ON USING FUZZY ARITHMETIC TO SOLVE PROBLEMS WITH UNCERTAIN MODEL PARAMETERS

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ABSTRACT

Fuzzy arithmetic, based on Zadeh's extension principle, is presented as a tool to solve engineering problems with uncertain model parameters. Fuzzy numbers are introduced, and different concepts to practically implement the uncertainty of the parameters are discussed. As an example, a rather simple but typical problem of engineering mechanics is considered. It consists of determining the displacements in a two-component massless rod under tensile load with uncertain elasticity parameters. The application of fuzzy arithmetic directly to the traditional techniques for the numerical solution of the engineering problem, however, turns out to be impracticable in all circumstances. In contrast to the use of exclusively crisp numbers, the results for the calculations including fuzzy numbers usually differ to a large extent depending on the solution technique applied. The uncertainties expressed in the different calculation results are then basically twofold. On the one hand, uncertainty is caused by the presence of parameters with fuzzy value, on the other hand, an additional, undesirable uncertainty is artificially created by the solution technique itself. This fuzzy-specific effect of artificial uncertainties is discussed and some concepts for its reduction are presented.

KEYWORDS

Uncertainty, fuzzy arithmetic, finite element method.

INTRODUCTION

To achieve reliable results for the numerical solution of engineering problems, exact values for the parameters of the problem equations should be available. In practice, however, exact values can often not be provided. The model parameters usually exhibit variability, e.g. due to irregularities in fabrication when considering the geometrical dimensions or the physical properties of a material. Thus, the results obtained for solutions that just use some specific crisp value for the uncertain parameters cannot be considered to be representative for the whole spectrum of possible results. To solve this limitation, the application of fuzzy set theory [5] proves to be a practical approach. More specifically, the uncertainties in the model parameters can be taken into account by representing the effects of scatter by fuzzy numbers with their shape derived from experimental data. By this technique, one can demonstrate how initially assumed uncertainties are processed through the calculation procedure leading finally to fuzzy results that reflect the reliability of the problem solution. Additionally, the fuzzy results allow the computation of a crisp value as the most likely result for the problem which in general differs from the result achieved by an initially non-fuzzy approach using only crisp parameters.

The implementation of fuzzy numbers and fuzzy arithmetical operations, however, turns out to be a non-trivial problem. Though the generalized mathematical operations for fuzzy numbers can theoretically be defined making use of Zadeh's extension principle [6], real-world application of fuzzy arithmetic raises two fundamental problems: first, the problem of practically implementing fuzzy numbers with arbitrarily shaped membership functions to be successfully handled by fuzzy arithmetical operations, and second, the necessity of considering the degree of dependency between fuzzy numbers, to reduce artificial uncertainties which prove to be a characteristic phenomenon of binary fuzzy arithmetical operations.

FUZZY ARITHMETIC

Definition of fuzzy numbers

Basically, fuzzy numbers can be considered as a special class of fuzzy sets showing some specific properties [4]. The fuzzy sets themselves result from a generalization of conventional sets by allowing elements of a universal set not only to entirely belong or not to belong to a specific set, but also to belong to the set to a certain degree [5, 7]. Thus, fuzzy sets can be expressed by the elements x of a universal set Ω with a certain degree of membership $\mu(x) \in [0,1]$ assigned. The elements x belonging to conventional sets, instead, are characterized by degrees of membership that can only be equal to zero or unity, i.e. by a membership function $\mu(x) \in \{0,1\}$. On this basis, closed intervals and crisp numbers of the form

$$\begin{aligned} [a,b] &= \{x \mid a \leq x \leq b\} \\ c &= \{x \mid x = c\}\,, \end{aligned} \qquad x \in \mathbb{R}\,, \qquad (1)$$

can be considered as conventional subsets of the universal set $\mathbb{R}$ which can also be expressed by

$$\begin{aligned} \mu_{[a,b]}(x) &= \begin{cases} 1 & \text{for} \quad a \leq x \leq b \\ 0 & \text{else} \end{cases} \\ \mu_c(x) &= \begin{cases} 1 & \text{for} \quad x = c \\ 0 & \text{else} \end{cases} \end{aligned} \qquad (2)$$

when the membership function $\mu(x) \in \{0,1\}$, $x \in \mathbb{R}$, is used (Figure 1).

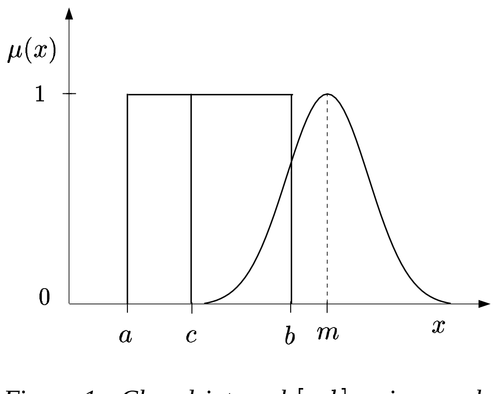

Figure 1: Closed interval $[a,b]$, crisp number c and fuzzy number of Gaussian shape with mean value m expressed by their membership functions.

Fuzzy numbers, instead, are defined as convex fuzzy sets over the universal set $\mathbb{R}$ with their membership functions $\mu(x) \in [0,1]$ where $\mu(x) = 1$ is true only for one single value $x = m \in \mathbb{R}$. As an example, symmetric fuzzy numbers of Gaussian shape are defined by the membership function

$$\mu(x) = \mathrm{e}^{-\dfrac{(x-m)^2}{2\,\sigma^2}}\,, \qquad (3)$$

where m and σ denote the mean value and the standard deviation of the Gaussian distribution (Figure 1).

To define unary or binary arithmetical operations with fuzzy numbers, two concepts can be applied. On the one hand, the fuzzy numbers can be decomposed into sets of intervals for different degrees of membership. The arithmetic for fuzzy numbers can thus be reduced to interval arithmetic [1, 4]. On the other hand, Zadeh's extension principle according to Eq. (5) can be applied which extends the evaluation of arithmetical functions from crisp to fuzzy-valued operands. Explicitely, if $\tilde{a}$ and $\tilde{b}$ are fuzzy numbers defined by the membership functions $\mu_{\tilde{a}}(x)$, $x \in \mathbb{R}$, and $\mu_{\tilde{b}}(y)$, $y \in \mathbb{R}$, the result of the binary operation

$$\tilde{c} = f\left(\tilde{a}, \tilde{b}\right) \qquad (4)$$

for an arbitrary function f is determined by

$$\mu_{\tilde{c}}(z) = \sup_{z=f(x,y)} \min\left\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\right\}\ . \qquad (5)$$

Implementation of fuzzy numbers

To guarantee the successful inclusion of uncertainties into the solution procedures of engineering problems, the fuzzy numbers that are used to represent the uncertain model parameters must be implemented in an appropriate form that complies at least with the following requirements:

- The form should allow the comprehension of fuzzy numbers with arbitrarily shaped membership functions, considering especially the case of fuzzy numbers with their shape derived from measured data.
- The form should allow a practical realization of the fuzzy arithmetical operations between fuzzy numbers by avoiding any loss of information in the uncertainty.

In the ensuing, two different concepts of implementing fuzzy numbers are presented: the often applied concept of using triangular fuzzy numbers and the more promising approach of using discretized fuzzy numbers.

Considering a definite uncertain parameter a, measured data for the parameter are assumed to be available from which a normalized distribution function $q_a(x)$ can be derived that expresses the frequency of occurrence of a certain measured value x for the parameter a within the interval Δx. In most cases, these data approximately show Gaussian distribution. The uncertainty in the parameter a can then be modeled by a fuzzy number $\tilde{a}$ with the membership function $\mu_{\tilde{a}}(x)$ of the form

$$\mu_{\tilde{a}}(x) = \mathrm{e}^{-\dfrac{(x-m_a)^2}{2\,\sigma_a^2}} , \qquad (6)$$

where m_a and σ_a are the mean value and the standard deviation of the Gaussian distribution (Figure 2).

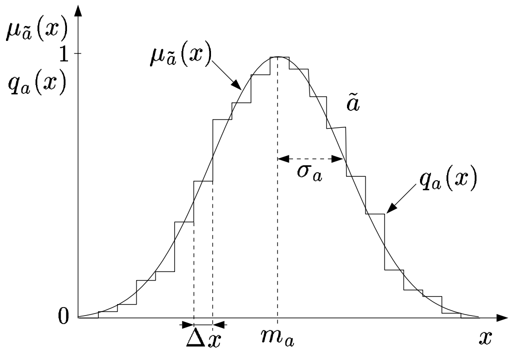

Figure 2: Normalized distribution function $q_a(x)$ for the uncertain parameter a approximated by the membership function $\mu_{\tilde{a}}(x)$ of the fuzzy number $\tilde{a}$.

Triangular fuzzy numbers

Up to now, fuzzy numbers have primarily been implemented as L-R fuzzy numbers [3]. In this form, the membership function of a fuzzy number is characterized by its ascending left and its descending right branch which on their part are expressed by parameterized functions that belong to a definite class of basis functions. For reasons of simplicity, linear functions are mostly chosen as the basis functions. The resulting special type of L-R fuzzy numbers can then be referred to as triangular fuzzy numbers (TFN) [4].
The original fuzzy number $\tilde{a}$ with the membership function $\mu_{\tilde{a}}(x)$ in Eq. (6) can be approximated by a symmetric triangular fuzzy number $\tilde{a}_{\mathrm{t}}$ with the membership function $\mu_{\tilde{a}_{\mathrm{t}}}(x)$ that can be obtained by postulating

$$\mu_{\tilde{a}_{\mathrm{t}}}(m_a) = \mu_{\tilde{a}}(m_a) = 1 \qquad (7)$$

and

$$\int_{-\infty}^{+\infty} \mu_{\tilde{a}_{\mathrm{t}}}(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \mu_{\tilde{a}}(x)\,\mathrm{d}x \,. \qquad (8)$$

The membership function $\mu_{\tilde{a}_{\mathrm{t}}}$ of the triangular fuzzy number is then defined by

$$\mu_{\tilde{a}_{\mathrm{t}}}(x) = \max\left\{0\,,\,1-\frac{|x-m_a|}{\delta}\right\} \qquad (9)$$

with

$$\delta = \sqrt{2\pi}\,\sigma_a\,, \qquad (10)$$

which can also be expressed in the short form

$$\tilde{a}_{\mathrm{t}} = \langle\, m_a-\delta\,,\, m_a\,,\, m_a+\delta\,\rangle_{\mathrm{TFN}} \,. \qquad (11)$$

The membership functions $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{a}_{\mathrm{t}}}(x)$ of the original fuzzy number $\tilde{a}$ and its approximation $\tilde{a}_{\mathrm{t}}$ are illustrated in Figure 3.

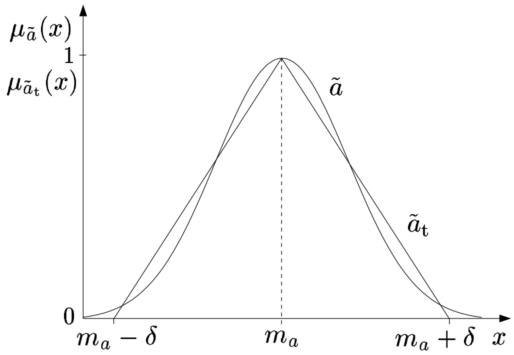

Figure 3: Original fuzzy number $\tilde{a}$ with Gaussian shape and its linear approximation as a triangular fuzzy number $\tilde{a}_{\mathrm{t}}$.

Obviously, the use of triangular fuzzy numbers shows two major advantages. First, the very simple way of implementation with linear functions and only three parameters that can even be reduced to two parameters in the case of symmetric fuzzy numbers. And second, the quite uncomplicated realization of the elementary fuzzy arithmetical operations leading again to triangular fuzzy numbers [3]. With respect to the application of fuzzy arithmetic to solve engineering problems with uncertainties, the disadvantages of this concept, however, are of higher weight. The triangular fuzzy numbers are just a rough approximation of the really existing uncertainty as can be seen from Figure 3. Furthermore, to successfully perform even the nonlinear elementary arithmetical operations, as multiplication and division, the membership functions of the resulting fuzzy numbers have to be approximated by linear functions after each of those operations to avoid a change of the type of basis function. Within the solution procedure of an engineering problem with uncertain parameters, these multiple approximations cause an enormous loss of information of the initially induced uncertainty.

Discretized fuzzy numbers

Motivated by the general practice of sampling analog signals for computer-based signal processing, the membership functions of arbitrary shape can be discretized, leading to discrete fuzzy sets for which the fuzzy arithmetical operations can then be defined using Zadeh's extension principle.

Basically, two different ways of obtaining discretized fuzzy numbers seem to be possible: discretizing the membership functions by subdividing either the abscissa or the ordinate into intervals of definite length. Splitting up the abscissa, i.e. the x-axis, however, turns out unsuitable due to some undesirable effects that can be observed when evaluating fuzzy arithmetical operations for such fuzzy numbers. As the major effect, the resulting fuzzy sets show shapes that vary depending on how the x-axis is subdivided. Thus, if the intervals are chosen rather large to reduce computational efforts, the fuzzy sets might show no convexity any more and can consequently no longer be considered as fuzzy numbers although, strictly speaking, they still are.

Those problems can effectively be avoided if the μ-axis is subdivided into a number of n segments, equally spaced by $\Delta\mu = 1/n$ as illustrated in Figure 4. The fuzzy number $\tilde{a}$ can then be approximated by the discrete fuzzy number $\tilde{a}_\mathrm{d}$ which can be expressed in the form

$$\tilde{a}_\mathrm{d} = \Big\{ (x_0^{(\mathrm{a})}, \mu_0), \ldots, (x_n^{(\mathrm{a})}, \mu_n), \quad (x_0^{(\mathrm{d})}, \mu_0), \ldots, (x_n^{(\mathrm{d})}, \mu_n) \Big\} \tag{12}$$

$$\begin{aligned} &\mu_i = \mu_{i-1} + \Delta\mu\,, \quad i = 1, \ldots, n\,, \\ &\mu_0 = 0 \quad \text{and} \quad \mu_n = 1\,. \end{aligned} \tag{13}$$

Introducing the strictly monotonous functions $\mu_{\tilde{a}}^{(\mathrm{a})}(x)$ and $\mu_{\tilde{a}}^{(\mathrm{d})}(x)$ to denote the ascending and the descending branch of the membership function $\mu_{\tilde{a}}(x)$ in the form

$$\mu_{\tilde{a}}(x) = \begin{cases} \mu_{\tilde{a}}^{(\mathrm{a})}(x)\,, & x \in \left\{ x \,\middle|\, \dfrac{\mathrm{d}\mu_{\tilde{a}}(x)}{\mathrm{d}x} \geq 0 \right\} \\[2ex] \mu_{\tilde{a}}^{(\mathrm{d})}(x)\,, & x \in \left\{ x \,\middle|\, \dfrac{\mathrm{d}\mu_{\tilde{a}}(x)}{\mathrm{d}x} \leq 0 \right\}\,, \end{cases} \tag{14}$$

the elements $x_i^{(\mathrm{a})}$ and $x_i^{(\mathrm{d})}$ of the support of the discrete fuzzy number $\tilde{a}_\mathrm{d}$ are finally given by

$$\left.\begin{aligned} x_i^{(\mathrm{a})} &= \left(\mu_{\tilde{a}}^{(\mathrm{a})}\right)^{-1}(\mu_i) \\ x_i^{(\mathrm{d})} &= \left(\mu_{\tilde{a}}^{(\mathrm{d})}\right)^{-1}(\mu_i) \end{aligned}\right. \quad i = 0, 1, \ldots, n\,. \tag{15}$$

In case of Gaussian fuzzy numbers, the elements for $i = 0$ are to be neglected.

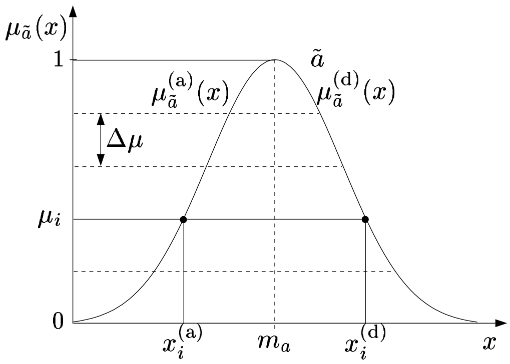

Figure 4: Approximation of the fuzzy number $\tilde{a}$ by the discrete fuzzy number $\tilde{a}_\mathrm{d}$.

Using discretized fuzzy numbers of type $\tilde{a}_\mathrm{d}$ in Eqs. (12) and (13) to represent fuzzy numbers like $\tilde{a}$, arithmetical operations with fuzzy numbers can successfully be implemented by defining the operations to be executed separately for the elements of each degree of membership μ_i [3]. Out of the number of possible combinations between these elements, the number of valid combinations that finally lead to the proper results can be determined by making use of Zadeh's extension principle.

For example, for the elementary arithmetical operation of adding two strictly positive fuzzy numbers $\tilde{a} > 0$ and $\tilde{b} > 0$ using their discrete forms $\tilde{a}_\mathrm{d}$ and $\tilde{b}_\mathrm{d}$, the discrete form $\tilde{c}_\mathrm{d}$ of the resulting sum $\tilde{c} = \tilde{a} + \tilde{b}$ can be obtained by combining only the elements of the ascending branches of both fuzzy numbers on the one side and the elements of the descending branches on the other. Explicitly, if $\tilde{a}$ and $\tilde{b}$ are defined according to Eq. (12) over x and y, $\tilde{c}_\mathrm{d}$ is given over z by

$$\tilde{c}_\mathrm{d} = \Big\{ (z_0^{(\mathrm{a})}, \mu_0), \ldots, (z_n^{(\mathrm{a})}, \mu_n), \quad (z_0^{(\mathrm{d})}, \mu_0), \ldots, (z_n^{(\mathrm{d})}, \mu_n) \Big\} \tag{16}$$

with

$$\begin{aligned} &z_i^{(\mathrm{a})} = x_i^{(\mathrm{a})} + y_i^{(\mathrm{a})} \quad \text{and} \quad z_i^{(\mathrm{d})} = x_i^{(\mathrm{d})} + y_i^{(\mathrm{d})}\,, \\ &i = 0, 1, \ldots, n\,. \end{aligned} \tag{17}$$

In general, however, the specific appearance of the formula in Eqs. (16) and (17) depends on the type of arithmetical operation to be realized and on whether the domain covered by the fuzzy numbers is strictly positive, strictly negative or both positive and negative.

The presented principle of implementing fuzzy numbers and fuzzy arithmetical operations is very similar to the well-known method of decomposing a fuzzy number into a number of α-cuts, $\alpha \in [0, 1]$, for which the arithmetical operations can then be defined using interval arithmetic [4]. The major advantage of the present method, however, is its extensibility to the handling of fuzzy sets with incoherent α-cuts. Such fuzzy sets are the results of some rather problematic arithmetical operations, e.g. the division by a fuzzy number with its support containing both positive and negative elements. With respect to the convexity of fuzzy numbers as defined on the basis of coherent α-cuts, the results of such operations would not be considered as fuzzy numbers any more. With the implementation presented above, however, those results can still be characterized by an ascending and a descending branch (not necessarily a left and a right one) and can thus be used for further processing. Research work on the handling of those degenerated fuzzy numbers is currently in process, and it already shows some promising results.

As an example, the fuzzy number $\tilde{d}$ with a membership function of Gaussian shape ($m_d = 1$, $\sigma_d = 1$) and its inverse $1/\tilde{d}$ are shown in Figure 5.

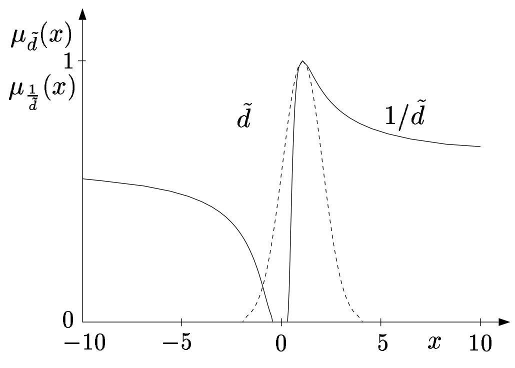

Figure 5: Fuzzy number $\tilde{d}$ (Gaussian shape, $m_d = 1$, $\sigma_d = 1$) [– – –] and its inverse $1/\tilde{d}$ [—].

ENGINEERING PROBLEM

As a rather simple but typical engineering problem, a two-component massless rod under tensile load is considered where the displacements u_1 and u_2 of the rod's components are to be determined (Figure 6). The components of the rod are characterized by the length parameters $l^{(1)}$ and $l^{(2)}$, the cross sections $A^{(1)}$ and $A^{(2)}$ and the Young's moduli $E^{(1)}$ and $E^{(2)}$ quantifying the elasticity of the components. The external loading consists of the tensile force F acting at the end of the rod. To determine the displacements u_1 and u_2, the finite element method can be applied, leading finally to the equation

$$\underbrace{\begin{bmatrix} c^{(1)} + c^{(2)} & -c^{(2)} \\ -c^{(2)} & c^{(2)} \end{bmatrix}}_{\boldsymbol{K}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\boldsymbol{u}} = \underbrace{\begin{bmatrix} 0 \\ F \end{bmatrix}}_{\boldsymbol{F}} \tag{18}$$

with the stiffness parameters

$$c^{(i)} = \frac{E^{(i)} A^{(i)}}{l^{(i)}} , \quad i = 1, 2 . \tag{19}$$

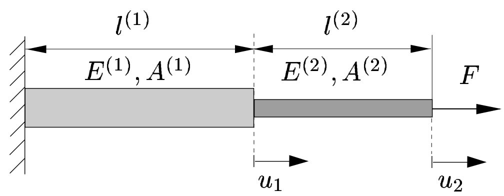

Figure 6: Two-component rod under tensile load.

When material uncertainty in elasticity is considered, the Young's moduli $E^{(1)}$ and $E^{(2)}$ are no longer crisp, but behave as fuzzy parameters $\widetilde{E}^{(1)}$ and $\widetilde{E}^{(2)}$ with their values given by fuzzy numbers. Thus, the global stiffness matrix $\boldsymbol{K}$ in Eq. (18) becomes a fuzzy-valued matrix $\widetilde{\boldsymbol{K}}$ with the fuzzy stiffness parameters

$$\tilde{c}^{(i)} = \frac{\widetilde{E}^{(i)} A^{(i)}}{l^{(i)}} , \quad i = 1, 2 . \tag{20}$$

Solving the system equation (18) for the unknown fuzzy-valued displacement vector $\widetilde{\boldsymbol{u}}$, one obtains

$$\widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{K}}^{-1} \boldsymbol{F} , \tag{21}$$

which finally leads to the following expressions for $\tilde{u}_1$ and $\tilde{u}_2$:

$$\tilde{u}_1 = \frac{F}{\tilde{c}^{(1)}} \tag{22}$$

$$\tilde{u}_2 = F \left(\frac{1}{\tilde{c}^{(1)}} + \frac{1}{\tilde{c}^{(2)}} \right) . \tag{23}$$

Since this solution is achieved by consecutive symbolic simplification of Eq. (21), this expression constitutes the canonical solution of Eq. (18), and will be refered to as the ”exact solution”.

When considering the usual case of a global stiffness matrix $\widetilde{\boldsymbol{K}}$ with a large dimension, symbolic simplification of the system solution is impractical. For this reason, the finite element problem is usually solved numerically using special computer programs. In the following, two different ways of numerical solution shall be presented.
In the first method, the finite element problem can be solved according to Eq. (21), i.e. by determining at first the inverse global stiffness matrix $\widetilde{\boldsymbol{K}}^{-1}$ and then forming the matrix product $\widetilde{\boldsymbol{K}}^{-1}\boldsymbol{F}$. This procedure leads to

$$\tilde{u}_1 = \left(\left(\tilde{c}^{(1)} + \tilde{c}^{(2)} \right) \tilde{c}^{(2)} - \left(\tilde{c}^{(2)} \right)^2 \right)^{-1} \tilde{c}^{(2)}\, F \tag{24}$$

$$\tilde{u}_2 = \left(\left(\tilde{c}^{(1)} + \tilde{c}^{(2)} \right) \tilde{c}^{(2)} - \left(\tilde{c}^{(2)} \right)^2 \right)^{-1} \times \\ \times \left(\tilde{c}^{(1)} + \tilde{c}^{(2)} \right) F\,. \tag{25}$$

In the second method, since the global stiffness matrix is usually symmetric and positive definite, the finite element problem in Eq. (18) can be solved effectively by an $\boldsymbol{LDL}^{\mathrm{T}}$ decomposition of $\widetilde{\boldsymbol{K}}$ where $\boldsymbol{L}$ denotes a lower triangular matrix with diagonal terms being unity and $\boldsymbol{D}$ is a matrix of a diagonal form. The problem is then solved by forward and back substitution procedures according to

$$\boldsymbol{L}\,\boldsymbol{a} = \boldsymbol{F}\,, \quad \boldsymbol{D}\,\boldsymbol{b} = \boldsymbol{a}\,, \quad \boldsymbol{L}^{\mathrm{T}}\boldsymbol{u} = \boldsymbol{b}\,. \tag{26}$$

The advantages of the $\boldsymbol{LDL}^{\mathrm{T}}$ decomposition are the following:

1. For a constant matrix $\widetilde{\boldsymbol{K}}$ the cost intensive part of the calculation, namely the decomposition, can be performed at the outset for all right-hand sides $\boldsymbol{F}$. This proves to be especially useful for different load cases or transient calculations.

2. An often encountered band width structure of $\widetilde{\boldsymbol{K}}$ is preserved by $\boldsymbol{L}$ and can be stored in-place with $\boldsymbol{D}$ on the main diagonal.

Formulating this method for the problem considered, one obtains

$$u_1 = \left(\tilde{c}^{(2)} - \frac{\left(\tilde{c}^{(2)}\right)^2}{\tilde{c}^{(1)} + \tilde{c}^{(2)}} \right)^{-1} \frac{\tilde{c}^{(2)}}{\tilde{c}^{(1)} + \tilde{c}^{(2)}}\, F \tag{27}$$

$$u_2 = \left(\tilde{c}^{(2)} - \frac{\left(\tilde{c}^{(2)}\right)^2}{\tilde{c}^{(1)} + \tilde{c}^{(2)}} \right)^{-1} F\,. \tag{28}$$

As an example, the finite element problem is now solved for a definite parameter configuration, where the first component of the rod is assumed to be steel and the second aluminum with the geometry parameters and the external loading specified by

$$\begin{aligned} A^{(1)} &= 100\,\mathrm{mm}^2\,, \quad l^{(1)} = 500\,\mathrm{mm}\,, \\ A^{(2)} &= 75\,\mathrm{mm}^2\,, \quad l^{(2)} = 500\,\mathrm{mm}\,, \\ F &= 1000\,\mathrm{N}\,. \end{aligned} \tag{29}$$

The uncertain Young's moduli are considered to have membership functions of Gaussian shape with

$$\begin{aligned} \widetilde{E}^{(1)} \;:\quad & m_1 = 2.0 \cdot 10^5 \tfrac{\mathrm{N}}{\mathrm{mm}^2}\,, \\ & \sigma_1 = 5\% \cdot m_1\,, \\ \widetilde{E}^{(2)} \;:\quad & m_2 = 6.9 \cdot 10^4 \tfrac{\mathrm{N}}{\mathrm{mm}^2}\,, \\ & \sigma_2 = 5\% \cdot m_2\,. \end{aligned} \tag{30}$$

Finally, three different results $\tilde{u}_2^{(1)}$, $\tilde{u}_2^{(2)}$ and $\tilde{u}_2^{(3)}$ can be obtained for the fuzzy-valued displacement $\tilde{u}_2$ at the end of the rod, depending on the solution technique applied (Figure 7). This fuzzy-specific effect of getting different results for different solution procedures is explained and discussed in the next section.

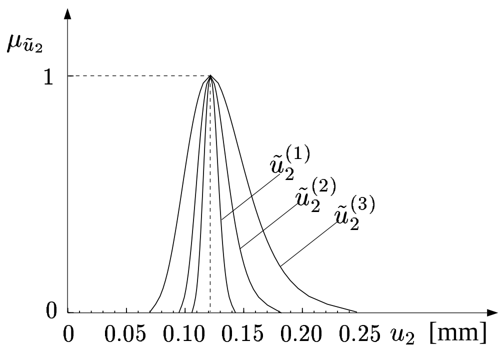

Figure 7: Fuzzy-valued displacement $\tilde{u}_2$ at the end of the rod. Exact solution: $\tilde{u}_2^{(1)}$, solution by $\boldsymbol{LDL}^T$ decomposition: $\tilde{u}_2^{(2)}$, solution by inversion: $\tilde{u}_2^{(3)}$.

When defuzzification of the fuzzy-valued result $\tilde{u}_2$ is applied to determine a crisp-valued equivalence u_2^* using the center-of-area method [2] with

$$u_2^* = \frac{\int_{-\infty}^{\infty} u_2\, \mu_{\tilde{u}_2}(u_2)}{\int_{-\infty}^{\infty} \mu_{\tilde{u}_2}(u_2)}\,, \tag{31}$$

the resulting crisp number u_2^* can be considered as the expected value for u_2 when uncertainty in the model parameters is assumed. As for the exact solution according Eq. (23), defuzzification of $\tilde{u}_2^{(1)}$ leads to

$$u_2^* = 0.1219\,\mathrm{mm}\ . \tag{32}$$

Considering the case when no uncertainty is implied, the non-fuzzy solution of the problem with the crisp Young's moduli

$$\begin{aligned} E^{(1)} &= m_1 = 2.0 \cdot 10^5 \,\tfrac{\mathrm{N}}{\mathrm{mm}^2} \\ E^{(2)} &= m_2 = 6.9 \cdot 10^4 \,\tfrac{\mathrm{N}}{\mathrm{mm}^2} \end{aligned} \tag{33}$$

results in the crisp displacement

$$\hat{u}_2 = 0.1216\,\mathrm{mm} \tag{34}$$

which equals the value with the maximum degree of membership of the support of $\tilde{u}_2$, i.e. where $\mu_{\tilde{u}_2}(\hat{u}_2) = 1$. Thus, due to the generally asymmetric shape of the fuzzy-valued result, caused by the nonlinear arithmetical operations in the solution procedure, the expected result u_2^* for the problem under the assumption of a given symmetric uncertainty in the elasticity parameters differs from the result of the crisp solution with no uncertainty assumed by 0.25 %. Although this difference is not very significant in this simple example, it more and more becomes an important factor when real-world applications are considered and the dimension of the corresponding finite element problem increases.

DISCUSSION

The reason for the observed fuzzy-specific effect of getting different results depending on the solution procedure will become clearer when considering the following simple example where triangular fuzzy numbers in the form

$$\langle \text{ lower bound },\ \text{peak value},\ \text{upper bound } \rangle_{\text{TFN}}$$

according to Eq. (11) are used for reasons of simplicity.
Let $\tilde{a}$ be a variable of fuzzy value given by the triangular fuzzy number

$$\tilde{a} = \langle 1, 2, 4 \rangle_{\text{TFN}} \tag{35}$$

and let $\tilde{b}$ be another fuzzy variable of different origin given by the triangular fuzzy number

$$\tilde{b} = \langle 1, 2, 4 \rangle_{\text{TFN}} \tag{36}$$

with the same mean value and the same uncertainty range as the variable $\tilde{a}$. Evaluating the expression

$$\tilde{y} = \frac{\tilde{a} + \tilde{b}}{\tilde{a}} \tag{37}$$

by using fuzzy arithmetic for triangular fuzzy numbers [4] then leads to

$$\tilde{y}_1 = \langle -1, 2, 5 \rangle_{\text{TFN}} \tag{38}$$

However, by applying some preceding symbolic calculation to the expression in Eq. (37) one obtains

$$\tilde{y} = \frac{\tilde{a}}{\tilde{a}} + \frac{\tilde{b}}{\tilde{a}} \tag{39}$$

which can be rewritten as

$$\tilde{y} = 1 + \frac{\tilde{b}}{\tilde{a}} \tag{40}$$

when the a priori knowledge is included that the quotient of two identical variables must be equal to a crisp unity. The evaluation of Eq. (40), instead, results in

$$\tilde{y}_2 = \langle 0.5, 2, 3.5 \rangle_{\text{TFN}} \tag{41}$$

which represents a fuzzy number with the same mean value as $\tilde{y}_1$ but with only half the range of uncertainty.
The reason for the different results of $\tilde{y}_1$ and $\tilde{y}_2$ must be seen in the light of uncertainties of different origin. Whereas the uncertainties in $\tilde{y}_2$ are just natural ones, arising from the initial assumption of fuzzy parameters, the numerical result $\tilde{y}_1$ for the non-simplified expression in Eq. (37) includes additional, artificial uncertainties induced by the special solution technique itself. Explicitly, both fractions in Eq. (39) are numerically treated the same way leading to a fuzzy unity due to $\tilde{a} = \tilde{b}$. Thus, in contrast to a symbolically preprocessed solution, a purely numerical solution technique is only capable of handling the feature "equality of values", but not "identity of variables". In other words, the conventional fuzzy arithmetic does only lead to results free of artificial uncertainties if unary arithmetical operation are carried out or for binary arithmetical operations if the operands are completely independent, i.e. if they stem from fuzzy parameters of different origin. In case of completely dependent operands, another fuzzy arithmetic has to be defined which is not based on Zadeh's extension principle.
As for the fuzzy finite element problem, the results for the exact solution expressed by $\tilde{u}_2^{(1)}$ in Figure 7 are free of any artificial uncertainty since this

solution completely fulfills the requirement mentioned above. Using the solution with inversion of the global stiffness matrix $\widetilde{\boldsymbol{K}}$, instead, the results are characterized by artificial uncertainties that are about as four times as large as the natural ones. These artificial uncertainties can effectively be reduced, although they cannot be avoided, when using the $\boldsymbol{LDL}^{\mathrm{T}}$ decomposition of $\widetilde{\boldsymbol{K}}$ for numerically solving the fuzzy finite element problem.

MODIFIED SOLUTION PROCEDURES

Since the $\boldsymbol{LDL}^{\mathrm{T}}$ decomposition proves to be the type of numerical solution technique with the lowest degree of artificial uncertainties, this method shall be used as the basis for further improvements which finally lead to a modified solution procedure. However, it is not possible to specify one globally valid solution technique for fuzzy finite element problems; there will instead be different modifications depending on the parameter configuration of the problem. The following are two practical modifications that could be adopted:

(1) In most cases the elasticity parameters of different elements, i.e. the fuzzy valued Young's moduli $\widetilde{E}^{(i)}$, can be assumed to be uncertain to the same percentile extent. Thus, the fuzzy valued and Gaussian shaped parameters $\tilde{c}^{(i)}$ of the global stiffness matrix $\widetilde{\boldsymbol{K}}$ can be rewritten as

$$\tilde{c}^{(i)} = m_i \cdot (1 + \tilde{\varepsilon}) \qquad (42)$$

where $\tilde{\varepsilon}$ represents a fuzzy zero with the mean value $m_\varepsilon = 0$ and the standard deviation σ_ε. Since $\tilde{\varepsilon}$ has the same value for all elements i, it can be factored out of the stiffness matrix. The $\boldsymbol{LDL}^{\mathrm{T}}$ decomposition, including most of the numerical operations to solve the problem, can then be performed for a crisp stiffness matrix avoiding any production of artificial uncertainties. By this, the critical fuzzy arithmetical operations can be reduced to a smaller number consisting of forward and back substitution according to Eq. (26).

(2) If at least some of the fuzzy parameters show identical percentile uncertainty, partitioning of the global stiffness matrix $\widetilde{\boldsymbol{K}}$ can be performed, enabling a partially crisp $\boldsymbol{LDL}^{\mathrm{T}}$ decomposition of the matrix as described above. Using this block elimination technique as a modified solution procedure, the ocurrence of artificial uncertainties can also be reduced.

CONCLUSION

The application of fuzzy arithmetic to solve engineering problems with uncertainties in the parameters is a powerful but also a problematic tool. Although the natural uncertainties associated with material variability can be considered to be acceptable, the artificial uncertainties that arise from computational aspects in the finite element procedure must be minimized. Thus, to achieve practical results, the different numerical techniques for solving finite element problems should not be applied in their common form, but should be modified with respect to a reduction of critical fuzzy arithmetical operations that can cause artificial uncertainties. Presently, these modifications of the solution techniques must be redefined for every special parameter configuration by some symbolic preprocessing of the solution schemes. One concept to solve this limitation is the development of self-modifying solution procedures. Another approach, instead, consists of introducing a new fuzzy arithmetic which provides a different realization of the operations between dependent and independent fuzzy numbers. Research activities in this field are currently in progress.

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