Chapter 11

Ex. 11.1 Suppose that we may write the power series $1+a_1u+a_2u^2+\cdots$ as the quotient of two polynomials $P(u)/Q(u)$. Show that we may assume that $P(0) = Q(0) = 1$.

Proof. Here $f(u) = 1 + a_1u + a_2u^2 + \cdots \in \mathbb{C}[[u]]$ is a formal series in the variable u.

We suppose that $f(u) = P(u)/Q(u)$, where we may assume, after simplification, that the two polynomials are relatively prime. Then $P(1)/Q(1) = 1$. Write $c = P(1) = Q(1) \in F$.

If $c = 0$, then $u \mid P(u)$ and $u \mid Q(u)$. This is impossible since $P \wedge Q = 1$. So $c \neq 0$.

Define $P_1(u) = (1/c)P(u), Q_1(u) = (1/c)Q(u)$. Then $f(u) = P_1(u)/Q_1(u)$ and $P_1(0) = Q_1(0) = 1$. If we replace P, Q by P_1, Q_1, then the pair (P_1, Q_1) has the required properties. □

Ex. 11.2 *Prove the converse to Proposition 11.1.1.*

Proof. If $N_s = \sum_{j=1}^e \beta_j^s - \sum_{i=1}^d \alpha_i^s$, where α_i, β_j are complex numbers, then

$$\begin{aligned}\sum_{s=1}^{\infty} \frac{N_s u^s}{s} &= \sum_{j=1}^{e} \left(\sum_{s=1}^{\infty} \frac{(\beta_j u)^s}{s} \right) - \sum_{i=1}^{d} \left(\sum_{s=1}^{\infty} \frac{(\alpha_i u)^s}{s} \right) \\ &= -\sum_{j=1}^{e} \ln(1-\beta_j u) + \sum_{i=1}^{d} \ln(1-\alpha_i u).\end{aligned}$$

Here u is a variable, and both members are formal polynomials in $\mathbb{C}[[u]]$, so we don't study convergence. Nevertheless, the left member has a radius of convergence at least q^{-n}, and the right member $\min_{i,j}(1/\beta_j|, 1/|\alpha_i|)$.

Therefore,

$$Z_f(u) = \exp\left(\sum_{s=1}^{\infty} \frac{N_s u^s}{s} \right) = \prod_{j=1}^{e} (1-\beta_j u)^{-1} \prod_{i=1}^{d} (1-\alpha_i u) = \frac{\prod_{i=1}^{d}(1-\alpha_i u)}{\prod_{j=1}^{e}(1-\beta_j u)}$$

is a rational fraction. □

Ex. 11.3 *Give the details of the proof that N_s is independant of the field F_s (see the concluding paragraph to section 1).*

Proof. Suppose that E and E' are two fields containing F both with q^s elements. We first show that there is a isomorphism $\sigma : E \to E'$ which fixes the elements of F, by showing that that both E and E' are isomorphic over F to $F[x]/(f(x))$ for some irreducible polynomial $f(x) \in F(x)$.

There is a primitive element $\alpha' \in E'$, i.e. such that $E' = F(\alpha')$. For example, take α' to be a primitive $q^s - 1$ root of unity : since α is a generator of E'^*, every element $\gamma \in E'^*$ is equal to α'^k for some integer k, thus $\gamma \in F(\alpha')$ (and $0 \in F(\alpha')$). This proves $E' \subset F(\alpha')$, and since $\alpha' \in E'$ and $F \subset E'$, $F(\alpha') \subset E'$, so $E' = F(\alpha')$.

Let $f(x) \in F[x]$ be the minimal polynomial of α' over F. Then

$$E' = F(\alpha') \simeq F(x)/(f(x)),$$

where the isomorphism $\sigma_1 : F(\alpha') \to F(x)/(f(x))$ maps α' to $\overline{x} = x + (f(x))$, and maps $a \in F$ on $\overline{a} = a + (f(x))$. Since α' is a root of $x^{q^s} - x$, $f(x) \mid x^{q^s} - x$.

E is a field with q^s elements, so we have $x^{q^s}-x = \prod_{\alpha\in E}(x-\alpha)$. Thus $f(x) \mid \prod_{\alpha\in E}(x-\alpha)$, where $\deg(f(x)) = s \geq 1$, so $f(\alpha) = 0$ for some $\alpha \in E$. The polynomial f being irreducible over F, f is the minimal polynomial of α over F, thus $F(\alpha) \simeq F[x]/(f(x))$ is a field with q^s elements. Since $F(\alpha) \subset E$, and $|F(\alpha)| = |E|$, we conclude $E = F(\alpha)$, therefore

$$E = F(\alpha) \simeq F(x)/(f(x)),$$

where the isomorphism $\sigma_2 : F(\alpha) \to F(x)/(f(x))$ maps α to $\overline{x} = x + (f(x))$, and maps $a \in F$ on $\overline{a} = a + (f(x))$.

Then $\sigma = \sigma_1^{-1} \circ \sigma_2 : E \to E'$ is an isomorphism, and $\sigma(a) = a$ for all $a \in F$.

We can now use the isomorphism σ to induce a map

$$\overline{\sigma} \left\{ \begin{array}{ccc} P^n(E) & \to & P^n(E') \\ [\alpha_0,\dots,\alpha_n] & \mapsto & [\sigma(\alpha_0),\dots,\sigma(\alpha_n)]. \end{array} \right.$$

Then $\overline{\sigma}$ is injective: if $[\sigma(\alpha_0),\dots,\sigma(\alpha_n)] = [\sigma(\beta_0),\dots,\sigma(\beta_n)]$, then there is $\lambda \in F^*$ such that $\beta_i = \lambda\sigma(\alpha_i) = \sigma(\lambda)\sigma(\alpha_i) = \sigma(\lambda\alpha_i,\ i = 0,\dots,n$, thus $\beta_i = \lambda\alpha_i$, which proves $[\alpha_0,\dots,\alpha_n] = [\beta_0,\dots,\beta_n]$.

If $[\gamma_0,\dots,\gamma_n]$ is any projective point of $P^n(E')$, then

$$[\gamma_0,\dots,\gamma_n] = \overline{\sigma}([\sigma^{-1}(\gamma_0),\dots,\sigma^{-1}(\gamma_n)]).$$

This proves that $\overline{\sigma}$ is surjective. So $\overline{\sigma}$ is a bijection.

Now take $f(y_0,\dots,y_n) \in F[y_0,\dots,y_n]$ an homogeneous polynomial, $\overline{H}_f(E)$ the corresponding projective hypersurface in $P^n(E)$, and $\overline{H}_f(E')$ the corresponding projective hypersurface in $P^n(E')$. We show that $\overline{\sigma}(\overline{H}_f(E)) = \overline{H}_f(E')$.

Since σ is a F-isomorphism, $\sigma(f(\alpha_0,\dots,\alpha_n)) = f(\sigma(\alpha_0),\dots,\sigma(\alpha_n)) \quad (\alpha_i \in E)$, and similarly $\sigma^{-1}(f(\beta_0,\dots,\beta_n)) = f(\sigma^{-1}(\beta_0),\dots,\sigma^{-1}(\beta_n)) \quad (\beta_i \in E')$, thus

$$\begin{aligned} [\alpha_0,\dots,\alpha_n] \in \overline{H}_f(E) &\Rightarrow f(\alpha_0,\dots,\alpha_n) = 0 \\ &\Rightarrow \sigma(f(\alpha_0,\dots,\alpha_n)) = \sigma(0) = 0 \\ &\Rightarrow f(\sigma(\alpha_0),\dots,\sigma(\alpha_0)) = 0 \\ &\Rightarrow \overline{\sigma}([\alpha_0,\dots,\alpha_n]) = [\sigma(\alpha_0),\dots,\sigma(\alpha_0)] \in \overline{H}_f(E'). \end{aligned}$$

This shows $\overline{\sigma}(\overline{H}_f(E)) \subset \overline{H}_f(E')$.

Conversely,

$$\begin{aligned} [\beta_0,\dots,\beta_n] \in \overline{H}_f(E') &\Rightarrow f(\beta_0,\dots,\beta_n) = 0 \\ &\Rightarrow \sigma^{-1}(f(\beta_0,\dots,\beta_n)) = \sigma(0) = 0 \\ &\Rightarrow f(\sigma^{-1}(\beta_0),\dots,\sigma^{-1}(\beta_0)) = 0 \\ &\Rightarrow \overline{\sigma}^{-1}([\beta_0,\dots,\beta_n]) = [\sigma^{-1}(\beta_0),\dots,\sigma^{-1}(\beta_0)] \in \overline{H}_f(E). \end{aligned}$$

If we define $\alpha_i = \sigma^{-1}(\beta_i),\ i = 0,\dots,n$, then $[\alpha_0,\dots,\alpha_n] \in \overline{H}_f(E)$, and $[\beta_0,\dots,\beta_n] = \overline{\sigma}([\alpha_0,\dots,\alpha_n]) \in \overline{\sigma}(\overline{H}_f(E))$. This shows $\overline{H}_f(E') \subset \overline{\sigma}(\overline{H}_f(E))$, and so

$$\overline{\sigma}(\overline{H}_f(E)) = \overline{H}_f(E').$$

Since $\overline{\sigma}$ is a bijection,

$$N_s = |\overline{H}_f(E)| = |\overline{H}_f(E') = N'_s.$$

So N_s is independent of the choice of the extension $F_s = \mathbb{F}_{q^s}$ of $F = \mathbb{F}_q$. $\square$

Ex. 11.4 *Calculate the zeta function of $x_0x_1 - x_2x_3 = 0$ over $\mathbb{F}_p$.*

Proof. Here $F = \mathbb{F}_p$, and $F_s = \mathbb{F}_{p^s}$.

To calculate N_s, we calculate the number of points at infinity (such that $x_0 = 0$), and the numbers of affine points of the curve $\overline{H}_f(\mathbb{F}_{p^s})$ associate to

$$f(x_0, x_1, x_2, x_3) = x_0x_1 - x_2x_3.$$

- To estimate le number of points at infinity, we calculate first the cardinality of the set

  $$U = \{(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in F_s^4 \mid \alpha_0\alpha_1 - \alpha_2\alpha_3 = 0,\ \alpha_0 = 0\}.$$

  Then α_1 takes an arbitrary value $a \in F_s$. Write

  $$U_a = \{(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in U \mid \alpha_1 = a\}.$$

  Then $U_a = \{(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in F_s^4 \mid \alpha_0 = 0,\ \alpha_1 = a,\ \alpha_2\alpha_3 = 0\}$, thus $U_a = A \cup B$, where

  $$\begin{aligned} A &= \{(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in U_a \mid \alpha_2 = 0\}, \\ B &= \{(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in U_a \mid \alpha_3 = 0\}. \end{aligned}$$

  Since $\alpha_0, \alpha_1, \alpha_3$ are fixed in A, the map $A \to F_s$ defined by $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \mapsto \alpha_3$ is a bijection, therefore $|A| = p^s$, and similarly $|B| = p^s$. But $A \cap B = \{(0,0,0,0)\}$, thus

  $$|U_a| = |A| + |B| - |A \cap B| = 2p^s - 1.$$

  Since U is the disjoint union of the U_a, thus

  $$|U| = \sum_{a \in F_s} |U_a| = \sum_{a \in F_s} (2p^s - 1) = 2p^{2s} - p^s.$$

  Therefore the number of projective points $[\alpha_0, \alpha_1, \alpha_2, \alpha_3] \in P^3(F_s)$ at infinity (such that $\alpha_0 = 0$) is

  $$N_\infty = \frac{|U| - 1}{p^s - 1} = \frac{2p^{2s} - p^s - 1}{p^s - 1} = 2p^s + 1.$$

- Now we calculate the number of points of the affine surface $H_f(\mathbb{F}_s)$ associate to the equation $y_1 = y_2y_3$ (where $y_i = x_i/x_0$).

  The maps

  $$u \left\{ \begin{array}{ccc} F_s^2 & \to & H_f(F_s) \\ (\beta, \gamma) & \mapsto & (\beta\gamma, \beta, \gamma) \end{array} \right. \qquad \left\{ \begin{array}{ccc} H_f(F_s) & \to & F_s^2 \\ (\alpha, \beta, \gamma) & \mapsto & (\beta, \gamma) \end{array} \right.$$

  satisfy $u \circ v = \mathrm{id}, v \circ u = \mathrm{id}$, so u is a bijection. With more informal words, the arbitrary choice of $\beta, \gamma \in F_s$ gives the affine point (α, β, γ), where $\alpha = \beta\gamma$.

  This gives $|H_f(F_s)| = p^{2s}$.

Therefore

$$N_s = |\overline{H}_f(F_s)| = p^{2s} + 2p^s + 1.$$

We obtain in $\mathbb{C}[[u]]$

$$\begin{aligned}\sum_{s=1}^{\infty} \frac{N_s u^s}{s} &= \sum_{s=1}^{\infty} \frac{(p^2u)^s}{s} + 2\sum_{s=1}^{\infty} \frac{(pu)^s}{s} + \sum_{s=1}^{\infty} \frac{u^s}{s} \\ &= -\ln(1-p^2u) - 2\ln(1-pu) - \ln(1-u).\end{aligned}$$

This gives

$$Z_f(u) = (1-p^2u)^{-1}(1-pu)^{-2}(1-u)^{-1}.$$

Note: The result for N_s is verified with the naive and very slow following code in Sage:

```
def N(p,s):
    Fs = GF(p^s)
    counter = 0
    for x in Fs:
        for y in Fs:
            for z in Fs:
                for t in Fs:
                    if x*y == z*t:
                        counter += 1
    return (counter - 1)//(p^s - 1)

p, s = 5, 3
print N(p,s), p^(2*s) + 2*p^s +1
```

15876 15876

There is a misprint in the "Selected Hints for the Exercises" in Ireland-Rosen p.371. □

Ex. 11.5 *Calculate as explicitly as possible the zeta function of* $a_0x_0^2+a_1x_1^2+\cdots+a_nx_n^2$ *over* $\mathbb{F}_q$*, where* q *is odd. The answer will depend on wether* n *is odd or even and whether* $q \equiv 1 \pmod 4$ *or* $q \equiv 3 \pmod 4$*.*

Proof. Since q is odd, there is a unique character χ of order 2 over $F = \mathbb{F}_q$, and a unique character of order 2 over $F_s = \mathbb{F}_{q^s}$. We first compute the number in $\mathbb{F}_q^{n+1}$ of solutions of the equation $f(x_0,\dots,x_n) = 0$, where $f(x_0,\dots,x_n) = a_0x_0^2 + \cdots + a_nx_n^2 \in F[x_0,\dots,x_n]$.

$$\begin{aligned}N(a_0x_0^2 + \cdots + a_nx_n^2 = 0) &= \sum_{a_0u_0+\cdots+a_nu_n=0} N(x_0^2 = u_0)\cdots N(x_n^2 = u_n) \\ &= \sum_{a_0u_0+\cdots+a_nu_n=0} (1+\chi(u_0))\cdots(1+\chi(u_n)) \\ &= \sum_{v_0+\cdots+v_n=0} (1+\chi(a_0)^{-1}\chi(v_0))\cdots(1+\chi(a_n^{-1})\chi(v_n)) \qquad (v_i = a_iu_i) \\ &= q^n + \chi(a_0^{-1})\cdots\chi(a_n^{-1})J_0(\chi,\chi,\cdots,\chi),\end{aligned}$$

Indeed $J_0(\varepsilon,\dots,\varepsilon) = q^{l-1}$, and $J_0(\chi_0,\dots,\chi_n) = 0$ if some but not all of the χ_i are trivial (generalization of Proposition 8.5.1).

We estimate $J_0(\chi,\dots,\chi)$, where there are $n+1$ entries of χ.

- If n is even, then $\chi^{n+1} = \chi \neq \varepsilon$, thus $J_0(\chi, \ldots, \chi) = 0$ (Proposition 8.5.1(d)), and so
$$N(a_0x_0^2 + \cdots + a_nx_n^2 = 0) = q^n,$$
and the number of projective points on the hypersurface is given by
$$N_1 = \frac{q^n - 1}{q - 1} = q^{n-1} + \cdots + q + 1.$$

- If n is odd, then $\chi^{n+1} = \varepsilon$, thus $J_0(\chi, \ldots, \chi) = \chi(-1)(q-1)J(\chi, \ldots, \chi)$, with n entries of χ (same Proposition).

  By Theorem 3 of chapter 8,
$$J(\chi, \ldots, \chi) = \frac{g(\chi)^n}{g(\chi)} = g(\chi)^{n-1}.$$
  Since $g(\chi)^2 = g(\chi)g(\chi)^{-1} = \chi(-1)q$ (Exercise 10.22),
$$\begin{aligned}\frac{1}{q-1}\, J_0(\chi, \ldots, \chi) &= \chi(-1)g(\chi)^{n-1}\\ &= \chi(-1)g(\chi)^{n-1}\\ &= \frac{\chi(-1)g(\chi)^{n+1}}{g(\chi)^2}\\ &= \frac{1}{q}\, g(\chi)^{n+1}.\end{aligned}$$
  Therefore
$$N(a_0x_0^2 + \cdots + a_nx_n^2 = 0) = q^n + \chi(a_0)^{-1} \cdots \chi(a_n)^{-1} \frac{q-1}{q} g(\chi)^{n_1},$$
  and
$$N_1 = q^{n-1} + \cdots + q + 1 + \frac{1}{q}\chi(a_0)^{-1} \cdots \chi(a_n)^{-1} g(\chi)^{n+1}.$$

To conclude this first part,
$$\begin{array}{ll} N_1 = q^{n-1} + \cdots + q + 1 & \text{if } n \text{ is even,}\\ N_1 = q^{n-1} + \cdots + q + 1 + \frac{1}{q}\chi(a_0)^{-1} \cdots \chi(a_n)^{-1} g(\chi)^{n+1} & \text{if } n \text{ is odd.}\end{array}$$

To compute N_s, we must replace q by q^s and χ by χ_s, the character of order 2 on F_s. Then
$$\begin{array}{ll} N_s = q^{s(n-1)} + \cdots + q^s + 1 & \text{if } n \text{ is even,}\\ N_s = q^{s(n-1)} + \cdots + q^s + 1 + \frac{1}{q^s}\chi_s(a_0)^{-1} \cdots \chi_s(a_n)^{-1} g(\chi_s)^{n+1} & \text{if } n \text{ is odd.}\end{array}$$

(These two results can also be obtained by using the equations (1) and (2) in Theorem 2 of Chapter 10.)

It remains to study χ_s in the odd case.

Since $\chi_s^2 = \varepsilon$, for all $\alpha \in F_s$, $\chi_s(\alpha)^{-1} = \chi_s(\alpha)$, and $\chi_s(\alpha) = -1 \in \mathbb{C}$ if $\alpha^{\frac{q^s-1}{2}} = -1 \in F_s, \chi_s(\alpha) = 1$ otherwise.

If $a \in F$, $a^{\frac{q-1}{2}} = \pm 1 = \varepsilon$. Since q is odd, $1 + q + \cdots + q^{s-1} \equiv s \pmod 2$, thus

$$a^{\frac{q^s-1}{2}} = a^{\frac{q-1}{2}(1+q+\cdots+q^{s-1})} = \varepsilon^{1+q+\cdots+q^{s-1}} = \varepsilon^s,$$

so

$$\chi_s(a) = \chi(a)^s \qquad (a \in F).$$

We know that $g(\chi_s)^2 = \chi_s(-1)q^s$ (Ex. 10.22), thus, as n is odd,

$$\begin{aligned} g(\chi_s)^{n+1} &= \left[g(\chi_s)^2\right]^{\frac{n+1}{2}} \\ &= \chi_s(-1)^{\frac{n+1}{2}} q^{s\frac{n+1}{2}}. \end{aligned}$$

If $q \equiv 1 \pmod 4$, then $(-1)^{\frac{q-1}{2}} = 1$, so -1 is a square in $\mathbb{F}_q$. In this case, -1 is a square in $\mathbb{F}_{q^s}$, and $\chi_s(-1) = 1$ for all $s \geq 1$. In this case, using $a_i \in F$,

$$\begin{aligned} N_s &= q^{s(n-1)} + \cdots + q^s + 1 + \chi_s(a_0) \cdots \chi_s(a_n) q^{s\frac{n-1}{2}} \\ &= q^{s(n-1)} + \cdots + q^s + 1 + [\chi(a_0) \cdots \chi(a_n)]^s q^{s\frac{n-1}{2}} \end{aligned}$$

If $q \equiv -1 \pmod 4$, then $\chi(-1) = (-1)^{\frac{q-1}{2}} = -1$, and

$$\chi_s(-1) = \chi(-1)^s = (-1)^s,$$

thus

$$\frac{1}{q^s} g(\chi_s)^{n+1} = (-1)^{s\frac{n+1}{2}} q^{s\frac{n-1}{2}}.$$

This gives for odd integers n, and $q \equiv -1 \pmod 4$,

$$\begin{aligned} N_s &= q^{s(n-1)} + \cdots + q^s + 1 + (-1)^{s\frac{n+1}{2}} \chi_s(a_0) \cdots \chi_s(a_n) q^{s\frac{n-1}{2}} \\ &= q^{s(n-1)} + \cdots + q^s + 1 + [(-1)^{\frac{n+1}{2}} \chi(a_0) \cdots \chi(a_n)]^s q^{s\frac{n-1}{2}}. \end{aligned}$$

To collect all these cases, we have proved

$$\begin{array}{lll} N_s = q^{s(n-1)} + \cdots + q^s + 1 & & \text{if } n \equiv 0 \pmod 2, \\ N_s = q^{s(n-1)} + \cdots + q^s + 1 + & [\chi(a_0) \cdots \chi(a_n)]^s\, q^{s\frac{n-1}{2}} & \text{if } n \equiv 1 \ (2), q \equiv +1 \ (4), \\ N_s = q^{s(n-1)} + \cdots + q^s + 1 + & [(-1)^{\frac{n+1}{2}} \chi(a_0) \cdots \chi(a_n)]^s\, q^{s\frac{n-1}{2}} & \text{if } n \equiv 1 \ (2), q \equiv -1 \ (4). \end{array}$$

If n is even this gives, as in paragraph 1,

$$Z_f(u) = (1 - q^{n-1}u)^{-1} \cdots (1 - qu)^{-1}(1 - u)^{-1}.$$

In the case $n \equiv 1 \ (2), q \equiv +1 \ (4)$, we write for simplicity $\varepsilon = \chi(a_0) \cdots \chi(a_n) = \pm 1$. Then

$$\begin{aligned} \sum_{s=1}^{\infty} \frac{N_s u^s}{s} &= \sum_{m=0}^{n-1} \left(\sum_{s=1}^{\infty} \frac{(q^m u)^s}{s} \right) + \sum_{s=1}^{\infty} \frac{(\varepsilon q^{\frac{n-1}{2}} u)^s}{s} \\ &= -\sum_{m=0}^{n-1} \ln(1 - q^m u) - \ln(1 - \varepsilon q^{\frac{n-1}{2}} u). \end{aligned}$$

Therefore

$$Z_f(u) = \left[\prod_{m=0}^{n-1} (1 - q^m u)^{-1} \right] (1 - \chi(a_0) \cdots \chi(a_n) q^{\frac{n-1}{2}} u)^{-1}.$$

(Same calculation in the last case, with $\varepsilon = (-1)^{\frac{n+1}{2}}\chi(a_0)\cdots\chi(a_n)$.)

We obtain

$$\begin{array}{lll} Z_f(u) = P(u) & & \text{if } n \equiv 0 \quad (2), \\ Z_f(u) = P(u)(1 - & \chi(a_0)\cdots\chi(a_n)q^{\frac{n-1}{2}}u)^{-1} & \text{if } n \equiv 1 \quad (2), q \equiv +1 \quad (4), \\ Z_f(u) = P(u)(1-(-1)^{\frac{n+1}{2}} & \chi(a_0)\cdots\chi(a_n)q^{\frac{n-1}{2}}u)^{-1} & \text{if } n \equiv 1 \quad (2), q \equiv -1 \quad (4), \end{array}$$

where $P(u) = (1-q^{n-1}u)^{-1}\cdots(1-qu)^{-1}(1-u)^{-1}$.

(These results are consistent with the example $N_s = q^{2s}+q^s+1+\chi_s(-1)q^s$ given in paragraph 1 for the surface defined by $-y_0^2+y_1^2+y_2^2+y_3^2=0$, where $n=3$ is odd.

$$\begin{aligned} Z_f(u) &= (1-q^2u)^{-1}(1-qu)^{-1}(1-u)^{-1}(1-\chi(-1)qu)^{-1} \\ &= \begin{cases} (1-q^2u)^{-1}(1-qu)^{-2}(1-u)^{-1} & \text{if } q \equiv 1 \pmod 4, \\ (1-q^2u)^{-1}(1-qu)^{-1}(1-u)^{-1}(1+qu)^{-1} & \text{if } q \equiv -1 \pmod 4).) \end{cases} \end{aligned}$$

□

Ex. 11.6 *Consider $x_0^3+x_1^3+x_2^3=0$ as an equation over F_4, the field with four elements. Show that there are nine points on the curve in $P^2(F_4)$. Calculate the zeta function. [Answer: $(1+2u)^2/((1-u)(1-4u))$.]*

Proof. Since $q=4\equiv 1 \pmod 3$, we can apply Theorem 2 of Chapter 10. Let χ be a character of order 3 over $F=\mathbb{F}_4$. The only other character of order 3 is then χ^2. Thus

$$N_1 = q+1+\frac{1}{q-1}\sum_{i,j,k} J_0(\chi^i,\chi^j,\chi^k),$$

where the sum is over all $(i,j,k)\in\{1,2\}^3$ such that $i+j+k\equiv 0 \pmod 3$, that is $(1,1,1)$ and $(2,2,2)$. Thus

$$N_1 = q+1+\frac{1}{q-1}\left(J_0(\chi,\chi,\chi)+J_0(\chi^2,\chi^2,\chi^2)\right).$$

Using $\frac{1}{q-1}J_0(\chi^k,\chi^k,\chi^k) = \frac{1}{q}g(\chi^k)^3$ for $k=1,2$, we obtain

$$N_1 = q+1+\frac{1}{q}\left(g(\chi)^3+g(\chi^2)^3\right).$$

Consider $\mathbb{F}_4 = \mathbb{F}_2[x]/(x^2+x+1)$, where $a=\overline{x} = x+(x^2+x+1)$ is a generator of $\mathbb{F}_4^*$. Then $\mathbb{F}_4=\{0,1,a,a^2=a+1\}$. We compute $g(\chi)$ for the character χ of order 3 defined by

t	0	1	a	a^2
$\chi(t)$	0	1	ω	ω^2

where $\omega = e^{\frac{2i\pi}{3}}$.

for each $t\in\mathbb{F}_4$, $\mathrm{tr}(a) = a+a^2\in\mathbb{F}_2$, so the traces are $\mathrm{tr}(1)=1+1=0, \mathrm{tr}(a) = a+a^2=1, \mathrm{tr}(a^2)=a^2+a^4=a^2+a=1$. Therefore

$$\begin{aligned} g(\chi) &= \sum_{t\in\mathbb{F}_4}\chi(t)\zeta_2^{\mathrm{tr}(t)} \\ &= \sum_{t\in\mathbb{F}_4}\chi(t)(-1)^{\mathrm{tr}(t)} \\ &= 1-\omega-\omega^2 \\ &= 2. \end{aligned}$$

(This is in accordance with $|g(\chi)| = q^{1/2} = 2$.) Then $g(\chi^2) = g(\chi^{-1}) = \chi(-1)\overline{g(\chi)} = g(\chi) = 2$. Therefore

$$\begin{aligned} N_1 &= q + 1 + \frac{1}{q} g(\chi)^3 + \frac{1}{q} g(\chi^2)^3 \\ &= 5 + \frac{1}{4}(8+8) \\ &= 9. \end{aligned}$$

There are nine points on the curve with equation $x_0^3 + x_1^3 + x_2^3 = 0$ in $P^2(F_4)$ (this is verified with a naive program in Sage).

Now we compute N_s. We must replace $q = 4$ by $q^s = 4^s$, and χ by χ_s, a character with order 3 on $F_s = \mathbb{F}_{4^s}$.

We obtain

$$N_s = q^s + 1 + \frac{1}{q^s}\left(g(\chi_s)^3 + g(\chi_s^2)^3\right).$$

Now we compute $g(\chi_s)^3$. By the generalization of Corollary of Proposition 8.3.3.,

$$g(\chi_s)^3 = q^s J(\chi_s, \chi_s),$$

thus

$$N_s = q^s + 1 + J(\chi_s, \chi_s) + J(\chi_s^2, \chi_s^2).$$

We know that $|J(\chi_s, \chi_s)|^2 = q^s = 4^s$ (generalization of Corollary of Theorem 1). Writing $J(\chi_s, \chi_s) = a + b\omega,\ a, b \in \mathbb{Z}$, we search the solutions of

$$|a + b\omega|^2 = a^2 - ab + b^2 = 4^s.$$

Since $\mathbb{Z}[\omega]$ is a PID, the factorization in primes is unique. Here 2 is a prime element of $\mathbb{Z}[\omega]$, and $(a + b\omega)(a + b\omega^2) = 2^{2s}$, therefore $a + b\omega = \varepsilon 2^k, a + b\omega^2 = \zeta 2^l$, where $l, k \in \mathbb{N}$ and ε, ζ are units. Moreover $2^k = |a + b\omega| = |a + b\omega^2| = 2^l$, so $k = l = s$. This shows that every solution $a + b\omega$ of $|a + b\omega|^2 = 4^s$ is associated to 2^s:

$$|a + b\omega|^2 = 4^s \iff a + b\omega \in \{-2^s, -1 - 2^s\omega, -2^s\omega, 2^s, 1 + 2^s\omega, 2^s\omega\}.$$

Moreover, we know that $a \equiv -1 \pmod 3$, $b \equiv 0 \pmod 3$ (generalization of Proposition 8.3.4.). Therefore

$$J(\chi_s, \chi_s) = a + b\omega = -(-2)^s,$$

and similarly $J(\chi_s^2, \chi_s^2) = -(-2)^s$. This gives

$$N_s = 4^s + 1 - 2(-2)^s.$$

For $s = 1$, we find anew $N_1 = 9$.

Then

$$\begin{aligned} \sum_{s=1}^{\infty} \frac{N_s u^s}{s} &= \sum_{s=1}^{\infty} \frac{(4u)^s}{s} + \sum_{s=1}^{\infty} \frac{u^s}{s} - 2\sum_{s=1}^{\infty} \frac{(-2u)^s}{s} \\ &= -\ln(1 - 4u) - \ln(1 - u) + 2\ln(1 + 2u). \end{aligned}$$

This gives

$$Z_f(u) = \frac{(1 + 2u)^2}{(1 - 4u)(1 - u)}.$$

This is the first example where Z_f has a zero, which satisfies the Riemann hypothesis for curves. $\square$

Ex. 11.7 *Try this exercise if you know a little projective geometry. Let N_s be the number of lines in $P_n(F_{p^s})$. Find N_s and calculate $\sum_{s=1}^{\infty} N_s u^s/s$. (The set of lines in projective space form an algebraic variety calles a Grassmannian variety. So do the set of planes three-dimensinal linear subspaces, etc.)*

Proof. Write $q = p^s$. The set of lines in $P_n(F_q)$ is in bijective correspondence with the set of planes of the vector space F_q^{n+1}. To count these planes, consider the set A of linearly independent pairs(u, v) of the space F_q^{n+1}, and B the set of planes of F_q^{n+1}, and

$$f \left\{ \begin{array}{ll} A & \to B \\ (u,v) & \mapsto \langle u, v\rangle. \end{array} \right.$$

The set of pre-images of a fixed plane P in B is the set of basis of this plane P. Thus, to obtain N_s, we divides the number of linearly independent pairs(u, v) of the space by the number of basis of a fixed plane. To build such a pair, we choose first a nonzero vector u, and then a vector v not on the line generated by u. Therefore

$$\begin{aligned} N_s &= \frac{(q^{n+1}-1)(q^{n+1}-q)}{(q^2-1)(q^2-q)} \\ &= \frac{(q^{n+1}-1)(q^n-1)}{(q^2-1)(q-1)}. \end{aligned}$$

□

- If $n = 2m + 1$ is odd, then

$$\begin{aligned} N_s &= \frac{q^{2m+2}-1}{q^2-1} \cdot \frac{q^{2m+1}-1}{q-1} \\ &= \sum_{k=0}^{m} q^{2k} \sum_{l=0}^{2} q^l \\ &= \sum_{k=0}^{m} \sum_{l=0}^{2m} q^{2k+l} \\ &= \sum_{r=0}^{4m} a_r q^r \qquad (r = 2k + l), \end{aligned}$$

where a_r is the cardinality of the set

$$A_r = \{(k,l) \in [\![0, m]\!] \times [\![0, 2m]\!] \mid 2k + l = r\}.$$

We note that $0 \le l = r - 2k \le 2m$ gives

$$\left\{ \begin{array}{rcl} \frac{r}{2} - m \le & k & \le \frac{r}{2}, \\ 0 \le & k & \le m, \end{array} \right.$$

that is

$$\max\left(0, \frac{r}{2} - m\right) \le k \le \min\left(\frac{r}{2}, m\right), \tag{1}$$

and each such k gives a unique pair $(k, l) = (k, r - 2k)$ in A_r.

  - If $0 \le r \le 2m$, then (1) $\iff 0 \le k \le \frac{r}{2}$, thus $a_r = \left\lfloor \frac{r}{2} \right\rfloor + 1$.

– If $2m < r \leq 4m$, then (1) $\iff \frac{r}{2} - m \leq k \leq m$, thus $a_r = 2m - \lceil \frac{r}{2} \rceil + 1$.

If n is odd, we have proved that

$$\begin{aligned} N_s &= \sum_{r=0}^{2m} \left(\left\lfloor \frac{r}{2} \right\rfloor + 1 \right) q^r + \sum_{r=2m+1}^{4m} \left(2m + 1 - \left\lceil \frac{r}{2} \right\rceil \right) q^r \\ &= \sum_{r=0}^{n-1} \left(\left\lfloor \frac{r}{2} \right\rfloor + 1 \right) p^{sr} + \sum_{r=n}^{2n-2} \left(n - \left\lceil \frac{r}{2} \right\rceil \right) p^{sr}. \end{aligned}$$

- If $n = 2m$ is even, then

$$\begin{aligned} N_s &= \frac{q^{2m} - 1}{q^2 - 1} \cdot \frac{q^{2m+1} - 1}{q - 1} \\ &= \sum_{k=0}^{m-1} q^{2k} \sum_{l=0}^{2m} q^l \\ &= \sum_{k=0}^{m-1} \sum_{l=0}^{2m} q^{2k+l} \\ &= \sum_{r=0}^{4m-2} b_r q^r \qquad (r = 2k + l), \end{aligned}$$

where b_r is the cardinality of the set

$$B_r = \{(k, l) \in [\![0, m-1]\!] \times [\![0, 2m]\!] \mid 2k + l = r\}.$$

Here $0 \leq l = r - 2k \leq 2m$ gives

$$\begin{cases} \frac{r}{2} - m \leq & k & \leq \frac{r}{2}, \\ 0 \leq & k & \leq m - 1, \end{cases}$$

that is

$$\max\left(0, \frac{r}{2} - m\right) \leq k \leq \min\left(\frac{r}{2}, m - 1\right), \tag{2}$$

and each such k gives a unique pair $(k, l) = (k, r - 2k)$ in B_r.

– If $0 \leq r \leq 2m - 1$, then (2) $\iff 0 \leq k \leq \frac{r}{2}$, thus $b_r = \lfloor \frac{r}{2} \rfloor + 1$.

– If $2m \leq r \leq 4m - 2$, then (2) $\iff \frac{r}{2} - m \leq k \leq m - 1$, thus $b_r = 2m - \lceil \frac{r}{2} \rceil$.

If n is odd, we have proved that

$$\begin{aligned} N_s &= \sum_{r=0}^{2m-1} \left(\left\lfloor \frac{r}{2} \right\rfloor + 1 \right) q^r + \sum_{r=2m}^{4m-2} \left(2m - \left\lceil \frac{r}{2} \right\rceil \right) q^r \\ &= \sum_{r=0}^{n-1} \left(\left\lfloor \frac{r}{2} \right\rfloor + 1 \right) p^{sr} + \sum_{r=n}^{2n-2} \left(n - \left\lceil \frac{r}{2} \right\rceil \right) p^{sr}. \end{aligned}$$

This is the same formula as in the odd case ! To conclude, for all dimension n,

$$N_s = \sum_{r=0}^{n-1} \left(\left\lfloor \frac{r}{2} \right\rfloor + 1 \right) p^{sr} + \sum_{r=n}^{2n-2} \left(n - \left\lceil \frac{r}{2} \right\rceil \right) p^{sr},$$

therefore

$$\sum_{s=1}^{\infty} \frac{N_s u^s}{s} = -\sum_{r=0}^{n-1} \left(\left\lfloor \frac{r}{2} \right\rfloor + 1 \right) \ln(1 - p^r u) - \sum_{r=n}^{2n-2} \left(n - \left\lceil \frac{r}{2} \right\rceil \right) \ln(1 - p^r u)$$

This gives the order of the poles p^{-r} of $Z(u) = \exp\left(\sum_{s=1}^{\infty} \frac{N_s u^s}{s}\right)$.

To verify the equality between the two formulas giving N_s, we test this equality with a Sage program.

```
def N(n,p,s):
     q = p^s
     num = (q^(n+1) - 1)*(q^(n+1) - q)
     den = (q^2 - 1)*(q^2-q)
     return num // den

 def M(n,p,s):
    q = p^s
    a = sum((floor(r/2) +1)*q^r for r in range(n))
    b = sum((n - ceil(r/2))*q^r for r in range(n,2*n-1))
    return a+b

 N(4,5,3),M(4,5,3)
```

(3845707062626, 3845707062626)

Ex. 11.8 *If f is a nonhomogeneous polynomial, we can consider the zeta function of the projective closure of the hypersurface defined by f (see Chapter 10). One way to calculate this is to count the number of points on $H_f(F_q)$ and then add to it the number of points at infinity. For example, consider $y^2 = x^3$ over F_{p^s}. Show that there is one point at infinity. The origin $(0,0)$ is clearly on this curve. If $x \neq 0$, write $(y/x)^2 = x$ and show that there are p^s more points on this curve. Altogether we have p^s points and the zeta function over F_p is $(1 - pu)^{-1}$.*

Proof. Consider the polynomial $f(x,y) = y^2 - x^3$ and $g(x,z) = y^2 - x$, and

$$\begin{aligned} \Gamma = H_f(F_q) &= \{(x,y) \in F_p^2 \mid y^2 = x^3\}, \\ \Gamma_1 = H_g(F_q) &= \{(x,y) \in F_q^2 \mid y^2 = x\}. \end{aligned}$$

Then

$$\varphi \left\{ \begin{array}{ccc} \Gamma \setminus \{(0,0)\} & \to & \Gamma_1 \setminus \{(0,0)\} \\ (x,y) & \mapsto & \left(x, \frac{y}{x}\right) \end{array} \right.$$

is defined, since $\left(\frac{y}{x}\right)^2 = x$ for $(x,y) \in \Gamma \setminus \{(0,0)\}$, thus $\left(x, \frac{y}{x}\right) \in \Gamma_1$. Moreover

$$\psi \left\{ \begin{array}{ccc} \Gamma_1 \setminus \{(0,0)\} & \to & \Gamma \setminus \{(0,0)\} \\ (x,y) & \mapsto & (x, xy) \end{array} \right.$$

is correctly defined, since for each $(x,y) \in \Gamma_1 \setminus \{(0,0)\}, y^2 = x$, then $x \neq 0$, thus $(xy)^2 = x^3$, and $(x,xy) \in \Gamma$, where $(x,xy) \neq (0,0)$.

Moreover ψ satisfies $\psi \circ \varphi = \mathrm{id}, \varphi \circ \psi = \mathrm{id}$:

$$(\psi \circ \varphi)(x,y) = \psi\left(x, \frac{y}{x}\right) = \left(x, x\frac{y}{x}\right) = (x,y) \qquad ((x,y) \in \Gamma \setminus \{(0,0)\}),$$
$$(\varphi \circ \psi)(x,y) = \varphi(x,xy) = \left(x, \frac{xy}{x}\right) = (x,y) \qquad ((x,y) \in \Gamma_1 \setminus \{(0,0)\}).$$

So φ is a bijection. This shows that $|\Gamma \setminus \{(0,0)\}| = |\Gamma_1 \setminus \{(0,0)\}|$, where $(0,0) \in \Gamma$ and $(0,0) \in \Gamma_1$, thus

$$|\Gamma_1| = |\Gamma|.$$

To count the points on Γ_1, we consider

$$\lambda \left\{ \begin{array}{ccc} F_q & \to & \Gamma_1 \\ y & \mapsto & (y^2, y). \end{array} \right.$$

Then λ is bijective, with inverse $\mu : (x,y) \mapsto y$. This show that

$$|\Gamma| = |\Gamma_1| = q = p^s.$$

Therefore the zeta function of the affine curve $y^2 = x^3$ over F_p is

$$Z_f(u) = (1 - pu)^{-1}.$$

But the projective completion $H_{\overline{f}}(F_q)$ of this curve has $p^s + 1$ points, with only one point at infinity, since $ty^2 = x^3$ has only one point $[t,x,y]$ satisfying $t = 0$, the point $[0,0,1]$.

The zeta function of the curve with homogeneous equation $\overline{f}(t,x,y) = ty^2 - x^3$ over F_p is

$$Z_{\overline{f}}(u) = (1-u)^{-1}(1-pu)^{-1}.$$

$\square$

Ex. 11.9 *Calculate the zeta function of $y^2 = x^3 + x^2$ over F_p.*

Proof. The curve Γ defined by the equation $y^2 = x^3 + x^2$ has a singularity at the origine, as in the previous exercise. The same method applies here: if we use $z = y/x$, then $z^2 = x + 1$.

Watch out! Here there are two points $(x,z) \in \Gamma_1$ such that $x = 0$, the points $(0,1)$ and $(0,-1)$ (here we assume that $p \neq 2$). The curve Γ_1 defined by the equation $z^2 = x+1$ is such that

$$\varphi \left\{ \begin{array}{ccc} \Gamma \setminus \{(0,0)\} & \to & \Gamma_1 \setminus \{(0,1),(0,-1)\} \\ (x,y) & \mapsto & \left(x, \frac{y}{x}\right) \end{array} \right.$$

is bijective, thus $|\Gamma| = |\Gamma_1| - 1$. Since each point of Γ_1 is determined by its coordinate z, $|\Gamma_1| = q = p^s$, and $|\Gamma| = p^s - 1$.

Therefore the zeta function of the affine curve $y^2 = x^3 + x^2$ over F_p is

$$Z_f(u) = (1-u)(1-pu)^{-1},$$

There is only one point p at infinity, given by $y^2 t = x^3 + x^2 t, t = 0$, i.e. $p = [0,0,1]$. Thus $N_s = p^s$, and the zeta function of the projective completion of Γ is

$$Z_{\overline{f}}(u) = (1-pu)^{-1}.$$

$\square$

The results of Ex.8 and Ex. 9 concern only singular cubics.

Ex. 11.10 *If $A \neq 0$ in F_q and $q \equiv 1 \pmod 3$, show that the zeta function of $y^2 = x^3+A$ over F_q has the form $Z(u) = (1+au+qu^2)/((1-u)(1-qu))$, where $a \in \mathbb{Z}$ and $|a| \leq 2q^{1/2}$.*

Proof. Here we compute the zeta function of the projective completion $\overline{H}_f(F_q)$, with equation $f(x,y,t) = y^2t = x^3 + At^3$. If $t = 0$, then $x = 0$, thus there is only one point $[0,1,0]$ at infinity (over F_q or over F_{q^s}).

We assume that the characteristic is not 2. Then q is odd, and so $q \equiv 1 \pmod 6$. Therefore, there are characters of order 2 and 3 on F_q. Write ρ the unique character of order 2, and write χ a character of order 3. As χ is a character of order 3, the characters whose order divides 3 are $\varepsilon, \chi, \chi^2$.

We compute first N_1. We write $N(y^2 = x^3 + A)$ for the number of points of the affine cubic over F_q, and N_1 for the number of points of the projective cubic, so that $N_1 = N(y^2 = x^3 + A) + 1$. We recall the results obtained in Ex. 8.15.

The map $x \mapsto -x$ is a bijection between the set of roots of $x^3 = b$ and the set of roots of $(-x)^3 = b$, so $N(x^3 = b) = N((-x)^3 = b) = N(x^3 = -b)$.

Using Prop. 8.1.5, we obtain, since $A \neq 0$,

$$\begin{aligned}
N(y^2 = x^3 + A) &= \sum_{a+b=A} N(y^2 = a)N(x^3 = -b)\\
&= \sum_{a+b=A} N(y^2 = a)N(x^3 = b)\\
&= \sum_{a+b=A} (1+\rho(a))(1+\chi(b)+\chi^2(b))\\
&= \sum_{i=0}^{1}\sum_{j=0}^{2}\sum_{a+b=A} \rho^i(a)\chi^j(b)\\
&= \sum_{i=0}^{1}\sum_{j=0}^{2} \rho(A)^i\chi(A)^j \sum_{a'+b'=1} \rho^i(a')\chi^j(b') \qquad (a = Aa', b = Ab')\\
&= \sum_{i=0}^{1}\sum_{j=0}^{2} \rho(A)^i\chi(A)^j J(\chi^j,\rho^i).
\end{aligned}$$

We know (generalization of Theorem 1, Chapter 8) that $J(\chi,\varepsilon) = J(\chi^2,\varepsilon) = J(\varepsilon,\rho) = 0$, and $J(\varepsilon,\varepsilon) = q$, so

$$N(y^2 = x^3 + A) = q + \rho(A)\chi(A)J(\chi,\rho) + \rho(A)\chi^2(A)J(\chi^2,\rho).$$

As $\chi^2(A) = \chi^{-1}(A) = \overline{\chi(A)}$, and as $\overline{\rho(A)} = \rho(A)$, then $J(\chi^2,\rho) = J(\overline{\chi},\overline{\rho}) = \overline{J(\chi,\rho)}$, and

$$N(y^2 = x^3 + A) = q + \pi + \bar{\pi}, \text{ where } \pi = \rho(A)\chi(A)J(\chi,\rho),$$

therefore

$$N_1 = q + 1 + \pi + \bar{\pi}, \text{ where } \pi = \rho(A)\chi(A)J(\chi,\rho).$$

Since the orders of χ, ρ, and $\chi\rho$ are $3, 2$ and 6, $\chi \neq \varepsilon, \rho \neq \varepsilon, \chi\rho \neq \varepsilon$, thus Theorem 1 of Chapter 6 gives

$$J(\chi,\rho) = \frac{g(\chi)g(\rho)}{g(\chi\rho)}, \qquad \pi = \rho(A)\chi(A)\frac{g(\chi)g(\rho)}{g(\chi\rho)}.$$

Write $\chi' = \chi \circ \mathrm{N}_{F_{q^s}/F_q}, \rho' = \rho \circ \mathrm{N}_{F_{q^s}/F_q}$. Then χ', ρ' are characters on F_{q^s}, and the orders of χ', ρ' are 3 and 2 (by properties (a), (b) of §3). The same reasoning in F_{q^s} gives

$$N_s = q^s + 1 + \pi' + \overline{\pi'}, \qquad \pi' = \rho'(A)\chi'(A)\frac{g(\chi')g(\rho')}{g(\chi'\rho')}.$$

Since $A \in F_q$, the property (c) of §3 gives $\chi'(A) = \chi(A)^s, \rho'(A) = \rho(A)^s$. Using the Hasse-Davenport Relation, and $(\chi\rho)' = \chi'\rho'$, we obtain

$$\begin{aligned}
\pi' &= \rho'(A)\chi'(A)\frac{g(\chi')g(\rho')}{g(\chi'\rho')} \\
&= -\rho(A)^s\chi(A)^s\frac{(-g(\chi))^s(-g(\rho))^s}{(-g(\chi\rho))^s} \\
&= (-1)^{s+1}\rho(A)^s\chi(A)^s\left[\frac{g(\chi)g(\rho)}{g(\chi\rho)}\right]^s \\
&= -\left[-\rho(A)\chi(A)\frac{g(\chi)g(\rho)}{g(\chi\rho)}\right]^s \\
&= -(-\pi)^s.
\end{aligned}$$

This gives N_s in the appropriate form:

$$N_s = q^s + 1 - (-\pi)^s - (-\overline{\pi})^s, \qquad \pi = \rho(A)\chi(A)J(\chi,\rho) = \rho(A)\chi(A)\frac{g(\chi)g(\rho)}{g(\chi\rho)}.$$

Using the converse to Proposition 11.1.1 given in Exercise 2, we obtain

$$Z_f(u) = \frac{(1+\pi u)(1+\overline{\pi}u)}{(1-u)(1-qu)}.$$

Note that $\pi\overline{\pi} = |\pi|^2 = q$ (by Exercise 10.22). Expanding the numerator, this gives

$$Z_f(u) = \frac{1 + au + qu^2}{(1-u)(1-qu)},$$

where $a = \pi + \overline{\pi}$.

For all $t \in F_q^*$, $\chi^3(t) = 1$, thus $\chi(t) \in \{1, \omega, \omega^2\} \subset \mathbb{Z}[\omega]$, and $\rho(t) = \pm 1$, therefore $\pi = \rho(A)\chi(A)\sum_{t \in F_q}\chi(t)\rho(t) \in \mathbb{Z}[\omega]$. Writing $\pi = u + v\omega,\ u, v \in Z$, we obtain $a = \pi + \overline{\pi} = 2u - v \in \mathbb{Z}$.

Moreover,

$$|a| \le |\pi| + |\overline{\pi}| = 2|\pi| = 2q^{1/2}.$$

To conclude,

$$Z_f(u) = \frac{1 + au + qu^2}{(1-u)(1-qu)}, \qquad a \in \mathbb{Z},\ |a| \le 2q^{1/2}.$$

□