

Solutions to Ireland, Rosen “A Classical Introduction to Modern Number Theory”

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Chapter 9

Ex. 9.1 If $\alpha \in \mathbb{Z}[\omega]$, show that α is congruent to either 0, 1, or -1 modulo $1 - \omega$.

Proof. Let $\lambda = 1 - \omega$, and $\alpha = a + b\omega \in D = \mathbb{Z}[\omega]$, $a, b \in \mathbb{Z}$.

$\omega \equiv 1 \pmod{\lambda}$, so $\alpha \equiv a + b \pmod{\lambda}$, $\alpha \equiv c$ with $c = a + b \in \mathbb{Z}$.

$c \equiv 0, 1, -1 \pmod{3}$, and since $\lambda \mid 3$, $z \equiv 0, 1, -1 \pmod{\lambda}$.

Every $\alpha \in D$ is congruent to either 0, 1, or -1 modulo $\lambda = 1 - \omega$.

The classes of 0, 1, -1 in $D/\lambda D$ are distinct. Indeed, $1 \not\equiv -1 \pmod{\lambda}$, if not $\lambda \mid 2$, so $2 = \lambda\lambda'$, $N(2) = N(\lambda)N(\lambda')$, thus $4 = 3N(\lambda')$, so $3 \mid 4$, which is nonsense.

$\pm 1 \equiv 0 \pmod{\lambda}$ implies $\lambda \mid 1$, so λ would be a unit, in contradiction with λ prime.

So there exist exactly three classes modulo λ in D : $|D/\lambda D| = 3 = N(\lambda)$.

□

Ex. 9.2 From now on we shall set $D = \mathbb{Z}[\omega]$ and $\lambda = 1 - \omega$. For μ in D show that we can write $\mu = (-1)^a \omega^b \lambda^c \pi_1^{a_1} \pi_2^{a_2} \cdots \pi_t^{a_t}$, where a, b, c , and the a_i are nonnegative integers and the π_i are primary primes.

Proof. Let S the set containing $\lambda = 1 - \omega$ and all primary primes.

We show that

- (a) every prime in D is associate to a prime in S ,
- (b) no two primes in S are associate.

Let π be a prime in D . There are three cases.

- If $N(\pi) = 3$, then π is associate to $\lambda \in S$, thus $\pi \in \{1 - \omega, -1\omega, -2 - \omega, 2 + \omega, 1 + 2\omega, -1 - 2\omega\}$, and no associate of λ is primary.
- If $N(\pi) = q^2$, where $q \equiv -1 \pmod{3}$, $q > 0$, is a rational prime, then π is associate to q (Proposition 9.1.2), and q is a primary prime. The primes associate to q are $q, -q, \omega q, -\omega q, -q - \omega q, q + \omega q$, so only q is primary.
- If $N(\pi) = p$, where $p \equiv 1 \pmod{4}$, then the proposition 9.1.4. shows that among the associates to π exactly one is primary.

Moreover, the norm of two primes belonging to two different cases are distinct, so two such primes are not associate.

By Theorem 3, Chapter 1, as $D = \mathbb{Z}[\omega]$ is a principal ideal domain, every $\mu \in D$ is of the form

$$\mu = u \prod_{\pi \in S} \pi^{e(\pi)},$$

where u is a unit, so $u = (-1)^a \omega^b$. Thus

$$\mu = (-1)^a \omega^b \lambda^c \pi_1^{a_1} \pi_2^{a_2} \cdots \pi_t^{a_t},$$

where the π are primary primes, and a, b, c and the a_i are nonnegative integers. \square

Ex. 9.3 Let γ a primary prime. To evaluate $\chi_\gamma(\mu)$ we see, by Exercise 2, that it is enough to evaluate $\chi_\gamma(-1), \chi_\gamma(\omega), \chi_\gamma(\lambda)$, and $\chi_\gamma(\pi)$, where π is a primary prime. Since $-1 = (-1)^3$ we have $\chi_\gamma(-1) = 1$. We now consider $\chi_\gamma(\omega)$. Let $\gamma = a + b\omega$ and set $a = 3m - 1$ and $b = 3n$. Show that $\chi_\gamma(\omega) = \omega^{m+n}$.

Proof. Let $\gamma = a + b\omega = 3m - 1 + 3n\omega$. Then $\chi_\gamma(\omega) = \omega^{\frac{N(\gamma)-1}{3}}$ (remark (b) of Theorem 1).

$$\begin{aligned} N(\gamma) - 1 &= (3m - 1)^2 + (3n)^2 - 3n(3m - 1) - 1 \\ &= 9m^2 - 6m + 9n^2 - 9nm + 3n \\ \frac{N(\gamma) - 1}{3} &= 3m^2 - 2m + 3n^2 - 3nm + n \equiv n + m \pmod{3} \end{aligned}$$

Thus, for $\gamma = a + b\omega = 3m - 1 + 3n\omega$,

$$\chi_\gamma(\omega) = \omega^{\frac{N(\gamma)-1}{3}} = \omega^{n+m}.$$

\square

Ex. 9.4 (continuation) Show that $\chi_\gamma(\omega) = 1, \omega$, or ω^2 according to whether γ is congruent to 8, 2, or 5 modulo 3λ . In particular, if q is a rational prime, $q \equiv 2 \pmod{3}$, then $\chi_q(\omega) = 1, \omega$, or ω^2 according to whether $q \equiv 8, 2$, or $5 \pmod{9}$. [Hint : $\gamma = a + b\omega = -1 + 3(m + n\omega)$, and so $\gamma \equiv -1 + 3(m + n) \pmod{3\lambda}$.]

Proof. $\lambda = 1 - \omega$, so $\omega \equiv 1 \pmod{\lambda}$. Thus

$$\begin{aligned} m + n\omega &\equiv m + n \pmod{\lambda} \\ 3(m + n\omega) &\equiv 3(m + n) \pmod{3\lambda} \\ \gamma = -1 + 3(m + n\omega) &\equiv -1 + 3(m + n) \pmod{3\lambda} \end{aligned}$$

Moreover $9 = 3\lambda\bar{\lambda} \equiv 0 \pmod{3\lambda}$, thus γ is congruent modulo 3λ to an integer between 0 and 8 of the form $3k - 1$: $\gamma \equiv 8, 2$ or $5 \pmod{3\lambda}$.

By Ex. 9.3, $\chi_\gamma(\omega) = 1 \iff m + n \equiv 0 \pmod{3}$, and $m + n \equiv 0 \pmod{3}$ implies $m + n = 3k, k \in \mathbb{Z}$, so $\gamma \equiv -1 + 9k \equiv -1 \equiv 8 \pmod{3\lambda}$.

Conversely, if $\gamma \equiv 8 \equiv -1 \pmod{3\lambda}$, then $3\lambda \mid 3(m + n)$, so $\lambda \mid m + n$, and $N(\lambda) \mid N(m + n)$, $3 \mid (m + n)^2$, thus $3 \mid m + n$, $m + n \equiv 0 \pmod{3}$, and so $\chi_\gamma(\omega) = 1$. The two other cases are similar, so we obtain

$$\begin{aligned}
\chi_\gamma(\omega) = 1 &\iff m+n \equiv 0 \pmod{3} \iff \gamma \equiv 8 \pmod{3\lambda}, \\
\chi_\gamma(\omega) = \omega &\iff m+n \equiv 1 \pmod{3} \iff \gamma \equiv 2 \pmod{3\lambda}, \\
\chi_\gamma(\omega) = \omega^2 &\iff m+n \equiv 2 \pmod{3} \iff \gamma \equiv 5 \pmod{3\lambda}.
\end{aligned}$$

If $\gamma = q$ is a rational prime, $q \equiv 8 \pmod{9}$ implies $q \equiv 8 \pmod{3\lambda}$, since $3\lambda \mid 9 = 3\lambda\bar{\lambda}$, thus $\chi_q(\omega) = 1$.

Conversely, if $\chi_q(\omega) = 1$, then $q \equiv 8 \pmod{3\lambda}$, $q - 8 = \mu(3\lambda)$, $\mu \in D$, therefore $(q - 8)^2 = N(\mu)3^3$, $3^3 \mid (q - 8)^2$, thus $3^2 \mid q - 8$ and so $q \equiv 8 \pmod{9}$. The two other cases are similar.

$$\begin{aligned}
\chi_q(\omega) = 1 &\iff q \equiv 8 \pmod{9}, \\
\chi_q(\omega) = \omega &\iff q \equiv 2 \pmod{9}, \\
\chi_q(\omega) = \omega^2 &\iff q \equiv 5 \pmod{9}.
\end{aligned}$$

□

Ex. 9.5 In the text we stated Eisenstein's result $\chi_\gamma(\lambda) = \omega^{2m}$. Show that $\chi_\gamma(3) = \omega^{2n}$.

Proof. Here $\gamma = (3m - 1) + 3n\omega$.

Note that $(1 - \omega)^2 = -3\omega$, thus $\chi_\gamma((1 - \omega)^2) = \chi_\gamma(-1)\chi_\gamma(3)\chi_\gamma(\omega)$.

Using Eisenstein's result (see a proof in Ex.24-26),

$$\chi_\gamma((1 - \omega)^2) = \chi_\gamma(\lambda^2) = \chi_\gamma(\lambda)^2 = \omega^{4m} = \omega^m.$$

As $-1 = (-1)^3$, $\chi_\gamma(-1) = 1$. Finally $\chi_\gamma(\omega) = \omega^{m+n}$ by Exercise 9.3. Thus

$$\omega^m = \chi_\gamma(3)\omega^{m+n}, \quad \chi_\gamma(3) = \omega^{-n} = \omega^{2n}.$$

In conclusion,

$$\chi_\gamma(3) = \omega^{2n}.$$

□

Ex. 9.6 Prove that

(a) $\chi_\gamma(\lambda) = 1$ for $\gamma \equiv 8, 8 + 3\omega, 8 + 6\omega \pmod{9}$.

(b) $\chi_\gamma(\lambda) = \omega$ for $\gamma \equiv 5, 5 + 3\omega, 5 + 6\omega \pmod{9}$.

(c) $\chi_\gamma(\lambda) = \omega^2$ for $\gamma \equiv 2, 2 + 3\omega, 2 + 6\omega \pmod{9}$.

Proof. Here $\gamma = -1 + 3(m + n\omega)$ is a primary prime, and $\chi_\gamma(\lambda) = \omega^{2m}$.

$$\begin{aligned}
\chi_\gamma(\lambda) = 1 &\iff m \equiv 0 \pmod{3} \Rightarrow \gamma \equiv 8 + 3n\omega \pmod{9} \Rightarrow \gamma \equiv 8, 8 + 3\omega, 8 + 6\omega \pmod{9} \\
\chi_\gamma(\lambda) = \omega &\iff m \equiv 1 \pmod{3} \Rightarrow \gamma \equiv 5 + 3n\omega \pmod{9} \Rightarrow \gamma \equiv 5, 5 + 3\omega, 5 + 6\omega \pmod{9} \\
\chi_\gamma(\lambda) = \omega^2 &\iff m \equiv 2 \pmod{3} \Rightarrow \gamma \equiv 2 + 3n\omega \pmod{9} \Rightarrow \gamma \equiv 2, 2 + 3\omega, 2 + 6\omega \pmod{9}
\end{aligned}$$

As $\chi_\gamma(\lambda) \in \{1, \omega, \omega^2\}$, these 9 cases are the only possibilities. Moreover these 9 cases are mutually exclusive, since 9 doesn't divide any difference. Thus the reciprocals are true.

$$\begin{aligned}\chi_\gamma(\lambda) = 1 &\iff \gamma \equiv 8, 8 + 3\omega, 8 + 6\omega \text{ [9]} \\ \chi_\gamma(\lambda) = \omega &\iff \gamma \equiv 5, 5 + 3\omega, 5 + 6\omega \text{ [9]} \\ \chi_\gamma(\lambda) = \omega^2 &\iff \gamma \equiv 2, 2 + 3\omega, 2 + 6\omega \text{ [9]}\end{aligned}$$

□

Ex. 9.7 Find primary primes associate to $1 - 2\omega$, $-7 - 3\omega$, and $3 - \omega$.

Proof. :

- $(1 - 2\omega)\omega = 2 + 3\omega \equiv 2 \pmod{3}$, so $2 + 3\omega$ is primary, and associate to $1 - 2\omega$.
 $N(2 + 3\omega) = 7$ and 7 is a rational prime, thus $2 + 3\omega$ is a primary prime.
- $-7 - 3\omega \equiv 2 \pmod{3}$.
 $N(-7 - 3\omega) = 37$ and 37 is a rational prime, thus $-7 - 3\omega$ is a primary prime.
- $(3 - \omega)\omega^2 = -4 - 3\omega \equiv 2 \pmod{3}$, so $-4 - 3\omega$ is primary, and associate to $3 - \omega$.
 $N(-4 - 3\omega) = 13$ and 13 is a rational prime, thus $-4 - 3\omega$ is a primary prime.

□

Ex. 9.8 Factor the following numbers into primes in D : 7, 21, 45, 22, and 143.

Proof. $7 = N(2 + 3\omega)$, thus $7 = (2 + 3\omega)(2 + 3\omega^2) = (2 + 3\omega)(-1 - 3\omega)$, where $2 + 3\omega$ and $-1 - 3\omega$ are primes in D , since their norm is a prime integer. Since these primes are primary, they are not associate.

$$21 = 3 \times 7 = -\omega^2 \lambda^2 (2 + 3\omega)(-1 - 3\omega) \text{ since } 3 = -\omega^2(1 - \omega)^2.$$

$$45 = 3^2 \times 5 = \omega \lambda^4 5, \text{ where } 5 \equiv 2 \pmod{3} \text{ is a primary prime in } D.$$

$$22 = 2 \times 11, \text{ where 2 and 11 are primes in } D.$$

$$143 = 11 \times 13 = 11(-4 - 3\omega)(-4 - 3\omega^2) = 11(-4 - 3\omega)(-1 + 3\omega).$$

□

Ex. 9.9 Show that $\bar{\alpha} \neq 0$, the residue class of α , is a cube in the field $D/\pi D$ iff $\alpha^{(N\pi-1)/3} \equiv 1 \pmod{\pi}$. Conclude that there are $(N\pi - 1)/3$ cubes in $(D/\pi D)^*$.

Solution 1:

Proof. Let π be a prime in D , $N\pi \neq 3$, and $\alpha \in D, \pi \nmid \alpha$.

$\bar{\alpha}$ is a cube in $(D/\pi D)^*$

$$\iff x^3 \equiv \alpha \pmod{\pi} \text{ has a solution in } D$$

$$\iff \chi_\pi(\alpha) = 1 \quad (\text{by Prop. 9.3.3(a)})$$

$$\iff \alpha^{\frac{N\pi-1}{3}} \equiv 1 \pmod{\pi}$$

$$\iff \bar{\alpha}^{\frac{N\pi-1}{3}} = \bar{1}.$$

The cubes in $(D/\pi D)^*$ are then the roots of the polynomial $f(x) = x^{\frac{N\pi-1}{3}} - \bar{1}$ in $D/\pi D$.

Let q be the cardinal of the field $D/\pi D$. Since $q = |D/\pi D| = N\pi$, $\frac{N\pi-1}{3} \mid q-1$, $f(x) \mid x^{q-1} - 1 \mid x^q - x$. By Corollary 2 of Proposition 7.1.1, f has $\deg(f) = \frac{N\pi-1}{3}$ roots. Conclusion: there are exactly $\frac{N\pi-1}{3}$ cubes in $(D/\pi D)^*$. \square

Solution 2:

Proof. Let $\varphi : (D/\pi D)^* \rightarrow (D/\pi D)^*$ be the group homomorphism defined by $\varphi(x) = x^3$. Then $\text{im}(\varphi)$ is the set of cubes in $(D/\pi D)^*$.

The equation $x^3 = \bar{1}$ has three distinct solutions $\bar{1}, \bar{\omega}, \bar{\omega}^2$ in $D/\pi D$ if $N\pi \neq 3$ (see the demonstration of Proposition 9.3.1).

Thus $\ker(\varphi) = \{\bar{1}, \bar{\omega}, \bar{\omega}^2\}$ and $|\ker(\varphi)| = 3$. Therefore $|\text{im}(\varphi)| = |(D/\pi D)^*|/|\ker(\varphi)| = (N\pi - 1)/3$. There exist exactly $\frac{N\pi-1}{3}$ cubes in $(D/\pi D)^*$. \square

Note: if $N\pi = 3$, that is to say, if π is associate to $1 - \omega$, $D/\pi D = \{\bar{0}, \bar{1}, \bar{2}\}$. As $\bar{1}^3 = \bar{1}, \bar{2}^3 = \bar{2}$, all the elements of $(D/\pi D)^*$ are cubes.

Ex. 9.10 What is the factorisation of $x^{24} - 1$ in $D/5D$.

Proof. $|(D/5D)^*| = N(5) - 1 = 24$, thus $x^{24} - 1 = \prod_{\alpha \in (D/5D)^*} (x - \alpha)$.

(where the $\alpha \in (D/5D)^*$ are of the form $\alpha = a + b[\omega]$, $0 \leq a < 5, 0 \leq b < 5, (a, b) \neq (0, 0)$). \square

Ex. 9.11 How many cubes are there in $D/5D$?

Proof. By Exercise 9.9, there exist $(N(5) - 1)/3 = 8$ cubes in $(D/5D)^*$ (and $0 = 0^3$ is a cube). \square

Ex. 9.12 Show that $\omega\lambda$ has order 8 in $D/5D$ and that $\omega^2\lambda$ has order 24. [Hint : Show first that $(\omega\lambda)^2$ has order 4.]

Proof. If $\alpha = (\omega\lambda)^2$, then

$$\alpha = (\omega\lambda)^2 = \omega^2(1 - \omega)^2 = \omega^2(1 + \omega^2 - 2\omega) = -3\omega^3 = -3.$$

So $\alpha^2 = 9 \equiv -1 \pmod{5}$, $\alpha^4 \equiv 1 \pmod{5}$ and $\alpha^2 \not\equiv 1 \pmod{5}$, thus the class of $\alpha = (\omega\lambda)^2$ has order 4 in $(D/5D)^*$, and this implies that $\omega\lambda$ has order 8.

Let $\beta = \omega^2\lambda$. $|(D/5D)^*| = 24$, thus $[\beta]^{24} = 1$ (where $[\beta]$ is the class of β in $D/5D$.)

To verify that $[\beta]$ has order 24, it is sufficient to show that $[\beta]^8 \neq 1, [\beta]^{12} \neq 1$.

$$\beta^8 = \omega^{16}\lambda^8 = \omega\lambda^8 = (\omega\lambda)^8\omega^2 \equiv \omega^2 \not\equiv 1 \pmod{5}.$$

$\beta^{12} = (\omega^2\lambda)^{12} = \lambda^{12} = (\omega\lambda)^{12} \equiv (\omega\lambda)^4 \equiv -1 \pmod{5}$ (since $(\omega\lambda)$ has order 8 in $D/5D$).

Conclusion: $\omega\lambda$ has order 8, $\omega\lambda^2$ has order 24 in $(D/5D)^*$. \square

Ex. 9.13 Show that π is a cube in $D/5D$ iff $\pi \equiv 1, 2, 3, 4, 1 + 2\omega, 2 + 4\omega, 3 + \omega, \text{ or } 4 + 3\omega \pmod{5}$.

Proof. Let $\pi \in D, [\pi] \neq 0$. Then $[\pi]$ is a cube in $D/5D$ iff $[\pi]^{(q^2-1)/3} = 1$, with $q = 5$, namely $[\pi]^8 = 1$ (Prop. 7.1.2, where $3 \mid q^2 - 1 = 24 = |(D/5D)^*|$).

By Exercise 9.12, the class of $\gamma = \omega\lambda$ has order 8, thus the 8 elements $[\gamma]^k, 0 \leq k \leq 7$ are distinct roots of the polynomial $x^8 - 1$, which has at most 8 roots. Therefore the subgroup of cubes in $(D/5D)^*$ is

$$\{1, [\gamma], [\gamma]^2, \dots, [\gamma]^7\}.$$

$\gamma = \omega(1 - \omega) = \omega + 1 + \omega = 1 + 2\omega$, so

$$\begin{aligned} \gamma^0 &= 1 \\ \gamma^1 &= 1 + 2\omega \\ \gamma^2 &\equiv -3 \equiv 2 \pmod{5} && (\text{Ex. 9.12}) \\ \gamma^3 &= -3 - 6\omega \equiv 2 + 4\omega \pmod{5} \\ \gamma^4 &\equiv -1 \equiv 4 \pmod{5} \\ \gamma^5 &\equiv -1 - 2\omega \equiv 4 + 3\omega \pmod{5} \\ \gamma^6 &\equiv 3 \pmod{5} \\ \gamma^7 &\equiv 3 + 6\omega \equiv 3 + \omega \pmod{5} \end{aligned}$$

Conclusion: If $\pi \not\equiv 0 \pmod{5}$, $\pi \equiv \alpha^3 \pmod{5}, \alpha \in D$ iff

$$\pi \equiv 1, 2, 3, 4, 1 + 2\omega, 2 + 4\omega, 3 + \omega, 4 + 3\omega \pmod{5}.$$

□

Ex. 9.14 For which primes $\pi \in D$ is $x^3 \equiv 5 \pmod{\pi}$ solvable?

Proof. If π is associate to 5, then $5^3 \equiv 0 \equiv 5 \pmod{\pi}$, so $x^3 \equiv 5 \pmod{\pi}$ is solvable.

If π is a primary prime not associate to 5, the Law of Cubic Reciprocity gives

$$\begin{aligned} 5 \equiv x^3 \pmod{\pi}, x \in D &\iff \chi_\pi(5) = 1 \\ &\iff \chi_5(\pi) = 1 \\ &\iff \pi \text{ is a cube in } D/5D \\ &\iff \pi \equiv 1, 2, 3, 4, 1 + 2\omega, 2 + 4\omega, 3 + \omega, 4 + 3\omega \pmod{5} \end{aligned}$$

(see Ex. 9.13)

Conclusion: the equation $5 \equiv x^3 \pmod{\pi}, x \in D$ is solvable iff the primary prime associate to π is congruent modulo 5 to 1, 2, 3, 4, $1 + 2\omega, 2 + 4\omega, 3 + \omega, 4 + 3\omega$ (or 0).

Examples:

- $q = 23$ is a primary prime congruent to 3 modulo 5, thus the equation $x^3 \equiv 5 \pmod{23}$ has a solution $x \in D$ ($x = 19$).

- $-4 - 3\omega$ is the primary prime associate to the prime $3 - \omega$, and $-4 - 3\omega \equiv 1 + 2\omega \pmod{5}$, thus the equation $x^3 \equiv 5 \pmod{3 - \omega}$ has a solution $a + b\omega \in \mathbb{Z}[\omega]$.

Indeed, $7^3 \equiv 8^3 \equiv 11^3 \equiv 5 \pmod{13}$, and $3 - \omega \mid 13$, so $7^3 \equiv 8^3 \equiv 11^3 \equiv 5 \pmod{3 - \omega}$. □

Ex. 9.15 Suppose that $p \equiv 1 \pmod{3}$ and that $p = \pi\bar{\pi}$, where π is a primary prime in D . Show that $x^3 \equiv a \pmod{p}$ is solvable in \mathbb{Z} iff $\chi_\pi(a) = 1$. We assume that $a \in \mathbb{Z}$.

Proof. Since $\pi \mid p$, if $x^3 \equiv a \pmod{p}$, $x \in \mathbb{Z}$, then $x^3 \equiv a \pmod{\pi}$, thus $\chi_\pi(a) = 1$.

Conversely, suppose that $\chi_\pi(a) = 1$. Then the equation $y^3 \equiv a \pmod{\pi}$ has a solution $y = u + v\omega$, $u, v \in \mathbb{Z}$. Moreover, the class of y has a representative $x \in \mathbb{Z}$ modulo π (see the proof of Proposition 9.2.1) :

$$y \equiv x \pmod{\pi}, x \in \mathbb{Z}.$$

So $x^3 \equiv a \pmod{\pi}$ has a solution $x \in \mathbb{Z}$.

Thus $\pi \mid x^3 - a$, $N(\pi) = p \mid (x^3 - a)^2$, therefore $p \mid x^3 - a$ in \mathbb{Z} , and so $x^3 \equiv a \pmod{p}$.

Conclusion: if $p \equiv 1 \pmod{3}$, $p = \pi\bar{\pi}$, where π is a primary prime, and $a \in \mathbb{Z}$,

$$\exists x \in \mathbb{Z}, x^3 \equiv a \pmod{p} \iff \chi_\pi(a) = 1.$$

In other words, $x^3 \equiv a \pmod{\pi}$ is solvable in D iff it is solvable in \mathbb{Z} . □

Ex. 9.16 Is $x^3 \equiv 2 - 3\omega \pmod{11}$ solvable? Since $D/11D$ has 121 elements this is hard to resolve by straightforward checking. Fill in the details of the following proof that it is not solvable. $\chi_\pi(2 - 3\omega) = \chi_{2-3\omega}(11)$ and so we shall have a solution iff $x^3 \equiv 11 \pmod{2 - 3\omega}$ is solvable. This congruence is solvable iff $x^3 \equiv 11 \pmod{7}$ is solvable in \mathbb{Z} . However, $x^3 \equiv a \pmod{7}$ is solvable in \mathbb{Z} iff $a \equiv 1$ or $6 \pmod{7}$.

Warning: false sentence, since

$$N(2 - 3\omega) = (2 - 3\omega)(2 - 3\omega^2) = 4 + 9 - 6(\omega + \omega^2) = 4 + 9 + 6 = 19 \text{ (and not 7!)}.$$

Proof. Since 19 is a rational prime, and since $\pi = 2 - 3\omega$ and 11 are primary primes, by the Law of Cubic Reciprocity, and by Exercise 9.15 (with $p = 11 \equiv 1 \pmod{3}$),

$$\begin{aligned} \exists x \in D, 2 - 3\omega \equiv x^3 [11] &\iff \chi_{11}(2 - 3\omega) = 1 \\ &\iff \chi_{2-3\omega}(11) = 1 \\ &\iff \exists x \in D, x^3 \equiv 11 [2 - 3\omega] \\ &\iff \exists x \in \mathbb{Z}, x^3 \equiv 11 [19] \end{aligned}$$

Moreover, by Proposition 7.1.2 (with $p = 19$, $d = (p - 1) \wedge 3 = 3$, $(p - 1)/d = 6$),

$$\exists x \in \mathbb{Z}, x^3 \equiv 11 [19] \iff 11^6 \equiv 1 \pmod{19},$$

which is true : $11^6 = 121^3 = (19 \times 6 + 7)^3 \equiv 49 \times 7 \equiv 11 \times 7 \equiv 77 \equiv 1 [19]$.

Conclusion: there exists $x \in D$ such that $2 - 3\omega \equiv x^3 \pmod{11}$.

With some computer code, we find a solution $x = 1 + 8\omega$ (and its associates $\omega^2 x = 7 - \omega$, $\omega x = -8 - 7\omega \equiv 3 + 4\omega \pmod{11}$) :

$$x^3 = (1 + 8\omega)^3 = 321 - 168\omega \equiv 2 - 3\omega \pmod{11}.$$

□

Note: The sentence becomes true if we replace $2 - 3\omega$ by the primary prime $2 + 3\omega$. Since $N(2 + 3\omega) = 7$, with the same reasoning,

$$\begin{aligned}\exists x \in D, 2 + 3\omega \equiv x^3 \pmod{11} &\iff \chi_{2+3\omega}(11) = 1 \\ &\iff \exists x \in D, x^3 \equiv 11 \pmod{2 + 3\omega} \\ &\iff \exists x \in \mathbb{Z}, x^3 \equiv 11 \equiv 4 \pmod{7} \\ &\iff 4^2 \equiv 1 \pmod{7}\end{aligned}$$

but $4^2 \equiv 2 \not\equiv 1 \pmod{7}$, so the equation $x^3 \equiv 2 + 3\omega \pmod{11}$ is not solvable.

($x^3 \equiv a \pmod{11}$ is solvable in \mathbb{Z} iff $a^{\frac{7-1}{3}} = a^2 \equiv 1 \pmod{7}$ iff $a \equiv \pm 1 \pmod{7}$.)

Ex. 9.17 An element $\gamma \in D$ is called primary if $\gamma \equiv 2 \pmod{3}$. If γ and ρ are primary, show that $-\gamma\rho$ is primary. If γ is primary, show that $\gamma = \pm\gamma_1\gamma_2 \dots \gamma_t$, where the γ_i are (not necessarily distinct) primary primes.

Proof. If $\gamma \equiv 2, \rho \equiv 2 \pmod{3}$, then $-\gamma\rho \equiv -2 \times 2 \equiv 2 \pmod{3}$, so $-\gamma\rho$ is primary.

By Ex. 9.2, γ can be written

$$\gamma = (-1)^a \omega^b \lambda^c \pi_1^{a_1} \dots \pi_t^{a_t},$$

where $\pi_i \equiv 2 \pmod{3}, a \in \{0, 1\}, b \in \{0, 1, 2\}$.

As $\pi_i \equiv -1 \pmod{3}$, and $\gamma \equiv -1 \pmod{3}$, we obtain $\omega^b \lambda^c \equiv \pm 1 \pmod{3}$. We prove that $b = c = 0$.

Note that $\lambda^2 = (1 - \omega)^2 = -3\omega \equiv 0 \pmod{3}$. If $c \geq 2$, we would obtain $\gamma \equiv 0 \pmod{3}$, in contradiction with the hypothesis, thus $c = 0$ or $c = 1$.

If $c = 1$,

$$\omega^b \lambda^c \in \{1 - \omega, \omega(1 - \omega), \omega^2(1 - \omega)\} = \{1 - \omega, 1 + 2\omega, -2 - \omega\}.$$

Since $1 - \omega \not\equiv \pm 1, 1 + 2\omega \not\equiv \pm 1, -2 - \omega \not\equiv \pm 1 \pmod{3}$, this is impossible, so $c = 0$.

Then $\omega^b \equiv \pm 1 \pmod{3}$, where $\omega^b \in \{1, \omega, -1 - \omega\}$. Since $\omega \not\equiv \pm 1 \pmod{3}$, and $-1 - \omega \not\equiv \pm 1 \pmod{3}$, then $\omega^b = 1, 0 \leq b \leq 2$, thus $b = 0$.

Finally, $\gamma = (-1)^a \pi_1^{a_1} \dots \pi_t^{a_t}$.

Conclusion: Every primary $\gamma \in D$ is under the form

$$\gamma = \pm\gamma_1\gamma_2 \dots \gamma_t,$$

where the γ_i are primary primes. □

Ex. 9.18 (continuation) If $\gamma = \pm\gamma_1\gamma_2 \dots \gamma_t$ is a primary decomposition of the primary element γ , define $\chi_\gamma(\alpha) = \chi_{\gamma_1}(\alpha)\chi_{\gamma_2}(\alpha) \dots \chi_{\gamma_t}(\alpha)$. Prove that $\chi_\gamma(\alpha) = \chi_\gamma(\beta)$ if $\alpha \equiv \beta \pmod{\gamma}$ and $\chi_\gamma(\alpha\beta) = \chi_\gamma(\alpha)\chi_\gamma(\beta)$. If ρ is primary, show that $\chi_\rho(\alpha)\chi_\gamma(\alpha) = \chi_{-\rho\gamma}(\alpha)$.

Proof. If $\alpha \equiv \beta \pmod{\gamma}$, then $\alpha \equiv \beta \pmod{\gamma_i}, 1 \leq i \leq t$, so $\chi_{\gamma_i}(\alpha) = \chi_{\gamma_i}(\beta)$, thus $\chi_\gamma(\alpha) = \chi_\gamma(\beta)$.

By Proposition 9.3.3,

$$\begin{aligned}\chi_\gamma(\alpha\beta) &= \chi_{\gamma_1}(\alpha\beta)\chi_{\gamma_2}(\alpha\beta) \dots \chi_{\gamma_t}(\alpha\beta) \\ &= \chi_{\gamma_1}(\alpha)\chi_{\gamma_2}(\alpha) \dots \chi_{\gamma_t}(\alpha)\chi_{\gamma_1}(\beta)\chi_{\gamma_2}(\beta) \dots \chi_{\gamma_t}(\beta) \\ &= \chi_\gamma(\alpha)\chi_\gamma(\beta)\end{aligned}$$

Finally, if $\rho = \pm \rho_1 \rho_2 \cdots \rho_l$ is primary, then $-\rho\gamma = \pm \rho_1 \rho_2 \cdots \rho_l \gamma_1 \gamma_2 \cdots \gamma_t$ is primary by Ex. 9.17, therefore

$$\chi_{-\rho\gamma}(\alpha) = (\chi_{\rho_1} \chi_{\rho_2} \cdots \chi_{\rho_l} \chi_{\gamma_1} \chi_{\gamma_2} \cdots \chi_{\gamma_t})(\alpha) = \chi_{\rho}(\alpha) \chi_{\gamma}(\alpha).$$

□

Note: The unit -1 is primary by definition, and -1 is the opposite of the empty product, so for all α in D , $\chi_{-1}(\alpha) = 1$ by definition. The result of the exercises remain true if we accept the unit -1 as a primary element.

Ex. 9.19 Suppose that $\gamma = A + B\omega$ is primary and that $A = 3M - 1$ and $B = 3N$. Prove that $\chi_{\gamma}(\omega) = \omega^{M+N}$ and that $\chi_{\gamma}(\lambda) = \omega^{2M}$.

Proof. We verify first that if $\gamma = -\gamma_1 \gamma_2$, with

$$\begin{aligned} \gamma &= A + B\omega, & A &= 3M - 1, & B &= 3N, \\ \gamma_1 &= A_1 + B_1\omega, & A_1 &= 3M_1 - 1, & B_1 &= 3N_1, \\ \gamma_2 &= A_2 + B_2\omega, & A_2 &= 3M_2 - 1, & B_2 &= 3N_2, \end{aligned}$$

then $M \equiv M_1 + M_2 \pmod{3}$, $N \equiv N_1 + N_2 \pmod{3}$.

$$-\gamma_1 \gamma_2 = -A_1 A_2 + B_1 B_2 + (-A_1 B_2 - A_2 B_1 + B_1 B_2)\omega = A + B\omega,$$

therefore

$$3M - 1 = A = -A_1 A_2 + B_1 B_2 \equiv 3(M_1 + M_2) - 1 \pmod{9},$$

thus $M \equiv M_1 + M_2 \pmod{3}$.

$$3N = B = -A_1 B_2 - A_2 B_1 + B_1 B_2 \equiv 3(N_1 + N_2) \pmod{9},$$

thus $N \equiv N_1 + N_2 \pmod{3}$.

By induction, if $\gamma = \pm \gamma_1 \gamma_2 \cdots \gamma_t = (-1)^{t-1} \gamma_1 \gamma_2 \cdots \gamma_t$, where $\gamma_i = A_i + B_i \omega$, $A_i = 3M_i - 1$, $B_i = 3N_i$, then

$$M \equiv M_1 + \cdots + M_t \pmod{3}, N \equiv N_1 + \cdots + N_t \pmod{3}.$$

By Exercise 9.3,

$$\begin{aligned} \chi_{\gamma}(\omega) &= \chi_{\gamma_1}(\omega) \cdots \chi_{\gamma_t}(\omega) \\ &= \omega^{M_1+N_1} \cdots \omega^{M_t+N_t} \\ &= \omega^{(M_1+\cdots+M_t)+(N_1+\cdots+N_t)} \\ &= \omega^{M+N}, \end{aligned}$$

and by Eisenstein's result,

$$\begin{aligned} \chi_{\gamma}(\lambda) &= \chi_{\gamma_1}(\lambda) \cdots \chi_{\gamma_t}(\lambda) \\ &= \omega^{2M_1} \cdots \omega^{2M_t} \\ &= \omega^{2(M_1+\cdots+M_t)} \\ &= \omega^{2M}. \end{aligned}$$

Conclusion: if $\gamma = 3M - 1 + 3N\omega$, then

$$\chi_{\gamma}(\omega) = \omega^{M+N}, \chi_{\gamma}(\lambda) = \omega^{2M}.$$

□

Ex. 9.20 If γ and ρ are primary, show that $\chi_\gamma(\rho) = \chi_\rho(\gamma)$.

Important note: The following solution assumes that $\chi_{\pi_2}(\pi_1) = \chi_{\pi_1}(\pi_2)$ for any pair π_1, π_2 of primary primes. But Theorem 1 uses the hypothesis $N(\pi_1) \neq N(\pi_2)$ to prove Cubic Reciprocity.

We can complete the proof in the case where $N(\pi_1) = N(\pi_2)$. Since π_1, π_2 are primary primes, then $\pi_1 = \pi_2$ or $\pi_1 = \overline{\pi_2}$.

In the case $\pi_1 = \pi_2$, then $\chi_{\pi_2}(\pi_1) = 0 = \chi_{\pi_1}(\pi_2)$.

To prove that $\left(\frac{\overline{\pi}}{\pi}\right) = \left(\frac{\pi}{\overline{\pi}}\right)$, we begin with a particular case of the proposition:

Lemma. Let $n \in \mathbb{Z}$ be a primary element in A , and let π be a primary prime such that $N(\pi) = p \equiv 1 \pmod{3}$. Then

$$\left(\frac{n}{\pi}\right)_3 = \left(\frac{\pi}{n}\right)_3.$$

Proof. If $p \mid n$, then $\left(\frac{n}{\pi}\right)_3 = 0 = \left(\frac{\pi}{n}\right)_3$. Now we assume that $p \nmid n$.

The decomposition of n is of the form

$$\begin{aligned} n &= \pm p_1 \cdots p_s q_1 \cdots q_r \quad (p_i \equiv 1 \pmod{3}, q_j \equiv -1 \pmod{3}) \\ &= \pm \pi_1 \overline{\pi_1} \cdots \pi_s \overline{\pi_s} q_1 \cdots q_r, \end{aligned}$$

where $\pi_i, \overline{\pi_i} (1 \leq i \leq s)$ and $q_j (1 \leq j \leq r)$ are primary prime.

Since $N(\pi_i) = p \neq p_i$ and $N(\pi) = p \neq N(q_j) = q_j^2$, Theorem 1 shows that

$$\begin{aligned} \left(\frac{n}{\pi}\right)_3 &= \left(\frac{\pi_1}{\pi}\right)_3 \left(\frac{\overline{\pi_1}}{\pi}\right)_3 \cdots \left(\frac{\pi_s}{\pi}\right)_3 \left(\frac{\overline{\pi_s}}{\pi}\right)_3 \left(\frac{q_1}{\pi}\right)_3 \cdots \left(\frac{q_r}{\pi}\right)_3 \\ &= \left(\frac{\pi}{\pi_1}\right)_3 \left(\frac{\pi}{\overline{\pi_1}}\right)_3 \cdots \left(\frac{\pi}{\pi_s}\right)_3 \left(\frac{\pi}{\overline{\pi_s}}\right)_3 \left(\frac{\pi}{q_1}\right)_3 \cdots \left(\frac{\pi}{q_r}\right)_3 \\ &= \left(\frac{\pi}{n}\right)_3. \end{aligned}$$

□

We can now remove the useless hypothesis $N(\pi_1) \neq N(\pi_2)$ in Theorem 1.

Proposition. Let π_1, π_2 be primary primes. Then

$$\left(\frac{\pi_2}{\pi_1}\right)_3 = \left(\frac{\pi_1}{\pi_2}\right)_3.$$

Proof. By theorem 1, it remains only the case where $N(\pi_1) = N(\pi_2)$.

If $\pi_1 = \pi_2$, then $\left(\frac{\pi_2}{\pi_1}\right)_3 = \left(\frac{\pi_1}{\pi_2}\right)_3 = 0$.

If $\pi_1 \neq \pi_2$, since π_1 et π_2 are primary, then π_1, π_2 are primes such that $N(\pi_1) = N(\pi_2) = p \equiv 1 \pmod{3}$, and $\pi_2 = \overline{\pi_1}$. Writing $\pi = \pi_1$, it is sufficient to prove

$$\left(\frac{\overline{\pi}}{\pi}\right)_3 = \left(\frac{\pi}{\overline{\pi}}\right)_3.$$

We use the “Evans’ trick” (see [Lemmermayer, Reciprocity Laws p. 215]). The element $n = -(\pi + \bar{\pi})$ is a rational integer, which is primary. The Lemma gives then

$$\begin{aligned}
\left(\frac{\bar{\pi}}{\pi}\right)_3 &= \left(\frac{\pi + \bar{\pi}}{\pi}\right)_3 \\
&= \left(\frac{-\pi - \bar{\pi}}{\pi}\right)_3 \\
&= \left(\frac{\pi}{-\pi - \bar{\pi}}\right)_3 \\
&= \left(\frac{-\bar{\pi}}{-\pi - \bar{\pi}}\right)_3 \\
&= \left(\frac{\bar{\pi}}{-\pi - \bar{\pi}}\right)_3 \\
&= \left(\frac{-\pi - \bar{\pi}}{\bar{\pi}}\right)_3 \\
&= \left(\frac{-\pi}{\bar{\pi}}\right)_3 \\
&= \left(\frac{\pi}{\bar{\pi}}\right)_3
\end{aligned}$$

□

We can now give the solution of Exercise 9.20.

Proof. ρ, γ are written

$$\begin{aligned}
\rho &= \pm \rho_1 \rho_2 \cdots \rho_l, \\
\gamma &= \pm \gamma_1 \gamma_2 \cdots \gamma_m,
\end{aligned}$$

where ρ_i, γ_i are primary primes. By the law of Cubic Reciprocity, we obtain

$$\begin{aligned}
\chi_\gamma(\rho) &= \prod_{j=1}^m \chi_{\gamma_j}(\rho) \\
&= \prod_{j=1}^m \prod_{i=1}^l \chi_{\gamma_j}(\rho_i) \\
&= \prod_{i=1}^l \prod_{j=1}^m \chi_{\gamma_j}(\rho_i) \\
&= \prod_{i=1}^l \prod_{j=1}^m \chi_{\rho_i}(\gamma_j) \\
&= \prod_{i=1}^l \chi_{\rho_i}(\gamma) \\
&= \chi_\rho(\gamma).
\end{aligned}$$

(if $\gamma = -1$, or $\rho = -1$, some products are empty, but the result remains true: $\chi_{-1}(\rho) = 1 = \chi_\rho(-1)$.)

□

Ex. 9.21 If γ is primary, show that there are infinitely many primary primes π such that $x^3 \equiv \gamma \pmod{\pi}$ is not solvable. Show also that there are infinitely many primary primes π such that $x^3 \equiv \omega \pmod{\pi}$ is not solvable and the same for $x^3 \equiv \lambda \pmod{\pi}$. (Hint: Imitate the proof of Theorem 3 of Chapter 5.)

Proof. a) As some primary elements of D may be cubes, by example $53 + 36\omega = (-1 + 3\omega)^3$, we must of course suppose that γ is not the cube of some element of D (in the contrary case $x^3 \equiv \gamma \pmod{\pi}$ is solvable for all prime π).

Note first that for all primes π in D , there exists $\sigma \in D$ such that $\chi_\pi(\sigma) = \omega$. Indeed, there exist $(N\pi - 1)/3$ cubes in $(D/\pi D)^*$, which has $N\pi - 1$ elements, so there exists an element $\bar{\tau} \in (D/\pi D)^*$ which is not a cube, therefore there exists $\tau \in D$ such that $\chi_\pi(\tau) \neq 1$. If $\chi_\pi(\tau) = \omega$, we put $\sigma = \tau$ and if $\chi_\pi(\tau) = \omega^2$, we put $\sigma = \tau^2$. In the two cases, $\chi_\pi(\sigma) = \omega$.

Let $\gamma \in D$, where γ is primary. Then $\gamma = \pm \gamma_1^{n_1} \gamma_2^{n_2} \cdots \gamma_p^{n_p}$, where the γ_i are distinct primary primes. Write $n_i = 3q_i + r_i$, $r_i \in \{0, 1, 2\}$. Then grouping in γ' the γ^{r_i} such that $r_i \neq 0$, we can write $\gamma = \delta^3 \gamma'$, $\gamma' = \gamma_1^{r_1} \gamma_2^{r_2} \cdots \gamma_l^{r_l}$, $r_i \in \{1, 2\}$, $\delta = \pm \gamma_1^{q_1} \cdots \gamma_p^{q_p} \in D$ (-1 is a cube). Since by hypothesis γ is not a cube, $l \geq 1$. Moreover the equation $x^3 \equiv \gamma \pmod{\pi}$ is solvable iff $x^3 \equiv \gamma' \pmod{\pi}$ is solvable. We may then suppose that

$$\gamma = \gamma_1^{r_1} \gamma_2^{r_2} \cdots \gamma_l^{r_l}, 1 \leq r_i \leq 2,$$

without cubic factors.

Note that the γ_i are not associate to $\lambda = 1 - \omega$ (see Ex. 9.17).

Let $A = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ a set (possibly empty) of distinct primary primes λ_i (therefore they are not associate), and not associate neither to γ_i , $1 \leq i \leq l$, nor to $\lambda = 1 - \omega$.

We will show that we can find a primary prime λ_{k+1} distinct of the λ_i with the same properties and such that the equation $x^3 \equiv \lambda \pmod{\lambda_{k+1}}$ is not solvable. This will prove the existence of infinitely many primes π such that the equation $x^3 \equiv \lambda \pmod{\pi}$ is not solvable.

Using the initial note, let $\sigma \in D$ such that $\chi_{\gamma_l}(\sigma) = \omega$. As D is a principal ideal domain, the Chinese Remainder Theorem is valid. Since $3 = \lambda \bar{\lambda} = -\omega^2 \lambda^2$ is relatively prime to γ_i, λ_i , there exists $\beta \in D$ such that

$$\begin{aligned} \beta &\equiv 2 \pmod{3}, \\ \beta &\equiv 1 \pmod{\lambda_i} & (1 \leq i \leq k), \\ \beta &\equiv 1 \pmod{\gamma_i} & (1 \leq i \leq l-1), \\ \beta &\equiv \sigma \pmod{\gamma_l}. \end{aligned}$$

The first equation show that β is primary, so $\beta = (-1)^{m-1} \beta_1 \cdots \beta_m$, where the β_i are primary primes.

By Exercise 9.20,

$$\chi_\beta(\gamma) = \chi_\beta(\gamma_1)^{r_1} \cdots \chi_\beta(\gamma_l)^{r_l} = \chi_{\gamma_1}(\beta)^{r_1} \cdots \chi_{\gamma_l}(\beta)^{r_l}.$$

As $\chi_{\gamma_i}(1) = 1$ ($1 \leq i \leq l-1$), and $\chi_{\gamma_l}(\beta) = \chi_{\gamma_l}(\sigma) = \omega$, we obtain $\chi_\beta(\gamma) = \omega^{r_l} \neq 1$, since $r_l = 1$ or $r_l = 2$.

By Exercise 9.18, $\chi_\rho(\alpha)\chi_\gamma(\alpha) = \chi_{-\rho\gamma}(\alpha)$, with primary ρ, γ , so by induction, as $\beta = (-1)^{m-1}\beta_1 \cdots \beta_m$,

$$\chi_\beta(\gamma) = \chi_{\beta_1}(\gamma) \cdots \chi_{\beta_m}(\gamma) \neq 1.$$

Thus there exists a subscript j such that $\chi_{\beta_j}(\gamma) \neq 1$.

We can then take $\lambda_{k+1} = \beta_j$. Indeed, since $\beta \equiv 1 [\lambda_i]$ and $\beta \not\equiv 0 [\gamma_i]$, β_j is distinct of the λ_i and γ_i , and β_j is not associate to λ since $\beta \equiv 2 \pmod{3}$.

As $\chi_{\lambda_{k+1}}(\gamma) \neq 1$, the equation $x^3 \equiv \gamma [\lambda_{k+1}]$ is not solvable, so λ_{k+1} is convenient.

Conclusion : if $\gamma \in D$ is primary and is not a cube in D , there exist infinitely many primes $\pi \in D$ such that the equation $x^3 \equiv \gamma [\pi]$ is not solvable.

- b) We show that $x^3 \equiv \omega [\pi]$ has no solution for infinitely many primes π .

To initialize the induction, we display such a prime π , namely $\pi = 2 + 3\omega$. Indeed, $N(\pi) = 4 + 9 - 6 = 7$, 7 is a rational prime, so π is a primary prime in D , of the form $\pi = 3m - 1 + 3n\omega$, with $n = m = 1$, so $\chi_\pi(\omega) = \omega^{m+n} = \omega^2 \neq 1$: the equation $x^3 \equiv \omega [\pi]$ is not solvable. Moreover π is not associate to $\lambda = 1 - \omega$.

Suppose now the existence of a set $A = \{\lambda_1, \lambda_2, \dots, \lambda_l\}, l \geq 1$, of distinct primary primes λ_i , not associate to λ and such the equation $x^3 \equiv \omega [\lambda_i]$ is not solvable for each i , $1 \leq i \leq l$. We will show that we can add a prime λ_{l+1} to the set A with the same properties.

Let

$$\beta = 3(-1)^{l-1}\lambda_1 \cdots \lambda_l - 1.$$

$(-1)^{l-1}\lambda_1 \cdots \lambda_l$ is primary, so $(-1)^{l-1}\lambda_1 \cdots \lambda_l = 3m - 1 + 3n\omega$, $m, n \in \mathbb{Z}$.

$\beta = 3(3m - 1 + 3n\omega) - 1 = 3(3m - 1) - 1 + 9n\omega = 3M - 1 + 3N\omega$, where $M = 3m - 1, N = 3n$. By Exercise 9.19,

$$\chi_\beta(\omega) = \omega^{M+N} = \omega^{3m-1+3n} = \omega^2 \neq 1.$$

As $\beta = \pm\beta_1 \cdots \beta_m$, where the β_i are primary primes, $\chi_\beta(\omega) = \chi_{\beta_1}(\omega) \cdots \chi_{\beta_m}(\omega) \neq 1$, so there exists a subscript i such that $\chi_{\beta_i}(\omega) \neq 1$.

Since $\beta = 3(-1)^{l-1}\lambda_1 \cdots \lambda_l - 1$, β_i is associate neither to λ_i nor to λ . Moreover $\chi_{\beta_i}(\omega) \neq 1$, thus the equation $x^3 \equiv \omega [\beta_i]$ is not solvable: $\lambda_{l+1} = \beta_i$ is convenient.

Conclusion: the equation $x^3 \equiv \omega [\pi]$ is not solvable for infinitely many primes π .

- c) We show that $x^3 \equiv \lambda [\pi]$ has no solution for infinitely many primes π .

To initialize the induction, we display such a prime π , namely $\pi = -4 + 3\omega$. Indeed, $N(\pi) = 16 + 9 + 12 = 37$, 37 is a rational prime, so π is a primary prime in D , of the form $\pi = 3m - 1 + 3n\omega$, with $m = -1, n = 1$, so $\chi_\pi(\lambda) = \omega^{2m} = \omega \neq 1$: the equation $x^3 \equiv \lambda [\pi]$ is not solvable.

Suppose now the existence of a set $A = \{\lambda_1, \lambda_2, \dots, \lambda_l\}, l \geq 1$, of distinct primary primes λ_i , not associate to λ and such the equation $x^3 \equiv \lambda [\lambda_i]$ is not solvable. We will show that we can add a prime λ_{l+1} to the set A with the same properties.

Let

$$\beta = 3(-1)^{l-1}\lambda_1 \cdots \lambda_l - 1.$$

$(-1)^{l-1}\lambda_1 \cdots \lambda_l$ is primary, so $(-1)^{l-1}\lambda_1 \cdots \lambda_l = 3m - 1 + 3n\omega$, $m, n \in \mathbb{Z}$.

$\beta = 3(3m - 1 + 3n\omega) - 1 = 3(3m - 1) - 1 + 9n\omega = 3M - 1 + 3N\omega$, where $M = 3m - 1, N = 3n$. By Exercise 9.19,

$$\chi_\beta(\lambda) = \omega^{2M} = \omega^{2(3m-1)} = \omega \neq 1.$$

As $\beta = \pm\beta_1 \cdots \beta_m$, where the β_i are primary primes, $\chi_\beta(\omega) = \chi_{\beta_1}(\omega) \cdots \chi_{\beta_m}(\omega) \neq 1$, so there exists a subscript i such that $\chi_{\beta_i}(\lambda) \neq 1$.

Since $\beta = 3(-1)^{l-1}\lambda_1 \cdots \lambda_l - 1$, β_i is associate neither to λ_i nor to λ . Moreover $\chi_{\beta_i}(\lambda) \neq 1$, thus the equation $x^3 \equiv \lambda [\beta_i]$ is not solvable : $\lambda_{l+1} = \beta_i$ is convenient.

Conclusion : the equation $x^3 \equiv \lambda [\pi]$ is not solvable for infinitely many primes π . \square

Ex. 9.22 (continuation) Show in general that if $\gamma \in D$ and $x^3 \equiv \gamma \pmod{\pi}$ is solvable for all but finitely many primary primes π , then γ is a cube in D .

Proof. Let $\gamma \in D$ and suppose that γ is not a cube in D . We will show that the equation $x^3 \equiv \gamma [\pi]$ is not solvable for infinitely primes $\pi \in D$.

By Exercise 9.2, we can write

$$\gamma = (-1)^u \omega^v \lambda^w \gamma_1^{n_1} \cdots \gamma_p^{n_p},$$

where the γ_i are distinct primary primes, not associate to λ . Let $v = 3q + b, w = 3q' + c, n_i = 3q_i + r_i$, with the remainders b, c, r_i in $\{0, 1, 2\}$. Grouping the factors with null remainders, we obtain $\gamma = \delta^3 \gamma', \gamma' = \omega^b \lambda^c \gamma_1^{r_1} \cdots \gamma_l^{r_l}$, with b, c, r_i in $\{1, 2\}, \delta \in D, l \geq 0$ (-1 is a cube).

Moreover the equation $x^3 \equiv \gamma [\pi]$ is solvable iff the equation $x^3 \equiv \gamma' [\pi]$ is solvable. So we may suppose that

$$\gamma = \omega^b \lambda^c \gamma_1^{r_1} \cdots \gamma_l^{r_l}, \quad b \in \{1, 2\}, c \in \{1, 2\}, r_i \in \{1, 2\},$$

without cubic factors.

- Case 1 : $l \geq 1$.

Let $A = \{\lambda_1, \dots, \lambda_k\}$ a possibly empty set of distinct primary primes λ_i , distinct of the γ_i , not associate to λ , and such that the equation $x^3 \equiv \gamma [\lambda_i]$ is not solvable. We will show that we can add a prime λ_{k+1} with the same properties.

Suppose that $l \geq 1$. We have proved in Ex. 9.21 that there exists $\sigma \in D$ such that $\chi_{\gamma_l}(\sigma) = \omega$. Since $9, \lambda_i, \gamma_j$ are relatively prime, there exists $\beta \in D$ such that

$$\begin{aligned} \beta &\equiv -1 [9] \\ \beta &\equiv 1 [\lambda_i], 1 \leq i \leq k \\ \beta &\equiv 1 [\gamma_i], 1 \leq i \leq l-1 \\ \beta &\equiv \sigma [\gamma_l] \end{aligned}$$

$\beta \equiv -1 [9]$, thus $\beta \equiv -1 [3]$: β is primary, of the form $\beta = 3M - 1 + 3N\omega$.

$\beta = 3M - 1 + 3N\omega \equiv -1 [9]$, so $3M + 3N\omega \equiv 0 [9]$, $M + N\omega \equiv 0 [3]$, thus $3 \mid M, 3 \mid N$.

By Exercise 9.18,

$$\begin{aligned}\chi_\beta(\omega) &= \omega^{M+N} = 1 \\ \chi_\beta(\lambda) &= \omega^{2M} = 1\end{aligned}$$

As β and γ_i are primary, $\chi_\beta(\gamma_i) = \chi_{\gamma_i}(\beta) = \chi_{\gamma_i}(1) = 1$ ($1 \leq i \leq l-1$).

$$\chi_\beta(\gamma) = \chi_\beta(\omega)^b \chi_\beta(\lambda)^c \chi_\beta(\gamma_1)^{r_1} \cdots \chi_\beta(\gamma_l)^{r_l} = \chi_\beta(\gamma_l)^{r_l} = \chi_{\gamma_l}(\beta)^{r_l} = \chi_{\gamma_l}(\sigma)^{r_l} = \omega^{r_l} \neq 1, \text{ since } r_l \in \{1, 2\}.$$

$\beta = \pm\beta_1 \cdots \beta_m$, with β_i primary primes, therefore

$$\chi_\beta(\gamma) = (\chi_{\beta_1} \cdots \chi_{\beta_m})(\gamma) \neq 1.$$

Thus there exists a subscript i such that $\chi_{\beta_i}(\gamma) \neq 1$, so $x^3 \equiv \gamma [\beta_i]$ is not solvable. Moreover $\beta \equiv 1 [\gamma_i]$, so β_i is not associate to any γ_j . Similarly, β_i is not associate to any γ_j , and $\beta \equiv -1 [9]$, therefore β_i is not associate to λ . So $\lambda_{k+1} = \beta_i$ is convenient.

There exist infinitely many π such that $x^3 \equiv \gamma [\pi]$ is not solvable.

- Case 2 : $l = 0$, so $\gamma = \omega^b \lambda^c$, $1 \leq b \leq 2, 1 \leq c \leq 2$.

$\pi_0 = 2 - 3\omega$ is a primary prime ($N(\pi_0) = 19$).

Let $A = \{\lambda_1, \dots, \lambda_k\}$ a possibly empty set of distinct primary primes $\lambda_i \neq \pi_0$ such that the equation $x^3 \equiv \gamma [\lambda_i]$ is not solvable. We will show that we can add a prime λ_{k+1} with the same properties.

Let $\beta = 9(-1)^{k-1} \lambda_1 \cdots \lambda_k + 2 - 3\omega$.

$\beta \equiv 2 [3]$: β is primary.

Moreover $(-1)^{k-1} \lambda_1 \cdots \lambda_k$ is primary, so

$$(-1)^{k-1} \lambda_1 \cdots \lambda_k = 3m - 1 + 3n\omega, m \in \mathbb{Z}, n \in \mathbb{Z}.$$

Then

$$\begin{aligned}\beta &= 9(3m - 1 + 3n\omega) + 2 - 3\omega \\ &= 27m - 7 + (27n - 3)\omega \\ &= 3(9m - 2) - 1 + 3(9n - 1)\omega \\ &= 3M - 1 + 3N\omega,\end{aligned}$$

where $M = 9m - 2, N = 9n - 1$. Therefore

$$\begin{aligned}\chi_\beta(\omega) &= \omega^{M+N} = \omega^{9m-2+9n-1} = 1 \\ \chi_\beta(\lambda) &= \omega^{2M} = \omega^{2(9m-2)} = \omega^2 \neq 1\end{aligned}$$

$\beta = \pm\beta_1 \cdots \beta_m$, where the β_i are primary primes.

$\chi_\beta(\gamma) = \chi_\beta(\omega)^b \chi_\beta(\lambda)^c = \omega^{2c} \neq 1$ since $c = 1$ or $c = 2$.

$$\chi_\beta(\gamma) = (\chi_{\beta_1} \cdots \chi_{\beta_m})(\gamma) \neq 1.$$

Thus there exists a subscript i such that $\chi_{\beta_i}(\gamma) \neq 1$, so $x^3 \equiv \gamma [\beta_i]$ is not solvable.

As $\beta_i \mid \beta = 9(-1)^{k-1}\lambda_1 \cdots \lambda_k + 2 - 3\omega$, if $\beta_i = \lambda_j$ for some subscript j , $\lambda_j \mid \pi_0 = 2 - 3\omega$, so $\lambda_j = \pi_0$, which is a contradiction, thus $\beta_i \notin A$. Similarly, if $\beta_i = \pi_0 = 2 - 3\omega$, then $\pi_0 \mid 9\lambda_1 \cdots \lambda_k$, and π_0 is relatively prime to λ , so $\pi_0 = \lambda_j$ for some subscript j : this is a contradiction, thus $\beta_i \neq \pi_0$. $\lambda_{k+1} = \beta_i$ is convenient.

So there exist infinitely many π such that $x^3 \equiv \gamma [\pi]$ is not solvable.

• Conclusion :

if γ is not a cube in D , there exist infinitely many primes π such that $x^3 \equiv \gamma [\pi]$ is not solvable.

By contraposition, if the equation $x^3 \equiv \gamma [\pi]$ is solvable for every prime π , at the exception perhaps of the primes in a finite set, then γ is a cube in D .

□

Ex. 9.23 Suppose that $p \equiv 1 \pmod{3}$. Use Exercise 5 to show that $x^3 \equiv 3 \pmod{p}$ is solvable in \mathbb{Z} iff p is of the form $4p = C^2 + 243B^2$.

Proof. Let p be a rational prime, $p \equiv 1 \pmod{3}$, then $p = \pi\bar{\pi}$, where $\pi \in D$ is a primary prime : $\pi = a + b\omega = 3m - 1 + 3n\omega$.

- Suppose that there exists $x \in \mathbb{Z}$ such that $x^3 \equiv 3 \pmod{p}$. Then $x^3 \equiv 3 \pmod{\pi}$, so $\chi_\pi(3) = 1$. By Exercise 9.5, $\omega^{2n} = \chi_\pi(3) = 1$, thus $3 \mid n$, therefore $9 \mid b = 3n$, namely $b = 9B, B \in \mathbb{Z}$.

$p = N\pi = a^2 + b^2 - ab, 4p = (2a - b)^2 + 3b^2 = C^2 + 243B^2$, where $C = 2a - b, B = b/9$. So there exists $C, B \in \mathbb{Z}$ such that $4p = C^2 + 243B^2$.

- Conversely, suppose that there exist $C, B \in \mathbb{Z}$ such that $4p = C^2 + 243B^2$.

As $4p = (2a - b)^2 + 3b^2 = C^2 + 3(9B)^2$, from the unicity proved in Exercise 8.13, we obtain $b = \pm 9B$, so $9 \mid b = 3n, 3 \mid n$, and $\chi_\pi(3) = \omega^{2n} = 1$.

Thus there exists $x \in D$ such that $x^3 \equiv 3 \pmod{\pi}$. As $p \equiv 1 \pmod{3}$, $D/\pi D = \{\bar{0}, \dots, \bar{p-1}\}$, so there exists $h \in \mathbb{Z}$ such that $x \equiv h \pmod{\pi}$, and $h^3 \equiv 3 \pmod{\pi}$.

Therefore $p = N\pi \mid N(h^3 - 3)$, namely $p \mid (h^3 - 3)^2$, where p is a rational prime, thus $p \mid h^3 - 3$: there exists $x \in \mathbb{Z}$ such that $x^3 \equiv 3 \pmod{p}$.

Moreover $4p = C^2 + 243B^2$ implies $p \equiv 1 \pmod{3}$.

$$(p \equiv 1 \pmod{3} \text{ and } \exists x \in \mathbb{Z}, x^3 \equiv 3 \pmod{p}) \iff \exists C \in \mathbb{Z}, \exists B \in \mathbb{Z}, 4p = C^2 + 243B^2.$$

□

Ex. 9.24 Let $\pi = a + b\omega$ be a complex primary element of $D = \mathbb{Z}[\omega]$. Put $a = 3m - 1, b = 3n, p = N(\pi)$.

(a) $(p - 1)/3 \equiv -2m + n \pmod{3}$.

(b) $(a^2 - 1)/3 \equiv m \pmod{3}$.

(c) $\chi_\pi(a) = \omega^m$.

(d) $\chi_\pi(a + b) = \omega^{2n}\chi_\pi(1 - \omega)$.

Lemma. Let $a \in \mathbb{Z}$, $a \equiv -1 \pmod{3}$, and $b \in \mathbb{Z}$ such that $a \wedge b = 1$. Then $\chi_a(b) = 1$.

Proof. (of Lemma.)

If q is a rational prime, $q \equiv 2 \pmod{3}$, and $q \wedge b = 1$, then $\chi_q(b) = 1$ (Prop. 9.3.4, Corollary).

If p is a rational prime, $p \equiv 1 \pmod{3}$ and $p \wedge b = 1$, then $p = \pi\bar{\pi}$, with π primary prime in D (and also $\bar{\pi}$), and by definition of χ_p , $\chi_p(b) = \chi_\pi(b)\chi_{\bar{\pi}}(b)$.

As $\chi_{\bar{\pi}}(b) = \chi_{\bar{\pi}}(\bar{b}) = \overline{\chi_\pi(b)}$ (Prop. 9.3.4(b)), then $\chi_p(b) = \chi_\pi(b)\chi_{\bar{\pi}}(b) = \chi_\pi(b)\overline{\chi_\pi(b)} = 1$.

a has a decomposition in prime factors of the form :

$$a = \pm q_1 q_2 \cdots q_k p_1 p_2 \cdots p_l = \pm q_1 q_2 \cdots q_k \pi_1 \bar{\pi}_1 \pi_2 \bar{\pi}_2 \cdots \pi_l \bar{\pi}_l,$$

where $q_i \equiv -1, p_j \equiv 1 \pmod{3}$, and the π_k are primary primes (since all these elements are primary, the symbol \pm is $(-1)^{k-1}$). Thus, by definition of χ_a ,

$$\chi_a(b) = \chi_{q_1}(b) \cdots \chi_{q_k}(b) \chi_{\pi_1}(b) \chi_{\bar{\pi}_1}(b) \cdots \chi_{\pi_l}(b) \chi_{\bar{\pi}_l}(b) = 1.$$

The result remains true if $a = -1$: then, by definition, $\chi_a(b) = 1$.

□

Proof. (of Ex 9.24.) By hypothesis, π is a primary element, so $\pi = 3m - 1 + 3n\pi$, $m, n \in \mathbb{Z}$. We don't suppose in this proof that π is a prime element, so $p = N(\pi)$ is not necessarily prime.

(a) $p - 1 = (3m - 1)^2 + (3n)^2 - 3n(3m - 1) - 1 \equiv -6m + 3n \pmod{9}$, thus

$$\frac{p-1}{3} \equiv -2m + n \pmod{3}.$$

(b) $a^2 - 1 = (3m - 1)^2 - 1 \equiv -6m \pmod{9}$, thus

$$\frac{a^2 - 1}{3} \equiv m \pmod{3}.$$

(c) As π, a are primary, by Exercise 9.20, $\chi_\pi(a) = \chi_a(\pi)$.

Since $\pi \equiv b\omega \pmod{a}$, $\chi_a(\pi) = \chi_a(b)\chi_a(\omega)$.

By Exercise 9.18, as $a = 3m - 1$, $\chi_a(\omega) = \omega^{M+N}$, where $M = m, N = 0$, so

$$\chi_a(\omega) = \omega^m.$$

Here a is relatively prime to b in \mathbb{Z} : if a rational prime r divides a, b , then $r \mid \pi$ in D , thus $r \mid \bar{\pi}$, so $r^2 \mid \pi\bar{\pi} = p$ in D , thus $r^2 \mid p$ in \mathbb{Z} , which is absurd. The Lemma gives then $\chi_a(b) = 1$.

We conclude that $\chi_a(b) = 1, \chi_a(\omega) = \omega^m$, so $\chi_\pi(a) = \chi_a(\pi) = \chi_a(b)\chi_a(\omega) = \omega^m$.

$$\chi_\pi(a) = \omega^m.$$

(d)

$$a + b = [(a + b)\omega]\omega^{-1},$$

and

$$(a + b)\omega = (a + b\omega) + a\omega - a \equiv a(\omega - 1) \pmod{\pi},$$

thus

$$a + b \equiv -a(1 - \omega)\omega^{-1} \pmod{\pi},$$

$$\chi_{\pi}(a + b) = \chi_{\pi}(1 - \omega)\chi_{\pi}(a)\chi_{\pi}(\omega)^{-1},$$

$\chi_{\pi}(a) = \omega^m$ by (c), and $\chi_{\pi}(\omega) = \omega^{m+n}$ (as in Ex. 9.3), thus

$$\chi_{\pi}(a + b) = \omega^{2n}\chi_{\pi}(1 - \omega).$$

□

Ex. 9.25 Show that $\chi_{a+b}(\pi)$ may be computed as follows.

(a) $\chi_{a+b}(\pi) = \chi_{a+b}(1 - \omega).$

(b) $\chi_{a+b}(\pi) = \omega^{2(m+n)}.$

Proof. (a) $\pi = a + b\omega$ and $a \equiv -b \pmod{a + b}$, thus $\pi \equiv -b(1 - \omega) \pmod{a + b}$. So

$$\chi_{a+b}(\pi) = \chi_{a+b}(b)\chi_{a+b}(1 - \omega).$$

Since $a \wedge b = 1$, $(a + b) \wedge b = 1$: as in Ex. 9.24, $\chi_{a+b}(b) = 1$. So

$$\chi_{a+b}(\pi) = \chi_{a+b}(1 - \omega).$$

(b) Since the character χ_{a+b} has order 3,

$$\begin{aligned} \chi_{a+b}(1 - \omega) &= (\chi_{a+b}((1 - \omega)^2))^2 \\ &= (\chi_{a+b}(-3\omega))^2 \\ &= [\chi_{a+b}(3)\chi_{a+b}(\omega)]^2 \end{aligned}$$

$$\chi_{a+b}(3) = 1 \text{ because } (a + b) \wedge 3 = (3(m + n) - 1) \wedge 3 = 1.$$

$$\chi_{a+b}(\omega) = \omega^{m+n} \text{ (Ex. 9.19).}$$

Conclusion :

$$\chi_{a+b}(1 - \omega) = \omega^{2(m+n)}.$$

□

Ex. 9.26 Combine the previous two exercises to conclude that $\chi_{\pi}(1 - \omega) = \omega^{2m}$.

Proof. Since π and $a + b$ are primary elements of D , by Exercise 9.20,

$$\chi_{\pi}(a + b) = \chi_{a+b}(\pi).$$

By Exercises 9.24 and 9.25,

$$\chi_{\pi}(a + b) = \omega^{2n}\chi_{\pi}(1 - \omega)$$

$$\chi_{a+b}(\pi) = \omega^{2(m+n)}$$

Thus $\omega^{2n}\chi_{\pi}(1 - \omega) = \omega^{2(m+n)}$. Consequently

$$\chi_{\pi}(1 - \omega) = \omega^{2m}.$$

□

Ex. 9.27 Let $\pi = a + bi$ be a primary irreducible in $\mathbb{Z}[i]$, $b \neq 0$. Show

$$(a) \quad a \equiv (-1)^{(p-1)/4} \pmod{4}, p = N(\pi).$$

$$(b) \quad b \equiv (-1)^{(p-1)/4} - 1 \pmod{4}.$$

(Wrong sentence for (b) in an older edition.)

Proof. Let $\pi = a + bi$ a primary prime in $\mathbb{Z}[i]$, $b \neq 0$, such that $p = N(\pi)$:

$$p = \pi \bar{\pi} = a^2 + b^2 \equiv 1 \pmod{4}.$$

By Lemma 6, Section 7, a is odd, b even, and

$$(a \equiv 1 \pmod{4}, b \equiv 0 \pmod{4}) \text{ or } (a \equiv 3 \pmod{4}, b \equiv 2 \pmod{4}).$$

- (a) • Case 1 : $a \equiv 1 \pmod{4}, b \equiv 0 \pmod{4}$. Then $a = 4A + 1, b = 4B$, $A, B \in \mathbb{Z}$, so $(a^2 + b^2 - 1)/4 = 4A^2 + 4B^2 + 2A$ is even :
 $(-1)^{(p-1)/4} = (-1)^{(a^2+b^2-1)/4} = 1$, and $a \equiv 1 \pmod{4}$, thus $a \equiv (-1)^{(p-1)/4} \pmod{4}$.
 • Case 2 : $a \equiv 3 \pmod{4}, b \equiv 2 \pmod{4}$.
 $a = 4A + 3, b = 4B + 2, a^2 + b^2 - 1 = 16A^2 + 24A + 9 + 16B^2 + 16B + 4 - 1 \equiv 4 \pmod{8}$,
 so $(a^2 + b^2 - 1)/4 \equiv 1 \pmod{2}$, $(-1)^{(p-1)/4} = (-1)^{(a^2+b^2-1)/4} = -1$, and $a \equiv -1 \pmod{4}$,
 thus $a \equiv (-1)^{(p-1)/4} \pmod{4}$.

In both cases,

$$a \equiv (-1)^{(p-1)/4} \pmod{4}.$$

(b) In every case, $b \equiv a - 1 \pmod{4}$, thus

$$b \equiv (-1)^{(p-1)/4} - 1 \pmod{4}.$$

In other words, for all primary primes $\pi = a + bi$ such that $N(\pi) = p$,

$$\begin{aligned} p \equiv 1 \pmod{8} &\iff \pi \equiv 1 \pmod{4}, \\ p \equiv 5 \pmod{8} &\iff \pi \equiv 3 + 2i \pmod{4}. \end{aligned}$$

□

Ex. 9.28 The notation being as in Exercise 27 show $\chi_\pi(\bar{\pi}) = \chi_\pi(2)\chi_\pi(a)$.

Proof. $\pi = a + bi, \bar{\pi} = a - bi = 2a - \pi \equiv 2a \pmod{\pi}$, thus, by Proposition 9.8.3 (e) :

$$\chi_\pi(\bar{\pi}) = \chi_\pi(2a) = \chi_\pi(2)\chi_\pi(a).$$

□

Ex. 9.29 By Exercise 9.27, $a(-1)^{(p-1)/4}$ is primary. Use biquadratic reciprocity to show $\chi_\pi(a(-1)^{(p-1)/4}) = (-1)^{(a^2-1)/8}$.

Proof. $a \equiv (-1)^{(p-1)/4} \pmod{4}$ [4] (Ex. 9.27(a)), $a(-1)^{(p-1)/4} \equiv 1 \pmod{4}$, thus $a(-1)^{(p-1)/4}$ is primary (if $a \neq \pm 1$).

If $a = \pm 1$ is a unit, $a(-1)^{(p-1)/4} = 1$ and $\chi_\pi(a(-1)^{(p-1)/4}) = 1 = (-1)^{(a^2-1)/8}$, so we can suppose that a is not a unit.

As $a(-1)^{(p-1)/4} \equiv 1 \pmod{4}$, the Law of Biquadratic Reciprocity (Prop. 9.9.8) gives

$$\begin{aligned}\chi_\pi(a(-1)^{(p-1)/4}) &= \chi_{a(-1)^{(p-1)/4}}(\pi) \\ &= \chi_a(\pi) \quad (\text{Prop. 9.8.3(f)}) \\ &= \chi_a(a + bi) \\ &= \chi_a(bi) \\ &= \chi_a(b)\chi_a(i).\end{aligned}$$

As $a \wedge b = 1$ (since $p = a^2 + b^2$), $\chi_a(b) = 1$ (Prop. 9.8.5, with $a \neq 1$), so

$$\chi_\pi(a(-1)^{(p-1)/4}) = \chi_a(i).$$

If $a \equiv 1 \pmod{4}$, Proposition 8.9.6 gives $\chi_a(i) = (-1)^{(a-1)/4}$. Write $a = 4A + 1$, $A \in \mathbb{Z}$. Then

$$(-1)^{(a^2-1)/8} = (-1)^{2A^2+A} = (-1)^A = (-1)^{(a-1)/4} = \chi_a(i).$$

If $a \equiv -1 \pmod{4}$, then $\chi_a(i) = \chi_{-a}(i) = (-1)^{(-a-1)/4}$ by the same proposition. Write $a = 4A - 1$, $A \in \mathbb{Z}$. Then

$$(-1)^{(a^2-1)/8} = (-1)^{2A^2-A} = (-1)^{-A} = (-1)^{(-a-1)/4} = \chi_a(i).$$

So, for each odd a , $a \neq \pm 1$,

$$\chi_a(i) = i^{(a^2-1)/8}.$$

Conclusion : if $\pi = a + bi$ is a primary irreducible such that $N(\pi) = p$, then

$$\chi_\pi(a(-1)^{(p-1)/4}) = (-1)^{(a^2-1)/8}.$$

□

Ex. 9.30 Use the preceding two exercises to show $\chi_\pi(\bar{\pi}) = \chi_\pi(2)(-1)^{(a^2-1)/8}$.

Proof. By Exercises 9.28, 9.29, and $\chi_\pi(-1) = (-1)^{(a-1)/2}$ (Prop. 9.8.3(d)),

$$\begin{aligned}\chi_\pi(\bar{\pi}) &= \chi_\pi(2)\chi_\pi(a) \\ &= \chi_\pi(2)\chi_\pi(a(-1)^{(p-1)/4})(\chi_\pi(-1))^{(p-1)/4} \\ &= \chi_\pi(2)(-1)^{(a^2-1)/8}((-1)^{(a-1)/2})^{(p-1)/4} \\ &= \chi_\pi(-2)(-1)^{(a^2-1)/8}((-1)^{(a-1)/2})^{(p+3)/4} \\ &= \chi_\pi(-2)(-1)^{(a^2-1)/8}(-1)^{((a-1)/2)((p+3)/4)}.\end{aligned}$$

If $a \equiv 1 \pmod{4}$, then $(-1)^{(a-1)/2} = 1$.

If $a \equiv 3 \pmod{4}$, then $b \equiv 2 \pmod{4}$:

$$a = 4A + 3, b = 4B + 2, p + 3 = a^2 + b^2 + 3 = (4A + 3)^2 + (4B + 2)^2 + 3 \equiv 0 \pmod{8},$$

so $(p+3)/4 \equiv 0 \pmod{2}$.

In both cases $(-1)^{((a-1)/2)((p+3)/4)} = 1$, and so

$$\chi_\pi(\bar{\pi}) = \chi_\pi(-2)(-1)^{(a^2-1)/8}.$$

□

Ex. 9.31 Let p be prime, $p \equiv 1 \pmod{4}$. Show that $p = a^2 + b^2$ where a and b are uniquely determined by the conditions $a \equiv 1 \pmod{4}$, $b \equiv -((p-1)/2)!a \pmod{p}$.

Proof. Recall the following lemma :

Lemma :

Let p be a prime, $p \equiv 1 \pmod{4}$, then $\left[\left(\frac{p-1}{2}\right)!\right]^2 \equiv -1 \pmod{p}$.

By Wilson's theorem (Prop. 4.1.1, Corollary), $(p-1)! \equiv -1 \pmod{p}$.

$$\begin{aligned} -1 &\equiv (p-1)! = 1.2.\dots \left(\frac{p-1}{2}\right) \left(\frac{p+1}{2}\right) \dots (p-2)(p-1) \\ &\equiv 1.2.\dots \frac{p-1}{2} \left[-\left(\frac{p-1}{2}\right)\right] \dots (-2)(-1) \\ &\equiv (-1)^{(p-1)/2} \left[\left(\frac{p-1}{2}\right)!\right]^2 \\ &\equiv \left[\left(\frac{p-1}{2}\right)!\right]^2 \pmod{p}, \end{aligned}$$

since $p \equiv 1 \pmod{4}$.

- We show that there exists a pair $a, b \in \mathbb{Z}$ which verifies the sentence.

By lemma 5 section 7, as $p \equiv 1 \pmod{4}$, there exists an irreducible π such that $N(\pi) = p$, and we can choose π such that $\pi = A + Bi$ is primary (lemma 7 section 7), so A is odd.

If $A \equiv 1 \pmod{4}$, we take $a = A$, and if $A \equiv 3 \pmod{4}$, we take $a = -A$: then $a \equiv 1 \pmod{4}$.

Let $u = \left(\frac{p-1}{2}\right)!$. Then $0 \equiv p = A^2 + B^2 \pmod{p}$, $B^2 \equiv -A^2 \equiv (uA)^2 \pmod{p}$.

$p \mid (B - uA)(B + uA)$, thus $B \equiv \pm uA \pmod{p}$.

Since $a = \pm A$, $B \equiv \pm ua \pmod{p}$.

If $B \equiv -ua \pmod{p}$, we take $b = B$, if not $b = -B$.

Then a, b are such that $p = a^2 + b^2$, $a \equiv 1 \pmod{4}$, $b \equiv -((p-1)/2)!a \pmod{p}$.

- Unicity of the pair (a, b) such that

$$p = a^2 + b^2, a \equiv 1 \pmod{4}, b \equiv -((p-1)/2)!a \pmod{p}.$$

Suppose that c, d are such that $p = c^2 + d^2$, $c \equiv 1 \pmod{4}$, $d \equiv -((p-1)/2)!c \pmod{p}$.

Let $\pi = a + ib, \lambda = c + id$. As $p = N\pi = N\lambda$ is a rational prime, π and λ are primes in D , and $p = \pi\bar{\pi} = \lambda\bar{\lambda}$, thus λ is associate to π or $\bar{\pi}$:

$$\lambda \in \{\pi, -\pi, i\pi, -i\pi, \bar{\pi}, -\bar{\pi}, i\bar{\pi}, -i\bar{\pi}\}.$$

As a, c are odd, and b, d even, it remains only the possibilities $\lambda = \pm\pi, \lambda = \pm\bar{\pi}$, thus $c = \pm a$. Moreover $a \equiv c \equiv 1 \pmod{4}$, thus $a = c$, and $d \equiv -((p-1)/2)!c \equiv -((p-1)/2)!a \equiv b \pmod{p}$.

$p = a^2 + b^2 = a^2 + d^2$, so $d = \pm b$, and $d \equiv b \pmod{p}$.

If $d = -b$, then $p \mid 2b$, thus $p \mid b$, and also $p \mid a$, so $p^2 \mid a^2 + b^2 = p$: this is impossible. So $a = b, c = d$. Unicity is proved.

Conclusion : if $p \equiv 1 \pmod{4}$, there exists an unique pair a, b such that

$$p = a^2 + b^2, a \equiv 1 \pmod{4}, b \equiv -((p-1)/2)!a \pmod{p}.$$

□

Ex. 9.32 Let p be a prime, $p \equiv 1 \pmod{4}$ and write $p = \pi\bar{\pi}, \pi \in \mathbb{Z}[i]$. Show $\chi_p(1+i) = i^{(p-1)/4}$.

Proof.

$$\begin{aligned} \chi_p(1+i) &= \chi_\pi(1+i)\chi_{\bar{\pi}}(1+i) \\ &= \chi_\pi(1+i)\overline{\chi_\pi(1-i)} \quad (\text{Prop. 9.8.3(c)}) \\ &= \frac{\chi_\pi(1+i)}{\chi_\pi(1-i)} = \chi_\pi(i) \quad (\text{since } (1-i)i = 1+i) \\ &= i^{\frac{p-1}{4}}. \end{aligned}$$

The last equality is a consequence of the definition of χ_π : $\chi_\pi(i) \equiv i^{\frac{p-1}{4}} \pmod{\pi}$, and the classes of $1, i, i^2, i^3$ modulo π are distinct. □

Ex. 9.33 Let q be a positive prime, $q \equiv 3 \pmod{4}$. Show $\chi_q(1+i) = i^{(q+1)/4}$. [Hint : $(1+i)^{q-1} \equiv -i \pmod{q}$.]

The sentence is false and must be replaced by

$$\chi_q(1+i) = (-i)^{(q+1)/4} = i^{-(q+1)/4}.$$

We verify this on the example $q = 11$:

$$\begin{aligned} \chi_q(1+i) &\equiv (1+i)^{(q^2-1)/4} \\ &\equiv (1+i)^{30} \\ &\equiv -2^{15}i \equiv -32i \equiv i \pmod{11}, \end{aligned}$$

so $\chi_{11}(1+i) = i$, and $i^{(-q-1)/4} = i^{-3} = i$ (but $i^{(q+1)/4} = -i$).

Proof. Write $q = 4k + 3, k \in \mathbb{N}$.

As $(1+i)^2 = 2i$, $(1+i)^{q-1} = (2i)^{(q-1)/2}$.

$2^{(q-1)/2} \equiv \left(\frac{2}{q}\right) [q]$ et $\left(\frac{2}{q}\right) = (-1)^{(q^2-1)/8} = (-1)^{2k^2+3k+1} = (-1)^{k+1}$

$i^{(q-1)/2} = i^{2k+1} = (-1)^k i$.

So

$$(1+i)^{q-1} \equiv -i [q].$$

$N(q) = q^2$, so $\chi_q(1+i) \equiv (1+i)^{(q^2-1)/4} = [(1+i)^{q-1}]^{(q+1)/4} \equiv (-i)^{(q+1)/4} [q] :$

$$\chi_q(1+i) = (-i)^{(q+1)/4} = i^{(-q-1)/4}.$$

□

Ex. 9.34 Let $\pi = a + bi$ be a primary irreducible, $(a, b) = 1$. Show

(a) if $\pi \equiv 1 \pmod{4}$, then $\chi_\pi(a) = i^{(a-1)/2}$.

(b) if $\pi \equiv 3 + 2i \pmod{4}$, then $\chi_\pi(a) = -i^{(-a-1)/2}$.

Proof. Let $\pi = a + bi$ be a primary irreducible, with $a \wedge b = 1$, so $b \neq 0$: we can apply the result of Exercise 9.29 :

$$\chi_\pi(a(-1)^{(p-1)/4}) = (-1)^{(a^2-1)/8}.$$

(a) Suppose that $\pi \equiv 1 [4]$.

Then $a \equiv 1 [4], b \equiv 0 [4], a = 4A + 1, b = 4B, A, B \in \mathbb{Z}$.

As $\chi_\pi(-1) = (-1)^{(a-1)/2}$,

$$\chi_\pi(a) = (-1)^{\frac{a-1}{2} \frac{p-1}{4}} (-1)^{\frac{a^2-1}{8}},$$

where

$$p = N\pi = a^2 + b^2, (-1)^{(p-1)/4} = (-1)^{\frac{a^2-1}{4} + \frac{b^2}{4}} = (-1)^{4A^2+2A+4B^2} = 1,$$

thus $(-1)^{\frac{a-1}{2} \frac{p-1}{4}} = 1$.

$$\chi_\pi(a) = (-1)^{(a^2-1)/8} = (-1)^{2A^2+A} = (-1)^A = (-1)^{(a-1)/4} = i^{(a-1)/2}.$$

Conclusion : if $\pi \equiv 1 [4]$, $\chi_\pi(a) = i^{(a-1)/2}$.

(b) Suppose that $\pi \equiv 3 + 2i [4]$.

Then $a \equiv 3 [4], b \equiv 2 [4], a = 4A + 3, b = 4B + 2, A, B \in \mathbb{Z}$. As in (a),

$$\chi_\pi(a) = (-1)^{\frac{a-1}{2} \frac{p-1}{4}} (-1)^{\frac{a^2-1}{8}},$$

where $a^2 + b^2 - 1 = 16A^2 + 24A + 16B^2 + 16B + 12 \equiv 4 [8]$, so $\frac{a^2+b^2-1}{4} \equiv 1 [2]$,

thus $(-1)^{(p-1)/4} = (-1)^{(a^2+b^2-1)/4} = -1$.

$$(-1)^{\frac{a-1}{2} \frac{p-1}{4}} = (-1)^{\frac{a-1}{2}} = (-1)^{2A+1} = -1,$$

$$\frac{a^2-1}{8} = 2A^2 + 3A + 1, (-1)^{(a^2-1)/8} = (-1)^{3A+1} = (-1)^{A+1} = (-1)^{(a+1)/4},$$

$$\chi_\pi(a) = -(-1)^{(a+1)/4} = -i^{(a+1)/2}.$$

Moreover

$$\frac{a+1}{2} \equiv \frac{-a-1}{2} \pmod{4} \iff a+1 \equiv -a-1 \pmod{8} \iff 2a \equiv -2 \pmod{8} \iff a \equiv 3 \pmod{4},$$

thus $i^{(a+1)/2} = i^{(-a-1)/2}$.

Conclusion : if $\pi \equiv 3 + 2i \pmod{4}$, $\chi_\pi(a) = -i^{(-a-1)/2}$.

□

Ex. 9.35 If $\pi = a + bi$ is as in Exercise 9.34 show $\chi_\pi(a)\chi_\pi(1+i) = i^{(3(a+b-1))/4}$. [Hint: $a(1+i) = a+b+i(a+bi)$. Generalize Exercises 32 and 33 to any integer $\equiv 1 \pmod{4}$ and use Proposition 9.9.8. Note $a+b \equiv 1 \pmod{4}$.]

Proof. We give a generalization of Exercises 9.32 and 9.33 : if $n \equiv 1 \pmod{4}$, $n \neq 1$, then $\chi_n(1+i) = i^{(n-1)/4}$.

By Exercises 9.32 and 9.33, we know that if $p \equiv 1 \pmod{4}$ is a rational prime, then

$$\chi_p(1+i) = i^{(p-1)/4},$$

and if $q \equiv 3 \pmod{4}$, in other words $-q \equiv 1 \pmod{4}$, where q is a rational prime, then

$$\chi_{-q}(1+i) = \chi_q(1+i) = i^{(-q-1)/4}.$$

Let $n \in \mathbb{Z}$, $n \equiv 1 \pmod{4}$, $n \neq 1$.

If $n > 0$, $n = q_1 q_2 \cdots q_k p_1 p_2 \cdots p_l$, where $q_i \equiv -1 \pmod{4}$, $p_i \equiv 1 \pmod{4}$, thus k is odd.

If $n < 0$, $n = -q_1 q_2 \cdots q_k p_1 p_2 \cdots p_l$, with k odd. In both cases,

$$n = (-q_1)(-q_2) \cdots (-q_k)p_1 p_2 \cdots p_l,$$

so we can write

$$n = s_1 s_2 \cdots s_N, \quad \text{where } s_i = -q_i, 1 \leq i \leq k, s_i = p_{i-k}, k+1 \leq i \leq k+l = N,$$

where $s_i \equiv 1 \pmod{4}$, $1 \leq i \leq N$.

$$\begin{aligned} \chi_n(1+i) &= \chi_{-q_1}(1+i) \cdots \chi_{-q_k}(1+i) \chi_{p_1}(1+i) \cdots \chi_{p_l}(1+i) \\ &= i^{(-q_1-1)/4} \cdots i^{(-q_k-1)/4} i^{(p_1-1)/4} \cdots i^{(p_l-1)/4} \\ &= i^{(s_1-1)/4} \cdots i^{(s_N-1)/4} \\ &= i^{\sum_{i=1}^N \frac{s_i-1}{4}} \\ &= i^{(n-1)/4}, \end{aligned}$$

the last equality resulting of Exercise 9.44.

Conclusion : if $n \in \mathbb{Z}$, $n \equiv 1 \pmod{4}$, $n \neq 1$, then $\chi_n(1+i) = i^{(n-1)/4}$.

Let $\pi = a + bi$, $a \wedge b = 1$ a primary irreducible. As $a(1+i) = a+b+i(a+bi)$, $a(1+i) \equiv a+b \pmod{\pi}$, so

$$\chi_\pi(a)\chi_\pi(1+i) = \chi_\pi(a+b).$$

As $\pi = a + bi$ is primary, $a + b \equiv 1 \pmod{4}$.

If $a + b = 1$, then $\chi_\pi(a)\chi_\pi(1+i) = \chi_\pi(a+b) = 1 = i^{3(a+b-1)/4}$. If not, the Law of Biquadratic Reciprocity (Proposition 9.9.8) gives

$$\chi_\pi(a+b) = \chi_{a+b}(\pi).$$

Now $b \equiv -a \pmod{a+b}$, so $a + bi \equiv a(1-i) \equiv -ia(1+i) \pmod{a+b}$. Therefore

$$\chi_{a+b}(\pi) = \chi_{a+b}(-1)\chi_{a+b}(a)\chi_{a+b}(i)\chi_{a+b}(1+i).$$

Since $n \equiv 1 \pmod{4}$, $\chi_n(i) = (-1)^{(n-1)/4}$ (Prop. 9.8.6), thus

$$\chi_n(-1) = \chi_n(i^2) = (-1)^{\frac{n-1}{2}} = 1.$$

Consequently, since $a + b \equiv 1 \pmod{4}$, $\chi_{a+b}(-1) = 1$.

As $a \wedge b = 1$, $(a+b) \wedge a = 1$, thus $\chi_{a+b}(a) = 1$ (Prop 9.8.5).

$a + b \equiv 1 \pmod{4}$, thus $\chi_{a+b}(i) = (-1)^{(a+b-1)/4}$ (Prop. 9.8.6).

From the first part of this proof, $\chi_{a+b}(1+i) = i^{(a+b-1)/4}$, so

$$\begin{aligned} \chi_{a+b}(\pi) &= \chi_{a+b}(-1)\chi_{a+b}(a)\chi_{a+b}(i)\chi_{a+b}(1+i) \\ &= (-1)^{(a+b-1)/4} i^{(a+b-1)/4} \\ &= i^{(a+b-1)/2} i^{(a+b-1)/4} \\ &= i^{3(a+b-1)/4} \end{aligned}$$

Conclusion : if $\pi = a + bi$ is a primary irreducible, such that $a \wedge b = 1$, then

$$\chi_\pi(a)\chi_\pi(1+i) = i^{3(a+b-1)/4}$$

□

Ex. 9.36 Remove the restriction $(a, b) = 1$ in Exercise 9.34.

Proof. Suppose that $q = a \wedge b > 1$. Then $a = qa'$, $b = qb'$, $a', b' \in \mathbb{Z}$, so $\pi = q(a' + ib')$.

As π is irreducible, and as q is not an unit, $u = a' + b'i$ is an unit, and so $\pi = uq$ is associate to q : the rational integer q is then a prime in D , so a rational prime $q \equiv 3 \pmod{4}$.

If $u = \pm i$, then $\pi = \pm q = a + bi$ is such that b is odd, in contradiction with π primary. Thus $u = \pm 1$, and $\pi = \varepsilon q$, $\varepsilon = \pm 1$. As π is primary, $\varepsilon = -1$, so $\pi = -q$.

Then $\chi_\pi(a) = \chi_{-q}(-q) = 0$, the result of Ex. 34 is false if $b = 0$.

Conclusion : if $\pi = a + bi$ is a primary irreducible, and $b \neq 0$, then

- (a) if $\pi \equiv 1 \pmod{4}$, $\chi_\pi(a) = i^{(a-1)/2}$,
- (b) if $\pi \equiv 3 + 2i \pmod{4}$, $\chi_\pi(a) = -i^{(-a-1)/2}$.

□

Ex. 9.37 Combine Exercises 32, 33, 34, and 35 to show $\chi_\pi(1+i) = i^{(a-b-b^2-1)/4}$. Show that this result implies Exercise 26 of Chapter 5 “the biquadratic character of 2”).

Lemma. If $\pi = a + bi$ is a primary prime, then

$$\chi_\pi(i) = i^{\frac{-a+1}{2}}.$$

Proof. (of Lemma.) Let $\pi = a + bi$ a primary prime in $\mathbb{Z}[i]$.

- If $\pi = -q$, where $q \equiv 3 \pmod{4}$, $q > 0$ is a rational prime, then $a = -q, b = 0$. By definition of the quartic character,

$$\chi_q(i) = i^{\frac{N(q)-1}{4}} = i^{\frac{q^2-1}{4}}.$$

Write $-q = a = 4k + 1$, $k \in \mathbb{Z}$. Then

$$\begin{aligned} \frac{q^2-1}{4} &= 4k^2 + 2k \\ &\equiv 2k = \frac{a-1}{2} \pmod{4}. \end{aligned}$$

Therefore

$$\chi_{-q}(i) = \chi_q(i) = i^{\frac{q^2-1}{4}} = i^{\frac{a-1}{2}} = \left(\frac{1}{i}\right)^{\frac{-a+1}{2}} = (-i)^{\frac{-a+1}{2}} = (-1)^{\frac{-a+1}{2}} i^{\frac{-a+1}{2}} = i^{\frac{-a+1}{2}},$$

since $(-1)^{\frac{-a+1}{2}} = (-1)^{-2k} = 1$.

Suppose now that $N(\pi) = p$, where $p \equiv 1 \pmod{4}$ is a rational prime. Then

$$\chi_\pi(i) = i^{\frac{N(\pi)-1}{4}} = i^{\frac{p-1}{4}}.$$

Since $\pi = a + bi$ is primary, there are two cases.

- If $a \equiv 1 \pmod{4}$, $b \equiv 0 \pmod{4}$, then $a = 4A + 1, b = 4B$, $A, B \in \mathbb{Z}$.

$$\begin{aligned} \frac{p-1}{4} &= \frac{a^2 + b^2 - 1}{4} \\ &= \frac{16A^2 + 8A + 16B^2}{4} \\ &= 4A^2 + 2A + 4B^2 \\ &\equiv 2A = \frac{a-1}{2} \end{aligned}$$

Therefore

$$\chi_\pi(i) = i^{\frac{p-1}{4}} = i^{\frac{a-1}{2}} = \left(\frac{1}{i}\right)^{\frac{-a+1}{2}} = (-i)^{\frac{-a+1}{2}} = (-1)^{\frac{-a+1}{2}} i^{\frac{-a+1}{2}} = i^{\frac{-a+1}{2}},$$

since $(-1)^{\frac{-a+1}{2}} = (-1)^{-2A} = 1$.

- If $a \equiv 3 \pmod{4}, b \equiv 2 \pmod{4}$, then $a = 4A - 1, b = 4B + 2, A, B \in \mathbb{Z}$.

$$\begin{aligned} \frac{p-1}{4} &= \frac{a^2 + b^2 - 1}{4} \\ &= \frac{16A^2 - 8A + 16B^2 + 16B + 4}{4} \\ &= 4A^2 - 2A + 4B^2 + 4B + 1 \\ &\equiv -2A + 1 = \frac{-a+1}{2} \pmod{4} \end{aligned}$$

Therefore $\chi_\pi(i) = (-1)^{\frac{-a+1}{4}}$.

The equality $\chi_\pi(i) = (-1)^{\frac{-a+1}{4}}$ is verified for all primary primes π .

□

Proof. (of Ex.9.37) Let $\pi = a + ib$ be a primary irreducible in $\mathbb{Z}[i]$.

- If $b = 0$, then $\pi = a \in \mathbb{Z}$. As π is primary, $\pi = -q, q \equiv 3 \pmod{4}$, where q is a rational prime, so $a = -q, b = 0$. By Ex. 9.32 (or its generalization 9.35),

$$\chi_\pi(1+i) = \chi_{-q}(1+i) = i^{(-q-1)/4} = i^{(a-b-b^2-1)/4}.$$

- If $b \neq 0$, then $a \wedge b = 1$ (see Ex. 9.36), and by Ex. 9.35,

$$\chi_\pi(a)\chi_\pi(1+i) = i^{3(a+b-1)/4}.$$

- If $\pi \equiv 1 [4], a \equiv 1 [4], b \equiv 0 [4] : a = 4A + 1, b = 4B, A, B \in \mathbb{Z}$.
By Ex. 9.34(a),

$$\chi_\pi(a) = i^{(a-1)/2}, \chi_\pi(a)^{-1} = (-i)^{(a-1)/2} = i^{(a-1)/2}.$$

$$\begin{aligned} \chi_\pi(1+i) &= i^{3\frac{a+b-1}{4} - 2\frac{a-1}{4}} \\ &= i^{\frac{a+3b-1}{4}} \\ &= i^{\frac{a-b-b^2-1}{4}}, \end{aligned}$$

$$\text{since } \left(\frac{a+3b-1}{4}\right) - \left(\frac{a-b-b^2-1}{4}\right) = b + \frac{b^2}{4} = 4B + 4B^2 \equiv 0 [4].$$

- If $\pi \equiv 3 + 2i [4], a \equiv 3 [4], b \equiv 2 [4] : a = 4A - 1, b = 4B + 2, A, B \in \mathbb{Z}$.
By Ex. 9.34(b),

$$\chi_\pi(a) = -i^{(-a-1)/2}, \chi_\pi(a)^{-1} = -i^{(a+1)/2} = i^{(a-3)/2},$$

so

$$\chi_\pi(1+i) = i^{(3a+3b-3+2a-6)/4} = i^{(5a+3b-9)/4}.$$

$$\text{Now } \frac{1}{4}[(a-b-b^2-1) - (5a+3b-9)] = \frac{1}{4}(-4a-4b-b^2+8) = -a-b+2-\frac{b^2}{4} = -4A+1-4B-2+2-(2B+1)^2 \equiv 0 [4],$$

$$\text{thus } \chi_\pi(1+i) = i^{(a-b-b^2-1)/4}.$$

Conclusion : if $\pi = a + ib$ is primary irreducible, then

$$\chi_\pi(1+i) = i^{(a-b-b^2-1)/4}$$

Second part : the biquadratic character of 2 (see Ex. 5.25 to 5.28).

Let $p \equiv 1 \pmod{4}$. Then $p = N(\pi)$, where $\pi = a + bi$ is a primary prime.

We show first that $\chi_\pi(2) = i^{\frac{ab}{2}}$.

Since $2 = i^3(1+i)^2$, the first part of the exercise, and the Lemma, give

$$\begin{aligned}\chi_\pi(2) &= \chi_\pi(i)^3 \chi_\pi(1+i)^2 \\ &= i^{\frac{3(-a+1)}{2}} i^{\frac{a-b-b^2-1}{2}} \\ &= i^{1-a-(b+1)\frac{b}{2}}\end{aligned}$$

Since π is primry, $a \equiv b+1 \equiv -b+1 \pmod{4}$, therefore

$$\begin{aligned}1-a-(b+1)\frac{b}{2} &\equiv -b-(b+1)\frac{b}{2} \\ &\equiv \frac{b}{2}(-b-3) \\ &\equiv \frac{b}{2}(-b+1) \\ &\equiv \frac{ab}{2} \pmod{4},\end{aligned}$$

so $\chi_\pi(2) = i^{\frac{ab}{2}}$.

Now we show that p is of the form $p = A^2 + 64b^2$ if and only if $p \equiv 1 \pmod{4}$ and if $x^4 \equiv 2$ has a solution $x \in \mathbb{Z}$.

If $p = A^2 + 64B^2 = A^2 + (8B)^2$, then the prime number p is a sum of two squares, and $p \neq 2$, therefore $p \equiv 1 \pmod{4}$. Since $p = A^2 + 64b^2$, A is odd. Put $b = 8B$, and $a = A$ if $A \equiv 1 \pmod{4}$, $a = -A$ if $A \equiv -1 \pmod{4}$. Then $\pi = a + bi$ is such that $N(\pi) = a^2 + b^2 = p$, and $a \equiv 1, b \equiv 0 \pmod{4}$, therefore π is a primary prime. Then

$$\chi_\pi(2) = i^{\frac{ab}{2}} = i^{4aB} = 1.$$

Therefore there exists $\alpha \in D$ such that $2 \equiv \alpha^4 \pmod{\pi}$. As $D/\pi D$ is the set of classes of $0, 1, \dots, p-1$, there exists $x \in \mathbb{Z}$ such that $x \equiv \alpha \pmod{\pi}$, so $2 \equiv x^4 \pmod{\pi}$.

Then $p = N(\pi) \mid N(x^4 - 2) = (x^4 - 2)^2$, thus $p \mid x^4 - 2$, in other words $2 \equiv x^4 \pmod{p}$.

Conversely, suppose that $p \equiv 1 \pmod{4}$ and that 2 is a biquadratic residue modulo p . As $p \equiv 1 \pmod{4}$, $p = \pi\bar{\pi}$, where $\pi = a + bi$ is a primary prime. Since $2 \equiv x^4 \pmod{p}$ for some $x \in \mathbb{Z}$, then $2 \equiv x^4 \pmod{\pi}$, so $\chi_\pi(2) = 1$. Moreover

$$1 = \chi_\pi(2) = i^{\frac{ab}{2}}.$$

Since a is odd, $8 \mid b$, therefore $p = A^2 + 64b^2$, where $A = a, B = b/8$.

Conclusion :

$$\exists(A, B) \in \mathbb{Z}^2, p = A^2 + 64B^2 \iff (p \equiv 1 \pmod{4} \text{ and } \exists x \in \mathbb{Z}, x^4 \equiv 2 \pmod{p}).$$

□

Ex. 9.38 Prove part (d) of Proposition 9.8.3.

Proposition 9.8.3(d) If π is a primary irreducible then $\chi_\pi(-1) = (-1)^{(a-1)/2}$, where $\pi = a + bi$.

Proof. Let $\pi = a + bi$ a primary irreducible. Then a is odd, and b is even, and $N(\pi) = a^2 + b^2$. Then

$$\chi_\pi(-1) = (-1)^{\frac{N(\pi)-1}{4}} = (-1)^{\frac{a^2-1}{4} + \frac{b^2}{4}} = [(-1)^{\frac{a+1}{2}}]^{\frac{a-1}{2}} (-1)^{\frac{b^2}{4}}.$$

By Lemma 6, section 7, $a \equiv 1 \pmod{4}$, $b \equiv 0 \pmod{4}$, or $a \equiv 3 \pmod{4}$, $b \equiv 2 \pmod{4}$.

- If $a \equiv 1 \pmod{4}$, $b \equiv 0 \pmod{4}$, then $(-1)^{\frac{a+1}{2}} = -1$, $(-1)^{\frac{b^2}{4}} = +1$, so

$$\chi_\pi(-1) = (-1)^{\frac{a-1}{2}}.$$

- If $a \equiv 3 \pmod{4}$, $b \equiv 2 \pmod{4}$, then $(-1)^{\frac{a+1}{2}} = 1$, $(-1)^{\frac{b^2}{4}} = -1$, so

$$\chi_\pi(-1) = -1 = (-1)^{\frac{a-1}{2}}.$$

Conclusion : if π is a primary irreducible in $\mathbb{Z}[i]$, then

$$\chi_\pi(-1) = (-1)^{(a-1)/2}.$$

□

Ex. 9.39 Let $p \equiv 1 \pmod{6}$ and write $4p = A^2 + 27B^2$, $A \equiv 1 \pmod{3}$. Put $m = (p-1)/6$. Show $\binom{3m}{m} \equiv -1 \pmod{p} \iff 2 \mid B$.

Proof. Let p a rational prime, $p \equiv 1 \pmod{6}$. As $p \equiv 1 \pmod{3}$, we know from Theorem 2, Chapter 8, that there are integers A and B such that $4p = A^2 + 27B^2$, $A \equiv 1 \pmod{3}$, and that A is uniquely determined by these conditions.

Then A, B are of opposite parity. If we take $a = \frac{A+3B}{2}$, $b = 3B$, then $A = 2a - b$, $B = \frac{b}{3}$, and $4p = (2a - b)^2 + 3b^2$, so $p = a^2 - ab + b^2$. If $\pi = a + b\omega$, then $N(\pi) = p$. Since $A = 2a - b \equiv 1 \pmod{3}$, and $b = 3B \equiv 0 \pmod{3}$, then $a \equiv -1 \pmod{3}$, so π is a primary prime.

- Suppose that $2 \mid B$. Since $p = a^2 - ab + b^2$ is odd, and $b = 3B$,

$$2 \mid B \iff 2 \mid b \iff (b \equiv 0 \pmod{2}, a \equiv 1 \pmod{2}) \iff \pi \equiv 1 \pmod{2}.$$

By Proposition 9.6.1,

$$\pi \equiv 1 \pmod{2} \iff x^3 - 2 \text{ is solvable in } D \iff \chi_\pi(2) = 1.$$

Therefore

$$2 \mid B \iff \chi_\pi(2) = 1.$$

Here χ_π is of order 3, so $\chi_\pi^2 \neq \varepsilon$. By Exercise 8.6,

$$J(\chi_\pi, \chi_\pi) = \chi_\pi(2)^{-2} J(\chi_\pi, \rho),$$

where ρ is the Legendre's character.

In this case, $2 \mid B$, $\chi_\pi(2) = 1$, so $J(\chi_\pi, \chi_\pi) = J(\chi_\pi, \rho)$, and by Lemma 1 section 4, where $p \equiv 1 \pmod{3}$ and $p = N(\pi)$,

$$\pi = a + b\omega = J(\chi_\pi, \chi_\pi) = J(\chi_\pi, \rho).$$

By Exercise 8.15,

$$N(y^2 = x^3 + 1) = p + A,$$

and the Exercise 8.27(b) gives

$$N(y^2 = x^3 + 1) = N(y^2 + x^3 = 1) = p + 2 \operatorname{Re} J(\chi_\pi, \rho).$$

thus

$$A = 2 \operatorname{Re} J(\chi_\pi, \rho) = 2 \operatorname{Re} \pi = 2a - b.$$

Moreover, since $J(\chi_\pi, \rho) = \pi = a + b\omega$, by Exercise 8.27(c),

$$2a - b \equiv -\binom{(p-1)/2}{(p-1)/3}.$$

Therefore

$$-A \equiv \binom{(p-1)/2}{(p-1)/3} = \binom{(p-1)/2}{(p-1)/2 - (p-1)/6} = \binom{(p-1)/2}{(p-1)/6} = \binom{3m}{m} \pmod{p},$$

where $m = (p-1)/6$. Since $A \equiv 1 \pmod{3}$,

$$\binom{3m}{m} \equiv -1 \pmod{p}.$$

• Conversely, suppose that $\binom{3m}{m} \equiv -1 \pmod{p}$. Then $A = 2a - b \equiv -\binom{3m}{m} \pmod{p}$. Write $J(\chi_\pi, \rho) = c + d\omega$. By Exercise 8.27(c), $2c - d \equiv -\binom{3m}{m} \pmod{p}$. thus

$$2a - b \equiv 2c - d \pmod{p}.$$

Since $|J(\chi_\pi, \rho)| = \sqrt{p}$,

$$4p = (2a - b)^2 + 3b^2 = (2c - d)^2 + 3d^2,$$

thus $d \equiv \pm b \pmod{p}$.

By Exercise 8.6,

$$\pi = J(\chi_\pi, \chi_\pi) = \chi_\pi(2)^{-2} J(\chi_\pi, \rho),$$

Here χ_π is of order 3, therefore $\chi_\pi(2)^{-2} = \chi_\pi(2) \in \{1, \omega, \omega^2\}$, so

$$\pi = J(\chi_\pi, \chi_\pi) = \chi_\pi(2) J(\chi_\pi, \rho).$$

If $\chi_\pi(2) = \omega$, then $a + b\omega = \omega(c + d\omega) = -d + \omega(c - d)$. Then $a = -d \equiv \pm b \pmod{p}$. As $a \equiv -b\omega \pmod{\pi}$, we would have $-b\omega \equiv \pm b \pmod{\pi}$. Here $\pi \nmid b$, otherwise $p = N(\pi) \mid N(b) = b^2$, so $p \mid b$, and $p = a^2 - ab + b^2$, so $p \mid a$, and $p^2 \mid p$, which is a nonsense. Therefore $\pi \mid \omega \pm 1$, where π is a primary prime : it's impossible : $\omega + 1$ is a unit and $\omega - 1$ is prime, so $\pi \mid \omega - 1 = -\lambda$ implies that π and λ are associate, in contradiction with $N(\pi) = p \neq 3 = N(\pi)$.

If $\chi_\pi(2) = \omega^2$, then $a + b\omega = \omega^2(c + d\omega) = (d - c) - \omega c$, so $a = d - c, b = -c$.

Reasoning modulo $\bar{\pi} = a + b\omega^2 = (a - b) + b\omega$, where $\bar{\pi} \mid \pi\bar{\pi} = p$, we obtain

$$d = a - b \equiv -b\omega \pmod{\bar{\pi}},$$

where $d \equiv \pm b \pmod{\bar{\pi}}$, so $-b\omega \equiv \pm b \pmod{\bar{\pi}}$. Since $N(\bar{\pi}) = p$, we obtain the same contradiction as above.

So $\chi_\pi(2) = 1$, and the previously proved equivalence $2 \mid B \iff \chi_\pi(2) = 1$ show that $2 \mid B$.

Conclusion :

$$\binom{(p-1)/2}{(p-1)/6} \equiv -1 \pmod{p} \iff 2 \mid B.$$

□

Ex. 9.40 Let $p \equiv 1 \pmod{6}$, and put $p = \pi\bar{\pi}$ where π is primary. Write $\pi = a + b\omega$ and show

(a) If $\chi_\pi(2) = \omega$ then $2b - a \equiv -\binom{3m}{m} \pmod{p}$.

(b) If $\chi_\pi(2) = \omega^2$ then $a + b \equiv \binom{3m}{m} \pmod{p}$.

(c) If $\chi_\pi(2) = \omega$ put $A = 2a - b, B = b/3$. Show $(A - 9B)/2 \equiv \binom{3m}{m} \pmod{p}$.

(d) If $\chi_\pi(2) = \omega^2$ put $2a - b = A$ and $B = -b/3$. Show $(A - 9B)/2 \equiv \binom{3m}{m} \pmod{p}$.

(e) Show that the “normalization” of B in (c) and (d) is equivalent to $A \equiv B \pmod{4}$.
[Recall $\chi_\pi(2) \equiv \pi \pmod{2}$ by cubic reciprocity.]

Proof. Here $p = 6m + 1, m \in \mathbb{Z}$, and $p = \pi\bar{\pi}$, where $\pi = a + b\omega$ is a primary prime.

We have proved in Exercise 39 that

$$\pi = J(\chi_\pi, \chi_\pi) = \chi_\pi(2)J(\chi_\pi, \rho). \quad (1)$$

Write $J(\chi_\pi, \rho) = c + d\omega$. The Exercise 8.27(c) shows that

$$2c - d \equiv -\binom{3m}{m} \pmod{p}. \quad (2)$$

(a) If $\chi_\pi(2) = \omega$, then (1) gives

$$a + b\omega = \omega(c + d\omega) = -d + \omega(c - d),$$

so $a = -d, b = c - d$, therefore the equality (2) gives

$$2b - a = 2(c - d) + d = 2c - d \equiv -\binom{3m}{m} \pmod{p}.$$

(b) If $\chi_\pi(2) = \omega^2$, then

$$a + b\omega = \omega^2(c + d\omega) = d - c - c\omega,$$

so $a = d - c, b = -c$, and

$$a + b = d - 2c \equiv \binom{3m}{m} \pmod{p}.$$

(c) Suppose that $\chi_\pi(2) = \omega$, and put $A = 2a - b, B = b/3$, so

$$4p = A^2 + 27B^2, \quad A \equiv 1 \pmod{3},$$

which shows that A, B have opposite parities. Then, by part (a),

$$\begin{aligned} \frac{A - 9B}{2} &= \frac{2a - b - 3b}{2} \\ &= a - 2b \\ &\equiv \binom{3m}{m} \pmod{p} \end{aligned}$$

(d) Suppose that $\chi_\pi(2) = \omega^2$, and put $A = 2a - b, B = -b/3$, so we have again

$$4p = A^2 + 27B^2, \quad A \equiv 1 \pmod{3}.$$

In this case, by part (b)

$$\begin{aligned} \frac{A - 9B}{2} &= \frac{2a - b + 3b}{2} \\ &= a + b \\ &\equiv \binom{3m}{m} \pmod{p} \end{aligned}$$

(e) The conditions $4p = A^2 + 27B^2, A \equiv 1 \pmod{3}$, determine A, B , except the sign of B . So $4p = A^2 + 27B^2 = (2a - b)^2 + 3b^2$, implies $A = 2a - b$ and $B = \pm \frac{b}{3}$.

By Exercise 39, since A, B have same parity, the condition A, B odd is equivalent to $\chi_\pi(2) \in \{\omega, \omega^2\}$. We choose this sign of B so that

$$\frac{A - 9B}{2} \equiv \binom{3m}{m} \pmod{p}.$$

By parts (d) and (e), where A, B are odd, this choice is given by $B = b/3$ if $\chi_\pi(2) = \omega$, and $B = -b/3$ if $\chi_\pi(2) = \omega^2$. We show that these conditions are equivalent to $A \equiv B \pmod{4}$.

• If $\chi_\pi(2) = \omega$, then $A = 2a - b, B = b/3$.

By cubic reciprocity, $\chi_\pi(2) \equiv \pi \pmod{2}$ (see section 6). Here $\chi_\pi(2) = \omega$, so $\omega \equiv a + b\omega \pmod{2}$, therefore $a \equiv 0 \pmod{2}, b \equiv 1 \pmod{2}$,

$$A = 2a - b \equiv -b \equiv \frac{b}{3} = B \pmod{4},$$

so $A \equiv B \pmod{4}$.

• If $\chi_\pi(2) = \omega^2$, then $A = 2a - b, B = -b/3$. In this case,

$$\omega^2 = -1 - \omega \equiv a + b\omega \pmod{2},$$

therefore $a \equiv 1 \equiv b \pmod{2}$, and

$$A = 2a - b \equiv 2 - b \equiv b \equiv -\frac{b}{3} = B \pmod{4}.$$

In both cases, the choice of the sign of B implies that $A \equiv B \pmod{4}$.

Conversely, suppose that $A \equiv B \pmod{4}$. Write $B = \varepsilon \frac{b}{3}$, where $\varepsilon = \pm 1$. Then $A \equiv B \pmod{4}$ gives

$$2a - b \equiv \varepsilon \frac{b}{3} \equiv -\varepsilon b \pmod{4},$$

thus $a \equiv \frac{1-\varepsilon}{2}b \pmod{2}$. Then

$$\begin{aligned}\chi_\pi(2) &\equiv \pi = a + b\omega \\ &\equiv b \left(\frac{1-\varepsilon}{2} + \omega \right) \pmod{2}\end{aligned}$$

If $\chi_\pi(2) = \omega$, since $b = 3B$ is odd, $\frac{1-\varepsilon}{2} \equiv 0 \pmod{2}$, therefore $\varepsilon = 1$, and $B = \frac{b}{3}$.

If $\chi_\pi(2) = \omega^2 = -1 - \omega$, $\frac{1-\varepsilon}{2} \equiv 1 \pmod{2}$, therefore $\varepsilon = -1$, and $B = -\frac{b}{3}$.

The normalisation given in parts (c) and (d) for the choice of the sign of B is equivalent to $A \equiv B \pmod{4}$ (where A, B are odd).

□

Ex. 9.41 Let $p \equiv 1 \pmod{6}$, $4p = A^2 + 27B^2$, $A \equiv 1 \pmod{3}$, A and B odd. Put $\pi = a + b\omega$, $2a - b = A$, $b = 3B$. Let χ_π be the cubic residue character.

- (a) If $\chi_\pi(2) = \omega$ show $N(x^3 + 2y^3 = 1) = p + 1 + 2b - a \equiv 0 \pmod{2}$.
- (b) If $\chi_\pi(2) = \omega^2$ show $N(x^3 + 2y^3 = 1) = p + 1 - a - b \equiv 0 \pmod{2}$.
- (c) Show that if $A \equiv B \pmod{4}$, then assuming $\chi_\pi(2) \neq 1$, one has $\chi_\pi(2) = \omega$.
- (d) If $\chi_\pi(2) \neq 1$, $A \equiv B \pmod{4}$ then

$$2^{(p-1)/3} \equiv (-A - 3B)/6B \equiv (A + 9B)/(A - 9B) \pmod{\pi}.$$

(This generalization of Euler's criterion is due to E. Lehmer [174]. See also K. Williams [243].)

Proof. With the help of Theorem 1, Chapter 8, we obtain, writing $\chi_\pi(2) = \omega^k$,

$$\begin{aligned}
N(x^3 + 2y^3 = 1) &= \sum_{a+2b=1} N(x^3 = a)N(y^3 = b) \\
&= \sum_{a+2b=1} \left(\sum_{i=0}^2 \chi_\pi^i(a) \right) \left(\sum_{j=0}^2 \chi_\pi^j(b) \right) \\
&= \sum_{i=0}^2 \sum_{j=0}^2 \sum_{a+2b=1} \chi_\pi^i(a) \chi_\pi^j(b) \\
&= \sum_{i=0}^2 \sum_{j=0}^2 \sum_{a+b'=1} \chi_\pi^i(a) \chi_\pi^j(2^{-1}b') \\
&= \sum_{i=0}^2 \sum_{j=0}^2 \chi_\pi(2)^{-j} J(\chi_\pi^i, \chi_\pi^j) \\
&= \sum_{i=0}^2 \sum_{j=0}^2 \omega^{-kj} J(\chi_\pi^i, \chi_\pi^j) \\
&= p + \omega^{-k} J(\chi_\pi^2, \chi_\pi) + \omega^{-2k} J(\chi_\pi, \chi_\pi^2) \\
&\quad + \omega^{-k} J(\chi_\pi, \chi_\pi) + \omega^{-2k} J(\chi_\pi^2, \chi_\pi^2) \\
&= p - \omega^{-k} \chi_\pi(-1) - \omega^{-2k} \chi_\pi(-1)^2 + 2 \operatorname{Re}(\omega^{-k} J(\chi_\pi, \chi_\pi)) \\
&= p - \omega^{-k} - \omega^{-2k} + 2 \operatorname{Re}(\omega^{-k} J(\chi_\pi, \chi_\pi)).
\end{aligned}$$

(a) If $\chi_\pi(2) = \omega$, then $k = 1$. Using $\chi_\pi^2 = \chi_\pi^{-1} = \overline{\chi_\pi}$, we obtain

$$\begin{aligned}
N(x^3 + 2y^3 = 1) &= p + 1 + 2 \operatorname{Re}(\omega^2 J(\chi_\pi, \chi_\pi)) \\
&= p + 1 + 2 \operatorname{Re}(\omega^2 \pi),
\end{aligned}$$

since $J(\chi_\pi, \chi_\pi) = \pi$ (Lemma 1, section 4).

$$\begin{aligned}
\omega^2 \pi &= \omega^2(a + b\omega) = b - a - \omega a, \\
2 \operatorname{Re}(\omega^2 \pi) &= (b - a - \omega a) + (b - a - \omega^2 a) = 2b - 2a + a = 2b - a,
\end{aligned}$$

therefore

$$N(x^3 + 2y^3 = 1) = p + 1 + 2b - a \quad (\text{if } \chi_\pi(2) = \omega).$$

Since in then case $\chi_\pi(2) = \omega$, then $a \equiv 0 \pmod{2}$ (see Ex. 39, part (e)), so $p + 1 + 2b - a \equiv 0 \pmod{2}$.

(b) If $\chi_\pi(2) = \omega^2 = \omega^{-1}$, then $k = -1$, and

$$N(x^3 + 2y^3 = 1) = p + 1 + 2 \operatorname{Re}(\omega \pi),$$

with

$$\begin{aligned}
\omega \pi &= \omega(a + b\omega) = -b + (a - b)\omega, \\
2 \operatorname{Re}(\omega \pi) &= (-b + (a - b)\omega) + (-b + (a - b)\omega^2) = -2b - (a - b) = -a - b,
\end{aligned}$$

therefore

$$N(x^3 + 2y^3 = 1) = p + 1 - a - b \quad (\text{if } \chi_\pi(2) = \omega^2).$$

(c) Suppose that $A \equiv B \pmod{4}$, and $\chi_\pi(2) \neq 1$. By hypothesis, $b = 3B$, and this implies by Exercise 40 (e) that $\chi_\pi(2) = \omega$ (if not, $\chi_\pi(2) = \omega_2$, and $A \equiv B \pmod{4}$ gives $B = -b/3$).

(d) Suppose that $\chi_\pi(2) \neq 1, A \equiv B \pmod{4}$. By part (c), $\chi_\pi(2) = \omega$.

Since $2a - b = A, B = b/3$, then $a = \frac{A+3B}{2}, b = 3B$.

Starting from $a + b\omega \equiv 0 \pmod{\pi}$, we obtain

$$3B\omega \equiv -\frac{A+3B}{2} \pmod{\pi}.$$

Since $pa = a^2 - ab + b^2$, a is relatively prime with p , therefore $\pi \nmid b = 1$, so $\pi \nmid B = 1$, and $\pi \nmid 6 = 1$, since $p \equiv 1 \pmod{6}$, thus

$$\chi_\pi(2) = \omega \equiv \frac{-A-3B}{6B} \pmod{\pi},$$

where we must read in this fraction the product of $A+3B$ by the inverse modulo p of $6B$. By definition, using $N(\pi) = p$,

$$\chi_\pi(2) \equiv 2^{\frac{p-1}{3}} \pmod{\pi},$$

so

$$2^{\frac{p-1}{3}} \equiv \frac{-A-3B}{6B} \pmod{\pi}.$$

Moreover, since $4p = A^2 + 27B^2$, $A^2 + 27B^2 \equiv 0 \pmod{p}$, therefore

$$6B(A+9B) + (A+3B)(A-9B) \equiv 0 \pmod{p}.$$

If $p \mid A-9B$, since $p \nmid 6B$, this equality implies that $p \mid A+9B$, therefore $p \mid (A-9B) + (A+9B) = 2A$, which is false. Therefore $A-9B \not\equiv 0 \pmod{p}$, and

$$2^{\frac{p-1}{3}} \equiv \frac{-A-3B}{6B} \equiv \frac{A+9B}{A-9B} \pmod{\pi}.$$

□

Note : By a usual argument, if $h \in \mathbb{Z}$, $2^{\frac{p-1}{3}} \equiv h \pmod{\pi} \iff 2^{\frac{p-1}{3}} \equiv h \pmod{p}$. Note that the hypothesis $\chi_\pi(2) \neq 1$ means that 2 is not a cubic residue modulo p , which is equivalent to A, B odd by Exercise 39. We can conclude

Suppose that $p \equiv 1 \pmod{6}$, and let (A, B) be the unique solution of $4p = A^2 + 27B^2$ such that $A \equiv 1 \pmod{3}$, and $B \equiv A \pmod{4}$ if B odd, and $B > 0$ otherwise.

If B is even, then 2 is a cubic residue modulo p , and $2^{\frac{p-1}{3}} = 1$.

If B is odd, then 2 is not a cubic residue modulo p , and B satisfies $B \equiv A \pmod{4}$.

Writing $a = \frac{A+3B}{2}, b = 3B$, and $\pi = a + b\omega$, then $\chi_\pi(2) = \omega$, and

$$2^{\frac{p-1}{3}} \equiv \frac{A+9B}{A-9B} \pmod{p}.$$

The three roots of $x^3 - 1$ in \mathbb{F}_p are $1, \frac{A+9B}{A-9B}, \frac{A-9B}{A+9B}$. Here 2 is not a cubic residue modulo p , and $2^{\frac{p-1}{3}}$ is also a cubic root of unity modulo p , so $2^{\frac{p-1}{3}} \equiv \frac{A+9B}{A-9B} \pmod{p}$. The proposition explicits the choice of the sign of B which gives $2^{\frac{p-1}{3}} \equiv \frac{A-9B}{A+9B} \pmod{p}$.

Numerical example : Let p be the prime 967. If we decompose p on the form $p = \pi\bar{\pi}$, we obtain $\pi = a + b\omega = -34 - 27\omega$. To obtain these result without tries, I find $k = 682$ such that $p \mid k^2 + 3$ with the Tonelli-Shanks algorithm, and I compute $\gcd(p, k+1+2\omega) = a + b\omega$, where $a + b\omega$ is primary, with a small Python program using the class of elements in $\mathbb{Z}[\omega]$ and the Euclid algorithm in $\mathbb{Z}[\omega]$. This gives the decompositions

$$967 = p = a^2 - ab + b^2 = 34^2 - 34 \times 27 + 27^2,$$

and

$$3868 = 4p = (2a - b)^2 + 3b^2 = A^2 + 27B^2 = 41^2 + 27 \times 9^2,$$

where $A \equiv 1 \pmod{3}$, and I choose the sign of B such that $B \equiv A \pmod{4}$. We obtain $A = -41, B = -9$, and a, b must verify $A = 2a - b, B = b/3$.

Then $\chi_\pi(2) = \omega$, where $\pi = -41 - 9\omega$. In \mathbb{F}_{967} , the cubic roots of unity modulo p are $1, 142, 824$: $142^3 \equiv 824^3 \equiv 1 \pmod{967}$.

Here $(A + 9B)(A - 9B)^{-1} = 142$, and we verify with a fast exponentiation that $2^{\frac{p-1}{3}} = 2^{322} \equiv 142 \pmod{967}$.

I give here an extract of a table obtained with this program, which for each p gives A, B such that $4p = A^2 + 27B^2$, $A \equiv 1 \pmod{3}$ and such that $B \equiv A \pmod{4}$ if A, B odd, and $\pi = a + b\omega$ satisfies $\chi_\pi(2) = \omega$ (or $\chi_\pi(2) = 1$ if A, B even, which corresponds to the case $a \equiv 1, b \equiv 0 \pmod{2}$).

p	A	B	$\pi = a + b\omega$	$a \% 2$	$b \% 2$	$2^{\frac{p-1}{3}} \% p$	$\frac{A-9B}{A+9B}$	$\chi_\pi(2)$
787	31	-9	$2 - 27\omega$	0	1	379	379	ω
811	-56	-2	$-31 - 6\omega$	1	0	1	130	1
823	-5	11	$14 + 33\omega$	0	1	648	648	ω
829	7	11	$20 + 33\omega$	0	1	125	125	ω
853	-35	9	$-4 + 27\omega$	0	1	632	632	ω
859	13	-11	$-10 - 33\omega$	0	1	260	260	ω
877	-59	1	$-28 + 3\omega$	0	1	594	594	ω
883	-47	-7	$-34 - 21\omega$	0	1	545	545	ω
907	19	11	$26 + 33\omega$	0	1	384	384	ω
919	52	-6	$17 - 18\omega$	1	0	1	52	1
937	61	1	$32 + 3\omega$	0	1	614	614	ω
967	-41	-9	$-34 - 27\omega$	0	1	142	142	ω
991	61	-3	$26 - 9\omega$	0	1	113	113	ω
997	10	-12	$-13 - 36\omega$	1	0	1	692	1

As a verification I compute $\chi_\pi(2)$ with a fast exponentiation in $\mathbb{Z}[\omega]$: $\chi_\pi(2) = 2^{\frac{p-1}{3}} \pmod{\pi}$.

We obtain the primary prime μ such that $N(\mu) = p, \chi_\mu(2) = \omega^2$ by taking the conjugate of π . For instance, with $p = 787$, $\pi = 2 - 27\omega$ satisfies $\chi_\pi(2) = \omega$, therefore $\chi_{\bar{\pi}}(2) = \chi_{29+27\omega} = \omega^2$.

The lines where $\chi_\pi(2) = 1$, corresponding to the case where A, B are even (or equivalently a odd, b even), give the decomposition $p = x^2 + 27y^2$, ($x = A/2, y = B/2$). For instance $997 = 5^2 + 27 \times 6^2$. If p is prime,

$$\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, p = x^2 + 27y^2 \iff p \equiv 1 \pmod{3} \text{ and } \exists a \in \mathbb{Z}, 2 \equiv a^3 \pmod{p}.$$

Ex. 9.42 The notation being as in Section 12 show that the minimal polynomial of $g(\chi_\pi)$ is $x^3 - 3px - Ap$.

Note : we must read “the minimal polynomial of $G = g(\chi_\pi) + \overline{g(\chi_\pi)}$ is $x^3 - 3px - Ap$ ”.

Proof. Write $f(x) = \sum_{i=0}^3 a_i x^i = x^3 - 3px - Ap$.

Then $a_3 = 1$, $p \mid a_0 = Ap$, $p \mid a_1 = -3p$, $p \mid a_2 = 0$.

Moreover, since $4p = A^2 + 27B^2$, $p \nmid A$, therefore $p^2 \nmid a_0$.

The Eisenstein's Irreducibility Criterion (Ex. 6.23) shows that $f(x)$ is irreducible over \mathbb{Q} . By section 12, G is a root of f , so f is the minimal polynomial of G . \square

Ex. 9.43 Find the local maxima and minima of $x^3 - 3px - Ap$ and show that each of the intervals $(-2\sqrt{p}, -\sqrt{p})$, $(-\sqrt{p}, \sqrt{p})$, $(\sqrt{p}, 2\sqrt{p})$ contains exactly one of the values $2\text{Re}(\omega^k g(\chi_\pi))$, $k = 0, 1, 2$.

Proof. Write $\chi = \chi_\pi$, and for $k \in \{0, 1, 2\}$,

$$G_k = 2\text{Re}(\omega^k g(\chi)) = \omega^k g(\chi) + \overline{\omega^k g(\chi)},$$

so $G = G_0$. As in section 12, since $g(\chi)^3 = p\pi$, and $|g(\chi)|^2 = p$,

$$\begin{aligned} G_k^3 &= g(\chi)^3 + \overline{g(\chi)}^3 + 3\omega^{2k} g(\chi)^2 \overline{g(\chi)} + 3\omega^k g(\chi) \overline{\omega^{2k} g(\chi)}^2 \\ &= p\pi + p\overline{\pi} + 3g(\chi)\overline{g(\chi)}(\omega^k g(\chi) + \overline{\omega^k g(\chi)}) \\ &= 3pG_k + p(2a - b) \\ &= 3pG_k + pA \end{aligned}$$

So G_0, G_1, G_2 are the three roots of $f(x) = x^3 - 3px - Ap$.

$f'(x) = 3(x^2 - p) < 0$ iff $-\sqrt{p} < x < \sqrt{p}$. f is decreasing on $[-\sqrt{p}, \sqrt{p}]$, and increasing on $] -\infty, -\sqrt{p}[$, and on $[\sqrt{p}, +\infty[$.

Since $4p = A^2 + 27B^2$, $|A| < 2\sqrt{p}$, therefore

$$\begin{aligned} f(\sqrt{p}) &= p\sqrt{p} - 3p\sqrt{p} - Ap \\ &= -p(2\sqrt{p} + A) < 0, \end{aligned}$$

and

$$\begin{aligned} f(-\sqrt{p}) &= -p\sqrt{p} + 3p\sqrt{p} - Ap \\ &= p(2\sqrt{p} - A) > 0. \end{aligned}$$

\square

Since $\lim_{x \rightarrow -\infty} f(x) = -\infty$ and $\lim_{x \rightarrow +\infty} f(x) = +\infty$, the intermediate value theorem shows that f has a unique root in each of the intervals $] -\infty, -\sqrt{p}[$, $[-\sqrt{p}, \sqrt{p}]$, $[\sqrt{p}, +\infty[$.

Moreover

$$\begin{aligned} f(2\sqrt{p}) &= 8p\sqrt{p} - 6p\sqrt{p} - Ap = p(2\sqrt{p} - A) > 0, \\ f(-2\sqrt{p}) &= -8p\sqrt{p} + 6p\sqrt{p} - Ap = p(-2\sqrt{p} - A) < 0, \end{aligned}$$

therefore f has a unique root in each of the intervals $] -2\sqrt{p}, -\sqrt{p}[$, $[-\sqrt{p}, \sqrt{p}]$, $[\sqrt{p}, 2\sqrt{p}[$.

Ex. 9.44 Let $n \in \mathbb{Z}$, $n = s_1 \cdots s_t$, $n \equiv 1 \pmod{4}$, $i = 1, \dots, t$. Show $(n-1)/4 \equiv \sum_{i=1}^t (s_i-1)/4 \pmod{4}$.

Proof. If $n = st$, $s \equiv 1$, $t \equiv 1 \pmod{4}$, then $s = 4k+1$, $t = 4l+1$, $k, l \in \mathbb{Z}$, so

$$n = (4k+1)(4l+1) = 16kl + 4k + 4l + 1, \frac{n-1}{4} = 4kl + k + l \equiv k + l = \frac{s-1}{4} + \frac{t-1}{4} \pmod{4}.$$

Reasoning by induction on t , suppose that every product of t factors $n = s_1 s_2 \cdots s_t$, where $s_i \equiv 1 \pmod{4}$ verifies

$$\frac{n-1}{4} \equiv \sum_{i=1}^t \frac{s_i-1}{4} \pmod{4}.$$

If $n' = s_1 s_2 \cdots s_t s_{t+1} = n s_{t+1}$, $s_i \equiv 1 \pmod{4}$, then $n \equiv 1$, $s_{t+1} \equiv 1 \pmod{4}$, so

$$\frac{n'-1}{4} \equiv \frac{n-1}{4} + \frac{s_{t+1}-1}{4} \equiv \sum_{i=1}^t \frac{s_i-1}{4} + \frac{s_{t+1}-1}{4} \equiv \sum_{i=1}^{t+1} \frac{s_i-1}{4} \pmod{4}.$$

Conclusion : if $n = s_1 s_2 \cdots s_t$, $s_i \equiv 1 \pmod{4}$, alors $\frac{n-1}{4} \equiv \sum_{i=1}^t \frac{s_i-1}{4} \pmod{4}$. □

Ex. 9.45 Let $\pi = a + bi \in \mathbb{Z}[i]$ and $q \equiv 3 \pmod{4}$ a rational prime. Show $\pi^q \equiv \bar{\pi} \pmod{4}$.

Proof. Let $\pi = a + bi \in \mathbb{Z}[i]$, and $q \equiv 3 \pmod{4}$ a rational prime.

As $\binom{q}{k} \equiv 0 \pmod{q}$ for $1 \leq k \leq q-1$, the Fermat's Little Theorem gives

$$\begin{aligned} \pi^q &= (a + bi)^q \\ &\equiv a^q + b^q i^q \pmod{q} \\ &\equiv a + bi^3 \pmod{q} \\ &= a - bi \\ &= \bar{\pi} \end{aligned}$$

Conclusion : $\pi^q \equiv \bar{\pi} \pmod{q}$ ($\pi \in \mathbb{Z}[i]$, and $q \equiv 3 \pmod{4}$) □