24-703/12-703 Numerical Methods in Engineering S25 Homework 4
Due 19 February 2025, 10:00AM

Problem 1 (15 points) The solution to the ordinary differential equation

$$\frac{d^2 y}{dx^2} + k^2 y = 0$$

with boundary conditions

$$y(0) = 0 \text{ and } y(1) = 0$$

Reduces to finding the values of *k* that satisfy

$$\sin k = 0.$$

When this governing equation is the spatial part of the wave equation, *k* is the wavevector, related to a wavelength *l* through $k = 2\pi / l$.

As discussed in class, this problem can be solved by using a finite difference approach to generate the eigenvalue problem

$$(A - \lambda I)y = 0,$$

where $\lambda = k^2 \Delta x^2$. For a discretization using *n* nodes and a total length of 1 [i.e., $\Delta x = 1/(n-1)$], A is an *n* -2 by *n* -2 matrix:

$$A = \begin{pmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & & \\ & -1 & 2 & -1 & & & & & \\ & & -1 & 2 & -1 & & & & \\ & & & & \cdots & \cdots & & & \\ & & & & & -1 & 2 & -1 & \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 2 \end{pmatrix}$$

Using a math program of your choice (e.g., Matlab, Mathematica), plot all the values of *l* generated by finding the eigenvalues of A for *n* values between 3 and 20. Discuss the convergence to the exact values. How many nodes are needed to get the three highest wavelengths to within 0.001 of the exact values? Include any program that you wrote to do this problem in your submission.

Problem 2 (15 points) Consider the mass-spring system as shown in Fig. 1.

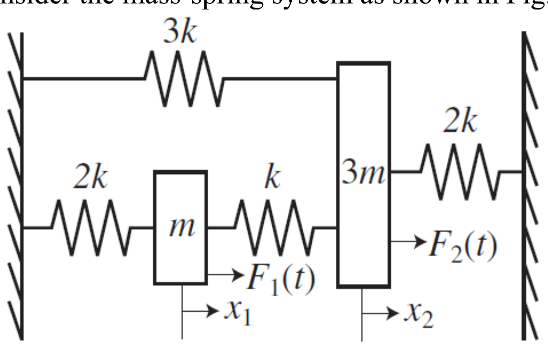

Figure 1: Mass-spring system.

(a) Determine the equations of motion for the two masses.

(b) For $k = m = 1$, determine the mass and stiffness matrices, and from these obtain the dynamical matrix.

(c) Use the power method to find the natural frequencies of the system. You can either do the calculations by hand or with an Excel spreadsheet. Show all your calculations and be sure to perform enough iterations to get a converged solution.

(d) Convert the equations of motion from (a) into first-order equations and implement these equations into the RK4.cpp program. Find and plot x_1 and x_2 for $0 < t < 50$, $m = k = 1$, initial conditions $x_1(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$, and $x_2(0) = 1$, and forcing functions $F_1(t) = 2e^{-t/2}$ and $F_2(t) = 0$.

Problem 3 (10 points) As a means to damp the motion of the masses in Fig. 1, a damper is added to the system, as shown in Fig. 2(a). The damper exerts a force proportional to the velocity that is opposite to the direction of motion, with magnitude $c\dot{x}_2$. For example, the equation of motion for the mass in Fig. 2(b) is $m\ddot{x}_a = -kx_a - c\dot{x}_a$.

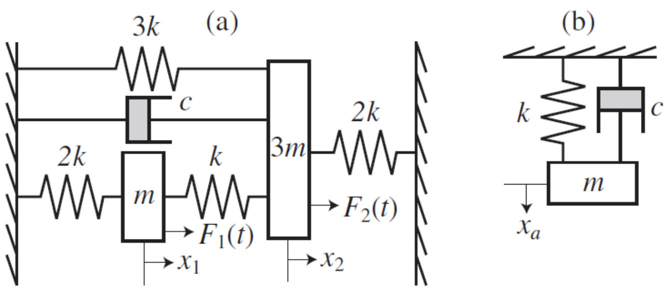

Figure 2: (a) Mass-spring-damper system. (b) Simple mass-spring-damper system.

(a) Find the equations of motion for the two masses in Fig. 2(a) and convert them into first-order equations.

(b) Implement the equations from (a) into RK4.cpp. Find and plot x_1 and x_2 for $0 < t < 50$, $m = k = 1$, and initial conditions $x_1(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$, and $x_2(0) = 1$, and forcing functions $F_1(t) = F_2(t) = 0$, for c =0, 0.1, 1, and 10. What is the effect of c on the motion of the masses?