Problem 3

part a

$F(x) = \theta\sigma^{\theta}x^{-\theta-1} \qquad x \geq \sigma$

MLE for σ & θ Expressed in terms of x_i

$x_{min} \geq \sigma$ The MLE of σ is limited to the smallest value of x

lowest value of x

the max possible value of σ is x_{min}

$$p(x_i|\Theta) = p(x_i|\sigma,\theta)$$

$$L(\theta) = \prod_{i=1}^{n} p(x_i|\sigma,\theta) = \prod_{i=1}^{n} (\theta\sigma^{\theta}x^{-\theta-1})$$

$$l(\theta) = \log(L(\theta)) = \log\left(\prod_{i=1}^{n} \theta\sigma^{\theta}x^{-\theta-1}\right)$$

$$l(\theta) = \sum_{i=1}^{n} \log\left(\theta\sigma^{\theta}x^{-\theta-1}\right)$$

$$l(\theta) = \sum_{i=1}^{n} \log(\theta) + \sum_{i=1}^{n} \theta\log(\sigma) + \sum_{i=1}^{n}(-\theta\log(x)) - \sum_{i=1}^{n}\log(x)$$

$$\frac{\partial l}{\partial\theta} = \sum_{i=1}^{n}\frac{1}{\theta} + \sum_{i=1}^{n}\log(\sigma) - \sum_{i=1}^{n}\log(x)$$

$$0 = \frac{\partial l}{\partial\theta} \qquad 0 = \sum_{i=1}^{n}\frac{1}{\theta} + \sum_{i=1}^{n}\log(\sigma) - \sum_{i=1}^{n}\log(x)$$

$$\left[\sum_{i=1}^{n}\log(x) - \sum_{i=1}^{n}\log(\sigma)\right]^{-1} = \left(\frac{n}{\theta}\right)^{-1}$$

$$\cancel{n}\frac{\theta}{\cancel{n}} = \left[\sum_{i=1}^{n}\log(x) - \sum_{i=1}^{n}\log(\sigma)\right]^{-1} n$$

$$\theta = \frac{n}{\sum_{i=1}^{n}\log(x) - \sum_{i=1}^{n}\log(\sigma)}$$

part a continued

$$\Theta = \frac{n}{\sum_{i=1}^{n} \log(x) - \sum_{i=1}^{n} \log(\theta)}$$

$x_i \geq \theta$

$$\Theta = \frac{n}{\sum_{i=1}^{n} \log(x) - \sum_{i=1}^{n} \log(x_i)}$$

x_{min}

$$\Theta = \frac{n}{\sum_{i=1}^{n} \log(x) - n \log(x_i)}$$

MLE for Θ in terms of x_i

ML & AI Hmk 1

Problem 3

b. $x_i = 13, 2, 16, 5, 11, 16, 18, 5, 8, 15$

$\ln(x) \sim N(\mu, \sigma^2) \quad \mu \to \lambda \quad \sigma^2 \to \xi^2$

MLE $\quad p(x|\lambda, \xi^2) = \frac{1}{x\xi\sqrt{2\pi}} \exp\left(\frac{-(\ln(x)-\lambda)^2}{2\xi^2}\right)$

$$L(\theta) = \prod_{i=1}^{n}\left[\frac{1}{x\xi\sqrt{2\pi}} \exp\left(\frac{-(\ln(x)-\lambda)^2}{2\xi^2}\right)\right]$$

$$\ell(\theta) = \log(L(\theta)) = \log\left[\prod_{i=1}^{n}\frac{1}{x\xi\sqrt{2\pi}} \exp\left(\frac{-(\ln(x)-\lambda)^2}{2\xi^2}\right)\right]$$

$$\ell(\theta) = \sum_{i=1}^{n}\log\left(\frac{1}{x\xi\sqrt{2\pi}} \exp\left(\frac{-(\ln(x)-\lambda)^2}{2\xi^2}\right)\right)$$

$$\ell(\theta) = \sum_{i=1}^{n}\log\left(\frac{1}{x\xi\sqrt{2\pi}}\right) + \sum_{i=1}^{n}\log\left[\exp\left(\frac{-(\ln(x)-\lambda)^2}{2\xi^2}\right)\right]$$

$$\ell(\theta) = \cancelto{0}{\sum_{i=1}^{n}\log(1)} - \sum_{i=1}^{n}\log(x\xi\sqrt{2\pi}) + \sum_{i=1}^{n}\left(\frac{-(\ln(x)-\lambda)^2}{2\xi^2}\right)$$

$$\ell(\theta) = \sum_{i=1}^{n}\left[-\log(x\xi\sqrt{2\pi}) + \left(\frac{-(\ln(x)-\lambda)^2}{2\xi^2}\right)\right]$$

$$\lambda: \frac{\partial\theta}{\partial l} = \frac{\partial\theta}{\partial l}\sum_{i=1}^{n}\left[\cancelto{0}{-\log(x\xi\sqrt{2\pi})} + \left(\frac{-(\ln(x)-\lambda)^2}{2\xi^2}\right)\right]$$

$$\lambda: \frac{\partial\theta}{\partial l} = \frac{\partial\theta}{\partial l}\sum_{i=1}^{n}\left[\frac{-(\ln(x)-\lambda)^2}{2\xi^2}\right]$$

Problem 3 Part 2

6.
$$\frac{\partial \theta}{\partial l} = \frac{\partial \theta}{\partial l} \sum_{i=1}^{n} \left[\frac{-(\ln(x)-\lambda)^2}{2\xi^2} \right]$$

$$0 = \frac{\partial \theta}{\partial l} \sum_{i=1}^{n} \left[\frac{-(\ln(x)-\lambda)^2}{2\xi^2} \right]$$

$$0 = \frac{-2(\ln(x))}{2\xi^2} \sum_{i=1}^{n} (\ln(x) - \lambda)$$

$$0 = \sum_{i=1}^{n} \ln(x) - \sum_{i=1}^{n} \lambda$$

$$\frac{n\lambda}{n} = \frac{\sum_{i=1}^{n} \ln(x)}{n}$$

$$\lambda = \frac{\sum_{i=1}^{n} \ln(x)}{n}$$

$$\lambda = \frac{\ln(5)+\ln(5)+\ln(16)+\ln(16)+\ln(13)+\ln(18)+\ln(2)+\ln(11)+\ln(8)+\ln(15)}{10}$$

$$\lambda = \frac{14.2194 + 7.8785}{10} = 2.2098$$

b. $\xi : \frac{\partial\theta}{\partial\xi} \sum_{i=1}^{n} \left[-\log(x\xi\sqrt{2\pi}) + \left(\frac{-(\ln(x)-\lambda)^2}{2\xi^2} \right) \right]$

$$\frac{\partial\theta}{\partial\xi} = \frac{\partial\theta}{\partial\xi} \sum_{i=1}^{n} -\log(x\xi\sqrt{2\pi}) + \frac{\partial\theta}{\partial\xi} \sum_{i=1}^{n} \left(\frac{-(\ln(x)-\lambda)^2}{2\xi^2} \right)$$

$$\frac{\partial\theta}{\partial\xi} = \sum_{i=1}^{n} - \frac{1}{\cancel{x}\xi\cancel{\sqrt{2\pi}}}(\cancel{x}\cancel{\sqrt{2\pi}}) + \sum_{i=1}^{n} \left[\frac{\cancel{-}(\ln(x)-\lambda)^2(\cancel{-}\cancel{2})}{\cancel{2}} \right] \xi^{-3}$$

$$\frac{\partial\theta}{\partial\xi} = -\sum_{i=1}^{n} \frac{1}{\xi} + \sum_{i=1}^{n} (\ln(x)-\lambda)^2 \xi^{-3}$$

$$0 = -\sum_{i=1}^{n} \frac{1}{\xi} + \sum_{i=1}^{n} (\ln(x)-\lambda)^2 \xi^{-3}$$

$$\xi^3 \sum_{i=1}^{n} \frac{1}{\xi} = \left[\sum_{i=1}^{n} (\ln(x)-\lambda)^2 \xi^{-3} \right] \xi^3$$

$$\frac{n\xi^2}{n} = \frac{\sum_{i=1}^{n} (\ln(x)-\lambda)^2}{n}$$

$$\xi^2 = \frac{\sum_{i=1}^{n} (\ln(x)-\lambda)^2}{n}$$

1.35433 $\left[\begin{array}{l} (\ln(16) - 2.2098)^2 \times 2) \\ (\ln(5) - 2.2098)^2 \times 2) \end{array}\right.$

2.46175 $\left[\begin{array}{l} (\ln(13) - 2.2098)^2 \\ (\ln(2) - 2.2098)^2 \\ (\ln(11) - 2.2098)^2 \end{array}\right.$

.7284245 $\left[\begin{array}{l} (\ln(18) - 2.2098)^2 \\ (\ln(8) - 2.2098)^2 \\ + (\ln(15) - 2.2098)^2 \end{array}\right.$

4.5445

$$\xi^2 = \frac{4.5445}{10}$$

$$\xi^2 = 0.45445$$

Problem 3

part c

$F(x) = (x;\theta) = \frac{x}{\theta^2} e^{-\frac{x}{\theta}} \qquad x > 0$

MLE For $\theta \qquad \theta > 0$

$p(x_i|\theta) = \frac{x}{\theta^2} e^{-\frac{x}{\theta}}$

$L(\theta) = \prod_{i=1}^{n} \left(\frac{x}{\theta^2} \exp\left(-\frac{x}{\theta}\right)\right)$

$l(\theta) = \log(L(\theta)) = \log\left(\prod_{i=1}^{n} \frac{x}{\theta^2} \exp\left(-\frac{x}{\theta}\right)\right)$

$l(\theta) = \sum_{i=1}^{n} \log\left(\frac{x}{\theta^2}\right) + \sum_{i=1}^{n} \log\left(\exp\left(-\frac{x}{\theta}\right)\right)$

$l(\theta) = \sum_{i=1}^{n} \log(x) - \sum_{i=1}^{n} \log(\theta^2) + \sum_{i=1}^{n} -\frac{x}{\theta}$

$l(\theta) = \sum_{i=1}^{n} \log(x) - \sum_{i=1}^{n} 2\log(\theta) + \sum_{i=1}^{n} -x\theta^{-1}$

$\frac{\partial \theta}{\partial l} = -\sum_{i=1}^{n} \frac{2}{\theta} + \sum_{i=1}^{n} x\theta^{-2}$

$0 = -\sum_{i=1}^{n} \frac{2}{\theta} + \sum_{i=1}^{n} x\theta^{-2}$

$\sum_{i=1}^{n} \frac{2}{\theta} = \sum_{i=1}^{n} x\theta^{-2}$

$\theta^{\cancel{2}} \frac{2n}{\cancel{\theta}} = \sum_{i=1}^{n} x\theta^{-2} \; \theta^2$

$\frac{2n\theta}{2n} = \frac{\sum_{i=1}^{n} x}{2n}$

MLE : $\theta \qquad \theta = \frac{\sum_{i=1}^{n} x}{2n}$