24-703/12-703 Numerical Methods in Engineering S25 Homework 6
Due 14 April 2025, 10:00AM

The time-dependent temperature profile in a one-dimensional beam is described by the parabolic partial differential equation

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}.$$

In Problems 1 and 2, you will solve this PDE with boundary conditions $T(0, t) = T(L, t) = 0$ and initial condition $T(x,0) = 100\left[1 - 4(x - 0.5)^2\right]$ in two different ways. Take $\alpha = 0.01$ and $L = 1$. You can use any computational tools that you like (e.g., C++, Matlab).

Problem 1 (15 points) Analytical and approximate solutions through Fourier series analysis

The analytical solution for $L = 1$ is:

$$T(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\exp\left[-(n\pi)^2 \alpha t\right].$$

By applying the initial condition, the B_n values are obtained from

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x),$$

which is a Fourier sine series over (0,1).

(a) Evaluate the B_n coefficients analytically. Plot B_n as a function of n for $1 \leq n \leq 10$.

(b) Using the result of (a), plot $T(x, t)$ for $t = 0.1$, 1, and 10 for an upper limit on the solution summation of (i) 3 and (ii) 5. For a given time, put the results for each of (i) and (ii) on the same plot.

(c) Take $\Delta x = 0.05$ and obtain all the B_n values up to $n = 10$ using the rectangle rule for integration. Plot the results with those from part (a). How does the error introduced by the numerical integration change as n is increased?

(d) Plot $T(x, t)$ for $t = 0.1$, 1, and 10 and upper limits on the summation of (i) 3 and (ii) 5 based on the results of (c). Compare the solutions to the results from (b).

Problem 2 (10 points) Finite Difference Solution

We showed in our previous lecture that this problem can be solved explicitly by:

$$T_i^{j+1} = r\left(T_{i+1}^j + T_{i-1}^j\right) + (1 - 2r)T_i^j,$$

where the i index corresponds to the spatial location and the j index corresponds to time. This solution is stable when $r = \alpha\Delta t / (\Delta x)^2 < 0.5$.

(a) Determine Δt for $r = 0.1$ and $\Delta x = 0.05$.
(b) Using any method you like, find and plot $T(x, t)$ for $t = 0.1$, 1, and 10 on the same graph.
(c) Compare your results from (b) to those from 1(d). Which numerical method agrees better with the exact solution? Which method is more computationally expensive for a given Δx?