24-703/12-703 Numerical Methods in Engineering S25 Homework 5
Due 07 April 2025, 10:00AM

Problem 1 (15 points) The n-point Gaussian quadrature formula is

$$\int_{-1}^{1} f(t)dt \cong \sum_{i=1}^{n} w_i f(t_i),$$

which will be exact for a polynomial of order $2n - 1$. We considered the 2-point ($n = 2$) case in class and found that

$$w_1 = w_2 = 1, \quad t_1 = -\frac{1}{\sqrt{3}}, \quad t_2 = \frac{1}{\sqrt{3}}.$$

In this problem you will work through an approach for finding the w_i and t_i values for a given n.

(a) It turns out that the required t_i values for the n-point formula are the roots of the nth order Legendre polynomial. The Legendre polynomials are defined recursively as

$$(n+1)L_{n+1}(t) - (2n+1)tL_n(t) + nL_{n-1}(t) = 0,$$

with

$$L_0(t) = 1 \text{ and } L_1(t) = t.$$

Find $L_n(t)$ for all n values between 2 and 6 by hand. Make it clear that you worked through the steps, and didn't copy the results from another source.

(b) Use your Newton's method code from HW#3 to find the roots of $L_n(t)$ for all n values between 1 and 6. Show the values graphically (the details of the plot are up to you).

(c) In class we generated a system of four non-linear equations for w_1, w_2, t_1, and t_2 for the $n = 2$ case. Generalize that approach to show that the $2n$ equations for the n-point quadrature are

$$\begin{aligned} w_1 t_1^k + \cdots + w_n t_n^k &= 0 \qquad for\ \ k = 1, 3, \cdots, 2n-1 \\ &= \frac{2}{k+1} \qquad for\ \ k = 0, 2, \cdots, 2n-2 \end{aligned}$$

(d) Once the t_i values are specified from (b), the system of equations from (c) becomes linear in the weights w_i. Solve the system of equations using LU.cpp for all n values between 1 and 6. Note that you will need to choose n independent equations from the $2n$ total equations. Are there any restrictions on the equations you choose? Augment your graphical presentation from part (b) to include information about the weights. Summarize all the t_i and w_i data in a table.

Problem 2 (5 points) Verify that the $n = 4$ formula from Problem 1 is correct by evaluating

$$\int_{-1}^{1} f(t)dt \quad for \quad f(t) = \sum_{i=1}^{7} \frac{t^i}{i(i+1)}$$

using quadrature and comparing to the exact answer.

Problem 3 (10 points) The perimeter, P, of an ellipse with major axis a and minor axis b ($a \geq b$) is

$$P = 4a\int_0^{\pi/2}\left(1 - k^2\sin^2\theta\right)^{1/2} d\theta \quad where \quad k^2 = \frac{a^2 - b^2}{a^2}.$$

(a) Apply the n-point (n = 1, 2, 3, 4, 5, 6) Gaussian quadrature formulas to find P for a = 1 and b = 0.25, 0.5, and 0.75. Summarize your results in a table.

(b) Using the Gaussian quadrature formula that gives you a suitably small error, obtain and plot P/a vs. b/a for $0 \leq b/a \leq 1$.

(c) The perimeter can be approximated by

$$P \approx 2^{1/2}\pi\left(a^2 + b^2\right)^{1/2}.$$

Compare the results from (b) to this approximation. When is this approximation valid? **BONUS (5 points)**: Can you derive this approximation?