

Halin's Infinite Ray Theorems: Complexity and Reverse Mathematics*

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Abstract

Halin [1965] proved that if a graph has n many pairwise disjoint rays for each n then it has infinitely many pairwise disjoint rays. We analyze the complexity of this and other similar results in terms of computable and proof theoretic complexity. The statement of Halin's theorem and the construction proving it seem very much like standard versions of compactness arguments such as König's Lemma. Those results, while not computable, are relatively simple. They only use arithmetic procedures or, equivalently, finitely many iterations of the Turing jump. We show that several Halin type theorems are much more complicated. They are among the theorems of hyperarithmetic analysis. Such theorems imply the ability to iterate the Turing jump along any computable well ordering. Several important logical principles in this class have been extensively studied beginning with work of Kreisel, H. Friedman, Steel and others in the 1960s and 1970s. Until now, only one purely mathematical example was known. Our work provides many more and so answers Question 30 of Montalbán's Open Questions in Reverse Mathematics [2011]. Some of these theorems including ones in Halin [1965] are also shown to have unusual proof theoretic strength as well.

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1 Introduction

In this paper we analyze the complexity of several results in infinite graph theory. These theorems are said to be ones of Halin type or, more generally, of ubiquity theory. The classical example is a theorem of Halin [11]: If a countable graph G contains, for each n , a sequence $\langle R_0, \dots, R_{n-1} \rangle$ of disjoint rays (a ray is a sequence $\langle x_i \mid i \in N \rangle$ of distinct vertices such that there is an edge between each x_i and x_{i+1}) then it contains an infinite such sequence of rays. (Note: As will be described in Definition 3.2, when we talk about disjoint rays we always mean pairwise disjoint.) Halin actually deals with arbitrary graphs and formulates the result differently. The uncountable cases, however, are essentially just counting arguments. We deal only with countable structures but discuss his formulation in §5. This standard formulation of his theorem seems like a typical compactness theorem going from arbitrarily large finite collections of objects to an infinite collection. The archetypical example here is König's Lemma: If a finitely branching tree has paths of length n for every n then it has a branch, i.e. an infinite path. In outline, a modern proof of Halin's theorem for countable graphs (due to Andreae, see [6, Theorem 8.2.5(i)]) seems much like that of König's Lemma (and many others in infinite graph theory). The construction of the desired sequence of rays proceeds by a recursion through the natural numbers in which each step is a simple procedure. While the procedure is much more delicate than for König's Lemma, it is basically of the same complexity. It uses Menger's theorem for finite graphs at each step but this represents a computable procedure (for finite graphs). The other parts of the step depend on the same type of information as in König's Lemma. They ask, for example, if various sets (computable in the given graph) are nonempty or infinite. Nonetheless, we prove that the complexity of this construction and theorem are much higher than that for König's Lemma or other applications of compactness.

We follow two well established procedures for measuring the complexity of constructions and theorems. The first is basically computability theoretic. It has its formal beginnings in the 1950s but has much earlier roots in constructive or computable mathematics reaching back to antiquity. The measuring rod here is relative computability. We say a set A of natural numbers is (Turing) computable from a set B , $A \leq_T B$, if there is an algorithm (say on a Turing machine or any other reasonable model of general computation) that, when given access to all membership facts about B (an oracle for B) computes membership in A . The standard hierarchies of complexity here are based on iterations of the Turing jump. This operator takes B to B' , the halting problem relativized to B , i.e. the set of programs e with oracle for B , Φ_e^B , such that Φ_e^B halts on input e . The undecidability of the halting problem tells us that $B <_T B'$. For example, if the tree of König's Lemma is computable in B then there is a branch computable in the double jump B'' of B .

The second approach is proof theoretic. It measures the complexity of a theorem by the logical strength of the axioms needed to prove it. This approach also has a long history but the formal subject, now called reverse mathematics, starts with H. Friedman's work in the 1970s (e.g. [7, 8]). One compares axiomatic systems S and T by saying that T is stronger than S , $T \vdash S$ (T proves S) if one can prove every sentence $\Theta \in S$ from the axioms of T . The goal here is to characterize to the extent possible the axioms needed to prove a given mathematical theorem Θ . To this end, one begins with a weak base theory. Then one wants to find a system S such that not only does $S \vdash \Theta$ but also Θ (with the weak

base theory) proves all the axioms of S . Hence the name reverse mathematics as we seek to prove the “axioms” of S from the theorem Θ . Typically, the systems here are formalized in arithmetic with quantification over sets as well as numbers. The standard base theory (RCA_0) corresponds to the axioms needed to do computable constructions. Stronger systems are then usually generated by adding comprehension axioms which assert the existence of specific families of sets. For example, a very important system is ACA_0 . It is equivalent in the sense of reverse mathematics just described to König’s Lemma. Formally, it asserts that every subset of the natural numbers defined by a formula that quantifies only over numbers (and not sets) exists. This is also equivalent to asserting that for every set B , B' exists.

The early decades of reverse mathematics were marked by a large variety of results characterizing a wide array of theorems and constructions as being one of five specific levels of complexity including RCA_0 and ACA_0 . Each of these systems (Simpson’s “big five”) have corresponding specific recursion theoretic construction principles. In more recent decades, there has been a proliferation of results placing theorems and constructions outside the big five. Sometimes these are inserted linearly and sometimes with incomparabilities. They are now collectively often called the “zoo” of reverse mathematics. (See <https://rmzoo.math.uconn.edu/diagrams/> for pictures.)

Theorems and constructions in combinatorics in general, and graph theory in particular, have been a rich source of such denizens of this zoo. Almost all of them have fallen below ACA_0 (König’s Lemma) and so have the objects they seek constructible computably in finitely many iterations of the Turing jump. Ramsey theory, in particular, has provided a very large class of constructions and theorems of distinct complexity. One example of the infinite version of a classical theorem of finite graph theory that is computationally and reverse mathematically strictly stronger than ACA_0 is König’s Duality Theorem (KDT) for countable graphs. (Every bipartite graph has a matching and a cover consisting of one vertex from each edge of the matching.) The proofs of this theorem for infinite graphs (Podewski and Steffens [20] for countable and Aharoni [1] for arbitrary ones) are not just technically difficult but explicitly used both transfinite recursions and well orderings of all subsets of the given graph. These techniques lie far beyond ACA_0 . Aharoni, Magidor and Shore [2] proved that this theorem is of great computational strength in that there are computable graphs for which the required matching and cover compute all the iterations of the Turing jump through all computable well-orderings. They also showed that it was strong reverse mathematically as it implied ATR_0 , the standard system above ACA_0 used to deal with such transfinite recursions. Some of the lemmas used in each of the then known proofs were shown to be equivalent to the next and final of the big five systems, $\Pi_1^1\text{-CA}_0$ and of corresponding computational strength. Simpson [27] later provided a new proof of the theorem using logical methods that avoided these lemmas and showed that the theorem itself is equivalent to ATR_0 and so strictly weaker than the lemmas both computationally and in terms of reverse mathematics.

The situation for the theorems of Halin type that we study here is quite different. The standard proofs do not seem to use such strong methods. Nonetheless, as we mentioned above, the theorems are much stronger than ACA_0 with some versions not even provable in ATR_0 . We prove that these theorems occupy a few houses in the area of the reverse mathematics zoo devoted to what are called theorems (or theories) of hyperarithmetic analysis, THAs (Definition 3.13). Computationally, for each computable well ordering α , there is a computable instance of any THA which has all of its required objects Turing above $0^{(\alpha)}$,

the α th iteration of the Turing jump. On the other hand, they are computationally and proof theoretically much weaker than ATR_0 and so KDT . The point here is that there is a single computable graph such that the matching and cover required by KDT lies above $0^{(\alpha)}$ for all the computable well-orderings α , while for each computable instance of a THA there is a computable well-ordering α such that $0^{(\alpha)}$ computes the desired object. In our cases, the instances are graphs with arbitrarily many disjoint rays and the desired object is an infinite sequence of disjoint rays. (The general usage of terms like instances and solutions of a theorem or principle is described in Remark 3.18.)

Beginning with work of Kreisel [14], H. Friedman [9], Steel [29] and others in the 1960s and 1970s and continuing into the last decade (by Montalbán [15, 16], Neeman [18, 19] and others), several axiomatic systems and logical theorems were found to be THAs and proven to lie in a number of distinct classes in terms of proof theoretic complexity. Until now, however, there has been only one mathematical but not logical example, i.e. one not mentioning classes of first order formulas or their syntactic complexity. This was a result (INDEC) about indecomposability of linear orderings in Jullien's thesis [13] (see Rosenstein [22, Lemma 10.3]). It was shown to be a THA by Montalbán [15].

The natural quest then became to find out if there are any other THAs in the standard mathematical literature. The issue was raised explicitly in Montalbán's "Open Questions in Reverse Mathematics" [17, Q30]. As our answer, we provide many examples (Theorem 4.1 and Theorem 5.8). Most of them are provable in a well known system above ACA_0 gotten by adding on a weak form of the axiom of choice ($\Sigma_1^1\text{-AC}_0$) at times with additional induction.

Several of the basic Halin type theorems (the IRT_{XYZ} defined after Definition 3.4) have versions (the IRT_{XYZ}^* of Definition 3.15) like those appearing in the original papers that show that there are always families of disjoint rays of maximal cardinality which are of the same computational strength as the basic versions (Proposition 3.17). On the other hand, the IRT_{XYZ}^* are strictly stronger proof theoretically than the IRT_{XYZ} because they imply more induction than is available in $\Sigma_1^1\text{-AC}_0$ (Theorem 5.3 and Corollary 5.4). Two of the variations we consider are as yet open problems of graph theory ([4] and Bowler, personal communication). We show (Theorem 5.8) that if we restrict the class of graphs to directed forests the principles are not only provable in, but reverse mathematically equivalent to, $\Sigma_1^1\text{-AC}_0 + \text{I}\Sigma_1^1$. Note that as $\text{ATR}_0 \not\vdash \text{I}\Sigma_1^1$ [28, IX.4.7], these theorems are not provable even in ATR_0 or from KDT (Corollary 5.9). We do not know of other mathematical but nonlogical theorems of this strength. Other versions that require maximal sets of rays (Definition 5.13) are much stronger and, in fact, equivalent to $\Pi_1^1\text{-CA}_0$ (Theorem 5.14).

2 Outline of Paper

Section 3 discusses basic concepts and background information from graph theory (§3.1), computability theory (§3.2) and formal syntax, semantics and axiom systems (§3.3). In particular, §3.1 states Halin's theorem IRT and some variants IRT_{XYZ} . These variants allow for graphs to be directed, rays to be double and disjointness for edges rather than vertices. Some additional ones, IRT_{XYZ}^* , that ask for sets of rays of maximal cardinality are introduced in Definition 3.15. These principles are the main targets of our analysis.

§3.2 describes the transfinite iterations of the Turing jump (Definitions 3.7 and 3.8).

Along with the syntax and semantics of §3.3, these are used to define the theorems (theories) of hyperarithmetic analysis (THAs) (Definition 3.13) which is our main object of study.

Section 4 proves (Theorem 4.11) that all sixteen variants of IRT are computationally very complex principles that imply closure under transfinite iterations of the Turing jump. A direct proof for IRT can be read off from those of Theorem 4.10 and Theorem 4.11 by ignoring the other cases. A stand alone treatment is in [3, §4]. Upper bounds on the computational complexity for ten of these variants are then provided that prove that they are THAs (Theorem 4.1). Again, simpler proofs for IRT are in [3, §4]. As mentioned in §1, of the remaining three versions of IRT, two are still open problems in graph theory. We do, however, have an analysis of their restrictions to special classes of graphs in Theorem 4.24 and §5.1. The last of the variations, IRT_{UED} , has been proven more recently by Bowler, Carmesin, Pott [4] using more sophisticated methods than the other results but we have yet to fully analyze the complexity of their construction.

Section 5.1 proves that the $\text{IRT}_{\text{XYZ}}^*$ all imply a specific instance of $\text{I}\Sigma_1^1$ that is strong enough to let them prove the consistency of $\Sigma_1^1\text{-AC}_0$ and so by Gödel's second incompleteness theorem they cannot be proved in $\Sigma_1^1\text{-AC}_0$ (Corollary 5.4).

As for proving full Σ_1^1 induction from an $\text{IRT}_{\text{XYZ}}^*$, we are in much the same situation mentioned above for $\Sigma_1^1\text{-AC}_0$ and IRT_{XYZ} . In particular, $\text{IRT}_{\text{DVD}}^*$ and $\text{IRT}_{\text{DED}}^*$ for directed forests each proves $\text{I}\Sigma_1^1$ as well as $\Sigma_1^1\text{-AC}_0$ (Theorems 5.7 and 5.8) and so are equivalent to $\text{I}\Sigma_1^1 + \Sigma_1^1\text{-AC}_0$. As before, this shows that they are strictly stronger than $\Sigma_1^1\text{-AC}_0$ (Corollary 5.9). Indeed, as mentioned in §1, they are not even provable in ATR_0 and we do not know of any other mathematical theorem with this level of reverse mathematical strength.

The second variation of maximality, MIRT_{XYZ} , studied in §5.2 is also mentioned in the original Halin paper [11]. It asks for a set of disjoint rays which is maximal in the sense of set containment. Of course, this follows immediately from Zorn's Lemma for all graphs. For countable graphs we provide a reverse mathematical analysis, showing that each of the MIRT_{XYZ} is equivalent to $\Pi_1^1\text{-CA}_0$ (Theorem 5.14).

In §6, we discuss the reverse mathematical relationships between the variations of Halin's theorem which are THAs and previously studied THAs as well as one new logical THA, finite- $\Sigma_1^1\text{-AC}_0$ (Definition 6.2). Basically, all the $\text{IRT}_{\text{XYZ}}^*$ (and so $\text{IRT}_{\text{XYZ}} + \text{I}\Sigma_1^1$) imply H. Friedman's ABW_0 (Definition 6.4) by Theorem 6.7 and finite- $\Sigma_1^1\text{-AC}_0$ (Theorem 6.3). On the other hand, none of them are implied by ABW_0 (Theorem 6.10) or by $\Delta_1^1\text{-CA}_0$ (Definition 6.8 and Theorem 6.9). $\text{ABW}_0 + \text{I}\Sigma_1^1$ does, however, imply finite- $\Sigma_1^1\text{-AC}_0$ which is not implied by weak (unique)- $\Sigma_1^1\text{-AC}_0$ (Goh [10]). Figure 4 summarizes many of the known relations with references.

In the penultimate section (§7) we continue the study begun in §4 of the only use of $\Sigma_1^1\text{-AC}_0$ in each of our proofs of IRT_{XYZ} . It consists of SCR_{XYZ} (Definition 4.12) which says we can go from the hypothesis that there are arbitrarily many disjoint rays to a sequence $\langle X_k \rangle_k$ in which each X_k is a sequence of k many disjoint rays. We also continue to analyze the weakenings WIRT_{XYZ} (Definition 4.12) of IRT_{XYZ} which each take the existence of such a sequence $\langle X_k \rangle_k$ as its hypothesis in place of there being arbitrarily many disjoint rays. For example, for all the IRT_{XYZ} which are consequences of $\Sigma_1^1\text{-AC}_0$ and so are THAs, IRT_{XYZ} is equivalent to SCR_{XYZ} over RCA_0 (Corollary 7.2) and so all of them are also THAs. For the same choices of XYZ, ACA_0 proves WIRT_{XYZ} over RCA_0 . While a natural strengthening of WIRT_{XYZ} does imply ACA_0 and indeed is equivalent to it (Theorem 7.5), we do not know

if WIRT_{XYZ} itself implies ACA_0 . All we can prove is that it is not a consequence of RCA_0 (Theorem 7.6).

In the last section (§8), we mention some open problems.

3 Basic Notions and Background

We begin with basic notions and terminology from graph theory (§3.1). At times we use formalizations that are clearly equivalent to more standard ones but are easier to work with computationally or proof-theoretically. The following two subsections supply information about standard computational and logical/proof theoretic notions. Note that we denote the set of natural numbers by N when we may be thinking of them in a model of arithmetic as in §3.3 and by \mathbb{N} when we emphasize that we specifically want the standard natural numbers.

Additional explanations can be found in [3]. Basic references for terminology, background and standard results not explicitly stated are Diestel [6] for graph theory; Rogers [21] and Sacks [23] for computability theory; and Simpson [28] for reverse mathematics.

3.1 Graph Theoretic Notions

Definition 3.1. A *graph* H is a pair $\langle V, E \rangle$ consisting of a set V (of *vertices*) and a set E of unordered pairs $\{u, v\}$ with $u \neq v$ from V (called *edges*). These structures are also called *undirected graphs* (or here *U-graphs*). A structure H of the form $\langle V, E \rangle$ as above is a *directed graph* (or here *D-graph*) if E consists of ordered pairs $\langle u, v \rangle$ of vertices with $u \neq v$. To handle both cases simultaneously, we often use X to stand for undirected (U) or directed (D). We then use (u, v) to stand for the appropriate kind of edge, i.e. $\{u, v\}$ or $\langle u, v \rangle$.

An *X-subgraph* of the X -graph H is an X -graph $H' = \langle V', E' \rangle$ such that $V' \subseteq V$ and $E' \subseteq E$. It is an *induced X-subgraph* if $E' = \{(u, v) \mid u, v \in V' \wedge (u, v) \in E\}$.

Definition 3.2. An *X-ray* in H is a pair consisting of an X -subgraph $H' = \langle V', E' \rangle$ of H and an isomorphism f from N with edges $(n, n+1)$ for $n \in N$ to H' . (Note that this implies that the range of f is the set V' .) We say that the *ray begins at* $f(0)$. We also describe this situation by saying that H contains the *X-ray* $\langle H', f \rangle$. We sometimes abuse notation by saying that the sequence $\langle f(n) \rangle$ of vertices is an *X-ray* in H . A *tail* of an *X-ray* is a final segment of said *X-ray*. Similarly we consider *double X-rays* where the isomorphism f is from the integers $\mathcal{Z} = \{-n, n \mid n \in N\}$ with edges $(z, z+1)$ for $z \in \mathcal{Z}$. A *tail* of a double *X-ray* R is a final segment of R or an initial segment of R considered in reverse order.

We use Z -ray to stand for either a (single) ray ($Z = S$) or double ray ($Z = D$) and so we have, in general, Z - X -rays or just Z -rays if the type of graph (U or D) is already established. For brevity, when we describe rays we will often only list their vertices in order instead of defining H' and f explicitly.

H contains k many Z - X -rays for $k \in N$ if there is a sequence $\langle H_i, f_i \rangle_{i < k}$ such that each $\langle H_i, f_i \rangle$ is a Z - X -ray in H (with $H_i = \langle V_i, E_i \rangle$). H contains k many disjoint (or vertex-disjoint) Z - X -rays if the V_i are pairwise disjoint. H contains k many edge-disjoint Z - X -rays if the E_i are pairwise disjoint. We often use Y to stand for either vertex (V) or edge (E) as in the following definitions.

An X -graph H contains arbitrarily many Y -disjoint Z - X -rays if it contains k many such rays for every $k \in N$.

An X -graph H contains infinitely many Y -disjoint Z - X -rays if there is an X -subgraph $H' = \langle V', E' \rangle$ of H and a sequence $\langle H_i, f_i \rangle_{i \in N}$ such that each $\langle H_i, f_i \rangle$ is a Z - X -ray in H (with $H_i = \langle V_i, E_i \rangle$) such that the V_i or E_i , respectively for $Y = V, E$, are pairwise disjoint and $V' = \bigcup V_i$ and $E' = \bigcup E_i$.

Definition 3.3. An X -path P in an X -graph H is defined similarly to single rays except that the domain of f is a proper initial segment of N instead of N itself. Thus they are finite sequences of distinct vertices with edges between successive vertices in the sequence. If $P = \langle x_0, \dots, x_n \rangle$ is a path, we say it is a *path of length n* between x_0 and x_n . Our notation for truncating and combining paths $P = \langle x_0, \dots, x_n \rangle$, $Q = \langle y_0, \dots, y_m \rangle$ and $R = \langle z_0, \dots, z_l \rangle$ is as follows: $x_i P = \langle x_i, \dots, x_n \rangle$, $P x_i = \langle x_0, \dots, x_i \rangle$, and we use concatenation in the natural way, e.g., if the union of Px , xQy and yR is a path, we denote it by $PxQyR$. We treat rays as we do paths in this notation, as long as it makes sense, writing, for example, $x_i R$ for the ray which is gotten by starting R at an element x_i of R ; Rx_i is the path which is the initial segment of R ending in x_i and we use concatenation as for paths as well.

The starting point of the work in this paper is a theorem of Halin [11] that we call the infinite ray theorem as expressed in [6, Theorem 8.2.5(i)].

Definition 3.4 (Halin's Theorem). IRT, *the infinite ray theorem*, is the principle that every graph H which contains arbitrarily many disjoint rays contains infinitely many disjoint rays.

The versions of Halin's theorem which we consider in this paper allow for H to be an undirected or a directed graph and for the disjointness requirement to be vertex or edge. We also allow the rays to be single or double. The corresponding versions of Halin's Theorem are labeled as IRT_{XYZ} for appropriate values of X , Y and Z . We often state a theorem for several or all XYZ and then in the proof use “graph”, “edge” and “disjoint” unmodified with the intention that the proof can be read for any of the cases.

We also consider restrictions of these theorems to specific families of graphs. We need a few more notions to define them.

Definition 3.5. A *tree* is a graph T with a designated element r called its *root* such that for each vertex $v \neq r$ there is a unique path from r to v . A *branch* on (or in) T is a ray that begins at its root. We denote the set of its branches by $[T]$ and say that T is *well-founded* if $[T] = \emptyset$ and otherwise it is *ill-founded*. A *forest* is an *effective disjoint union* of trees, or more formally, a graph with a designated set R (of vertices called roots) such that for each vertex v there is a unique $r \in R$ such that there is a path from r to v and, moreover, there is only one such path. In general, the *effectiveness* we assume when we take *effective disjoint unions of graphs* means that we can effectively (i.e. computably) uniquely identify each vertex in the union with the original vertex (and the graph to which it belongs) which it represents in the disjoint union.

A *directed tree* is a directed graph $T = \langle V, E \rangle$ such that its *underlying graph* $\hat{T} = \langle V, \hat{E} \rangle$ where $\hat{E} = \{ \{u, v\} \mid \langle u, v \rangle \in E \vee \langle v, u \rangle \in E \}$ is a tree. A *directed forest* is a directed graph whose underlying graph is a forest.

Definition 3.6. An X -graph H is *locally finite* if, for each $u \in V$, the set $\{v \in E \mid (u, v) \in E \vee (v, u) \in E\}$ of *neighbors of u* is finite. A locally finite X -tree is also called *finitely branching*. (Note this does not mean there are finitely many branches in the tree.)

Of course, there are many well known equivalent definitions of trees and associated notions. We have given one possible set of graph-theoretic definitions. In the case of undirected graphs our definition is equivalent to all the standard ones. Note, however, we are restricting ourselves to what would (in set theory) be called countable trees with all nodes of finite rank. Thus, we typically think of trees as *subtrees of $N^{<N}$* , i.e. the sets of finite strings of numbers (as vertices) with an edge between σ and τ if and only if they differ by one being an extension of the other by one element, e.g. $\sigma^\frown k = \tau$.

There does not seem to be a standard definition of graphs being directed trees and associated notions. We have picked one that seems to be fairly common and works for the situations for which we consider them in Theorems 4.24, 5.7, and 5.8 and Corollary 5.9.

3.2 Computability Hierarchies

As all sets and structures actually studied in this paper are countable, for analyzing their complexity, we can think of all of them as being subsets of, or relations or functions on, N .

To set our notation, we fix an enumeration Φ_e^A of the partial computable functions with oracle A . We let A'_s be the set of the first s many numbers enumerated in a fixed one-one A -computable enumeration of $A' = \{e \mid \Phi_e^A(e) \text{ converges}\}$. We frequently use two basic general concepts: relativization and uniformity. The first typically entails noting that a theorem about computable sets or procedures holds when we relativize computable to computable in A for any A . The second typically says that when we specify some procedure that produces (from some inputs) computable sets or Turing functions then we can uniformly (i.e. by a fixed procedure) compute indices for the outputs from those of the inputs usually even when oracles are involved. We describe an important example of uniformity in Remark 3.9.

The most fundamental hierarchy measuring the complexity of sets and functions is given by the iterations of the Turing jump. While the finite iterations of the jump capture most construction techniques and theorems in graph theory (and most other areas of classical countable/separable mathematics), we will be interested in ones that go beyond such techniques and proofs. The basic idea is that we continue the hierarchy by iteration into the transfinite while still tying the iteration to computable procedures.

Definition 3.7. We represent well-orderings or *ordinals* α as well-ordered relations on N . Typically such *ordinal notations* are endowed with various additional structure such as identifying 0, successor and limit ordinals and specifying cofinal ω -sequences for the limit ordinals. If we have a representation of α then restricting the well-ordering to numbers in its domain provides representations of each ordinal $\beta < \alpha$. We generally simply work with ordinals and omit concerns about translating standard relations and procedures to the representation. An ordinal is computable (in a set X) if it has a computable (in X) representation. For a set X and ordinal (notation) α computable from X , we define the transfinite iterations $X^{(\beta)}$ of the Turing jump of X by transfinite induction on $\beta \leq \alpha$: $X^{(0)} = X$; $X^{(\beta+1)} = (X^{(\beta)})'$ and for a limit ordinal λ , $X^{(\lambda)} = \bigoplus\{X^{(\beta)} \mid \beta < \lambda\} = \bigcup\{\{\beta\} \times X^{(\beta)} \mid \beta < \lambda\}$ (or as the effective disjoint sum over the $X^{(\beta)}$ in the specified cofinal sequence in λ).

Definition 3.8. $\text{HYP}(X)$, the collection of all sets *hyperarithmetic in* X consists of those sets computable in some $X^{(\alpha)}$ for α an ordinal computable in X . We say that Y is *hyperarithmetic in* X or *hyperarithmetically reducible to* X , $Y \leq_h X$, if $Y \in \text{HYP}(X)$.

The computational strength of our graph theoretic theorems such as **IRT** is measured by this hierarchy as we will show that for every set X and every set Y hyperarithmetic in X , there is a graph G computable from X which satisfies the hypotheses of **IRT** but for which any collection of rays satisfying its conclusion computes Y . Thus they clearly go far beyond the common procedures computable in finitely many iterations of the Turing jump. We note one important well known basic fact relating the jumps of X to trees computable from X that we use in our proof that **IRT** has this computational complexity.

Proposition 3.9. *For any set X and any ordinal α computable from X , there is a sequence $\langle T_\beta \mid \beta < \alpha \rangle$ computable from X of trees (necessarily) computable from X such that each tree has exactly one branch P_β and P_β is of the same complexity as $X^{(\beta)}$, i.e. $P_\beta \equiv_T X^{(\beta)}$. The procedure for computing this sequence is uniform in X and the index for the program computing the well ordering α from X , i.e. there is one computable functional that when given an oracle for X , an index for α (i.e. the i such that Φ_i^X is the well ordering α) computes the whole sequence $\langle T_\beta \mid \beta < \alpha \rangle$ along with indices for each T_β for β in the domain of the ordering α and the reductions between P_β and $X^{(\beta)}$. (See, e.g. [25, Theorem 2.3]). We may also easily assure that the T_β are effectively disjoint so that their union is a forest.*

On the other hand, placing an upper bound on the strength of **IRT** requires analyzing its proof and the principles used in it. The relevant one is a form of the axiom of choice defined in the next subsection.

Some versions of the variations on **IRT** (see §5.2) that call for the maximality of the infinite set of disjoint rays are stronger both computationally and proof theoretically than the **IRT_{XYZ}** described above. Their computational strength is captured by a kind of jump operator that goes beyond all the hyperarithmetic ones. It captures the ability to tell if a computable ordering is a well-ordering.

Definition 3.10. The *hyperjump* of X , T^X , is the set $\{e \mid \Phi_e^X$ is (the characteristic function of) a subtree of $N^{<N}$ which is well-founded $\}$.

This operator is also often defined as \mathcal{O}^X in terms of a particular notation system and gives sets computably isomorphic to T^X uniformly in X [21, Corollary 16.XX(b)].

3.3 Logical and Axiomatic Hierarchies

The basic notions from logic that we need here are those of languages, structures and axiomatic systems and proofs. As we will deal only with countable sets and structures, we can assume that we are dealing just with the natural numbers with a way to define and use sets and functions on them. Thus, at the beginning, we have in mind the natural numbers \mathbb{N} along with the usual apparatus of the *language of (first order) arithmetic*, say $+, \times, <, =, 0$ and 1 along with the syntax of standard first order logic. A *structure* for this language is a set N along with elements for 0 and 1 , binary functions for $+$ and \times and a binary relation for $<$ along with the standard equality relation. We also need a way of talking about subsets

of (or functions on) the numbers. We follow the usual practice in reverse mathematics of using sets and defining functions in terms of their graphs. So we expand our language by adding on new classes of (second order or set) variables such as X and Y and the associated quantifiers $\forall X$ and $\exists Y$ along with a new relation symbol \in . The syntax is as usual with some caveats. The only terms are those of the first order structure (numerical terms) and the relation symbols $<$ and $=$ only apply to terms. The only atomic formulas using \in are of the form $t \in X$ for t a term and X a second order variable. So, in particular, following [28], we do not take equality for sets to be a primitive relation on this structure. The notation $A = B$ is defined by $\forall n(n \in A \leftrightarrow n \in B)$.

A structure for this language is one of the form $\mathcal{N} = \langle N, S, +, \times, <, 0, 1, \in \rangle$ where its restriction $\langle N, +, \times, <, 0, 1 \rangle$ is a structure for first order arithmetic and $S \subseteq 2^N$ is a specified nonempty collection of subsets of N disjoint from N . We call N the *first order part* of \mathcal{N} and S its *second order part*. The semantics here is as usual with the caveats that first order variables range over the first order part and second order ones over the second order part. Also the symbol \in always denotes the standard membership relation between elements of N and subsets of N that are in S .

Proofs are defined as usual to generate the provability notion \vdash used in §1 to define our notion of logical strength and equivalences of theories (sets of sentences often called axioms). Proof theoretic notions deal with all possible structures for the language and axiom systems to specify what we need in any particular argument. For most of our purposes and all of the computational ones, one can restrict attention to *standard models of arithmetic*, i.e. ones \mathcal{N} with $N = \mathbb{N}$ and some $S \subseteq 2^{\mathbb{N}}$ with the usual interpretations of the functions and relations. We generally abbreviate these structures as $\langle \mathbb{N}, S \rangle$ with $S \subseteq 2^{\mathbb{N}}$, or simply S , as all the functions and relations are then fixed.

We now define the standard weak base theory (RCA_0) discussed in §1 and describe several other common systems that we use. The formal details can be found in [28].

Each axiomatic subsystem of second order arithmetic that we consider contains the standard basic axioms for $+$, \times , and $<$ (which say that N is a discrete ordered semiring) and an Induction Axiom:

$$(I_0) \quad (\forall X)((0 \in X \wedge \forall n(n \in X \rightarrow n + 1 \in X)) \rightarrow \forall n(n \in X)).$$

Typically axiom systems for second order arithmetic are defined by adding various types of set existence axioms although at times additional induction axioms are used as well. In order to define them we need to specify various standard syntactic classes of formulas determined by quantifier complexity. As usual, we add to our language bounded quantifiers $\forall x < t$ and $\exists x < t$ for (first order) terms t defined in the standard way. We typically denote formulas by capital Greek letters except that the indexed Φ_e and Φ_e^X refer, as above, to our fixed enumeration of the Turing machines and associated partial functions.

Definition 3.11. The Σ_0^0 and Π_0^0 formulas of second order arithmetic are just the ones with only bounded quantifiers but we allow parameters for elements of either N or $S(\mathcal{N})$ when working with a structure \mathcal{N} . Proceeding inductively, a formula Φ is Σ_{n+1}^0 (Π_{n+1}^0) if it is of the form $\exists x\Psi$ ($\forall x\Psi$) where Ψ is Π_n^0 (Σ_n^0). We say Φ is *arithmetic* if it is Σ_n^0 or Π_n^0 for some $n \in \mathbb{N}$. It is Σ_1^1 (Π_1^1) if it is of the form $\exists X\Psi$ ($\forall X\Psi$) where Ψ is arithmetic. We say a set X is in one of these classes Γ relative to A (i.e. with A as a parameter) if there is a formula

$\Psi(n, A) \in \Gamma$ such that $n \in X \Leftrightarrow \Psi(n, A)$. If X is both Σ_n^i in A and Π_n^i in A it is called Δ_n^i in A .

We mention a few additional standard connections between the syntactic complexity of the definition of a set X and X 's properties in terms of computability and graph theoretic notions. They can all be found in [21].

Proposition 3.12. *The sets $A^{(n)}$ are Σ_n^0 in A . A set X is computable in A if and only if it is Δ_1^0 in A . More generally, it is computable in $A^{(n)}$ if and only if it is Δ_{n+1}^0 in A . It is hyperarithmetic in A if and only if it is Δ_1^1 in A . There is a computable function $f(e, n)$ such that if X is Σ_1^1 in A via the Σ_1^1 formula with code e then for every n , $\Phi_{f(e, n)}^A$ is (the characteristic function of) a tree T such that $n \in X \Leftrightarrow T$ has a branch.*

The first system for analyzing the proof theoretic strength of theorems and theories in reverse mathematics is just strong enough to prove the existence of the computable sets and so supplies us with all the usual computable functions such as pairing $\langle n, m \rangle$ or more generally those coding finite sequences as numbers and so a Gödel numbering of formulas. In particular, it provides the predicates defining the (codes e of) the partial computable functions Φ_e^X and the relations saying the computation $\Phi_e^X(n)$ halts in s many steps with output y . Thus we have the basic tools to define and discuss Turing reducibility and the Turing jump. It is our weak base theory and is assumed to be included in every system we consider.

(RCA_0) Recursive Comprehension Axioms: In addition to the ones mentioned above, its axioms include the schemes of recursive (generally called Δ_1^0) comprehension and Σ_1^0 induction:

- (Δ_1^0 -CA) $\forall n (\Phi(n) \leftrightarrow \Psi(n)) \rightarrow \exists X \forall n (n \in X \leftrightarrow \Phi(n))$ for all Σ_1^0 formulas Φ and Π_1^0 formulas Ψ in which X is not free.
- ($\text{I}\Sigma_1^0$) $(\Phi(0) \wedge \forall n (\Phi(n) \rightarrow \Phi(n+1))) \rightarrow \forall n \Phi(n)$ for all Σ_1^0 formulas Φ .

Note that these formulas may have free set or number variables. As usual, the existence assertion $\exists X \dots$ of the axiom is taken to mean that for each instantiation of the free variables (by numbers or sets, as appropriate, called *parameters*) there is an X as described. We take this for granted as well as the restriction that the X is not free in the rest of the formula in all of the set existence axioms of any of our systems.

The standard models of RCA_0 are just those whose second order part is closed under Turing reduction and disjoint union ($X \oplus Y = \{\langle 0, x \rangle \mid x \in X\} \cup \{\langle 1, y \rangle \mid y \in Y\}$). As suggested above, what are now often called the computable in A sets which are the Δ_1^0 in A sets, were originally called the sets recursive in A . Hence the terminology RCA_0 .

If we have some axiom scheme or principle ABC we typically denote the system formed by adding it to RCA_0 by ABC_0 . We next move up to the arithmetic comprehension axiom and its system.

- (ACA) $\exists X \forall n (n \in X \leftrightarrow \Phi(n))$ for every arithmetic formula Φ .

As mentioned above the $X^{(n)}$ are defined by a Σ_n^0 formula with X as a parameter. So one can show that ACA_0 is equivalent (over RCA_0) to the totality of the Turing jump operator, i.e. for every X , X' exists. Its standard models are those of RCA_0 whose second order part

is also closed under Turing jump. It is also equivalent (in the sense of reverse mathematics) to König's Lemma.

In general, we say one system of axioms S is *logically or reverse mathematically reducible* to another T over one R if $R \cup T \vdash \psi$ for every sentence $\psi \in S$. Note that S and/or T may be a single sentence or theorem. We say that S and T are *equivalent* over R if each is reducible to the other. If no system R is specified we assume that RCA_0 is intended.

The standard system between RCA_0 and ACA_0 is WKL_0 . It is characterized by the restriction of König's Lemma to trees that are subsets of $2^{< N}$, the tree of finite binary strings under extension.

The next system of the five basic ones after ACA_0 is ATR_0 . Its defining axiom says that arithmetic comprehension can be iterated along any countable well-order and so implies the existence of the sets hyperarithmetic in X for each X but is computationally stronger than this assumption.

We formally describe the computationally defined class of theorems/theories that are the main focus of this paper and include several variations of IRT. The definition is semantic, not axiomatic, and involves only standard models. (Indeed by Van Wesep [30, 2.2.2], there can be no axiomatic characterization of this class in second order arithmetic.)

Definition 3.13. A sentence (theory) T is a *theorem (theory) of hyperarithmetic analysis (THA)* if

1. For every $X \subseteq \mathbb{N}$, $\langle \mathbb{N}, \text{HYP}(X) \rangle \models T$ and
2. For every $S \subseteq 2^{\mathbb{N}}$, if $\langle \mathbb{N}, S \rangle \models T$ and $X \in S$ then $\text{HYP}(X) \subseteq S$.

It is worth pointing out some of the relations between THAs and ATR_0 . THAs are defined by only using standard models and iterations of the jump over true well orderings. ATR_0 talks about all models of RCA_0 and asserts the existence of iterates of the jump over all orderings that appear well-founded in the model. Thus for standard models it implies the second clause of the definition of THAs. On the other hand, there is a recursive linear order with no hyperarithmetic infinite descending sequence and so it seems well-founded in HYP but it has a well-founded part longer than every recursive ordinal. Thus iterating the jump along this ordering would yield a set strictly Turing above every hyperarithmetic set. In particular, HYP is not a model of ATR_0 which therefore is not a THA. (See e.g. [28, V.2.6].)

The last of the standard axiomatic systems, $\Pi_1^1\text{-CA}_0$, is characterized by the comprehension axiom for Π_1^1 formulas:

$$(\Pi_1^1\text{-CA}) \exists X \forall k (k \in X \leftrightarrow \Phi(k)) \text{ for every } \Pi_1^1 \text{ formula } \Phi(k).$$

Remark 3.14. The hyperjump, T^X , is clearly a Π_1^1 set with parameter X . In fact, every Π_1^1 set with parameter X is reducible to T^X . Indeed, there is a computable function $f(e, n)$ such that for every index e for a Π_1^1 formula $\Psi(n)$ with parameter X and every n , $\Psi(n) \Leftrightarrow f(e, n) \in T^X$ [21, Corollary 16.XX(b)]. Thus $\Pi_1^1\text{-CA}_0$ corresponds to closure under the hyperjump. By Theorem 5.14, it is equivalent to a version of IRT asking for a maximal set of disjoint rays.

For this paper, the most important other existence axiom is a restricted form of the axiom of choice.

$(\Sigma_1^1\text{-AC}) \forall n \exists X \Phi(n, X) \rightarrow \exists X \forall n \Phi(n, X^{[n]})$ where Φ is arithmetic and $X^{[n]} = \{m \mid \langle n, m \rangle \in X\}$ is the n th column of X .

A more common but clearly equivalent version of this axiom allows Φ to be Σ_1^1 . A variant commonly called weak- $\Sigma_1^1\text{-AC}$ (introduced in Definition 6.1 as unique- $\Sigma_1^1\text{-AC}$) requires the corresponding Φ to be arithmetic. To make these and other choice axioms uniform we have adopted the format required elsewhere and equivalent here to be used for all the variations. The system $\Sigma_1^1\text{-AC}_0$ is well known to be a THA (essentially in Kreisel [14]). Thus it is strictly stronger than ACA_0 . On the other hand, it is strictly weaker than ATR_0 [28, V.8.3-4]. This choice axiom plays a crucial role in our analysis because we provide the upper bound on the strength of most of our theorems by showing that they follow from $\Sigma_1^1\text{-AC}_0$. This provides the computational upper bound for being a THA as any consequence of a THA must also satisfy Definition 3.13(1). Thus the bulk of our proofs for the computational complexity of the theorems we study consist of showing that they imply Definition 3.13(2), i.e. closure under ‘‘hyperarithmetic in’’. We also introduce another variant, finite- $\Sigma_1^1\text{-AC}$ (Definition 6.2) which is a THA as well.

For those interested in the proof theory and so nonstandard models, we also at times explicitly consider the induction axiom at the same Σ_1^1 level.

$$(\text{I}\Sigma_1^1) (\Phi(0) \wedge \forall n(\Phi(n) \rightarrow \Phi(n+1))) \rightarrow \forall n \Phi(n) \text{ for every } \Sigma_1^1 \text{ formula } \Phi.$$

This axiom does not imply the existence of any infinite sets and is, of course, true in every standard model. Thus the readers interested only in the computational complexity of the Halin type theorems can safely ignore these considerations of induction assumptions. They are, however, closely tied to the original formulation of IRT in [11] (and its variants):

Definition 3.15. Let IRT_{XYZ}^* be the statement that every X -graph G has a set of Y -disjoint Z -rays of maximum cardinality, or more formally, the statement that for every X -graph G :

- there is no Z -ray in G , or
- there is some $n > 0$ and some R such that $\langle R^{[i]} \mid i < n \rangle$ is a sequence of Y -disjoint Z -rays in G , and there is no R such that $\langle R^{[i]} \mid i < n+1 \rangle$ is a sequence of Y -disjoint Z -rays in G , or
- there is some R such that $\langle R^{[i]} \mid i \in N \rangle$ is a sequence of Y -disjoint Z -rays in G .

When we talk about the cardinality of a (possibly empty) finite sequence $\langle R^{[i]} \mid i < n \rangle$ we mean the number n (which may be 0). Of course a sequence $\langle R^{[i]} \mid i \in N \rangle$ is said to have infinite cardinality.

The notation in Definition 3.15 is inspired by the well known variant ACA_0^* of ACA_0 which asserts the existence of $A^{(n)}$ for every set A and number n :

Definition 3.16.

$$\text{ACA}^* : (\forall A)(\forall n)(\exists W)(W^{[0]} = A \wedge (\forall i < n)(W^{[i+1]} = W^{[i']})).$$

Thus ACA_0^* asserts (in addition to ACA_0) particular instances of $\text{I}\Sigma_1^1$. So too (in addition to IRT_{XYZ}) do the IRT_{XYZ}^* :

Proposition 3.17. *For each choice of X , Y , Z , IRT_{XYZ}^* implies IRT_{XYZ} over RCA_0 and IRT_{XYZ} implies IRT_{XYZ}^* over $\text{RCA}_0 + \text{I}\Sigma_1^1$. Therefore IRT_{XYZ} and IRT_{XYZ}^* are equivalent over $\text{RCA}_0 + \text{I}\Sigma_1^1$. In particular, they have the same standard models.*

Proof. The first implication holds because if an X -graph has arbitrarily many Y -disjoint Z -rays, then any sequence of Y -disjoint Z -rays in the graph of maximum cardinality must be infinite. To prove the second, let G be an X -graph and $\Phi(n)$ be the Σ_1^1 formula which says that there is a sequence of length n of Y -disjoint Z -rays in G . If $\forall n(\Phi(n) \rightarrow \Phi(n+1))$, then by $\text{I}\Sigma_1^1$, $\forall n\Phi(n)$ holds and so by IRT_{XYZ} there is a sequence $\langle R_i \rangle_{i \in N}$ of Y -disjoint Z -rays in G as required. On the other hand, if there is an n such that $\Phi(n)$ holds but $\Phi(n+1)$ fails, then there is exactly one as $\Phi(n) \rightarrow \Phi(m)$ for $m < n$ and this n witnesses IRT_{XYZ}^* . \square

Indeed we will show (Theorem 5.3) that each IRT_{XYZ} implies ACA_0^* (over RCA_0) and that this has important consequences for their reverse mathematical strength.

Remark 3.18. It is in the nature of reverse mathematics that sentences and sets of sentences of second order arithmetic are often viewed in several different ways. In different contexts they may be seen as mathematical or logical principles, axioms, axiom schemes, theories, theorems or the like. We point out what may be a less familiar terminology that is currently popular. What might be seen as a typical axiom or theorem asserting that for every X of some sort there is a Y with some relation to X , i.e. a sentence of the form $\forall X(\Phi(X) \rightarrow \exists Y\Psi(X, Y))$ may be called a principle. With this terminology come the notions of an instance of the principle, i.e. an X satisfying Φ and a solution for X , i.e. a Y such that $\Psi(X, Y)$ holds.

4 IRT and Hyperarithmetic Analysis

In this section we shall prove the following.

Theorem 4.1. IRT_{XYS} , IRT_{XYS}^* , IRT_{UVD} and IRT_{UVD}^* are theorems of hyperarithmetic analysis.

IRT_{UVS} and IRT_{UVD} were proved by Halin [11, 12]. IRT_{UES} is an exercise in [6, 8.2.5(ii)]. IRT_{DVS} and IRT_{DES} may be folklore (and so, similarly for the corresponding IRT_{XYZ}^*).

In order to minimize repetition we first note some implications between variants of IRT over RCA_0 . The proofs of each of these reductions follow the same basic plan. To deduce IRT_{XYZ} from $\text{IRT}_{X'Y'Z'}$ we provide computable maps g , h and k which, provably in RCA_0 , take X -graphs G to X' -graphs G' , Y -disjoint Z -rays or sets of Y -disjoint Z -rays in G to Y' -disjoint Z' -rays or sets of Y' -disjoint Z' -rays in G' , and Y' -disjoint Z' -rays or sets of Y' -disjoint Z' -rays in G' to Y -disjoint Z -rays or sets of Y -disjoint Z -rays in G , respectively. These functions are designed to take witnesses of the hypothesis of IRT_{XYZ} in G to witnesses of the hypothesis of $\text{IRT}_{X'Y'Z'}$ in G' and witnesses to the conclusion of $\text{IRT}_{X'Y'Z'}$ in G' to witnesses to the conclusion of IRT_{XYZ} in G . Clearly it suffices to provide such computable maps to establish the desired reduction in RCA_0 . The constructions are straightforward and we omit the details which can be found in [3]. (Those familiar with Weihrauch reducibility will recognize that these arguments establish Weihrauch reductions between certain problems

corresponding to variants of IRT .) They also supply the same results for the corresponding $\text{IRT}_{\text{XYZ}}^*$.

Unless otherwise noted all definitions and proofs in this section are in RCA_0 .

Lemma 4.2. *Given an undirected graph G , we can uniformly compute a directed graph G' and mappings between Z -rays in G and Z -rays in G' which preserve Y -disjointness.*

Proof idea. For each edge $\{u, v\}$ in G , add two new vertices $x = x(u, v)$ and $y = y(u, v)$ and define five directed edges $\langle u, x \rangle, \langle v, x \rangle, \langle x, y \rangle, \langle y, u \rangle, \langle y, v \rangle$. \square

Proposition 4.3. IRT_{DYZ} implies IRT_{UYZ} , and $\text{IRT}_{\text{DYZ}}^*$ implies $\text{IRT}_{\text{UYZ}}^*$, for each value of Y and Z .

Lemma 4.4. *Given a directed graph G , we can uniformly compute a directed graph G' and mappings between Z -rays in G and Z -rays in G' which satisfy the following properties: if two Z -rays in G are vertex-disjoint, then the corresponding Z -rays in G' are edge-disjoint, and if two Z -rays in G' are edge-disjoint, then the corresponding Z -rays in G are vertex-disjoint.*

Proof idea. The set of vertices of G' is $\{x_i, x_o \mid x \in V\}$, where i and o stand for incoming and outgoing respectively. The set of edges of G' consists of $\langle u_o, v_i \rangle$ for each $\langle u, v \rangle \in E$, and $\langle x_i, x_o \rangle$ for each $x \in V$. \square

Proposition 4.5. IRT_{DEZ} implies IRT_{DVZ} , and $\text{IRT}_{\text{DEZ}}^*$ implies $\text{IRT}_{\text{DVZ}}^*$, for each value of Z .

Lemma 4.6. *Given a directed graph G , we can uniformly compute a directed graph G' and mappings between sets of Y -disjoint rays in G and sets of Y -disjoint double rays in G' which preserve cardinality.*

Proof idea. For each vertex x of G we add new vertices x_n for each $n < 0$ and edges $\langle x_{-1}, x \rangle$ and $\langle x_{n-1}, x_n \rangle$ for all $n < 0$. \square

Proposition 4.7. IRT_{DYD} implies IRT_{DYS} , and $\text{IRT}_{\text{DYD}}^*$ implies $\text{IRT}_{\text{DYS}}^*$, for each value of Y .

There does not seem to be an analog of the previous lemma for undirected graphs. Nonetheless we have a related fact for certain undirected graphs.

Lemma 4.8. *Suppose G is a connected (undirected) graph such that any two single rays in G share a vertex. Then G computably embeds into a graph G' such that any two double rays in G' must share a vertex. Furthermore, given a single ray in G , one can compute a double ray in G' and vice versa. All the aforementioned embeddings and computations are uniform.*

Proof idea. Let v_0 denote the $<_N$ -least vertex of G . Add new vertices v_i for $i > 0$ to G and add the edges $\{v_i, v_{i+1}\}$ for $i \in N$ to G . \square

Remark 4.9. Bowler, Carmesin, Pott [4, pg. 2 l. 3–7] describe an implication from IRT_{UVS} to IRT_{UES} which appears to use $\Sigma_1^1\text{-AC}_0$. It turns out that the graph-theoretic principle used to carry out the implication does not imply even ACA_0 over RCA_0 (and is hence much weaker than $\Sigma_1^1\text{-AC}_0$), but when combined with ACA_0 , yields a THA. It and several other principles with the same property (almost theorems/theories of hyperarithmetic analysis) are analyzed in Shore [26].

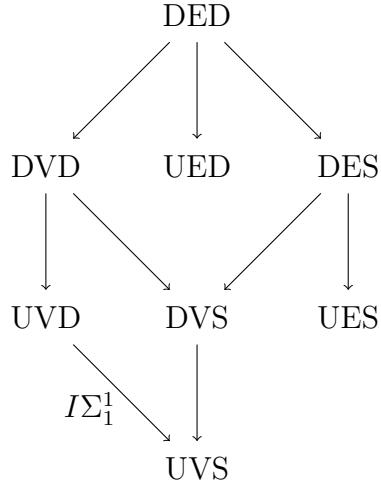


Figure 1: Known implications between variants of IRT. All implications are over RCA_0 , except for the implication from UVD to UVS (Theorem 5.12).

We are now ready to give lower bounds for the complexity of IRT_{XYZ} .

Theorem 4.10. IRT_{UYZ} (even for forests) implies ACA_0 . Therefore IRT_{UYZ}^* (even for forests), IRT_{DYZ} and IRT_{DYZ}^* each imply ACA_0 .

Proof. It suffices to prove the first statement because of Propositions 4.3 and 3.17. First we consider IRT_{UYS} . For each $n \in N$, consider the tree $T_n \subseteq N^{<N}$ consisting of all strings of the form $s^\frown 0^t$ such that some number below n is enumerated into \emptyset' at stage s , and either $t \leq s$ or $\emptyset'_s \upharpoonright n = \emptyset'_t \upharpoonright n$. Since RCA_0 proves that for each $n \in N$, there is some s such that $\emptyset' \upharpoonright n = \emptyset'_s \upharpoonright n$, it follows that each T_n has a unique branch $\{s^\frown 0^t \mid t \in N\}$ (for the least such s as above).

Consider the disjoint union $\bigsqcup_n T_n$. It satisfies the premise of IRT_{UYS} for forests. By IRT_{UYS} , there is a sequence $\langle R_i \rangle_i$ of disjoint rays in $\bigsqcup_n T_n$. Each R_i is contained in some T_n , and no two disjoint rays can lie in the same T_n . Therefore, we can, uniformly in i , extend or truncate R_i to the unique branch P_n of T_n . Hence $\langle P_n \rangle_n$ exists and computes longer and longer initial segments of \emptyset' , implying that \emptyset' exists as well. The above argument relativizes to prove that IRT_{UYS} implies ACA_0 .

In order to prove the desired result for IRT_{UYD} , apply IRT_{UYD} to the forest $\bigsqcup_n T'_n$, where T'_n is defined from T_n using Lemma 4.8. \square

Henceforth we will not explicitly mention uses of ACA_0 whenever we are assuming any IRT_{XYZ} or IRT_{XYZ}^* .

Theorem 4.11. Every standard model of IRT_{XYZ} or IRT_{XYZ}^* is closed under hyperarithmetic reduction.

Proof. By Propositions 4.3 and 3.17 it suffices to consider IRT_{UYZ} . Consider first a standard model \mathcal{M} of IRT_{UYS} . We shall show that, for each computable limit ordinal λ , if \mathcal{M} contains $\emptyset^{(\alpha)}$ for every $\alpha < \lambda$ then \mathcal{M} contains $\emptyset^{(\lambda)}$. (Again the desired result follows by relativization.)

By Proposition 3.9, there is a computable sequence $\langle T_\beta \rangle_{\beta < \lambda}$ of trees such that each tree has exactly one branch $P_\beta \equiv_T \emptyset^{(\beta)}$ with these reductions computed uniformly. Fix an increasing computable sequence $\langle \alpha_n \rangle_n$ which is cofinal in λ and consider the disjoint union $\bigsqcup_n T_{\alpha_n}$. Observe that $\bigsqcup_n T_{\alpha_n}$ satisfies the premise of IRT_{UYS} (in \mathcal{M}): for each n , $\emptyset^{(\alpha_n)}$ computes the branches P_{α_m} for $m \leq n$. By IRT_{UYS} , \mathcal{M} contains a sequence $\langle R_i \rangle_i$ of disjoint rays in $\bigsqcup_n T_{\alpha_n}$. As in the proof of Theorem 4.10, $\langle R_i \rangle_i$ computes a sequence of infinitely many distinct branches P_{α_n} , and hence a sequence of infinitely many distinct $\emptyset^{(\alpha_n)}$. Each $\emptyset^{(\alpha_n)}$ uniformly computes $\emptyset^{(\alpha_m)}$ for $m \leq n$, so $\langle R_i \rangle_i$ computes $\bigoplus_m \emptyset^{(\alpha_m)}$ as desired.

In order to prove the desired result for IRT_{UYD} , apply IRT_{UYD} to the forest $\bigsqcup_n T'_{\alpha_n}$, where T'_{α_n} is defined from T_{α_n} using Lemma 4.8. □

We turn now to upper bounds for the complexity of IRT_{XYS} and IRT_{UVD} . All proofs of IRT in the literature begin by (implicitly) collecting a sequence of finite sequences of disjoint rays in the given graph. We formalize this step as **SCR** (Strongly Collecting Rays) and the rest of the proof as **WIRT_{XYZ}** (Weak Infinite Ray Theorem):

Definition 4.12. Let SCR_{XYZ} be the assertion that if G is an X -graph with arbitrarily many Y -disjoint Z -rays, then there is a sequence of sets $\langle X_k \rangle_k$ such that for each $k \in N$, X_k is a set of k Y -disjoint Z -rays in G .

Let WIRT_{XYZ} be the assertion that if G is an X -graph and there is a sequence of sets $\langle X_k \rangle_k$ such that for each $k \in N$, X_k is a set of k Y -disjoint Z -rays in G , then G has infinitely many Y -disjoint Z -rays.

First observe that Lemmas 4.2, 4.4 and 4.6 imply

Proposition 4.13. SCR_{DYZ} implies SCR_{UYZ} , SCR_{DEZ} implies SCR_{DVZ} and SCR_{DYD} implies SCR_{DYS} . Analogous statements hold for **WIRT**.

It is clear that $\Sigma_1^1\text{-AC}_0$ implies SCR_{XYZ} and that $\text{SCR}_{\text{XYZ}} + \text{WIRT}_{\text{XYZ}}$ is equivalent to IRT_{XYZ} over RCA_0 . In all cases where IRT_{XYZ} was a known theorem before [4] (i.e. IRT_{XYS} and IRT_{UVD}), we shall prove that ACA_0 implies WIRT_{XYZ} . Since $\Sigma_1^1\text{-AC}_0$ implies ACA_0 (essentially due to [14]; see also the proof of Theorem 4.10), it follows that the corresponding IRT_{XYZ} are provable in $\Sigma_1^1\text{-AC}_0$.

Theorem 4.14. ACA_0 proves WIRT_{UVD} . Therefore $\Sigma_1^1\text{-AC}_0$ proves IRT_{UVD} .

Proof. The mathematical result is due to [11] and is Exercise 42 of Chapter 8 in [6]. The idea is to follow [6, Theorem 8.2.5(i)] except we need to grow our family in “two directions”. We shall use the following notation for decomposing double rays into single rays. If R and R_i are double rays we let $R_f = R$ and $R_{i,f} = R_i$ while R_b and $R_{i,b}$ are R and R_i , respectively, but with order reversed. We use d or d' to stand for one of f or b . For single rays R we use $*R$ to denote the reverse sequence of vertices. So, for example, if (c, d) is an edge in a double ray R , we have $*(cR_b)dR_f = R$.

Suppose we are given a graph G and a sequence $Y = \langle \langle S_j^n \rangle_{j < n} \rangle_{n > 0}$ such that for each n , $\langle S_j^n \rangle_{j < n}$ is a sequence of disjoint double rays. We shall construct, using a $(G \oplus Y)'$ -computable recursion on n , sequences $\langle R_i^n \rangle_{i < n}$ and $\langle P_i^n \rangle_{i < n}$ such that the R_i^n are disjoint double rays with

subpaths P_i^n of length $2n$ such that, for each $i < n$, P_i^{n+1} extends P_i^n by a new vertex at each end. Our required sequence of disjoint double rays is then $\langle \bigcup \{P_i^n \mid n \in N\} \rangle_{i \in N}$.

We begin our recursion by setting $R_0^1 = S_0^1$ and P_0^1 as any subpath of length 2. Future steps will use the following combinatorial lemma.

Lemma 4.15 (ACA₀). *Given disjoint double rays $\langle R_i \rangle_{i < n}$ with subpaths $\langle P_i \rangle_{i < n}$, as well as a sequence $\langle S_j \rangle_{j < 11n^2+1}$ of disjoint double rays, there is a sequence $\langle R'_i \rangle_{i < n+1}$ of disjoint double rays such that (1) R'_i extends P_i for each $i < n$; (2) for each $i < n+1$, R'_i has the form $QaPbR$ where P is a finite path while Q and R are each $(S_j)_f$ or $(S_j)_b$ for some $j < 11n^2+1$.*

Before we prove the lemma, we explain how we apply it. Suppose we are given $\langle R_i^n \rangle_{i < n}$ and $\langle P_i^n \rangle_{i < n}$ as above. For $i < n$, let $x_{i,b}$ be the first vertex in R_i^n before P_i^n and $x_{i,f}$ be the first vertex in R_i^n after P_i^n . Define $P_i^{n+1} = x_{i,b}P_i^n x_{i,f}$ for each $i < n$. Then, apply the lemma to $\langle R_i^n \rangle_{i < n}$, $\langle P_i^{n+1} \rangle_{i < n}$, and $\langle S_j^{11n^2+1} \rangle_{j < 11n^2+1}$. By (2), we may use $(G \oplus Y)'$ to search for $\langle R'_i \rangle_{i < n+1}$ as in the lemma. Define $R_i^{n+1} = R'_i$ for each i . To complete the recursion, define P_n^{n+1} to be any subpath of R_n^{n+1} of length $2n+2$.

It remains to prove the lemma. In the following we will frequently need to determine if and where two rays intersect, which can be done in ACA₀. We first discard from $\{S_j \mid j < 11n^2+1\}$ any S_j which intersects any P_i . As there are $n(2n+1) \leq 3n^2$ such vertices we have at least $8n^2+1$ many S_j remaining. We relabel these as S_j for $j < 8n^2+1$ and choose an edge $(c_{j,b}, c_{j,f})$ in each S_j .

Let $x_{i,b}$ and $x_{i,f}$ denote the endpoints of P_i . We have sets of disjoint single rays $\mathcal{R} = \{x_{i,f}R_{i,f}, x_{i,b}R_{i,b} \mid i < n\}$ and $\mathcal{S} = \{c_{j,f}S_{j,f}, c_{j,b}S_{j,b} \mid j < 8n^2+1\}$. Our goal for each $x_{i,d}R_{i,d} \in \mathcal{R}$, $i < n$ and $d \in \{b, f\}$, is to find a suitable replacement $R'_{\langle i,d \rangle}$ (beginning with the same vertex while maintaining the required disjointness) so that we can assemble the R'_i from $R'_{\langle i,b \rangle}$ and $R'_{\langle i,f \rangle}$ for $i < n$. We also want a double ray S_j disjoint from all the double rays R'_i with $i < n$ which will be R'_n .

To that end we first thin out \mathcal{S} as follows. Check if there is some $\langle i, d \rangle$ such that $R'_{\langle i,d \rangle}$ has not been defined and $x_{i,d}R_{i,d}$ intersects at most $2n$ many rays $c_{j,d'}S_{j,d'}$ (which have not been discarded). If there is no such $\langle i, d \rangle$, we end the procedure. Otherwise, find the least such $\langle i, d \rangle$ and (1) discard all rays $c_{j,d'}S_{j,d'}$ which intersect $x_{i,d}R_{i,d}$ from \mathcal{S} ; (2) define $R'_{\langle i,d \rangle} = x_{i,d}R_{i,d}$. Repeat the above check.

After the procedure ends, let I be the set of $\langle i, d \rangle$ for which $R'_{\langle i,d \rangle}$ has not been defined. Let m denote $|I|$. (Of course, $m \leq 2n$.) Observe that at most $(2n-m)(2n) \leq 4n^2$ many single rays were discarded from \mathcal{S} .

Next, for each $\langle i, d \rangle \in I$, let $z_{i,d}$ be the first vertex on $x_{i,d}R_{i,d}$ such that $x_{i,d}R_{i,d}z_{i,d}$ meets exactly m many rays in \mathcal{S} . (Each $z_{i,d}$ exists by construction of I . By RCA₀, $\{z_{i,d} \mid \langle i, d \rangle \in I\}$ exists.) Let F denote the finite set of vertices $\bigcup_{\langle i,d \rangle \in I} x_{i,d}R_{i,d}z_{i,d}$. Since F meets at most m^2 many single rays in \mathcal{S} , and $8n^2+1 > 4n^2+m^2$, there must be some double ray S_j , $j < 8n^2+1$ disjoint from F such that neither $c_{j,b}S_{j,b}$ nor $c_{j,f}S_{j,f}$ was discarded. We let one such double ray be R'_n . Before defining R'_i for $i < n$, we discard all single rays in \mathcal{S} not meeting F .

To complete the proof we shall use Menger's theorem for finite graphs. If A and B are disjoint sets of vertices in a graph, we say that P is an A - B path if P starts with some vertex in A and ends with some vertex in B . A set of vertices C separates A and B if every A - B path contains at least one vertex in C .

Theorem 4.16 (Menger, see [6, Theorem 3.3.1]). *Let H be a finite graph. If A and B are disjoint sets of vertices in H , then the minimum size of a set of vertices which separate A and B is equal to the maximum size of a set of disjoint A - B paths.*

One can check that Menger's theorem is provable in RCA_0 by following the first proof for it given in [6, Theorem 3.3.1].

For each $c_{j,d}S_{j,d}$ remaining in \mathcal{S} let $y_{j,d}$ be the first vertex on $c_{j,d}S_{j,d}$ such that $y_{j,d}S_{j,d}$ is disjoint from F . By ACA_0 , we may consider the following graph H and sets of vertices $A, B \subseteq H$:

$$A = \{x_{i,d} \mid \langle i, d \rangle \in I\}, \quad B = \{y_{j,d} \mid c_{j,d}S_{j,d} \in \mathcal{S}\}, \quad H = F \cup \bigcup_{c_{j,d}S_{j,d} \in \mathcal{S}} c_{j,d}S_{j,d}y_{j,d}.$$

We claim that A cannot be separated from B in H by fewer than m vertices. Suppose $C \subseteq H$ and $|C| < m$. Since $|I| = m$ and $\{x_{i,d}R_{i,d} : \langle i, d \rangle \in I\}$ is a disjoint collection of rays, C is disjoint from $x_{i,d}R_{i,d}$ for some $\langle i, d \rangle \in I$. Next, since $x_{i,d}R_{i,d}z_{i,d}$ meets m many disjoint rays in \mathcal{S} , there is some $c_{j,d'}S_{j,d'}$ in \mathcal{S} which meets $x_{i,d}R_{i,d}z_{i,d}$ (say at vertex z) but is disjoint from C . Then $x_{i,d}R_{i,d}z_{i,d}S_{j,d'}y_{j,d'}$ is a path in H from $x_{i,d}$ to $y_{j,d'}$ which does not meet C . This proves our claim.

By Menger's theorem, there is a set of m disjoint paths $P_{i,d}$ in H from each $x_{i,d}, \langle i, d \rangle \in I$, to a $y_{j,d'}$ in $c_{j,d'}S_{j,d'} \in \mathcal{S}$. We can now define $R'_{\langle i, d \rangle}$ for $\langle i, d \rangle \in I$ as the single ray beginning with $P_{i,d}$ and then continuing with $S_{j,d'}$ after $y_{j,d'}$. Finally, we can define the required double rays R'_i for $i < n$ as ${}^*R'_{\langle i, b \rangle}P_iR'_{\langle i, f \rangle}$ and check that these have all the desired properties. \square

A similar but simpler argument proves WIRT_{UVS} . We just state the crucial combinatorial lemma which we extracted from the proof in [6, Theorem 8.2.5(i)]. For details see [3].

Lemma 4.17 (ACA_0). *Suppose G is an undirected graph. Given vertex-disjoint rays $\langle R_i \rangle_{i < n}$ and vertex-disjoint rays $\langle S_j \rangle_{j < n^2 + 1}$ in G , there are vertex-disjoint rays $\langle R'_i \rangle_{i < n+1}$ such that (1) R_i and R'_i start at the same vertex for each $i < n$; (2) for each $i < n+1$, R'_i has the form PaS_j where P is a finite path, a is a vertex, and $j < n^2 + 1$.*

Theorem 4.18. ACA_0 proves WIRT_{UVS} .

The above theorem will be used in the proof of, and will be superseded by, Theorem 4.23. Indeed, our strategy for proving WIRT_{XES} is to reduce it to the problem of finding an infinite sequence of vertex-disjoint rays in a certain locally finite graph (see [4, pg. 21.3–7]). To carry out this reduction, we define the line graph:

Definition 4.19. The *line graph* $L(G)$ of an X -graph G is the X -graph with vertices the edges of G and edges $((u, v), (v, w))$, where (u, v) and (v, w) are edges in G .

Lemma 4.20. *Let G be an X -graph. There is a computable mapping from rays in G to rays in $L(G)$ such that if two rays in G are edge-disjoint, then their images are vertex-disjoint.*

Proof. Map x_0, x_1, x_2, \dots to $(x_0, x_1), (x_1, x_2), \dots$ \square

However, vertex-disjoint rays in $L(G)$ do not always yield edge-disjoint rays in G . An extreme counterexample is the (undirected) *star graph*. It consists of a single vertex with infinitely many neighbors. It does not contain any rays yet its line graph is isomorphic to the complete graph on N which contains infinitely many vertex-disjoint rays. Nonetheless, if G is locally finite, vertex-disjoint rays in $L(G)$ do correspond to edge-disjoint rays in G :

Lemma 4.21 (ACA₀). *Let G be a locally finite X -graph. There is a mapping from rays in $L(G)$ to rays in G such that if two rays in $L(G)$ are vertex-disjoint, then their images are edge-disjoint rays in G .*

Proof. Given a ray $R = e_0, e_1, \dots$ in $L(G)$, we construct a ray $S = y_0, y_1, \dots$ in G by recursion. Say $e_i = (u_i, v_i)$. Start by defining y_0 to be u_0 . Having defined y_n , we define y_{n+1} as follows. Let k_n be the largest k such that y_n is an endpoint of e_k . Such k exists because G is locally finite and R is a ray. We can find k_n by ACA₀. Then define y_{n+1} to be the endpoint of e_{k_n} other than y_n . This completes the recursion. Note that the k_n are strictly increasing: $e_{k_n} = (y_n, y_{n+1})$ as y_n is not an endpoint of e_{k_n+1} by the maximality of k_n . The next edge then includes y_{n+1} and so $k_{n+1} \geq k_n + 1$. Also note that if the graph is directed the e_i must be of the form $\langle x_i, x_{i+1} \rangle$ and the last occurrence of any x in an e_i must be as its first element. By construction S is infinite and contains no repeated vertices by the maximality requirement, hence it is a ray. Observe that every edge in S is a vertex in R , so the above mapping maps vertex-disjoint rays in $L(G)$ to edge-disjoint rays in G . \square

It remains to show that we can restrict our attention to locally finite graphs.

Lemma 4.22 (ACA₀). *Suppose that G is an X -graph and there is some family $\langle\langle R_j^k \rangle_{j < k} \rangle_{k > 0}$ of rays in G such that for each $k > 0$, the rays $\langle R_j^k \rangle_{j < k}$ are edge-disjoint. Then there is some locally finite X -subgraph G' of G and some family $\langle\langle S_j^k \rangle_{j < k} \rangle_{k > 0}$ of rays in G' such that for each $k > 0$, the rays $\langle S_j^k \rangle_{j < k}$ are edge-disjoint.*

Proof. Define the vertices of G' to be the vertices of G , say $\{v_i \mid i \in N\}$. We specify the set of edges E' of G' by providing a recursive construction of sets E_i of edges putting in a set of edges at each step. We guarantee that each E_i is a union of finitely many finite sets of edge-disjoint rays in G and that after stage k no edge with a vertex v_i for $i < k$ as an endpoint is ever put into E' .

Begin at stage 0 by putting all the edges in R_0^1 into E_1 . Proceeding recursively at stage $k > 0$ we have E_k and consider the edge-disjoint rays R_j^k , $j < k$. For each $j < k$, say $R_j^k = x_{j,0}^k, x_{j,1}^k, \dots$. Each v_i for $i < k$ appears at most once in R_j^k as R_j^k is a ray. For each $j < k$, since we have access to the set of vertices of R_j^k , we can decide whether R_j^k contains v_i and, if so, find the index n such that $v_i = x_{j,n}^k$. Call it $n_{i,j}^k$. If there is no such n , set $n_{i,j}^k = 0$. Define S_j^k to be the tail of R_j^k after $x_{j,\max_{i < k} n_{i,j}^k}^k$. We put all the edges in S_j^k , $j < k$, into E_{k+1} . Let $E' = \bigcup_k E_k$.

It remains to show that G' is locally finite. Consider any vertex v_k . No edge containing v_k as an endpoint is put in after stage k . On the other hand, E_k is the union of finitely many finite sets of edge-disjoint rays (all of which have been computed uniformly). Each set of edge-disjoint rays in this union has v_k appearing at most once in each of its rays. Thus at most two edges containing v_k appear in each of the finitely many rays in this set. Therefore

there are only finitely many edges containing v_k in each of the finite sets of edge-disjoint rays making up E_k . All in all, only finitely many edges in G' contain v_k . \square

Theorem 4.23. ACA_0 proves WIRT_{XES} . Therefore $\Sigma_1^1\text{-AC}_0$ proves IRT_{XES} .

Proof. The second statement follows from the first by the discussion before Theorem 4.14. Given a family $\langle\langle Q_j^k \rangle_{j < k} \rangle_{k > 0}$ such that for each k , the rays Q_j^k , $j < k$, are edge-disjoint, there is a locally finite subgraph H of G and a family $\langle\langle R_j^k \rangle_{j < k} \rangle_{k > 0}$ such that for each k , the R_j^k , $j < k$, are edge-disjoint rays in H (Lemma 4.22). By Lemma 4.20, there is a family $\langle\langle S_j^k \rangle_{j < k} \rangle_{k > 0}$ such that for each $k > 0$, the S_j^k , $j < k$, are vertex-disjoint rays in $L(H)$. By WIRT_{UVS} (which is provable in ACA_0 by Theorem 4.18), $L(H)$ has infinitely many vertex-disjoint rays. Finally by Lemma 4.21, H has infinitely many edge-disjoint rays. Hence G has infinitely many edge-disjoint rays. \square

Finally, we give a proof of IRT_{DED} for directed forests using $\Sigma_1^1\text{-AC}_0$ (recall that IRT_{DED} remains open). We will see that $\Sigma_1^1\text{-AC}_0$ and IRT_{DED} for directed forests are equivalent over $\text{RCA}_0 + \text{I}\Sigma_1^1$ but note that $\Sigma_1^1\text{-AC}_0$ does not imply $\text{I}\Sigma_1^1$. (See Theorem 5.8 and the comment following it).

Theorem 4.24. ACA_0 proves WIRT_{DED} for directed forests. Therefore $\Sigma_1^1\text{-AC}_0$ proves IRT_{DED} for directed forests.

Note the second statement follows from the first by the discussion before Theorem 4.14. We first prove two lemmas.

Lemma 4.25 (ACA₀). Let G be a directed forest and let $R_0 = \langle x_{0,i} \mid i \in \mathcal{Z} \rangle$, $R_1 = \langle x_{1,i} \mid i \in \mathcal{Z} \rangle$ be directed double rays in G . Suppose R_0 and R_1 have an edge $\langle u, v \rangle$ in common. Then there are vertices t and w such $tR_0w = tR_1w$ and R_0 , R_1 have no vertices in common outside of those in $tR_0w = tR_1w$. Note that we allow for the possibility that $t = -\infty$ and/or $w = +\infty$ in the sense that $(-\infty)R = R = R(+\infty)$ for any double ray R . We call $tR_0w = tR_1w$ the intersection of R_0 and R_1 .

Proof. Suppose R_0 , R_1 provide a counterexample. As they have an edge $\langle u, v \rangle$ in common, they lie in the same directed tree T in G and can be viewed as (undirected) double rays in \hat{T} (the underlying graph for T). As they form a counterexample to the lemma, there must be either a first $t \in R_0$ (i.e. earliest in the double ray R_0) such that $tR_0v = tR_1v$ or a last $w \in R_0$ such that $uR_0w = uR_1w$ but R_0 and R_1 have a vertex z in common outside the common interval. The situations are symmetric and we consider the second. The immediate successors x and y of w in R_0 and R_1 , respectively, must be different by our choice of w . Consider now the location of z in R_0 . If it is after x then the paths from w to z in R_0 and w to z in R_1 (both considered now as undirected graphs within \hat{T}) are different as the immediate successor of w in R_0 is x while in R_1 it is either y or a vertex in vR_1w . Thus there are two different paths in \hat{T} from w to z contradicting \hat{T} 's being a tree. If, on the other hand z is before u in R_0 , it must be before $t \in R_0$ and a similar argument provides different paths from t to z in R_0 and R_1 . \square

Lemma 4.26 (ACA₀). There is a computable function f such that given any sequence $\langle S_i \rangle_{i < n}$ of DED rays with subpaths $\langle P_i \rangle_{i < n}$ of length $2n$ in a directed tree T and sequence $\langle R_j \rangle_{j < f(n)}$

of DED rays in T , we can construct a sequence $\langle S'_i \rangle_{i \leq n}$ of DED rays with subpaths $\langle P'_i \rangle_{i \leq n}$ of length $2n + 2$ in T such that P'_i extends P_i at each end for $i < n$. Indeed, we may take $f(n) = 2n^2 + 2^{2n}n! + 1$.

Proof. First we remove all the R_j that contain an edge in any P_i at a cost of at most $2n^2$ many j . Consider any remaining R_j in the second sequence. By Lemma 4.25, its intersections with the S_i are intervals $Q_{j,i}$ of edges in R_j which are disjoint as the S_i are. By our first thinning of the R_j list, none of the $Q_{j,i}$ intersect any of the P_i so each $Q_{j,i}$ must lie entirely above or entirely below P_i . We associate to each R_j a label consisting of the set $C_j = \{i < n \mid Q_{j,i} \neq \emptyset\}$; the elements i of C_j in the order in which the $Q_{j,i}$ (for $i \in C_j$) appear in R_j (in the ordering of \mathcal{Z}) along with a $+$ or $-$ depending on which side of P_i it falls in S_i . We write $Q_{j,i}^s$ for the starting vertex of $Q_{j,i}$ and $Q_{j,i}^e$ for the ending one. As above, we allow the values $\pm\infty$ for these endpoints if the intervals are infinite. There are at most finitely many such labels. In particular, there are at most $2^n n! 2^n$ such labels. Thus, if we have $2^{2n}n! + 1$ many R_j left, at least two of them, say R_a and R_e have the same label say with set C .

Claim: $|C| < 2$.

For the sake of a contradiction, assume we have $k \neq l$ in C with k preceding l in the ordering of C in the label. Say R_a is the ray such that $Q_{a,k}$ is before $Q_{e,k}$ in S_k . Note that $Q_{a,k}$ and $Q_{e,k}$ are edge disjoint as R_a and R_e are. We consider two cases: (1) $Q_{a,l}$ is before $Q_{e,l}$ in S_l and (2) $Q_{e,l}$ is before $Q_{a,l}$ in S_l . We now produce, for each case, two vertices with two distinct sequences (i) and (ii) of adjacent edges in T connecting these two vertices. These sequences are illustrated in Figures 2 and 3. Note that by our assumptions on the orderings of the intervals $Q_{c,d}$ (for $c \in \{a, e\}$ and $d \in \{k, l\}$) as displayed, all of the starting or ending points of $Q_{c,d}$ that appear in our sequences are vertices in one of the rays (i.e. none are $\pm\infty$):

- (1i) Start at $Q_{a,k}^e$ in R_a and go to $Q_{a,l}^e$ then in S_l go to $Q_{e,l}^s$.
- (1ii) Start at $Q_{a,k}^e$ in S_k and go to $Q_{e,k}^s$ then go in R_e to $Q_{e,l}^s$.
- (2i) Start at $Q_{a,k}^s$ in S_k and go to $Q_{e,k}^e$ then in R_e go to $Q_{e,l}^e$ then in S_l go to $Q_{a,l}^e$.
- (2ii) Start at $Q_{a,k}^s$ in R_a and go to $Q_{a,l}^e$.

To see that the two sequences of vertices are different, note for (1) that (1i) contains an edge in $Q_{a,l}$ but (1ii) does not. For (2) note that (2i) contains an edge in $Q_{e,k}$ but (2ii) does not. We now, in each case, view the associated two distinct sequences of vertices with the same endpoints in the underlying (undirected) tree \hat{T} . The only way one can have such sequences in a tree is for one of the sequences to contain some vertices uvv in order. However, any three successive vertices in any of these sequences lie within one of the R_j or S_i or both and so cannot have two instances of the same vertex. The crucial point is that each $Q_{j,i}$ is in both R_j and S_i and has at least two vertices. Any transition along the sequence between an R_j and an S_i (in either order) goes through $Q_{j,i}$ and so any three consecutive vertices are all contained in one R_j or one S_i (or both).

Knowing now that $|C|$ is 0 or 1, we complete the proof of the Lemma. If $|C| = 0$, then both R_a and R_e are disjoint from all the S_i and so we may add on either one of them as S'_n

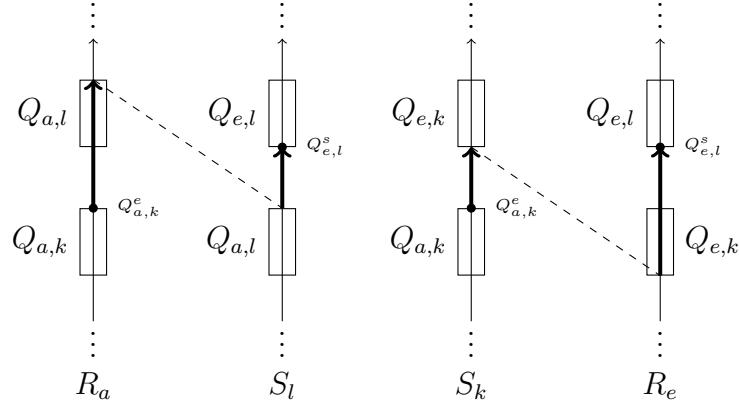


Figure 2: (1i) follows the thick arrows in R_a and S_l . (1ii) follows the thick arrows in S_k and R_e .

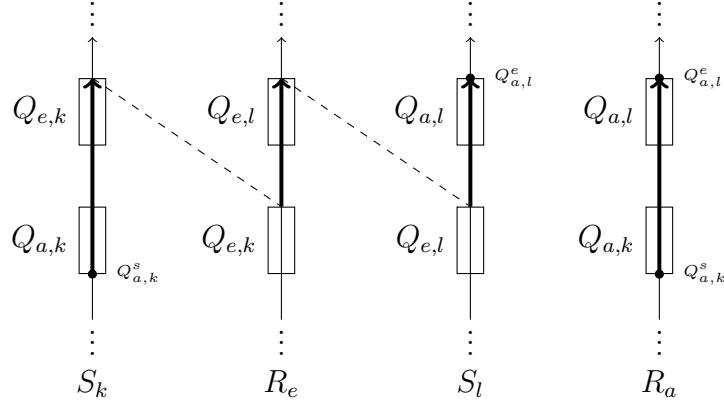


Figure 3: (2i) follows the thick arrows in S_k , R_e and S_l . (2ii) follows the thick arrow in R_a .

with P'_n an arbitrary subpath of length $2n + 2$ while keeping $S_i = S'_i$ and extending the P'_i appropriately for all $i < n$. Otherwise say $C = \{i\}$. Let c be the one of a or e such that $Q_{c,i}$ is closer to P_i . (Remember that they are both on the same side of this interval in S_i by our fixing the label.) Now replace the tail of S_i starting with $Q_{c,i}$ and going away from P_i by the tail of R_c starting with $Q_{c,i}$ and going in the same direction. Let this ray be S'_i . Note that it is disjoint from all the S_j , $j \neq i$ as it contains only edges that are in S_i or R_c neither of which share any edges with such S_j . It is also disjoint from R_d where d is the one of a, e which is not c since all the edges of S'_i are either in R_c or in S_i outside of $Q_{d,i}$ by our choice of $Q_{c,i}$ as closer to P_i . As R_d is also disjoint from all the S_j for $j \neq i$ by our fixing the label, we may define $S'_j = S_j$ and P_j appropriately for $j < n$, $j \neq i$ and $S'_n = R_d$ and choosing P'_n of length $2n + 2$ arbitrarily so as to get the sequence required in the Lemma. \square

Lemma 4.26 provides the inductive step for the following proof:

Proof of Theorem 4.24. Assume we are given a directed forest G and a sequence $\langle \langle R_{k,i} \rangle_{i < k} \rangle_{k \in N}$ such that, for each k , $\langle R_{k,i} \rangle_{i < k}$ is a sequence of k many DED rays in G . If there are infinitely

many of the trees making up G each of which contains some $R_{k,i}$ then we are done. So we may assume that all of them are in one directed tree T . We now wish to define $\langle\langle S_{k,s} \rangle_{k < s}\rangle_{s \in N}$ by recursion such that, for each s , $\langle S_{k,s} \rangle_{k < s}$ is a sequence of s many DED rays with subpaths $P_{k,s}$ of length $2s + 1$ such that for each $s \in N$, $P_{k,s+1}$ extends $P_{k,s}$ at each end so that the $\lim_s P_{k,s}$ form an infinite sequence of DED rays in T as required. Lemma 4.26 provides precisely the required inductive step for the construction since we have the required sequences of DED rays $\langle R_{f(n),i} \mid i < f(n) \rangle$ at each step n of the construction. Once again we just have to note that Lemma 4.26 provides witnesses for the double rays that are composed of finite paths in T and final segments in one direction or the other of some of the $R_{k,i}$ and so they can be found recursively in $(T \oplus \langle\langle R_{k,i} \rangle_{i < k}\rangle_{k \in N})'$. \square

For ease of reference we summarize our proof-theoretic upper bounds on IRT and WIRT in the following theorem.

Theorem 4.27. *The following principles are provable in ACA_0 : WIRT_{UVD} , WIRT_{XYS} , and WIRT_{DED} for directed forests. Therefore the following principles are provable in $\Sigma_1^1\text{-AC}_0$: IRT_{UVD} , IRT_{XYS} , and IRT_{DED} for directed forests.*

Proof. Apply Theorems 4.14, 4.23, Proposition 4.13, and Theorem 4.24. \square

5 Variations on Maximality

5.1 Maximum Cardinality Variants of IRT

We show below that $\text{IRT}_{\text{XYZ}}^*$ proves sufficient induction in order to transcend $\Sigma_1^1\text{-AC}_0$. This implies that $\text{IRT}_{\text{XYZ}}^*$ is strictly stronger than IRT_{XYZ} for certain choices of XYZ (Corollary 5.5). The connection between $\Sigma_1^1\text{-AC}_0$ and graphs is obtained by viewing the set of solutions of an arithmetic predicate as the set of (projections of) branches on a subtree of $N^{<N}$:

Lemma 5.1 ([28, V.5.4]). *If $A(X)$ is an arithmetic formula, ACA_0 proves that there is a tree $T \subseteq N^{<N}$ such that*

$$\begin{aligned} \forall X(A(X) \leftrightarrow \exists f(\langle X, f \rangle \in [T])) \\ \text{and } \forall X(\exists \text{ at most one } f)(\langle X, f \rangle \in [T]). \end{aligned}$$

In fact, as the proof in [28, V.5.4] shows, the required functions f are what are called the minimal Skolem functions and are arithmetically defined uniformly in X and the formula A .

The following easy corollary will be useful.

Lemma 5.2. *If $A(n, X)$ is an arithmetic formula, ACA_0 proves that there is a sequence of subtrees $\langle T_n \rangle_n$ of $N^{<N}$ such that for each $n \in N$,*

$$\begin{aligned} \forall X(A(n, X) \leftrightarrow \exists f(\langle X, f \rangle \in [T_n])) \\ \text{and } \forall X(\exists \text{ at most one } f)(\langle X, f \rangle \in [T_n]). \end{aligned}$$

Proof. Say that $B(Y)$ holds if and only if $A(Y(0), X)$ holds, where X is such that $Y = Y(0) \cap X$. Apply Lemma 5.1 to the arithmetic formula $B(Y)$ to obtain a tree $T \subseteq N^{< N}$. For each $n \in N$, define T_n to be the set of all σ such that $n \cap \sigma \in T$. It is straightforward to check that $\langle T_n \rangle_n$ satisfies the desired properties. \square

Theorem 5.3. $\text{IRT}_{\text{XYZ}}^*$ proves ACA_0^* .

Before proving the above theorem, we derive some corollaries:

Corollary 5.4. $\text{IRT}_{\text{XYZ}}^*$ proves the consistency of $\Sigma_1^1\text{-AC}_0$. Therefore it is not provable in $\Sigma_1^1\text{-AC}_0$.

Proof. Simpson [28, IX.4.6] proves that $\text{ACA}_0 + \text{I}\Sigma_1^1$ implies the consistency of $\Sigma_1^1\text{-AC}_0$. The only use of $\text{I}\Sigma_1^1$ in Simpson's proof is to establish ACA_0^* , so Simpson's proof shows that ACA_0^* implies the consistency of $\Sigma_1^1\text{-AC}_0$. The desired result then follows from Theorem 5.3 and Gödel's second incompleteness theorem. \square

Corollary 5.5. $\text{IRT}_{\text{XYZ}}^*$ is strictly stronger than IRT_{XYZ} for the following choices of XYZ: XYS and UVD.

Proof. We showed in §4 that the specified variants of IRT are provable in $\Sigma_1^1\text{-AC}_0$. On the other hand, none of the IRT^* are provable in $\Sigma_1^1\text{-AC}_0$ (Corollary 5.4). \square

We now prove Theorem 5.3:

Proof that $\text{IRT}_{\text{XYZ}}^$ implies ACA_0^* .* By Proposition 4.3, it suffices to prove the desired result for $\text{IRT}_{\text{UYS}}^*$. To prove ACA_0^* from $\text{IRT}_{\text{UYS}}^*$, begin by using Lemma 5.2 to define for each A a sequence of trees $\langle T_n \rangle_n$ such that for each n and W there is at most one f such that $\langle W, f \rangle \in [T_n]$ and

$$\begin{aligned} & (\exists f)(\langle W, f \rangle \in [T_n]) \\ \Leftrightarrow & W^{[0]} = A \wedge (\forall i \leq n)((W^{[i]})' = W^{[i+1]}) \wedge (\forall i > n)(W^{[i]} = \emptyset). \end{aligned}$$

We want to show that each T_n is ill-founded. Note that if $m < n$ and T_n is ill-founded, then so is T_m . Therefore it suffices to show that for cofinally many n , T_n is ill-founded.

Apply $\text{IRT}_{\text{UYS}}^*$ to the disjoint union $\bigsqcup_n T_n$ to obtain a collection C of Y -disjoint rays of maximum cardinality. We prove that C is infinite. Suppose not. Then there is some maximum m such that C contains a ray in T_m . A ray in T_m can be computably truncated or extended to a branch on T_m , so T_m is ill-founded. Hence T_{m+1} is ill-founded as well (by ACA_0). But then there is a collection of Y -disjoint rays in $\bigsqcup_n T_n$ which has cardinality greater than that of C , contradiction.

We have proved that C is infinite. Next we prove that each T_n has at most one branch. That would imply that each T_n contains at most one ray in C , so C contains rays in cofinally many T_n , as desired.

If T_n has two distinct branches $\langle W_0, f_0 \rangle$ and $\langle W_1, f_1 \rangle$, then $W_0 \neq W_1$ by the “at most one” condition in the definition of the T_n . Consider the least i such that $W_0^{[i]} \neq W_1^{[i]}$. Such i exists by ACA_0 . Note that $0 < i \leq n$ because $W_0^{[0]} = A = W_1^{[0]}$ and $W_0^{[i]} = \emptyset = W_1^{[i]}$ for $i > n$. But then $W_0^{[i-1]} = W_1^{[i-1]}$ and $(W_0^{[i-1]})' \neq (W_1^{[i-1]})'$, contradiction.

This proves that $\text{IRT}_{\text{UYS}}^*$ implies ACA_0^* . In order to prove that $\text{IRT}_{\text{UYD}}^*$ implies ACA_0^* , we modify the above proof by adding to each T_n a branch consisting of new vertices to form a tree S_n . Apply $\text{IRT}_{\text{UYD}}^*$ to $\bigsqcup_n S_n$ to obtain a collection C of Y -disjoint double rays of maximum cardinality. Following the above proof, we may prove that C is infinite and each S_n contains at most one double ray in C . So C contains double rays in cofinally many S_n , as desired. \square

Remark 5.6. The same proof shows that $\text{IRT}_{\text{XYZ}}^*$ implies the following induction scheme: Suppose $\langle T_n \rangle_n$ is a sequence of trees such that

1. T_0 has a unique branch;
2. for all n , the number of branches on T_{n+1} is the same as the number of branches on T_n .

Then for all n , there is a sequence $\langle P_m \rangle_{m < n}$ such that for each $m < n$, P_m is the unique branch on T_m . It also shows that $\text{IRT}_{\text{XYZ}}^*$ implies ACA_0^+ (i.e. closure under the ω -jump) and much more. Indeed, similar ideas prove in Theorem 6.3 that $\text{IRT}_{\text{XYZ}}^*$ implies unique- Σ_1^1 -AC $_0$ (Definition 6.1). $\text{IRT}_{\text{XYZ}}^*$ also implies a similar induction scheme analogous to finite- Σ_1^1 -AC $_0$ (Definition 6.2).

We can prove that even fragments of $\text{IRT}_{\text{DVD}}^*$ give more induction than the specific instances derived in Theorem 5.3.

Theorem 5.7. $\text{IRT}_{\text{DVD}}^*$ (even for directed forests) implies Σ_1^1 over RCA_0 .

Proof. Suppose $\Psi(n)$ is a Σ_1^1 formula such that $\Psi(0)$ and $\forall n(\Psi(n) \rightarrow \Psi(n+1))$ hold. Let $\langle S_i \rangle_{i \in N}$ be a sequence of subtrees of $N^{<N}$ such that S_i is ill-founded if and only if $\Psi(i)$ holds. Let $\langle T_n \rangle_{n \in N}$ be a sequence of subtrees of $N^{<N}$ such that for each n , T_n consists of all sequences $\langle \sigma_i : i \leq n \rangle$ where for each $i \leq n$, σ_i is a vertex in S_i . In T_n there is an edge between $\langle \sigma_i : i \leq n \rangle$ and $\langle \tau_i : i \leq n \rangle$ if and only if for all $i \leq n$, there is an edge between σ_i and τ_i in S_i . It is clear that T_n is ill-founded if and only if $(\forall i \leq n)\Psi(i)$ holds.

For each n , orient each edge in T_n towards its root and add a D -ray of new vertices which starts at its root. This forms a directed tree G_n . If G_n contains a double ray, T_n is ill-founded and $(\forall i \leq n)\Psi(i)$ holds. Furthermore, no two disjoint double rays can lie in the same G_n .

Let G be the directed forest $\bigsqcup_n G_n$. By $\text{IRT}_{\text{DVD}}^*$, there is a sequence $\langle R_i \rangle_i$ of disjoint double rays in G of maximum cardinality, so the sequence may be for $i < k$ for some k or $i \in N$. Since $\Psi(0)$ holds, $\langle R_i \rangle_i$ is nonempty. If $\langle R_i \rangle_i$ is finite, let n be maximal such that G_n contains some R_i . Then $\Psi(n)$ holds, so $\Psi(n+1)$ holds as well. It follows that G_{n+1} contains some double ray, which we can then add to $\langle R_i \rangle_i$ to obtain a larger sequence of disjoint double rays in G for the desired contradiction. Therefore $\langle R_i \rangle_i$ is infinite. Since each G_n contains at most one R_i , infinitely many G_n contain some R_i . Thus $\Psi(n)$ holds for all n . \square

In fact, we have the following equivalences:

Theorem 5.8. The following are equivalent (over RCA_0):

1. $\Sigma_1^1\text{-AC}_0 + \text{I}\Sigma_1^1$;
2. IRT_{DED} for directed forests + $\text{I}\Sigma_1^1$;
3. $\text{IRT}_{\text{DED}}^*$ for directed forests;
4. $\text{IRT}_{\text{DVD}}^*$ for directed forests;
5. IRT_{DVD} for directed forests + $\text{I}\Sigma_1^1$.

Proof. (1) \rightarrow (2) follows from Theorem 4.24. (2) \rightarrow (3) follows from the proof of Proposition 3.17. (3) \rightarrow (4) follows from the observation that the mapping of graphs defined in Lemma 4.4 sends a directed forest to a directed forest. (4) \rightarrow (5) follows from Theorem 5.7 and the proof of Proposition 3.17.

To prove (5) \rightarrow (1), suppose $A(n, X)$ is an arithmetic formula such that $\forall n \exists X A(n, X)$. By Lemma 5.2, there is a sequence $\langle T_n \rangle_n$ of subtrees of $N^{< N}$ such that

$$\begin{aligned} \forall n \forall X (A(n, X) \leftrightarrow \exists f (\langle X, f \rangle \in [T_n])) \\ \text{and } \forall X (\exists \text{ at most one } f) (\langle X, f \rangle \in [T_n]). \end{aligned}$$

By assumption on $A(n, X)$, each T_n is ill-founded. We use $\langle T_n \rangle_n$ to construct a sequence $\langle G_n \rangle_n$ of directed trees as we did in the proof of Theorem 5.7 to construct T_n from the S_i .

By $I\Sigma_1^1$, the directed forest $\bigsqcup_n G_n$ contains arbitrarily many disjoint double rays. Therefore $\bigsqcup_n G_n$ contains infinitely many disjoint double rays $\langle R_k \rangle_k$, by IRT_{DVD} . Note that any double ray in any G_n must contain the ray we added, so any two double rays in the same G_n must intersect. This implies that each R_k belongs to some distinct G_n . Therefore for every m , there is some k and some $n > m$ such that R_k is a double ray in G_n . When we remove the added ray from R_k we are left with a branch in T_n which is of the form $\langle X, f \rangle$ where X consists of witnesses X_i for $i < n$. \square

Since $\Sigma_1^1\text{-AC}_0$ (ATR_0 , even) does not prove $\text{I}\Sigma_1^1$ [28, IX.4.7], it follows that

Corollary 5.9. $\text{IRT}_{\text{DYD}}^*$ (even for directed forests) is not provable in ATR_0 , and strictly implies $\Sigma_1^1\text{-AC}_0$ over RCA_0 .

In the rest of this section we prove implications from $\text{IRT}_{\text{UYD}}^*$ for forests to $\text{IRT}_{\text{UYS}}^*$ for forests and from $\text{IRT}_{\text{UYD}}^*$ to IRT_{UYS} . These will be used in the next section. The following lemma will be used in the two subsequent proofs.

Lemma 5.10 (ACA₀). Suppose G is a graph and $\langle R_i \rangle_i$ is a (possibly finite) maximal sequence of Y -disjoint double rays in G . Let H be the induced subgraph of G consisting of all vertices which do not lie in any R_i . Then any two single rays in the same connected component of H must share a vertex.

Proof. If S_0 and S_1 are disjoint single rays in the same connected component of H , then we can construct a double ray disjoint from all R_i by connecting S_0 and S_1 : Start with a path between S_0 and S_1 of minimum length, then connect it to the tails of S_0 and S_1 which begin at the endpoints of the path. \square

Theorem 5.11. $\text{IRT}_{\text{UYD}}^*$ for forests implies $\text{IRT}_{\text{UYS}}^*$ for forests over RCA_0 . Therefore IRT_{UYD} for forests implies IRT_{UYS} for forests over $\text{RCA}_0 + \text{I}\Sigma_1^1$.

Proof. Let G be a forest. If G happens to have arbitrarily many disjoint double rays, then by $\text{IRT}_{\text{UYD}}^*$, G has infinitely many disjoint double rays. Therefore there is an infinite sequence of disjoint single rays in G . Such a sequence has maximum cardinality, so we are done in this case.

Suppose G does not have arbitrarily many disjoint double rays. By $\text{IRT}_{\text{UYD}}^*$ for forests, there is a sequence $\langle R_i \rangle_{i < j}$ of disjoint double rays in G of maximum cardinality (hence maximal). Let \mathcal{R} be the subgraph of G consisting of the union of all R_i . Let H be the induced subgraph of G consisting of all vertices which do not lie in \mathcal{R} . Decompose H into its connected components $\langle H_i \rangle_i$ (there may only be finitely many). For each i , define H'_i by adding a ray of new vertices to H_i , which begins at the $<_N$ -least vertex in H_i . Define H' to be the disjoint union $\bigsqcup_i H'_i$.

By $\text{IRT}_{\text{UYD}}^*$ for forests, there is a sequence of disjoint double rays in H' of maximum cardinality. If this sequence is infinite, then there is an infinite sequence of disjoint single rays in H because each double ray in the sequence has a tail which lies in H . This is a sequence of disjoint single rays of maximum cardinality in G , so we are done in this case.

Otherwise, suppose $\langle S_k \rangle_{k < l}$ is a disjoint sequence of double rays in H' of maximum cardinality. Consider the following disjoint sequence of single rays in G . First, for each $k < l$, consider the single ray formed by intersecting H and the double ray S_k . Second, for each $i < j$, we can split the double ray R_i into a pair of disjoint single rays in G . This yields a finite sequence $\langle Q_m \rangle_{m < n}$ of disjoint single rays in G .

We claim that $\langle Q_m \rangle_{m < n}$ is a sequence of disjoint single rays in G of maximum cardinality. Suppose there is a larger sequence of disjoint single rays in G . Since G is a forest, any two single rays in G which share infinitely many edges or vertices must share a tail. Therefore there is a single ray Q in this larger sequence which only shares finitely many edges and vertices with each Q_m . Then some tail of Q , say xQ , is vertex-disjoint from each Q_m . In particular, xQ is vertex-disjoint from each R_i , i.e. xQ lies in H . We may extend xQ to a double ray S in H' . We claim that S is disjoint from every S_k . Indeed, since xQ and $S_k \cap H$ are disjoint by construction, they cannot lie in the same connected component of H' (Lemma 5.10). This contradicts the maximality of l . \square

Theorem 5.12. $\text{IRT}_{\text{UVD}}^*$ implies IRT_{UVS} over RCA_0 . Therefore (1) IRT_{UVD} implies IRT_{UVS} over $\text{RCA}_0 + \text{I}\Sigma_1^1$; (2) if any standard model of RCA_0 satisfies IRT_{UVD} , then it satisfies IRT_{UVS} as well.

Proof. Let G be a graph which contains arbitrarily many disjoint single rays. By $\text{IRT}_{\text{UVD}}^*$, there is a sequence of disjoint double rays in G of maximum cardinality. If this sequence is infinite, then there are infinitely many disjoint single rays in G as desired. Otherwise, suppose that $\langle R_i \rangle_{i < j}$ is a sequence of disjoint double rays in G of maximum cardinality j . Following the proof of Theorem 5.11, define the graphs \mathcal{R} , H and H' . By $\text{IRT}_{\text{UVD}}^*$, there is a sequence of disjoint double rays in H' of maximum cardinality.

Case 1. If this sequence is infinite, then G contains infinitely many disjoint single rays because each double ray in the sequence has a tail which lies in H . In this case we are done.

Case 2. Otherwise, H' does not contain arbitrarily many disjoint double rays. Since any two single rays in the same H_i must intersect (Lemma 5.10), we can transform any collection of disjoint single rays in H into a collection of disjoint double rays in H' of equal cardinality by connecting each single ray to the $<_N$ -least vertex in its connected component H_i and then following the ray we added. It follows that H does not contain arbitrarily many disjoint single rays. Fix l such that H does not contain $l + 1$ many disjoint single rays.

Towards a contradiction, we construct a collection of $(j + 1)$ -many disjoint double rays in G as follows. Fix a collection \mathcal{S} of $l + 2j + 4j(j + 1)$ many disjoint single rays in G . First, at most $2j$ of these single rays lie in \mathcal{R} . In fact at most $2j$ of these single rays can have finite intersection with H , because given a collection of disjoint single rays each of which has finite intersection with H , we can obtain a collection of disjoint single rays in \mathcal{R} of the same cardinality by replacing each ray with an appropriate tail. Second, by reasoning analogous to the above, at most l of these single rays can have finite intersection with \mathcal{R} . Therefore, there are at least $4j(j + 1)$ many disjoint single rays in \mathcal{S} each of which have infinite intersection with both \mathcal{R} and H .

Next, choose an edge (u_i, v_i) in each R_i and split R_i into two single rays $u_i R_{i,b}$ and $v_i R_{i,f}$. By the pigeonhole principle, there is some single ray R of the form $u_i R_{i,b}$ or $v_i R_{i,f}$, and at least $2(j + 1)$ many disjoint single rays in \mathcal{S} , each of which have infinite intersection with both R and H . Call these rays $S_0, S_1, \dots, S_{2(j+1)-1}$. Discard all the other rays in \mathcal{S} . Below we describe how to connect pairs of single rays S_k using segments of R to form a collection of $(j + 1)$ -many disjoint double rays in G .

Let x_0, x_1, \dots denote the vertices of R . Since each single ray S_k has infinite intersection with R , by the pigeonhole principle, there is a pair of disjoint rays S_{k_0} and S_{l_0} such that for each tail R' of R , there is a vertex in $S_{k_0} \cap R'$ and a vertex in $S_{l_0} \cap R'$ such that no S_k intersects R between these two vertices. (Formally, we justify this by defining the following coloring recursively. Start from the first vertex in R which is also in some S_k . Search for the next vertex on R which intersects some S_l , $l \neq k$. Then we color 0 with the unordered pair $\{k, l\}$. Then we search for the next vertex on R which intersects some S_m , $m \neq l$ and color 1 with $\{l, m\}$, and so on. Some color $\{k_0, l_0\}$ must appear infinitely often.) Then we commit to connecting S_{k_0} and S_{l_0} (but we do not do so just yet). Applying the pigeonhole principle again, there is a pair of disjoint rays S_{k_1} and S_{l_1} (with k_1, l_1, k_0, l_0 all distinct) such that for each tail R' of R , there is a vertex x in $S_{k_1} \cap R'$ and a first vertex y in $S_{l_1} \cap R'$ (after x in R) such that no S_k , except perhaps S_{k_1}, S_{k_0} or S_{l_0} , intersects R between these two vertices. We may eliminate any elements of S_{k_1} by changing x (if necessary) to the last element of Ry in S_{k_1} . Again we commit to connecting S_{k_1} and S_{l_1} . Repeat this process until we have obtained $j + 1$ pairs of single rays. That is, when we have S_{k_i} and S_{l_i} for an $i < j$, we find $S_{k_{i+1}}$ and $S_{l_{i+1}}$ with k_{i+1} and l_{i+1} distinct from all previous k_m and l_m such that for each tail R' of R there is a vertex $x \in S_{k_{i+1}} \cap R'$ and a first $y \in S_{l_{i+1}} \cap R'$ after x in R such that no S_k , except perhaps S_{k_m} or S_{l_m} for $m \leq i$, intersects R between these two vertices. This process stops when we define k_j and l_j .

Finally, we connect these pairs of single rays in the opposite order in which we defined them: Start by picking some $x^j \in S_{k_j} \cap R$ and some $y^j \in S_{l_j} \cap R$. Then we define a double ray D_j by following ${}^*(S_{k_j}) Ry^j S_{l_j}$, i.e., $D_j := {}^*(x^j S_{k_j}) Ry^j S_{l_j}$. Having defined $D_j, D_{j-1}, \dots, D_{i+1}$, define $D_i := {}^*(x^i S_{k_i}) Ry^i S_{l_i}$, where $x^i \in S_{k_i} \cap R$ and $y^i \in S_{l_i} \cap R$ are chosen as follows: Consider a tail R' of R such that

the union of $x^j Ry^j, \dots, x^{i+1} Ry^{i+1}$ is disjoint from (1) R' ; (2) xS_{k_i} , for each $x \in S_{k_i} \cap R'$; (3) yS_{l_i} for each $y \in S_{l_i} \cap R'$. By choice of k_i and l_i , there are vertices $x^i \in S_{k_i} \cap R'$ and $y^i \in S_{l_i} \cap R'$ such that none of $S_{k_j}, \dots, S_{k_{i+1}}$ or $S_{l_j}, \dots, S_{l_{i+1}}$ intersect $x^i Ry^i$.

It is straightforward to check that each of $D_j, D_{j-1}, \dots, D_{i+1}$ is disjoint from D_i . This process yields disjoint double rays D_j, D_{j-1}, \dots, D_0 in G , contradicting the maximality of j . \square

5.2 Maximal Variants of IRT

Instead of sets of disjoint rays of maximum cardinality, we could consider sets of disjoint rays which are maximal with respect to set inclusion. For uncountable graphs, Halin [11] observed that any uncountable maximal set of disjoint rays is in fact of maximum cardinality (because rays are countable). This suggests another variant of IRT, which we call *maximal IRT*:

Definition 5.13. Let MIRT_{XYZ} be the statement that every X -graph G has a (possibly finite) sequence $\langle R_i \rangle_i$ of Y -disjoint Z -rays which is maximal, i.e., for any Z -ray R in G , there is some i such that R and R_i are not Y -disjoint.

We easily show below that MIRT_{XYZ} implies $\Pi_1^1\text{-CA}_0$, hence MIRT_{XYZ} is much stronger than IRT_{XYZ} or even IRT_{XYZ}^* . For the other direction, MIRT_{XYZ} immediately follows from Zorn's Lemma. This situation is reminiscent of a couple of other maximality results in graph theory often proved using Zorn's Lemma that are equivalent to $\Pi_1^1\text{-CA}_0$ (as in [2]). Our original proofs here of MIRT_{XYZ} from $\Pi_1^1\text{-CA}_0$ (described in [3]) as those in other cases were ad hoc. Formulations of versions of Zorn's Lemma that can be used as in the standard mathematical proofs of these results and are equivalent to $\Pi_1^1\text{-CA}_0$ will appear in [24]. The referee suggested a slicker approach not using Zorn's lemma than the ones we had. We give it below.

Theorem 5.14. $\Pi_1^1\text{-CA}_0$ is equivalent to MIRT_{XYZ} .

Proof that MIRT_{XYZ} implies $\Pi_1^1\text{-CA}_0$. We first prove that MIRT_{XYZ} implies ACA_0 by adapting the proof of Theorem 4.10. If we apply MIRT_{XYZ} instead of IRT_{XYZ} to any of the forests constructed in that proof, we obtain a sequence containing a Z -ray in each tree which constitutes the forest. This is more than sufficient for carrying out the remainder of the proof of Theorem 4.10.

To prove that MIRT_{UVS} implies $\Pi_1^1\text{-CA}_0$, suppose we are given a set A . Consider the disjoint union of all A -computable trees (this exists, by ACA_0). Any maximal sequence of Y -disjoint rays in this forest must contain a ray in each ill-founded A -computable tree. Hence its jump computes the hyperjump T^A . This shows that MIRT_{UVS} implies $\Pi_1^1\text{-CA}_0$. To prove that the other MIRT_{XYZ} imply $\Pi_1^1\text{-CA}_0$, it suffices to exhibit a computable procedure which takes trees $T \subseteq N^{<N}$ to X -graphs T' such that T is ill-founded if and only if T' contains a Z -ray. For MIRT_{UYD} , it suffices to apply the construction of Lemma 4.8 to each tree. For MIRT_{DYZ} , it suffices to orient each of the graphs we constructed above in the obvious way. \square

Proof that $\Pi_1^1\text{-CA}_0$ implies MIRT_{XYZ} . Suppose we are given an X -graph G in a model \mathcal{N} of MIRT_{XVZ} . By [28, VII.2.10] there is a set X such that $X^{[0]} = G$ and $B(X) = \langle N, \{X^{[i]} | i \in N\} \rangle$ is a β -model in \mathcal{N} , i.e. any Σ_1^1 formula Φ with parameters from $B(X)$, i.e. from among the $X^{[i]}$, has a witness in \mathcal{N} if and only some $X^{[j]}$ is such a witness. In \mathcal{N} we build, recursively in X'' , a sequence of disjoint Z -rays in G with each initial segment in $B(X)$. If there are no Z -rays in \mathcal{N} we are done. Otherwise, by our choice of X there is a j such that $X^{[j]}$ is a Z -ray let one such be R_0 . Suppose at stage n we have constructed a sequence $\langle R_i \rangle_{i < m_n}$ in $B(X)$ of disjoint Z -rays for some $m_n \leq n$. If there is a j such that $X^{[j]}$ is a Z -ray containing n as a vertex which is disjoint from the R_i for $i < m_n$, choose one as R_{m_n} and set $m_{n+1} = m_n + 1$. Otherwise, let $m_{n+1} = m_n$ and move on to stage $n+1$. This construction is clearly recursive in X'' . It produces in \mathcal{N} (a possibly finite) sequence $\langle R_i \rangle$ of disjoint Z -rays in G . We show that this sequence is maximal. If R is a Z -ray in \mathcal{N} which is disjoint from every R_i , then go to stage n of the construction, where n is a vertex of R . If we inserted some R_{m_n} during stage n , then R would not be disjoint from $\langle R_i \rangle$. Hence we did not insert any Z -ray during stage n . However, R is a Z -ray in \mathcal{N} that contains n and is disjoint from $\langle R_i \rangle_{i < m_n}$ contradicting $B(X)$ being a β -model in \mathcal{N} .

To prove MIRT_{XEZ} , at stage $n = (u, v)$, we ask for an $X^{[j]}$ which is a Z -ray R edge disjoint from the previous rays and has (u, v) as an edge. The rest of the proof is as above. \square

Remark 5.15. It is easy to see that the variants of IRT_{XYZ} asking for a set of rays which is both of maximum cardinality and maximal (MIRT_{XYZ}^*) for XYS and UVD are each equivalent to $\Pi_1^1\text{-CA}_0$.

6 Relationships Between IRT and Other Theories of Hyperarithmetic Analysis

In this section, we establish implications and nonimplications between variants of IRT and THAs other than $\Sigma_1^1\text{-AC}_0$. One such standard theory is as follows:

Definition 6.1. The theory *unique- $\Sigma_1^1\text{-AC}_0$* consists of RCA_0 and the principle

$$(\forall n)(\exists!X)A(n, X) \rightarrow (\exists Y)(\forall n)A(n, Y^{[n]})$$

for each arithmetic formula $A(n, X)$.

The above theory is typically known as weak- $\Sigma_1^1\text{-AC}_0$ (e.g., [28, VIII.4.12]). We deviate from this terminology to introduce a new choice principle where the requirement for unique solutions is replaced by one for finitely many solutions.

Definition 6.2. The theory *finite- $\Sigma_1^1\text{-AC}_0$* consists of RCA_0 and the principle

$$(\forall n)(\exists \text{ nonzero finitely many } X)A(n, X) \rightarrow (\exists Y)(\forall n)A(n, Y^{[n]})$$

for each arithmetic formula $A(n, X)$. Formally, “ $(\exists \text{ nonzero finitely many } X)A(n, X)$ ” means that there is a nonempty sequence $\langle X_i \rangle_{i < j}$ such that for each X , $A(n, X)$ holds if and only if $X = X_i$ for some $i < j$.

Similarly to $\Sigma_1^1\text{-AC}$, each of these two choice principles are equivalent to ones where A is allowed to be of the form $(\exists!Y)B(n, X, Y)$ or $(\exists \text{ nonzero finitely many } Y)B(n, X, Y)$, respectively. However, unlike $\Sigma_1^1\text{-AC}$ neither of these two principles is equivalent to the version where A is allowed to be Σ_1^1 . Not only would those versions fail to capture the idea that we are dealing with unique or finitely many witnesses and paths through trees but they should be stronger than the stated principles. It is easy to see, for example, that even the unique version with $A \Sigma_1^1$ implies $\Delta_1^1\text{-CA}_0$ (Definition 6.8) which is stronger than unique- $\Sigma_1^1\text{-AC}$ by van Wesep [30].

Since the THA $\Sigma_1^1\text{-AC}_0$ implies finite- $\Sigma_1^1\text{-AC}_0$ which in turn implies unique- $\Sigma_1^1\text{-AC}_0$ whose models are closed under hyperarithmetic reducibility by Proposition 3.9, it follows that finite- $\Sigma_1^1\text{-AC}_0$ is a THA (as is unique- $\Sigma_1^1\text{-AC}_0$). Goh [10] shows that finite- $\Sigma_1^1\text{-AC}_0$ is strictly stronger than unique- $\Sigma_1^1\text{-AC}_0$. We were led to study this version of choice by realizing that a variant of our original proof that $\text{IRT}_{\text{UVS}}^*$ implies unique- $\Sigma_1^1\text{-AC}_0$ worked for the finite version.

Theorem 6.3. $\text{IRT}_{\text{XYZ}}^*$ implies finite- $\Sigma_1^1\text{-AC}_0$ over RCA_0 . (It follows that IRT_{XYZ} implies finite- $\Sigma_1^1\text{-AC}_0$ over $\text{RCA}_0 + \text{I}\Sigma_1^1$, but this is superseded by Theorem 6.7 below.)

Proof. By Proposition 4.3 and Theorem 5.11, it suffices to prove that $\text{IRT}_{\text{UYS}}^*$ for forests implies finite- $\Sigma_1^1\text{-AC}_0$. By Lemma 5.2, it suffices to prove that for any sequence $\langle T_n \rangle_n$ of subtrees of $N^{<N}$ such that each T_n has finitely many branches, a sequence $\langle P_n \rangle_n$ exists with each $P_n \in [T_n]$. As in the proof of Theorem 5.7, we construct a sequence of trees $\langle S_n \rangle_n$ such that for each n , the branches on S_n are precisely those of the form $P_0 \oplus \dots \oplus P_n$ where P_i is a branch on T_i for $i \leq n$.

By $\text{IRT}_{\text{UYS}}^*$ for forests there is a sequence $\langle R_k \rangle_k$ of Y -disjoint rays in $\bigsqcup_n S_n$ of maximum cardinality. We claim that $\langle R_k \rangle_k$ is infinite. If not, let m be least such that there is no R_k in S_m . Then we can increase the cardinality of $\langle R_k \rangle_k$ by adding any ray R from S_m while maintaining disjointness by our choice of n . The point here is that if $R \cap R_k \neq \emptyset$ for any k then they are both in S_m as the trees S_n are disjoint and there are no edges between them. Therefore $\langle R_k \rangle_k$ has a ray in infinitely many S_n . Thus we may construct the desired sequence $\langle P_n \rangle_n$ recursively by searching at stage n for an R_k in S_m for some $m > n$ and take P_n to be the branch in T_n which shares a tail with the n th coordinate of R_k . \square

Another THA which follows from $\text{IRT}_{\text{XYZ}}^*$ is *arithmetic Bolzano-Weierstrass* (ABW_0).

Definition 6.4. The theory ABW_0 consists of RCA_0 and the following principle: If $A(X)$ is an arithmetic predicate on 2^N , either there is a finite sequence $\langle X_i \rangle_i$ which contains every X such that $A(X)$ holds or there is an X such that every one of its neighborhoods has two Y such that $A(Y)$ holds. Such an X is called an accumulation point of the class $\{X \mid A(X)\}$.

Friedman [8] introduced ABW_0 and asserted that it follows from $\Sigma_1^1\text{-AC}_0$ (with unrestricted induction). Conidis [5] proved Friedman's assertion and established relationships between ABW_0 and most then known THAs. Goh [10] shows that $\text{ABW}_0 + \text{I}\Sigma_1^1$ implies finite- $\Sigma_1^1\text{-AC}_0$. We do not know if ABW_0 is strictly stronger than finite- $\Sigma_1^1\text{-AC}_0$.

The following lemmas will be useful in deriving ABW_0 from $\text{IRT}_{\text{XYZ}}^*$. We first describe a connection between sets of solutions of arithmetic predicates and rays in trees.

Lemma 6.5 (ACA_0). *Suppose $A(X)$ is an arithmetic predicate. Then there is a tree $T \subseteq N^{<N}$ such that if there is a sequence of distinct solutions of $A(X)$, then there is a sequence of Y -disjoint single rays in T of the same cardinality, and vice versa.*

Proof. By Lemma 5.1, there is a tree $T \subseteq N^{<N}$ such that

$$\begin{aligned} \forall X(A(X) \leftrightarrow \exists f(\langle X, f \rangle \in [T])) \\ \text{and } \forall X(\exists \text{ at most one } f)(\langle X, f \rangle \in [T]). \end{aligned}$$

If $\langle X_i \rangle_i$ is a sequence of distinct solutions of $A(X)$, then, as the required f_i are arithmetic uniformly in the X_i , there is a sequence of distinct branches $\langle \langle X_i, f_i \rangle \rangle_i$ on T of the same cardinality.

By taking an appropriate tail of each branch, we obtain a sequence $\langle R_i \rangle_i$ of vertex-disjoint (hence edge-disjoint) single rays in T of the same cardinality with each one being a tail of $\langle X_i, f_i \rangle$: As no two distinct branches in a tree can have infinitely many vertices in common, simply take R_n to be the tail of $\langle X_n, f_n \rangle$ starting after all vertices it has in common with any $\langle X_i, f_i \rangle$, $i < n$.

Conversely, suppose there is a sequence $\langle R_i \rangle_i$ of Y -disjoint single rays in T . For each R_i , we define a branch on T which corresponds to it as follows. Let x be the vertex in R_i which is closest to the root of T . Then we can extend xR_i to the root to obtain a branch $\langle X_i, f_i \rangle$ on T . We claim that $\langle X_i \rangle_i$ is a sequence of distinct solutions of $A(X)$. For each $i \neq j$, since R_i and R_j are Y -disjoint, they cannot share a tail. So $\langle X_i, f_i \rangle$ and $\langle X_j, f_j \rangle$ must be distinct. Since for each X , there is at most one f such that $\langle X, f \rangle$ is a branch on T , it follows that $X_i \neq X_j$ as desired. \square

The second lemma is essentially the well-known fact that the Bolzano-Weierstrass theorem is provable in ACA_0 :

Lemma 6.6 ([28, III.2.7]). *ACA_0 proves that if $\langle X_n \rangle_n$ is a sequence of distinct elements of 2^N , then there is some Z which is an accumulation point of $\{X_n \mid n \in N\}$.*

Theorem 6.7. IRT_{XYZ}^* implies ABW_0 over RCA_0 . Therefore IRT_{XYZ} implies ABW_0 over $\text{RCA}_0 + \text{I}\Sigma_1^1$.

Proof. By Proposition 4.3, it suffices to show that the undirected variants of IRT^* imply ABW_0 . Suppose $A(X)$ is an arithmetic predicate on 2^N such that no finite sequence $\langle X_i \rangle_i$ contains every X such that $A(X)$ holds. By Lemma 6.5, there is a tree $T \subseteq N^{<N}$ such that for any sequence of distinct solutions of $A(X)$, there is a sequence of Y -disjoint single rays in T of the same cardinality, and vice versa.

By IRT_{UYS}^* , or by IRT_{UYD}^* and Theorem 5.11, there is a sequence of Y -disjoint single rays in T of maximum cardinality. This yields a sequence of distinct solutions of $A(X)$ of the same cardinality.

If this sequence is finite, then there is a solution Y of $A(X)$ not in the sequence by our assumption. Hence there is a sequence of distinct solutions of $A(X)$ of larger cardinality, which yields a sequence of Y -disjoint single rays in T of larger cardinality for the desired contradiction.

Thus there is an infinite sequence $\langle X_n \rangle_n$ of distinct solutions of A . By Lemma 6.6, there is an accumulation point of $\{X_n \mid n \in N\}$, which is of course an accumulation point of $\{X \mid A(X)\}$, as desired. \square

We now turn our attention to nonimplications. One prominent theory of hyperarithmetic analysis is the scheme of Δ_1^1 -comprehension (studied by Kreisel [14]):

Definition 6.8. The theory $\Delta_1^1\text{-CA}_0$ consists of RCA_0 and the principle

$$(\forall n)(\Phi(n) \leftrightarrow \neg\Psi(n)) \rightarrow \exists X(n \in X \leftrightarrow \Phi(n))$$

for all Σ_1^1 formulas Φ and Ψ .

Theorem 6.9. $\Delta_1^1\text{-CA}_0 \not\vdash \text{IRT}_{XYZ}, \text{IRT}_{XYZ}^*$.

Proof. Conidis [5, Theorem 3.1] constructed a standard model which satisfies $\Delta_1^1\text{-CA}_0$ but not ABW_0 . By Theorem 6.7, this model does not satisfy IRT_{XYZ}^* . Since standard models satisfy full induction, this model does not satisfy IRT_{XYZ} either (by Proposition 3.17). \square

Theorem 6.10. $\text{ABW}_0 \not\vdash \text{IRT}_{XYZ}, \text{IRT}_{XYZ}^*$.

Proof. By Propositions 4.3 and 3.17, it suffices to show that $\text{ABW}_0 \not\vdash \text{IRT}_{UYZ}$. Van Wesep [30, I.1] constructed a standard model \mathcal{N} which satisfies unique- $\Sigma_1^1\text{-AC}_0$ but not $\Delta_1^1\text{-CA}_0$. Conidis [5, Theorem 4.1], using the approach of [18], showed that \mathcal{N} satisfies ABW_0 . We show below that \mathcal{N} does not satisfy IRT_{UYZ} .

In order to define \mathcal{N} , van Wesep constructed a tree T^G and branches $\langle f_i^G \rangle_{i \in \mathbb{N}}$ of T^G such that (1) \mathcal{N} contains T^G and infinitely many (distinct) f_i^G (see [30, pg. 13 l. 1–11]); (2) \mathcal{N} does not contain any infinite sequence of distinct branches of T^G (see [30, pg. 12 l. 7–9] and Steel [29, Lemma 7].) Then T^G is an instance of IRT_{UYS} in \mathcal{N} which has no solution in \mathcal{N} . This shows that \mathcal{N} does not satisfy IRT_{UYS} for trees. (The reader who wants to follow the details of the proofs in [5] and [30, I.1] should look at the presentation of the basic methods in [18].)

Since \mathcal{N} is a standard model, it satisfies full induction. By Theorem 5.11, it follows that \mathcal{N} does not satisfy IRT_{UYD} for forests. \square

Figure 4 illustrates some of our results. In order to simplify the diagram, we have omitted all variants of IRT except IRT_{UVS} .

7 Isolating the Use of $\Sigma_1^1\text{-AC}_0$ in Proving IRT

In this section we present further proof-theoretic results on SCR and WIRT (Definition 4.12).

Proposition 7.1. SCR_{XYZ} implies ACA_0 .

Proof. By Proposition 4.13, it suffices to establish the desired result for the undirected variants of SCR . The proofs are almost identical to that of Theorem 4.10. There, we applied IRT_{UYZ} to forests $G = \bigsqcup_n T_n$, where each T_n contains a Z -ray, and no two Z -rays in T_n can be Y -disjoint. Any infinite sequence of Y -disjoint Z -rays in G must contain a Z -ray in cofinally many graphs T_n . Therefore from such a sequence we can uniformly compute Z -rays in cofinally many graphs T_n , which establishes ACA_0 by the construction of $\bigsqcup_n T_n$. If we assume SCR_{UYZ} instead of IRT_{UYZ} , we only have access to a sequence $\langle X_k \rangle_{k \in \mathbb{N}}$ such that for each k , X_k is a set of k Y -disjoint Z -rays in G . From such a sequence we can still uniformly compute Z -rays in cofinally many graphs T_n , because for any k , X_{k+1} must contain a Z -ray in some T_n , $n \geq k$. \square

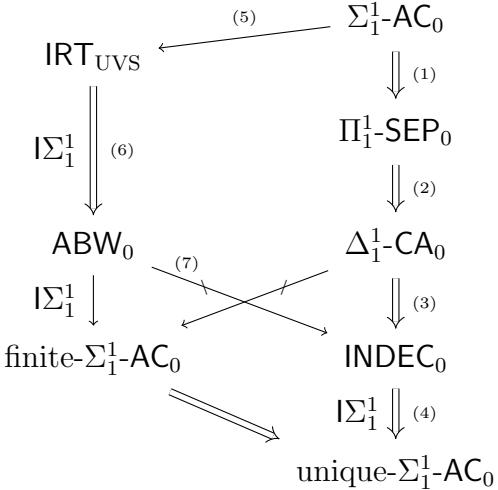


Figure 4: Partial zoo of theories of hyperarithmetic analysis. Single arrows indicate implication while double arrows indicate strict implication. The references for the above results are as follows: (1, 2) Montalbán [16, Theorems 2.1, 3.1]; (3, 4) Montalbán [15, Theorem 2.2], Neeman [18, Theorems 1.2, 1.3, 1.4], see also Neeman [19, Theorem 1.1]; (5) Theorem 4.18; (6) Theorems 6.7, 6.10; (7) Conidis [5, Theorem 4.1]. All results concerning finite- $\Sigma_1^1\text{-AC}_0$ are in Goh [10].

By Theorem 4.27, Proposition 7.1, and the observation that $\text{SCR}_{\text{XYZ}} + \text{WIRT}_{\text{XYZ}}$ and IRT_{XYZ} are equivalent over RCA_0 , we obtain

Corollary 7.2. *SCR_{XYZ} and IRT_{XYZ} are equivalent over RCA_0 for the following choices of XYZ: XYS and UVD.*

We now turn our attention to WIRT_{XYZ} . We know that WIRT_{XYS} and WIRT_{UVD} are provable in ACA_0 (Theorem 4.27). On the other hand WIRT_{DVD} and WIRT_{DED} are open, because $\Sigma_1^1\text{-AC}_0 + \text{WIRT}_{\text{XYZ}}$ implies IRT_{XYZ} , and IRT_{DVD} and IRT_{DED} are open (see comments after Theorem 4.1). We do not have an upper bound on the proof-theoretic strength of WIRT_{UED} (an upper bound on WIRT_{UED} would yield an upper bound on IRT_{UED} , which we do not currently have).

We do not know if any WIRT_{XYZ} is equivalent to ACA_0 . In an effort to clarify the situation, we define an apparent strengthening of WIRT_{XYZ} and show that it implies ACA_0 :

Definition 7.3. Let nonuniform- WIRT_{XYZ} be the assertion that if G is an X -graph and there is a sequence of Z -rays R_0, R_1, \dots in G such that for each k , there are i_0, \dots, i_k such that R_{i_0}, \dots, R_{i_k} are Y -disjoint, then G has infinitely many Y -disjoint Z -rays.

Every instance of WIRT_{XYZ} is also an instance of nonuniform- WIRT_{XYZ} , so nonuniform- WIRT_{XYZ} implies WIRT_{XYZ} . Conversely, we have

Proposition 7.4. $\text{ACA}_0 + \text{WIRT}_{\text{XYZ}}$ implies nonuniform- WIRT_{XYZ} .

Proof. Suppose G is an instance of nonuniform- WIRT_{XYZ} , i.e., G is an X -graph and $\langle R_n \rangle_{n \in N}$ is a sequence of Z -rays in G such that for each k , there are i_0, \dots, i_k such that R_{i_0}, \dots, R_{i_k} are

Y -disjoint. Then ACA_0 can find such i_0, \dots, i_k uniformly in k . Therefore by ACA_0 , G is an instance of WIRT_{XYZ} . By WIRT_{XYZ} , G has infinitely many Y -disjoint Z -rays as desired. \square

Theorem 7.5. *Nonuniform-WIRT_{XYZ} implies ACA_0 over RCA_0 . It follows that nonuniform-WIRT_{XYS} and nonuniform-WIRT_{UV} are both equivalent to ACA_0 over RCA_0 .*

Proof. By Theorem 4.27 and Proposition 7.4, ACA_0 implies nonuniform-WIRT_{XYS} and nonuniform-WIRT_{UV}. To show that nonuniform-WIRT_{XYZ} implies ACA_0 , it suffices to consider the undirected versions of nonuniform-WIRT (by Lemma 4.2). First, we prove that nonuniform-WIRT_{UYS} implies ACA_0 by constructing a computable instance of nonuniform-WIRT_{UVS} such that every nonuniform-WIRT_{UES} solution computes \emptyset' . (The desired result follows by relativization.) We use a variation of the graph used in the analogous result in Theorem 4.10.

Construction of $G = (V, E)$: $V = \{0^n \mid n > 0\} \cup \{n^\frown s^\frown 0^t \mid n > 0 \text{ and some number below } n \text{ is enumerated into } \emptyset' \text{ at stage } s, \text{ and either } t \leq s \text{ or } \emptyset'_s \upharpoonright n = \emptyset'_t \upharpoonright n\}$. $E = \{(0^n, 0^{n+1}) \mid n > 0\} \cup \{(n^\frown s^\frown 0^t, n^\frown s^\frown 0^{t+1}) \mid n^\frown s^\frown 0^t, n^\frown s^\frown 0^{t+1} \in V\} \cup \{(n^\frown s^\frown 0^t, 0) \mid n^\frown s^\frown 0^t \in V \text{ and } n^\frown s^\frown 0^{t+1} \notin V\}$. G is clearly computable.

Verification: It is clear that there is exactly one ray $R_{\langle n, s \rangle}$ in G beginning with $n^\frown s$ for $n^\frown s \in V$ and the sequence $\langle R_{\langle n, s \rangle} \rangle_{n^\frown s \in V}$ is also computable. Note that if $\emptyset' \upharpoonright n = \emptyset'_s \upharpoonright n$ then this ray is $\langle n^\frown s^\frown 0^t \rangle_{t \in N}$. Otherwise, it is $\langle n^\frown s^\frown 0^t \rangle_{n^\frown s^\frown 0^t \in V} \cap \langle 0^n \rangle_{n > 0}$. Next observe that for each $k \in N$, $\{\langle n, i \rangle \mid i < n \leq k \text{ and } i \in \emptyset'\}$ is Σ_1^0 and contained in $k \times k$ and so is a set by bounded Σ_1^0 comprehension [28, II.3.9]. Thus the finite function taking $n > l$ (the least number in \emptyset') to the last stage s_n at which an $i < n$ is enumerated in \emptyset' is also (coded by) a finite set. So we have, for each k , a sequence of V -disjoint rays $\langle R_{n, s_n} \rangle_{l < n \leq l+k}$ of length k as required for the hypothesis of nonuniform-WIRT_{UVS}.

Suppose then that S_i is the sequence of rays in a solution for nonuniform-WIRT_{UES}. We wish to compute \emptyset' from this solution. As the S_i are E -disjoint at most one of them contains the edge $(0, 00)$. So by eliminating that one, we can assume none of the S_i contain $(0, 00)$. If any of the remaining rays contain some edge of the form $(0^j, 0^{j+1})$ for $j > 0$ then (as it does not contain $(0, 00)$) it must contain $(0^k, 0^{k+1})$ for every $k \geq j$. Thus there can be at most one such ray among the remaining S_i and so we can discard it and assume there are no such rays in our list. No remaining ray can have 0 as its first vertex as if it did its second vertex would have to be of the form $n^\frown s^\frown 0^t$ with $n^\frown s^\frown 0^{t+1} \notin V$. Any continuation of this sequence would have to follow the $n^\frown s^\frown 0^r$ with r descending from t and so would have to terminate at $n^\frown s$ and not be a ray. Thus all the remaining S_i are of the form $\langle n_i^\frown s_i^\frown 0^{t+j} \rangle_{j \in N}$ for some t with $n_i \neq n_k$ for $i \neq k$. So the remaining S_i witness the conclusion of nonuniform-WIRT_{UVS} as desired.

As we can replace the first vertex of S_i by the sequence beginning with $n^\frown s$ and ending with its second vertex, we know that $\emptyset' \upharpoonright n_i = \emptyset'_s \upharpoonright n_i$. Since the sequence S_i and so that of the n_i is infinite, given any m we can find an $n_i > m$ and so compute $\emptyset' \upharpoonright m$ as $\emptyset'_{n_i} \upharpoonright m$ as required.

To show that nonuniform-WIRT_{UYD} implies ACA_0 , define G as above. Consider the graph G' gotten by adding on for each $n^\frown s \in V$ new vertices $x_{n, s, k}$ for $k > 0$ and edges $(n^\frown s, x_{n, s, 1})$ and $(x_{n, s, k}, x_{n, s, k+1})$ for $k > 0$. The witnesses for the hypothesis of nonuniform-WIRT_{UVS} in G supply ones for nonuniform-WIRT_{UV} by tacking on the $x_{n, s, k}$ before $n^\frown s$ in reverse order. The witnesses for the conclusion of nonuniform-WIRT_{UED} can be converted into ones for the

conclusion of nonuniform-WIRT_{UES} in G by removing the new vertices. So once again we can compute \emptyset' . \square

We are unable to show that WIRT_{XYZ} implies ACA₀, but we can prove the following:

Theorem 7.6. *WIRT_{XYZ} is not provable in RCA₀.*

Proof. By Proposition 4.13, it suffices to consider the undirected variants of WIRT. For all these variants it suffices to construct a computable graph G on \mathbb{N} and a computable sequence $\langle \langle X_i^k \rangle_{i < k} \rangle_{k \in \mathbb{N}}$ such that (1) for each $k \in \mathbb{N}$, the X_i^k for $i < k$ are pairwise vertex-disjoint double rays in G and (2) there is no computable sequence $\langle R_j \rangle_{j \in \mathbb{N}}$ of edge-disjoint single rays in G . It is clear that the G constructed for this C is a counterexample to each WIRT_{UYZ} in the standard model of RCA₀ with second order part the recursive sets. Of course, as this model is standard, WIRT_{XYZ} is not provable in RCA, RCA₀ plus induction for all formulas.

The computable construction will be a finite injury priority argument. At the end of stage s of our construction, for each $i < k \leq s$, we will have defined a path $P_{i,s+1}^k$ with lengths strictly increasing with s which is intended to be a segment of the double ray X_i^k . We think of these paths P_s as having domain a segment $[u, v]$ of \mathbb{Z} containing $[-s, s]$. Its endpoints are $P_s(u)$ and $P_s(v)$. The intention is that the $X_i^k = \bigcup_s P_{i,s}^k$ will be the desired double rays such that, for each k , the X_i^k for $i < k$ will be vertex-disjoint. We will also have put all numbers less than s in as vertices in at least one of these $P_{i,s}^k$. In future stages, we will not add any edges between vertices which are currently in any $P_{i,s}^k$ for $i < k$. Thus G will be a computable graph given by the union of the double rays $X_i^k = \bigcup_s P_{i,s}^k$. We let G_s be the graph defined so far, i.e. $\bigcup\{P_{i,s}^k \mid i < k < s\}$. We let $G_s^{>t}$ be its subgraph defined as the union of the $P_{0,s}^k, \dots, P_{k-1,s}^k$ for $k > t$ and similarly for $k < t$ and other interval notations.

We say that the *disjointness condition, d.c., holds at stage s* if for every $m < s$ and distinct n and $n' < m$, $P_{n,s}^m$ and $P_{n',s}^m$ are vertex-disjoint. Otherwise we say we have *violated the d.c.* Clearly, if we never violate the d.c. the X_n^m (for fixed m) are pairwise vertex-disjoint. We arrange the construction so that we obviously never violate the d.c.

So it suffices to also meet the following requirements:

Q_e : If R_0, R_1, \dots is a computable sequence of single rays in G defined by Φ_e , i.e. $\Phi_e(i, n) = R_i(n)$ then the R_i are not edge-disjoint.

The requirements Q_e are listed in order of priority. During our construction, if all else fails, we will attempt to satisfy each Q_e at some stage s by *merging* certain rays X_i^k and X_j^l using vertices x and $y \neq x$ which are endpoints of $P_{i,s}^k$ and $P_{j,s}^l$, respectively. We do this by adding the least new number r (*the merge point of this merger*) as a vertex of G as well as edges (x, r) and (y, r) which are appended to each $P_{r,s}^q$ with x or y , respectively, as an endpoint. We also ensure that $P_{i,t}^k$ and $P_{j,t}^l$ henceforth agree after the vertex r as they grow in the corresponding directions.

Without loss of generality, and to simplify notation later, we make the assumption that if $\Phi_{e,s}(i, u)$ is convergent for any e , i and u then so is $\Phi_{e,s}(i, u')$ for every $u' < u$.

Construction. At stage s of the construction, we are given a finite graph G_s consisting of, for each $k < s$, finite vertex-disjoint paths $P_{0,s}^k, \dots, P_{k-1,s}^k$ as described above. We let $f_s(P)$

be the final stage before s at which Q_e was initialized. For notational convenience when s and e are specified we simply write f for $f_s(e)$.

First, we act for the requirement Q_e of highest priority with $e < s$ which requires attention as described below. All requirements Q_e are initialized and unsatisfied at stage 0 and are initialized and declared to be unsatisfied whenever we act for a $Q_{e'}$ with $e' < e$.

We say that Q_e requires attention at stage s if Q_e is not satisfied and there are $a, u, x, b, v, y < s$ such that $u > 0$, $\Phi_{e,s}(a, u) \downarrow$, $\langle \Phi_{e,s}(a, n) \rangle_{n \leq u}$ is a path in G_s disjoint from $G_s^{< f}$ which can be extended in only one way to a maximal path in G_s and this extension eventually reaches an $x \notin G_s^{< f}$ which is an endpoint of some $P_{i,s}^k$ for $k \geq f$ and similarly for b, v and y for some $P_{j,s}^l$ such that x and y are not both endpoints of the same P_r^q . We also require that the merger using x and y would not violate the d.c. We then let $a(e, s)$ etc. be the associated witnesses for the least such computation. In this case, the actions for Q_e is to perform the merger using $x(e, s)$ and $y(e, s)$ as defined above and declare Q_e to be satisfied.

Finally, for each w , in turn, which is an endpoint of any the paths P_i^k as now defined we extend those paths by taking the least new number z which we append after w in each of these paths (and so add (w, z) as a new edge). For each $i < s$, in turn, we also take the next $2s + 1$ least new numbers and let the $P_i^s(n)$ be these numbers in order for $n \in [-s, s]$. This defines the $P_{i,s+1}^k$ for $i < k < s + 1$ and completes stage s of the construction. As promised we let $X_i^k = \bigcup_s P_{i,s}^k$.

Verification. It is clear that, for each $i < k \in \mathbb{N}$, X_i^k is a double ray and that G is a computable graph consisting of the union of these rays. It is also clear that by construction we never violate the d.c. and so for each k the X_i^k for $i < k$ are pairwise vertex disjoint as required. Thus we only need to prove that we meet each Q_e .

We now state a series of facts about G each of which follows immediately (or by simple inductions) from the construction and previous facts on the list.

Lemma 7.7. *Every vertex r which is a merge point has exactly three neighbors and they are the x and y used in the merger and the z added on after r in the final part of the action at the merger stage. In addition, no endpoint of any $P_{i,s}^k$ is a merge point and every vertex which is not a merge point has exactly two neighbors.*

If $u \in G_s$ and so $u \in P_{i,s}^k$ for some $i < k < s$, then for any $j < l$, $u \in X_j^l \Leftrightarrow u \in P_{j,s}^l$ and if so $l < s$.

If $(u, v) \in G_s$ then not both u and v are merge points. If neither are merge points then $\forall k \forall i < k (v \in P_{i,s}^k \Leftrightarrow u \in P_{i,s}^k)$. If one, say u , is a merge point r for a merger at some stage t necessarily less than s using some x and y with (r, z) the edge added on at the end of stage t , then the other (v) is x, y or z ; $\forall k \forall i < k (r \in P_{i,t+1}^k \Leftrightarrow x \in P_{i,t}^k \vee y \in P_{i,t}^k); \forall k \forall i < k (r \in P_{i,t+1}^k \Leftrightarrow r \in P_{i,s}^k \Leftrightarrow z \in P_{i,t+1}^k \Leftrightarrow z \in P_{i,s}^k)$.

Any ray in G which begins in G_s remains in G_s until it reaches an endpoint of some $P_{i,s}^k$.

Each requirement acts at most once after the last time it is initialized. So, by induction, each requirement acts and is initialized only finitely often. Thus there is an infinite sequence w_i such that at each w_i we act for some Q_e and we never act for any $Q_{e'}$ with $e' \leq e$ afterwards. The observation to make here is that if there were a last stage w at which we act for any Q_e then there would be no mergers of any X_i^k and X_j^l for $k, l > w$. In this case all the X_i^k , $i < k$ for $k > w$ would be disjoint. It is then easy to see from the observations above that at some stage after w we would act for a Q_e with $e > w$ such that $\Phi_e(0, n) = X_i^k(n)$ and

$\Phi_e(1, n) = X_j^l(n)$ for $k, l > w$.

If w is one of the w_i just defined, $z \in G^{<w}$ and $(z, z') \in G$ then $z' \notin G^{\geq w}$. The point to notice here is that by construction no merger at a stage $s > w$ can use any $x \in G^{<w}$ as every Q_e which can act after w is initialized at w by the choice of the w_i . Thus any path in G which starts with a $z \in G^{<w}$ never enters $G^{\geq w}$. \square

Suppose now that Φ_e is total and defines a sequence of edge-disjoint single rays $\langle R_i \rangle_{i \in \mathbb{N}}$ in G . By Lemma 7.7, we may choose a $w = w_c$ for some c after which no $Q_{e'}$ for $e' \leq e$ ever acts again.

If Q_e is not satisfied at the end of stage w , we argue that we act for it later to get a contradiction. Once a ray begins in $G^{<w}$, it must stay there by Lemma 7.7 and once beyond all the (finitely many) merge points in $G^{<w}$ (none can be put in after stage w and no ray has repeated vertices) it remains in some X_i^k for $k < w$ with which it shares a tail (as each vertex which is not a merge point has exactly two neighbors). So by the edge-disjointness of the R_i there is an a such that $\Phi_e(a, 0) = R_a(0)$ is not in $G^{<w}$. It starts in $G^{<w'}$ with $w' = w_{c'}$ for some $c' > c$ and so shares a tail with some P_j^k with $w_c \leq k < w_{c'}$. Similarly there is b such that R_b begins in some $G^{\geq w_d}$ with $d > c'$ and shares a tail with some X_j^l with $w_d \leq l$. So eventually we have a stage s such that $\Phi_{e,s}(a, u) \downarrow$ and $\Phi_{e,s}(b, v) \downarrow$ define paths in G_s which go beyond the points by which R_a and R_b share tails with X_i^k and X_j^l , respectively, and after which neither tail contains a merge point and so these paths have unique extensions to paths in G_s (determined by the appropriate tails of X_i^k and X_j^l) ending with the endpoints of $P_{i,s}^k$ and $P_{j,s}^l$, respectively. Finally, note that the merger of X_i^k and X_j^l at s would not violate d.c. as by Lemma 7.7 $P_{i,s}^k$ can share vertices only with $P_{n,s}^m$ with $w_c \leq m < w_{c'}$ and $P_{j,s}^l$ can share vertices only with $P_{n',s}^m$ with $m \geq w_d > w_{c'}$. Thus at stage s we would act to satisfy Q_e for the desired contradiction.

So Q_e is satisfied at w and was satisfied at some $s \leq w$. With the notations as at s , $\Phi_e(a, u)$ and $\Phi_e(b, v)$ define initial segments of R_a and R_b which can each be extended in only one way to maximal paths in G_s eventually reaching the vertices x and y , respectively, as described at s . The merger performed at s puts the merge point r in both $P_{i,s+1}^k$ and $P_{j,s+1}^l$ and so in R_a and R_b (as the successor of x and y , respectively, as by Lemma 7.7 r is the only neighbor in G of x other than its predecessor in R_a and similarly for y and R_b). We then add in (r, z) to G at stage s . By Lemma 7.7, the only neighbors of r are x , y and z so any continuation of R_a and R_b after r must produce a shared edge, as R_a can continue only with either (y, r) which is an edge of R_b or with (r, z) and R_b can continue only with (x, r) which is an edge of R_a or (r, z) . This yields the final contradiction. \square

8 Open Questions

In addition to the variations of the Halin type theorems investigated here that remain open problems of graph theory (IRT_{DVD} and IRT_{DED}) the most intriguing computational and reverse mathematical questions are about either separating the variants or providing additional reductions or equivalences among the IRT_{XYZ} and $\Sigma_1^1\text{-AC}_0$. Clearly the most important issue is deciding if any (or even all) the IRT_{XYZ} which are known to be provable in $\Sigma_1^1\text{-AC}_0$ are actually equivalent to it. We extend this problem to include the $\text{IRT}_{\text{XYZ}}^*$ and $\text{I}\Sigma_1^1$.

Question 8.1. Can one show that any of the IRT_{XYZ} which are provable in $\Sigma_1^1\text{-AC}_0$ (IRT_{XYS} and IRT_{UVD}) do not imply $\Sigma_1^1\text{-AC}_0$ over RCA_0 or even over $\text{RCA}_0 + \text{I}\Sigma_1^1$? An intermediate result might be that IRT_{XYZ}^* (for one of these versions) does not imply $\Sigma_1^1\text{-AC}_0$ over RCA_0 .

Should any of these IRT_{XYZ} be strictly weaker than $\Sigma_1^1\text{-AC}_0$, the question would then be to determine the relations among the IRT_{XYZ} and analogously the IRT_{XYZ}^* .

Question 8.2. Can any additional arrows be added to Figure 1 over RCA_0 or $\text{RCA}_0 + \text{I}\Sigma_1^1$? (This includes the question of whether $\text{RCA}_0 \vdash \text{IRT}_{UVD} \rightarrow \text{IRT}_{UVS}$.)

As we noted in Remark 4.9 there is an apparent additional reduction in Bowler, Carmesin, Pott [4, pg. 21. 3–7]. They use an intermediate reduction to locally finite graphs in the sense of relying on the fact that if a graph has arbitrarily many disjoint rays it has a locally finite subgraph with arbitrarily many disjoint rays. This is the principle to which that Remark refers. It plus ACA_0 is a THA but over RCA_0 it does not imply ACA_0 and is provably very weak (in the sense of being highly conservative over RCA_0). Shore [26] proves these results and further analyzes this and many similar principles some related to the IRT_{XYZ} and others to an array of classical logical principles.

Any reductions in RCA_0 as requested in the Question above would, of course, provide the analogous ones for the IRT_{XYZ}^* . However, it is possible that other implications can be proven for the IRT_{XYZ}^* :

Question 8.3. Can any implications of the form $\text{IRT}_{XYZ}^* \rightarrow \text{IRT}_{X'Y'Z}'^*$ be proven in RCA_0 other than the ones known to hold for the IRT versions?

Probably more challenging is the problem of separating the principles.

Question 8.4. Can one prove any nonimplication over RCA_0 or over $\text{RCA}_0 + \text{I}\Sigma_1^1$ for any pair of the IRT_{XYZ} ?

Of course, any such separation for the IRT_{XYZ} of Question 8.1 would answer a case of that question by proving that at least one of these principles is strictly weaker than $\Sigma_1^1\text{-AC}_0$. In addition, a separation by standard models or even ones over $\text{I}\Sigma_1^1$ for the IRT_{XYZ} would give nonimplication for the corresponding IRT_{XYZ}^* but it might be that nonstandard models could be used to separate one pair of versions but not the other.

The next natural question looks below ABW_0 in Figure 4.

Question 8.5. Can one prove that finite- $\Sigma_1^1\text{-AC}_0$ does not imply ABW_0 over RCA_0 or $\text{RCA}_0 + \text{I}\Sigma_1^1$?

The weaker versions, WIRT_{XYZ} , of the IRT_{XYZ} , prompt a question about ACA_0 .

Question 8.6. Do any of the WIRT_{XYZ} (especially the ones provable from ACA_0) imply ACA_0 ? An easier question might be whether they imply WKL_0 ?

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