**24-703/12-703 Numerical Methods in Engineering S25 Homework 2**
**Due 05 February 2025, 10:00AM**

The C++ code for the LU decomposition of a matrix based on the Crout reduction is available in the HW#2 folder as LU.cpp. In Problems 2 and 3, you will modify the code to solve variations on the conduction problem. Note that the LU decomposition part will not need to be altered, just the formulation of A and b.

**Problem 1** (10 points) Prove that the number of multiplcations/divisions needed to find the Crout LU decomposition of an $n \times n$ matrix is $n^3/3 - n/3$ and that the number of multiplications/divisions needed to solve an associated system of equations is $n^2$.

**Problem 2** (10 points) Instead of a temperature boundary condition, we may wish to specify the heat flow, $Q$, at one of the boundaries.

(a) Noting that

$$Q(x) = -k\frac{dT}{dx}$$

describes how you would implement such a boundary condition in the LU.cpp code. Hint: There is nothing that says that you can't put extra nodes outside of the solution domain.

(b) Obtain and plot $T(x)$ for $n = 20$ for the following conditions:

(i) $Q(0) = 0, \quad T(L) = 2, \quad q(x) = 1 + x$;

(ii) $T(0) = 5, \quad Q(L) = -4, \quad q(x) = \sin x$.

Take $L = 2$ and $k = 1$. Provide written explanations of how you modified the finite difference equations and the LU.cpp code. Do not submit your code. Compare your results to the analytical solutions.

(c) How might you formulate a problem that has no solution? What would break down in the numerical solution?

**Problem 3** (15 points) The steady heat conduction equation in the radial direction in a spherical system with no angular variations in temperature is

$$-\frac{1}{r^2}\left[\frac{d}{dr}\left(kr^2\frac{dT}{dr}\right)\right] = q(r). \tag{1}$$

Note that $q(r)$ gives the rate of energy generation per unit volume. In this problem you will consider heat conduction in a sphere that has an inner diameter $R_i$ and an outer diameter $R_o$, as shown in Fig. 1.

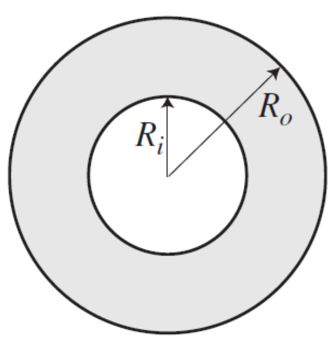


Figure 1: Heat transfer by conduction in a sphere.

(a) Derive the general analytical solution to Eq. (1) for boundary conditions $T(R_i) = T_i$ and $T(R_o) = T_o$, $q(r) = 0$, and a temperature-independent thermal conductivity.
(b) Determine the central difference equation corresponding to Eq. (1) for an internal node for a temperature-independent thermal conductivity and a general $q(r)$.
(c) Implement the finite difference equation from (b) into LU.cpp and obtain the temperature profile for $k = 5$, $R_i = 1$, $R_o = 5$, $T(R_i) = 5$, $T(R_o) = 30$, $q(r) = 0$. Use $n = 21$ nodes (i.e., $\Delta r = 0.2$). Compare your prediction to the analytical solution from (a).

**Problem 4** (5 points) Consider the system Ax = b with

$$A = \begin{pmatrix} 3.01 & 6.03 & 1.99 \\ 1.27 & 4.16 & -1.23 \\ 0.987 & -4.81 & 9.34 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Use LU.cpp to solve for x. Do not submit the code.
(b) Modify A such that $a_{11} = 3.00$ and $a_{31} = 0.99$. Use LU.cpp to solve the system, calling the solution x'.
(c) Discuss your results from (a) and (b) in the context of the inequality derived in class:

$$\frac{\|x - x'\|}{\|x'\|} \le Co(A)\frac{\|E\|}{\|A\|}.$$