strength of Ramsey's Theorem for Pairs. *J. Symbolic Logic* **66** (2001) 1–55.

[2] D. D. Dzhafarov. Cohesive avoidance and strong reductions. *Proc. Amer. Math. Soc.* **143** (2015) 869–876.

[3] D. R. Hirschfeldt. Slicing the Truth: On the Computable and Reverse Mathematics of Combinatorial Principles. Lecture Notes Series Volume **28**, Institute for Mathematical Sciences, National University of Singapore, 2014.

[4] D. R. Hirschfeldt, C. G. Jockusch, Jr. On Notions of Computability Theoretic Reduction between Π_2^1 Principles. To appear.

[5] D. R. Hirschfeldt, R. A. Shore. Combinatorial Principles Weaker than Ramsey's Theorem for Pairs. *J. Symbolic Logic* **72:1** (2007) 171–206.

[6] D. R. Hirschfeldt, R. A. Shore, T. A. Slaman. The Atomic Model Theorem and Type Omitting. *Trans. Amer. Math. Soc.* **361:11** (2009) 5805–5837.

[7] C. Jockusch, F. Stephan. A cohesive set which is not high. *Math. Log. Quart.* **39:1** (1993) 515–530.
[8] M. Lerman, R. Solomon, H. Towsner. Separating principles below Ramsey's Theorem for Pairs. *J. Math. Log.* **13:2** (2013) 1350007.
[9] L. Patey. Dominating the Erdös-Moser theorem in reverse mathematics. arXiv:1505.03425 (2015).
[10] L. Patey. Iterative forcing and hyperimmunity in reverse mathematics. http://www.ludovicpatey.com/media/research/iterative-forcing-extended.pdf (2015).
[11] R. A. Shore. Reverse mathematics: the playground of logic. *Bull. Symbolic Logic* **16:3** (2010) 378–402.
[12] S. G. Simpson. Subsystems of Second Order Arithmetic, 2$^{\text{nd}}$ edition. Perspectives in Logic, Cambridge University Press, Cambridge, 2009.
[13] W. Wang. The Definability Strength of Combinatorial Principles. arXiv:1408.1465 (2014).

Department of Mathematics, University of Connecticut, Storrs, CT 06269
E-mail address: damir@math.uconn.edu

Department of Mathematics, Cornell University, Ithaca, NY 14853
E-mail address: goh@math.cornell.edu

Department of Mathematics, Cornell University, Ithaca, NY 14853
E-mail address: shore@math.cornell.edu