Solutions to Ireland, Rosen "A Classical Introduction to Modern Number Theory"

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Chapter 4

Ex. 4.1 *Show that* 2 *is a primitive root modulo* 29.

Proof. Let $p = 29 : p - 1 = 2^2 \times 7$.

$2^4 = 16 \not\equiv 1[29]$

$2^{14} = 4^7 = 4 \times 16^3 = 64 \times 256 \equiv 6 \times (-34) = -204 \equiv 86 = 3 \times 29 - 1 \equiv -1[29]$

$2^{28} \equiv 1[29]$ and $2^d \not\equiv 1$ if $d \mid 28, d < 28$, hence 2 is a primitive element modulo 29. $\square$

Ex. 4.2 *Compute all primitive roots for* $p = 11, 13, 17,$ *and* 19.

Proof. • $p = 11$. Then $p - 1 = 10 = 2 \times 5$.

$2^2 = 4 \not\equiv 1 \pmod{11}$, and $2^5 = 32 \equiv -1 \not\equiv 1 \pmod{11}$, so 2 is a primitive element modulo 11.

The other primitive elements modulo 11 are congruent to the powers $2^i, i \wedge 10 = 1, 1 \le i < 10$, namely $2, 2^3, 2^7, 2^9$.

$2^7 \equiv 7 \pmod{11}, 2^9 \equiv 6 \pmod{11}$, so

$\{\overline{2}, \overline{8}, \overline{7}, \overline{6}\}$ is the set of the generators of $U(\mathbb{Z}/11\mathbb{Z})$.

Similarly :

• $p = 13 : \{2, 6, 11, 7\}$ is the set of the generators of $U(\mathbb{Z}/13\mathbb{Z})$.

• $p = 17 : \{3, 10, 5, 11, 14, 7, 12, 6\}$ is the set of the generators of $U(\mathbb{Z}/17\mathbb{Z})$.

• $p = 19 : \{2, 13, 14, 15, 3, 10\}$ is the set of the generators of $U(\mathbb{Z}/19\mathbb{Z})$.

I obtain these results with the direct orders in S.A.G.E. :

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p = 19; Fp = GF(p); a = Fp.multiplicative_generator()
print([a^k for k in range(1,p) if gcd(k,p-1) == 1])
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$\square$

Ex. 4.3 *Suppose that* a *is a primitive root modulo* p^n, p *an odd prime. Show that* a *is a primitive root modulo* p.

Proof. Suppose that a is a primitive root modulo p^n : then $\overline{a}$ is a generator of $U(\mathbb{Z}/p^n\mathbb{Z})$.

If a was not a primitive root modulo p, $\overline{a}$ is not a generator of $U(\mathbb{Z}/p\mathbb{Z})$, so there exists $b \in \mathbb{Z}, b \wedge p = 1$ such that $a^k \not\equiv b \pmod{p}$ for all $k \in \mathbb{Z}$. A fortiori $a^k \not\equiv b \pmod{p^n}$, and $b \wedge p^n = 1$, so $\overline{b} \in U(\mathbb{Z}/p^n\mathbb{Z})$ and $\overline{b} \notin \langle \overline{a} \rangle$ in $U(\mathbb{Z}/p^n\mathbb{Z})$, in contradiction with the hypothesis. So a is a primitive root modulo p.

(the reasoning on the orders of a, modulo p and modulo p^n, is possible, but not so easy.) $\square$

Ex. 4.4 *Consider a prime p of the form $4t+1$. Show that a is a primitive root modulo p iff $-a$ is a primitive root modulo p.*

Proof. Solution 1.

As. $p-1$ is even, $(-a)^{p-1} = a^{p-1} \equiv 1 \pmod p$.

If $(-a)^n \equiv 1 \pmod p$, with $n \in \mathbb{N}$, then $a^n \equiv (-1)^n \pmod p$.

If n is odd, then $a^n \equiv -1, a^{2n} \equiv 1 \pmod p$. As a is a primitive root modulo p, $p-1 \mid 2n$, $2t \mid n$, so n is even : this is a contradiction.

Consequently, n is even, and $a^n \equiv 1 \pmod p$, so $p-1 \mid n$, so the least $n \in \mathbb{N}^*$ such that $a^n \equiv 1 \pmod p$ is $p-1$: the order of a modulo p is $p-1$, a is a primitive root modulo p.

Reciprocally, if $-a$ is a primitive root modulo p, we apply the previous result at $-a$ to to obtain that $-(-a) = a$ is a primitive root.

Solution 2.

Let $p-1 = 2^{a_0} p_1^{a_1} \cdots p_k^{a_k}$ the decomposition of $p-1$ in prime factors.

As p_i is odd for $i = 1, 2, \cdots k$, $(p-1)/p_i$ is even, and a is primitive, so

$$\begin{aligned} (-a)^{(p-1)/p_i} &= a^{(p-1)/p_i} \not\equiv 1 \pmod p, \\ (-a)^{(p-1)/2} &= (-a)^{2k} = a^{2k} = a^{(p-1)/2} \not\equiv 1 \pmod p. \end{aligned}$$

So the order of a is $p-1$ modulo p (see Ex. 4.8) : a is a primitive element modulo p. ☐

Ex. 4.5 *Consider a prime p of the form $4t+3$. Show that a is a primitive root modulo p iff $-a$ has order $(p-1)/2$.*

Proof. Let a a primitive root modulo p.

As $a^{p-1} \equiv 1 \pmod p$, $p \mid (a^{(p-1)/2} - 1)(a^{(p-1)/2} + 1)$, so $p \mid a^{(p-1)/2} - 1$ or $p \mid a^{(p-1)/2} + 1$. As a is a primitive root modulo p, $a^{(p-1)/2} \not\equiv 1 \pmod p$, so

$$a^{(p-1)/2} \equiv -1 \pmod p.$$

Hence $(-a)^{(p-1)/2} = (-1)^{2t+1} a^{(p-1)/2} \equiv (-1) \times (-1) = 1 \pmod p$.

Suppose that $(-a)^n \equiv 1 \pmod p$, with $n \in \mathbb{N}$.

Then $a^{2n} = (-a)^{2n} \equiv 1 \pmod p$, so $p-1 \mid 2n$, $\frac{p-1}{2} \mid n$.

So $-a$ has order $(p-1)/2$ modulo p.

Reciprocally, suppose that $-a$ has order $(p-1)/2 = 2t+1$ modulo p. Let $2, p_1, \ldots p_k$ the prime factors of $p-1$, where p_i are odd.

$a^{(p-1)/2} = a^{2t+1} = -(-a)^{2t+1} = -(-a)^{(p-1)/2} \equiv -1$, so $a^{(p-1)/2} \not\equiv 1 \pmod 2$.

As $p-1$ is even, $(p-1)/p_i$ is even, so

$a^{(p-1)/p_i} = (-a)^{(p-1)/p_i} \not\equiv 1 \pmod p$ (since $-a$ has order $p-1$).

So the order of a is $p-1$ (see Ex. 4.8) : a is a primitive root modulo p. ☐

Ex. 4.6 *If $p = 2^{2^n} + 1$ is a Fermat prime, show that 3 is a primitive root modulo p.*

Proof. Solution 1 (with quadratic reciprocity).

Write $p = 2^k + 1$, with $k = 2^n$.

We suppose that $n > 0$, so $k \geq 2, p \geq 5$. As p is prime, $3^{p-1} \equiv 1 \pmod p$.

In other words, $3^{2^k} \equiv 1 \pmod p$: the order of 3 is a divisor of 2^k, a power of 2.

3 has order 2^k modulo p iff $3^{2^{k-1}} \not\equiv 1 \pmod p$. As $\left(3^{2^{k-1}}\right)^2 \equiv 1 \pmod p$, where p is prime, this is equivalent to $3^{2^{k-1}} \equiv -1 \pmod p$, which remains to prove.

$3^{2^{k-1}} = 3^{(p-1)/2} \equiv \left(\frac{3}{p}\right) \pmod p$.

As the result is true for $p = 5$, we can suppose $n \geq 2$. From the law of quadratic reciprocity :

$$\left(\frac{3}{p}\right)\left(\frac{p}{3}\right) = (-1)^{(p-1)/2} = (-1)^{2^{k-1}} = 1.$$

So $\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right)$

$$\begin{aligned} p = 2^{2^n} + 1 &\equiv (-1)^{2^n} + 1 \pmod 3 \\ &\equiv 2 \equiv -1 \pmod 3, \end{aligned}$$

so $\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right) = -1$, that is to say

$$3^{2^{k-1}} \equiv -1 \pmod p.$$

The order of 3 modulo $p = 2^{2^n} + 1$ is $p - 1 = 2^{2^n}$: 3 is a primitive root modulo p.

(On the other hand, if 3 is of order $p-1$ modulo p, then p is prime, so

$$F_n = 2^{2^n} + 1 \text{ is prime } \iff 3^{(F_n-1)/2} = 3^{2^{2^n-1}} \equiv -1 \pmod{F_n}.)$$

Solution 2 (without quadratic reciprocity, with the hint of chapter 4).

As above, if if we suppose that 3 is not a primitive root modulo p, then $3^{2^{n-1}} \equiv 1 \pmod p$, so $n \geq 2$, and $(-3)^{(p-1)/2} = 3^{2^{n-1}} \equiv 1 \pmod p$, so -3 is a square modulo p : there exists $a \in \mathbb{Z}$ such that $-3 \equiv a^2 \pmod p$.

As $2 \wedge p = 1$, there exists $u \in \mathbb{Z}$ such that $2u \equiv -1 + a \pmod p$ ($\overline{u}$ is similar to $\omega = \frac{-1+i\sqrt{3}}{2} \in \mathbb{C}$). Then

$$\begin{aligned} 8u^3 &\equiv (-1+a)^3 \\ &\equiv -1 + 3a - 3a^2 + a^3 \\ &\equiv -1 + 3a + 9 - 3a \\ &\equiv 8 \pmod p \end{aligned}$$

As $p \wedge 2 = p \wedge 8 = 1, u^3 \equiv 1 \pmod p$. Moreover, if $u \equiv 1 \pmod 3$, then $a \equiv 3 \pmod p$, $-3 \equiv 9 \pmod p, p \mid 12$, so $p = 2$ or $p = 3$, in contradiction with $p \geq 5$. So the order of u modulo p is 3 : $(\mathbb{Z}/p\mathbb{Z})^*$ contains an element $\overline{u}$ of order 3. So $3 \mid p-1$, $p \equiv 1 \pmod 3$, but $p \equiv (-1)^{2^n} + 1 \equiv 2 \equiv -1 \pmod 3$: this is a contradiction, so 3 is a primitive root modulo $p = 2^{2^n} + 1$. □

Ex. 4.7 *Suppose that p is a prime of the form $8t+3$ and that $q = (p-1)/2$ is also a prime. Show that 2 is a primitive root modulo p.*

Proof. The first examples of such couples (q,p) are $(5,11), (29,59), (41,83), (53,107), (89,179)$.

$p = 2q + 1 = 8t + 3$ and p, q are prime numbers.

From Fermat's little theorem, $2^{p-1} \equiv 1 \pmod p$, so $2^{2q} \equiv 1 \pmod p$.

The order of 2 modulo p divides $2q$: to prove that the order of 2 is $2q = p-1$, it is suffisant to prove

$$2^2 \not\equiv 1 \pmod p, \quad 2^q \not\equiv 1 \pmod p.$$

If $2^2 \equiv 1 \pmod p$, then $p \mid 3$, $p = 3$ and $q = 1$: q is not a prime, so $2^2 \not\equiv 1 \pmod p$.

If $2^q = 2^{(p-1)/2} \equiv 1 \pmod p$, then 2 is a square modulo p (prop. 4.2.1) : there exists $a \in \mathbb{Z}$ such that $2 \equiv a^2 \pmod p$.

From the complementary case of law of quadratic reciprocity (see next chapter, prop. 5.1.3), 2 is a square modulo p iff

$$1 = \left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}.$$

Yet $p \equiv 3 \pmod 8$, so $p^2 \equiv 1 \pmod{16}$, $\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} = -1$, so 2 is not a square modulo p. This is a contradiction, so $2^q \not\equiv 1 \pmod p$: 2 is a primitive root modulo p. $\square$

Ex. 4.8 *Let p be an odd prime. Show that a is a primitive root modulo p iff $a^{(p-1)/q} \not\equiv 1$ $\pmod p$ for all prime divisors q of $p-1$.*

Proof. • If a is a primitive root, then $a^k \not\equiv 1$ for all $k, 1 \le k < p-1$, so $a^{(p-1)/q} \not\equiv 1$ $\pmod p$ for all prime divisors q of $p-1$.

• In the other direction, suppose $a^{(p-1)/q} \not\equiv 1 \pmod p$ for all prime divisors q of $p-1$.

Let δ the order of a, and $p-1 = q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k}$ the decomposition of $p-1$ in prime factors. As $\delta \mid p-1, \delta = q_1^{b_1} p_2^{b_2} \cdots q_k^{b_k}$, with $b_i \le a_i, i = 1, 2, \ldots, k$. If $b_i < a_i$ for some index i, then $\delta \mid (p-1)/q_i$, so $a^{(p-1)/q_i} \equiv 1 \pmod p$, which is in contradiction with the hypothesis. Thus $b_i = a_i$ for all i, and $\delta = q-1$: a is a primitive root modulo p. $\square$

Ex. 4.9 *Show that the product of all the primitive roots modulo p is congruent to $(-1)^{\phi(p-1)}$ modulo p.*

Proof. Here we suppose p prime, $p > 2$. Let g a primitive root modulo p. $U(\mathbb{Z}/p\mathbb{Z})$ is cyclic, generated by $\overline{g}$:

$$U(\mathbb{Z}/p\mathbb{Z}) = \{\overline{1}, \overline{g}, \overline{g}^2, \ldots, \overline{g}^{p-2}\}, \qquad \overline{g}^{p-1} = \overline{1}.$$

$\overline{g}^k$ is a primitive element iff $k \wedge (p-1) = 1$, so the product of primitive elements in $U(\mathbb{Z}/p\mathbb{Z})$ is

$$\overline{P} = \prod_{\substack{k \wedge (p-1) = 1 \\ 1 \le k < p-1}} \overline{g}^k.$$

so $\overline{P} = \overline{g}^S$, where $S = \sum_{\substack{k \wedge (p-1) = 1 \\ 1 \le k < p-1}} k$.

From Ex. 2.22, we know that for $n \ge 2$,

$$\sum_{\substack{k \wedge n = 1 \\ 1 \le k < n}} k = \frac{1}{2} n \phi(n).$$

So $S = \sum_{\substack{k \wedge (p-1) = 1 \\ 1 \le k < p-1}} k = \frac{1}{2}(p-1)\phi(p-1)$.

As $p > 2$, $p-1$ is even. $(\overline{g}^{(p-1)/2})^2 = \overline{g}^{p-1} = \overline{1}$, and $\overline{g}^{(p-1)/2} \ne \overline{1}$. As $\mathbb{Z}/p\mathbb{Z}$ is a field, $\overline{g}^{(p-1)/2} = -\overline{1}$.

Thus $\overline{P} = (-\overline{1})^{\phi(p-1)}$: so the product P of all the primitive roots modulo p is such that

$$P \equiv (-1)^{\phi(p-1)} \pmod p.$$

$\square$

Ex. 4.10 *Show that the sum of all the primitive roots modulo p is congruent to $\mu(p-1)$ modulo p.*

Proof. Notation : $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is the field with p elements, $|x|$ the multiplicative order of an element $x \in \mathbb{F}_p^*$, $\mathbb{N}^* = \{1, 2, 3, \ldots\}$.

Let

$$\psi : \left\{ \begin{array}{ccc} \mathbb{N}^* & \to & \mathbb{F}_p \\ n & \mapsto & \psi(n) = \sum\limits_{d \in \mathbb{F}_p^*, |d| = n} d \end{array} \right.$$

$\psi(n)$ is the sum of the elements with order n in $\mathbb{F}_p^*$. So $\psi(n) = 0$ if $n \nmid p-1$, and $S = \psi(p-1)$ is the sought sum of all the primitive roots modulo p.

We compute for all $n \in \mathbb{N}^*$

$$f(n) = \sum_{d|n} \psi(d).$$

$f(n)$ is the sum of elements whose order divides n, in other worlds the sum of the roots of $x^n - 1$. This sum is, up to the sign, the coefficient of x^{n-1}, so is null, except in the case $n = 1$, where the sum of the unique root 1 of $x - 1$ is 1. So

$$f(1) = 1, \qquad \forall n > 1, f(n) = 0,$$

($f = \chi_{\{1\}}$ is the characteristic function of $\{1\}$).

From the Möbius inversion formula, for all $n \in \mathbb{N}^*, \psi(n) = \sum_{d|m} \mu\left(\frac{n}{d}\right) f(d)$, so

$$\psi(p-1) = \sum_{d|p-1} \mu\left(\frac{p-1}{d}\right) f(d) = \mu(p-1).$$

Conclusion :

$$S = \sum_{d \in \mathbb{F}_p^*, |d| = p-1} d = \mu(p-1) :$$

the sum of all the primitive roots modulo p is congruent to $\mu(p-1)$ modulo p. □

Ex. 4.11 *Prove that $1^k + 2^k + \ldots + (p-1)^k \equiv 0 \pmod p$ if $p - 1 \nmid k$, and $-1 \pmod p$ if $p - 1 \mid k$.*

Proof. Let $S_k = 1^k + 2^k + \cdots + (p-1)^k$.

Let g a primitive root modulo p : $\overline{g}$ a generator of $\mathbb{F}_p^*$.

As $(\overline{1}, \overline{g}, \overline{g}^2, \ldots, \overline{g}^{p-2})$ is a permutation of $(\overline{1}, \overline{2}, \ldots, \overline{p-1})$,

$$\begin{aligned} \overline{S_k} &= \overline{1}^k + \overline{2}^k + \cdots + \overline{p-1}^k \\ &= \sum_{i=0}^{p-2} \overline{g}^{ki} = \left\{ \begin{array}{ll} \overline{p-1} = -\overline{1} & \text{if} \quad p-1 \mid k \\ \frac{\overline{g}^{(p-1)k} - 1}{\overline{g}^k - 1} = \overline{0} & \text{if} \quad p-1 \nmid k \end{array} \right. \end{aligned}$$

since $p - 1 \mid k \iff \overline{g}^k = \overline{1}$.

Conclusion :

$$\begin{aligned} 1^k + 2^k + \cdots + (p-1)^k &\equiv 0 \pmod p \text{ if } p-1 \nmid k \\ 1^k + 2^k + \cdots + (p-1)^k &\equiv -1 \pmod p \text{ if } p-1 \mid k \end{aligned}$$

□

Ex. 4.12 *Use the existence of a primitive root to give another proof of Wilson's theorem*$(p-1)! \equiv -1 \pmod{p}$.

Proof. As the result is trivial if $p = 2$, we suppose that p is an odd prime.

Let g a primitive root modulo p : $\overline{g}$ a generator of $\mathbb{F}_p^*$.

As $(\overline{g}^{(p-1)/2})^2 = \overline{g}^{p-1} = \overline{1}$, and $\overline{g}^{(p-1)/2} \neq 1$ in the field $\mathbb{F}_p^*$, then $\overline{g}^{(p-1)/2} = -1$, and $(\overline{1}, \overline{g}, \overline{g}^2, \ldots, \overline{g}^{p-2})$ is a permutation of $(\overline{1}, \overline{2}, \ldots, \overline{p-1})$, so

$$\begin{aligned}
\overline{(p-1)!} &= \prod_{k=0}^{p-2} \overline{g}^k \\
&= \overline{g}^{\sum_{k=0}^{p-2} k} \\
&= \overline{g}^{(p-2)(p-1)/2} \\
&= \left(\overline{g}^{(p-1)/2}\right)^{p-2} \\
&= (-\overline{1})^{p-2} \\
&= -1.
\end{aligned}$$

Hence $(p-1)! \equiv -1 \pmod{p}$ for each prime p. □

Ex. 4.13 *Let G be a finite cyclic group and $g \in G$ a generator. Show that all the other generators are of the form g^k, where $(k, n) = 1$, n being the order of G.*

Proof. Suppose $G = \langle g \rangle$, with $\operatorname{Card} G = n$, so the order of g is n.

Let x another generator of G, then $x = g^k$, and $g = x^l$, $k, l \in \mathbb{Z}$, so $g = g^{kl}, g^{kl-1} = e : n \mid kl - 1$, then $kl - 1 = qn, q \in \mathbb{Z}$, so $n \wedge k = 1$.

Reciprocally, if $u \wedge k = 1$, there exist $u, v \in \mathbb{Z}$ such that $un + vk = 1$, so $g = g^{un+vk} = (g^n)^u (g^k) v = x^v \in \langle x \rangle$, so $G \subset \langle x \rangle$, $G = \langle x \rangle$: x is a generator of G.

Conclusion : if g is a generator of G, all the other generators are the elements g^k, where $k \wedge n = 1$, $n = |G|$. □

Ex. 4.14 *Let A be a finite abelian group and $a, b \in A$ elements of order m and n, respectively. If $(m, n) = 1$, prove that ab has order mn.*

Proof. Suppose $|a| = m, |b| = n, m \wedge n = 1$.

• If $(ab)^k = e$, then $a^k = b^{-k}$, so $a^{kn} = b^{-kn} = (b^n)^{-k} = e$, so $m \mid kn$, with $m \wedge n = 1$, so $m \wedge k$.

Similarly, $b^{km} = a^{-km} = (a^m)^{-k} = e$, so $n \mid km, n \wedge m = 1 : n \mid k$.

As $n \mid k, m \mid k, n \wedge m = 1, nm \mid k$.

• Reciprocally, if $nm \mid k, nm = qnm, q \in \mathbb{Z}$, so $(ab)^k = a^k b^k = (a^m)^{qn} (b^n)^{qm} = e$.

$$\forall k \in \mathbb{Z},\ (ab)^k = e \iff nm \mid k.$$

So $|ab| = nm$. □

Ex. 4.15 *Let K be a field and $G \subset K^*$ a finite subgroup of the multiplicative group of K. Extend the arguments used in the proof of Theorem 4.1 to show that G is cyclic.*

Solution 1.

Proof. Let $n = |G|$. From Lagrange's theorem, $a^n = 1$ for all $a \in G$, so the polynomial $x^n - 1 \in K[x]$ has exactly n roots in G, and so

$$\forall x \in K, x \in G \iff x^n = 1.$$

If $d \mid n$, the polynomial $x^d - 1 \in K[x]$ has exactly d roots in K otherwise $x^n - 1 = (x^d - 1)g(x), g(x) \in K[x]$, and $\deg(g) = n - d$ has at most $n - d$ roots, so $x^n - 1$ would have less than n roots in K. As $x_0^d = 1 \Rightarrow x_0^n = 1$, all these roots are in G : $x^d - 1$ has d roots in G.

Let $\psi(d)$ the number of elements in G of order d ($\psi(d) = 0$ if $d \nmid n$). Then $\sum_{c|d} \psi(c) = d$. Applying the Möbius inversion theorem, $\psi(d) = \sum_{c|d} \mu(c)d/c = \Phi(d)$ (Prop. 2.2.5), in particular, $\psi(n) = \phi(n) > 1$ if $n > 2$. Since a group of order 2 is cyclic, we have shown in all cases the existence of an element of order n in G, so G is cyclic.

(variation : $\psi(d) = 0$ if there exists no element of order d, and $\psi(d) = \phi(d)$ otherwise : see Ex.4.13. So $\psi(d) \leq \phi(d)$ for all $d \mid n$. As $\sum_{d|n} \psi(d) = \sum_{d|n} \phi(d) = n$, $\psi(d) = \phi(d)$ for all $d \mid n$. So there exists in G an element of order n, and G is cyclic.) $\square$

Solution 2.

Proof. Let $n = |G| = p_1^{a_1} \cdots p_k^{a_k}$. From Lagrange's theorem, $y^n = 1$ for all $y \in G$.

$p(x) = x^{n/p_1} - 1 \in K[x]$ has at most $n/p_1 < n$ roots in K^*, a fortiori in G, so there exists $a \in G$ such that $a^{n/p_1} \neq 1$.

Let $c_1 = a^{n/p_1^{a_1}} = a^{p_2^{a_2} \cdots p_k^{a_k}}$. Then $c_1^{p_1^{a_1}} = 1$ and $c_1^{p_1^{a_1-1}} = a^{n/p_1} \neq 1$, so $|c_1| = p_1^{a_1}$.

Similarly, there exist $c_2, \ldots, c_k$ with respective orders $|c_i| = p_i^{a_i}$.

From exercise 4.14, we obtain by induction that $c = c_1 \cdots c_k$ has order $p_1^{a_1} \cdots p_k^{a_k} = n$, so G is cyclic. $\square$

Ex. 4.16 *Calculate the solutions to* $x^3 \equiv 1 \pmod{19}$ *and* $x^4 \equiv 1 \pmod{17}$.

Proof. Here we note a the class of a in $\mathbb{Z}/p\mathbb{Z}$.

Let $x \in \mathbb{F}_{19}$. $x^3 - 1 = 0 \iff x - 1 = 0$ or $x^2 + x + 1 = 0$.

$$\begin{aligned} x^2 + x + 1 = 0 &\iff (x + 10) - 99 = 0 \\ &\iff (x + 10)^2 - 4 = 0 \\ &\iff (x + 8)(x + 12) = 0 \end{aligned}$$

So, for all $x \in \mathbb{Z}$,

$$x^3 \equiv 1 \pmod{19} \iff x \equiv 1, 7, 11 \pmod{19}.$$

Let $x \in \mathbb{F}_{17}$.

$$\begin{aligned} x^4 = 1 &\iff x^2 = 1 \text{ or } x^2 = -1 = 4^2 \\ &\iff x = \pm 1 \text{ or } x = \pm 4 \end{aligned}$$

So, for all $x \in \mathbb{Z}$,

$$x^4 \equiv 1 \pmod{17} \iff x \equiv -1, 1, -4, 4 \pmod{17}.$$

Alternatively, we can take primitives roots modulo 19 and 17.

2 is a primitive root modulo 19, Let $x = 2^k \in \mathbb{F}_{19}$.

$$\begin{aligned} x^3 = 1 &\iff 2^{3k} = 1 \\ &\iff 18 \mid 3k \\ &\iff 6 \mid k \\ &\iff x = 1, 2^6 = 7, 2^{12} = 11 \end{aligned}$$

3 is a primitive root modulo 17. Let $x = 3^k \in \mathbb{F}_{17}$.

$$\begin{aligned} x^4 = 1 &\iff 3^{4k} = 1 \\ &\iff 16 \mid 4k \\ &\iff 4 \mid k \\ &\iff x = 1, 3^4 = -4, 3^8 = -1, 3^{12} = 4 \end{aligned}$$

□

Ex. 4.17 *Use the fact that* 2 *is a primitive root modulo* 29 *to find the seven solutions to* $x^7 \equiv 1 \pmod{29}$*.*

Proof. Let $x \in \mathbb{Z}$, then $x \equiv 2^k \pmod{29}, k \in \mathbb{N}$.

$$\begin{aligned} x^7 \equiv 1 \pmod{29} &\iff 2^{7k} \equiv 1 \pmod{29} \\ &\iff 28 \mid 7k \\ &\iff 4 \mid k \end{aligned}$$

So the group cyclic S of the roots of $x^7 - 1$ in $\mathbb{F}_{29}$ are

$$S = \{1, 2^4, 2^8, 2^{12}, 2^{16}, 2^{20}, 2^{24}\},$$

$$S = \{1, 16, 24, 7, 25, 23, 20\}.$$

□

Ex. 4.18 *Solve the congruence* $1 + x + \cdots + x^6 \equiv 0 \pmod{29}$*.*

Proof. As $(1 + x + \cdots + x^6)(1 - x) = 1 - x^7$,

$$1 + x + \cdots + x^6 \equiv 0 \pmod{29} \iff \left\{ \begin{array}{rcll} x^7 & \equiv & 1 & \pmod{29} \\ x & \not\equiv & 1 & \pmod{29} \end{array} \right.$$

From Ex. 4.17, the solutions are congruent to $2^4, 2^8, 2^{12}, 2^{16}, 2^{20}, 2^{24}$ modulo 29. □

Ex. 4.19 *Determine the numbers* a *such that* $x^3 \equiv a \pmod{p}$ *is solvable for* $p = 7, 11, 13$*.*

Proof. (a) If $p = 7$, then $3 \mid p - 1, d = 3 \wedge (p-1) = 3$. From Prop. 4.2.1,

$$\exists x \in \mathbb{Z},\ a \equiv x^3 \pmod{7} \iff a \equiv 0 \pmod{7} \text{ or } a^{(p-1)/3} = a^2 \equiv 1 \pmod{7}.$$

So the numbers a such that $x^3 \equiv a \pmod{7}$ is solvable are congruent at $0, 1, -1$ modulo 7.

(b) If $p = 11$, then $d = 3 \wedge (p-1) = 1$. With the same proposition,

$$\exists x \in \mathbb{Z},\ a \equiv x^3 \pmod{11} \iff a \equiv 0 \pmod{11} \text{ or } a^{p-1} = a^6 \equiv 1 \pmod{11}.$$

So all integers a are cube modulo 11, in only one way.

For an alternative proof, the application

$$f : \left\{ \begin{array}{ccc} \mathbb{F}_{11}^* & \to & \mathbb{F}_{11}^* \\ x & \mapsto & x^3 \end{array} \right.$$

f is a bijection. Indeed,

- f is a group homomorphism,
- $x^3 = 1 \Rightarrow (x^3)^7 = 1 \Rightarrow x = 1$ so $\ker(f) = \{1\}$,
- $f : \mathbb{F}_{11}^* \to \mathbb{F}_{11}^*$ is injective and $\mathbb{F}_{11}^*$ is finite, so f is bijective.

In $\mathbb{F}_{11}$, $0 = 0^3, 1 = 1^3, 2 = 7^3, 3 = 9^3, 4 = 5^3, 5 = 3^3, 6 = 8^3, 7 = 6^3, 8 = 2^3, 9 = 4^3, 10 = 10^3$.

(c) If $p = 13$, then $3 \mid p-1, 3 \wedge (p-1) = 3$, so

$$\begin{aligned} \exists x \in \mathbb{Z},\ a \equiv x^3 \pmod{13} &\iff a \equiv 0 \pmod{13} \text{ or } a^{(p-1)/3} = a^4 \equiv 1 \pmod{13} \\ &\iff a \equiv 0, 1, -1, 5, -5 \pmod{13} \end{aligned}$$

($5 \equiv 8^3 \pmod{13}$.)

□

Ex. 4.20 *Let p be a prime, and d a divisor of $p-1$. Show that dth powers form a subgroup of $U(\mathbb{Z}/p\mathbb{Z})$ of order $(p-1)/d$. Calculate this subgroup for $p = 11, d = 5$, for $p = 17, d = 4$, and for $p = 19, d = 6$.*

Proof. Here p is a prime number, and $d \mid p-1$. Let

$$f : \left\{ \begin{array}{ccc} \mathbb{F}_p^* & \to & \mathbb{F}_p^* \\ x & \to & x^d \end{array} \right.$$

Then f is a group homomorphism, and $\mathrm{im}(f)$ is the set of dth powers, and consequently is a subgroup of $U(\mathbb{F}_p) = \mathbb{F}_p^*$. $\ker(f)$ is the group of the roots of $x^d - 1$. As $d \mid p-1$, the polynomial $x^d - 1$ has exactly d roots (Prop. 4.1.2), so $|\ker(f)| = d$.

As $\mathrm{im}(f) \simeq \mathbb{F}_p^* / \ker(f)$,

$$|\mathrm{im}(f)| = |\mathbb{F}_p^*| / |\ker(f)| = (p-1)/d.$$

So there exist exactly $(p-1)/d$ dth powers in $(\mathbb{Z}/p\mathbb{Z})^*$.

From Prop. 4.2.1, as $d \mid p-1, d \wedge p-1$, for all $x \in \mathbb{F}_p^*$,

$$x \in \mathrm{im}(f) \iff x^{(p-1)/d} = 1.$$

So the group of dth powers is the group of the roots of $x^{(p-1)/d} - 1$.
- If $p = 11, d = 5$, $\mathrm{im}(f) = \{1, -1\}$.
- If $p = 17, d = 4$, $x \in \mathrm{im}(f) \iff x^4 = 1 : \mathrm{im}(f) = \{1, -1, 4, -4\}$.
- If $p = 19, d = 6$, $x \in \mathrm{im}(f) \iff x^3 = 1 : \mathrm{im}(f) = \{1, 7, 7^2 = 11\}$,

where $7 \equiv 2^6 \pmod{19}$.

□

Ex. 4.21 *If g is a primitive root modulo p, and $d|p-1$, show that $g^{(p-1)/d}$ has order d. Show also that a is a dth power iff $a \equiv g^{kd} \pmod p$ for some k. Do Exercises 16-20 making use of those observations.*

Proof. Let $x = \overline{g}^{(p-1)/d} \in \mathbb{F}_p^*$, where g is a primitive root modulo p. For all $k \in \mathbb{Z}$,

$$\begin{aligned} x^k = 1 &\iff g^{k\frac{p-1}{d}} = 1 \\ &\iff p-1 \mid k\frac{p-1}{d} \\ &\iff d \mid k \end{aligned}$$

So the ordre of $\overline{g}^{(p-1)/d}$ is d.

• If $\overline{a} = \overline{g}^{kd}$, then $\overline{a} = x^d$, where $x = \overline{g}^k$, so $\overline{a}$ is a dth power.

• If $\overline{a} \neq \overline{0}$ is a dth power, $\overline{a} = x^d, x \in \mathbb{F}_p^*$. As $x \in \langle \overline{g} \rangle$, $x = \overline{g}^k$, so $\overline{a} = \overline{g}^{kd}$.

So, if $a \not\equiv 0 \pmod p$, a is a dth power iff $a \equiv g^{kd} \pmod p$ for some k.

By example (Ex. 4.20), 2 is a primitive root modulo 19, so the 6th powers modulo 19 are $2^0 = 1, 2^6 = 7, 2^{12} = 11$. □

Ex. 4.22 *If a has order 3 modulo p, show that $1+a$ has order 6.*

Proof. If a has order 3 modulo p, then $0 \equiv a^3 - 1 = (a-1)(a^2+a+1) \pmod p$, with $a \not\equiv 1 \pmod p$, so $a^2 + a + 1 \equiv 0 \pmod p$. Thus

$$\begin{aligned} (1+a)^3 &\equiv 1 + 3a + 3a^2 + a^3 \\ &\equiv 1 + 3a + 3(-1-a) + 1 \\ &\equiv -1 \pmod p \end{aligned}$$

So $(1+a)^6 \equiv 1 \pmod p$.

$(1+a)^2 \equiv 1 + 2a + a^2 = 1 + 2a + (-1-a) \equiv a \not\equiv 1 \pmod p$.

So $(1+a)^6 \equiv 1, (1+a)^2 \not\equiv 1, (1+a)^3 \not\equiv 1 \pmod p$, so the order of $1+a$ divides 6, but doesn't divides 2 or 3, so $1+a$ has order 6 modulo p. □

Ex. 4.23 *Show that $x^2 \equiv -1 \pmod p$ has a solution iff $p \equiv 1 \pmod 4$, and that $x^4 \equiv -1 \pmod p$ has a solution iff $p \equiv 1 \pmod 8$.*

Proof. If $x^2 \equiv -1 \pmod p$, then $\overline{x}$ has order 4 in $\mathbb{F}_p^*$, hence from Lagrange's theorem, $4 \mid p-1$.

Reciprocally, suppose $4 \mid p-1$, so $p = 4k+1, k \in \mathbb{N}^*$. From proposition 4.2.1, as $2 \mid p-1$, -1 is a square modulo p iff $(-1)^{(p-1)/2} \equiv 1 \pmod p$, which is true because $(-1)^{(p-1)/2} = (-1)^{2k} = 1$.

If $x^4 \equiv -1 \pmod p$, then $\overline{x}^8 = 1 \in \mathbb{F}_p^*$, and $\overline{x}^4 \neq 1$, so x has order 8 in $\mathbb{F}_p^*$, so $8 \mid p-1$.

Reciprocally, if $p \equiv 1 \pmod 8$, $p = 8K+1, K \in \mathbb{N}^*$. From Prop.4.2.1, as $4 \mid p-1$, there exists $x \in \mathbb{Z}$ such that $-1 = x^4$ iff $(-1)^{(p-1)/4} \equiv 1 \pmod 8$, which is true because $(-1)^{(p-1)/4} = (-1)^{2K} = 1$.

Conclusion :

$$\exists x \in \mathbb{Z},\ x^4 \equiv -1 \pmod p \iff p \equiv 1 \pmod 8.$$

□

Ex. 4.24 *Show that* $ax^m + by^n \equiv c \pmod p$ *has the same number of solutions as* $ax^{m'} + by^{n'} \equiv c \pmod p$, *where* $m' = (m, p-1)$ *and* $n' = (n, p-1)$.

Proof. If $a \wedge b \nmid c$, the two equations have no solution. So we can suppose $a \wedge b \mid c$, and after division by $\delta = a \wedge b$, we obtain an equation $a'x^m + b'y^n = c'$, $a' = a/\delta, b' = b\delta, c' = c\delta$, and $a' \wedge b' = 1$. So it remains to prove that $ax^m + by^n \equiv c \pmod p$ has the same number of solutions as $ax^{m'} + by^{n'} \equiv c \pmod p$ when $a \wedge b = 1$.

In this case the equation $au + bv = c$ has solutions. Let N the number of solutions $(\overline{x}, \overline{y})$ of the equation $\overline{a}\,\overline{x}^m + \overline{b}\,\overline{y}^n = \overline{c}$,$N'$ the number of solutions $(\overline{x}, \overline{y})$ of the equation $\overline{a}\,\overline{x}^{m'} + \overline{b}\,\overline{y}^{n'} = \overline{c}$. Then

$$\begin{aligned} N &= \mathrm{Card}\{(\overline{x}, \overline{y}) \in \mathbb{F}_p \times \mathbb{F}_p \mid \overline{a}\,\overline{x}^m + \overline{b}\,\overline{y}^n = \overline{c}\} \\ &= \sum_{\overline{a}\overline{u} + \overline{b}\overline{v} = \overline{c}} \mathrm{Card}\{(\overline{x}, \overline{y}) \in \mathbb{F}_p \times \mathbb{F}_p \mid \overline{x}^m = \overline{u}, \overline{y}^n = \overline{v}\} \\ &= \sum_{\overline{a}\overline{u} + \overline{b}\overline{v} = \overline{c}} \mathrm{Card}\{\overline{x} \in \mathbb{F}_p \mid \overline{x}^m = \overline{u}\} \times \mathrm{Card}\{\overline{y} \in \mathbb{F}_p \mid \overline{y}^n = \overline{v}\}. \end{aligned}$$

The same is true for N', so it is suffisant to prove that

$$\mathrm{Card}\{\overline{x} \in \mathbb{F}_p \mid \overline{x}^m = \overline{u}\} = \mathrm{Card}\{\overline{x} \in \mathbb{F}_p \mid \overline{x}^{m'} = \overline{u}\},$$

where $m' = m \wedge (p-1)$, and a similar equality for the equation $\overline{y}^n = \overline{v}$.

Let $\overline{g}$ a generator of $\mathbb{F}_p^*$. Write $\overline{u} = \overline{g}^r, r \in \mathbb{N}$.

$$\begin{aligned} \exists \overline{x} \in \mathbb{F}_p,\ \overline{x}^m = \overline{u} &\iff \exists k \in \mathbb{Z},\ \overline{g}^{mk} = \overline{g}^r \\ &\iff \exists k \in \mathbb{Z},\ p-1 \mid mk - r \\ &\iff \exists k \in \mathbb{Z}, \exists l \in \mathbb{Z},\ r = mk + l(p-1) \\ &\iff m \wedge (p-1) \mid r \end{aligned}$$

So

$$\{\overline{x} \in \mathbb{F}_p \mid \overline{x}^m = \overline{u}\} \neq \emptyset \iff m \wedge (p-1) \mid r,$$

and similarly

$$\{\overline{x} \in \mathbb{F}_p \mid \overline{x}^{m'} = \overline{u}\} \neq \emptyset \iff m' \wedge (p-1) \mid r.$$

Since $m' \wedge (p-1) = (m \wedge (p-1)) \wedge (p-1) = m \wedge (p-1)$, these two conditions are equivalent, so these two sets are empty for the same values of $\overline{u}$.

Let $\overline{u}$ is such that $\{\overline{x} \in \mathbb{F}_p \mid \overline{x}^m = \overline{u}\} \neq \emptyset$, and x_0 a fixed solution of $\overline{x}^m = \overline{u}$.

Write $\overline{x} = \overline{g}^k, \overline{x_0} = g^{k_0}$. Let $d = m \wedge (p-1) (= m')$.

$$\begin{aligned} \overline{x}^m = u &\iff \overline{x}^m = \overline{x_0}^m \\ &\iff \overline{g}^{mk} = \overline{g}^{mk_0} \\ &\iff p-1 \mid m(k-k_0) \\ &\iff \frac{p-1}{d} \mid \frac{m}{d}(k-k_0) \\ &\iff \frac{p-1}{d} \mid k-k_0 \\ &\iff \exists j \in \mathbb{Z}, k = k_0 + j\frac{p-1}{d} \end{aligned}$$

As g is a primitive root modulo p, the distinct solutions are $x_0, x_0 g^{\frac{p-1}{d}}, \ldots, x_0 g^{k\frac{p-1}{d}}, \ldots x_0 g^{(d-1)\frac{p-1}{d}}$, so in this case

$$\mathrm{Card}\{\overline{x} \in \mathbb{F}_p \mid \overline{x}^m = \overline{u}\} = d = m \wedge (p-1).$$

As $m' \wedge (p-1) = m \wedge (p-1)$,

$$\mathrm{Card}\{\overline{x} \in \mathbb{F}_p \mid \overline{x}^m = \overline{u}\} = \mathrm{Card}\{\overline{x} \in \mathbb{F}_p \mid \overline{x}^{m'} = \overline{u}\}.$$

So $N = N' : ax^m + by^n \equiv c \pmod p$ has the same number of solutions as $ax^{m'} + by^{n'} \equiv c \pmod p$, where $m' = (m, p-1)$ and $n' = (n, p-1)$. $\square$

Ex. 4.25 *Prove Propositions 4.2.2 and 4.2.4.*

Proposition 4.2.2. *Suppose that a is odd, $e \geq 3$, and consider the congruence $x^n \equiv a \pmod{2^e}$. If n is odd, a solution always exists and it is unique.*

If n is even, a solution exists iff $a \equiv 1 \pmod 4, a^{2^{e-2}/d} \equiv 1 \pmod{2^e}$, where $d = (n, 2^{e-2})$. When a solution exists there are exactly $2d$ solutions.

Proof. We suppose that a is odd and $e \geq 3$.

From Theorem 2', we know that $\{(-1)^a 5^b \mid 0 \leq a \leq 1, 0 \leq b \leq 2^{e-2}\}$ constitutes a reduced residue system modulo 2^e, so we can write

$$\begin{aligned} a &\equiv (-1)^s 5^t \pmod{2^e}, 0 \leq s \leq 1, 0 \leq t \leq 2^{e-2}, \\ x &\equiv (-1)^y 5^z \pmod{2^e}, 0 \leq y \leq 1, 0 \leq z \leq 2^{e-2}. \end{aligned}$$

For all $x \in \mathbb{Z}$,

$$x^n \equiv a \pmod{2^e} \iff (-1)^{ny} 5^{nz} \equiv (-1)^s 5^t \pmod{2^e}$$

Then $(-1)^{ny} \equiv (-1)^s \pmod 4, ny \equiv s \pmod 2, (-1)^{ny} = (-1)^s$, so $5^{nz} \equiv 5^t \pmod{2^e}$.

Reciprocally, if $ny \equiv s \pmod 2$ and $5^{nz} \equiv 5^t \pmod{2^e}$, then $x^n \equiv a \pmod{2^e}$, so

$$x^n \equiv a \pmod{2^e} \iff \left\{ \begin{array}{lll} ny & \equiv & s \pmod 2 \\ 5^{nz} & \equiv & 5^t \pmod{2^e} \end{array} \right. \iff \left\{ \begin{array}{lll} ny & \equiv & s \pmod 2 \\ nz & \equiv & t \pmod{2^{e-2}} \end{array} \right.$$

since the order of 5 modulo 2^e is 2^{e-2}.

• Suppose that n is an odd integer. Then

$$\left\{ \begin{array}{lll} ny & \equiv & s \pmod 2 \\ nz & \equiv & t \pmod{2^{e-2}} \end{array} \right. \iff \left\{ \begin{array}{lll} y & \equiv & s \pmod 2 \\ z & \equiv & n't \pmod{2^{e-2}} \end{array} \right.$$

where n' is an inverse of n modulo $2^{e-2} : nn' \equiv 1 \pmod{2^{e-2}}$.

So $x^n \equiv a \pmod{2^e}$ has an unique solution modulo 2^e.

• Suppose that n is an even integer.

Then $\left\{ \begin{array}{lll} ny & \equiv & s \pmod 2 \\ nz & \equiv & t \pmod{2^{e-2}} \end{array} \right.$ implies $s \equiv 0 \pmod 2$ and $d = n \wedge 2^{e-2} \mid t$.

Then $a \equiv (-1)^s 5^t \equiv 5^t \pmod{2^e}$, so $a \equiv 1 \pmod 4$.

Hence $a^{\frac{2^{e-2}}{d}} \equiv \left(5^{2^{e-2}}\right)^{\frac{t}{d}} \equiv 1 \pmod{2^e}$, since 5 has order 2^{e-2}, and $d \mid t$.

So, if n is even, and $d = n \wedge 2^{e-2}$,

$$\exists x \in \mathbb{Z},\ x^n \equiv a \pmod{2^e} \Rightarrow \left\{ \begin{array}{lll} a & \equiv & 1 \pmod 4 \\ a^{\frac{2^{e-2}}{d}} & \equiv & 1 \pmod{2^e} \end{array} \right.$$

Reciprocally, suppose that $\left\{\begin{array}{lcl} a & \equiv & 1 \pmod 4 \\ a^{\frac{2^{e-2}}{d}} & \equiv & 1 \pmod{2^e} \end{array}\right.$. Then $a \equiv (-1)^s 5^t \pmod{2^e}$ implies $a \equiv (-1)^s \pmod 4$, so s is even, and $a \equiv 5^t \pmod{2^e}$.

Therefore $5^{t\frac{2^{e-2}}{d}} \equiv 1 \pmod{2^e}$, which implies $2^{e-2} \mid t\frac{2^{e-2}}{d}$, so $d \mid t$.

$$\begin{aligned}
\exists x \in \mathbb{Z},\ x^n \equiv a \pmod{2^e} &\iff \exists y \in \mathbb{Z},\ \exists z \in \mathbb{Z},\ \left\{\begin{array}{lcll} ny & \equiv & s & \pmod 2 \\ nz & \equiv & t & \pmod{2^{e-2}} \end{array}\right. \\
&\iff \exists z \in \mathbb{Z},\ nz \equiv t \pmod{2^{e-2}} \qquad \text{(since } n, s \text{ even)} \\
&\iff \exists z \in \mathbb{Z},\ 2^{e-2} \mid nz - t \\
&\iff \exists z \in \mathbb{Z},\ \frac{2^{e-2}}{d} \mid \frac{n}{d}z - \frac{t}{d} \\
&\iff \exists z \in \mathbb{Z},\ \exists q \in \mathbb{Z},\ q\frac{2^{e-2}}{d} + z\frac{n}{d} = \frac{t}{d}
\end{aligned}$$

As $\frac{2^{e-2}}{d} \wedge \frac{n}{d} = 1$, there exists a solution (q, z_0) of this last equation, where $0 \leq z_0 < \frac{2^{e-2}}{d}$, and so $x_0 = 5^{z_0}$ is a particular solution of $x^n \equiv a \pmod{2^e}$, therefore

$$\exists x \in \mathbb{Z},\ x^n \equiv a \pmod{2^e} \iff \left\{\begin{array}{lcll} a & \equiv & 1 & \pmod 4 \\ a^{\frac{2^{e-2}}{d}} & \equiv & 1 & \pmod{2^e} \end{array}\right.$$

If there exists a particular solution $x_0 \equiv (-1)^{y_0} 5^{z_0}$, then

$$\begin{aligned}
x^n \equiv a \pmod{2^e} &\iff x^n \equiv x_0^n \pmod{2^e} \\
&\iff \left\{\begin{array}{lcll} ny & \equiv & ny_0 & \pmod 2 \\ nz & \equiv & nz_0 & \pmod{2^{e-2}} \end{array}\right. \\
&\iff n(z - z_0) \equiv 0 \pmod{2^{e-2}} \qquad \text{(since } n \text{ even)} \\
&\iff \frac{2^{e-2}}{d} \mid \frac{n}{d}(z - z_0) \\
&\iff \frac{2^{e-2}}{d} \mid z - z_0, \qquad \left(\text{since } \frac{2^{e-2}}{d} \wedge \frac{n}{d} = 1\right) \\
&\iff \exists k \in \mathbb{Z},\ z = z_0 + k\frac{2^{e-2}}{d}
\end{aligned}$$

As the order of 5 modulo 2^e is 2^{e-2}, the solutions of $x^n \equiv a \pmod{2^e}$ are

$$x_k = (-1)^y 5^{z_0 + k\frac{2^{e-2}}{d}},\ 0 \leq y < 2,\ 0 \leq k < d,$$

so there are exactly $2d$ solutions modulo 2^e. $\square$

Proposition 4.2.4. *Let 2^l be the highest power of 2 dividing n. Suppose that a is odd and that $x^n \equiv a \pmod{2^{2l+1}}$ is solvable. Then $x^n \equiv a \pmod{2^e}$ is solvable for all $e \geq 2l+1$, and consequently for all $e \geq 1$). Moreover, all these congruences have the same number of solutions.*

Proof. We suppose that a is odd, and that $x^n \equiv a \pmod{2^{2l+1}}$ is solvable. l is such that $n = 2^l n'$, where n' is an odd integer.

Let the induction hypothesis be, for a fixed integer $m \geq 2l+1$,

$$\exists x_0 \in \mathbb{Z},\ x_0^n \equiv a \pmod{2^m}.$$

Let $x_1 = x_0 + b2^{m-l}$: we show that for an appropriate choice of $b \in \{0,1\}$, $x_1^n \equiv a \pmod{2^{m+1}}$.

$x_1^n = x_0^n + nb2^{m-l}x_0^{n-1} + 2^{2m-2l}A,\ A \in \mathbb{Z}$.

Since $m \geq 2l+1, 2m-2l \geq m+1$, so

$$x_1^n \equiv x_0^n + nb2^{m-l}x_0^{n-1} \pmod{2^{m+1}}.$$

$$\begin{aligned} x_1^n \equiv a \pmod{2^{m+1}} &\iff (x_0^n - a) + n'bx_0^{n-1}2^m \equiv 0 \pmod{2^{n+1}} \\ &\iff \frac{x_0^n - a}{2^m} + n'bx_0^{n-1} \equiv 0 \pmod 2 \end{aligned}$$

As a is odd, and $x_0^n \equiv a \pmod{2^m}, m \geq 1$, x_0 is odd, and n' is odd, so there exists an unique $b \in \{0,1\}$ such that $\frac{x_0^n-a}{2^m} + n'bx_0^{n-1} \equiv 0 \pmod 2$. So there exists $x_1 \in \mathbb{Z}$ such that $x_1^b \equiv a \pmod{2^{m+1}}$, and the induction is completed. Therefore, $x^n \equiv a \pmod{2^e}$ is solvable for all $e \geq 2l+1$, and consequently for all $e \geq 1$).

From the Proposition 4.2.2., with the hypothesis $e \geq 3$, we know that the number of solutions of the solvable equation $x^n \equiv a \pmod{2^e}, e \geq 2l+1$, is 1 if n is odd, $2(n \wedge 2^{e-2})$ if n is even.

If n is even, $l \geq 1$, $e \geq 2l+1 \geq 3$. Since $e \geq 2l+1$, and $n = 2^l n'$ for an odd n', $l \leq \frac{e-1}{2} \leq e-2$, so $n \wedge 2^{e-2} = n'2^l \wedge 2^{e-2} = 2^l$, and the number of solutions is 2^{l+1}, independent of $e \geq 2l+1$.

Conclusion : under the hypothesis $x^n \equiv a \pmod{2^{2l+1}}$, where $l = \mathrm{ord}_2(n)$, then $x^n \equiv a \pmod{2^e}$ is solvable for all $e \geq 1$, and all these congruences have the same number of solutions for $e \geq 2l+1, e \geq 3$. ☐

Chapter 5

Ex. 5.1 *Use Gauss' lemma to determine* $\left(\frac{5}{7}\right), \left(\frac{3}{11}\right), \left(\frac{6}{13}\right), \left(\frac{-1}{p}\right)$.

Proof. • $a = 5, p = 7$.

The array of values of the least residues modulo $p = 7$, for $1 \leq k \leq (p-1)/2$.

$k \mod 7$	1	2	3
$5k \mod 7$	-2	3	1

So the number of negative least residues is $\mu = 1$, and $\left(\frac{5}{7}\right) = (-1)^\mu = -1$.

• $a = 3, p = 11$.

$k \mod 11$	1	2	3	4	5
$3k \mod 11$	3	-5	-2	1	4

So $\mu = 2$, $\left(\frac{3}{11}\right) = (-1)^\mu = 1$.

• $a = 6, p = 13$.

$k \mod 13$	1	2	3	4	5	6
$6k \mod 13$	6	-1	5	-2	4	-3

So $\mu = 3$, $\left(\frac{6}{13}\right) = (-1)^\mu = -1$.

• If $a = -1$, and p an odd prime, the values of the least residues of $-k$ modulo p for $k = 1, 2, \ldots, (p-1)/2$ are $-k$, all negative. So the number of negative least residues is $\mu = (p-1)/2$, and $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$. ☐

Ex. 5.2 *Show that the number of solutions to* $x^2 \equiv a \pmod p$ *is equal to* $1 + (a/p)$.

Proof. Let N the number of solutions of $x^2 \equiv a \pmod p$.

• If $\left(\frac{a}{p}\right) = 0$, then $p \mid a$, $a \equiv 0 \pmod p$, so the unique solution of $x^2 \equiv a = 0$ is $x \equiv 0 \pmod p$, so $N = 1 = 1 + \left(\frac{a}{p}\right)$.

• If $\left(\frac{a}{p}\right) = -1$, then $N = 0 = 1 + \left(\frac{a}{p}\right)$.

• If $\left(\frac{a}{p}\right) = 1$, then $x^2 \equiv a \pmod p$ has a solution x_0, and $x^2 \equiv a \pmod p \iff x^2 \equiv x_0^2 \pmod p \equiv p \mid (x - x_0)(x + x_0) \equiv x \equiv \pm x_0 \pmod p$, so $N = 2 = 1 + \left(\frac{a}{p}\right)$. □

Ex. 5.3 *Suppose* $p \not| a$. *Show that the number of solutions to* $ax^2 + bx + c \equiv 0 \pmod p$ *is equal to* $1 + ((b^2 - 4ac)/p)$.

Proof. Here p is an odd prime number, and $p \nmid a$. Let N be the number of solutions of $ax^2 + bx + c \equiv 0 \pmod p$

For $\overline{x} \in \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$,

$$\begin{aligned}\overline{a}\overline{x}^2 + \overline{b}\overline{x} + \overline{c} &= \overline{a}\left(\overline{x}^2 + \frac{\overline{b}}{\overline{a}}\,\overline{x} + \frac{\overline{c}}{\overline{a}}\right)\\ &= \overline{a}\left(\left(\overline{x} + \frac{\overline{b}}{2\overline{a}}\right)^2 - \frac{\overline{b}^2 - 4\overline{a}\overline{c}}{4\overline{a}^2}\right)\end{aligned}$$

Let $\Delta = b^2 - 4ac$. Then N is the number of solutions of $\left(\overline{x} + \frac{\overline{b}}{2\overline{a}}\right)^2 - \frac{\overline{\Delta}}{4\overline{a}^2} = \overline{0}$ in $\mathbb{F}_p$. As in Ex.5.2, $N = 1$ if $\overline{\Delta} = \overline{0}$, $N = 0$ if $\overline{\Delta}$ is not a square in $\mathbb{F}_p^*$, otherwise $\overline{\Delta} = \delta^2, \delta \in \mathbb{F}_p^*$, and the solutions are $\overline{x} = (-\overline{b} \pm \overline{\delta})/2\overline{a}$, so $N = 2$. In the three cases, $N = 1 + \left(\frac{\Delta}{p}\right)$. □

Ex. 5.4 *Prove that* $\sum_{a=1}^{p-1}(a/p) = 0$.

Proof. Here p is an odd prime (the result is false if $p = 2$). In the interval $[1, p-1]$, there exist $(p-1)/2$ residues, and $(p-1)/2$ nonresidues (Prop. 5.1.2., Corollary 1), so $\sum_{a=1}^{p-1}(a/p) = 0$. □

Proof. As an alternative proof, let $S = \sum\limits_{a=1}^{p-1} \left(\frac{a}{p}\right)$, and b a nonresidue modulo p : $\left(\frac{b}{p}\right) = -1$ (such a b exists if $p \neq 2$). As $a \mapsto ab$ is a bijection from $\mathbb{F}_p^*$ to itself,

$$\left(\frac{b}{p}\right)S = \sum_{a=1}^{p-1}\left(\frac{ab}{p}\right) = \sum_{c=1}^{p-1}\left(\frac{c}{p}\right) = S,$$

so $-S = S, S = 0$. □

Ex. 5.5 *Prove that* $\sum_{x=1}^{p-1}((ax+b)/p) = 0$ *provided that* $p \nmid a$.

There is a mistake in the sentence : we must read

Prove that $\sum_{x=0}^{p-1}((ax+b)/p) = 0$ *provided that* $p \nmid a$.

By example,

$$\sum_{x=1}^{5-1}\left(\frac{x+1}{5}\right) = \left(\frac{2}{5}\right) + \left(\frac{3}{5}\right) + \left(\frac{4}{5}\right) = -1 \neq 0.$$

Proof. From exercise 5.3, as $\left(\frac{0}{p}\right) = 0$, we know that

$$\sum_{\overline{x}\in\mathbb{F}_p} \left(\frac{x}{p}\right) = \sum_{x=0}^{p-1} \left(\frac{x}{p}\right) = \sum_{x=1}^{p-1} \left(\frac{x}{p}\right) = 0.$$

(This sum is well defined, since $\left(\frac{x}{p}\right)$ depends only of $\overline{x}$: $x \equiv x' \pmod p \Rightarrow \left(\frac{x}{p}\right) = \left(\frac{x'}{p}\right)$.)
As $\overline{a} \neq \overline{0}$ in $\mathbb{F}_p$, $f : \left\{ \begin{array}{ccc} \mathbb{F}_p & \to & \mathbb{F}_p \\ x & \mapsto & \overline{a}x + \overline{b} \end{array} \right.$ is a bijection. Thus

$$\begin{aligned} \sum_{x=0}^{p-1} \left(\frac{ax+b}{p}\right) &= \sum_{x\in\mathbb{F}_p} \left(\frac{f(x)}{p}\right) \\ &= \sum_{y\in\mathbb{F}_p} \left(\frac{y}{p}\right) \qquad (y = f(x)) \\ &= 0 \end{aligned}$$

□

Ex. 5.6 *Show that the number of solutions to* $x^2 - y^2 \equiv a \pmod p$ *is given by:*

$$\sum_{y=0}^{p-1} \left(1 + \left(\frac{y^2+a}{p}\right)\right).$$

Proof. Let $S = \{(\overline{x}, \overline{y}) \in \mathbb{F}_p^2 \mid \overline{x}^2 - \overline{y}^2 = \overline{a}\}$. From Ex.5.2,

$$\begin{aligned} |S| &= \sum_{\overline{y}\in\mathbb{F}_p} \mathrm{Card}\ \{\overline{x} \in \mathbb{F}_p \mid \overline{x}^2 = \overline{y}^2 + \overline{a}\} \\ &= \sum_{y=0}^{p-1} \left(1 + \left(\frac{y^2+a}{p}\right)\right). \end{aligned}$$

□

Ex. 5.7 *By calculating directly show that the number of solutions to* $x^2 - y^2 \equiv a \pmod p$ *is* $p-1$ *if* $p \nmid a$*, and* $2p-1$ *if* $p \mid a$*. (Hint. Use the change of variables* $u = x+y, v = x-y$*.)*

Proof. Let $S = \{(\overline{x}, \overline{y}) \in \mathbb{F}_p^2 \mid \overline{x}^2 - \overline{y}^2 = \overline{a}\}$, and $T = \{(\overline{u}, \overline{v}) \in \mathbb{F}_p^2 \mid \overline{u}\,\overline{v} = \overline{a}\}$. Then $f : \left\{ \begin{array}{ccc} S & \to & T \\ (\overline{x}, \overline{y}) & \mapsto & (\overline{x}+\overline{y}, \overline{x}-\overline{y}) \end{array} \right.$ is well defined (if $(\overline{x}, \overline{y}) \in S, (\overline{x}-\overline{y})(\overline{x}+\overline{y}) = a$, so $(\overline{x}+\overline{y}, \overline{x}-\overline{y}) \in T$). Moreover f is a bijection, with inverse $(\overline{u}, \overline{v}) \mapsto ((\overline{u}+\overline{v})/2, (\overline{u}-\overline{v})/2)$, so $|S| = |T|$.

We compute $|T|$.

• Suppose $p \nmid a$, so $\overline{a} \neq \overline{0}$. For $\overline{v} \neq 0$, there is no solution, and for each $\overline{v} \neq 0$, we obtain the unique solution $(\overline{a}\,\overline{v}^{-1}, \overline{v})$, so there exist $p-1$ solutions.

• Suppose $p \mid a$. The solutions of $\overline{uv} = \overline{0}$ are $(\overline{0}, \overline{0})$, $(\overline{0}, \overline{v})$ for each $\overline{v} \neq \overline{0}$, $(\overline{u}, \overline{0})$ for each $\overline{v} \neq \overline{0}$, that is to say $N = 1 + (p-1) + (p-1) = 2p-1$ solutions.

Conclusion :

$$\begin{aligned}\text{Card } \{(\overline{x},\overline{y}) \in \mathbb{F}_p^2 \mid \overline{x}^2 - \overline{y}^2 = \overline{a}\} &= p-1 \quad \text{if } p \nmid a\\ &= 2p-1 \quad \text{if } p \mid a\end{aligned}$$

□

Ex. 5.8 *Combining the results of Ex. 5.6 and 5.7 show that:*

$$\sum_{y=0}^{p-1} \left(\frac{y^2+a}{p}\right) = \begin{cases} -1 & \text{if } p \nmid a\\ p-1 & \text{if } p \mid a\end{cases}$$

Proof. Let $S = \{(\overline{x},\overline{y}) \in \mathbb{F}_p^2 \mid \overline{x}^2 - \overline{y}^2 = \overline{a}\}$.

We obtain in Ex 5.6, $|S| = \sum_{y=0}^{p-1} \left(1 + \left(\frac{y^2+a}{p}\right)\right)$, and in Ex. 5.7. , $|S| = p-1$ if $p \nmid a$, $|S| = 2p-1$ if $p \mid a$.

So

$$S - p = \sum_{y=0}^{p-1} \left(\frac{y^2+a}{p}\right) = \begin{cases} -1 & \text{if } p \nmid a\\ p-1 & \text{if } p \mid a\end{cases}$$

□

Ex. 5.9 *Prove that* $1^2 3^2 \cdots (p-2)^2 \equiv (-1)^{(p+1)/2} \pmod p$ *using Wilson's theorem.*

Proof. Here p is an odd prime.

From Wilson's theorem, as $k(p-k) \equiv -k^2 \pmod p$ for $k = 1, 2, \ldots, p-1$,

$$\begin{aligned}-1 &\equiv (p-1)!\\ &\equiv \left[1 \times 2 \times \cdots \times k \times \cdots \times \left(\frac{p-1}{2}\right)\right] \times \left[\left(\frac{p+1}{2}\right) \times \cdots \times (p-k) \cdots \times (p-2) \times (p-1)\right]\\ &\equiv \prod_{k=1}^{(p-1)/2} k(p-k)\\ &\equiv (-1)^{(p-1)/2} \prod_{k=1}^{(p-1)/2} k^2\\ &\equiv (-1)^{(p-1)/2} \left[\left(\frac{p-1}{2}\right)!\right]^2 \pmod p\end{aligned}$$

So

$$\left[\left(\frac{p-1}{2}\right)!\right]^2 \equiv (-1)^{(p+1)/2} \pmod p.$$

Moreover, from Wilson' theorem and Fermat's little theorem,

$$\begin{aligned}1^2 2^2 3^2 \cdots (p-1)^2 &= [(p-1)!]^2 \equiv 1 \pmod p\\ 2^2 4^2 \cdots (p-1)^2 &= \left(2^{p-1}\right)^2 \left[\left(\frac{p-1}{2}\right)!\right]^2 \equiv \left[\left(\frac{p-1}{2}\right)!\right]^2 \pmod p\end{aligned}$$

Thus

$$1^2 3^2 \cdots (p-2)^2 \left[\left(\frac{p-1}{2}\right)!\right]^2 \equiv 1 \pmod{p}.$$

which gives

$$1^2 3^2 \cdots (p-2)^2 \equiv (-1)^{(p+1)/2} \pmod{p}.$$

$\square$

Ex. 5.10 *Let $r_1, r_2, \ldots, r_{(p-1)/2}$ be the quadratic residues between* 1 *and p. Show that their product is congruent to* $1 \pmod p$ *if* $p \equiv 3 \pmod 4$, *and to* -1 *if* $p \equiv 1 \pmod 4$.

Proof. We proved in Ex. 5.9 that

$$\left[\left(\frac{p-1}{2}\right)!\right]^2 \equiv (-1)^{(p+1)/2} \pmod{p}.$$

The application $f : \left\{ \begin{array}{ccc} \{\overline{1}, \overline{2}, \ldots, \overline{(p-1)/2}\} & \mapsto & \{\overline{r_1}, \overline{r_2}, \ldots, \overline{r_{(p-1)/2}}\} \\ x & \mapsto & x^2 \end{array} \right.$ is a bijection, so

$$\prod_{i=1}^{(p-1)/2} r_i \equiv \left[\left(\frac{p-1}{2}\right)!\right]^2 \pmod{p},$$

so

$$\prod_{i=1}^{(p-1)/2} r_i \equiv (-1)^{(p+1)/2} \pmod{p}.$$

That is to say, the product of the quadratic residues between 1 and p is congruent to 1 $\pmod p$ if $p \equiv 3 \pmod 4$, and to -1 if $p \equiv 1 \pmod 4$. $\square$

Ex. 5.11 *Suppose that* $p \equiv 3 \pmod 4$, *and that $q = 2p+1$ is also prime. Prove that $2^p - 1$ is not prime. (Hint : Use the quadratic character of 2 to show that $q \mid 2^p - 1$) One must assume that $p > 3$.*

Proof. The result is false if $p = 3$, so we must suppose $p > 3$.

$p = 4k+3$ for an integer k, so $q = 2p+1 = 8k+7 \equiv -1 \pmod 8$. Thus

$$\left(\frac{2}{q}\right) = (-1)^{(q^2-1)/8} = 1.$$

So $2^{(q-1)/2} \equiv 1 \pmod q$, $2^p \equiv 1 \pmod q$, so $q \mid 2^p - 1$.

Moreover, as $p > 3$, $q = 2p+1 < 2^p - 1$

$(2p+1 < 2^p - 1 \iff 2p < 2^p - 2 \iff p+1 < 2^{p-1}$.

$4+1 < 2^{4-1}$ and for all $k \geq 4$, $k+1 < 2^{k-1}$ implies $k+2 < 2^{k-1}+1 \leq 2^k$, and $4+1 < 2^{4-1}$, so by induction $k+1 < 2^{k-1}$ for all $k > 3$.

So $q \mid 2^p - 1$ with $1 < q < 2^p - 1$: $2^p - 1$ is composite.

Conclusion : if $p \equiv 3 \pmod 4, p > 3$ is prime, and $q = 2p+1$ is also prime, then $2^p - 1$ is not a prime.

For instance, le Mersenne's number $2^{11} - 1 = 2047$ is not a prime : $2047 = 23 \times 89$. $\square$

Ex. 5.12 *Let $f(x) \in \mathbb{Z}[x]$. We say that a prime p divides $f(x)$ if there's an integer n such that $p \mid f(n)$. Describe the prime divisors of x^2+1 and x^2-2.*

Proof. p divides x^2+1 iff there exists $a \in \mathbb{Z}$ such that $-1 \equiv a^2 \pmod p$, iff $p=2$ or $\left(\frac{-1}{p}\right) = 1$ iff $p=2$ or $p \equiv 1 \pmod 4$.

p divides x^2-2 iff there exists $a \in \mathbb{Z}$ such that $2 \equiv a^2 \pmod p$, iff $p=2$ or $\left(\frac{2}{p}\right)=1$ iff $p=2$ or $p \equiv \pm 1 \pmod 8$. □

Ex. 5.13 *Show that any prime divisor of x^4-x^2+1 is congruent to 1 modulo 12.*

Proof. • As $a^6+1 = (a^2+1)(a^4-a^2+1)$, $p \mid a^4-a^2+1$ implies $p \mid a^6+1$, so $\left(\frac{-1}{p}\right)=1$ and $p \equiv 1 \pmod 4$.

• $p \mid 4a^4-4a^2+4 = (2a-1)^2+3$, so $\left(\frac{-3}{p}\right)=1$.

As $-3 \equiv 1 \pmod 4$, $\left(\frac{-3}{p}\right) = \left(\frac{p}{3}\right)$, so $\left(\frac{p}{3}\right)=1$, thus $p \equiv 1 \pmod 3$.

$4 \mid p-1$ and $3 \mid p-1$, thus $12 \mid p-1$:

$$p \equiv 1 \pmod{12}.$$

□

Ex. 5.14 *Use the fact that $U(\mathbb{Z}/p\mathbb{Z})$ is cyclic to give a direct proof that $(-3/p)=1$ when $p \equiv 1 \pmod 3$.[Hint : There is a ρ in $U(\mathbb{Z}/p\mathbb{Z})$ of order 3. Show that $(2\rho+1)^2 = -3$.]*

Proof. Suppose that $p \equiv 1 \pmod 3$. Let g a generator of $\mathbb{F}_p^*$. Then g has order $p-1$, thus $\rho = g^{(p-1)/3}$ has order 3. As $\rho^3 = 1, \rho \neq 1$, then $\rho^2+\rho+1=0$.

$$\begin{aligned}(2\rho+1)^2 &= 4\rho^2+4\rho+1\\ &= 4(\rho^2+\rho+1)-3\\ &= -3.\end{aligned}$$

Thus $\left(\frac{-3}{p}\right)=1$. □

The inverse form of this proposition is also true for an odd prime p : if $\left(\frac{-3}{p}\right)=1$, then there exists $a \in \mathbb{F}_p^*$ such that $-\overline{3} = a^2$. $\rho = \frac{-1+a}{2}$ has order 3. Indeed $\rho^2 = \frac{1+a^2-2a}{4} = \frac{-2-2a}{4} = \frac{-1-a}{2}$, so

$$\begin{aligned}1+\rho+\rho^2 &= 1 + \frac{-1+a}{2} + \frac{-1-a}{2}\\ &= 0\end{aligned}$$

so $\rho \neq 1, \rho^3 = 1$. The group $\mathbb{F}_p^*$ contains an element of order 3, thus from Lagrange's theorem $3 \mid p-1$: $p \equiv 1 \pmod 3$.

Ex. 5.15 *If $p \equiv 1 \pmod 5$, show directly that $(5/p)=1$ by the method of Ex. 5.14. [Hint : Let ρ be an element of $U(\mathbb{Z}/p\mathbb{Z})$) of order 5. Show that $(\rho+\rho^4)^2+(\rho+\rho^4)-\overline{1}=\overline{0}$, etc.]*

Proof. Let g a generator of $\mathbb{F}_p^*$. g has order $p-1$, thus $\rho = g^{(p-1)/5}$ has order 5.

Let

$$\left\{ \begin{array}{lcl} \alpha & = & \rho + \rho^4 \\ \beta & = & \rho^2 + \rho^3 \end{array} \right.$$

As $0 = \rho^5 - 1 = (\rho - 1)(1 + \rho + \rho^2 + \rho^3 + \rho^4)$ and $\rho \neq 1$, then $1 + \rho + \rho^2 + \rho^3 + \rho^4 = 0$, thus

$$\begin{aligned} \alpha + \beta &= -1 \\ \alpha\beta &= \rho^3 + \rho^4 + \rho + \rho^2 = -1 \end{aligned}$$

So α, β are the roots in $\mathbb{F}_p$ of $x^2 + x - 1 : \alpha^2 + \alpha - 1 = 0$.

Thus $4\alpha^2 + 4\alpha - 4 = (2\alpha + 1)^2 - 5 = 0 : \overline{5}$ is a square in $\mathbb{F}_p^*$ and $\left(\frac{5}{p}\right) = 1$. $\square$

Ex. 5.16 *Using quadratic reciprocity find the primes for which* 7 *is quadratic residue. Do the same for* 15.

Proof. 7 is a quadratic residue for 2 and for the odd primes such that $\left(\frac{7}{p}\right) = 1$.

From the law of quadratic reciprocity,

$$\left(\frac{7}{p}\right) = 1 \iff (-1)^{(p-1)/2} \left(\frac{p}{7}\right) = 1$$

iff either $p \equiv 1 \pmod 4$ and $\left(\frac{p}{7}\right) = 1$, or $p \equiv -1 \pmod 4$ and $\left(\frac{p}{7}\right) = -1$.

In the first case , $p \equiv 1 \pmod 4, p \equiv 1, 4, 2 \pmod 7$, which gives $p \equiv 1, -3, 9 \pmod{28}$.

In the second case, $p \equiv -1 \pmod 4, p \equiv -1, -4, -2 \pmod 7$, which gives $p \equiv -1, 3, -9 \pmod{28}$.

Conclusion : the primes for which 7 is a quadratic residue are 2 and the odd primes p such that

$$\left(\frac{7}{p}\right) = 1 \iff p \equiv \pm 1, \pm 3, \pm 9 \pmod{28}.$$

$\square$

15 is a quadratic residue for 2 and for the odd primes such that $\left(\frac{15}{p}\right) = 1$.

$$\left(\frac{15}{p}\right) = 1 \iff \left(\frac{3}{p}\right) = \left(\frac{5}{p}\right) = 1 \text{ or } \left(\frac{3}{p}\right) = \left(\frac{5}{p}\right) = -1$$

From the examples of theorem 2, we know that

$$\left(\frac{3}{p}\right) = 1 \iff p \equiv 1, -1 \pmod{12}, \qquad \left(\frac{3}{p}\right) = -1 \iff p \equiv 5, -5 \pmod{12},$$

$$\left(\frac{5}{p}\right) = 1 \iff p \equiv 1, -1 \pmod 5, \qquad \left(\frac{5}{p}\right) = -1 \iff p \equiv 2, -2 \pmod 5.$$

As $5 \wedge 12 = 1$, there exist 8 cases, all possible, which give

$$\left(\frac{15}{p}\right) = 1 \iff p \equiv \pm 1, \pm 7, \pm 11, \pm 17 \pmod{60}.$$

For instance, the primes $2, 7, 11, 17, 43, 53, 59, 61, 67, 137, \ldots$ are suitable.

Ex. 5.17 *Supply the details to the proof of Proposition 5.2.1 and to the corollary to the lemma following it.*

Proposition 5.2.1

(a) $(a_1/b) = (a_2/b)$ *if* $a_1 \equiv a_2 \pmod b$.

(b) $(a_1a_2/b) = (a_1/b)(a_2/b)$.

(c) $(a/b_1b_2) = (a/b_1)(a/b_2)$.

Proof. (a) Let $b = p_1p_2\cdots p_m$, where the p_i are not necessarily distinct primes. For each prime p_i, $(a_1, p_i) = (a_2, p_i)$ (Prop. 5.1.2 (c)), so $\prod_i (a_1, p_i) = \prod_i (a_2, p_i)$, thus $(a_1/b) = (a_2/b)$.

(b) From Prop. 5.1.2(b),
$(a_1a_2/b) = \prod_i (a_1a_2/p_i) = \prod_i (a_1/p_i)(a_2/p_i) = \prod_i (a_1/p_i)\prod_i (a_2/p_i) = (a_1/b)(a_2/b)$.

(c) Let $b_1 = p_1p_2\cdots p_m, b_2 = q_1q_2\cdots q_l$. Then $b_1b_2 = p_1p_2\cdots p_m q_1q_2\cdots q_l = \prod_{i=1}^{m+l} r_i$, where $r_i = p_i$ for $i = 1,\ldots,m$, $r_i = q_{i-m}$ for $i = m+1,\ldots,m+l$. Then
$(a/b_1b_2) = \prod_{i=1}^{m+l}(a/r_i) = \prod_{i=1}^{m}(a/p_i)\prod_{j=1}^{l}(a/q_i) = (a/b_1)(a/b_2)$.

□

Lemma. *Let r and s be odd integers. Then*

(a) $(rs-1)/2 \equiv ((r-1)/2) + ((s-1)/2) \pmod 2$.

(b) $(r^2s^2-1)/8 \equiv ((r^2-1)/8) + ((s^2-1)/8) \pmod 2$.

(Proof in the book.)

Corollary. *Let $r_1, r_2, \ldots, r_m$ be odd integers. Then*

(a) $\sum_{i=1}^m (r_i-1)/2 \equiv (r_1r_2\cdots r_m - 1)/2 \pmod 2$.

(b) $\sum_{i=1}^m (r_i^2-1)/8 \equiv (r_1^2r_2^2\cdots r_m^2 - 1)/8 \pmod 2$.

Proof. Let $\mathcal{P}(m)$ the proposition defined by

$$\mathcal{P}(m) \iff \sum_{i=1}^m (r_i-1)/2 \equiv (r_1r_2\cdots r_m - 1)/2 \pmod 2.$$

Then $\mathcal{P}(1) \iff (r_1-1)/2 \equiv (r_1-1)/2 \pmod 2$ is true, and $\mathcal{P}(2)$ is part (a) of the lemma. If we make the induction hypothesis $\mathcal{P}(m)$, then

$$\begin{aligned}\sum_{i=1}^{m+1}(r_i-1)/2 &= \sum_{i=1}^{m}(r_i-1)/2 + (r_{m+1}-1)/2\\ &\equiv (r_1r_2\cdots r_m - 1)/2 + (r_{m+1}-1)/2 \pmod 2\\ &\equiv (r_1r_2\cdots r_m r_{m+1} - 1)/2 \pmod 2,\end{aligned}$$

where the last congruence is a consequence of the part (a) of the Lemma : the induction is completed, and $\mathcal{P}(m)$ is true for all $m \geq 1$.

The proof of part (b) is similar. □

Ex. 5.18 *Let D be a square-free integer that is also odd and positive. Show that there's an integer b prime to D such that $(b/D) = -1$.*

Proof. Let $D = p_1 p_2 \cdots p_k$, where the p_i are distinct odd primes.

Let s a nonresidue modulo p_k. From Chinese remainder theorem, as $p_i \wedge p_j = 1$ if $i \neq j$, there exists an integer b such that

$$b \equiv 1 \pmod{p_1}, b \equiv 1 \pmod{p_2}, \cdots, b \equiv 1 \pmod{p_{k-1}}, b \equiv s \pmod{p_k}.$$

Then $(b/p_i) = 1,\ i = 1, 2, \ldots k-1$, $(b/p_k) = -1$, so $b \wedge p_i = 1$ for all $i = 1, 2, \ldots, k$. Then $b \wedge D = b \wedge p_1 \cdots p_k = 1$, and

$$\left(\frac{b}{D}\right) = \prod_{i=1}^{k} \left(\frac{b}{p_i}\right) = \left(\frac{b}{p_k}\right) = -1.$$

□

Ex. 5.19 *Let D be as in Exercise 18. Show that $\sum (a/D) = 0$, where the sum is over a reduced residue system modulo D. Conclude that exactly one half of the elements in $U(\mathbb{Z}/D\mathbb{Z})$ satisfy $(a/D) = 1$.*

Proof. Let b such that $(b/D) = 1$: the existence of b comes from Ex 5.18.

Let $S = \sum_{a \in A} (a/D)$, where A is reduced residue system modulo D. As two reduced system modulo D represent the same elements in $U(\mathbb{Z}/D\mathbb{Z})$, the sum is independent of the reduced residue system A : we can write

$$S = \sum_{\overline{a} \in U(\mathbb{Z}/D\mathbb{Z})} (a/D).$$

As $b \wedge D = 1$, we know from Ex. 3.6 that $B = bA = \{ba \mid a \in A\}$ is also a reduced system modulo D. In other words, the application $U(Z/DZ) \to U(\mathbb{Z}/D\mathbb{Z}), \overline{a} \mapsto \overline{a}\overline{b}$ is a bijection, so

$$\left(\frac{b}{D}\right) S = \sum_{\overline{a} \in U(\mathbb{Z}/D\mathbb{Z})} \left(\frac{b}{D}\right)\left(\frac{a}{D}\right) = \sum_{\overline{a} \in U(\mathbb{Z}/D\mathbb{Z})} \left(\frac{ba}{D}\right) = \sum_{\overline{c} \in U(\mathbb{Z}/D\mathbb{Z})} \left(\frac{c}{D}\right) = S \qquad (\overline{c} = \overline{a}\overline{b}).$$

As $(b/D) = -1$, $-S = S$, so $S = 0$.

Since $(a/D) = \pm 1$, one half of the elements in $U(\mathbb{Z}/D\mathbb{Z})$ satisfy $(a/D) = 1$, and one half of the elements in $U(\mathbb{Z}/D\mathbb{Z})$ satisfy $(a/D) = -1$. □

Ex. 5.20 *(continuation) Let $a_1, a_2, \ldots, a_{\phi(D)/2}$ be integers between 1 and D such that $(a_i, D) = 1$ and $(a_i/D) = 1$. Prove that D is a quadratic residue modulo a prime $p \not| D$, $p \equiv 1 \pmod 4$ iff $p \equiv a_i \pmod D$ for some i.*

Proof. From Ex. 5.19 we know that there exist exactly $\phi(D)/2$ integers a_i between 1 and D such that $a_i \wedge D = 1$ and $(a_i/D) = 1$. So $\{\overline{a_1}, \ldots, \overline{a_{\phi(D)/2}}\}$ is the set of all $\overline{a} \in U(\mathbb{Z}/D\mathbb{Z})$ such that $(a/D) = 1$.

Let $D = p_1 p_2 \cdots p_k$, with distinct p_i, and p a prime number, $p \equiv 1 \pmod 4, p \notin \{p_1, \ldots, p_k\}$ (so $p = 4k+1, k \in \mathbb{N}$).

($\Leftarrow$) Suppose that $p \equiv a_i$ for some $i,\ 1 \le i \le \phi(D)/2$, then $(p/D) = (a_i/D) = 1$, so (Prop. 5.2.2)

$$\left(\frac{D}{p}\right) = (-1)^{\frac{p-1}{2}\frac{D-1}{2}} \left(\frac{p}{D}\right) = (-1)^{2k\left(\frac{D-1}{2}\right)} \left(\frac{p}{D}\right) = \left(\frac{p}{D}\right) = 1.$$

($\Rightarrow$) Suppose that D is a quadratic residue modulo p. Then $(D/p) = 1$, so

$$\left(\frac{p}{D}\right) = (-1)^{\frac{p-1}{2}\frac{D-1}{2}} \left(\frac{D}{p}\right) = 1.$$

Thus $\overline{p} \in \{\overline{a_1}, \ldots, \overline{a_{\phi(D)/2}}\}$ since $\{\overline{a_1}, \ldots, \overline{a_{\phi(D)/2}}\}$ is the set of all $\overline{a} \in U(\mathbb{Z}/D\mathbb{Z})$ such that $(a/D) = 1$. Consequently $p \equiv a_i \pmod{D}$ for some i. $\square$

Ex. 5.21 *Apply the method of Ex. 5.19 and 5.20 to find those primes for which* 21 *is a quadratic residue.*

Proof. Let $D = 21 = 3 \times 7$ (D is positive, odd and square-free). We first search the $\phi(D)/2 = 6$ integers $a,\ 1 \le a \le 21$, such that $(a/D) = 1$.

$$\left(\frac{a}{21}\right) = 1 \iff \left(\frac{a}{3}\right) = \left(\frac{a}{7}\right) = 1 \text{ or } \left(\frac{a}{3}\right) = \left(\frac{a}{7}\right) = -1.$$

The first case is equivalent to $a \equiv 1 \pmod 3, a \equiv 1, 2, 4 \pmod 7$, that is $a \equiv 1, 16, 4 \pmod{21}$.

The second case gives $a \equiv -1 \pmod 3, a \equiv -1, -2, -4 \pmod 7$, that is $a \equiv -1, -16, -4 \pmod{21}$, or equivalently $a \equiv 20, 5, 17 \pmod{21}$.

So $A = \{1, 4, 5, 16, 17, 20\}$ is the set of the integers a such that $1 \le a \le 21$, $(a/D) = 1$.

As $(21/3) = (21/7) = 0$, 21 is not a quadratic residue modulo 3 or 7.

• $p \equiv 1 \pmod 4$.

From Ex.5.20, we know that $D = 21$ is a quadratic residue modulo an odd prime p, $p \ne 3, p \ne 7$, $p \equiv 1 \pmod 4$, iff $p \equiv a \pmod D$ for some $a \in A$.

• $p \equiv -1 \pmod 4$.

As $D = 21 \equiv 1 \pmod 4$, $\left(\frac{D}{p}\right)\left(\frac{p}{D}\right) = (-1)^{\frac{p-1}{2}\frac{D-1}{2}} = 1$, so the same reasoning as in Ex. 5.20 show that D is a quadratic residue modulo 21 iff $p \equiv a, a \in A$.

Conclusion : 21 is a quadratic residue for 2, and for the primes p such that

$$p \equiv 1, 4, 5, 16, 17, 20 \pmod{21}.$$

$\square$

Ex. 5.22 *Use the Jacobi symbol to determine* $(113/997)$*,* $(215/761)$*,* $(514/1093)$*, and* $(401/757)$*.*

Proof. $(113/997) = (997/113) = (93/11) = (113/93) = (20/93) = (2^2/93)(5/93) = (5/93) = (93/5) = (3/5) = (5/3) = (2/3) = -1$.

$(215/761) = (761/215) = (116/215) = (2^2/215)(29/215) = (29/215) = (215/29) = (12/29) = (2^2/29)(3/29) = (3/29) = (29/3) = (2/3) = -1$.

$(514/1093) = (2/1093)(257/1093) = -(257/1093) = -(1093/57) = -(65/257) = -(257/65) = -(62/65) = -(2/65)(31/65) = -(31/65) = -(65/31) = -(3/31) = (31/3) = (1/3) = 1$.

$(401/757) = (757/401) = (356/401) = (401/89) = (45/89) = (89/45) = (44/45) = (2^2/45)(11/45) = (11/45) = (45/11) = (1/11) = 1$. $\square$

Ex. 5.23 *Suppose that $p \equiv 1 \pmod 4$. Show that there exist integers s and t such that $pt = 1 + s^2$. Conclude that p is not a prime in $\mathbb{Z}[i]$. Remember that $\mathbb{Z}[i]$ has unique factorization.*

Proof. As $p \equiv 1 \pmod 4$, then $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = 1 :\ -1$ is a square modulo p.

So $-1 \equiv s^2 \pmod p, s \in \mathbb{Z}$: there exist $s \in \mathbb{Z}, t \in \mathbb{Z}$ such that $pt = 1 + s^2$.

In $\mathbb{Z}[i]$, $p | (s+i)(s-i)$.

If p was a prime in $\mathbb{Z}[i]$, then $p \mid s+i$ ou $p \mid s-i$.

This implies $s \pm i = (a+bi)p, (a,b) \in \mathbb{Z}^2$, thus $\pm 1 = bp, p \mid 1 :$ it's impossible.

Conclusion : if $p \equiv 1 \pmod 4$, p is not a prime in $\mathbb{Z}[i]$. □

Ex. 5.24 *If $p \equiv 1 \pmod 4$, show that p is a sum of two squares, i.e. $p = a^2 + b^2$ with $a, b \in \mathbb{Z}$.(Hint : $p = \alpha\beta$, with α and β being non units in $\mathbb{Z}[i]$. Remember that $\mathbb{Z}[i]$ has unique factorisation.)*

Proof. $\mathbb{Z}[i]$ is a principal ideal domain, thus p prime is in $\mathbb{Z}[i]$ iff p is irreducible in $\mathbb{Z}[i]$.

If $p \equiv 1 \pmod 4$, p is not a prime from Ex.5.23, so it is not irreducible :

$p = \alpha\beta,\ \alpha, \beta \in \mathbb{Z}[i], N(\alpha) > 1, N(\beta) > 1$ (where $N(a+bi) = a^2 + b^2$ is the complex norm).

$N(p) = p^2 = N(u)N(v), 1 < N(u) < p^2$

Thus $N(u) = p$, that is $p = a^2 + b^2$, where $u = a + bi$.

Conclusion : if p is prime in $\mathbb{N}$, $p \equiv 1 \pmod 4$, then $p = a^2 + b^2,\ a, b \in \mathbb{Z}$, p is a sum of two squares.

□

Ex. 5.25 *An integer is called a biquadratic residue modulo p if it is congruent to a fourth power. Using the identity $x^4 + 4 = ((x+1)^2 + 1)((x-1)^2 + 1)$ show that -4 is a biquadratic residue modulo p iff $p \equiv 1 \pmod 4$.*

Proof. $x^4 + 4 = (x^4 + 4x^2 + 4) - 4x^2 = (x^2 + 2)^2 - 4x^2 = (x^2 + 2 - 2x)(x^2 + 2 + 2x)$, so

$$x^4 + 4 = ((x-1)^2 + 1)((x+1)^2 + 1).$$

If $-4 \equiv x^4\ [p]$, then $p \mid (x+1)^2 + 1$ or $p \mid (x-1)^2 + 1$

In the two cases, -1 is a quadratic residue modulo p, thus $\left(\frac{-1}{p}\right) = 1 : p \equiv 1\ [4]$.

Reciprocally, if $p \equiv 1\ [4]$, $\left(\frac{-1}{p}\right) = 1$, then it exists an integer a such that $-1 \equiv a^2\ [p]$.

Let $x = a - 1$. Then $p \mid (x+1)^2 + 1$, thus $p \mid x^4 + 4 :\ -4$ is a biquadratic residue modulo p.

Conclusion :

$$\exists x \in \mathbb{Z},\ x^4 \equiv -4\ [p] \iff p \equiv 1\ [4].$$

□

Ex. 5.26 *This exercise and Ex. 5.27 and 5.28 give Dirichlet's beautiful proof that 2 is a biquadratic residue modulo p iff p can be written in the form $A^2 + 64B^2$, where $A, B \in \mathbb{Z}$. Suppose that $p \equiv 1 \pmod 4$. Then $p = a^2 + b^2$ by Ex. 5.24. Take a to be odd. Prove the following statements:*

(a) $(a/p) = 1$.

(b) $((a+b)/p) = (-1)^{((a+b)^2 - 1)/8}$.

(c) $(a+b)^2 \equiv 2ab \pmod p$

(d) $(a+b)^{(p-1)/2} \equiv (2ab)^{(p-1)/4} \pmod p$.

Proof. Let p a prime number, $p \equiv 1\ [4] : p = 4k+1, k \in \mathbb{N}^*$.

Then $p = a^2 + b^2$ (Ex. 5.24).

As a, b are not of the same parity, up to exchange a and b, we will suppose that a is odd (then b is even).

(a)

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} = a^{2k}\ [p].$$

Using the law of quadratic reciprocity for Jacobi's symbol (Proposition 5.2.2), where a, p are odd numbers :

$$\left(\frac{a}{p}\right) = \left(\frac{p}{a}\right)(-1)^{\frac{p-1}{2}\frac{a-1}{2}} = \left(\frac{p}{a}\right),$$

since $p \equiv 1\ [4]$.

If $a = p_1 p_2 \cdots p_l$ is the decomposition of a in prime factors, with not necessarily distinct primes , then

$$\left(\frac{p}{a}\right) = \left(\frac{p}{p_1}\right)\left(\frac{p}{p_2}\right)\cdots\left(\frac{p}{p_l}\right).$$

Since $p = a^2 + b^2$, $p \equiv b^2\ [p_i]$, thus $\left(\frac{p}{p_i}\right) = 1$ for all i.

$$\left(\frac{a}{p}\right) = \left(\frac{p}{a}\right) = 1.$$

(b) $a+b$ is odd, and $p \equiv 1\ [4]$, thus

$$\left(\frac{a+b}{p}\right) = \left(\frac{p}{a+b}\right) = \left(\frac{2^2 p}{a+b}\right) = \left(\frac{2}{a+b}\right)\left(\frac{2p}{a+b}\right).$$

If $a+b = q_1 q_2 \cdots q_l$, as $2p = (a+b)^2 + (a-b)^2$, $2p \equiv (a-b)^2\ [q_i]$, thus $\left(\frac{2p}{q_i}\right) = 1$.

$$\left(\frac{2p}{a+b}\right) = \left(\frac{2p}{q_1}\right)\cdots\left(\frac{2p}{q_l}\right) = 1.$$

Moreover $\left(\frac{2}{a+b}\right) = (-1)^{\frac{(a+b)^2-1}{8}}$, so

$$\left(\frac{a+b}{p}\right) = (-1)^{\frac{(a+b)^2-1}{8}}.$$

(c) $(a+b)^2 = a^2 + b^2 + 2ab = p + 2ab \equiv 2ab\ [p]$

(d)$[(a+b)^2]^{\frac{p-1}{4}} \equiv (2ab)^{\frac{p-1}{4}}\ [p]$, thus

$$(a+b)^{\frac{p-1}{2}} \equiv (2ab)^{\frac{p-1}{4}}\ [p].$$

$\square$

Ex. 5.27 *Suppose that f is such that $b \equiv af \pmod p$. Show that $f^2 \equiv -1 \pmod p$, and that $2^{(p-1)/4} \equiv f^{ab/2} \pmod p$.*

Proof. Let f such as $b \equiv af$ $[p]$.
This is equivalent to $\overline{f} = \overline{b}\overline{a}^{-1}$ dans $\mathbb{F}_p^*$.
As $\overline{a}^2 = -\overline{b}^2$, $\overline{f}^2 = -\overline{1}$: $f^2 \equiv -1$ $[p]$.
We deduce from Ex. 5.26 (d) and (b) that

$$\begin{aligned}(2ab)^{\frac{p-1}{4}} \equiv (a+b)^{\frac{p-1}{2}} &= \left(\frac{a+b}{p}\right)\\ &\equiv (-1)^{\frac{(a+b)^2-1}{8}}\\ &\equiv (f^2)^{\frac{(a+b)^2-1}{8}}\\ &\equiv f^{\frac{(a+b)^2-1}{4}} = f^{\frac{a^2+b^2-1+2ab}{4}}\\ &\equiv f^{\frac{p-1}{4}} f^{\frac{ab}{2}} \pmod p\end{aligned}$$

Since $a^{\frac{p-1}{2}} = \left(\frac{a}{p}\right) = 1$ from Ex. 5.26(a)), then

$$(ab)^{\frac{p-1}{4}} \equiv (a^2 f)^{\frac{p-1}{4}} \equiv a^{\frac{p-1}{2}} f^{\frac{p-1}{4}} \equiv f^{\frac{p-1}{4}}\ [p],$$

so

$$2^{\frac{p-1}{4}} f^{\frac{p-1}{4}} \equiv f^{\frac{ab}{2}} f^{\frac{p-1}{4}}\ [p].$$

As $f^{\frac{p-1}{4}} \not\equiv 0$ $[p]$,

$$2^{\frac{p-1}{4}} \equiv f^{\frac{ab}{2}}\ [p].$$

$\square$

Ex. 5.28 *Show that $x^4 \equiv 2 \pmod p$ has a solution for $p \equiv 1 \pmod 4$ iff p is of the form $A^2 + 64B^2$.*

Proof. If $p \equiv 1$ $[4]$ and if there exists $x \in \mathbb{Z}$ such that $x^4 \equiv 2$ $[p]$, then

$$2^{\frac{p-1}{4}} \equiv x^{p-1} \equiv 1\ [p].$$

From Ex. 5.27, where $p = a^2 + b^2, a$ odd, we know that

$$f^{\frac{ab}{2}} \equiv 2^{\frac{p-1}{4}} \equiv 1\ [p].$$

Since $f^2 \equiv -1$ $[p]$, the order of f modulo p is 4, thus $4 \mid \frac{ab}{2}$, so $8 \mid ab$.
As a is odd, $8|b$, then $p = A^2 + 64B^2$ (with $A = a, B = b/8$).

Reciprocaly, if $p = A^2 + 64B^2$, then $p \equiv A^2 \equiv 1$ $[4]$.
Let $a = A, b = 8B$. Then

$$2^{\frac{p-1}{4}} \equiv f^{\frac{ab}{2}} \equiv f^{4AB} \equiv (-1)^{2AB} \equiv 1\ [p].$$

As $2^{\frac{p-1}{4}} \equiv 1$ $[p]$, $x^4 \equiv 2$ $[p]$ has a solution in $\mathbb{Z}$ (Prop. 4.2.1) : 2 is a biquadratic residue modulo p.
Conclusion :

$$\exists A \in \mathbb{Z}, \exists B \in \mathbb{Z}, p = A^2 + 64B^2 \iff (p \equiv 1\ [4] \text{ and } \exists x \in \mathbb{Z},\ x^4 \equiv 2\ [p]).$$

Remark : the equation $x^4 \equiv 2\ [p]$ has also solutions if $p \equiv -1\ [8]$.

Indeed, the equation $x^4 \equiv 2\ [p]$ has a solution in $\mathbb{Z}$ iff $2^{\frac{p-1}{d}} = 1$, where $d = 4 \wedge (p-1) = 2$, thus iff $2^{\frac{p-1}{2}} \equiv 1\ [p]$, which is true as $\left(\frac{2}{p}\right) = 1$.

For instance, $8^4 \equiv 2 \pmod{23}$, with $23 \equiv -1 \pmod 8$. □

Ex. 5.29 *Let (RR) be the number of pairs $(n, n+1)$ in the set $1, 2, 3, \ldots, p-1$ such that n and $n+1$ are both quadratic residues modulo p. Let (NR) be the number of pairs $(n, n+1)$ in the set $1, 2, 3, \ldots, p-1$ such that n is a quadratic nonresidue and $n+1$ is a quadratic residue. Similarly, define (RN) and (NN). Determine the sums $(RR)+(RN)$, $(NR)+(NN)$, $(RR)+(NR)$, and $(RN)+(NN)$.*

Proof. Let $E = [\![1, p-2]\!]$. Then $|E| = p-2$.

Write RR the set of integers $n \in E$ such that n and $n+1$ are both a quadratic residues, and $(RR) = |RR|$ its cardinality, and similar definitions for RN, NR, NN.

As $E = RR \cup RN \cup NR \cup NN$ (disjoint union),

$$(RR) + (RN) + (NR) + (NN) = |E| = p - 2.$$

• The union $RR \cup RN$ is the set of $n \in E$ such that n is a quadratic residue. Its cardinality is the number of quadratic residues in $[\![1, p-2]\!]$, that is the number of quadratic residues in $[\![1, p-1]\!]$, minus s, where $s = 1$ if $p - 1 \equiv -1$ is a residue, $s = 0$ otherwise. Since $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$, we obtain $s = \frac{1+(-1)^{\frac{p-1}{2}}}{2}$, and the total number of quadratic residues is $(p-1)/2$, thus

$$(RR) + (RN) = \frac{p-1}{2} - s = \frac{p-1}{2} - \frac{1 + (-1)^{\frac{p-1}{2}}}{2} = \frac{1}{2}(p - 2 - (-1)^{\frac{p-1}{2}}).$$

• Similarly, $(NR)+(NN)$ is the number of quadratic nonresidues in $[\![1, p-1]\!]$, minus t, where $t = 1$ if $p-1$ is a quadratic nonresidue, $t = 0$ otherwise : $t = \frac{1-(-1)^{\frac{p-1}{2}}}{2}$, so

$$(NR) + (NN) = \frac{1}{2}(p - 2 + (-1)^{\frac{p-1}{2}})$$

(the sum of these two results is indeed $p - 2 = |E|$).

• Since 1 is a residue, $(RR) + (NR)$ is the number of residues in $[\![1, p-1]\!]$, minus 1 :

$$(RR) + (NR) = \frac{p-1}{2} - 1.$$

• $(RN) + (NN)$ is the number of nonresidues in $[2, p-1]$, equal to the number of residues in $[\![1, p-1]\!]$:

$$(RN) + (NN) = \frac{p-1}{2}.$$

□

Ex. 5.30 *Show that* $(RR)+(NN)-(RN)-(NR)=\sum_{n=1}^{p-1}(n(n+1)/p)$. *Evaluate this sum and show that it is equal to* -1. *(Hint : The result of Exercise 8 is useful.)*

Proof. Let χ be the characteristic function of $RR\cup NN$: if $1\le n\le p-1$, $\chi(n)=1$ if $n, n+1$ are both quadratic residues, or if $n, n+1$ are both quadratic nonresidues. Then

$$\chi(n)=\frac{1}{2}\left(1+\left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right)\right)$$

(if $\chi(n)=1$, $\left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right)=1$, and $\left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right)=-1$ otherwise.)

Similarly, let χ' be the characteristic function of the complement $RN\cup NR$: $\chi(n)=1$ if exactly one of the integer $n, n+1$ is a residue, 0 otherwise. Then $\chi'(n)=1-\chi(n)$, so that

$$\chi'(n)=\frac{1}{2}\left(1-\left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right)\right).$$

Since

$$|(RR)\cup(NN)|=\sum_{n=1}^{p-1}\chi(n)$$
$$|(RN)\cup(NR)|=\sum_{n=1}^{p-1}\chi'(n),$$

we obtain

$$\begin{aligned}(RR)+(NN)-(RN)-(NR)&=\sum_{n=1}^{p-1}(\chi(n)-\chi'(n))\\&=\frac{1}{2}\sum_{n=1}^{p-1}\left(1+\left(\frac{n(n+1)}{p}\right)\right)-\left(1-\left(\frac{n(n+1)}{p}\right)\right)\\&=\sum_{n=1}^{p-1}\left(\frac{n(n+1)}{p}\right)\end{aligned}$$

To evaluate this sum S, note that $4n(n+1)=(2n+1)^2-1$, so

$$S=\sum_{n=1}^{p-1}\left(\frac{n(n+1)}{p}\right)=\sum_{n=1}^{p-1}\left(\frac{4n(n+1)}{p}\right)=\sum_{n=1}^{p-1}\left(\frac{(2n+1)^2-1}{p}\right).$$

This sum can be written $S=\sum_{\overline{n}\in\mathbb{F}_p^*}((2n+1)^2-1)/p)=\sum_{\overline{n}\in\mathbb{F}_p}((2n+1)^2-1)/p)$, since $(0/p)=0$. As $f:\mathbb{F}_p\to\mathbb{F}_p, \overline{n}\mapsto(2\overline{n}+1)$ is a bijection (2 is invertible in $\mathbb{F}_p^*$),

$$\sum_{\overline{n}\in\mathbb{F}_p}\left(\frac{(2n+1)^2-1}{p}\right)=\sum_{\overline{y}\in\mathbb{F}_p}\left(\frac{y^2-1}{p}\right)\qquad(y=2n+1).$$

As $p\nmid 1$, the evaluation of this last sum is given in Exercise 5.8 : $S=-1$, so

$$(RR)+(NN)-(RN)-(NR)=\sum_{n=1}^{p-1}\left(\frac{n(n+1)}{p}\right)=-1.$$

$\square$

Ex. 5.31 *Use the results of Exercises 29 and 30 to show that* $(RR) = \frac{1}{4}(p - 4 - \varepsilon)$, *where* $\varepsilon = (-1)^{(p-1)/2}$

Proof. To summarize the results of the Ex. 5.29 and 5.30,

$$(a)(RR) + (RN) + (NR) + (NN) = p - 2$$
$$(b)(RR) + (NN) - (RN) - (NR) = -1$$

and

$$(c)(RR) + (RN) = \frac{1}{2}\left(p - 2 - (-1)^{\frac{p-1}{2}}\right)$$
$$(d)(RR) + (NR) = \frac{p-1}{2} - 1$$

The sum of (a) and (b) gives

$$(e)(RR) + (NN) = \frac{p-3}{2}.$$

The sum of (c),(d),(e) gives (using (a))

$$2(RR) + p - 2 = \frac{p-2}{2} + \frac{p-1}{2} + \frac{p-3}{2} - 1 - \frac{(-1)^{\frac{p-1}{2}}}{2},$$

so

$$2(RR) = \frac{p-1}{2} + \frac{p-3}{2} - \frac{p-2}{2} - 1 - \frac{(-1)^{\frac{p-1}{2}}}{2} = \frac{p}{2} - 2 - \frac{(-1)^{\frac{p-1}{2}}}{2},$$
$$(RR) = \frac{1}{4}(p - 4 - \varepsilon), \text{ where } \varepsilon = (-1)^{\frac{p-1}{2}}.$$

□

Ex. 5.32 *If p is an odd prime, show that* $(2/p) = \prod_{j=1}^{(p-1)/2} 2\cos(2\pi j/p)$. *Use this to give another proof to Proposition 5.1.3.*

Proof. Let p an odd prime number, and $\zeta = e^{2i\pi/p}$: then $\zeta^p = 1$.

Let

$$P = \prod_{j=0}^{p-1}(\zeta^j + \zeta^{-j}) = \prod_{j=0}^{p-1} 2\cos(2\pi j/p).$$

$$\begin{aligned} P &= \zeta^0\zeta^{-1}\cdots\zeta^{-(p-1)}\prod_{j=0}^{p-1}(\zeta^{2j}+1) \\ &= (\zeta^p)^{-(p-1)/2}\prod_{j=0}^{p-1}(\zeta^{2j}+1) \\ &= \prod_{j=0}^{p-1}(\zeta^{2j}+1) \end{aligned}$$

As ζ^j depends only of the class $\overline{j} \in \mathbb{F}_p$, this product can be written

$$P = \prod_{\overline{j} \in \mathbb{F}_p} (\zeta^{2j} + 1) = \prod_{\overline{k} \in \mathbb{F}_p} (\zeta^k + 1) \qquad (k = 2j),$$

since $f : \mathbb{F}_p \to \mathbb{F}_p, x \mapsto 2x$ is a bijection. So

$$P = \prod_{k=0}^{p-1} (\zeta^k + 1).$$

Since $\zeta^0 = 1, \zeta, \cdots, \zeta^{p-1}$ are the roots of the polynomial $f(x) = x^p - 1$, then $1 + \zeta^0, \cdots, 1 + \zeta^{p-1}$ are the roots of $g(x) = (x-1)^p - 1 = f(x-1)$, so $g(x) = \prod_{k=0}^{p-1}(x - (1 + \zeta^k))$.

As $g(0) = (-1)^p - 1 = -2 = (-1 - \zeta^0) \cdots (-1 - \zeta^{p-1}) = -\prod_{k=0}^{p-1}(\zeta^k + 1)$, we obtain

$$P = \prod_{j=0}^{p-1} 2\cos(2\pi j/p) = \prod_{k=0}^{p-1} (\zeta^k + 1) = 2,$$

so

$$\prod_{j=1}^{p-1} 2\cos(2\pi j/p) = 1.$$

$$\begin{aligned}
1 &= \prod_{j=1}^{p-1} 2\cos(2\pi j/p) \\
&= \prod_{j=1}^{(p-1)/2} 2\cos(2\pi j/p) \prod_{j=(p+1)/2}^{p-1} 2\cos(2\pi j/p) \\
&= \prod_{j=1}^{(p-1)/2} 2\cos(2\pi j/p) \prod_{k=1}^{(p-1)/2} 2\cos(2\pi - 2\pi k/p) \qquad (k = p - j)
\end{aligned}$$

As $\cos(2\pi - \alpha) = \cos(\alpha)$,

$$1 = \left(\prod_{j=1}^{(p-1)/2} 2\cos(2\pi j/p) \right)^2, \text{ so } \prod_{j=1}^{(p-1)/2} 2\cos(2\pi j/p) = \pm 1$$

Case 1 : if $1 \le j \le p/4, 0 \le 2\pi j/p < \pi/2$, so $\cos(2\pi j/p) > 0$,

case 2 : if $p/4 < j \le (p-1)/2, \pi/2 < 2\pi j/p < \pi$, so $\cos(2\pi j/p) < 0$.

In the first case, $2 \le 2j \le (p-1)/2$: the least residue of $2j$ is positive. In the second case $p/2 < 2j \le p - 1$: the least residue of $2j$ is negative.

Let μ the number of negative least residues of the integer $2j$, $1 \le j \le (p-1)/2$: we know from Gauss' Lemma that $(2/p) = (-1)^\mu$. As μ is also the number of j, $1 \le j \le (p-1)/2$ such that $\cos(2\pi j/p) > 0$,

$$\prod_{j=1}^{(p-1)/2} 2\cos(2\pi j/p) = (-1)^\mu = \left(\frac{2}{p}\right).$$

If $p \equiv 1\ [8]$, $p = 8q+1, q \in \mathbb{N}$. For $1 \le j \le (p-1)/2$,

$$\cos(2\pi j/p) < 0 \iff p/4 \le j \le (p-1)/2 \iff 2q+1 \le j \le 4q,$$

so $\mu = 2q$ and $(2/p) = (-1)^\mu = 1$.

If $p \equiv -1\ [8]$, $p = 8q-1, q \in \mathbb{N}^*$.

$$\cos(2\pi j/p) < 0 \iff p/4 \le j \le (p-1)/2 \iff 2q \le j \le 4q-1,$$

so $\mu = 2q$ and $(2/p) = (-1)^\mu = 1$.

If $p \equiv 3\ [8]$, $p = 8q+3, q \in \mathbb{N}$.

$$\cos(2\pi j/p) < 0 \iff p/4 \le j \le (p-1)/2 \iff 2q+1 \le j \le 4q+1,$$

so $\mu = 2q+1$ and $(2/p) = (-1)^\mu = 1$.

If $p \equiv -3\ [8]$, $p = 8q-3, q \in \mathbb{N}^*$,

$$\cos(2\pi j/p) < 0 \iff p/4 \le j \le (p-1)/2 \iff 2q \le j \le 4q-2,$$

so $\mu = 2q-1$ and $(2/p) = (-1)^\mu = 1$. □

Ex. 5.33 *Use Proposition 5.3.2 to derive the quadratic character of -1.*

Proof. Let $f(z) = e^{2\pi i z} - e^{-2\pi i z}$. If p is an odd prime, $a \in \mathbb{Z}$, and $p \nmid a$, we know from Prop. 5.3.2 that

$$\prod_{l=1}^{(p-1)/2} f\left(\frac{la}{p}\right) = \left(\frac{a}{p}\right) \prod_{l=1}^{(p-1)/2} f\left(\frac{l}{p}\right).$$

For $a = -1$, as $f(-z) = -f(z)$,

$$\begin{aligned}\left(\frac{-1}{p}\right) \prod_{l=1}^{(p-1)/2} f\left(\frac{l}{p}\right) &= \prod_{l=1}^{(p-1)/2} f\left(\frac{-l}{p}\right)\\ &= (-1)^{(p-1)/2} \prod_{l=1}^{(p-1)/2} f\left(\frac{l}{p}\right)\end{aligned}$$

Moreover $f(z) = 0 \iff e^{4\pi i z} = 1 \iff 4\pi i z = 2ki\pi, k \in \mathbb{Z} \iff z = k/2, k \in \mathbb{Z}$, so, if $l \in \mathbb{Z}$, $f\left(\frac{l}{p}\right) = 0 \iff l/p = k/2, k \in \mathbb{Z} \iff p \mid 2l \iff p \mid l$. For $1 \le l < p$, this is impossible, so $\prod_{l=1}^{(p-1)/2} f\left(\frac{l}{p}\right) \ne 0$. Consequently,

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}.$$

□

Ex. 5.34 *If p is an odd prime distinct from 3, show that*

$$\left(\frac{3}{p}\right) = \prod_{j=1}^{(p-1)/2} \left(3 - 4\sin^2\left(\frac{2\pi j}{p}\right)\right).$$

Proof. Let p an odd prime number, $p \neq 3$ and $\zeta = e^{2i\pi/p}$.

$$\begin{aligned}
3 - 4\sin^2\left(\frac{2\pi j}{p}\right) &= 3 - 4\left(\frac{\zeta^j - \zeta^{-j}}{2i}\right)^2 \\
&= 3 + \zeta^{2j} + \zeta^{-2j} - 21 \\
&= 1 + \zeta^{2j} + \zeta^{-2j} \\
&= 1 + 2\cos\left(\frac{4\pi j}{p}\right)
\end{aligned}$$

(Or $\cos(2\alpha) = 1 - 2\sin^2\alpha$, so $3 - 4\sin^2\alpha = 1 + 2\cos\alpha$.)

Let

$$P = \prod_{j=1}^{p-1}\left(3 - 4\sin^2\left(\frac{2\pi j}{p}\right)\right) = \prod_{\overline{j}\in\mathbb{F}_p^*}\left(3 - 4\sin^2\left(\frac{2\pi j}{p}\right)\right).$$

Then

$$\begin{aligned}
P &= \prod_{\overline{j}\in\mathbb{F}_p^*}\left(1 + \zeta^{2j} + \zeta^{-2j}\right) \\
&= \prod_{\overline{k}\in\mathbb{F}_p^*}\left(1 + \zeta^{k} + \zeta^{-k}\right) \qquad (k = 2j)
\end{aligned}$$

since $f : \mathbb{F}_p \to \mathbb{F}_p, \overline{j} \mapsto 2\overline{j}$ is a bijection. So

$$\begin{aligned}
P &= \prod_{k=0}^{p-1}\zeta^{-k}\left(1 + \zeta^{k} + \zeta^{2k}\right) \\
&= 3\prod_{k=1}^{p-1}\zeta^{-k}\left(1 + \zeta^{k} + \zeta^{2k}\right) \\
&= 3\prod_{k=0}^{p-1}\zeta^{-k}\ \frac{\prod_{k=1}^{p-1}(1 - \zeta^{3k})}{\prod_{k=1}^{p-1}(1 - \zeta^{k})}
\end{aligned}$$

$\prod_{k=0}^{p-1}\zeta^{-k} = (\zeta^p)^{-(p-1)/2} = 1$. Morover, $\prod_{k=1}^{p-1}(1 - \zeta^{3k}) = \prod_{k=1}^{p-1}(1 - \zeta^{k})$, since $\overline{k} \mapsto 3\overline{k}$ is a bijection in $\mathbb{F}_p^*$, so $P = 3$, and consequently

$$\begin{aligned}
1 &= \prod_{j=1}^{p-1}\left(3 - 4\sin^2\left(\frac{2\pi j}{p}\right)\right) \\
&= \prod_{j=1}^{(p-1)/2}\left(3 - 4\sin^2\left(\frac{2\pi j}{p}\right)\right)\prod_{j=(p+1)/2}^{p-1}\left(3 - 4\sin^2\left(\frac{2\pi j}{p}\right)\right) \\
&= \prod_{j=1}^{(p-1)/2}\left(3 - 4\sin^2\left(\frac{2\pi j}{p}\right)\right)\prod_{k=1}^{(p-1)/2}\left(3 - 4\sin^2\left(\frac{2\pi(k-j)}{p}\right)\right) \qquad (k = p - j) \\
&= \left[\prod_{j=1}^{(p-1)/2}\left(3 - 4\sin^2\left(\frac{2\pi j}{p}\right)\right)\right]^2
\end{aligned}$$

So $\prod_{j=1}^{(p-1)/2}\left(3 - 4\sin^2\left(\frac{2\pi j}{p}\right)\right) = \pm 1$.

Let ν the number of negative factors in this product.

If $1 \le j \le (p-1)/2$, then $0 < 4\pi j/p < 2\pi$.

$$\begin{aligned} 1 + 2\cos\frac{4\pi j}{p} < 0 &\iff \cos\frac{4\pi j}{p} < \cos\frac{2\pi}{3} \\ &\iff \frac{2\pi}{3} < \frac{4\pi j}{p} < \frac{4\pi}{3} \\ &\iff \frac{p}{6} < j < \frac{p}{3} \\ &\iff \frac{p}{2} < 3j < p \end{aligned}$$

Let μ the number of integers $j, 1 \le j \le (p-1)/2$ such their least remainder is negative. Since $3 \le 3j \le 3(p-1)/2$ and $3j \ne p/2$, these j are such that $\frac{p}{2} < 3j < p$, so $\mu = \nu$. Therefore

$$\prod_{j=1}^{(p-1)/2} \left(3 - 4\sin^2\left(\frac{2\pi j}{p}\right)\right) = (-1)^\nu = (-1)^\mu = \left(\frac{3}{p}\right).$$

$\square$

Ex. 5.35 *Use the preceding exercise to show that* 3 *is a square modulo* p *iff* p *is congruent to* 1 *or* -1 *modulo* 12.

Proof. We know from Ex. 5.34 that $\nu = \mathrm{Card}\,\{j \in [1,(p-1)/2] \mid p/2 \le 3j < p\} = \mu$. ν is the number of j such that $p/6 \le j < p/3$, so $\nu = \lfloor p/3 \rfloor - \lfloor p/6 \rfloor$.

If $p = 12k+1$, $\nu = \lfloor p/3 \rfloor - \lfloor p/6 \rfloor = 4k - 2k = 2k : (3/p) = (-1)^\nu = 1$

If $p = 12k+5$, $\nu = \lfloor p/3 \rfloor - \lfloor p/6 \rfloor = 4k + 1 - 2k = 2k+1 : (3/p) = (-1)^\nu = -1$

If $p = 12k-5$, $\nu = \lfloor p/3 \rfloor - \lfloor p/6 \rfloor = 4k - 2 - (2k-1) = 2k-1 : (3/p) = (-1)^\nu = -1$

If $p = 12k-1$, $\nu = \lfloor p/3 \rfloor - \lfloor p/6 \rfloor = 4k - 1 - (2k-1) = 2k : (3/p) = (-1)^\nu = -1$

3 is a square modulo p $(p \ne 2, p \ne 3)$ iff p is congruent to 1 or -1 modulo 12. $\square$

Ex. 5.36 *Show that part (c) of Proposition 5.2.2 is true if* a *is negative and* b *is positive (both still odd).*

As said by Adam Michalik, the Jacobi symbol $\left(\frac{a}{b}\right)$ only defined for positive b, so the question, which concerns $\left(\frac{a}{b}\right)\left(\frac{b}{a}\right), a < 0$ makes no sense.

To give sense to this question, we must substitute the Kronecker symbol to the Jacobi symbol. The Kronecker symbol (not defined in Ireland-Rosen) is the usual extension of Jacobi symbol (see for instance [Henri Cohen] A course in computational algebraic number theory, [Henri Cohen] Number theory (vol. 1), or [Harvey Cohn] Advanced number theory).

We define Kronecker (or Kronecker-Jacobi) symbol $\left(\frac{a}{b}\right)$ for any a and b in $\mathbb{Z}$ in the following way.

(1) If $b = 0$, then $\left(\frac{a}{0}\right) = 1$ if $a = \pm 1$, and is equal to 0 otherwise.

(2) For $b \ne 0$, write $b = \prod p$, where the p are not necessarily distinct primes (including 2), or $p = -1$ to take care of the sign. Then we set

$$\left(\frac{a}{b}\right) = \prod \left(\frac{a}{p}\right),$$

where $\left(\frac{a}{p}\right)$ is the Legendre symbol defined above for $p > 2$, and where we define

$$\left(\frac{a}{2}\right) = \begin{cases} 0 & \text{if } a \text{ is even} \\ (-1)^{(a^2-1)/8} & \text{if } a \text{ is odd,} \end{cases}$$

and also

$$\left(\frac{a}{-1}\right) = \begin{cases} 1 & \text{if } a \geq 0 \\ -1 & \text{if } a < 0 \end{cases}$$

Proof. Suppose $a < 0, b > 0$, both odd. Let $a = -A, A > 0, A = p_1 p_2 \cdots p_k$, where the p_i are not necessarily distinct primes. Then

$$\begin{aligned} \left(\frac{a}{b}\right)\left(\frac{b}{a}\right) &= \left(\frac{-A}{b}\right)\left(\frac{b}{-A}\right) \\ \left(\frac{-A}{b}\right) &= \left(\frac{-1}{b}\right)\left(\frac{A}{b}\right) = (-1)^{(b-1)/2}\left(\frac{A}{b}\right) \\ \left(\frac{b}{-A}\right) &= \left(\frac{b}{-1}\right)\left(\frac{b}{p_1}\right)\cdots\left(\frac{b}{p_k}\right) = \left(\frac{b}{A}\right) \end{aligned}$$

so, from Prop. 5.2.2, as A, b are odd and positive,

$$\begin{aligned} \left(\frac{a}{b}\right)\left(\frac{b}{a}\right) &= (-1)^{\frac{b-1}{2}}\left(\frac{A}{b}\right)\left(\frac{b}{A}\right) \\ &= (-1)^{\frac{b-1}{2}}(-1)^{\frac{A-1}{2}\frac{b-1}{2}} \\ &= (-1)^{\frac{b-1}{2}[1+\frac{-a-1}{2}]} \\ &= (-1)^{\frac{b-1}{2}\frac{1-a}{2}} \\ &= (-1)^{\frac{b-1}{2}\frac{a-1}{2}} \end{aligned}$$

So the law of quadratic reciprocity remains valid for the Kronecker symbol when a is negative ($b > 0$, a, b both odd). $\square$

Ex. 5.37 *Show that if a is negative, then $p \equiv q \pmod{4a}$, $p \nmid a$ implies $(a/p) = (a/q)$.*

Proof. Write $a = -A, A > 0$. As $p \equiv q \pmod{4a}$, we know from Prop. 5.3.3. (b) that $(A/p) = (A/q)$.

Moreover,

$$\begin{aligned} \left(\frac{a}{p}\right) &= \left(\frac{-A}{p}\right) = (-1)^{(p-1)/2}\left(\frac{A}{p}\right) \\ \left(\frac{a}{q}\right) &= \left(\frac{-A}{q}\right) = (-1^{(q-1)/2}\left(\frac{A}{q}\right) \end{aligned}$$

As $p \equiv q \pmod{4a}$, $p = q + 4ak, k \in \mathbb{Z}$, so

$$(-1)^{(p-1)/2} = (-1)^{(q+4ak-1)/2} = (-1)^{(q-1)/2},$$

so $(a/p) = (a/q)$. $\square$

Ex. 5.38 *Let p be an odd prime. Derive the quadratic character of 2 modulo p by verifying the following steps, involving the Jacobi symbol:*

$$\left(\frac{2}{p}\right) = \left(\frac{8-p}{p}\right) = \left(\frac{p}{p-8}\right) = \left(\frac{8}{p-8}\right) = \left(\frac{2}{p-8}\right).$$

Generalize the argument to show that

$$\left(\frac{a}{p}\right) = \left(\frac{a}{p-4a}\right), \qquad a > 0, p \nmid a.$$

(As in Ex. 5.36, since $8-p$ or $p-8$ is negative, we interpret (a/b) as the Kronecker symbol : see definition in Ex. 5.36.)

Proof. As $(2^2/p) = 1$ and $8 - p \equiv 8 \pmod p$,

$$\left(\frac{2}{p}\right) = \left(\frac{2^2}{p}\right)\left(\frac{2}{p}\right) = \left(\frac{8}{p}\right) = \left(\frac{8-p}{p}\right).$$

As p and $8-p$ are odd numbers and $p > 0$, from the extension of the law of quadratic reciprocity to $a < 0$ proved in Ex. 5.36, we obtain

$$\left(\frac{8-p}{p}\right) = (-1)^{\frac{7-p}{2}\frac{p-1}{2}}\left(\frac{p}{8-p}\right).$$

Moreover

$$(7-p)(p-1) \equiv (-1-p)(p-1) = 1 - p^2 \pmod 8$$

As $p = 2k+1$ is odd, $p^2 = 4k^2+4k+1 = 8\frac{k(k+1)}{2} + 1 \equiv 1 \pmod 8$, so $(7-p)(p-1) \equiv 0 \pmod 8$ and $\frac{7-p}{2}\frac{p-1}{2}$ is even, so

$$\left(\frac{8-p}{p}\right) = \left(\frac{p}{8-p}\right).$$

As $p > 0$, $\left(\frac{p}{-1}\right) = 1$, thus $\left(\frac{p}{8-p}\right) = \left(\frac{p}{-1}\right)\left(\frac{p}{p-8}\right) = \left(\frac{p}{p-8}\right)$ (with the same argument, this is also true for the 3 odd primes such that $8-p>0$), so

$$\left(\frac{8-p}{p}\right) = \left(\frac{p}{p-8}\right).$$

□

As $p \equiv 8 \pmod{p-8}$, $\left(\frac{p}{p-8}\right) = \left(\frac{8}{p-8}\right)$, and since $8 = 2^2 \times 2$, $\left(\frac{8}{p-8}\right) = \left(\frac{2}{p-8}\right)$. We have proved for all odd primes p that

$$\left(\frac{2}{p}\right) = \left(\frac{8-p}{p}\right) = \left(\frac{p}{p-8}\right) = \left(\frac{8}{p-8}\right) = \left(\frac{2}{p-8}\right).$$

The preceding arguments remain valid if we replace the odd prime p by any odd positive integer. So with an immediate induction, we see that for all $k \in \mathbb{N}$,

$$\left(\frac{2}{p}\right) = \left(\frac{2}{p-8k}\right).$$

So the quadratic character of 2 modulo p depends only of the class of p modulo 8.

If $p \equiv 1 \pmod 8$, $\left(\frac{2}{p}\right) = \left(\frac{2}{1}\right) = 1$.

If $p \equiv -1 \pmod 8$, $\left(\frac{2}{p}\right) = \left(\frac{2}{-1}\right) = 1$.

If $p \equiv \pm 3 \pmod 8$, $\left(\frac{2}{p}\right) = \left(\frac{2}{\pm 3}\right) = -1$.

Generalization : let $a > 0$ and p an odd positive integer such that $p \wedge a = 1$ (not necessarily prime).

$$\left(\frac{a}{p}\right) = \left(\frac{4ap}{p}\right) = \left(\frac{4a-p}{p}\right) = (-1)^{\frac{4a-p-1}{2}\frac{p-1}{2}} \left(\frac{p}{4a-p}\right).$$

$(4a-p-1)(p-1) = 4a(p-1) + 1 - p^2 \equiv 0 \pmod 8$, so

$$\left(\frac{a}{p}\right) = \left(\frac{p}{4a-p}\right).$$

As $\left(\frac{p}{-1}\right) = 1$,

$$\left(\frac{p}{4a-p}\right) = \left(\frac{p}{p-4a}\right).$$

Since $p \equiv 4a \pmod p - 4a$, and 4 is a square,

$$\left(\frac{p}{p-4a}\right) \equiv \left(\frac{4a}{p-4a}\right) = \left(\frac{a}{p-4a}\right).$$

We have proved

$$\left(\frac{a}{p}\right) = \left(\frac{4a-p}{p}\right) = \left(\frac{p}{p-4a}\right) = \left(\frac{4a}{p-4a}\right) = \left(\frac{a}{p-4a}\right).$$

By induction, for all $k \geq 0$, $\left(\frac{a}{p}\right) = \left(\frac{a}{p-4a}\right)$, so $\left(\frac{a}{p}\right)$ depends only of the class of p modulo $4a$.

Chapter 6

Ex. 6.1 *Show that $\sqrt{2} + \sqrt{3}$ is an algebraic integer.*

Proof. Let $x = \sqrt{2} + \sqrt{3}$. Then $x^2 = 5 + 2\sqrt{6}$.

$(x^2 - 5)^2 = (2\sqrt{6})^2 = 24$, so $x^4 - 10x^2 + 1 = 0$: x is an algebraic integer. □

Ex. 6.2 *Let α be an algebraic number. Show that there's an integer n such that $n\alpha$ is an algebraic integer.*

(0 is a valid answer to this sentence ! More seriously, we search a *positive* integer n.)

Proof. Let α an algebraic number. By definition, there exist $a_0, a_1, \cdots, a_n \in \mathbb{Z}, a_n \neq 0$, such that

$$a_n \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_k \alpha^k + \cdots + a_0 = 0.$$

(Up to multiply this equation by -1, we can suppose that $a_n > 0$).

Multiplying by a_n^{n-1}, we obtain

$$a_n^n \alpha^n + a_n^{n-1} a_{n-1} \alpha^{n-1} + \cdots + a_n^{n-1} a_k \alpha^k + \cdots + a_n^{n-1} a_0 = 0.$$

So
$$(a_n\alpha)^n + a_{n-1}(a_n\alpha)^{n-1} + \cdots + a_n^{n-k-1}a_k(a_n\alpha)^k + \cdots + a_n^{n-1}a_0 = 0.$$

Soit $p(x) = x^n + \sum\limits_{k=0}^{n-1} a_n^{n-k-1}a_k x^k$. Then $p(x) \in \mathbb{Z}[x]$, $p(x)$ is monic, and $p(a_n x) = 0$. So $a_n x$ is an algebraic integer, with $m = a_n \in \mathbb{N}^*$.

Conclusion : if α is an algebraic number, there exists an integer $m > 0$ such that $m\alpha$ is an algebraic integer. $\square$

Ex. 6.3 *If α and β are algebraic integers, prove that any solution to $f(x) = x^2 + \alpha x + \beta = 0$ is an algebraic integer. Generalize this result.*

Proof. Let γ a root of $x^2 + \alpha x + \beta$, where α, β verify :
$$\alpha^n + r_1\alpha^{n-1} + \cdots + r_n = 0, \quad r_i \in \mathbb{Z},$$
$$\beta^m + s_1\beta^{m-1} + \cdots + s_m = 0, \quad s_j \in \mathbb{Z}.$$

Let V the set of linear combinations with integer coefficients of
$$\alpha^i\beta^j\gamma^k, 0 \le i < n, 0 \le j < m, 0 \le k < 2.$$

Then V if a finitely generated $\mathbb{Z}$-module.

Moreover, for all $\delta \in V, \gamma\delta \in V$. Indeed, every $\delta \in V$ is a linear combination with coefficients in $\mathbb{Z}$ of $\alpha^i\beta^j, \alpha^i\beta^j\gamma$, and
$$\begin{aligned}\gamma(\alpha^i\beta^j) &= \alpha^i\beta^j\gamma \in V\\ \gamma(\alpha^i\beta^j\gamma) &= \alpha^i\beta^j\gamma^2 = \alpha^i\beta^j(-\alpha\gamma - \beta) = -\alpha^{i+1}\beta^j\gamma - \alpha^i\beta^{j+1} \in V.\end{aligned}$$

(if $i + 1 = n$, we replace $\alpha^{i+1} = \alpha^n$ by $-\sum\limits_{k=1}^{n-1} r_k\alpha^{n-k}$, and a similar replacement if if $j + 1 = m$.)

As for each $\gamma \in V$, where V if a finitely generated $\mathbb{Z}$-module, $x\gamma \in V$, so γ is an algebraic integer (Proposition 6.1.4).

More generaly, if $\gamma^n + \alpha_1\gamma^{n-1} + \cdots + \alpha_n = 0$, where the α_i are algebraic integers, then x is an algebraic integer. $\square$

Ex. 6.4 *A polynomial $f(x) \in \mathbb{Z}[x]$ is said to be primitive if the greatest common divisor of its coefficients is 1. Prove that product of primitive polynomials is also primitive.*

Solution 1

Proof. Let $p(x) = \sum_{i=0}^n a_i x^i, q(x) = \sum_{j=0}^m b_j x^j$ two primitive polynomials, and p a prime number. There exist a coefficient of $p(x)$ (and of $q(x)$) not divisible by p. Let
$$\begin{aligned}i_0 &= \min\{i \in [0, n] \mid a_i \not\equiv 0\ [p]\}\\ j_0 &= \min\{j \in [0, m] \mid b_j \not\equiv 0\ [p]\}\end{aligned}$$

Let $p(x)q(x) = \sum_{k=0}^{n+m} c_k x^k$. Then $c_k = \sum\limits_{i+j=k} a_i b_j,\ k = 0, \ldots n + m$. Then
$$c_{i_0+j_0} = \sum_{i+j=i_0+j_0} a_i b_j.$$

• If $i < i_0$, then $a_i \equiv 0 \pmod p$.
• If $i > i_0$, then $j < j_0$ and $b_j \equiv 0 \pmod p$.

In the two cases $a_i b_j \equiv 0 \pmod p$, so $c_{i_0+j_0} \equiv a_{i_0} b_{j_0} \pmod p$, so $c_{j_0} \not\equiv 0 \pmod p$: as it's true for all primes p, the polynomial $p(x)q(x)$ is primitive. $\square$

Solution 2

Proof. Let

$$\varphi : \begin{cases} \mathbb{Z}[x] & \to & \mathbb{F}_p[x] \\ p(x) = a_0 + \cdots + a_n x^n & \mapsto & \overline{p}(x) = \overline{a_0} + \cdots + \overline{a_n} x^n, \end{cases}$$

where $\overline{a_i}$ is the class of a_i in $\mathbb{F}_p$. φ is a ring homomorphism.

As $\mathbb{F}_p[x]$ is an integrity domain, if $p(x), q(x)$ are both primitive,

$$\overline{p(x)} \neq 0, \overline{q(x)} \neq 0 \Rightarrow \overline{p(x)q(x)} = \overline{p(x)}\,\overline{q(x)} \neq 0.$$

As $\overline{p(x)q(x)} \neq 0$ in all fields $\mathbb{F}_p$, $p(x)q(x)$ is a primitive polynomial. $\square$

Ex. 6.5 *Let α be an algebraic integer and $f(x) \in \mathbb{Q}[x]$ be the monic polynomial of least degree such that $f(\alpha) = 0$. Use Exercise 6.4 to show that $f(x) \in \mathbb{Z}[x]$.*

Proof. As α is an algebraic integer, there exists a monic polynomial $h(x) \in \mathbb{Z}[x]$ such that $h(\alpha) = 0$. As $f(x) \in \mathbb{Q}[x]$ is the minimal polynomial of α, and $h(\alpha) = 0$, $f(x)$ divides $h(x)$ in $\mathbb{Q}[x]$.

(quick reminder : $h(x) = q(x)f(x) + r(x),\ q(x), r(x) \in \mathbb{Q}[x], \deg(r(x)) < \deg(f(x))$ or $r(x) = 0$. As $r(\alpha) = 0$ and $f(x) \in \mathbb{Q}[x]$ is the monic polynomial of least degree such that $f(\alpha) = 0$, $r = 0$ so $f(x) \mid h(x)$).

So there exists $g(x) \in \mathbb{Q}[x]$ such that $h(x) = f(x)g(x)$. As $h(x), f(x)$ are both monic, $g(x)$ is also monic.

Let $d \in \mathbb{Z}, d \neq 0$ such that $df(x) = \sum_{i=0}^m a_i x^i \in \mathbb{Z}[x]$, and $c = a_1 \wedge a_2 \wedge \cdots \wedge a_m$, $a_i = cb_i$, with $b_1 \wedge b_2 \wedge \cdots \wedge b_m = 1$, so $f(x) = \frac{c}{d} f_1(x)$, with f_1 is primitive. Similarly $g(x) = \frac{s}{t} g_1(x)$, $s, t \in \mathbb{Z}$, $g_1(x)$ primitive.

So $h(x) = \frac{cs}{dt} f_1(x) f_2(x) = \frac{u}{v} f_1(x) f_2(x)$, where $u \wedge v = 1$. The polynomial $f_1(x)f_2(x) = \sum_{k=0}^r c_k x^k$ is primitive (Ex. 6.4). As $vh(x)(x) = uf_1(x)f_2(x)$, $c \mid uc_k$, and $u \wedge v = 1$, thus $v \mid c_k, k = 0, 1, \ldots, r$. As $c_1 \wedge \cdots c_k = 1$, $v \mid 1$, so $v = \pm 1$. $h(x) = uf_1(x)f_2(x)$ is monic, thus $u = 1$, and f_1, f_2 are monic. From $f(x) = \frac{c}{d} f_1(x)$ we deduce $\frac{c}{d} = 1$ and $f(x) = f_1(x) \in \mathbb{Z}[x]$.

Conclusion : if $f(x)$ is the minimal polynomial of an algebraic integer α, $f \in \mathbb{Z}[x]$. $\square$

Ex. 6.6 *Let $x^2 + mx + n \in \mathbb{Z}[x]$ be irreducible, and α be a root. Show that $\mathbb{Q}[\alpha] = \{r + s\alpha : r, s \in \mathbb{Q}\}$ is a ring (in fact, it is a field). Let $m^2 - 4n = D_0^2 D$, where D is square-free. Show that $\mathbb{Q}[\alpha] = \mathbb{Q}[\sqrt{D}]$.*

Proof. By definition, for all $z \in \mathbb{C}, z \in \mathbb{Q}[\alpha] \iff \exists P \in \mathbb{Q}[x], z = P(\alpha)$.

The Euclidean division gives $P = Q_1(x^2 + mx + n) + R(x),\ Q_1, R \in \mathbb{Q}[x], \deg(R) < 2$, so $R = rx + s,\ r, s \in \mathbb{Q}$. So $z = Q_1(\alpha)(\alpha^2 + m\alpha + n) + r\alpha + s = r\alpha + s$:

$$\mathbb{Q}(\alpha) = \{z \in \mathbb{C} \mid \exists r \in \mathbb{Q}, \exists s \in \mathbb{Q},\ z = r + s\alpha\}.$$

• $\mathbb{Q}[\alpha] \subset \mathbb{C}$, where $(\mathbb{C}, +, \times)$ is a field. $1 \in \mathbb{Q}[\alpha]$ ($1 = P_0(\alpha)$, where P_0 is the constant polynomial 1).

• Let $\beta, \gamma \in \mathbb{Q}[\alpha] : \beta = P(\alpha), \gamma = Q(\alpha)$, where P, Q are in $\mathbb{Q}[x]$. Then $\alpha - \beta = P(\alpha) - Q(\alpha) = R(\alpha)$, where $R = P - Q \in \mathbb{Q}[x]$, and $\alpha\beta = P(\alpha)Q(\alpha) = S(\alpha)$, where $S = PQ \in \mathbb{Q}[x]$. So $\alpha - \beta \in \mathbb{Q}[\alpha], \alpha\beta \in \mathbb{Q}[\alpha]$. So $\mathbb{Q}[\alpha]$ is a subring of $(\mathbb{C}, +, \times)$.

• Let $\beta = P(\alpha) \in \mathbb{Q}[\alpha), P \in \mathbb{Q}[x]$ and $\beta \neq 0$. As $\beta \neq 0$, $Q = x^2 + mx + n \nmid P$.

Let $D \in \mathbb{Q}[x]$ such that $D \mid P, D \mid Q$. As Q is irreducible by hypothesis, $D = \lambda$ or $D = \lambda Q, \lambda \in \mathbb{C}^*$ (D is an associate of 1 or Q). If $D = \lambda Q$, then $Q \mid D$, and $D \mid P$, so $Q \mid P$. Since $Q(\alpha) = 0$, this implies $\beta = P(\alpha) = 0$, in contradiction with the definition of β. So $D = \lambda \mid 1$. Therefore $P \wedge Q = 1$.

From Bézout's theorem, there exist polynomials $U, V \in \mathbb{Q}[x]$ such that $UP + VQ = 1$. As $\mathbb{Q}(\alpha) = 0$, $U(\alpha)P(\alpha) = 1$ and $\gamma = U(\alpha) \in \mathbb{Q}[\alpha]$ is such that $\gamma\beta = 1$. Therefore $\mathbb{Q}[\alpha]$ is a subfield of $(\mathbb{C}, +, \times)$ (and $\mathbb{Q}(\alpha) = \mathbb{Q}[\alpha]$).

As $x^2 + mx + n$ is irreducible, $\Delta = m^2 - 4n \neq 0$ (if not, $x^2 + mx + n = (x + m/2)^2 - (m^2 - 4n)/4 = (x + m/2)^2$ is not irreducible). So $\Delta \in \mathbb{Z} \setminus \{0\}$ can be written $\Delta = m^2 - 4n = D_0^2 D$, where D is square-free (positive or negative), $D \neq 0, D_0 \neq 0$.

$\alpha = -\frac{m}{2} + \varepsilon\frac{\sqrt{\Delta}}{2}$, $\varepsilon = \pm 1$, so $\alpha = -\frac{m}{2} + \varepsilon D_0 \frac{\sqrt{D}}{2}$, thus $\alpha \in \mathbb{Q}[\sqrt{D}]$ and $\mathbb{Q}[\alpha] \subset \mathbb{Q}[\sqrt{D}]$.

As $D_0 \neq 0$, $\sqrt{D} = \varepsilon\frac{2\alpha + m}{D_0} \in \mathbb{Q}[\alpha]$, so $\mathbb{Q}[\sqrt{D}] \subset \mathbb{Q}[\alpha)]$:

$$\mathbb{Q}[\alpha] = \mathbb{Q}[\sqrt{D}].$$

□

Ex. 6.7 *(continuation) If* $D \equiv 2, 3 \pmod 4$*, show that all the algebraic integers in* $\mathbb{Q}[\sqrt{D}]$ *have the form* $a + b\sqrt{D}$*, where* $a, b \in \mathbb{Z}$*. If* $D \equiv 1 \pmod 4$*, show that all the algebraic integers in* $\mathbb{Q}[\sqrt{D}]$ *have the form* $a + b((-1 + \sqrt{D})/2)$*, where* $a, b \in \mathbb{Z}$*.*

Proof. (We write $\overline{\mathbb{Z}}$ the ring of algebraic integers in $\mathbb{C}$, and $\mathcal{O}_K$ (or $\mathbb{Z}_K$) the ring of algebraic integers in the field K.)

If $D = 1$, $\mathbb{Q}[\sqrt{D}] = \mathbb{Q}$. If $D \neq 1$, as D is square-free, D in not a square, so $\sqrt{D}$ is irrational.

Let $\gamma = r + s\sqrt{D} \in \mathbb{Q}[\sqrt{D}]$ ($r, s \in \mathbb{Q}$) an algebraic integer of $\mathbb{Q}[\sqrt{D}]$ ($D \in \mathbb{Z}, D$ square-free). $(\gamma - r)^2 = s^2 D$, so $\gamma^2 - 2r\gamma + r^2 - Ds^2 = 0$. γ is a root of

$$p(x) = x^2 - 2rx + r^2 - Ds^2.$$

If $s = 0$, then the minimal polynomial of γ is $x - r$. As $r = \gamma$ is an algebraic integer and $r \in \mathbb{Q}$, then $r \in \mathbb{Z}$. In this case $r \in \mathbb{Z}$ and $s = 0$.

If $s \neq 0$, $\gamma \notin \mathbb{Q}$, so no polynom of degree $d \leq 1$ has the root γ. Thus the minimal polynomial of γ is $p(x)$. From Exercise 6.5, $p(x) \in \mathbb{Z}[x]$, so (in the two cases $s = 0, s \neq 0$)

$$2r \in \mathbb{Z}, r^2 - Ds^2 \in \mathbb{Z}.$$

Reciprocally, if $2r \in \mathbb{Z}, r^2 - Ds^2 \in \mathbb{Z}$, then $p(x) \in \mathbb{Z}[x]$ and $p(\gamma) = 0$, thus γ is an algebraic integer.

If $r, s \in \mathbb{Q}$, $D \neq 1$ square-free,

$$r + s\sqrt{D} \in \overline{\mathbb{Z}} \iff 2r \in \mathbb{Z},\ r^2 - Ds^2 \in \mathbb{Z}.$$

Let $\gamma = r + s\sqrt{D} \in \overline{\mathbb{Z}}$. We can write

$$r = \frac{a}{d}, s = \frac{b}{d}, \qquad a, b, d \in \mathbb{Z}, d \geq 1, d \wedge a \wedge b = 1.$$

Then
$$n = \frac{2a}{d} \in \mathbb{Z}, \quad m = \frac{a^2 - Db^2}{d^2} \in \mathbb{Z}.$$
As D is square-free, $D \not\equiv 0 \pmod 4$.

- Case 1 : $D \equiv 2, 3 \pmod 4$.

  $n^2 - 4m = \frac{4Db^2}{d^2}$, so $d \mid 2a, d^2 \mid 4Db^2$.

  If $2 \mid d$, $4 \mid a^2 - Db^2$, $a^2 \equiv Db^2 \pmod 4$. As $d \wedge a \wedge b = 1$, and $2 \mid d$, a or b is odd, and $a^2 \equiv Db^2 \pmod 4, D \not\equiv 0 \pmod 4$, implies that a and b are both odd. Then $a^2 \equiv b^2 \equiv 1 \pmod 4$, so $D \equiv 1 \pmod 4$: this is in contradiction with the hypothesis $D \equiv 2, 3 \pmod 4$. So d is an odd number.

  Consequently, $d \mid p, d^2 \mid Dq^2$. If $p \in \mathbb{N}$ is a prime factor of d, $p \mid d, p \mid a$,and $d \wedge a \wedge b = 1$, so $p \nmid b$, and since $p^2 \mid Db^2$, $p^2 \mid D$, in contradiction with D square-free. So $d \geq 1$ has no prime factor : $d = 1$ and $r = a, s = b \in \mathbb{Z}$. Reciprocally , any $\gamma = a + b\sqrt{D}, a, b \in \mathbb{Z}$ is an algebraic integer, so
$$\mathcal{O}_{\mathbb{Q}[\sqrt{D}]} = \overline{\mathbb{Z}} \cap \mathbb{Q}[\sqrt{D}] = \{a + b\sqrt{D} \mid a, b \in \mathbb{Z}\}.$$

- Case 2 : $D \equiv 1 \pmod 4$.

  Then $r = \frac{n}{2}, n \in \mathbb{Z}$. Write $s = \frac{u}{v}, u \wedge v = 1, v \geq 1$.

  $m = r^2 - Ds^2 = \frac{n^2}{4} - D\frac{u^2}{v^2} \in \mathbb{Z}$, $4D\frac{u^2}{v^2} = n^2 - 4m \in \mathbb{Z}$, so $v^2 \mid 4Du^2$. Since $u \wedge v = 1, u^2 \wedge v^2 = 1$, so $v^2 \mid 4D$. As D is square-free, v has no odd prime factor, so $v = 2^k$. Since D is odd, $k \leq 1$ and $v = 1$ or $v = 2$. So r, s are both half-integers: $r = n/2, s = n'/2,\ n, n' \in \mathbb{Z}$.

  $4m = n^2 - Dn'^2$, thus $n^2 \equiv n'^2 \pmod 4$, so n, n' have the same parity. Let $a = \frac{n+n'}{2} \in \mathbb{Z}, b = n' \in \mathbb{Z}$. Then $n = 2a - b, n' = b$ and $\gamma = \frac{n}{2} + \frac{n'}{2}\sqrt{D} = a - \frac{b}{2} + \frac{b}{2}\sqrt{D} = a + b\left(\frac{-1+\sqrt{D}}{2}\right)$.

  Reciprocally, $\frac{-1+\sqrt{D}}{2}$ is a root of $x^2 + x + \frac{1-D^2}{4} \in \mathbb{Z}[x]$, so every $a + b\left(\frac{-1+\sqrt{D}}{2}\right)$ is an algebraic integer.
$$\mathcal{O}_{\mathbb{Q}[\sqrt{D}]} = \overline{\mathbb{Z}} \cap \mathbb{Q}[\sqrt{D}] = \{a + b\left(\frac{-1+\sqrt{D}}{2}\right) \mid a, b \in \mathbb{Z}\}.$$

□

Ex. 6.8 *Let $\omega = e^{2\pi i/3}$, ω satisfies $x^3 - 1 = 0$. Show that $(2\omega + 1)^2 = -3$, and use this to determine $(-3/p)$ by the method of section 2.*

Proof. As $\omega^2 + \omega + 1 = 0$,$(2\omega + 1)^2 = 4\omega^2 + 4\omega + 1 = -4 + 1 = -3$. Let $\alpha = 2\omega + 1$,so $\alpha^2 = -3$
$$\begin{aligned}\left(\frac{-3}{p}\right) &\equiv (-3)^{(p-1)/2} \pmod p\\ &\equiv \alpha^{p-1} \pmod p\end{aligned}$$
$$\alpha^p = \left(\frac{-3}{p}\right)\alpha.$$

From Prop. 6.1.6,

$$\begin{aligned}\alpha^p &= (2\omega+1)^p\\ &\equiv 2^p\omega^p + 1 \pmod p\\ &\equiv 2\omega^p + 1 \pmod p\end{aligned}$$

- If $p \equiv 0 \pmod 3$, $\left(\frac{-3}{p}\right) = 0$.
- If $p \equiv 1 \pmod 3$, $\omega^p = \omega$, so $\alpha^p \equiv \alpha \pmod p$.
  $\left(\frac{-3}{p}\right)\alpha \equiv \alpha \pmod p$, thus $\left(\frac{-3}{p}\right)\alpha^2 \equiv \alpha^2 \pmod p$, $\left(\frac{-3}{p}\right)3 \equiv 3 \pmod p$. As $p \wedge 3 = 1$, $\left(\frac{-3}{p}\right) \equiv 1 \pmod p$. Since $\left(\frac{-3}{p}\right) = \pm 1$, $\left(\frac{-3}{p}\right) = 1$.
- If $p \equiv -1 \pmod 3$,

$$\begin{aligned}\alpha^p &\equiv 2\omega^p + 1 \pmod p\\ &\equiv 2\omega^2 + 2 = 2(-1-\omega) + 1 = -2\omega - 1 = -\alpha \pmod p.\end{aligned}$$

  $\left(\frac{-3}{p}\right)\alpha \equiv -\alpha \pmod p$, thus $\left(\frac{-3}{p}\right)\alpha^2 \equiv -\alpha^2 \pmod p$, $\left(\frac{-3}{p}\right)3 \equiv -3 \pmod p$. As $p \wedge 3 = 1$, $\left(\frac{-3}{p}\right) \equiv -1 \pmod p$. Since $\left(\frac{-3}{p}\right) = \pm 1$, $\left(\frac{-3}{p}\right) = -1$.

Conclusion :

$$\begin{aligned} p \equiv 0[3] &\iff \left(\frac{-3}{p}\right) = 0\\ p \equiv 1[3] &\iff \left(\frac{-3}{p}\right) = 1\\ p \equiv -1[3] &\iff \left(\frac{-3}{p}\right) = -1\end{aligned}$$

In other words, $\left(\frac{-3}{p}\right) = \left(\frac{p}{3}\right)$.

Remark : $\alpha = 2\omega + 1 = \omega - \omega^2 = g$, the quadratic Gauss sum for $p = 3$. □

Ex. 6.9 *Verify Proposition 6.3.2 explicitly for $p = 3, 5$, i.e., write out the Gauss sum longhand and square.*

Proof.

- p=3. Let $\omega = e^{2i\pi/3}$. Let $g = \sum_{t=0}^{2}(t/3)\omega^t$ the quadratic Gauss sum. Then $g = \omega - \omega^2$.
  As $1 + \omega + \omega^2 = 0$, $g^2 = (\omega - \omega^2)^2 = \omega^2 - 2\omega^3 + \omega^4 = \omega^2 - 2 + \omega = -3$:

$$g^2 = -3.$$

- p=5. Let $\zeta = e^{2i\pi/5}$.
  $g = \sum_{t=0}^{4}(t/3)\zeta^t = \zeta - \zeta^2 - \zeta^3 + \zeta^4$.
  Then $g = \alpha - \beta$, where $\alpha = \zeta + \zeta^4, \beta = \zeta^2 + \zeta^3$.
  $\alpha + \beta = \zeta + \zeta^4 + \zeta^2 + \zeta^3 = -1$.
  $\alpha\beta = \zeta^3 + \zeta^4 + \zeta^6 + \zeta^7 = \zeta^3 + \zeta^4 + \zeta + \zeta^2 = -1$

So α, β are the two roots of $x^2 + x - 1$.

$$\begin{aligned} g^2 &= (\alpha - \beta)^2 \\ &= \alpha^2 + \beta^2 - 2\alpha\beta \\ &= (\alpha + \beta)^2 - 4\alpha\beta \\ &= (-1)^2 - 4(-1) \\ &= 5. \end{aligned}$$

Remark : here we know explicitely g :
if $p = 3$, $g = \omega - \omega^2 = i\sqrt{3}$.
If $p = 5$, $g = \alpha - \beta = (-1 + \sqrt{5})/2 - (-1 - \sqrt{5})/2 = \sqrt{5}$.

□

Ex. 6.10 *What is* $\sum_{a=1}^{p-1} g_a$*?*

Proof. From Prop. 6.3.1 and Lemma 2,

$$\sum_{a=1}^{p-1} g_a = g_1 \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) = 0.$$

□

Ex. 6.11 *By evaluating* $\sum_t (1 + (t/p))\zeta^t$ *in two ways, prove that* $g = \sum_t \zeta^{t^2}$.

Proof. For $a \in \mathbb{F}_p$, Write $N[x^2 = a]$ the number of solutions of the equation $x^2 = a$ in $\mathbb{F}_p$. We know from Ex. 5.2 that $N[x^2 = a] = 1 + (a/p)$. So

$$\begin{aligned} \sum_{t=0}^{p-1} \zeta^{t^2} &= \sum_{\bar{t} \in \mathbb{F}_p} \zeta^{t^2} \\ &= \sum_{\bar{t} \in \mathbb{F}_p} N[x^2 = t]\zeta^t \\ &= \sum_{\bar{t} \in \mathbb{F}_p} \left(1 + \left(\frac{t}{p}\right)\right) \zeta^t \\ &= \sum_{\bar{t} \in \mathbb{F}_p} \zeta^t + \sum_{\bar{t} \in \mathbb{F}_p} \left(\frac{t}{p}\right)\zeta^t \\ &= \sum_{t=0}^{p-1} \left(\frac{t}{p}\right)\zeta^t \\ &= g \end{aligned}$$

□

Ex. 6.12 *Write $\psi_a(t) = \zeta^{at}$. Show that*

(a) $\overline{\psi_a(t)} = \psi_a(-t) = \psi_{-a}(t)$

(b) $(1/p)\sum_a \psi_a(t-s) = \delta(t,s)$

Proof. (a) Let $a \in \mathbb{Z}$. As $\overline{\zeta} = \zeta^{-1}$,

$$\begin{aligned}\overline{\psi_a(t)} &= \overline{\zeta^{at}} = \zeta^{-at}\\ &= \zeta^{a(-t)} = \zeta^{(-a)t}\\ &= \psi_a(-t) = \psi_{-a}(t)\end{aligned}$$

$$\overline{\psi_a(t)} = \psi_a(-t) = \psi_{-a}(t)$$

(b) From Corollary of Lemma 1 :

$$\frac{1}{p}\sum_{a=0}^{p-1} \psi_a(t-s) = \frac{1}{p}\sum_{a=0}^{p-1} \zeta^{a(t-s)} = \delta(t,s)$$

$$\frac{1}{p}\sum_{a} \psi_a(t-s) = \delta(t,s).$$

□

Ex. 6.13 *Let f be a function from $\mathbb{Z}$ to the complex numbers. Suppose that p is a prime and that $f(n+p) = f(n)$ for all $n \in \mathbb{Z}$. Let $\hat{f}(a) = p^{-1}\sum_t f(t)\psi_{-a}(t)$. Prove that $f(t) = \sum_a \hat{f}(a)\psi_a(t)$. This result is directly analogous to a result in the theory of Fourier series.*

Proof. Let $\hat{f}(a) = p^{-1}\sum_t f(t)\psi_{-a}(t)$. Then

$$\begin{aligned}\sum_{a=0}^{p-1} \hat{f}(a)\psi_a(t) &= \sum_{a=0}^{p-1} p^{-1}\sum_{s=0}^{p-1} f(s)\psi_{-a}(s)\psi_a(t)\\ &= p^{-1}\sum_{s=0}^{p-1} f(s)\sum_{a=0}^{p-1} f(s)\psi_{-a}(s)\psi_a(t)\\ &= p^{-1}\sum_{s=0}^{p-1} f(s)\sum_{a=0}^{p-1} f(s)\psi_a(t-s)\\ &= \sum_{s=0}^{p-1} f(s)\delta(s,t)\\ &= f(t)\end{aligned}$$

□

Ex. 6.14 *In Ex. 13 take f to be the Legendre symbol and show that $\hat{f}(a) = p^{-1}g_{-a}$.*

Proof. Here $f(a) = \left(\frac{a}{p}\right)$. Then $\hat{f}(a) = p^{-1}\sum_{t=0}^{p-1}\left(\frac{t}{p}\right)\zeta^{-at} = p^{-1}g_{-a}$. □

Ex. 6.15 *Show that*

$$\left|\sum_{t=n}^{m}\left(\frac{t}{p}\right)\right| < \sqrt{p}\log p.$$

The inequality holds for the sum over any range.

Lemma. If $0 \le x \le \frac{\pi}{2}, \sin x \ge \frac{2}{\pi}x$.

Proof. As $-\sin$ is a convex function on $[0,\pi/2]$, the graph of sin is above any chord, and the chord between the points $(0,0)$ and $(\pi/2,1)$ has equation $y=(2/\pi)x$, we conclude that $\sin x \ge \frac{2}{\pi}$ for $0 \le x \le \pi/2$. □

Proof. Let $S = \sum\limits_{t=n}^{m}\left(\frac{t}{p}\right)g$ with $n \le m$. Then $|S| = \sqrt{p}\left|\sum\limits_{t=n}^{m}\left(\frac{t}{p}\right)\right|$. As $(t/p)g = g_t$,

$$\begin{aligned}
S &= \sum_{t=m}^{n} g_t\\
&= \sum_{t=m}^{n}\sum_{s=0}^{p-1}\left(\frac{s}{p}\right)\zeta^{ts}\\
&= \sum_{s=0}^{p-1}\left(\frac{s}{p}\right)\zeta^{ms}\sum_{t=m}^{n}\zeta^{(t-m)s}\\
&= \sum_{s=0}^{p-1}\left(\frac{s}{p}\right)\zeta^{ms}\sum_{u=0}^{n-m}\zeta^{us} \qquad (u=t-m)\\
&= \sum_{s=1}^{p-1}\left(\frac{s}{p}\right)\zeta^{ms}\frac{\zeta^{(n-m+1)s}-1}{\zeta^{s}-1}
\end{aligned}$$

(since for $s=0$, the sum $\sum_{u=0}^{n-m}\zeta^{us} = n-m+1$ and $\sum_{s=0}^{p-1}\left(\frac{s}{p}\right)=0$). So

$$\begin{aligned}
S &= \sum_{s=1}^{p-1}\left(\frac{s}{p}\right)\frac{\zeta^{(n+1)s}-\zeta^{ms}}{\zeta^{s}-1}\\
&= \sum_{s=1}^{p-1}\left(\frac{s}{p}\right)\frac{\zeta^{\frac{n+m+1}{2}s}}{\zeta^{\frac{s}{2}}}\,\frac{\zeta^{\frac{n-m+1}{2}s}-\zeta^{\frac{-n+m-1}{2}s}}{\zeta^{\frac{s}{2}}-\zeta^{\frac{-s}{2}}}\\
&= \sum_{s=1}^{p-1}\left(\frac{s}{p}\right)\zeta^{\frac{n+m}{2}s}\,\frac{\sin\left((n-m+1)s\frac{\pi}{p}\right)}{\sin\left(s\frac{\pi}{p}\right)}
\end{aligned}$$

As $\sin(x) \ge \frac{2}{\pi}x$ for $x \in [0,\frac{\pi}{2}]$, for all s, $1 \le s < \frac{p}{2}$, $0 \le \frac{s\pi}{p} \le \frac{\pi}{2}$, so

$$\left|\frac{\sin\left((n-m+1)s\frac{\pi}{p}\right)}{\sin\left(s\frac{\pi}{p}\right)}\right| \le \frac{1}{\frac{2}{\pi}\left(s\frac{\pi}{p}\right)} = \frac{p}{2s} \qquad (s=1,2,\ldots,(p-1)/2).$$

Since $\left(\frac{s}{p}\right)\zeta^{ts}$ depends only of the class of s, we can replace in the preceding calculation the values $s=1,2,\ldots,p-1$ by $s=-(p-1)/2,\ldots,-1,1,\ldots,(p-1)/2$, so

$$S = \sum_{s=1}^{(p-1)/2}\left(\frac{s}{p}\right)\zeta^{\frac{n+m}{2}s}\,\frac{\sin\left((n-m+1)s\frac{\pi}{p}\right)}{\sin\left(s\frac{\pi}{p}\right)} + \sum_{s=-(p-1)/2}^{-1}\left(\frac{s}{p}\right)\zeta^{\frac{n+m}{2}s}\,\frac{\sin\left((n-m+1)s\frac{\pi}{p}\right)}{\sin\left(s\frac{\pi}{p}\right)}.$$

As sin is an odd function,

$$S = \sum_{s=1}^{(p-1)/2} \left(\frac{s}{p}\right) \zeta^{\frac{n+m}{2}s} \frac{\sin\left((n-m+1)s\frac{\pi}{p}\right)}{\sin\left(s\frac{\pi}{p}\right)} + \sum_{s=1}^{(p-1)/2} \left(\frac{-s}{p}\right) \zeta^{-\frac{n+m}{2}s} \frac{\sin\left((n-m+1)s\frac{\pi}{p}\right)}{\sin\left(s\frac{\pi}{p}\right)}.$$

Thus

$$|S| \le 2 \sum_{s=1}^{(p-1)/2} \frac{p}{2s} = p \sum_{s=1}^{(p-1)/2} \frac{1}{s}.$$

As $S = \sum\limits_{t=n}^{m} \left(\frac{t}{p}\right) g$ and $|g| = \sqrt{p}$,

$$\left| \sum_{t=n}^{m} \left(\frac{t}{p}\right) \right| \le \sqrt{p} \sum_{s=1}^{(p-1)/2} \frac{1}{s}.$$

It remains to do a sufficient estimation of the harmonic sum. We prove by induction that for all $n \ge 1$,

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \le \log(2n+1).$$

As $1 \le \log(3)$, this proposition is true for $n = 1$. Suppose that is it true for $n - 1$:

$$1 + \frac{1}{2} + \cdots + \frac{1}{n-1} \le \log(2n-1).$$

Then

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \le \frac{1}{n} + \log(2n-1).$$

If we prove that $\frac{1}{n} + \log(2n-1) \le \log(2n+1)$, the induction is done.

Let $u(x) = \log(2x+1) - \log(2x-1) - \frac{1}{x}, x > \frac{1}{2}$.

$$\begin{aligned} u'(x) &= \frac{2}{2x+1} - \frac{2}{2x-1} - \frac{1}{x^2} \\ &= \frac{-4}{4x^2-1} + \frac{1}{x^2} \\ &= \frac{-1}{(4x^2-1)x^2} < 0 \end{aligned}$$

As $u(x) = \log\left(\frac{2x+1}{2x-1}\right) - \frac{1}{x}$, $\lim\limits_{x \to +\infty} u(x) = 0$. Moreover u is a decreasing function, so for all $x > 1/2, u(x) > 0$, and for all $n \in \mathbb{N}, n \ge 1$,

$$\frac{1}{n} + \log(2n-1) \le \log(2n+1).$$

We have proved by induction that for all $n \ge 1$,

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \le \log(2n+1).$$

If $n = (p-1)/2$, where p is an odd prime ($p \ge 3$),

$$\sum_{s=1}^{(p-1)/2} \frac{1}{s} \le \log p.$$

Conclusion :

$$\left| \sum_{t=n}^{m} \left(\frac{t}{p}\right) \right| < \sqrt{p} \log p.$$

$\square$

Ex. 6.16 *Let α be an algebraic number with minimal polynomial $f(x)$. Show that $f(x)$ does not have repeated roots in $\mathbb{C}$.*

Proof. Let γ a repeated root of $f(x)$. Then $f(\gamma) = f'(\gamma) = 0$, so $x - \gamma$ is a common factor of f and f'. Thus $f \wedge f' \neq 1$ ($\deg(f \wedge f') \geq 1$). Since $f \wedge f' \mid f$ and f is irreducible (with f , $f \wedge f'$ monic), we conclude $f \wedge f' = f$, so $f \mid f'$. In $\mathbb{C}$, this is impossible since $\deg(f) \geq 1$, so $f' \neq 0$, and $\deg(f') < \deg(f)$. $f(x)$ does not have repeated roots in $\mathbb{C}$. $\square$

Ex. 6.17 *Show that the minimal polynomial for $\sqrt[3]{2}$ is $x^3 - 2$.*

Proof. Let $f(x) = x^3 - 2$. Then $f(\sqrt[3]{2}) = 0$. If $f(x)$ was not irreducible, then $f(x) = u(x)v(x)$, with $1 \leq \deg(u) \leq \deg(v) \leq 2, \deg(u) + \deg(v) = \deg(f) = 3$, so $\deg(u) = 1, \deg(v) = 2$.

Then $f(x) = (ax + b)(cx^2 + dx + e),\ a, b, c, d, e \in \mathbb{Q}$. Let $w = -b/a$. Then $f(w) = w^3 - 2 = 0$ and $w \in \mathbb{Q}$, so there exist $p, q \in \mathbb{Z}$, such that $w = p/q, p \wedge q = 1$.

Thus $p^3 = 2q^3$, so p^3 is even, thefore p is even : $p = 2p', p' \in \mathbb{Z}$.

$8p'^3 = 2q^3$, $4p'^3 = q^3$, so q^3 is even, which implies that q is even. Then $2 \mid p \wedge q = 1$: this is a contradiction.

So $f(\sqrt[3]{2}) = 0$, and f is monic, irreducible : f is the minimal polynomial of $\sqrt[3]{2}$ on $\mathbb{Q}$. $\square$

Ex. 6.18 *Show that there exist algebraic numbers of arbitrarily high degree.*

Proof. As $1 + x + \cdots + x^{p-1}$ is irreducible on $\mathbb{Q}[x]$ (Prop. 6.4.1), the numbers $\zeta_p = e^{2i\pi/p}$, with p prime number, are algebraic numbers of arbitrary large degree. $\square$

Ex. 6.19 *Find the conjugates of* $\cos(2\pi/5)$.

Proof. Let $\gamma = \cos(2\pi/5)$, $\zeta = e^{2i\pi/5}$ and $\alpha = \zeta + \zeta^4, \beta = \zeta^2 + \zeta^3$.

Then $\gamma = \frac{\zeta + \zeta^{-1}}{2} = \frac{\zeta + \zeta^4}{2} = \frac{\alpha}{2}$.

$\alpha + \beta = \zeta + \zeta^4 + \zeta^2 + \zeta^3 = -1$.

$\alpha\beta = \zeta^3 + \zeta^4 + \zeta^6 + \zeta^7 = \zeta^3 + \zeta^4 + \zeta + \zeta^2 = -1$

So α, β are the two roots of $x^2 + x - 1$:

$\alpha^2 + \alpha - 1 = 0$, so $4(\alpha/2)^2 + 2(\alpha/2) - 1 = 0$: $\gamma = \alpha/2$ is a root of

$$f(x) = 4x^2 + 2x - 1.$$

As $\Delta = 4 \times 5$,the two roots of f are irrational. $\deg(f) = 2$ and f has no root in $\mathbb{Q}$, so $f(x)$ is irreducible in $\mathbb{Q}[x]$. So the minimal polynomial of $\gamma = \cos(2\pi/5)$ is $f(x) = 4x^2 + 2x - 1$. The other root of f is $\beta/2 = (\zeta^2 + \zeta^3)/2 = \cos(4pi/5)$.

Conclusion : the conjugates of $\gamma = \cos(2\pi/5)$ are $\gamma = \cos(2\pi/5)$ and $\cos(4\pi/5)$. $\square$

Ex. 6.20 *Let F be a subfield of $\mathbb{C}$ which is a finite-dimensional vector space over $\mathbb{Q}$ of degree n. Show that every element of F is algebraic of degree at most n.*

Proof. Let $\alpha \in F$, with $\dim_{\mathbb{Q}} F = n$. Any subset of $n + 1$ vectors in F is linearly dependent, so $\{1, \alpha, \alpha^2, \cdots, \alpha^n\}$ is linearly dependent.

Thus there exists $(a_0, \ldots, a_n) \in \mathbb{Q}^{n+1}, (a_0, \ldots, a_n) \neq (0, 0, \ldots, 0)$ such that $a_0 + a_1\alpha + \cdots + a_n\alpha^n = 0$.

Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$. Then $f(x) \in \mathbb{Q}[x], f(x) \neq 0$ and $f(\alpha) = 0, \deg(f(x)) \leq n$. So every element of F is algebraic of degree at most n. $\square$

Ex. 6.21 *Let $f(x) = \sum_{n=0}^\infty a_n x^n/n!$ and $g(x) = \sum_{n=0}^\infty b_n x^n/n!$ be power series with a_n and b_n integers. If p is a prime such that $p|a_i$ for $i = 0, \ldots, p-1$, show that each coefficient c_t of the product $f(x)g(x) = \sum_{n=0}^\infty c_n x^n$ for $t = 0, \ldots, p-1$ may be written in the form $p(A/B)$, $p \not| B$.*

Proof. Let $k \in \mathbb{N}, 0 \le k \le p-1$.

$$\begin{aligned} c_k &= \sum_{i+j=k} \frac{a_i}{i!}\frac{b_j}{j!} \\ &= \sum_{i=0}^k \frac{a_i}{i!}\frac{b_{k-i}}{(k-i)!} \\ &= \frac{1}{k!}\sum_{i=0}^k \binom{k}{i} a_i b_{k-i} \end{aligned}$$

As $k! \wedge p = 1$, and $\sum_{i=0}^k \binom{k}{i} a_i b_{k-i} \equiv 0 \pmod p$ for $k = 0, 1, \ldots, p-1$,
$c_k = p(A/B), p \wedge B = 1$. □

Ex. 6.22 *Show that the relation $\varepsilon \equiv 1 \pmod p$ in Proposition 6.4.4 can also be achieved by replacing x by $1+t$ instead of e^z.*

Warning : incomplete solution.

Proof. We know from Prop. 6.4.3 that

$$g(\zeta) = \varepsilon \prod_{k=1}^{(p-1)/2} (\zeta^{2k-1} - \zeta^{-(2k-1)}),$$

where $\varepsilon = \pm 1$. Let

$$f(x) = \sum_{j=1}^{p-1} \chi(j) x^j - \varepsilon \prod_{k=1}^{(p-1)/2} (x^{2k-1} - x^{p-(2k-1)}).$$

Then $f(0) = 0$ and $f(\zeta) = 0$, so $(x^p - 1)$ divides $f(x)$. As $f(x) \in \mathbb{Z}[x]$ and $x^p - 1 \in \mathbb{Z}[x]$ is monic, $f(x) = (x^p - 1)h(x), h(x) \in \mathbb{Z}[x]$. If we replace x by $1+t$, we obtain

$$f(1+t) = \sum_{j=1}^{p-1} \chi(j)(1+t)^j - \varepsilon \prod_{k=1}^{(p-1)/2} \left((1+t)^{2k-1} - (1+t)^{p-(2k-1)}\right).$$

We compute the coefficient of $t^{(p-1)/2}$ in the polynomial $f(1+t)$:

$$\begin{aligned} \sum_{j=1}^{p-1} \chi(j)(1+t)^j &= \sum_{j=1}^{p-1} \chi(j) \sum_{i=1}^{j} \binom{j}{i} t^i \\ &= \sum_{i=1}^{p-1}\sum_{j=i}^{p-1} \chi(j)\binom{j}{i} t^i \end{aligned}$$

So the coefficient of $t^{(p-1)/2}$ in $\sum\limits_{j=1}^{p-1}\chi(j)(1+t)^j$ is $\sum\limits_{j=(p-1)/2}^{p-1}\chi(j)\binom{j}{(p-1)/2}$.

$$\begin{aligned}\prod_{k=1}^{(p-1)/2}\left((1+t)^{2k-1}-(1+t)^{p-(2k-1)}\right) &= \prod_{k=1}^{(p-1)/2}\big((1+(2k-1)t)-(1+(p-(2k-1))t+t^2u(t)\big)\\ &= \prod_{k=1}^{(p-1)/2}\big((4k-2-p)t+t^2v(t)\big)\\ &= t^{(p-1)/2}\prod_{k=1}^{(p-1)/2}(4k-2-p)+t^{(p-1)/2}w(t),\end{aligned}$$

where $u(t),v(t),w(t)$ are polynomials. So the coefficient of $t^{(p-1)/2}$ in $f(1+t)$ is

$$c_{(p-1)/2}=\sum_{j=(p-1)/2}^{p-1}\chi(j)\binom{j}{(p-1)/2}-\varepsilon\prod_{k=1}^{(p-1)/2}(4k-2-p).$$

Furthermore,

$$\begin{aligned}f(1+t) &= ((1+t)^p-1)\,h(1+t)\\ &= \left[\sum_{i=1}^{p}\binom{p}{i}t^i\right]h(1+t)\\ &= \left[\sum_{i=1}^{p}i!\binom{p}{i}\frac{t^i}{i!}\right]h(1+t)\\ &= \left[\sum_{i=0}^{p}a_i\frac{t^i}{i!}\right]h(1+t),\end{aligned}$$

where $a_0=0, a_i=i!\binom{p}{i}=\frac{p!}{(p-i)!}$, so $p\mid a_i, i=0,\ldots,p-1$: the conditions of Ex.21 are verified, so $f(1+t)=\sum_{i=0}^{p-1}c_it^i$ is such that $c_{(p-1)/2}=p(A/B),\ p\nmid B$. Equating these two evaluations of $c_{(p-1)/2}$, we obtain

$$\sum_{j=(p-1)/2}^{p-1}\chi(j)\binom{j}{(p-1)/2}-\varepsilon\prod_{k=1}^{(p-1)/2}(4k-2-p)=p\frac{A}{B},\quad p\nmid B.$$

Multiplying by $B(p-1)!/2$, we obtain, as $p\nmid B$,

$$\begin{aligned}\frac{(p-1)!}{2}\sum_{j=(p-1)/2}^{p-1}\chi(j)\binom{j}{(p-1)/2} &\equiv \varepsilon\frac{(p-1)!}{2}\prod_{k=1}^{(p-1)/2}(4k-2)\\ &\equiv \varepsilon(2\cdot4\cdot6\cdots(p-1))\prod_{k=1}^{(p-1)/2}(2k-1)\quad\equiv\varepsilon(p-1)!\\ &\equiv -\varepsilon\pmod p\end{aligned}$$

To prove that $\varepsilon = +1$, it remains to prove

$$\frac{(p-1)!}{2} \sum_{j=(p-1)/2}^{p-1} \chi(j) \binom{j}{(p-1)/2} \equiv -1 \pmod p \qquad ??$$

I haven't found such a direct proof, but if we know from Prop. 6.4.4. that $\varepsilon = 1$, then this proposition is certainly true (morever, I have verified this proposition on S.A.G.E. for the primes $p < 100$). I will appreciate an answer. $\square$

Ex. 6.23 *If $f(x) = x^n + a_1 x^{n-1} + \ldots + a_n$, $a_i \in \mathbb{Z}$, and p is prime such that $p \mid a_i$ for $i = 1, \ldots, n$, and $p^2 \nmid a_n$, show that $f(x)$ is irreducible over $\mathbb{Q}$ (Eisenstein's irreducibility criterion).*

Lemma. If $f \in \mathbb{Z}[x], \deg(f) \geq 1$, is not irreducible in $\mathbb{Q}[x]$, then there exist $g, h \in \mathbb{Z}[x], \deg(g) \geq 1, \deg(h) \geq 1$ such that $f = gh$.

Proof. (lemma) Suppose that $f(x) = \sum_{k=0}^{n} a_k x^k, a_k \in \mathbb{Z}$, is not irreducible in $\mathbb{Q}[x]$.

Then $f(x) = f_1(x) f_2(x)$, with $f_1, f_2 \in \mathbb{Q}[X]$, and $\deg(f_1) \geq 1, \deg(f_2) \geq 1$. As in Ex. 6.5, we can write $f_1(x) = \lambda p(x), f_2(x) = \mu q(x)$ where $\lambda, \mu \in \mathbb{Q}$, and $p, q \in \mathbb{Z}[X]$ are primitive. Let $\nu = \lambda\mu \in \mathbb{Q}$: write $\nu = u/v, u \wedge v = 1, v \geq 1$. Then $r(x) = p(x)q(x) = \sum_{k=0}^{n} c_k x^k$ is primitive (Ex. 6.4), and $f(x) = \frac{u}{v} r(x) = \frac{u}{v} p(x) q(x)$.

As $vf(x) = ur(x)$, $v \mid uc_i$, $i = 0, 1, \ldots, n$, with $u \wedge v = 1$, so $u \mid c_i$ for all i. The polynomial r being primitive, $v \mid 1$, so $v = \varepsilon = \pm 1$.

Let $g(x) = \varepsilon u p(x), h(x) = q(x)$.Then $g, h \in \mathbb{Z}[x]$, $\deg(g) \geq 1, \deg(h) \geq 1$, and $f = gh$ is the product of two non constant polynomials in $\mathbb{Z}[x]$.

$\square$

Proof. (Ex. 6.23)

Let

$$\varphi : \left\{ \begin{array}{ccc} \mathbb{Z}[x] & \to & \mathbb{F}_p[x] \\ p(x) = a_0 + \cdots + a_n x^n & \mapsto & \overline{p}(x) = \overline{a_0} + \cdots + \overline{a_n} x^n, \end{array} \right.$$

where $\overline{a_i}$ is the class of a_i in $\mathbb{F}_p$. φ is a ring homomorphism.

We show that $f(x) = g(x)h(x)$, $g, h \in \mathbb{Z}[x], \deg(g) \geq 1, \deg(h) \geq 1$ is impossible. Indeed in such a situation,

$$\overline{f}(x) = x^n = \overline{g}(x)\overline{h}(x).$$

As the only irreducible factor of x^n is x, the unicity of the decomposition of a polynomial in irreducible factors in $\mathbb{F}_p[x]$ gives

$$\overline{g}(x) = \lambda x^i,\ \overline{h}(x) = \mu x^j,\ \lambda, \mu \in \mathbb{F}_p, i, j \in \mathbb{N}.$$

As $\deg(\overline{g}) \leq \deg(g), \deg(\overline{h}) \leq \deg(h)$ and $\deg(\overline{g}) + \deg(\overline{h}) = n = \deg(f) + \deg(g)$, this implies that $i = \deg(\overline{f}) = \deg(f)$, $j = \deg(\overline{g}) = \deg(g)$, so $i \geq 1, j \geq 1$. Therefore $p \mid g(0), p \mid h(0)$, so $p^2 \mid a_n = g(0)h(0)$, which is in contradiction with the hypothesis.

From the lemma we deduce that $f(x)$ is irreducible in $\mathbb{Q}[x]$. $\square$