Solutions to Ireland, Rosen "A Classical Introduction to Modern Number Theory"

Richard Ganaye

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Chapter 9

Ex. 9.1 *If* $\alpha \in \mathbb{Z}[\omega]$*, show that* α *is congruent to either* $0, 1,$ *or* -1 *modulo* $1-\omega$*.*

Proof. Let $\lambda = 1 - \omega$, and $\alpha = a + b\omega \in D = \mathbb{Z}[\omega], a, b \in \mathbb{Z}$.

$\omega \equiv 1 \pmod{\lambda}$, so $\alpha \equiv a + b \pmod{\lambda}$, $\alpha \equiv c$ with $c = a + b \in \mathbb{Z}$.

$c \equiv 0, 1, -1 \pmod 3$, and since $\lambda \mid 3$, $c \equiv 0, 1, -1 \pmod{\lambda}$.

Every $\alpha \in D$ is congruent to either $0, 1$, or -1 modulo $\lambda = 1 - \omega$.

The classes of $0, 1, -1$ in $D/\lambda D$ are distinct. Indeed, $1 \not\equiv -1 \pmod{\lambda}$, if not $\lambda \mid 2$, so $2 = \lambda\lambda'$, $N(2) = N(\lambda)N(\lambda')$, thus $4 = 3N(\lambda')$, so $3 \mid 4$, which is nonsense.

$\pm 1 \equiv 0 \pmod{\lambda}$ implies $\lambda \mid 1$, so λ would be a unit, in contradiction with λ prime.

So there exist exactly three classes modulo λ in D : $|D/\lambda D| = 3 = N(\lambda)$.

□

Ex. 9.2 *From now on we shall set* $D = \mathbb{Z}[\omega]$ *and* $\lambda = 1 - \omega$*. For* μ *in* D *show that we can write* $\mu = (-1)^a \omega^b \lambda^c \pi_1^{a_1} \pi_2^{a_2} \cdots \pi_t^{a_t}$*, where* $a, b, c,$ *and the* a_i *are nonnegative integers and the* π_i *are primary primes.*

Proof. Let S the set containing $\lambda = 1 - \omega$ and all primary primes.

We show that

(a) every prime in D is associate to a prime in S,

(b) no two primes in S are associate.

Let π be a prime in D. There are three cases.

- If $\mathrm{N}(\pi) = 3$, then π is associate to $\lambda \in S$, thus $\pi \in \{1 - \omega, -1 + \omega, -2 - \omega, 2 + \omega, 1 + 2\omega, -1 - 2\omega\}$, and no associate of λ is primary.
- If $\mathrm{N}(\pi) = q^2$, where $q \equiv -1 \pmod 3, q > 0$, is a rational prime, then π is associate to q (Proposition 9.1.2), and q is a primary prime. The primes associate to q are $q, -q, \omega q, -\omega q, -q - \omega q, q + \omega q$, so only q is primary.
- If $\mathrm{N}(\pi) = p$, where $p \equiv 1 \pmod 3$, then the proposition 9.1.4. shows that among the associates to π exactly one is primary.

Moreover, the norm of two primes belonging to two different cases are distinct, so two such primes are not associate.

By Theorem 3, Chapter 1, as $D = \mathbb{Z}[\omega]$ is a principal ideal domain, every $\mu \in D$ is of the form

$$\mu = u \prod_{\pi \in S} \pi^{e(\pi)},$$

where u is a unit, so $u = (-1)^a \omega^b$. Thus

$$\mu = (-1)^a \omega^b \lambda^c \pi_1^{a_1} \pi_2^{a_2} \cdots \pi_t^{a_t},$$

where the π are primary primes, and a, b, c and the a_i are nonnegative integers. □

Ex. 9.3 *Let γ a primary prime. To evaluate $\chi_\gamma(\mu)$ we see, by Exercise 2, that it is enough to evaluate $\chi_\gamma(-1), \chi_\gamma(\omega), \chi_\gamma(\lambda)$, and $\chi_\gamma(\pi)$, where π is a primary prime. Since $-1 = (-1)^3$ we have $\chi_\gamma(-1) = 1$. We now consider $\chi_\gamma(\omega)$. Let $\gamma = a + b\omega$ and set $a = 3m - 1$ and $b = 3n$. Show that $\chi_\gamma(\omega) = \omega^{m+n}$.*

Proof. Let $\gamma = a + b\omega = 3m - 1 + 3n\omega$. Then $\chi_\gamma(\omega) = \omega^{\frac{N(\gamma)-1}{3}}$ (remark (b) of Theorem 1).

$$\begin{aligned} N(\gamma) - 1 &= (3m-1)^2 + (3n)^2 - 3n(3m-1) - 1 \\ &= 9m^2 - 6m + 9n^2 - 9nm + 3n \\ \frac{N(\gamma)-1}{3} &= 3m^2 - 2m + 3n^2 - 3nm + n \equiv n + m \ [3] \end{aligned}$$

Thus, for $\gamma = a + b\omega = 3m - 1 + 3n\omega$,

$$\chi_\gamma(\omega) = \omega^{\frac{N(\gamma)-1}{3}} = \omega^{n+m}.$$

□

Ex. 9.4 *(continuation) Show that $\chi_\gamma(\omega) = 1, \omega$, or ω^2 according to whether γ is congruent to 8,2, or 5 modulo 3λ. In particular, if q is a rational prime, $q \equiv 2 \pmod 3$, then $\chi_q(\omega) = 1, \omega$, or ω^2 according to whether $q \equiv 8, 2$, or $5 \pmod 9$. [Hint : $\gamma = a + b\omega = -1 + 3(m + n\omega)$, and so $\gamma \equiv -1 + 3(m+n) \pmod{3\lambda}$.]*

Proof. $\lambda = 1 - \omega$, so $\omega \equiv 1 \pmod{\lambda}$. Thus

$$\begin{aligned} m + n\omega &\equiv m + n \pmod{\lambda} \\ 3(m + n\omega) &\equiv 3(m+n) \pmod{3\lambda} \\ \gamma = -1 + 3(m + n\omega) &\equiv -1 + 3(m+n) \pmod{3\lambda} \end{aligned}$$

Moreover $9 = 3\lambda\bar{\lambda} \equiv 0 \pmod{3\lambda}$, thus γ is congruent modulo 3λ to an integer between 0 and 8 of the form $3k - 1$: $\gamma \equiv 8, 2$ or $5 \pmod{3\lambda}$.

By Ex. 9.3, $\chi_\gamma(\omega) = 1 \iff m + n \equiv 0$ [3], and $m + n \equiv 0$ [3] implies $m + n = 3k, k \in \mathbb{Z}$, so $\gamma \equiv -1 + 9k \equiv -1 \equiv 8$ [3λ].

Conversely, if $\gamma \equiv 8 \equiv -1$ [3λ], then $3\lambda \mid 3(m+n)$, so $\lambda \mid m+n$, and $\mathrm{N}(\lambda) \mid \mathrm{N}(m+n)$, $3 \mid (m+n)^2$, thus $3 \mid m+n$, $m+n \equiv 0$ [3], and so $\chi_\gamma(\omega) = 1$. The two other cases are similar, so we obtain

$$\begin{aligned}
\chi_\gamma(\omega) = 1 &\iff m+n \equiv 0\ [3] \iff \gamma \equiv 8\ [3\lambda],\\
\chi_\gamma(\omega) = \omega &\iff m+n \equiv 1\ [3] \iff \gamma \equiv 2\ [3\lambda],\\
\chi_\gamma(\omega) = \omega^2 &\iff m+n \equiv 2\ [3] \iff \gamma \equiv 5\ [3\lambda].
\end{aligned}$$

If $\gamma = q$ is a rational prime, $q \equiv 8$ [9] implies $q \equiv 8$ $[3\lambda]$, since $3\lambda \mid 9 = 3\lambda\bar{\lambda}$, thus $\chi_q(\omega) = 1$.

Conversely, if $\chi_q(\omega) = 1$, then $q \equiv 8$ $[3\lambda]$, $q - 8 = \mu(3\lambda), \mu \in D$, therefore

$(q-8)^2 = N(\mu)3^3, 3^3 \mid (q-8)^2$, thus $3^2 \mid q - 8$ and so $q \equiv 8$ [9]. The two other cases are similar.

$$\begin{aligned}
\chi_q(\omega) = 1 &\iff q \equiv 8\ [9],\\
\chi_q(\omega) = \omega &\iff q \equiv 2\ [9],\\
\chi_q(\omega) = \omega^2 &\iff q \equiv 5\ [9].
\end{aligned}$$

□

Ex. 9.5 *In the text we stated Eisenstein's result $\chi_\gamma(\lambda) = \omega^{2m}$. Show that $\chi_\gamma(3) = \omega^{2n}$.*

Proof. Here $\gamma = (3m-1) + 3n\omega$.

Note that $(1-\omega)^2 = -3\omega$, thus $\chi_\gamma((1-\omega)^2) = \chi_\gamma(-1)\chi_\gamma(3)\chi_\gamma(\omega)$.

Using Eisenstein's result (see a proof in Ex.24-26),

$$\chi_\gamma((1-\omega)^2) = \chi_\gamma(\lambda^2) = \chi_\gamma(\lambda)^2 = \omega^{4m} = \omega^m.$$

As $-1 = (-1)^3, \chi_\gamma(-1) = 1$. Finally $\chi_\gamma(\omega) = \omega^{m+n}$ by Exercise 9.3. Thus

$$\omega^m = \chi_\gamma(3)\omega^{m+n}, \qquad \chi_\gamma(3) = \omega^{-n} = \omega^{2n}.$$

In conclusion,

$$\chi_\gamma(3) = \omega^{2n}.$$

□

Ex. 9.6 *Prove that*

(a) $\chi_\gamma(\lambda) = 1$ *for* $\gamma \equiv 8, 8+3\omega, 8+6\omega$ [9].

(b) $\chi_\gamma(\lambda) = \omega$ *for* $\gamma \equiv 5, 5+3\omega, 5+6\omega$ [9].

(c) $\chi_\gamma(\lambda) = \omega^2$ *for* $\gamma \equiv 2, 2+3\omega, 2+6\omega$ [9].

Proof. Here $\gamma = -1 + 3(m + n\omega)$ is a primary prime, and $\chi_\gamma(\lambda) = \omega^{2m}$.

$$\begin{aligned}
\chi_\gamma(\lambda) = 1 &\iff m \equiv 0\ [3] \Rightarrow \gamma \equiv 8 + 3n\omega\ [9] \Rightarrow \gamma \equiv 8, 8+3\omega, 8+6\omega\ [9]\\
\chi_\gamma(\lambda) = \omega &\iff m \equiv 2\ [3] \Rightarrow \gamma \equiv 5 + 3n\omega\ [9] \Rightarrow \gamma \equiv 5, 5+3\omega, 5+6\omega\ [9]\\
\chi_\gamma(\lambda) = \omega^2 &\iff m \equiv 1\ [3] \Rightarrow \gamma \equiv 2 + 3n\omega\ [9] \Rightarrow \gamma \equiv 2, 2+3\omega, 2+6\omega\ [9]
\end{aligned}$$

As $\chi_\gamma(\lambda) \in \{1, \omega, \omega^2\}$, these 9 cases are the only possibilities. Moreover these 9 cases are mutually exclusive, since 9 doesn't divide any difference. Thus the reciprocals are true.

$$\begin{aligned}
\chi_\gamma(\lambda) = 1 &\iff \gamma \equiv 8, 8+3\omega, 8+6\omega \ [9] \\
\chi_\gamma(\lambda) = \omega &\iff \gamma \equiv 5, 5+3\omega, 5+6\omega \ [9] \\
\chi_\gamma(\lambda) = \omega^2 &\iff \gamma \equiv 2, 2+3\omega, 2+6\omega \ [9]
\end{aligned}$$

$\square$

Ex. 9.7 *Find primary primes associate to* $1-2\omega, -7-3\omega$, *and* $3-\omega$.

Proof. :

- $(1-2\omega)\omega = 2+3\omega \equiv 2 \pmod 3$, so $2+3\omega$ is primary, and associate to $1-2\omega$. $\mathrm{N}(2+3\omega) = 7$ and 7 is a rational prime, thus $2+3\omega$ is a primary prime.
- $-7-3\omega \equiv 2 \pmod 3$.

  $\mathrm{N}(-7-3\omega) = 37$ and 37 is a rational prime, thus $-7-3\omega$ is a primary prime.
- $(3-\omega)\omega^2 = -4-3\omega \equiv 2 \pmod 3$, so $-4-3\omega$ is primary, and associate to $3-\omega$.

  $\mathrm{N}(-4-3\omega) = 13$ and 13 is a rational prime, thus $-4-3\omega$ is a primary prime.

$\square$

Ex. 9.8 *Factor the following numbers into primes in* D : $7, 21, 45, 22$, *and* 143.

Proof. $7 = \mathrm{N}(2+3\omega)$, thus $7 = (2+3\omega)(2+3\omega^2) = (2+3\omega)(-1-3\omega)$, where $2+3\omega$ and $-1-3\omega$ are primes in D, since their norm is a prime integer. Since these primes are primary, they are not associate.

$21 = 3 \times 7 = -\omega^2\lambda^2(2+3\omega)(-1-3\omega)$ since $3 = -\omega^2(1-\omega)^2$.

$45 = 3^2 \times 5 = \omega\lambda^4 5$, where $5 \equiv 2 \pmod 3$ is a primary prime in D.

$22 = 2 \times 11$, where 2 and 11 are primes in D.

$143 = 11 \times 13 = 11(-4-3\omega)(-4-3\omega^2) = 11(-4-3\omega)(-1+3\omega)$. $\square$

Ex. 9.9 *Show that* $\overline{\alpha} \neq 0$, *the residue class of* α, *is a cube in the field* $D/\pi D$ *iff* $\alpha^{(N\pi-1)/3} \equiv 1 \pmod \pi$. *Conclude that there are* $(N\pi - 1)/3$ *cubes in* $(D/\pi D)^*$.

Solution 1:

Proof. Let π be a prime in D, $N\pi \neq 3$, and $\alpha \in D, \pi \nmid \alpha$.

$\overline{\alpha}$ is a cube in $(D/\pi D)^*$

$\iff x^3 \equiv \alpha \pmod \pi$ has a solution in D

$\iff \chi_\pi(\alpha) = 1$ (by Prop. 9.3.3(a))

$\iff \alpha^{\frac{N\pi-1}{3}} \equiv 1 \pmod \pi$

$\iff \overline{\alpha}^{\frac{N\pi-1}{3}} = \overline{1}$.

The cubes in $(D/\pi D)^*$ are then the roots of the polynomial $f(x) = x^{\frac{N\pi-1}{3}} - \overline{1}$ in $D/\pi D$.

Let q be the cardinal of the field $D/\pi D$. Since $q = |D/\pi D| = N\pi$, $\frac{N\pi-1}{3} \mid q-1$, $f(x) \mid x^{q-1}-1 \mid x^q - x$. By Corollary 2 of Proposition 7.1.1, f has $\deg(f) = \frac{N\pi-1}{3}$ roots.

Conclusion: there are exactly $\frac{N\pi-1}{3}$ cubes in $(D/\pi D)^*$. □

Solution 2:

Proof. Let $\varphi : (D/\pi D)^* \to (D/\pi D)^*$ be the group homomorphism defined by $\varphi(x) = x^3$.

Then $\mathrm{im}(\varphi)$ is the set of cubes in $(D/\pi D)^*$.

The equation $x^3 = \overline{1}$ has three distinct solutions $\overline{1}, \overline{\omega}, \overline{\omega}^2$ in $D/\pi D$ if $N\pi \neq 3$ (see the demonstration of Proposition 9.3.1).

Thus $\ker(\varphi) = \{\overline{1}, \overline{\omega}, \overline{\omega}^2\}$ and $|\ker(\varphi)| = 3$. Therefore $|\mathrm{im}(\varphi)| = |(D/\pi D)^*|/|\ker(\varphi)| = (N\pi - 1)/3$. There exist exactly $\frac{N\pi-1}{3}$ cubes in $(D/\pi D)^*$. □

Note: if $N\pi = 3$, that is to say, if π is associate to $1-\omega$, $D/\pi D = \{\overline{0}, \overline{1}, \overline{2}\}$. As $\overline{1}^3 = \overline{1}, \overline{2}^3 = \overline{2}$, all the elements of $(D/\pi D)^*$ are cubes.

Ex. 9.10 *What is the factorisation of $x^{24}-1$ in $D/5D$.*

Proof. $|(D/5D)^*| = N(5) - 1 = 24$, thus $x^{24} - 1 = \prod_{\alpha \in (D/5D)^*} (x - \alpha)$.

(where the $\alpha \in (D/5D)^*$ are of the form $\alpha = a + b\,[\omega]$, $0 \le a < 5, 0 \le b < 5, (a,b) \neq (0,0)$). □

Ex. 9.11 *How many cubes are there in $D/5D$?*

Proof. By Exercise 9.9, there exist $(N(5)-1)/3 = 8$ cubes in $(D/5D)^*$ (and $0 = 0^3$ is a cube). □

Ex. 9.12 *Show that $\omega\lambda$ has order 8 in $D/5D$ and that $\omega^2\lambda$ has order 24. [Hint : Show first that $(\omega\lambda)^2$ has order 4.]*

Proof. If $\alpha = (\omega\lambda)^2$, then

$$\alpha = (\omega\lambda)^2 = \omega^2(1-\omega)^2 = \omega^2(1+\omega^2-2\omega) = -3\omega^3 = -3.$$

So $\alpha^2 = 9 \equiv -1 \pmod 5, \alpha^4 \equiv 1 \pmod 5$ and $\alpha^2 \not\equiv 1 \pmod 5$, thus the class of $\alpha = (\omega\lambda)^2$ has order 4 in $(D/5D)^*$, and this implies that $\omega\lambda$ has order 8.

Let $\beta = \omega^2\lambda$. $|(D/5D)^*| = 24$, thus $[\beta]^{24} = 1$ (where $[\beta]$ is the class of β in $D/5D$.)

To verify that $[\beta]$ has order 24, it is sufficient to show that $[\beta]^8 \neq 1, [\beta]^{12} \neq 1$.

$\beta^8 = \omega^{16}\lambda^8 = \omega\lambda^8 = (\omega\lambda)^8\omega^2 \equiv \omega^2 \not\equiv 1 \pmod 5$.

$\beta^{12} = (\omega^2\lambda)^{12} = \lambda^{12} = (\omega\lambda)^{12} \equiv (\omega\lambda)^4 \equiv -1 \pmod 5$ (since $(\omega\lambda)$ has order 8 in $D/5D$).

Conclusion: $\omega\lambda$ has order 8, $\omega\lambda^2$ has order 24 in $(D/5D)^*$. □

Ex. 9.13 *Show that π is a cube in $D/5D$ iff $\pi \equiv 1, 2, 3, 4, 1+2\omega, 2+4\omega, 3+\omega,$ or $4+3\omega \pmod 5$.*

Proof. Let $\pi \in D, [\pi] \neq 0$. Then $[\pi]$ is a cube in $D/5D$ iff $[\pi]^{(q^2-1)/3} = 1$, with $q = 5$, namely $[\pi]^8 = 1$ (Prop. 7.1.2, where $3 \mid q^2 - 1 = 24 = |(D/5D)^*|$).

By Exercise 9.12, the class of $\gamma = \omega\lambda$ has order 8, thus the 8 elements $[\gamma]^k, 0 \leq k \leq 7$ are distinct roots of the polynomial $x^8 - 1$, which has at most 8 roots. Therefore the subgroup of cubes in $(D/5D)^*$ is

$$\{1, [\gamma], [\gamma]^2, \ldots, [\gamma]^7\}.$$

$\gamma = \omega(1-\omega) = \omega + 1 + \omega = 1 + 2\omega$, so

$$\begin{aligned}
\gamma^0 &= 1\\
\gamma^1 &= 1 + 2\omega\\
\gamma^2 &\equiv -3 \equiv 2\ [5] \qquad \text{(Ex. 9.12)}\\
\gamma^3 &= -3 - 6\omega \equiv 2 + 4\omega\ [5]\\
\gamma^4 &\equiv -1 \equiv 4\ [5]\\
\gamma^5 &\equiv -1 - 2\omega \equiv 4 + 3\omega\ [5]\\
\gamma^6 &\equiv 3\ [5]\\
\gamma^7 &\equiv 3 + 6\omega \equiv 3 + \omega\ [5]
\end{aligned}$$

Conclusion: If $\pi \not\equiv 0 \pmod 5$, $\pi \equiv \alpha^3 \pmod 5, \alpha \in D$ iff

$$\pi \equiv 1, 2, 3, 4, 1+2\omega, 2+4\omega, 3+\omega, 4+3\omega\ [5].$$

□

Ex. 9.14 *For which primes $\pi \in D$ is $x^3 \equiv 5 \pmod \pi$ solvable ?*

Proof. If π is associate to 5, then $5^3 \equiv 0 \equiv 5 \pmod \pi$, so $x^3 \equiv 5 \pmod \pi$ is solvable.

If π is a primary prime not associate to 5, the Law of Cubic Reciprocity gives

$$\begin{aligned}
5 \equiv x^3\ [\pi], x \in D &\iff \chi_\pi(5) = 1\\
&\iff \chi_5(\pi) = 1\\
&\iff \pi \text{ is a cube in } D/5D\\
&\iff \pi \equiv 1, 2, 3, 4, 1+2\omega, 2+4\omega, 3+\omega, 4+3\omega\ [5]
\end{aligned}$$

(see Ex. 9.13)

Conclusion: the equation $5 \equiv x^3\ [\pi], x \in D$ is solvable iff the primary prime associate to π is congruent modulo 5 to $1, 2, 3, 4, 1+2\omega, 2+4\omega, 3+\omega, 4+3\omega$ (or 0).

Examples:

• $q = 23$ is a primary prime congruent to 3 modulo 5, thus the equation $x^3 \equiv 5 \pmod{23}$ has a solution $x \in D$ ($x = 19$).

• $-4 - 3\omega$ is the primary prime associate to the prime $3 - \omega$, and $-4 - 3\omega \equiv 1 + 2\omega \pmod 5$, thus the equation $x^3 \equiv 5 \pmod{3 - \omega}$ has a solution $a + b\omega \in \mathbb{Z}[\omega]$.

Indeed, $7^3 \equiv 8^3 \equiv 11^3 \equiv 5 \pmod{13}$, and $3 - \omega \mid 13$, so $7^3 \equiv 8^3 \equiv 11^3 \equiv 5 \pmod{3 - \omega}$. □

Ex. 9.15 *Suppose that $p \equiv 1 \pmod 3$ and that $p = \pi\overline{\pi}$, where π is a primary prime in D. Show that $x^3 \equiv a \pmod p$ is solvable in $\mathbb{Z}$ iff $\chi_\pi(a) = 1$. We assume that $a \in \mathbb{Z}$.*

Proof. Since $\pi \mid p$, if $x^3 \equiv a \pmod p$, $x \in \mathbb{Z}$, then $x^3 \equiv a \pmod \pi$, thus $\chi_\pi(a) = 1$.

Conversely, suppose that $\chi_\pi(a) = 1$. Then the equation $y^3 \equiv a \pmod \pi$ has a solution $y = u + v\omega$, $u, v \in \mathbb{Z}$. Moreover, the class of y has a representative $x \in \mathbb{Z}$ modulo π (see the proof of Proposition 9.2.1) :

$$y \equiv x \pmod \pi, x \in \mathbb{Z}.$$

So $x^3 \equiv a \pmod \pi$ has a solution $x \in \mathbb{Z}$.

Thus $\pi \mid x^3 - a, N(\pi) = p \mid (x^3-a)^2$, therefore $p \mid x^3 - a$ in $\mathbb{Z}$, and so $x^3 \equiv a \pmod p$.

Conclusion: if $p \equiv 1 \pmod 3$, $p = \pi\overline{\pi}$, where π is a primary prime, and $a \in \mathbb{Z}$,

$$\exists x \in \mathbb{Z},\ x^3 \equiv a \pmod p \iff \chi_\pi(a) = 1.$$

In other words, $x^3 \equiv a \pmod \pi$ is solvable in D iff it is solvable in $\mathbb{Z}$. $\square$

Ex. 9.16 *Is $x^3 \equiv 2 - 3\omega \pmod{11}$ solvable ? Since $D/11D$ has 121 elements this is hard to resolve by straightforward checking. Fill in the details of the following proof that it is not solvable. $\chi_\pi(2 - 3\omega) = \chi_{2-3\omega}(11)$ and so we shall have a solution iff $x^3 \equiv 11 \pmod{2 - 3\omega}$ is solvable. This congruence is solvable iff $x^3 = 11 \pmod 7$ is solvable in $\mathbb{Z}$. However, $x^3 \equiv a \pmod 7$ is solvable in $\mathbb{Z}$ iff $a \equiv 1$ or $6 \pmod 7$.*

Warning: false sentence, since

$$N(2 - 3\omega) = (2 - 3\omega)(2 - 3\omega^2) = 4 + 9 - 6(\omega + \omega^2) = 4 + 9 + 6 = 19 \text{ (and not 7!)}.$$

Proof. Since 19 is a rational prime, and since $\pi = 2 - 3\omega$ and 11 are primary primes, by the Law of Cubic Reciprocity, and by Exercise 9.15 (with $p = 19 \equiv 1 \pmod 3$),

$$\begin{aligned} \exists x \in D,\ 2 - 3\omega \equiv x^3\ [11] &\iff \chi_{11}(2 - 3\omega) = 1 \\ &\iff \chi_{2-3\omega}(11) = 1 \\ &\iff \exists x \in D,\ x^3 \equiv 11\ [2 - 3\omega] \\ &\iff \exists x \in \mathbb{Z},\ x^3 \equiv 11\ [19] \end{aligned}$$

Moreover, by Proposition 7.1.2 (with $p = 19$, $d = (p - 1) \wedge 3 = 3, (p - 1)/d = 6$),

$$\exists x \in \mathbb{Z},\ x^3 \equiv 11\ [19] \iff 11^6 \equiv 1 \pmod{19},$$

which is true : $11^6 = 121^3 = (19 \times 6 + 7)^3 \equiv 49 \times 7 \equiv 11 \times 7 \equiv 77 \equiv 1\ [19]$.

Conclusion: there exists $x \in D$ such that $2 - 3\omega \equiv x^3 \pmod{11}$.

With some computer code, we find a solution $x = 1 + 8\omega$ (and its associates $\omega^2 x = 7 - \omega, \omega x = -8 - 7\omega \equiv 3 + 4\omega \pmod{11}$) :

$$x^3 = (1 + 8\omega)^3 = 321 - 168\omega \equiv 2 - 3\omega \pmod{11}.$$

$\square$

Note: The sentence becomes true if we replace $2-3\omega$ by the primary prime $2+3\omega$. Since $N(2+3\omega)=7$, with the same reasoning,

$$\begin{aligned}\exists x \in D,\ 2+3\omega \equiv x^3\ [11] &\iff \chi_{2+3\omega}(11)=1\\ &\iff \exists x\in D,\ x^3 \equiv 11\ [2+3\omega]\\ &\iff \exists x \in \mathbb{Z},\ x^3\equiv 11 \equiv 4\ [7]\\ &\iff 4^2 \equiv 1 \pmod 7\end{aligned}$$

but $4^2 \equiv 2 \not\equiv 1 \pmod 7$, so the equation $x^3 \equiv 2+3\omega \pmod{11}$ is not solvable.

($x^3 \equiv a \pmod{11}$ is solvable in $\mathbb{Z}$ iff $a^{\frac{7-1}{3}} = a^2 \equiv 1 \pmod 7$ iff $a \equiv \pm 1 \pmod 7$.)

Ex. 9.17 *An element $\gamma \in D$ is called primary if $\gamma \equiv 2 \pmod 3$. If γ and ρ are primary, show that $-\gamma\rho$ is primary. If γ is primary, show that $\gamma = \pm\gamma_1\gamma_2\ldots\gamma_t$, where the γ_i are (not necessarily distinct) primary primes.*

Proof. If $\gamma \equiv 2, \rho \equiv 2 \pmod 3$, then $-\gamma\rho \equiv -2\times 2 \equiv 2 \pmod 3$, so $-\gamma\rho$ is primary.

By Ex. 9.2, γ can be written

$$\gamma = (-1)^a\omega^b\lambda^c\pi_1^{a_1}\cdots\pi_t^{a_t},$$

where $\pi_i \equiv 2 \pmod 3, a\in\{0,1\}, b\in\{0,1,2\}$.

As $\pi_i \equiv -1 \pmod 3$, and $\gamma \equiv -1 \pmod 3$, we obtain $\omega^b\lambda^c \equiv \pm 1 \pmod 3$. We prove that $b=c=0$.

Note that $\lambda^2 = (1-\omega)^2 = -3\omega \equiv 0 \pmod 3$. If $c \geq 2$, we would obtain $\gamma \equiv 0 \pmod 3$, in contradiction with the hypothesis, thus $c=0$ or $c=1$.

If $c=1$,

$$\omega^b\lambda^c \in \{1-\omega, \omega(1-\omega), \omega^2(1-\omega)\} = \{1-\omega, 1+2\omega, -2-\omega\}.$$

Since $1-\omega \not\equiv \pm1, 1+2\omega \not\equiv \pm 1, -2-\omega \not\equiv \pm 1 \pmod 3$, this is impossible, so $c=0$.

Then $\omega^b \equiv \pm 1 \pmod 3$, where $\omega^b \in \{1,\omega,-1-\omega\}$. Since $\omega \not\equiv \pm 1 \pmod 3$, and $-1-\omega \not\equiv \pm 1 \pmod 3$, then $\omega^b = 1, 0 \leq b \leq 2$, thus $b=0$.

Finally, $\gamma = (-1)^a\pi_1^{a_1}\cdots\pi_t^{a_t}$.

Conclusion: Every primary $\gamma \in D$ is under the form

$$\gamma = \pm\gamma_1\gamma_2\cdots\gamma_t,$$

where the γ_i are primary primes.

□

Ex. 9.18 *(continuation) If $\gamma = \pm\gamma_1\gamma_2\cdots\gamma_t$ is a primary decomposition of the primary element γ, define $\chi_\gamma(\alpha) = \chi_{\gamma_1}(\alpha)\chi_{\gamma_2}(\alpha)\cdots\chi_{\gamma_t}(\alpha)$. Prove that $\chi_\gamma(\alpha) = \chi_\gamma(\beta)$ if $\alpha\equiv\beta \pmod\gamma$ and $\chi_\gamma(\alpha\beta) = \chi_\gamma(\alpha)\chi_\gamma(\beta)$. If ρ is primary, show that $\chi_\rho(\alpha)\chi_\gamma(\alpha) = \chi_{-\rho\gamma}(\alpha)$.*

Proof. If $\alpha \equiv \beta\ [\gamma]$, then $\alpha\equiv\beta \pmod{\gamma_i}, 1 \leq i \leq t$, so $\chi_{\gamma_i}(\alpha) = \chi_{\gamma_i}(\beta)$, thus $\chi_\gamma(\alpha) = \chi_\gamma(\beta)$.

By Proposition 9.3.3,

$$\begin{aligned}\chi_\gamma(\alpha\beta) &= \chi_{\gamma_1}(\alpha\beta)\chi_{\gamma_2}(\alpha\beta)\cdots\chi_{\gamma_t}(\alpha\beta)\\ &= \chi_{\gamma_1}(\alpha)\chi_{\gamma_2}(\alpha)\cdots\chi_{\gamma_t}(\alpha)\chi_{\gamma_1}(\beta)\chi_{\gamma_2}(\beta)\cdots\chi_{\gamma_t}(\beta)\\ &= \chi_\gamma(\alpha)\chi_\gamma(\beta)\end{aligned}$$

Finally, if $\rho = \pm\rho_1\rho_2\cdots\rho_l$ is primary, then $-\rho\gamma = \pm\rho_1\rho_2\cdots\rho_l\gamma_1\gamma_2\cdots\gamma_t$ is primary by Ex. 9.17, therefore

$$\chi_{-\rho\gamma}(\alpha) = (\chi_{\rho_1}\chi_{\rho_2}\cdots\chi_{\rho_l}\chi_{\gamma_1}\chi_{\gamma_2}\cdots\chi_{\gamma_t})(\alpha) = \chi_\rho(\alpha)\chi_\gamma(\alpha).$$

□

Note: The unit -1 is primary by definition, and -1 is the opposite of the empty product, so for all α in D, $\chi_{-1}(\alpha) = 1$ by definition. The result of the exercises remain true if we accept the unit -1 as a primary element.

Ex. 9.19 *Suppose that $\gamma = A + B\omega$ is primary and that $A = 3M - 1$ and $B = 3N$. Prove that $\chi_\gamma(\omega) = \omega^{M+N}$ and that $\chi_\gamma(\lambda) = \omega^{2M}$.*

Proof. We verify first that if $\gamma = -\gamma_1\gamma_2$, with

$$\begin{array}{lll} \gamma = A + B\omega, & A = 3M - 1, & B = 3N, \\ \gamma_1 = A_1 + B_1\omega, & A_1 = 3M_1 - 1, & B_1 = 3N_1, \\ \gamma_2 = A_2 + B_2\omega, & A_2 = 3M_2 - 1, & B_2 = 3N_2, \end{array}$$

then $M \equiv M_1 + M_2 \pmod 3, N \equiv N_1 + N_2 \pmod 3$.

$$-\gamma_1\gamma_2 = -A_1A_2 + B_1B_2 + (-A_1B_2 - A_2B_1 + B_1B_2)\omega = A + B\omega,$$

therefore

$$3M - 1 = A = -A_1A_2 + B_1B_2 \equiv 3(M_1 + M_2) - 1 \pmod 9,$$

thus $M \equiv M_1 + M_2 \pmod 3$.

$$3N = B = -A_1B_2 - A_2B_1 + B_1B_2 \equiv 3(N_1 + N_2) \pmod 9,$$

thus $N \equiv N_1 + N_2 \pmod 3$.

By induction, if $\gamma = \pm\gamma_1\gamma_2\cdots\gamma_t = (-1)^{t-1}\gamma_1\gamma_2\cdots\gamma_t$, where $\gamma_i = A_i + B_i\omega, A_i = 3M_i - 1, B_i = 3N_i$, then

$$M \equiv M_1 + \cdots + M_t \pmod 3, N \equiv N_1 + \cdots + N_t \pmod 3.$$

By Exercise 9.3,

$$\begin{aligned} \chi_\gamma(\omega) &= \chi_{\gamma_1}(\omega)\cdots\chi_{\gamma_t}(\omega) \\ &= \omega^{M_1+N_1}\cdots\omega^{M_t+N_t} \\ &= \omega^{(M_1+\cdots+M_t)+(N_1+\cdots+N_t)} \\ &= \omega^{M+N}, \end{aligned}$$

and by Eisenstein's result,

$$\begin{aligned} \chi_\gamma(\lambda) &= \chi_{\gamma_1}(\lambda)\cdots\chi_{\gamma_t}(\lambda) \\ &= \omega^{2M_1}\cdots\omega^{2M_t} \\ &= \omega^{2(M_1+\cdots+M_t)} \\ &= \omega^{2M}. \end{aligned}$$

Conclusion: if $\gamma = 3M - 1 + 3N\omega$, then

$$\chi_\gamma(\omega) = \omega^{M+N}, \chi_\gamma(\lambda) = \omega^{2M}.$$

□

Ex. 9.20 *If γ and ρ are primary, show that $\chi_\gamma(\rho) = \chi_\rho(\gamma)$.*

Important note: The following solution assumes that $\chi_{\pi_2}(\pi_1) = \chi_{\pi_1}(\pi_2)$ for any pair π_1, π_2 of primary primes. But Theorem 1 uses the hypothesis $N(\pi_1) \neq N(\pi_2)$ to prove Cubic Reciprocity.

We can complete the proof in the case where $N(\pi_1) = N(\pi_2)$. Since π_1, π_2 are primary primes, then $\pi_1 = \pi_2$ or $\pi_1 = \overline{\pi_2}$.

In the case $\pi_1 = \pi_2$, then $\chi_{\pi_2}(\pi_1) = 0 = \chi_{\pi_1}(\pi_2)$.

To prove that $\left(\frac{\overline{\pi}}{\pi}\right) = \left(\frac{\pi}{\overline{\pi}}\right)$, we begin with a particular case of the proposition:

Lemma. *Let $n \in \mathbb{Z}$ be a primary element in A, and let π be a primary prime such that $N(\pi) = p \equiv 1 \pmod 3$. Then*

$$\left(\frac{n}{\pi}\right)_3 = \left(\frac{\pi}{n}\right)_3.$$

Proof. If $p \mid n$, then $\left(\frac{n}{\pi}\right)_3 = 0 = \left(\frac{\pi}{n}\right)_3$. Now we assume that $p \wedge n = 1$.

The decomposition of n is of the form

$$\begin{aligned} n &= \pm p_1 \cdots p_s q_1 \cdots q_r \ (p_i \equiv 1\ [3], q_j \equiv -1\ [3]) \\ &= \pm \pi_1 \overline{\pi_1} \cdots \pi_s \overline{\pi}_s q_1 \cdots q_r, \end{aligned}$$

where $\pi_i, \overline{\pi_i} (1 \leq i \leq s)$ and $q_j (1 \leq j \leq r)$ are primary prime.

Since $N(\pi_i) = p_i \neq p$ and $N(\pi) = p \neq N(q_j) = q_j^2$, Theorem 1 shows that

$$\begin{aligned} \left(\frac{n}{\pi}\right)_3 &= \left(\frac{\pi_1}{\pi}\right)_3 \left(\frac{\overline{\pi_1}}{\pi}\right)_3 \cdots \left(\frac{\pi_s}{\pi}\right)_3 \left(\frac{\overline{\pi_s}}{\pi}\right)_3 \left(\frac{q_1}{\pi}\right)_3 \cdots \left(\frac{q_r}{\pi}\right)_3 \\ &= \left(\frac{\pi}{\pi_1}\right)_3 \left(\frac{\pi}{\overline{\pi_1}}\right)_3 \cdots \left(\frac{\pi}{\pi_s}\right)_3 \left(\frac{\pi}{\overline{\pi_s}}\right)_3 \left(\frac{\pi}{q_1}\right)_3 \cdots \left(\frac{\pi}{q_r}\right)_3 \\ &= \left(\frac{\pi}{n}\right)_3. \end{aligned}$$

□

We can now remove the useless hypothesis $N(\pi_1) \neq N(\pi_2)$ in Theorem 1.

Proposition. Let π_1, π_2 be primary primes. Then

$$\left(\frac{\pi_2}{\pi_1}\right)_3 = \left(\frac{\pi_1}{\pi_2}\right)_3.$$

Proof. By theorem 1, it remains only the case where $N(\pi_1) = N(\pi_2)$.

If $\pi_1 = \pi_2$, then $\left(\frac{\pi_2}{\pi_1}\right)_3 = \left(\frac{\pi_1}{\pi_2}\right)_3 = 0$.

If $\pi_1 \neq \pi_2$, since π_1 et π_2 are primary, then π_1, π_2 are primes such that $N(\pi_1) = N(\pi_2) = p \equiv 1 \pmod 3$, and $\pi_2 = \overline{\pi_1}$. Writing $\pi = \pi_1$, it is sufficient to prove

$$\left(\frac{\overline{\pi}}{\pi}\right)_3 = \left(\frac{\pi}{\overline{\pi}}\right)_3.$$

We use the "Evans' trick" (see [Lemmermayer, Reciprocity Laws p. 215]). The element $n = -(\pi + \overline{\pi})$ is a rationnal integer, which is primary. The Lemma gives then

$$\begin{aligned}
\left(\frac{\overline{\pi}}{\pi}\right)_3 &= \left(\frac{\pi + \overline{\pi}}{\pi}\right)_3\\
&= \left(\frac{-\pi - \overline{\pi}}{\pi}\right)_3\\
&= \left(\frac{\pi}{-\pi - \overline{\pi}}\right)_3\\
&= \left(\frac{-\overline{\pi}}{-\pi - \overline{\pi}}\right)_3\\
&= \left(\frac{\overline{\pi}}{-\pi - \overline{\pi}}\right)_3\\
&= \left(\frac{-\pi - \overline{\pi}}{\overline{\pi}}\right)_3\\
&= \left(\frac{-\pi}{\overline{\pi}}\right)_3\\
&= \left(\frac{\pi}{\overline{\pi}}\right)_3
\end{aligned}$$

□

We can now give the solution of Exercise 9.20.

Proof. ρ, γ are written

$$\begin{aligned}
\rho &= \pm \rho_1 \rho_2 \cdots \rho_l,\\
\gamma &= \pm \gamma_1 \gamma_2 \cdots \gamma_m,
\end{aligned}$$

where ρ_i, γ_i are primary primes. By the law of Cubic Reciprocity, we obtain

$$\begin{aligned}
\chi_\gamma(\rho) &= \prod_{j=1}^{m} \chi_{\gamma_j}(\rho)\\
&= \prod_{j=1}^{m} \prod_{i=1}^{l} \chi_{\gamma_j}(\rho_i)\\
&= \prod_{i=1}^{l} \prod_{j=1}^{m} \chi_{\gamma_j}(\rho_i)\\
&= \prod_{i=1}^{l} \prod_{j=1}^{m} \chi_{\rho_i}(\gamma_j)\\
&= \prod_{i=1}^{l} \chi_{\rho_i}(\gamma)\\
&= \chi_\rho(\gamma).
\end{aligned}$$

(if $\gamma = -1$, or $\rho = -1$, some products are empty, but the result remains true: $\chi_{-1}(\rho) = 1 = \chi_\rho(-1)$.)

□

Ex. 9.21 *If γ is primary, show that there are infinitely many primary primes π such that $x^3 \equiv \gamma \pmod{\pi}$ is not solvable. Show also that there are infinitely many primary primes π such that $x^3 \equiv \omega \pmod{\pi}$ is not solvable and the same for $x^3 \equiv \lambda \pmod{\pi}$. (Hint: Imitate the proof of Theorem 3 of Chapter 5.)*

Proof. a) As some primary elements of D may be cubes, by example $53 + 36\omega = (-1+3\omega)^3$, we must of course suppose that γ is not the cube of some element of D (in the contrary case $x^3 \equiv \gamma \pmod{\pi}$ is solvable for all prime π).

Note first that for all primes π in D, there exists $\sigma \in D$ such that $\chi_\pi(\sigma) = \omega$. Indeed, there exist $(N\pi - 1)/3$ cubes in $(D/\pi D)^*$, which has $N\pi - 1$ elements, so there exists an element $\overline{\tau} \in (D/\pi D)^*$ which is not a cube, therefore there exists $\tau \in D$ such that $\chi_\pi(\tau) \neq 1$. If $\chi_\pi(\tau) = \omega$, we put $\sigma = \tau$ and if $\chi_\pi(\tau) = \omega^2$, we put $\sigma = \tau^2$. In the two cases, $\chi_\pi(\sigma) = \omega$.

Let $\gamma \in D$, where γ is primary. Then $\gamma = \pm\gamma_1^{n_1}\gamma_2^{n_2}\cdots\gamma_p^{n_p}$, where the γ_i are distinct primary primes. Write $n_i = 3q_i + r_i$, $r_i \in \{0,1,2\}$. Then grouping in γ' the γ^{r_i} such that $r_i \neq 0$, we can write $\gamma = \delta^3\gamma', \gamma' = \gamma_1^{r_1}\gamma_2^{r_2}\cdots\gamma_l^{r_l}, r_i \in \{1,2\}, \delta = \pm\gamma_1^{q_1}\cdots\gamma_p^{q_p} \in D$ (-1 is a cube). Since by hypothesis γ is not a cube, $l \geq 1$. Moreover the equation $x^3 \equiv \gamma \pmod{\pi}$ is solvable iff $x^3 \equiv \gamma' \pmod{\pi}$ is solvable. We may then suppose that

$$\gamma = \gamma_1^{r_1}\gamma_2^{r_2}\cdots\gamma_l^{r_l}, 1 \leq r_i \leq 2,$$

without cubic factors.

Note that the γ_i are not associate to $\lambda = 1 - \omega$ (see Ex. 9.17).

Let $A = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ a set (possibly empty) of distinct primary primes λ_i (therefore they are not associate), and not associate neither to $\gamma_i, 1 \leq i \leq l$, nor to $\lambda = 1 - \omega$.

We will show that we can find a primary prime λ_{k+1} distinct of the λ_i with the same properties and such that the equation $x^3 \equiv \lambda \pmod{\lambda_{k+1}}$ is not solvable. This will prove the existence of infinitely many primes π such that the equation $x^3 \equiv \lambda \pmod{\pi}$ is not solvable.

Using the initial note, let $\sigma \in D$ such that $\chi_{\gamma_l}(\sigma) = \omega$. As D is a principal ideal domain, the Chinese Remainder Theorem is valid. Since $3 = \lambda\overline{\lambda} = -\omega^2\lambda^2$ is relatively prime to γ_i, λ_i, there exists $\beta \in D$ such that

$$\begin{aligned}
&\beta \equiv 2\ [3], \\
&\beta \equiv 1\ [\lambda_i] \qquad (1 \leq i \leq k), \\
&\beta \equiv 1\ [\gamma_i] \qquad (1 \leq i \leq l-1), \\
&\beta \equiv \sigma\ [\gamma_l].
\end{aligned}$$

The first equation show that β is primary, so $\beta = (-1)^{m-1}\beta_1 \ldots \beta_m$, where the β_i are primary primes.

By Exercise 9.20,

$$\chi_\beta(\gamma) = \chi_\beta(\gamma_1)^{r_1}\cdots\chi_\beta(\gamma_l)^{r_l} = \chi_{\gamma_1}(\beta)^{r_1}\cdots\chi_{\gamma_l}(\beta)^{r_l}.$$

As $\chi_{\gamma_i}(1) = 1$ $(1 \leq i \leq l-1)$, and $\chi_{\gamma_l}(\beta) = \chi_{\gamma_l}(\sigma) = \omega$, we obtain $\chi_\beta(\gamma) = \omega^{r_l} \neq 1$, since $r_l = 1$ or $r_l = 2$.

By Exercise 9.18, $\chi_\rho(\alpha)\chi_\gamma(\alpha) = \chi_{-\rho\gamma}(\alpha)$, with primary ρ, γ, so by induction, as $\beta = (-1)^{m-1}\beta_1 \cdots \beta_m$,

$$\chi_\beta(\gamma) = \chi_{\beta_1}(\gamma) \cdots \chi_{\beta_m}(\gamma) \neq 1.$$

Thus there exists a subscript j such that $\chi_{\beta_j}(\gamma) \neq 1$.

We can then take $\lambda_{k+1} = \beta_j$. Indeed, since $\beta \equiv 1\ [\lambda_i]$ and $\beta \not\equiv 0\ [\gamma_i]$, β_j is distinct of the λ_i and γ_i, and β_j is not associate to λ since $\beta \equiv 2 \pmod 3$.

As $\chi_{\lambda_{k+1}}(\gamma) \neq 1$, the equation $x^3 \equiv \gamma\ [\lambda_{k+1}]$ is not solvable, so λ_{k+1} is convenient.

Conclusion : if $\gamma \in D$ is primary and is not a cube in D, there exist infinitely many primes $\pi \in D$ such that the equation $x^3 \equiv \gamma\ [\pi]$ is not solvable.

b) We show that $x^3 \equiv \omega\ [\pi]$ has no solution for infinitely many primes π.

To initialize the induction, we display such a prime π, namely $\pi = 2 + 3\omega$. Indeed, $N(\pi) = 4 + 9 - 6 = 7$, 7 is a rational prime, so π is a primary prime in D, of the form $\pi = 3m - 1 + 3n\omega$, with $n = m = 1$, so $\chi_\pi(\omega) = \omega^{m+n} = \omega^2 \neq 1$: the equation $x^3 \equiv \omega\ [\pi]$ is not solvable. Moreover π is not associate to $\lambda = 1 - \omega$.

Suppose now the existence of a set $A = \{\lambda_1, \lambda_2, \ldots, \lambda_l\}, l \geq 1$, of distinct primary primes λ_i, not associate to λ and such the equation $x^3 \equiv \omega\ [\lambda_i]$ is not solvable for each $i,\ 1 \leq i \leq l$. We will show that we can add a prime λ_{l+1} to the set A with the same properties.

Let

$$\beta = 3(-1)^{l-1}\lambda_1 \cdots \lambda_l - 1.$$

$(-1)^{l-1}\lambda_1 \cdots \lambda_l$ is primary, so $(-1)^{l-1}\lambda_1 \cdots \lambda_l = 3m - 1 + 3n\omega,\ m, n \in \mathbb{Z}$.

$\beta = 3(3m - 1 + 3n\omega) - 1 = 3(3m - 1) - 1 + 9n\omega = 3M - 1 + 3N\omega$, where $M = 3m - 1, N = 3n$. By Exercise 9.19,

$$\chi_\beta(\omega) = \omega^{M+N} = \omega^{3m-1+3n} = \omega^2 \neq 1.$$

As $\beta = \pm\beta_1 \cdots \beta_m$, where the β_i are primary primes, $\chi_\beta(\omega) = \chi_{\beta_1}(\omega) \cdots \chi_{\beta_m}(\omega) \neq 1$, so there exists a subscript i such that $\chi_{\beta_i}(\omega) \neq 1$.

Since $\beta = 3(-1)^{l-1}\lambda_1 \cdots \lambda_l - 1$, β_i is associate neither to λ_i nor to λ. Moreover $\chi_{\beta_i}(\omega) \neq 1$, thus the equation $x^3 \equiv \omega\ [\beta_i]$ is not solvable: $\lambda_{l+1} = \beta_i$ is convenient.

Conclusion: the equation $x^3 \equiv \omega\ [\pi]$ is not solvable for infinitely many primes π.

c) We show that $x^3 \equiv \lambda\ [\pi]$ has no solution for infinitely many primes π.

To initialize the induction, we display such a prime π, namely $\pi = -4 + 3\omega$. Indeed, $N(\pi) = 16 + 9 + 12 = 37$, 37 is a rational prime, so π is a primary prime in D, of the form $\pi = 3m - 1 + 3n\omega$, with $m = -1, n = 1$, so $\chi_\pi(\lambda) = \omega^{2m} = \omega \neq 1$: the equation $x^3 \equiv \lambda\ [\pi]$ is not solvable.

Suppose now the existence of a set $A = \{\lambda_1, \lambda_2, \ldots, \lambda_l\}, l \geq 1$, of distinct primary primes λ_i, not associate to λ and such the equation $x^3 \equiv \lambda\ [\lambda_i]$ is not solvable. We will show that we can add a prime λ_{l+1} to the set A with the same properties.

Let

$$\beta = 3(-1)^{l-1}\lambda_1 \cdots \lambda_l - 1.$$

$(-1)^{l-1}\lambda_1 \cdots \lambda_l$ is primary, so $(-1)^{l-1}\lambda_1 \cdots \lambda_l = 3m - 1 + 3n\omega,\ m, n \in \mathbb{Z}$.

$\beta = 3(3m-1+3n\omega) - 1 = 3(3m-1) - 1 + 9n\omega = 3M - 1 + 3N\omega$, where $M = 3m-1, N = 3n$. By Exercise 9.19,

$$\chi_\beta(\lambda) = \omega^{2M} = \omega^{2(3m-1)} = \omega \neq 1.$$

As $\beta = \pm\beta_1 \cdots \beta_m$, where the β_i are primary primes, $\chi_\beta(\omega) = \chi_{\beta_1}(\omega) \cdots \chi_{\beta_m}(\omega) \neq 1$, so there exists a subscript i such that $\chi_{\beta_i}(\lambda) \neq 1$.

Since $\beta = 3(-1)^{l-1}\lambda_1 \cdots \lambda_l - 1$, β_i is associate neither to λ_i nor to λ. Moreover $\chi_{\beta_i}(\lambda) \neq 1$, thus the equation $x^3 \equiv \lambda\ [\beta_i]$ is not solvable : $\lambda_{l+1} = \beta_i$ is convenient.

Conclusion : the equation $x^3 \equiv \lambda\ [\pi]$ is not solvable for infinitely many primes π.

□

Ex. 9.22 *(continuation) Show in general that if $\gamma \in D$ and $x^3 \equiv \gamma \pmod{\pi}$ is solvable for all but finitely many primary primes π, then γ is a cube in D.*

Proof. Let $\gamma \in D$ and suppose that γ is not a cube in D. We will show that the equation $x^3 \equiv \gamma\ [\pi]$ is not solvable for infinitely primes $\pi \in D$.

By Exercise 9.2, we can write

$$\gamma = (-1)^u \omega^v \lambda^w \gamma_1^{n_1} \cdots \gamma_p^{n_p},$$

where the γ_i are distinct primary primes, not associate to λ. Let $v = 3q + b, w = 3q' + c, n_i = 3q_i + r_i$, with the remainders b, c, r_i in $\{0, 1, 2\}$. Grouping the factors with null remainders, we obtain $\gamma = \delta^3\gamma', \gamma' = \omega^b\lambda^c\gamma_1^{r_1} \cdots \gamma_l^{r_l}$, with b, c, r_i in $\{1, 2\}, \delta \in D, l \geq 0$ (-1 is a cube).

Moreover the equation $x^3 \equiv \gamma\ [\pi]$ is solvable iff the equation $x^3 \equiv \gamma'\ [\pi]$ is solvable. So we may suppose that

$$\gamma = \omega^b\lambda^c\gamma_1^{r_1} \cdots \gamma_l^{r_l}, \qquad b \in \{1,2\}, c \in \{1,2\}, r_i \in \{1,2\},$$

without cubic factors.

- Case 1 : $l \geq 1$.

  Let $A = \{\lambda_1, \ldots, \lambda_k\}$ a possibly empty set of distinct primary primes λ_i, distinct of the γ_i, not associate to λ, and such that the equation $x^3 \equiv \gamma\ [\lambda_i]$ is not solvable. We will show that we can add a prime λ_{k+1} with the same properties.

  Suppose that $l \geq 1$. We have proved in Ex. 9.21 that there exists $\sigma \in D$ such that $\chi_{\gamma_l}(\sigma) = \omega$. Since $9, \lambda_i, \gamma_j$ are relatively prime, there exists $\beta \in D$ such that

  $$\begin{aligned} &\beta \equiv -1\ [9] \\ &\beta \equiv 1\ [\lambda_i], 1 \leq i \leq k \\ &\beta \equiv 1\ [\gamma_i], 1 \leq i \leq l-1 \\ &\beta \equiv \sigma\ [\gamma_l] \end{aligned}$$

  $\beta \equiv -1\ [9]$, thus $\beta \equiv -1\ [3]$: β is primary, of the form $\beta = 3M - 1 + 3N\omega$.

  $\beta = 3M - 1 + 3N\omega \equiv -1\ [9]$, so $3M + 3N\omega \equiv 0\ [9]$, $M + N\omega \equiv 0\ [3]$, thus $3 \mid M, 3 \mid N$.

By Exercise 9.18,

$$\begin{aligned}\chi_\beta(\omega) &= \omega^{M+N} = 1\\ \chi_\beta(\lambda) &= \omega^{2M} = 1\end{aligned}$$

As β and γ_i are primary, $\chi_\beta(\gamma_i) = \chi_{\gamma_i}(\beta) = \chi_{\gamma_i}(1) = 1$ $(1 \le i \le l-1)$.

$\chi_\beta(\gamma) = \chi_\beta(\omega)^b \chi_\beta(\lambda)^c \chi_\beta(\gamma_1)^{r_1} \cdots \chi_\beta(\gamma_l)^{r_l} = \chi_\beta(\gamma_l)^{r_l} = \chi_{\gamma_l}(\beta)^{r_l} = \chi_{\gamma_l}(\sigma)^{r_l} = \omega^{r_l} \ne 1$, since $r_l \in \{1,2\}$.

$\beta = \pm\beta_1 \cdots \beta_m$, with β_i primary primes, therefore

$$\chi_\beta(\gamma) = (\chi_{\beta_1} \cdots \chi_{\beta_m})(\gamma) \ne 1.$$

Thus there exists a subscript i such that $\chi_{\beta_i}(\gamma) \ne 1$, so $x^3 \equiv \gamma\ [\beta_i]$ is not solvable. Moreover $\beta \equiv 1\ [\gamma_i]$, so β_i is not associate to any γ_j. Similarly, β_i is not associate to any γ_j, and $\beta \equiv -1\ [9]$, therefore β_i is not associate to λ. So $\lambda_{k+1} = \beta_i$ is convenient.

There exist infinitely many π such that $x^3 \equiv \gamma\ [\pi]$ is not solvable.

- Case 2 : $l = 0$, so $\gamma = \omega^b \lambda^c,\ 1 \le b \le 2, 1 \le c \le 2$.

  $\pi_0 = 2 - 3\omega$ is a primary prime ($N(\pi_0) = 19$).

  Let $A = \{\lambda_1, \ldots, \lambda_k\}$ a possibly empty set of distinct primary primes $\lambda_i \ne \pi_0$ such that the equation $x^3 \equiv \gamma\ [\lambda_i]$ is not solvable. We will show that we can add a prime λ_{k+1} with the same properties.

  Let $\beta = 9(-1)^{k-1}\lambda_1 \cdots \lambda_k + 2 - 3\omega$.

  $\beta \equiv 2\ [3]$: β is primary.

  Moreover $(-1)^{k-1}\lambda_1 \cdots \lambda_k$ is primary, so

$$(-1)^{k-1}\lambda_1 \cdots \lambda_k = 3m - 1 + 3n\omega, m \in \mathbb{Z}, n \in \mathbb{Z}.$$

  Then

$$\begin{aligned}\beta &= 9(3m - 1 + 3n\omega) + 2 - 3\omega\\ &= 27m - 7 + (27n - 3)\omega\\ &= 3(9m-2) - 1 + 3(9n-1)\omega\\ &= 3M - 1 + 3N\omega,\end{aligned}$$

  where $M = 9m - 2, N = 9n - 1$. Therefore

$$\begin{aligned}\chi_\beta(\omega) &= \omega^{M+N} = \omega^{9m-2+9n-1} = 1\\ \chi_\beta(\lambda) &= \omega^{2M} = \omega^{2(9m-2)} = \omega^2 \ne 1\end{aligned}$$

  $\beta = \pm\beta_1 \cdots \beta_m$, where the β_i are primary primes.

  $\chi_\beta(\gamma) = \chi_\beta(\omega)^b \chi_\beta(\lambda)^c = \omega^{2c} \ne 1$ since $c = 1$ or $c = 2$.

$$\chi_\beta(\gamma) = (\chi_{\beta_1} \cdots \chi_{\beta_m})(\gamma) \ne 1.$$

  Thus there exists a subscript i such that $\chi_{\beta_i}(\gamma) \ne 1$, so $x^3 \equiv \gamma\ [\beta_i]$ is not solvable.

As $\beta_i \mid \beta = 9(-1)^{k-1}\lambda_1 \cdots \lambda_k + 2 - 3\omega$, if $\beta_i = \lambda_j$ for some subscript j, $\lambda_j \mid \pi_0 = 2 - 3\omega$, so $\lambda_j = \pi_0$, which is a contradiction, thus $\beta_i \notin A$. Similarly, if $\beta_i = \pi_0 = 2 - 3\omega$, then $\pi_0 \mid 9\lambda_1 \cdots \lambda_k$, and π_0 is relatively prime to λ, so $\pi_0 = \lambda_j$ for some subscript j : this is a contradiction, thus $\beta_i \neq \pi_0$. $\lambda_{k+1} = \beta_i$ is convenient.

So there exist infinitely many π such that $x^3 \equiv \gamma\ [\pi]$ is not solvable.

• Conclusion :

if γ is not a cube in D, there exist infinitely many primes π such that $x^3 \equiv \gamma\ [\pi]$ is not sovable.

By contraposition, if the equation $x^3 \equiv \gamma\ [\pi]$ is solvable for every prime π, at the exception perhaps of the primes in a finite set, then γ is a cube in D.

□

Ex. 9.23 *Suppose that $p \equiv 1 \pmod 3$. Use Exercise 5 to show that $x^3 \equiv 3 \pmod p$ is solvable in $\mathbb{Z}$ iff p is of the form $4p = C^2 + 243B^2$.*

Proof. Let p be a rational prime, $p \equiv 1 \pmod 3$, then $p = \pi\overline{\pi}$, where $\pi \in D$ is a primary prime : $\pi = a + b\omega = 3m - 1 + 3n\omega$.

- Suppose that there exists $x \in \mathbb{Z}$ such that $x^3 \equiv 3 \pmod p$. Then $x^3 \equiv 3 \pmod \pi$, so $\chi_\pi(3) = 1$. By Exercise 9.5, $\omega^{2n} = \chi_\pi(3) = 1$, thus $3 \mid n$, therefore $9 \mid b = 3n$, namely $b = 9B, B \in \mathbb{Z}$.

  $p = N\pi = a^2 + b^2 - ab, 4p = (2a-b)^2 + 3b^2 = C^2 + 243B^2$, where $C = 2a - b, B = b/9$. So there exists $C, B \in \mathbb{Z}$ such that $4p = C^2 + 243B^2$.

- Conversely, suppose that there exist $C, B \in \mathbb{Z}$ such that $4p = C^2 + 243B^2$.

  As $4p = (2a - b)^2 + 3b^2 = C^2 + 3(9B)^2$, from the unicity proved in Exercise 8.13, we obtain $b = \pm 9B$, so $9 \mid b = 3n, 3 \mid n$, and $\chi_\pi(3) = \omega^{2n} = 1$.

  Thus there exists $x \in D$ such that $x^3 \equiv 3 \pmod \pi$. As $p \equiv 1 \pmod 3$, $D/\pi D = \{\overline{0}, \ldots, \overline{p-1}\}$, so there exists $h \in \mathbb{Z}$ such that $x \equiv h \pmod \pi$, and $h^3 \equiv 3 \pmod \pi$. Therefore $p = N\pi \mid N(h^3 - 3)$, namely $p \mid (h^3 - 3)^2$, where p is a rational prime, thus $p \mid h^3 - 3$: there exists $x \in \mathbb{Z}$ such that $x^3 \equiv 3 \pmod p$.

  Moreover $4p = C^2 + 243B^2$ implies $p \equiv 1 \pmod 3$.

$$(p \equiv 1\ [3] \text{ and } \exists x \in \mathbb{Z},\ x^3 \equiv 3\ [p]) \iff \exists C \in \mathbb{Z}, \exists B \in \mathbb{Z},\ 4p = C^2 + 243B^2.$$

□

Ex. 9.24 *Let $\pi = a + b\omega$ be a complex primary element of $D = \mathbb{Z}[\omega]$. Put $a = 3m - 1, b = 3n, p = N(\pi)$.*

(a) $(p - 1)/3 \equiv -2m + n \pmod 3$.

(b) $(a^2 - 1)/3 \equiv m \pmod 3$.

(c) $\chi_\pi(a) = \omega^m$.

(d) $\chi_\pi(a + b) = \omega^{2n}\chi_\pi(1 - \omega)$.

Lemma. *Let $a \in \mathbb{Z}$, $a \equiv -1 \pmod 3$, and $b \in \mathbb{Z}$ such that $a \wedge b = 1$. Then $\chi_a(b) = 1$.*

Proof. (of Lemma.)

If q is a rational prime, $q \equiv 2 \pmod 3$, and $q \wedge b = 1$, then $\chi_q(b) = 1$ (Prop. 9.3.4, Corollary).

If p is a rational prime, $p \equiv 1 \pmod 3$ and $p \wedge b = 1$, then $p = \pi\overline{\pi}$, with π primary prime in D (and also $\overline{\pi}$), and by definition of χ_p, $\chi_p(b) = \chi_\pi(b)\chi_{\overline{\pi}}(b)$.

As $\chi_{\overline{\pi}}(b) = \chi_{\overline{\pi}}(\overline{b}) = \overline{\chi_\pi(b)}$ (Prop. 9.3.4(b)), then $\chi_p(b) = \chi_\pi(b)\chi_{\overline{\pi}}(b) = \chi_\pi(b)\overline{\chi_\pi(b)} = 1$.

a has a decomposition in prime factors of the form :

$$a = \pm q_1 q_2 \cdots q_k p_1 p_2 \cdots p_l = \pm q_1 q_2 \cdots q_k \pi_1 \overline{\pi_1} \pi_2 \overline{\pi_2} \cdots \pi_l \overline{\pi_l},$$

where $q_i \equiv -1, p_j \equiv 1 \pmod 3$, and the π_k are primary primes (since all these elements are primary, the symbol $\pm$ is $(-1)^{k-1}$). Thus, by definition of χ_a,

$$\chi_a(b) = \chi_{q_1}(b) \cdots \chi_{q_k}(b)\chi_{\pi_1}(b)\chi_{\overline{\pi_1}}(b) \cdots \chi_{\pi_l}(b)\chi_{\overline{\pi_l}}(b) = 1.$$

The result remains true if $a = -1$: then, by definition, $\chi_a(b) = 1$.

$\square$

Proof. (of Ex 9.24.) By hypothesis, π is a primary element, so $\pi = 3m - 1 + 3n\pi$, $m, n \in \mathbb{Z}$. We don't suppose in this proof that π is a prime element, so $p = N(\pi)$ is not necessarily prime.

(a) $p - 1 = (3m-1)^2 + (3n)^2 - 3n(3m-1) - 1 \equiv -6m + 3n \pmod 9$, thus

$$\frac{p-1}{3} \equiv -2m + n \pmod 3.$$

(b) $a^2 - 1 = (3m-1)^2 - 1 \equiv -6m \pmod 9$, thus

$$\frac{a^2-1}{3} \equiv m \pmod 3.$$

(c) As π, a are primary, by Exercise 9.20, $\chi_\pi(a) = \chi_a(\pi)$.

Since $\pi \equiv b\omega \pmod a$, $\chi_a(\pi) = \chi_a(b)\chi_a(\omega)$.

By Exercise 9.18, as $a = 3m - 1$, $\chi_a(\omega) = \omega^{M+N}$, where $M = m, N = 0$, so

$$\chi_a(\omega) = \omega^m.$$

Here a is relatively prime to b in $\mathbb{Z}$: if a rational prime r divides a, b, then $r \mid \pi$ in D, thus $r \mid \overline{\pi}$, so $r^2 \mid \pi\overline{\pi} = p$ in D, thus $r^2 \mid p$ in $\mathbb{Z}$, which is absurd. The Lemma gives then $\chi_a(b) = 1$.

We conclude that $\chi_a(b) = 1, \chi_a(\omega) = \omega^m$, so $\chi_\pi(a) = \chi_a(\pi) = \chi_a(b)\chi_a(\omega) = \omega^m$.

$$\chi_\pi(a) = \omega^m.$$

(d)
$$a+b=[(a+b)\omega]\omega^{-1},$$
and
$$(a+b)\omega=(a+b\omega)+a\omega-a\equiv a(\omega-1) \pmod{\pi},$$
thus
$$a+b\equiv a(1-\omega)\omega^{-1}\ [\pi],$$
$$\chi_\pi(a+b)=\chi_\pi(1-\omega)\chi_\pi(a)\chi_\pi(\omega)^{-1},$$
$\chi_\pi(a)=\omega^m$ by (c), and $\chi_\pi(\omega)=\omega^{m+n}$(as in Ex. 9.3), thus
$$\chi_\pi(a+b)=\omega^{2n}\chi_\pi(1-\omega).$$
□

Ex. 9.25 *Show that $\chi_{a+b}(\pi)$ may be computed as follows.*

(a) $\chi_{a+b}(\pi)=\chi_{a+b}(1-\omega)$.

(b) $\chi_{a+b}(\pi)=\omega^{2(m+n)}$.

Proof. (a) $\pi=a+b\omega$ and $a\equiv -b \pmod{a+b}$, thus $\pi\equiv -b(1-\omega) \pmod{a+b}$. So
$$\chi_{a+b}(\pi)=\chi_{a+b}(b)\chi_{a+b}(1-\omega).$$
Since $a\wedge b=1$, $(a+b)\wedge b=1$: as in Ex. 9.24, $\chi_{a+b}(b)=1$. So
$$\chi_{a+b}(\pi)=\chi_{a+b}(1-\omega).$$

(b) Since the character χ_{a+b} has order 3,
$$\begin{aligned}\chi_{a+b}(1-\omega)&=(\chi_{a+b}((1-\omega)^2))^2\\&=(\chi_{a+b}(-3\omega))^2\\&=[\chi_{a+b}(3)\chi_{a+b}(\omega)]^2\end{aligned}$$
$\chi_{a+b}(3)=1$ because $(a+b)\wedge 3=(3(m+n)-1)\wedge 3=1$.
$\chi_{a+b}(\omega)=\omega^{m+n}$ (Ex. 9.19).
Conclusion :
$$\chi_{a+b}(1-\omega)=\omega^{2(m+n)}.$$
□

Ex. 9.26 *Combine the previous two exercises to conclude that $\chi_\pi(1-\omega)=\omega^{2m}$.*

Proof. Since π and $a+b$ are primary elements of D, by Exercise 9.20,
$$\chi_\pi(a+b)=\chi_{a+b}(\pi).$$
By Exercises 9.24 and 9.25,
$$\begin{aligned}\chi_\pi(a+b)&=\omega^{2n}\chi_\pi(1-\omega)\\ \chi_{a+b}(\pi)&=\omega^{2(m+n)}\end{aligned}$$
Thus $\omega^{2n}\chi_\pi(1-\omega)=\omega^{2(m+n)}$. Consequently
$$\chi_\pi(1-\omega)=\omega^{2m}.$$
□

Ex. 9.27 *Let $\pi = a + bi$ be a primary irreducible in $\mathbb{Z}[i], b \neq 0$. Show*

(a) $a \equiv (-1)^{(p-1)/4} \pmod 4$, $p = N(\pi)$.

(b) $b \equiv (-1)^{(p-1)/4} - 1 \pmod 4$.

(Wrong sentence for (b) in the edition 1990.)

Proof. Let $\pi = a + bi$ be a primary prime in $\mathbb{Z}[i]$, $b \neq 0$, such that $p = N(\pi)$. Then

$$p = \pi\bar{\pi} = a^2 + b^2 \equiv 1\ [4].$$

By Lemma 6, Section 7, a is odd, b even, and

$$(a \equiv 1\ [4], b \equiv 0\ [4]) \text{ or } (a \equiv 3\ [4], b \equiv 2\ [4]).$$

(a)
- Case 1: $a \equiv 1\ [4], b \equiv 0[4]$. Then $a = 4A + 1, b = 4B$, $A, B \in \mathbb{Z}$, so $(a^2 + b^2 - 1)/4 = 4A^2 + 4B^2 + 2A$ is even :
$(-1)^{(p-1)/4} = (-1)^{(a^2+b^2-1)/4} = 1$, and $a \equiv 1[4]$, thus $a \equiv (-1)^{(p-1)/4}\ [4]$.
- Case 2: $a \equiv 3\ [4], b \equiv 2 \pmod 4$.
$a = 4A+3, b = 4B+2, a^2+b^2-1 = 16A^2+24A+9+16B^2+16B+4-1 \equiv 4\ [8]$, so $(a^2+b^2-1)/4 \equiv 1\ [2], (-1)^{(p-1)/4} = (-1)^{(a^2+b^2-1)/4} = -1$, and $a \equiv -1\ [4]$, thus $a \equiv (-1)^{(p-1)/4}\ [4]$.

In both cases,

$$a \equiv (-1)^{(p-1)/4}\ [4].$$

(b) In every case, $b \equiv a - 1\ [4]$, thus

$$b \equiv (-1)^{(p-1)/4} - 1\ [4].$$

In other words, for all primary primes $\pi = a + bi$ such that $N(\pi) = p$,

$$\begin{aligned} p \equiv 1\ [8] &\iff \pi \equiv 1\ [4], \\ p \equiv 5\ [8] &\iff \pi \equiv 3 + 2i\ [4]. \end{aligned}$$

□

Ex. 9.28 *The notation being as in Exercise 27 show $\chi_\pi(\overline{\pi}) = \chi_\pi(2)\chi_\pi(a)$.*

Proof. $\pi = a + bi, \overline{\pi} = a - bi = 2a - \pi \equiv 2a\ [\pi]$, thus, by Proposition 9.8.3 (e) :

$$\chi_\pi(\overline{\pi}) = \chi_\pi(2a) = \chi_\pi(2)\chi_\pi(a).$$

□

Ex. 9.29 *By Exercise 9.27, $a(-1)^{(p-1)/4}$ is primary. Use biquadratic reciprocity to show $\chi_\pi(a(-1)^{(p-1)/4}) = (-1)^{(a^2-1)/8}$.*

Proof. $a \equiv (-1)^{(p-1)/4}$ [4] (Ex. 9.27(a)), $a(-1)^{(p-1)/4} \equiv 1$ [4], thus $a(-1)^{(p-1)/4}$ is primary (if $a \neq \pm 1$).

If $a = \pm 1$ is an unit, $a(-1)^{(p-1)/4} = 1$ and $\chi_\pi(a(-1)^{(p-1)/4}) = 1 = (-1)^{(a^2-1)/8}$, so we can suppose that a is not an unit.

As $a(-1)^{(p-1)/4} \equiv 1 \pmod 4$, the Law of Biquadratic Reciprocity (Prop. 9.9.8) gives

$$\begin{aligned}
\chi_\pi(a(-1)^{(p-1)/4}) &= \chi_{a(-1)^{(p-1)/4}}(\pi)\\
&= \chi_a(\pi) \qquad \text{(Prop.9.8.3(f))}\\
&= \chi_a(a+bi)\\
&= \chi_a(bi)\\
&= \chi_a(b)\chi_a(i).
\end{aligned}$$

As $a \wedge b = 1$ (since $p = a^2 + b^2$), $\chi_a(b) = 1$ (Prop. 9.8.5, with $a \neq 1$), so

$$\chi_\pi(a(-1)^{(p-1)/4}) = \chi_a(i).$$

If $a \equiv 1$ [4], Proposition 9.8.6 gives $\chi_a(i) = (-1)^{(a-1)/4}$. Write $a = 4A+1,\ A \in \mathbb{Z}$. Then

$$(-1)^{(a^2-1)/8} = (-1)^{2A^2+A} = (-1)^A = (-1)^{(a-1)/4} = \chi_a(i).$$

If $a \equiv -1$ [4], then $\chi_a(i) = \chi_{-a}(i) = (-1)^{(-a-1)/4}$ by the same proposition. Write $a = 4A-1,\ A \in \mathbb{Z}$. Then

$$(-1)^{(a^2-1)/8} = (-1)^{2A^2-A} = (-1)^{-A} = (-1)^{(-a-1)/4} = \chi_a(i).$$

So, for each odd a, $a \neq \pm 1$,

$$\chi_a(i) = i^{(a^2-1)/8}.$$

Conclusion : if $\pi = a + bi$ is a primary irreducible such that $N(\pi) = p$, then

$$\chi_\pi(a(-1)^{(p-1)/4}) = (-1)^{(a^2-1)/8}.$$

□

Ex. 9.30 *Use the preceding two exercises to show $\chi_\pi(\overline{\pi}) = \chi_\pi(2)(-1)^{(a^2-1)/8}$.*

Proof. By Exercises 9.28, 9.29, and $\chi_\pi(-1) = (-1)^{(a-1)/2}$ (Prop. 9.8.3(d)),

$$\begin{aligned}
\chi_\pi(\overline{\pi}) &= \chi_\pi(2)\chi_\pi(a)\\
&= \chi_\pi(2)\chi_\pi(a(-1)^{(p-1)/4})(\chi_\pi(-1))^{(p-1)/4}\\
&= \chi_\pi(2)(-1)^{(a^2-1)/8}((-1)^{(a-1)/2})^{(p-1)/4}\\
&= \chi_\pi(-2)(-1)^{(a^2-1)/8}((-1)^{(a-1)/2})^{(p+3)/4}\\
&= \chi_\pi(-2)(-1)^{(a^2-1)/8}(-1)^{((a-1)/2)\,((p+3)/4)}.
\end{aligned}$$

If $a \equiv 1 \pmod 4$, then $(-1)^{(a-1)/2} = 1$.

If $a \equiv 3 \pmod 4$, then $b \equiv 2$ [4] :

$$a = 4A+3, b = 4B+2, p+3 = a^2+b^2+3 = (4A+3)^2 + (4B+2)^2 + 3 \equiv 0 \ [8],$$

so $(p+3)/4 \equiv 0$ [2].

In both cases $(-1)^{((a-1)/2)\,((p+3)/4)} = 1$, and so

$$\chi_\pi(\overline{\pi}) = \chi_\pi(-2)(-1)^{(a^2-1)/8}.$$

$\square$

Ex. 9.31 *Let p be prime, $p \equiv 1 \pmod 4$. Show that $p = a^2 + b^2$ where a and b are uniquely determined by the conditions $a \equiv 1 \pmod 4, b \equiv -((p-1)/2)!a \pmod p$.*

Proof. Recall the following lemma :

Lemma :

Let p be prime, $p \equiv 1$ [4], then $\left[\left(\frac{p-1}{2}\right)!\right]^2 \equiv -1$ $[p]$.

By Wilson's theorem (Prop. 4.1.1, Corollary), $(p-1)! \equiv -1$ $[p]$.

$$\begin{aligned}
-1 \equiv (p-1)! &= 1.2.\cdots.\left(\frac{p-1}{2}\right)\left(\frac{p+1}{2}\right)\cdots(p-2)(p-1)\\
&\equiv 1.2.\cdots\frac{p-1}{2}\left[-\left(\frac{p-1}{2}\right)\right]\cdots(-2)(-1)\\
&\equiv (-1)^{(p-1)/2}\left[\left(\frac{p-1}{2}\right)!\right]^2\\
&\equiv \left[\left(\frac{p-1}{2}\right)!\right]^2 \quad [p],
\end{aligned}$$

since $p \equiv 1$ [4].

- We show that there exists a pair $a, b \in \mathbb{Z}$ which verifies the sentence.

  By lemma 5 section 7, as $p \equiv 1$ [4], there exists an irreducible π such that $N(\pi) = p$, and we can choose π such that $\pi = A + Bi$ is primary (lemma 7 section 7), so A is odd.

  If $A \equiv 1 \pmod 4$, we take $a = A$, and if $A \equiv 3 \pmod 4$, we take $a = -A$: then $a \equiv 1 \pmod 4$.

  Let $u = \left(\frac{p-1}{2}\right)!$. Then $0 \equiv p = A^2 + B^2 \pmod p$, $B^2 \equiv -A^2 \equiv (uA)^2 \pmod p$.

  $p \mid (B - uA)(B + uA)$, thus $B \equiv \pm uA \pmod p$.

  Since $a = \pm A$, $B \equiv \pm ua \pmod p$.

  If $B \equiv -ua \pmod p$, we take $b = B$, if not $b = -B$.

  Then a, b are such that $p = a^2 + b^2, a \equiv 1$ [4]$, b \equiv -((p-1)/2)!\,a$ $[p]$.

- Unicity of the pair (a, b) such that

  $$p = a^2 + b^2, a \equiv 1\ [4], b \equiv -((p-1)/2)!\,a\ [p].$$

  Suppose that c, d are such that $p = c^2 + d^2, c \equiv 1$ [4]$, d \equiv -((p-1)/2)!c$ $[p]$.

Let $\pi = a + ib, \lambda = c + id$. As $p = N\pi = N\lambda$ is a rational prime, π and λ are primes in D, and $p = \pi\overline{\pi} = \lambda\overline{\lambda}$, thus λ is associate to π or $\overline{\pi}$. :

$$\lambda \in \{\pi, -\pi, i\pi, -i\pi, \overline{\pi}, -\overline{\pi}, i\overline{\pi}, -i\overline{\pi}\}.$$

As a, c are odd, and b, d even, it remains only the possibilities $\lambda = \pm\pi, \lambda = \pm\overline{\pi}$, thus $c = \pm a$. Moreover $a \equiv c \equiv 1$ [4], thus $a = c$, and $d \equiv -((p-1)/2)!c \equiv -((p-1)/2)!a \equiv b$ [p].

$p = a^2 + b^2 = a^2 + d^2$, so $d = \pm b$, and $d \equiv b$ [p].

If $d = -b$, then $p \mid 2b$, thus $p \mid b$, and also $p \mid a$, so $p^2 \mid a^2 + b^2 = p$: this is impossible. So $a = c, b = d$. Unicity is proved.

Conclusion : if $p \equiv 1$ [4], there exists an unique pair a, b such that

$$p = a^2 + b^2, a \equiv 1 \pmod 4, b \equiv -((p-1)/2)!a \pmod p.$$

$\square$

Ex. 9.32 *Let p be a prime, $p \equiv 1 \pmod 4$ and write $p = \pi\overline{\pi}, \pi \in \mathbb{Z}[i]$. Show $\chi_p(1+i) = i^{(p-1)/4}$.*

Proof.

$$\begin{aligned}
\chi_p(1+i) &= \chi_\pi(1+i)\chi_{\overline{\pi}}(1+i) \\
&= \chi_\pi(1+i)\overline{\chi_\pi(1-i)} \qquad \text{(Prop. 9.8.3(c))} \\
&= \frac{\chi_\pi(1+i)}{\chi_\pi(1-i)} = \chi_\pi(i) \qquad \text{(since } (1-i)i = 1+i) \\
&= i^{\frac{p-1}{4}}.
\end{aligned}$$

The last equality is a consequence of the definition of χ_π : $\chi_\pi(i) \equiv i^{\frac{p-1}{4}} \pmod \pi$, and the classes of $1, i, i^2, i^3$ modulo π are distinct. $\square$

Ex. 9.33 *Let q be a positive prime, $q \equiv 3 \pmod 4$. Show $\chi_q(1+i) = i^{(q+1)/4}$. [Hint : $(1+i)^{q-1} \equiv -i \pmod q$.]*

The sentence is false and must be replaced by

$$\chi_q(1+i) = (-i)^{(q+1)/4} = i^{-(q+1)/4}.$$

We verify this on the example $q = 11$:

$$\begin{aligned}
\chi_q(1+i) &\equiv (1+i)^{(q^2-1)/4} \\
&\equiv (1+i)^{30} \\
&\equiv -2^{15}i \equiv -32i \equiv i \pmod{11},
\end{aligned}$$

so $\chi_{11}(1+i) = i$, and $i^{(-q-1)/4} = i^{-3} = i$ (but $i^{(q+1)/4} = -i$).

Proof. Write $q = 4k+3, k \in \mathbb{N}$.

As $(1+i)^2 = 2i$, $(1+i)^{q-1} = (2i)^{(q-1)/2}$.

$2^{(q-1)/2} \equiv \left(\frac{2}{q}\right)$ $[q]$ and $\left(\frac{2}{q}\right) = (-1)^{(q^2-1)/8} = (-1)^{2k^2+3k+1} = (-1)^{k+1}$

$i^{(q-1)/2} = i^{2k+1} = (-1)^k i$.

So

$$(1+i)^{q-1} \equiv -i \ [q].$$

$N(q) = q^2$, so $\chi_q(1+i) \equiv (1+i)^{(q^2-1)/4} = [(1+i)^{q-1}]^{(q+1)/4} \equiv (-i)^{(q+1)/4}$ $[q]$:

$$\chi_q(1+i) = (-i)^{(q+1)/4} = i^{(-q-1)/4}.$$

$\square$

Ex. 9.34 *Let $\pi = a+bi$ be a primary irreducible, $(a,b)=1$. Show*

(a) if $\pi \equiv 1 \pmod 4$, then $\chi_\pi(a) = i^{(a-1)/2}$.

(b) if $\pi \equiv 3+2i \pmod 4$, then $\chi_\pi(a) = -i^{(-a-1)/2}$.

Proof. Let $\pi = a+bi$ be a primary irreducible, with $a \wedge b = 1$, so $b \neq 0$. We can apply the result of Exercise 9.29:

$$\chi_\pi(a(-1)^{(p-1)/4}) = (-1)^{(a^2-1)/8}.$$

(a) Suppose that $\pi \equiv 1$ $[4]$.

Then $a \equiv 1$ $[4], b \equiv 0$ $[4], a = 4A+1, b = 4B,\ A, B \in \mathbb{Z}$.

As $\chi_\pi(-1) = (-1)^{(a-1)/2}$,

$$\chi_\pi(a) = (-1)^{\frac{a-1}{2}\frac{p-1}{4}}(-1)^{\frac{a^2-1}{8}},$$

where

$$p = N\pi = a^2+b^2, (-1)^{(p-1)/4} = (-1)^{\frac{a^2-1}{4}+\frac{b^2}{4}} = (-1)^{4A^2+2A+4B^2} = 1,$$

thus $(-1)^{\frac{a-1}{2}\frac{p-1}{4}} = 1$.

$$\chi_\pi(a) = (-1)^{(a^2-1)/8} = (-1)^{2A^2+A} = (-1)^A = (-1)^{(a-1)/4} = i^{(a-1)/2}.$$

Conclusion: if $\pi \equiv 1$ $[4]$, $\chi_\pi(a) = i^{(a-1)/2}$.

(b) Suppose that $\pi \equiv 3+2i$ $[4]$.

Then $a \equiv 3$ $[4], b \equiv 2$ $[4], a = 4A+3, b = 4B+2,\ A, B \in \mathbb{Z}$. As in (a),

$$\chi_\pi(a) = (-1)^{\frac{a-1}{2}\frac{p-1}{4}}(-1)^{\frac{a^2-1}{8}},$$

where $a^2+b^2-1 = 16A^2+24A+16B^2+16B+12 \equiv 4$ $[8]$, so $\frac{a^2+b^2-1}{4} \equiv 1$ $[2]$, thus $(-1)^{(p-1)/4} = (-1)^{(a^2+b^2-1)/4} = -1$.

$$(-1)^{\frac{a-1}{2}\frac{p-1}{4}} = (-1)^{\frac{a-1}{2}} = (-1)^{2A+1} = -1,$$

$$\frac{a^2-1}{8} = 2A^2+3A+1, (-1)^{(a^2-1)/8} = (-1)^{3A+1} = (-1)^{A+1} = (-1)^{(a+1)/4},$$

$$\chi_\pi(a) = -(-1)^{(a+1)/4} = -i^{(a+1)/2}.$$

Moreover

$$\frac{a+1}{2} \equiv \frac{-a-1}{2}\ [4] \iff a+1 \equiv -a-1\ [8] \iff 2a \equiv -2\ [8] \iff a \equiv 3\ [4],$$

thus $i^{(a+1)/2} = i^{(-a-1)/2}$.

Conclusion : if $\pi \equiv 3+2i\ [4]$, $\chi_\pi(a) = -i^{(-a-1)/2}$.

$\square$

Ex. 9.35 *If $\pi = a+bi$ is as in Exercise 9.34 show $\chi_\pi(a)\chi_\pi(1+i) = i^{(3(a+b-1))/4}$. [Hint: $a(1+i) = a+b+i(a+bi)$. Generalize Exercises 32 and 33 to any integer $\equiv 1 \pmod 4$ and use Proposition 9.9.8. Note $a+b \equiv 1 \pmod 4$.]*

Proof. We give a generalization of Exercises 9.32 and 9.33 : if $n \equiv 1\ [4], n \neq 1$, then $\chi_n(1+i) = i^{(n-1)/4}$.

By Exercises 9.32 and 9.33, we know that if $p \equiv 1\ [4]$ is a rational prime, then

$$\chi_p(1+i) = i^{(p-1)/4},$$

and if $q \equiv 3\ [4]$, in other words $-q \equiv 1\ [4]$, where q is a rational prime, then

$$\chi_{-q}(1+i) = \chi_q(1+i) = i^{(-q-1)/4}.$$

Let $n \in \mathbb{Z}, n \equiv 1\ [4], n \neq 1$.

If $n > 0$, $n = q_1q_2\cdots q_kp_1p_2\cdots p_l$, where $q_i \equiv -1\ [4], p_i \equiv 1\ [4]$, thus k is even.

If $n < 0$, $n = -q_1q_2\cdots q_kp_1p_2\cdots p_l$, with k odd. In both cases,

$$n = (-q_1)(-q_2)\cdots(-q_k)p_1p_2\cdots p_l,$$

so we can write

$$n = s_1s_2\cdots s_N, \qquad \text{where } s_i = -q_i, 1 \le i \le k, s_i = p_{i-k},\ k+1 \le i \le k+l = N,$$

where $s_i \equiv 1\ [4],\ 1 \le i \le N$.

$$\begin{aligned}
\chi_n(1+i) &= \chi_{-q_1}(1+i)\cdots\chi_{-q_k}(1+i)\chi_{p_1}(1+i)\cdots\chi_{p_l}(1+i)\\
&= i^{(-q_1-1)/4}\cdots i^{(-q_k-1)/4} i^{(p_1-1)/4}\cdots i^{(p_l-1)/4}\\
&= i^{(s_1-1)/4}\cdots i^{(s_N-1)/4}\\
&= i^{\sum_{i=1}^N \frac{s_i-1}{4}}\\
&= i^{(n-1)/4},
\end{aligned}$$

the last equality resulting of Exercise 9.44.

Conclusion : if $n \in \mathbb{Z}, n \equiv 1\ [4], n \neq 1$, then $\chi_n(1+i) = i^{(n-1)/4}$.

Let $\pi = a+bi, a \wedge b = 1$ a primary irreducible. As $a(1+i) = a+b+i(a+bi)$, $a(1+i) \equiv a+b\ [\pi]$, so

$$\chi_\pi(a)\chi_\pi(1+i) = \chi_\pi(a+b).$$

As $\pi = a + bi$ is primary, $a + b \equiv 1$ [4].

If $a + b = 1$, then $\chi_\pi(a)\chi_\pi(1 + i) = \chi_\pi(a + b) = 1 = i^{3(a+b-1)/4}$. If not, the Law of Biquadratic Reciprocity (Proposition 9.9.8) gives

$$\chi_\pi(a + b) = \chi_{a+b}(\pi).$$

Now $b \equiv -a \pmod{a + b}$, so $a + bi \equiv a(1 - i) \equiv -ia(1 + i) \pmod{a + b}$. Therefore

$$\chi_{a+b}(\pi) = \chi_{a+b}(-1)\chi_{a+b}(a)\chi_{a+b}(i)\chi_{a+b}(1 + i).$$

Since $n \equiv 1$ [4], $\chi_n(i) = (-1)^{(n-1)/4}$ (Prop.9.8.6), thus

$$\chi_n(-1) = \chi_n(i^2) = (-1)^{\frac{n-1}{2}} = 1.$$

Consequently, since $a + b \equiv 1$ [4], $\chi_{a+b}(-1) = 1$.

As $a \wedge b = 1, (a + b) \wedge a = 1$, thus $\chi_{a+b}(a) = 1$ (Prop 9.8.5).

$a + b \equiv 1$ [4], thus $\chi_{a+b}(i) = (-1)^{(a+b-1)/4}$ (Prop. 9.8.6).

From the first part of this proof, $\chi_{a+b}(1 + i) = i^{(a+b-1)/4}$, so

$$\begin{aligned}
\chi_{a+b}(\pi) &= \chi_{a+b}(-1)\chi_{a+b}(a)\chi_{a+b}(i)\chi_{a+b}(1 + i)\\
&= (-1)^{(a+b-1)/4} i^{(a+b-1)/4}\\
&= i^{(a+b-1)/2} i^{(a+b-1)/4}\\
&= i^{3(a+b-1)/4}
\end{aligned}$$

Conclusion : if $\pi = a + bi$ is a primary irreducible, such that $a \wedge b = 1$, then

$$\chi_\pi(a)\chi_\pi(1 + i) = i^{3(a+b-1)/4}$$

□

Ex. 9.36 *Remove the restriction* $(a, b) = 1$ *in Exercise 9.34.*

Proof. Suppose that $q = a \wedge b > 1$. Then $a = qa', b = qb',\ a', b' \in \mathbb{Z}$, so $\pi = q(a' + ib')$.

As π is irreducible, and as q is not an unit, $u = a' + b'i$ is an unit, and so $\pi = uq$ is associate to q : the rational integer q is then a prime in D, so a rational prime $q \equiv 3 \pmod 4$.

If $u = \pm i$, then $\pi = \pm q = a + bi$ is such that b is odd, in contradiction with π primary. Thus $u = \pm 1$, and $\pi = \varepsilon q, \varepsilon = \pm 1$. As π is primary, $\varepsilon = -1$, so $\pi = -q$.

Then $\chi_\pi(a) = \chi_{-q}(-q) = 0$, the result of Ex. 34 is false if $b = 0$.

Conclusion : if $\pi = a + bi$ is a primary irreducible, and $b \neq 0$, then

(a) if $\pi \equiv 1$ [4], $\chi_\pi(a) = i^{(a-1)/2}$,

(b) if $\pi \equiv 3 + 2i$ [4], $\chi_\pi(a) = -i^{(-a-1)/2}$.

□

Ex. 9.37 *Combine Exercises 32, 33, 34, and 35 to show $\chi_\pi(1+i) = i^{(a-b-b^2-1)/4}$. Show that this result implies Exercise 26 of Chapter 5 "the biquadratic character of 2").*

Lemma. If $\pi = a+bi$ is a primary prime, then

$$\chi_\pi(i) = i^{\frac{-a+1}{2}}.$$

Proof. (of Lemma.) Let $\pi = a+bi$ a primary prime in $\mathbb{Z}[i]$.

- If $\pi = -q$, where $q \equiv 3 \pmod 4, q > 0$ is a rational prime, then $a = -q, b = 0$. By definition of the quartic character,

$$\chi_q(i) = i^{\frac{N(q)-1}{4}} = i^{\frac{q^2-1}{4}}.$$

  Write $-q = a = 4k+1,\ k \in \mathbb{Z}$. Then

$$\begin{aligned}\frac{q^2-1}{4} &= 4k^2 + 2k\\ &\equiv 2k = \frac{a-1}{2} \pmod 4.\end{aligned}$$

  Therefore

$$\chi_{-q}(i) = \chi_q(i) = i^{\frac{q^2-1}{4}} = i^{\frac{a-1}{2}} = \left(\frac{1}{i}\right)^{\frac{-a+1}{2}} = (-i)^{\frac{-a+1}{2}} = (-1)^{\frac{-a+1}{2}} i^{\frac{-a+1}{2}} = i^{\frac{-a+1}{2}},$$

  since $(-1)^{\frac{-a+1}{2}} = (-1)^{-2k} = 1$.

  Suppose now that $N(\pi) = p$, where $p \equiv 1 \pmod 4$ is a rational prime. Then

$$\chi_\pi(i) = i^{\frac{N(\pi)-1}{4}} = i^{\frac{p-1}{4}}.$$

  Since $\pi = a+bi$ is primary, there are two cases.

- If $a \equiv 1 \pmod 4, b \equiv 0 \pmod 4$, then $a = 4A+1, b = 4B,\ A, B \in \mathbb{Z}$.

$$\begin{aligned}\frac{p-1}{4} &= \frac{a^2+b^2-1}{4}\\ &= \frac{16A^2+8A+16B^2}{4}\\ &= 4A^2+2A+4B^2\\ &\equiv 2A = \frac{a-1}{2}\end{aligned}$$

  Therefore

$$\chi_\pi(i) = i^{\frac{p-1}{4}} = i^{\frac{a-1}{2}} = \left(\frac{1}{i}\right)^{\frac{-a+1}{2}} = (-i)^{\frac{-a+1}{2}} = (-1)^{\frac{-a+1}{2}} i^{\frac{-a+1}{2}} = i^{\frac{-a+1}{2}},$$

  since $(-1)^{\frac{-a+1}{2}} = (-1)^{-2A} = 1$.

- If $a \equiv 3 \pmod 4, b \equiv 2 \pmod 4$, then $a = 4A - 1, B = 4B + 2,\ A, B \in \mathbb{Z}$.

$$\begin{aligned}\frac{p-1}{4} &= \frac{a^2+b^2-1}{4}\\ &= \frac{16A^2 - 8A + 16B^2 + 16B + 4}{4}\\ &= 4A^2 - 2A + 4B^2 4B + 1\\ &\equiv -2A + 1 = \frac{-a+1}{2} \pmod 4\end{aligned}$$

Therefore $\chi_\pi(i) = (-1)^{\frac{-a+1}{4}}$.

The equality $\chi_\pi(i) = (-1)^{\frac{-a+1}{4}}$ is verified for all primary primes π.

$\square$

Proof. (of Ex.9.37) Let $\pi = a + ib$ be a primary irreducible in $\mathbb{Z}[i]$.

- If $b = 0$, then $\pi = a \in \mathbb{Z}$. As π is primary, $\pi = -q, q \equiv 3 \pmod 4$, where q is a rational prime, so $a = -q, b = 0$. By Ex. 9.32 (or its generalization 9.35),

$$\chi_\pi(1+i) = \chi_{-q}(1+i) = i^{(-q-1)/4} = i^{(a-b-b^2-1)/4}.$$

- If $b \neq 0$, then $a \wedge b = 1$ (see Ex. 9.36), and by Ex. 9.35,

$$\chi_\pi(a)\chi_\pi(1+i) = i^{3(a+b-1)/4}.$$

  - If $\pi \equiv 1\ [4]$, $a \equiv 1\ [4], b \equiv 0\ [4] :\ a = 4A + 1, b = 4B,\ A, B \in \mathbb{Z}$.
    By Ex. 9.34(a),

$$\chi_\pi(a) = i^{(a-1)/2}, \chi_\pi(a)^{-1} = (-i)^{(a-1)/2} = i^{(a-1)/2}.$$

$$\begin{aligned}\chi_\pi(1+i) &= i^{3\frac{a+b-1}{4} - 2\frac{a-1}{4}}\\ &= i^{\frac{a+3b-1}{4}}\\ &= i^{\frac{a-b-b^2-1}{4}},\end{aligned}$$

    since $\left(\frac{a+3b-1}{4}\right) - \left(\frac{a-b-b^2-1}{4}\right) = b + \frac{b^2}{4} = 4B + 4B^2 \equiv 0\ [4]$.
  - If $\pi \equiv 3 + 2i\ [4]$, $a \equiv 3[4], b \equiv 2\ [4] :\ a = 4A - 1, b = 4B + 2,\ A, B \in \mathbb{Z}$.
    By Ex. 9.34(b),

$$\chi_\pi(a) = -i^{(-a-1)/2}, \chi_\pi(a)^{-1} = -i^{(a+1)/2} = i^{(a-3)/2},$$

    so

$$\chi_\pi(1+i) = i^{(3a+3b-3+2a-6)/4} = i^{(5a+3b-9)/4}.$$

    Now $\frac{1}{4}[(a-b-b^2-1)-(5a+3b-9)] = \frac{1}{4}(-4a-4b-b^2+8) = -a-b+2-\frac{b^2}{4} = -4A + 1 - 4B - 2 + 2 - (2B+1)^2 \equiv 0\ [4]$,
    thus $\chi_\pi(1+i) = i^{(a-b-b^2-1)/4}$.

Conclusion : if $\pi = a + ib$ is primary irreducible, then

$$\chi_\pi(1+i) = i^{(a-b-b^2-1)/4}$$

Second part : the biquadratic character of 2 (see Ex. 5.25 to 5.28).
Let $p \equiv 1$ [4]. Then $p = N(\pi)$, where $\pi = a + bi$ is a primary prime.
We show first that $\chi_\pi(2) = i^{\frac{ab}{2}}$.
Since $2 = i^3(1+i)^2$, the first part of the exercise, and the Lemma, give

$$\begin{aligned}\chi_\pi(2) &= \chi_\pi(i)^3 \chi_\pi(1+i)^2 \\ &= i^{\frac{3(-a+1)}{2}} i^{\frac{a-b-b^2-1}{2}} \\ &= i^{1-a-(b+1)\frac{b}{2}}\end{aligned}$$

Since π is primary, $a \equiv b + 1 \equiv -b + 1 \pmod 4$, therefore

$$\begin{aligned}1 - a - (b+1)\frac{b}{2} &\equiv -b - (b+1)\frac{b}{2} \\ &\equiv \frac{b}{2}(-b-3) \\ &\equiv \frac{b}{2}(-b+1) \\ &\equiv \frac{ab}{2} \qquad \pmod 4,\end{aligned}$$

so $\chi_\pi(2) = i^{\frac{ab}{2}}$.

Now we show that p is of the form $p = A^2 + 64B^2$ if and only if $p \equiv 1 \pmod 4$ and if $x^4 \equiv 2$ has a solution $x \in \mathbb{Z}$.

If $p = A^2 + 64B^2 = A^2 + (8B)^2$, then the prime number p is a sum of two squares, and $p \neq 2$, therefore $p \equiv 1 \pmod 4$. Since $p = A^2 + 64B^2$, A is odd. Put $b = 8B$, and $a = A$ if $A \equiv 1 \pmod 4$, $a = -A$ if $A \equiv -1 \pmod 4$. Then $\pi = a + bi$ is such that $N(\pi) = a^2 + b^2 = p$, and $a \equiv 1, b \equiv 0 \pmod 4$, therefore π is a primary prime. Then

$$\chi_\pi(2) = i^{\frac{ab}{2}} = i^{4aB} = 1.$$

Therefore there exists $\alpha \in D$ such that $2 \equiv \alpha^4 \pmod \pi$. As $D/\pi D$ is the set of classes of $0, 1, \cdots, p-1$, there exists $x \in \mathbb{Z}$ such that $x \equiv \alpha \pmod \pi$, so $2 \equiv x^4 \pmod \pi$.

Then $p = N(\pi) \mid N(x^4 - 2) = (x^4 - 2)^2$, thus $p \mid x^4 - 2$, in other words $2 \equiv x^4 \pmod p$.

Conversely, suppose that $p \equiv 1 \pmod 4$ and that 2 is a biquadratic residue modulo p. As $p \equiv 1 \pmod 4$, $p = \pi\overline{\pi}$, where $\pi = a + bi$ is a primary prime. Since $2 \equiv x^4 \pmod p$ for some $x \in \mathbb{Z}$, then $2 \equiv x^4 \pmod \pi$, so $\chi_\pi(2) = 1$. Moreover

$$1 = \chi_\pi(2) = i^{\frac{ab}{2}}.$$

Since a is odd, $8 \mid b$, therefore $p = A^2 + 64B^2$, where $A = a, B = b/8$.

Conclusion :

$$\exists (A, B) \in \mathbb{Z}^2,\ p = A^2 + 64B^2 \iff (p \equiv 1\ [4] \text{ and } \exists x \in \mathbb{Z},\ x^4 \equiv 2\ [p]).$$

□

Ex. 9.38 *Prove part (d) of Proposition 9.8.3.*

Proposition 9.8.3(d) *If π is a primary irreducible then $\chi_\pi(-1) = (-1)^{(a-1)/2}$, where $\pi = a + bi$.*

Proof. Let $\pi = a + bi$ a primary irreducible. Then a is odd, and b is even, and $N(\pi) = a^2 + b^2$. Then

$$\chi_\pi(-1) = (-1)^{\frac{N(\pi)-1}{4}} = (-1)^{\frac{a^2-1}{4}+\frac{b^2}{4}} = [(-1)^{\frac{a+1}{2}}]^{\frac{a-1}{2}}(-1)^{\frac{b^2}{4}}.$$

By Lemma 6, section 7, $a \equiv 1\ [4], b \equiv 0\ [4]$, or $a \equiv 3[4], b \equiv 2[4]$.

- If $a \equiv 1\ [4], b \equiv 0\ [4]$, then $(-1)^{\frac{a+1}{2}} = -1, (-1)^{\frac{b^2}{4}} = +1$, so
$$\chi_\pi(-1) = (-1)^{\frac{a-1}{2}}.$$
- If $a \equiv 3\ [4], b \equiv 2\ [4]$, then $(-1)^{\frac{a+1}{2}} = 1, (-1)^{\frac{b^2}{4}} = -1$, so
$$\chi_\pi(-1) = -1 = (-1)^{\frac{a-1}{2}}.$$

Conclusion: if π is a primary irreducible in $\mathbb{Z}[i]$, then

$$\chi_\pi(-1) = (-1)^{(a-1)/2}.$$

$\square$

Ex. 9.39 *Let $p \equiv 1 \pmod 6$ and write $4p = A^2 + 27B^2$, $A \equiv 1 \pmod 3$. Put $m = (p-1)/6$. Show $\binom{3m}{m} \equiv -1 \pmod p \iff 2 \mid B$.*

Proof. Let p be a rational prime, $p \equiv 1 \pmod 6$. As $p \equiv 1 \pmod 3$, we know from Theorem 2, Chapter 8, that there are integers A and B such that $4p = A^2 + 27B^2, A \equiv 1 \pmod 3$, and that A is uniquely determined by these conditions.

Then A, B have same parities. If we take $a = \frac{A+3B}{2}, b = 3B$, then $A = 2a - b, B = \frac{b}{3}$, and $4p = (2a-b)^2 + 3b^2$, so $p = a^2 - ab + b^2$. If $\pi = a + b\omega$, then $N(\pi) = p$. Since $A = 2a - b \equiv 1\ [3]$, and $b = 3B \equiv 0\ [3]$, then $a \equiv -1\ [3]$, so π is a primary prime.

- Suppose that $2 \mid B$. Since $p = a^2 - ab + b^2$ is odd, and $b = 3B$,

$$2 \mid B \iff 2 \mid b \iff (b \equiv 0\ [2], a \equiv 1\ [2]) \iff \pi \equiv 1\ [2].$$

By Proposition 9.6.1,

$$\pi \equiv 1\ [2] \iff x^3 - 2 \text{ is solvable in } D \iff \chi_\pi(2) = 1.$$

Therefore

$$2 \mid B \iff \chi_\pi(2) = 1.$$

Here χ_π is of order 3, so $\chi_\pi^2 \neq \varepsilon$. By Exercise 8.6,

$$J(\chi_\pi, \chi_\pi) = \chi_\pi(2)^{-2} J(\chi_\pi, \rho),$$

where ρ is the Legendre's character.

In this case, $2 \mid B$, $\chi_\pi(2) = 1$, so $J(\chi_\pi, \chi_\pi) = J(\chi_\pi, \rho)$, and by Lemma 1 section 4, where $p \equiv 1$ [3] and $p = N(\pi)$,

$$\pi = a + b\omega = J(\chi_\pi, \chi_\pi) = J(\chi_\pi, \rho).$$

By Exercise 8.15,

$$N(y^2 = x^3 + 1) = p + A,$$

and the Exercise 8.27(b) gives

$$N(y^2 = x^3 + 1) = N(y^2 + x^3 = 1) = p + 2 \operatorname{Re} J(\chi_\pi, \rho).$$

thus

$$A = 2 \operatorname{Re} J(\chi_\pi, \rho) = 2 \operatorname{Re} \pi = 2a - b.$$

Moreover, since $J(\chi_\pi, \rho) = \pi = a + b\omega$, by Exercise 8.27(c),

$$2a - b \equiv -\binom{(p-1)/2}{(p-1)/3}.$$

Therefore

$$-A \equiv \binom{(p-1)/2}{(p-1)/3} = \binom{(p-1)/2}{(p-1)/2 - (p-1)/6} = \binom{(p-1)/2}{(p-1)/6} = \binom{3m}{m} \pmod p,$$

where $m = (p-1)/6$. Since $A \equiv 1 \pmod 3$,

$$\binom{3m}{m} \equiv -1 \pmod p.$$

• Conversely, suppose that $\binom{3m}{m} \equiv -1 \pmod p$. Then $A = 2a - b \equiv -\binom{3m}{m} \pmod p$. Write $J(\chi_\pi, \rho) = c + d\omega$. By Exercise 8.27(c), $2c - d \equiv -\binom{3m}{m} \pmod p$. thus

$$2a - b \equiv 2c - d \pmod p.$$

Since $|J(\chi_\pi, \rho)| = \sqrt{p}$,

$$4p = (2a - b)^2 + 3b^2 = (2c - d)^2 + 3d^2,$$

thus $d \equiv \pm b \pmod p$.

By Exercise 8.6,

$$\pi = J(\chi_\pi, \chi_\pi) = \chi_\pi(2)^{-2} J(\chi_\pi, \rho),$$

Here χ_π is of order 3, therefore $\chi_\pi(2)^{-2} = \chi_\pi(2) \in \{1, \omega, \omega^2\}$, so

$$\pi = J(\chi_\pi, \chi_\pi) = \chi_\pi(2) J(\chi_\pi, \rho).$$

If $\chi_\pi(2) = \omega$, then $a + b\omega = \omega(c + d\omega) = -d + \omega(c - d)$. Then $a = -d \equiv \pm b \pmod p$. As $a \equiv -b\omega \pmod \pi$, we would have $-b\omega \equiv \pm b \pmod \pi$. Here $\pi \nmid b$, otherwise $p = N(\pi) \mid N(b) = b^2$, so $p \mid b$, and $p = a^2 - ab + b^2$, so $p \mid a$, and $p^2 \mid p$, which is a nonsense. Therefore $\pi \mid \omega \pm 1$, where π is a primary prime: it's impossible. Indeed $\omega + 1$ is a unit and $\omega - 1$ is prime, so $\pi \mid \omega - 1 = -\lambda$ implies that π and λ are associate, in contradiction with $N(\pi) = p \neq 3 = N(\lambda)$.

If $\chi_\pi(2) = \omega^2$, then $a + b\omega = \omega^2(c + d\omega) = (d - c) - \omega c$, so $a = d - c, b = -c$.

Reasoning modulo $\overline{\pi} = a + b\omega^2 = (a-b) + b\omega$, where $\overline{\pi} \mid \pi\overline{\pi} = p$, we obtain

$$d = a - b \equiv -b\omega \pmod{\overline{\pi}},$$

where $d \equiv \pm b \pmod{\overline{\pi}}$, so $-b\omega \equiv \pm b \pmod{\overline{\pi}}$. Since $N(\overline{\pi}) = p$, we obtain the same contradiction as above.

So $\chi_\pi(2) = 1$, and the previously proved equivalence $2 \mid B \iff \chi_\pi(2) = 1$ show that $2 \mid B$.

Conclusion:

$$\binom{(p-1)/2}{(p-1)/6} \equiv -1 \pmod{p} \iff 2 \mid B.$$

$\square$

Ex. 9.40 *Let $p \equiv 1 \pmod 6$, and put $p = \pi\overline{\pi}$ where π is primary. Write $\pi = a + b\omega$ and show*

(a) If $\chi_\pi(2) = \omega$ then $2b - a \equiv -\binom{3m}{m} \pmod{p}$.

(b) If $\chi_\pi(2) = \omega^2$ then $a + b \equiv \binom{3m}{m} \pmod{p}$.

(c) If $\chi_\pi(2) = \omega$ put $A = 2a - b, B = b/3$. Show $(A - 9B)/2 \equiv \binom{3m}{m} \pmod{p}$.

(d) If $\chi_\pi(2) = \omega^2$ put $2a - b = A$ and $B = -b/3$. Show $(A - 9B)/2 \equiv \binom{3m}{m} \pmod{p}$.

(e) Show that the "normalization" of B in (c) and (d) is equivalent to $A \equiv B \pmod 4$. [Recall $\chi_\pi(2) \equiv \pi \pmod 2$ by cubic reciprocity.]

Proof. Here $p = 6m + 1, m \in \mathbb{Z}$, and $p = \pi\overline{\pi}$, where $\pi = a + b\omega$ is a primary prime.

We have proved in Exercise 39 that

$$\pi = J(\chi_\pi, \chi_\pi) = \chi_\pi(2) J(\chi_\pi, \rho). \tag{1}$$

Write $J(\chi_\pi, \rho) = c + d\omega$. The Exercise 8.27(c) shows that

$$2c - d \equiv -\binom{3m}{m} \pmod{p}. \tag{2}$$

(a) If $\chi_\pi(2) = \omega$, then (1) gives

$$a + b\omega = \omega(c + d\omega) = -d + \omega(c - d),$$

so $a = -d, b = c - d$, therefore the equality (2) gives

$$2b - a = 2(c - d) + d = 2c - d \equiv -\binom{3m}{m} \pmod{p}.$$

(b) If $\chi_\pi(2) = \omega^2$, then

$$a + b\omega = \omega^2(c + d\omega) = d - c - c\omega,$$

so $a = d - c, b = -c$, and

$$a + b = d - 2c \equiv \binom{3m}{m} \pmod{p}.$$

(c) Suppose that $\chi_\pi(2) = \omega$, and put $A = 2a - b, B = b/3$, so

$$4p = A^2 + 27B^2, \quad A \equiv 1\ [3],$$

which shows that A, B have same parities. Then, by part (a),

$$\begin{aligned}\frac{A - 9B}{2} &= \frac{2a - b - 3b}{2}\\ &= a - 2b\\ &\equiv \binom{3m}{m} \pmod p\end{aligned}$$

(d) Suppose that $\chi_\pi(2) = \omega^2$, and put $A = 2a - b, B = -b/3$, so we have again

$$4p = A^2 + 27B^2, \quad A \equiv 1\ [3].$$

In this case, by part (b)

$$\begin{aligned}\frac{A - 9B}{2} &= \frac{2a - b + 3b}{2}\\ &= a + b\\ &\equiv \binom{3m}{m} \pmod p\end{aligned}$$

(e) The conditions $4p = A^2 + 27B^2,\ A \equiv 1\ [3]$, determine A, B, except the sign of B. So $4p = A^2 + 27B^2 = (2a - b)^2 + 3b^2$, implies $A = 2a - b$ and $B = \pm\frac{b}{3}$.

By Exercise 39, since A, B have same parity, the condition A, B odd is equivalent to $\chi_\pi(2) \in \{\omega, \omega^2\}$. We choose this sign of B so that

$$\frac{A - 9B}{2} \equiv \binom{3m}{m} \pmod p.$$

By parts (d) and (e), where A, B are odd, this choice is given by $B = b/3$ if $\chi_\pi(2) = \omega$, and $B = -b/3$ if $\chi_\pi(2) = \omega^2$. We show that these conditions are equivalent to $A \equiv B \pmod 4$.

- If $\chi_\pi(2) = \omega$, then $A = 2a - b, B = b/3$.

By cubic reciprocity, $\chi_\pi(2) \equiv \pi \pmod 2$ (see section 6). Here $\chi_\pi(2) = \omega$, so $\omega \equiv a + b\omega \pmod 2$, therefore $a \equiv 0 \pmod 2, b \equiv 1 \pmod 2$,

$$A = 2a - b \equiv -b \equiv \frac{b}{3} = B \pmod 4,$$

so $A \equiv B \pmod 4$.

- If $\chi_\pi(2) = \omega^2$, then $A = 2a - b, B = -b/3$. In this case,

$$\omega^2 = -1 - \omega \equiv a + b\omega \pmod 2,$$

therefore $a \equiv 1 \equiv b \pmod 2$, and

$$A = 2a - b \equiv 2 - b \equiv b \equiv -\frac{b}{3} = B \pmod 4.$$

In both cases, the choice of the sign of B implies that $A \equiv B \pmod 4$.

Conversely, suppose that $A \equiv B \pmod 4$. Write $B = \varepsilon \frac{b}{3}$, where $\varepsilon = \pm 1$. Then $A \equiv B \pmod 4$ gives

$$2a - b \equiv \varepsilon \frac{b}{3} \equiv -\varepsilon b \pmod 4,$$

thus $a \equiv \frac{1-\varepsilon}{2} b \pmod 2$. Then

$$\begin{aligned} \chi_\pi(2) &\equiv \pi = a + b\omega \\ &\equiv b\left(\frac{1-\varepsilon}{2} + \omega\right) \pmod 2 \end{aligned}$$

If $\chi_\pi(2) = \omega$, since $b = 3B$ is odd, $\frac{1-\varepsilon}{2} \equiv 0 \pmod 2$, therefore $\varepsilon = 1$, and $B = \frac{b}{3}$.
If $\chi_\pi(2) = \omega^2 = -1 - \omega$, $\frac{1-\varepsilon}{2} \equiv 1 \pmod 2$, therefore $\varepsilon = -1$, and $B = -\frac{b}{3}$.
The normalisation given in parts (c) and (d) for the choice of the sign of B is equivalent to $A \equiv B \pmod 4$ (where A, B are odd).

$\square$

Ex. 9.41 *Let $p \equiv 1 \pmod 6, 4p = A^2 + 27B^2, A \equiv 1 \pmod 3$, A and B odd. Put $\pi = a + b\omega, 2a - b = A, b = 3B$. Let χ_π be the cubic residue character.*

(a) If $\chi_\pi(2) = \omega$ show $N(x^3 + 2y^3 = 1) = p + 1 + 2b - a \equiv 0 \pmod 2$.

(b) If $\chi_\pi(2) = \omega^2$ show $N(x^3 + 2y^3 = 1) = p + 1 - a - b \equiv 0 \pmod 2$.

(c) Show that if $A \equiv B \pmod 4$, then assuming $\chi_\pi(2) \neq 1$, one has $\chi_\pi(2) = \omega$.

(d) If $\chi_\pi(2) \neq 1, A \equiv B \pmod 4$ then

$$2^{(p-1)/3} \equiv (-A - 3B)/6B \equiv (A + 9B)/(A - 9B) \pmod \pi.$$

(This generalization of Euler's criterion is due to E.Lehmer [174]. See also K.Williams [243].)

Proof. With the help of Theorem 1, Chapter 8, we obtain, writing $\chi_\pi(2) = \omega^k$,

$$\begin{aligned}
N(x^3 + 2y^3 = 1) &= \sum_{a+2b=1} N(x^3 = a) N(y^3 = b)\\
&= \sum_{a+2b=1} \left(\sum_{i=0}^{2} \chi_\pi^i(a)\right)\left(\sum_{j=0}^{2} \chi_\pi^j(b)\right)\\
&= \sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{a+2b=1} \chi_\pi^i(a)\chi_\pi^j(b)\\
&= \sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{a+b'=1} \chi_\pi^i(a)\chi_\pi^j(2^{-1}b')\\
&= \sum_{i=0}^{2}\sum_{j=0}^{2} \chi_\pi(2)^{-j} J(\chi_\pi^i, \chi_\pi^j)\\
&= \sum_{i=0}^{2}\sum_{j=0}^{2} \omega^{-kj} J(\chi_\pi^i, \chi_\pi^j)\\
&= p + \omega^{-k} J(\chi_\pi^2, \chi_\pi) + \omega^{-2k} J(\chi_\pi, \chi_\pi^2)\\
&\qquad + \omega^{-k} J(\chi_\pi, \chi_\pi) + \omega^{-2k} J(\chi_\pi^2, \chi_\pi^2)\\
&= p - \omega^{-k}\chi_\pi(-1) - \omega^{-2k}\chi_\pi^2(-1) + 2\,\mathrm{Re}\,(\omega^{-k} J(\chi_\pi, \chi_\pi)\\
&= p - \omega^{-k} - \omega^{-2k} + 2\,\mathrm{Re}\,(\omega^{-k} J(\chi_\pi, \chi_\pi)).
\end{aligned}$$

(a) If $\chi_\pi(2) = \omega$, then $k = 1$. Using $\chi_\pi^2 = \chi_\pi^{-1} = \overline{\chi_\pi}$, we obtain

$$\begin{aligned}
N(x^3 + 2y^3 = 1) &= p + 1 + 2\,\mathrm{Re}\,(\omega^2 J(\chi_\pi, \chi_\pi))\\
&= p + 1 + 2\,\mathrm{Re}\,(\omega^2 \pi),
\end{aligned}$$

since $J(\chi_\pi, \chi_\pi) = \pi$ (Lemma 1, section 4).

$$\begin{aligned}
\omega^2\pi &= \omega^2(a + b\omega) = b - a - \omega a,\\
2\,\mathrm{Re}\,(\omega^2\pi) &= (b - a - \omega a) + (b - a - \omega^2 a) = 2b - 2a + a = 2b - a,
\end{aligned}$$

therefore

$$N(x^3 + 2y^3 = 1) = p + 1 + 2b - a \qquad (\text{if } \chi_\pi(2) = \omega).$$

Since case is $\chi_\pi(2) = \omega$, then $a \equiv 0 \pmod 2$ (see Ex. 40, part (e)), so $p+1+2b-a \equiv 0 \pmod 2$.

(b) If $\chi_\pi(2) = \omega^2 = \omega^{-1}$, then $k = -1$, and

$$N(x^3 + 2y^3 = 1) = p + 1 + 2\,\mathrm{Re}\,(\omega\pi),$$

with

$$\begin{aligned}
\omega\pi &= \omega(a + b\omega) = -b + (a - b)\omega,\\
2\,\mathrm{Re}\,(\omega\pi) &= (-b + (a - b)\omega) + (-b + (a - b)\omega^2) = -2b - (a - b) = -a - b,
\end{aligned}$$

therefore

$$N(x^3 + 2y^3 = 1) = p + 1 - a - b \qquad (\text{if } \chi_\pi(2) = \omega^2).$$

Since case is $\chi_\pi(2) = \omega^2$, then $a \equiv 1 \equiv b \pmod 2$ (see Ex. 40, part (e)), so $p + 1 - a - b \equiv 0 \pmod 2$.

(c) Suppose that $A \equiv B \pmod 4$, and $\chi_\pi(2) \neq 1$. By hypothesis, $b = 3B$, and this implies by Exercise 40 (e) that $\chi_\pi(2) = \omega$ (if not, $\chi_\pi(2) = \omega^2$, and $A \equiv B \pmod 4$ gives $B = -b/3$).

(d) Suppose that $\chi_\pi(2) \neq 1, A \equiv B \pmod 4$. By part (c), $\chi_\pi(2) = \omega$.

Since $2a - b = A, B = b/3$, then $a = \frac{A+3B}{2}, b = 3B$.

Starting from $a + b\omega \equiv 0 \pmod \pi$, we obtain

$$3B\omega \equiv -\frac{A + 3B}{2} \pmod \pi.$$

Since $p = a^2 - ab + b^2$, a is relatively prime with p, therefore $\pi \wedge b = 1$, so $\pi \wedge B = 1$, and $\pi \wedge 6 = 1$, since $p \equiv 1 \pmod 6$, thus

$$\chi_\pi(2) = \omega \equiv \frac{-A - 3B}{6B} \pmod \pi,$$

where we must read in this fraction the product of $A + 3B$ by the inverse modulo p of $6B$. By definition, using $N(\pi) = p$,

$$\chi_\pi(2) \equiv 2^{\frac{p-1}{3}} \pmod \pi,$$

so

$$2^{\frac{p-1}{3}} \equiv \frac{-A - 3B}{6B} \pmod \pi.$$

Moreover, since $4p = A^2 + 27B^2$, $A^2 + 27B^2 \equiv 0 \pmod p$, therefore

$$6B(A + 9B) + (A + 3B)(A - 9B) \equiv 0 \pmod p.$$

If $p \mid A - 9B$, since $p \nmid 6B$, this equality implies that $p \mid A + 9B$, therefore $p \mid (A - 9B) + (A + 9B) = 2A$, which is false. Therefore $A - 9B \not\equiv 0 \pmod p$, and

$$2^{\frac{p-1}{3}} \equiv \frac{-A - 3B}{6B} \equiv \frac{A + 9B}{A - 9B} \pmod \pi.$$

□

Note : By a usual argument, if $h \in \mathbb{Z}$, $2^{\frac{p-1}{3}} \equiv h \pmod \pi \iff 2^{\frac{p-1}{3}} \equiv h \pmod p$. Note that the hypothesis $\chi_\pi(2) \neq 1$ means that 2 is not a cubic residue modulo p, which is equivalent to A, B odd by Exercise 39. We can conclude

Suppose that $p \equiv 1 \pmod 6$, and let (A, B) be the unique solution of $4p = A^2 + 27B^2$ such that $A \equiv 1 \pmod 3$, and $B \equiv A \pmod 4$ if B odd, and $B > 0$ otherwise.
If B is even, then 2 is a cubic residue modulo p, and $2^{\frac{p-1}{3}} = 1$.
If B is odd, then 2 is not a cubic residue modulo p, and B satisfies $B \equiv A \pmod 4$. Writing $a = \frac{A+3B}{2}, b = 3B$, and $\pi = a + b\omega$, then $\chi_\pi(2) = \omega$, and

$$2^{\frac{p-1}{3}} \equiv \frac{A + 9B}{A - 9B} \pmod p.$$

The three roots of x^3-1 in $\mathbb{F}_p$ are $1, \frac{A+9B}{A-9B}, \frac{A-9B}{A+9B}$. Here 2 is not a cubic residue modulo p, and $2^{\frac{p-1}{3}}$ is also a cubic root of unity modulo p, so $2^{\frac{p-1}{3}} \equiv \frac{A\pm 9B}{A\mp 9B} \pmod p$. The proposition explicits the choice of the sign of B which gives $2^{\frac{p-1}{3}} \equiv \frac{A+9B}{A-9B} \pmod p$.

Numerical example : Let p be the prime 967. If we decompose p on the form $p = \pi\overline{\pi}$, we obtain $\pi = a + b\omega = -34 - 27\omega$. To obtain these result without tries, I find $k = 682$ such that $p \mid k^2+3$ with the Tonelli-Shanks algorithm, and I compute $\gcd(p, k+1+2\omega) = a+b\omega$, where $a+b\omega$ is primary, with a small Python program using the class of elements in $\mathbb{Z}[\omega]$ and the Euclid algorithm in $\mathbb{Z}[\omega]$. This gives the decompositions

$$967 = p = a^2 - ab + b^2 = 34^2 - 34 \times 27 + 27^2,$$

and

$$3868 = 4p = (2a-b)^2 + 3b^2 = A^2 + 27B^2 = 41^2 + 27 \times 9^2,$$

where $A \equiv 1 \pmod 3$, and I choose the sign of B such that $B \equiv A \pmod 4$. We obtain $A = -41, B = -9$, and a, b must verify $A = 2a - b, B = b/3$.

Then $\chi_\pi(2) = \omega$, where $\pi = -41 - 9\omega$. In $\mathbb{F}_{967}$, the cubic roots of unity modulo p are $1, 142, 824$: $142^3 \equiv 824^3 \equiv 1 \pmod{967}$.

Here $(A+9B)(A-9B)^{-1} = 142$, and we verify with a fast exponentiation that $2^{\frac{p-1}{3}} = 2^{322} \equiv 142 \pmod{967}$.

I give here an extract of a table obtained with this program, which for each p gives A, B such that $4p = A^2 + 27B^2, A \equiv 1 \pmod 3$ and such that $B \equiv A \pmod 4$ if A, B odd, and $\pi = a + b\omega$ satisfies $\chi_\pi(2) = \omega$ (or $\chi_\pi(2) = 1$ if A, B even, which corresponds to the case $a \equiv 1, b \equiv 0 \pmod 2$).

p	A	B	$\pi = a + b\omega$	$a\%2$	$b\%2$	$2^{\frac{p-1}{3}}$ % p	$\frac{A-9B}{A+9B}$	$\chi_\pi(2)$
787	31	-9	$2-27\omega$	0	1	379	379	ω
811	-56	-2	$-31-6\omega$	1	0	1	130	1
823	-5	11	$14+33\omega$	0	1	648	648	ω
829	7	11	$20+33\omega$	0	1	125	125	ω
853	-35	9	$-4+27\omega$	0	1	632	632	ω
859	13	-11	$-10-33\omega$	0	1	260	260	ω
877	-59	1	$-28+3\omega$	0	1	594	594	ω
883	-47	-7	$-34-21\omega$	0	1	545	545	ω
907	19	11	$26+33\omega$	0	1	384	384	ω
919	52	-6	$17-18\omega$	1	0	1	52	1
937	61	1	$32+3\omega$	0	1	614	614	ω
967	-41	-9	$-34-27\omega$	0	1	142	142	ω
991	61	-3	$26-9\omega$	0	1	113	113	ω
997	10	-12	$-13-36\omega$	1	0	1	692	1

As a verification I compute $\chi_\pi(2)$ with a fast exponentiation in $\mathbb{Z}[\omega]$: $\chi_\pi(2) = 2^{\frac{p-1}{3}} \pmod \pi$.

We obtain the primary prime μ such that $N(\mu) = p, \chi_\mu(2) = \omega^2$ by taking the conjugate of π. For instance, with $p = 787$, $\pi = 2 - 27\omega$ satisfies $\chi_\pi(2) = \omega$, therefore $\chi_{\overline{\pi}}(2) = \chi_{29+27\omega} = \omega^2$.

The lines where $\chi_\pi(2) = 1$, corresponding to the case where A, B are even (or equivalently a odd, b even), give the decomposition $p = x^2 + 27y^2$, $(x = A/2, y = B/2)$. For instance $997 = 5^2 + 27 \times 6^2$. If p is prime,

$$\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z},\ p = x^2 + 27y^2 \iff p \equiv 1 \pmod 3 \text{ and } \exists a \in \mathbb{Z},\ 2 \equiv a^3 \pmod p.$$

Ex. 9.42 *The notation being as in Section 12 show that the minimal polynomial of* $g(\chi_\pi)$ *is* $x^3 - 3px - Ap$.

Note : we must read "the minimal polynomial of $G = g(\chi_\pi) + \overline{g(\chi_\pi)}$ is $x^3 - 3px - Ap$".

Proof. Write $f(x) = \sum_{i=0}^{3} a_i x^i = x^3 - 3px - Ap$.

Then $a_3 = 1$, $p \mid a_0 = Ap, p \mid a_1 = -3p, p \mid a_2 = 0$.

Moreover, since $4p = A^2 + 27B^2$, $p \nmid A$, therefore $p^2 \nmid a_0$.

The Eisenstein's Irreducibility Criterion (Ex. 6.23) shows that $f(x)$ is irreducible over $\mathbb{Q}$. By section 12, G is a root of f, so f is the minimal polynomial of G. $\square$

Ex. 9.43 *Find the local maxima and minima of* $x^3 - 3px - Ap$ *and show that each of the intervals* $(-2\sqrt{p}, -\sqrt{p}), (-\sqrt{p}, \sqrt{p}), (\sqrt{p}, 2\sqrt{p})$ *contains exactly one of the values* $2\mathrm{Re}(\omega^k g(\chi_\pi))$, $k = 0, 1, 2$.

Proof. Write $\chi = \chi_\pi$, and for $k \in \{0, 1, 2\}$,

$$G_k = 2\mathrm{Re}(\omega^k g(\chi)) = \omega^k g(\chi) + \overline{\omega}^k \overline{g(\chi)},$$

so $G = G_0$. As in section 12, since $g(\chi)^3 = p\pi$, and $|g(\chi)|^2 = p$,

$$\begin{aligned} G_k^3 &= g(\chi)^3 + \overline{g(\chi)}^3 + 3\omega^{2k} g(\chi)^2 \overline{\omega}^k \overline{g(\chi)} + 3\omega^k g(\chi) \overline{\omega}^{2k} \overline{g(\chi)}^2 \\ &= p\pi + p\overline{\pi} + 3g(\chi)\overline{g(\chi)}(\omega^k g(\chi) + \overline{\omega}^k \overline{g(\chi)}) \\ &= 3pG_k + p(2a - b) \\ &= 3pG_k + pA \end{aligned}$$

So G_0, G_1, G_2 are the three roots of $f(x) = x^3 - 3px - Ap$.

$f'(x) = 3(x^2 - p) < 0$ iff $-\sqrt{p} < x < \sqrt{p}$. f is decreasing on $[-\sqrt{p}, \sqrt{p}]$, and increasing on $]-\infty, -\sqrt{p}[$, and on $[\sqrt{p}, +\infty[$.

Since $4p = A^2 + 27B^2$, $|A| < 2\sqrt{p}$, therefore

$$\begin{aligned} f(\sqrt{p}) &= p\sqrt{p} - 3p\sqrt{p} - Ap \\ &= -p(2\sqrt{p} + A) < 0, \end{aligned}$$

and

$$\begin{aligned} f(-\sqrt{p}) &= -p\sqrt{p} + 3p\sqrt{p} - Ap \\ &= p(2\sqrt{p} - A) > 0. \end{aligned}$$

$\square$

Since $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to +\infty} f(x) = +\infty$, the intermediate value theorem shows that f has a unique root in each of the intervals $]-\infty, -\sqrt{p}[,]-\sqrt{p}, \sqrt{p}[, [\sqrt{p}, +\infty[$.

Moreover

$$\begin{aligned} f(2\sqrt{p}) &= 8p\sqrt{p} - 6p\sqrt{p} - Ap = p(2\sqrt{p} - A) > 0, \\ f(-2\sqrt{p}) &= -8p\sqrt{p} + 6p\sqrt{p} - Ap = p(-2\sqrt{p} - A) < 0, \end{aligned}$$

therefore f has a unique root in each of the intervals $]-2\sqrt{p}, -\sqrt{p}[,]-\sqrt{p}, \sqrt{p}[, [\sqrt{p}, 2\sqrt{p}[$.

Ex. 9.44 *Let $n \in \mathbb{Z}$, $n = s_1 \cdots s_t, n \equiv 1 \pmod 4, i = 1, \ldots, t$. Show $(n-1)/4 \equiv \sum_{i=1}^{t} (s_i - 1)/4 \pmod 4$.*

Proof. If $n = st, s \equiv 1, t \equiv 1$ [4], then $s = 4k+1, t = 4l+1, k, t \in \mathbb{Z}$, so

$$n = (4k+1)(4l+1) = 16kl + 4k + 4l + 1, \frac{n-1}{4} = 4kl + k + l \equiv k + l = \frac{s-1}{4} + \frac{l-1}{4} \ [4].$$

Reasoning by induction on t, suppose that every product of t factors $n = s_1 s_2 \cdots s_t$, where $s_i \equiv 1$ [4] verifies

$$\frac{n-1}{4} \equiv \sum_{i=1}^{t} \frac{s_i - 1}{4} [4].$$

If $n' = s_1 s_2 \cdots s_t s_{t+1} = n s_{t+1}, s_i \equiv 1[4]$, then $n \equiv 1, s_{t+1} \equiv 1$ [4], so

$$\frac{n'-1}{4} \equiv \frac{n-1}{4} + \frac{s_{t+1}-1}{4} \equiv \sum_{i=1}^{t} \frac{s_i-1}{4} + \frac{s_{t+1}-1}{4} \equiv \sum_{i=1}^{t+1} \frac{s_i-1}{4} \ [4].$$

Conclusion : if $n = s_1 s_2 \cdots s_t, s_i \equiv 1[4]$, then $\frac{n-1}{4} \equiv \sum\limits_{i=1}^{t} \frac{s_i-1}{4} [4]$. $\square$

Ex. 9.45 *Let $\pi = a + bi \in \mathbb{Z}[i]$ and $q \equiv 3$ [4] a rational prime. Show $\pi^q \equiv \overline{\pi}$ [q].*

Proof. Let $\pi = a + bi \in \mathbb{Z}[i]$, and $q \equiv 3$ [4] a rational prime.

As $\binom{q}{k} \equiv 0 \pmod q$ for $1 \le k \le q-1$, the Fermat's Little Theorem gives

$$\begin{aligned}
\pi^q &= (a+bi)^q \\
&\equiv a^q + b^q i^q \ [q] \\
&\equiv a + bi^3 \ [q] \\
&= a - bi \\
&= \overline{\pi}
\end{aligned}$$

Conclusion : $\pi^q \equiv \bar{\pi}$ $[q]$ ($\pi \in \mathbb{Z}[i]$, and $q \equiv 3$ [4]) $\square$