

# Solutions to Ireland, Rosen “A Classical Introduction to Modern Number Theory”

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## Chapter 7

**Ex. 7.1** Use the method of Theorem 1 to show that a finite subgroup of the multiplicative group of a field is cyclic.

A solution is already given in Ex. 4.15

**Ex. 7.2** Find the finite subgroups of  $\mathbb{R}^*$  and  $\mathbb{C}^*$  and show directly that they are cyclic.

*Proof.* If  $G$  is a finite subgroup of  $\mathbb{R}$  or  $\mathbb{C}$ , and  $n = |G|$ , then from Lagrange’s Theorem,  $x^n = 1$  for all  $x \in G$ .

- If  $G$  is a finite subgroup of  $\mathbb{R}^*$ , then the solutions of  $x^n = 1$  are in  $\{-1, 1\}$ , so  $\{1\} \subset G \subset \{-1, 1\}$  :  $G = \{1\}$  or  $G = \{-1, 1\}$ , both cyclic.
- If  $G$  is a finite subgroup of  $\mathbb{C}^*$ , then  $G \subset \mathbb{U}_n = \{e^{2ik\pi/n} \mid 0 \leq k \leq n-1\}$ . As  $|G| = |\mathbb{U}_n| = n$ , then  $G = \mathbb{U}_n \simeq \mathbb{Z}/n\mathbb{Z}$  is cyclic.  $\square$

**Ex. 7.3** Let  $F$  a field with  $q$  elements and suppose that  $q \equiv 1 \pmod{n}$ . Show that for  $\alpha \in F^*$ , the equation  $x^n = \alpha$  has either no solutions or  $n$  solutions.

*Proof.* This is a particular case of Prop. 7.1.2., where  $d = n \wedge (q-1) = n$  : the equation  $x^n = \alpha$  has solutions iff  $\alpha^{(q-1)/n} = 1$ . In this case, there are exactly  $d = n$  solutions.

We give here a direct proof.

Let  $g$  a generator of  $F^*$ . Write  $x = g^y, \alpha = g^a$ . Then

$$x^n = \alpha \iff g^{ny} = g^a \iff q-1 \mid ny - a.$$

Suppose that there exists  $x \in F$  such that  $x^n = \alpha$ . Then there exists  $y \in \mathbb{Z}$  such that  $q-1 \mid ny - a$ . Since  $n \mid q-1$ , then  $n \mid a$ .

$$q-1 \mid ny - a \iff \frac{q-1}{n} \mid y - \frac{a}{n} \iff y = \frac{a}{n} + k \frac{q-1}{n}, k \in \mathbb{Z}.$$

As  $\frac{a}{n} + (k+n) \frac{q-1}{n} = \frac{a}{n} + k \frac{q-1}{n}, k \in \mathbb{Z}$ , the values  $k = 0, 1, \dots, n-1$  are sufficient :

$$x^n = \alpha \iff y = \frac{a}{n} + k \frac{q-1}{n}, k \in \{0, 1, \dots, n-1\}.$$

Moreover, these solutions are all distinct : if  $k, l \in \{0, 1, \dots, n-1\}$ ,

$$\begin{aligned} g^{\frac{a}{n} + k \frac{q-1}{n}} &= g^{\frac{a}{n} + l \frac{q-1}{n}} \Rightarrow g^{(k-l) \frac{q-1}{n}} = 1 \\ &\Rightarrow q-1 \mid (k-l) \frac{q-1}{n} \\ &\Rightarrow n \mid k-l \\ &\Rightarrow k \equiv l \pmod{n} \Rightarrow k = l. \end{aligned}$$

Conclusion : if  $F$  is a field with  $q$  elements and  $n \mid q-1$ , the equation  $x^n = \alpha$  has either no solutions or  $n$  solutions in  $F$ .

Remark :

$$\exists x \in F^*, x^n = \alpha \iff n \mid a \iff \alpha^{(q-1)/n} = 1.$$

Indeed, if  $x^n = \alpha$  has a solution, we have proved that  $n \mid a$ , thus  $\alpha^{(q-1)/n} = (g^{a/n})^{q-1} = 1$ .

Reciprocally, if  $\alpha^{(q-1)/n} = 1$ ,  $g^{a \cdot (q-1)/n} = 1$ , thus  $q-1 \mid a(q-1)/n$ , so  $n \mid a : \alpha = x^n$ , with  $x = g^{n/a}$ .  $\square$

**Ex. 7.4** (continuation) Show that the set of  $\alpha \in F^*$  such that  $x^n = \alpha$  is solvable is a subgroup with  $(q-1)/n$  elements.

*Proof.* Here  $n \mid q-1$ .

Let  $\varphi = F^* \rightarrow F^*$  the application defined by  $\varphi(x) = x^n$ .  $\varphi$  is a morphism of groups, and  $\ker \varphi$  is the set of solutions of  $x^n = 1$ . As  $n \mid q-1$ ,  $x^n = 1$  has exactly  $n$  solutions (Prop 7.1.1, Corollary 2, or Ex 7.3 with  $\alpha = 1$ ). So  $|\ker \varphi| = n$ .

Thus  $\text{Im} \varphi \simeq F^*/\ker \varphi$  is a subgroup with cardinality  $|F^*|/|\ker \varphi| = (q-1)/n$ , and  $\text{Im} \varphi$  is the set of  $\alpha$  such that  $x^n = \alpha$  is solvable.

Conclusion : the set of  $\alpha \in F^*$  such that  $x^n = \alpha$  is solvable is a subgroup with  $(q-1)/n$  elements.  $\square$

**Ex. 7.5** (continuation) Let  $K$  be a field containing  $F$  such that  $[K : F] = n$ . For all  $\alpha \in F^*$ , show that the equation  $x^n = \alpha$  has  $n$  solutions in  $K$ . [Hint: Show that  $q^n - 1$  is divisible by  $n(q-1)$  and use the fact that  $\alpha^{q-1} = 1$ .]

*Proof.* As  $q \equiv 1 \pmod{n}$ ,  $\frac{q^n - 1}{q - 1} = 1 + q + \dots + q^{n-1} \equiv 0 \pmod{n}$ , then  $n \mid \frac{q^n - 1}{q - 1}$  :

$$q^n - 1 = kn(q - 1), k \in \mathbb{N}.$$

Since  $\alpha \in F^*$ ,  $\alpha^{q-1} = 1$ , so

$$\alpha^{(q^n - 1)/n} = (\alpha^{q-1})^k = 1.$$

As  $|K| = q^n$ , Prop. 7.1.2 (or the final remark in Ex. 7.3) show that there exists  $x \in K^*$  such that  $x^n = \alpha$ . Then, from Ex. 7.3, we know that there exist  $n$  solutions in  $K$ .

Conclusion : if  $[K : F] = n$ , the equation  $x^n = \alpha$  has  $n$  solutions in  $K$ .  $\square$

**Ex. 7.6** Let  $K \supset F$  be finite fields with  $[K : F] = 3$ . Show that if  $\alpha \in F$  is not a square in  $F$ , it is not a square in  $K$ .

*Proof.* Let  $q = |F|$ . Then  $|K| = q^3$ .

If the characteristic of  $F$  is 2,  $q = 2^k$ , and for all  $x \in F$ ,  $x = x^q = (x^{2^{k-1}})^2$ . So all elements in  $F$  or  $K$  are squares. We can now suppose that the characteristic of  $F$  is not 2, and consequently  $1 \neq -1$  in  $F$ .

As  $\alpha$  is not a square in  $F$ ,  $\alpha^{(q-1)/2} \neq 1$  (Prop. 7.1.2). From  $0 = \alpha^{q-1} - 1 = (\alpha^{(q-1)/2} - 1)(\alpha^{(q-1)/2} + 1)$ , we deduce  $\alpha^{(q-1)/2} = -1$ . Then

$$\alpha^{(q^3-1)/2} = (\alpha^{(q-1)/2})^{q^2+q+1} = (-1)^{q^2+q+1} = -1,$$

since  $q^2 + q + 1$  is always odd.

$\alpha^{(q^3-1)/2} \neq 1$  : this implies (Prop. 7.1.2) that  $\alpha$  is not a square in  $K$ .  $\square$

**Ex. 7.7** Generalize Exercise 6 by showing that if  $\alpha$  is not a square in  $F$ , it is not a square in any extension of odd degree and is a square in every extension of even degree.

*Proof.* Write  $q = [K : F]$ , and  $q = \text{Card } F$ .

As  $\alpha$  is not a square in  $F$ , the characteristic of  $F$  is not 2 (see Ex.7.6), and  $\alpha^{(q-1)/2} \neq 1$ . Since  $\alpha^{q-1} = 1$ ,  $\alpha^{(q-1)/2} = -1$ .

$$\alpha^{(q^n-1)/2} = (\alpha^{(q-1)/2})^{1+q+\dots+q^{n-1}} = (-1)^{1+q+\dots+q^{n-1}}.$$

• If  $n$  is odd,  $1+q+\dots+q^{n-1} \equiv 1 \pmod{2}$ , thus  $\alpha^{(q^n-1)/2} = -1 \neq 1$ , and consequently  $\alpha$  is not a square in  $K$ .

• If  $n$  is even, as  $q$  is odd ( $\text{char}(F) \neq 2$ ),  $1+q+\dots+q^{n-1} \equiv 0 \pmod{2}$ , thus  $\alpha^{(q^n-1)/2} = 1$ , so  $\alpha$  is a square in  $K$ .  $\square$

**Ex. 7.8** In a field with  $2^n$  elements, what is the subgroup of squares.

Let  $F$  a field with  $q = 2^n$  elements.

### Proof 1

*Proof.*  $d = (q-1) \wedge 2 = (2^n-1) \wedge 2 = 1$ , thus each  $\alpha \in F^*$  verifies  $\alpha^{(q-1)/d} = \alpha^{q-1} = 1$ . Theorem 7.1.2 show that  $\alpha$  is a square in  $F$ , of exactly one root.  $\square$

### Proof 2

*Proof.* For all  $x \in F$ ,  $x = x^q = (x^{2^{n-1}})^2$ . So all elements in  $F$  or  $K$  are squares.  $\square$

**Ex. 7.9** If  $K \supset F$  are finite fields,  $|F| = q$ ,  $q \equiv 1 \pmod{n}$ , and  $x^n = \alpha$  is not solvable in  $F$ , show that  $x^n = \alpha$  is not solvable in  $K$  if  $(n, [K : F]) = 1$ .

*Proof.* Let  $k = [K : F]$ . From hypothesis,  $k \wedge n = 1$ , so there exist integers  $u, v$  such that  $uk + vn = 1$ .

As  $n \mid q-1$ ,  $n \wedge (q-1) = n$ , so the hypothesis " $x^n = \alpha$  is not solvable in  $F$ " implies that  $\alpha^{(q-1)/n} \neq 1$  (Prop. 7.1.2).

Write  $\omega = \alpha^{(q-1)/n}$ , so  $\omega \neq 1$  and  $\omega^n = 1$ .

As  $n \mid q-1$ ,  $n \mid q^k-1$  and

$$\alpha^{(q^k-1)/n} = (\alpha^{(q-1)/n})^{1+q+q^2+\dots+q^{k-1}} = \omega^{1+q+q^2+\dots+q^{k-1}}.$$

Moreover  $1+q+\dots+q^{k-1} \equiv k \pmod{n}$ , and  $\omega^n = 1$ , so  $\alpha^{(q^k-1)/n} = \omega^k$ .

If  $\omega^k = 1$ , then  $\omega = \omega^{uk+vn} = (\omega^k)^u (\omega^n)^v = 1$ , which is in contradiction with  $\omega = \alpha^{(q-1)/n} \neq 1$ .

So  $\alpha^{(q^k-1)/n} = \omega^k \neq 1$ , and consequently the equation  $x^n = \alpha$  has no solution in  $K$ .  $\square$

**Ex. 7.10** If  $K \supset F$  be finite fields and  $[K : F] = 2$ . For  $\beta \in K$ , show that  $\beta^{1+q} \in F$  and moreover that every element in  $F$  is of the form  $\beta^{1+q}$  for some  $\beta \in K$ .

*Proof.* If  $\beta = 0$ ,  $\beta^{1+q} = 0 \in F$ , and if  $\beta \in K^*$ ,  $\beta^{q^2-1} = 1$ , so  $(\beta^{1+q})^{q-1} = 1$ , thus  $\beta^{1+q} \in F$  (Prop. 7.1.1, Corollary 1).

Let  $g$  a generator of  $K^* : K^* = \{1, g, g^2, \dots, g^{q^2-2}\}$ .

For every integer  $k \in \mathbb{Z}$ ,

$$g^k \in F^* \iff (g^k)^{q-1} = 1 \iff g^{k(q-1)} = 1 \iff q^2 - 1 \mid k(q-1) \iff q+1 \mid k.$$

Thus  $F^* = \{1, g^{q+1}, g^{2(q+1)}, \dots, g^{(q-2)(q+1)}\}$ . If  $\alpha \in F^*$ , there exists  $i, 0 \leq i \leq q-1$  such that  $\alpha = g^{i(q+1)}$ . If we write  $\beta = g^i$ , then  $\alpha = \beta^{1+q}$  (and for  $\alpha = 0$ , we take  $\beta = 0$ ).

Conclusion : if  $K$  is a quadratic extension of  $F$  ( $F, K$  finite fields), every element in  $F$  is of the form  $\beta^{1+q}$  for some  $\beta \in K$ .  $\square$

**Ex. 7.11** With the situation being that of Exercise 10 suppose that  $\alpha \in F$  has order  $q-1$ . Show that there is a  $\beta \in K$  with order  $q^2-1$  such that  $\beta^{1+q} = \alpha$ .

Write  $|a|$  the order of an element  $a$  in a group  $G$ . We recall the following lemma :

**Lemma** If  $|a| = d$ , then for all  $i \in \mathbb{Z}$ ,  $|a^i| = \frac{d}{d \wedge i}$ .

*Proof.* Indeed, for all  $k \in \mathbb{Z}$ ,

$$(a^i)^k = e \iff a^{ik} = e \iff d \mid ik \iff \frac{d}{d \wedge i} \mid \frac{i}{d \wedge i} k \iff \frac{d}{d \wedge i} \mid k.$$

$\square$

*Proof.* (Ex. 7.11)

Let  $\alpha \in F^*$  with  $|\alpha| = q-1$ , and  $g$  a generator of  $K^*$ , so  $|g| = q^2-1$ . We know from exercise 7.10 that there exists an integer  $i$  such that  $\alpha = g^{i(q+1)}$ .

Let  $h = g^{q+1}$ . As  $h^{q-1} = 1$ , then  $h \in F^*$ , and since  $|g| = q^2-1$ ,  $|h| = q-1$ , so  $h$  is a generator of  $F^*$ .

Note that for all  $s \in \mathbb{Z}$ ,  $\alpha = g^{(i+s(q-1))(q+1)}$ , since  $g^{q^2-1} = 1$ .

We will show that we can choose  $s$  such that  $j = i + s(q-1)$  is relatively prime with  $q+1$ . Then  $j$  is such that  $\alpha = g^{j(q+1)} = h^j$ .

$i$  is odd : if not  $\alpha$  is an element of the subgroup of squares in  $F^*$ , so its order divides  $(q-1)/2$ , in contradiction with  $|\alpha| = q-1$ .

$(q-1) \wedge (q+1) \mid 2$ . Since  $i-1$  is even, there exist integers  $s, t$  verifying the Bézout's equation

$$i-1 = t(q+1) - s(q-1).$$

Then  $j = i + s(q - 1) = 1 + t(q + 1)$  is relatively prime with  $q + 1 : j \wedge (q + 1) = 1$ .

Moreover, as  $\alpha = h^j$ , with  $|\alpha| = |h| = q - 1$ , the lemme implies that

$$q - 1 = |\alpha| = \frac{q - 1}{(q - 1) \wedge j},$$

so  $(q - 1) \wedge j = 1$ . As  $(q + 1) \wedge j = 1$  and  $(q - 1) \wedge j = 1$ , then  $(q^2 - 1) \wedge j = 1$ .

Let  $\beta = g^j$  : then  $\alpha = \beta^{1+q}$ , and using the lemma :

$$|\beta| = |g^j| = \frac{q^2 - 1}{(q^2 - 1) \wedge j} = q^2 - 1.$$

Conclusion : there exists a  $\beta \in K^*$  with order  $q^2 - 1$  such that  $\beta^{1+q} = \alpha$ .  $\square$

**Ex. 7.12** Use Proposition 7.2.1 to show that given a field  $k$  and a polynomial  $f(x) \in k[x]$  there is a field  $K \supset k$  such that  $[K : k]$  is finite and  $f(x) = a(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$  in  $K[x]$ .

*Proof.* We show by induction on the degree  $n$  of  $f$  that for all polynomials  $f \in k[x]$  with  $\deg(f) = n \geq 1$ , there exists a field extension  $K$  such that  $[K : k]$  is finite, and  $f(x)$  splits in linear factors on  $K$ .

If  $n = 1$ ,  $f(x) = ax + b = a(x - \alpha_0)$ , where  $\alpha_0 = -b/a$  :  $K = k$  is suitable.

Suppose that the property is true for all polynomials of degree less than  $n$  on an arbitrary field  $k$ .

Let  $f(x) \in k[x]$ ,  $\deg(f) = n$ . From proposition 7.2.1. applied to an irreducible factor of  $f$ , there exists a field  $L$ ,  $[L : k] < \infty$  and  $\alpha \in L$  such that  $f(\alpha_1) = 0$ . Then  $f(x) = (x - \alpha_1)g(x)$ ,  $g(x) \in L[x]$ .

Applying the induction hypothesis in the field  $L$  on the polynomial  $g \in L[x]$  with  $\deg(g) = n - 1$ , we obtain a field  $K$ ,  $[K : L] < \infty$  such that  $g(x) = a(x - \alpha_2) \cdots (x - \alpha_n)$  with  $\alpha_i \in K$ . So  $f(x) = a(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$  splits in linear factors in  $K$ . The induction is achieved.  $\square$

**Ex. 7.13** Apply Exercise 7.12 to  $k = \mathbb{Z}/p\mathbb{Z}$  and  $f(x) = x^{p^n} - x$  to obtain another proof of Theorem 2.

*Proof.* Let  $f(x) = x^{p^n} - x$ . We know from Ex. 7.12 that there exists a finite extension  $K$  of  $\mathbb{F}_p$  such that  $f$  splits in linear factors on  $K$  :

$$f(x) = \prod_{k=1}^{p^n} (x - \alpha_k), \quad \alpha_1, \dots, \alpha_{p^n} \in K.$$

The set  $k = \{\alpha_1, \dots, \alpha_{p^n}\} \subset K$  of the roots of  $x^{p^n} - x$  is a subfield of  $K$  : indeed, if  $\alpha, \beta \in k$ ,

- (a)  $f(1) = 0$ , so  $1 \in k$
- (b)  $(\alpha - \beta)^{p^n} = \alpha^{p^n} - \beta^{p^n} = \alpha - \beta$ , so  $\alpha - \beta \in k$ .
- (c)  $(\alpha\beta)^{p^n} = \alpha^{p^n} \beta^{p^n} = \alpha\beta$ , so  $\alpha\beta \in k$ .
- (d)  $(\alpha^{-1})^{p^n} = (\alpha^{p^n})^{-1} = \alpha^{-1}$ , so  $\alpha^{-1} \in k$  if  $\alpha \neq 0$ .

As  $f'(x) = -1$ ,  $f(x) \wedge f'(x) = 1$ , so  $f$  has no multiple root, so the cardinality of  $k$  is  $p^n$ .

Let  $g(x) \in \mathbb{F}_p[x]$  a factor of  $f(x)$ , irreducible in  $\mathbb{F}_p[x]$ , with  $d = \deg(g)$ . As  $g \mid f$ ,  $g$  splits in linear factors in  $k[x]$ . Let  $\alpha$  a root of  $g(x)$  in  $k$ . As  $g$  is irreducible on  $\mathbb{F}_p$ ,  $d = \deg(g) = [\mathbb{F}_p[\alpha] : \mathbb{F}_p]$ . Moreover  $n = [k : \mathbb{F}_p] = [k : \mathbb{F}_p[\alpha]] [\mathbb{F}_p[\alpha] : \mathbb{F}_p]$ , so  $d \mid n$ .

Reciprocally, suppose that  $g$  is any irreducible polynomial in  $\mathbb{F}_p[x]$ , with  $d = \deg(g) \mid n$ . Then  $K_0 = \mathbb{F}_p[x]/\langle g \rangle$  contains a root  $\alpha$  of  $g$ , and  $[K_0 : \mathbb{F}_p] = \deg(g) = d$ , so  $\alpha^{p^d} = \alpha$ .

As  $d \mid n$ , then  $p^d - 1 \mid p^n - 1$  and  $x^{p^d} - 1 \mid x^{p^n} - 1$  (Lemma 2,3 in section 1), so

$$x^{p^d} - x \mid x^{p^n} - x.$$

$f(\alpha) = \alpha^{p^n} - \alpha = 0$  and  $g$  is the minimal polynomial of  $\alpha$ , so  $g \mid f$ .

Conclusion :

$$x^{p^n} - x = \prod_{d \mid n} F_d(x),$$

where  $F_d(x)$  is the product of the monic irreducible polynomial of degree  $d$ . □

**Ex. 7.14** Let  $F$  be a field with  $q$  elements and  $n$  a positive integer. Show that there exist irreducible polynomials in  $F[x]$  of degree  $n$ .

*Proof.* Let  $F = \mathbb{F}_q$  a field with  $q = p^m$  elements, and  $n$  a positive integer.

From Theorem 2 Corollary 3, there exists an irreducible polynomial  $f(x) \in \mathbb{F}_p[x]$  of degree  $nm$ . Let  $g$  an irreducible factor of  $f$  in  $\mathbb{F}_q[x]$ , and  $\alpha$  a root of  $g$  in an extension of  $\mathbb{F}_q$ .

We show that  $\mathbb{F}_q \subset \mathbb{F}_p[\alpha]$ .

$\mathbb{F}_q$  and  $\mathbb{F}_p[\alpha]$  are two subfield of the same finite field  $\mathbb{F}_q[\alpha]$ . Moreover,  $|\mathbb{F}_q| = p^m$ , and  $|\mathbb{F}_p[\alpha]| = p^{nm}$ . As  $m \mid n$ ,  $\mathbb{F}_q \subset \mathbb{F}_p[\alpha]$ .

Indeed, for all  $\gamma \in \mathbb{F}_q[\alpha]$ ,

$$\gamma \in \mathbb{F}_q \Rightarrow \gamma^{p^m} = \gamma \Rightarrow \gamma^{p^{mn}} = \gamma \Rightarrow \gamma \in \mathbb{F}_p[\alpha].$$

So  $\mathbb{F}_q \subset \mathbb{F}_p[\alpha]$ .

We show that  $\mathbb{F}_q[\alpha] = \mathbb{F}_p[\alpha]$ .

As  $\mathbb{F}_p \subset \mathbb{F}_q$ ,  $\mathbb{F}_p[\alpha] \subset \mathbb{F}_q[\alpha]$ .

Let  $\beta \in \mathbb{F}_q[\alpha] : \beta = \sum_{i=1}^k a_i \alpha^i$ , where  $a_i \in \mathbb{F}_q \subset \mathbb{F}_p[\alpha]$ , so  $a_i = p_i(\alpha), p_i \in \mathbb{F}_p[\alpha]$ .

Consequently

$$\beta = \sum_{i=1}^k p_i(\alpha) \alpha^i \in \mathbb{F}_p[\alpha],$$

so  $\mathbb{F}_q[\alpha] = \mathbb{F}_p[\alpha]$ .

$$nm = [\mathbb{F}_p[\alpha] : \mathbb{F}_p] = [\mathbb{F}_q[\alpha] : \mathbb{F}_p] = [\mathbb{F}_q[\alpha] : \mathbb{F}_q] \times [\mathbb{F}_q : \mathbb{F}_p] = [\mathbb{F}_q[\alpha] : \mathbb{F}_q] \times m.$$

Thus  $[\mathbb{F}_q[\alpha] : \mathbb{F}_q] = n$ , and  $g$  is the minimal polynomial of  $\alpha$  on  $\mathbb{F}_q$ , so  $\deg(g) = n$ .

Conclusion : if  $F$  is a field with  $q = p^m$  elements, there exist irreducible polynomials in  $F[x]$  of degree  $n$  for all positive integers  $n$ . □

**Ex. 7.15** Let  $x^n - 1 \in F[x]$ , where  $F$  is a finite field with  $q$  elements. Suppose that  $(q, n) = 1$ . Show that  $x^n - 1$  splits into linear factors in some extension field and that the least degree of such a field is the smallest integer  $f$  such that  $q^f \equiv 1 \pmod{n}$ .

*Proof.* From exercise 7.12, we know that  $x^n - 1$  splits into linear factors in some extension field  $K$ , with  $[K : F] < \infty$  :

$$u(x) = x^n - 1 = (x - \zeta_0)(x - \zeta_1) \cdots (x - \zeta_{n-1}), \quad \zeta_i \in K.$$

$u'(x) \wedge u(x) = nx^{n-1} \wedge (x^n - 1) = 1$ , since  $x(nx^{n-1}) - n(x^n - 1) = n$ , and  $n \neq 0$  in the field  $F$ , since we know from the hypothesis  $q \wedge n = 1$  that the characteristic  $p$  doesn't divide  $n$ . So the  $n$  roots of  $x^n - 1$  are distinct.

The set  $G = \{x \in K \mid x^n = 1\}$  is a subgroup of  $K^*$ , thus  $G$  is cyclic of order  $n$ . Let  $\zeta$  a generator of  $G$ . Then

$$x^n - 1 = (x - 1)(x - \zeta)(x - \zeta^2) \cdots (x - \zeta^{n-1}).$$

Let  $p(x)$  the minimal polynomial of  $\zeta$  on  $F$ , and  $f$  the degree of  $p$  :

$$f = \deg(p) = [F[\zeta] : F].$$

So  $\text{Card } F[\zeta] = q^f$ , and since  $\zeta \in F[\zeta]^*$ ,  $\zeta^{q^f - 1} - 1 = 0$ . As the order of  $\zeta$  in the group  $G$  is  $n$ ,  $n \mid q^f - 1$ , namely  $q^f \equiv 1 \pmod{n}$ .

Let  $k$  any positive integer such that  $q^k \equiv 1 \pmod{n}$ .

Then  $n \mid q^k - 1$ , so  $\zeta^{q^k - 1} - 1 = 0$ ,  $\zeta^{q^k} - \zeta = 0$ . Let  $L$  an extension of  $K$  such that  $x^{q^k} - x$  splits in linear factors in  $L$ . As  $\zeta^{q^k} - \zeta = 0$ ,  $\zeta$  belongs to the subfield  $M$  of  $L$  with cardinality  $q^k$ , such that  $[M : F] = k$ . Thus  $F[\zeta] \subset M$ , so  $f = [F[\zeta] : F] \leq k = [M : F]$ .

$f = [F[\zeta] : F]$  is the smallest  $k \in \mathbb{N}^*$  such that  $q^k \equiv 1 \pmod{n}$ .

If  $K$  is any extension of  $F$  containing the roots of  $x^n - 1$ , then  $K \supset F[\zeta]$ , where  $\zeta$  is a primitive root of unity, so  $[K : F] \geq [F[\zeta] : F] = f$ .

Conclusion : the minimal degree of a extension  $K \supset F$  containing the roots of  $x^n - 1$ , with  $n \wedge q = 1$ , is the smallest positive integer  $f$  such that  $q^f \equiv 1 \pmod{n}$ , the order of  $q$  modulo  $n$ .  $\square$

**Ex. 7.16** Calculate the monic irreducible polynomials of degree 4 in  $\mathbb{Z}/2\mathbb{Z}[x]$ .

*Proof.* Write  $F_d$  the product of irreducible monic polynomials in  $\mathbb{F}_2[x]$ .

Theorem 2 gives

$$x^{16} - x = x^{2^4} - x = \prod_{d \mid 4} F_d(x) = F_1(x)F_2(x)F_4(x)$$

and

$$x^4 - x = x^{2^2} - x = \prod_{d \mid 2} F_d(x) = F_1(x)F_2(x)$$

$$\text{so } F_4(x) = \frac{x^{16} - x}{x^4 - x} = \frac{x^{15} - 1}{x^3 - 1} = x^{12} + x^9 + x^6 + x^3 + 1$$

$$F_4(x) = (x^4 + x^3 + x^2 + x + 1)(x^4 + x + 1)(x^4 + x^3 + 1)$$

Among the 16 monic polynomials of degree 4 in  $\mathbb{F}_2[x]$ , 3 are irreducible :

$$P_1(x) = x^4 + x^3 + x^2 + x + 1,$$

$$P_2(x) = x^4 + x + 1$$

$$P_3(x) = x^4 + x^3 + 1$$

With sage :

```
sage: A = PolynomialRing(GF(2), 'x')
sage: x = A.gen()
sage: f = (x^16-x)/(x^4-x)
sage: factor(f)
(x^4 + x + 1) * (x^4 + x^3 + 1) * (x^4 + x^3 + x^2 + x + 1)
```

□

**Ex. 7.17** Let  $q$  and  $p$  be distinct odd primes. Show that the number of monic irreducibles of degree  $q$  in  $\mathbb{Z}/p\mathbb{Z}$  is  $q^{-1}(p^q - p)$ .

*Proof.* From Theorem 2 Corollary 2, we know that the number of irreducible polynomials on  $\mathbb{F}_p$  of degree  $q$  is given by

$$N_q = \frac{1}{q} \sum_{d|q} \mu\left(\frac{q}{d}\right) p^d.$$

As  $q$  is prime,  $d$  takes the values 1,  $q$ , with  $\mu(1) = 1, \mu(q) = -1$ , so

$$N_q = \frac{p^q - p}{q}.$$

□

**Ex. 7.18** Let  $p$  be a prime with  $p \equiv 3 \pmod{4}$ . Show that the residue classes modulo  $p$  in  $\mathbb{Z}[i]$  form a field with  $p^2$  elements.

*Proof.* If  $p$  is a prime rational integer, with  $p \equiv 3 \pmod{4}$ , then  $p$  is a prime in  $\mathbb{Z}[i]$ .

Indeed,  $p$  is irreducible : if  $p = uv$ ,  $u, v \in \mathbb{Z}[i]$ , where  $u = c + di, v$  are not units, then  $p^2 = N(u)N(v)$ ,  $N(u) > 1, N(v) > 1$ , so  $p = N(u) = u\bar{u} = c^2 + d^2$ .

As  $c^2 \equiv 0, 1 \pmod{4}, d^2 \equiv 0, 1 \pmod{4}$ , so  $p \equiv 1 \pmod{4}$ , which is in contradiction with the hypothesis.

So  $p$  is irreducible in  $\mathbb{Z}[i]$ , and since  $\mathbb{Z}[i]$  is a principal ideal domain,  $p$  is prime in  $\mathbb{Z}[i]$ , thus  $\mathbb{Z}[i]/(p)$  is a field.

Let  $z = a + bi \in \mathbb{Z}[i]$ . The Euclidean division of  $a, b$  by  $p$  gives

$$a = qp + r, \quad 0 \leq r < p, \quad b = q'p + s, \quad 0 \leq s < p,$$

so

$$z \equiv r + is \pmod{p}, \quad 0 \leq r < p, 0 \leq s < p.$$

Let's verify that these  $p^2$  elements are in different classes of congruences modulo  $p$ .

If  $r + is \equiv r' + is' \pmod{p}$ , then  $(r - r')/p + i(s - s')/p \in \mathbb{Z}[i]$ , so  $r \equiv r', s \equiv s' \pmod{p}$ .

As  $r, r', s, s'$  are between 0 and  $p - 1$ ,  $r = r', s = s'$ .

So the cardinality of the field  $\mathbb{Z}[i]/(p)$  is  $p^2$ .

□



**Ex. 7.19** Let  $F$  be a finite field with  $q$  elements. If  $f(x) \in F[x]$  has degree  $t$ , put  $|f| = q^t$ . Verify the formal identity  $\sum_f |f|^{-s} = (1 - q^{1-s})^{-1}$ . The sum is over all monic polynomials.

*Proof.* Let  $U$  the set of monic polynomials in  $\mathbb{F}_q[x]$ , and  $U_t$  the set of monic polynomials of degree  $t$ , and  $s \in \mathbb{C}$ . Then  $U = \coprod_{t \in \mathbb{N}} U_t$ , so

$$\begin{aligned} \sum_{f \in U} |f|^{-s} &= \sum_{t=0}^{\infty} \sum_{f \in U_t} |f|^{-s} \\ &= \sum_{t=0}^{\infty} \frac{1}{q^{ts}} \sum_{f \in U_t} 1 \end{aligned}$$

As  $\sum_{f \in U_t} 1 = \text{Card}(U_t) = q^t$ , then, for  $\text{Re}(s) > 1$

$$\begin{aligned} \sum_{f \in U} |f|^{-s} &= \sum_{t=0}^{\infty} \frac{1}{q^{t(s-1)}} \\ &= \frac{1}{1 - \frac{1}{q^{s-1}}} \\ &= (1 - q^{1-s})^{-1} \end{aligned}$$

As  $\left| \frac{1}{q^{t(s-1)}} \right| = \frac{1}{q^{t(\text{Re}(s)-1)}}$ , the serie is absolutely convergent for  $\text{Re}(s) > 1$ . This justifies the grouping of terms in this sum.

Conclusion : if  $\text{Re}(s) > 1$ ,

$$\sum_{f \in U} |f|^{-s} = (1 - q^{1-s})^{-1},$$

where  $U$  is the set of monic polynomials in  $\mathbb{F}_q[x]$ . □

**Ex. 7.20** With the notation of Exercise 19 let  $d(f)$  be the number of monic divisors of  $f$  and  $\sigma(f) = \sum_{g|f} |g|$ , where the sum is over the monic divisors of  $f$ . Verify the following identities :

$$(a) \sum_f d(f) |f|^{-s} = (1 - q^{1-s})^{-2}$$

$$(b) \sum \sigma(f) |f|^{-s} = (1 - q^{1-s})^{-1} (1 - q^{2-s})^{-1}$$

*Proof.* (a) With the notation of 7.19, for  $s \in \mathbb{C}, \text{Re}(s) > 1$ ,  $\sum_{f \in U} |f|^{-s}$  is absolutely convergent and

$$(1 - q^{1-s})^{-1} = \sum_{f \in U} |f|^{-s}$$

Then

$$\begin{aligned} (1 - q^{1-s})^{-2} &= \sum_{f \in U} |f|^{-s} \sum_{g \in U} |g|^{-s} \\ &= \sum_{(f,g) \in U^2} |fg|^{-s} \\ &= \sum_{h \in U} \sum_{g \in U, g|h} |h|^{-s}, \end{aligned}$$

indeed, the application

$$\varphi : \begin{cases} U \times U & \rightarrow \{ (h, g) \in U \times U, g \mid h \} \\ (f, g) & \mapsto (fg, g) \end{cases}$$

is a bijection.

So

$$\begin{aligned} (1 - q^{1-s})^{-2} &= \sum_{h \in U} |h|^{-s} \text{Card}\{g \in U, g \mid h\} \\ &= \sum_{h \in U} |h|^{-s} d(h) \\ &= \sum_{f \in U} d(f) |f|^{-s} \end{aligned}$$

(b) Similarly,

$$\begin{aligned} (1 - q^{1-s})^{-1} (1 - q^{2-s})^{-1} &= \sum_{f \in U} |f|^{-s} \sum_{g \in U} |g|^{-s+1} \\ &= \sum_{(f, g) \in U^2} |g| |fg|^{-s} \\ &= \sum_{h \in U} \sum_{g \in U, g \mid h} |g| |h|^{-s} \\ &= \sum_{h \in U} |h|^{-s} \sum_{g \in U, g \mid h} |g| \\ &= \sum_{h \in U} \sigma(h) |h|^{-s} \\ &= \sum_{f \in U} \sigma(f) |f|^{-s} \end{aligned}$$

□

**Ex. 7.21** Let  $F$  be a field with  $q = p^n$  elements. For  $\alpha \in F$  set  $f(x) = (x - \alpha)(x - \alpha^p)(x - \alpha^{p^2}) \cdots (x - \alpha^{p^{n-1}})$ . Show that  $f(x) \in \mathbb{Z}/p\mathbb{Z}[x]$ . In particular,  $\alpha + \alpha^p + \cdots + \alpha^{p^{n-1}}$  and  $\alpha\alpha^p\alpha^{p^2} \cdots \alpha^{p^{n-1}}$  are in  $\mathbb{Z}/p\mathbb{Z}$ .

*Proof.* Let  $F : \begin{cases} \mathbb{F}_q & \rightarrow \mathbb{F}_q \\ x & \mapsto x^p \end{cases}$ .

As the characteristic of  $\mathbb{F}_q$  is  $p$ ,  $(x + y)^p = x^p + y^p$  et  $(xy)^p = x^p y^p$ , and each homomorphism of field is injective,  $F$  is a field automorphism (Frobenius automorphism).

For every automorphism  $H$  in  $\mathbb{F}_q$ , and every polynomial  $p(x) = \sum a_i x^i \in \mathbb{F}_q[x]$ , write  $(H.p)(x) = \sum_i H(a_i) x^i$ . Then for all  $(p, q) \in \mathbb{F}_q[x]^2$ ,  $H.(pq) = (H.p)(H.q)$ .

With this notation,

$$\begin{aligned} f(x) &= (x - \alpha)(x - F\alpha)(x - F^2\alpha) \cdots (x - F^{n-1}\alpha), \\ (H.f)(x) &= (x - F\alpha)(x - F^2\alpha)(x - F^3\alpha) \cdots (x - F^n\alpha). \end{aligned}$$

Since  $\alpha \in \mathbb{F}_{p^n}$ ,  $F^n \alpha = \alpha^{p^n} = \alpha$ , thus

$$H.f = f.$$

In other words, if  $f(x) = \sum_i a_i x^i$ , then for all  $i$ ,  $H(a_i) = a_i$ , so  $a_i^p = a_i$ , thus  $a_i \in \mathbb{F}_p$ , and  $f \in \mathbb{F}_p[x]$ . In particular, the coefficients  $a_{n-1} = \alpha + \alpha^p + \cdots + \alpha^{p^{n-1}}$ ,  $a_0 = \alpha \alpha^p \alpha^{p^2} \cdots \alpha^{p^{n-1}}$  are in  $\mathbb{F}_p$ .  $\square$

**Ex. 7.22** (continuation) Set  $\text{tr}(\alpha) = \alpha + \alpha^p + \cdots + \alpha^{p^{n-1}}$ . Prove that

$$(a) \text{tr}(\alpha) + \text{tr}(\beta) = \text{tr}(\alpha + \beta).$$

$$(b) \text{tr}(a\alpha) = a \text{tr}(\alpha) \text{ for } a \in \mathbb{Z}/p\mathbb{Z}.$$

$$(c) \text{ There is an } \alpha \in F \text{ such that } \text{tr}(\alpha) \neq 0.$$

*Proof.* Let  $F$  the Frobenius automorphism of  $\mathbb{F}_q$  introduced in Ex.7.21.

(a),(b) : If  $x, y \in \mathbb{F}_q$ , and  $a \in \mathbb{F}_p$ , then  $a^p = a$ , so  $F(x + y) = (x + y)^p = x^p + y^p = F(x) + F(y)$ , and  $F(ax) = a^p x^p = ax^p = aF(x)$ , so  $F$  is  $\mathbb{F}_p$ -linear, and also  $\text{tr} = I + F + F^2 + \cdots + F^{n-1}$ .

(c) The polynomial  $p(x) = x + x^p + x^{p^2} + \cdots + x^{p^{n-1}}$  has degree  $p^{n-1}$ , so  $p(x)$  has at most  $p^{n-1}$  roots in  $\mathbb{F}_q$ , and  $|\mathbb{F}_q| = p^n > \deg(p) = p^{n-1}$ . Therefore there exist in  $\mathbb{F}_q$  some element  $\alpha$  which is not a root of  $p(x)$ , and so  $\text{tr}(\alpha) = p(\alpha) \neq 0$ .  $\square$

**Ex. 7.23** (continuation) For  $\alpha \in F$  consider the polynomial  $x^p - x - \alpha \in F[x]$ . Show that this polynomial is either irreducible or the product of linear factors. Prove that the latter alternative holds iff  $\text{tr}(\alpha) = 0$ .

*Proof.* Let  $f(x) = x^p - x - \alpha \in F[x]$ . There exists an extension  $K \supset F$  with finite degree on  $F$  which contains a root  $\gamma$  of  $f$ .

As  $\gamma^p - \gamma - \alpha = 0$ , then for all  $i \in \mathbb{F}_p$ ,

$$(\gamma + i)^p - (\gamma + i) - \alpha = (\gamma^p - \gamma - \alpha) + i^p - i = 0.$$

So  $f$  has  $n$  distinct roots in  $K$  :  $\gamma, \gamma + 1, \dots, \gamma + p - 1$ , and so

$$f(x) = (x - \gamma)(x - \gamma - 1) \cdots (x - \gamma - (p - 1)).$$

$F[\gamma]$  contains all roots of  $f$ .

- If  $\gamma \in F$ ,  $f(x)$  splits in linear factors in  $F$ .  $f(x)$  is not irreducible, since  $\deg(f) = p > 1$ .

- If  $\gamma \notin F$ , we will show that  $f$  is irreducible in  $F[x]$ .

If not, then  $f(x) = g(x)h(x)$  is the product of two polynomials  $g, h \in F[x]$  such that  $1 \leq \deg(g) \leq p - 1$ .

The unicity of the decomposition in irreducible factors in  $F[\gamma][x]$  shows that

$$g(x) = \prod_{i \in A} (x - \gamma - i),$$

where  $A$  is a subset of  $\mathbb{F}_p$ , with  $A \neq \emptyset, A \neq \mathbb{F}_p$ . As  $g(x) \in F[x]$ ,  $\sum_{i \in A} (\gamma + i) = k\gamma + l \in \mathbb{F}_p$ ,

where  $1 \leq k = |A| \leq p - 1$  and  $l = \sum_{i \in A} i \in \mathbb{F}_p$ .

So  $k\gamma \in \mathbb{F}_p$ . Since  $\gamma \notin \mathbb{F}_p$ ,  $k$  is not invertible in  $\mathbb{F}_p$ , in contradiction with  $1 \leq k \leq p-1$ . Consequently,  $f(x)$  is irreducible.

We conclude that  $x^p - x - \alpha \in F[x]$  is irreducible iff  $\gamma \notin F$ .

Let  $F$  the Frobenius automorphism of  $K$  (cf. Ex. 7.21).

$$\alpha = F(\gamma) - \gamma, F(\alpha) = F^2(\gamma) - F(\gamma), \dots, F^{n-1}(\alpha) = F^n(\gamma) - F^{n-1}(\gamma).$$

The sum of these equalities gives

$$\text{tr}(\alpha) = \alpha + F(\alpha) + \dots + F^{n-1}(\alpha) = F^n(\gamma) - \gamma = \gamma^{p^n} - \gamma.$$

As the cardinality of  $F$  is  $q = p^n$ ,

$$\gamma \in F \iff \gamma^{p^n} - \gamma = 0 \iff \text{tr}(\alpha) = 0.$$

Conclusion :  $x^p - x - \alpha$  is irreducible iff  $\text{tr}(\alpha) \neq 0$ . If  $\text{tr}(\alpha) = 0$ ,  $x^p - x - \alpha$  splits in linear factors in  $F[x]$ .  $\square$

**Ex. 7.24** Suppose that  $f(x) \in \mathbb{Z}/p\mathbb{Z}[x]$  has the property that  $f(x+y) = f(x) + f(y) \in \mathbb{Z}/p\mathbb{Z}[x, y]$ . Show that  $f(x)$  must be of the form  $a_0x + a_1x^p + a_2x^{p^2} + \dots + a_mx^{p^m}$ .

**Lemma** If the prime number  $p$  divides all binomial coefficients  $\binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n-1}$ , then  $n$  is a power of  $p$ .

*Proof.* Let  $u(x) = (x+1)^n - x^n - 1 \in \mathbb{F}_p[x]$ . Then  $f(x) = \sum_{k=1}^{n-1} \binom{n}{k} x^k = 0$ .

Write  $n = p^a q$ , with  $p \nmid q = 1$ . With a reductio ad absurdum, suppose that  $q > 1$ . Then

$$f(x) = 0 = (x+1)^{p^a q} - x^{p^a q} - 1 = (x^{p^a} + 1)^q - x^{p^a q} - 1 = \sum_{k=1}^{q-1} \binom{q}{k} x^{kp^a}.$$

Consequently, the coefficient of  $x^{p^a}$  is null, so  $p \mid q$  : this is absurd. Therefore  $q = 1$  and  $n = p^a$ .  $\square$

*Proof.* (Ex. 7.24)

Suppose that  $f \in \mathbb{F}_p[x]$  verify in  $\mathbb{F}_p[x, y]$  the equality  $f(x+y) = f(x) + f(y)$ .

Write  $f(x) = \sum_{k=1}^d c_k x^k$ .

$$\begin{aligned} 0 = f(x+y) - f(x) - f(y) &= \sum_{n=0}^d c_n [(x+y)^n - x^n - y^n] \\ &= \sum_{n=0}^d \sum_{k=1}^{n-1} c_n \binom{n}{k} x^k y^{n-k} \end{aligned}$$

So for all  $n$ , for all  $k$ ,  $1 \leq k \leq n-1$ ,  $c_n \binom{n}{k} = 0$  in  $\mathbb{F}_p$ .

From the lemma, if  $n$  is not a power of  $p$ , there exists a  $k$ ,  $1 \leq k \leq n-1$  such that  $\binom{n}{k} \not\equiv 0 \pmod{p}$ , so  $c_n = 0$ . If we write  $a_k = c_{p^k}$ , then  $f(x)$  is of the form

$$f(x) = a_0x + a_1x^p + a_2x^{p^2} + \dots + a_mx^{p^m}.$$

$\square$

## Chapter 8

**Ex. 8.1** Let  $p$  be a prime and  $d = (m, p-1)$ . Prove that  $N(x^m = a) = \sum \chi(a)$ , the sum being over all  $\chi$  such that  $\chi^d = \varepsilon$ .

*Proof.* Let  $d = m \wedge (p-1)$ . we prove that  $N(x^m = a) = N(x^d = a)$  for all  $d \in \mathbb{F}_p$ .

- If  $a = 0$ , 0 is the only root of  $x^m - a$  or  $x^d - a$ , so  $N(x^m = a) = N(x^d = a) = 1$ .
- If  $a \in \mathbb{F}_p^*$  and  $x^n = a$  has a solution, then we know from the demonstration of Proposition 4.2.1 that  $N(x^n - a) = d = N(x^d - a)$ .
- If  $a \in \mathbb{F}_p^*$  and  $x^n = a$  has no solution, then (Prop. 4.2.1)  $a^{(p-1)/d} \neq 1$ , so  $x^d = a$  has no solution :  $N(x^n - a) = 0 = N(x^d - a)$ .

Using Prop. 8.1.5, as  $d \mid n$ , we obtain

$$N(x^n = a) = N(x^d = a) = \sum_{\chi^d = \varepsilon} \chi(a).$$

□

**Ex. 8.2, false sentence.** With the notation of Exercise 1 show that  $N(x^m = a) = N(x^d = a)$  and conclude that if  $d_i = (m_i, p-1)$ , then  $\sum_i a_i x_i^{m_i} = b$  and  $\sum_i a_i x_i^{d_i} = b$  have the same number of solutions.

*This result is false. I give a counterexample with  $p = 5$  :  $x + x^3 = 0 \in \mathbb{F}_5[x]$  has 3 solutions 0, 2, -2. As  $3 \wedge (p-1) = 3 \wedge 4 = 1$ , the reduced equation is  $x + x = 0$ , which has an unique solution 0. The true sentence is :*

**Ex. 8.2** With the notation of Exercise 1 show that  $N(x^m = a) = N(x^d = a)$  and conclude that if  $d_i = (m_i, p-1)$ , then  $\sum_i a_i x_i^{m_i} = b$  and  $\sum_i a_i x_i^{d_i} = b$  have the same number of solutions.

*Proof.* From Ex. 8.1, we know that

$$N(x^m = a) = \sum_{\chi^d = \varepsilon} \chi(a) = N(x^d = a).$$

Using this result, we obtain

$$\begin{aligned} N\left(\sum_{i=1}^l a_i x_i^{m_i} = b\right) &= \sum_{a_1 u_1 + \dots + a_l u_l = b} \prod_{i=1}^l N(x^{m_i} = u_i) \\ &= \sum_{a_1 u_1 + \dots + a_l u_l = b} \prod_{i=1}^l N(x^{d_i} = u_i) \\ &= N\left(\sum_{i=1}^l a_i x_i^{d_i} = b\right) \end{aligned}$$

□

**Ex. 8.3** Let  $\chi$  be a non trivial multiplicative character of  $\mathbb{F}_p$  and  $\rho$  be the character of order 2. Show that  $\sum_t \chi(1 - t^2) = J(\chi, \rho)$ . [Hint: Evaluate  $J(\chi, \rho)$  using the relation  $N(x^2 = a) = 1 + \rho(a)$ .]

*Proof.*

$$\begin{aligned} J(\chi, \rho) &= \sum_{a+b=1} \chi(a)\rho(b) \\ &= \sum_{a+b=1} \chi(a)(N(x^2 = b) - 1) \\ &= \sum_{a+b=1} \chi(a)N(x^2 = b) - \sum_{a+b=1} \chi(a) \end{aligned}$$

As  $\chi \neq \varepsilon$ ,

$$\sum_{a+b=1} \chi(a) = \sum_{a \in \mathbb{F}_p} \chi(a) = 0.$$

Let  $C = \{x^2 \mid x \in \mathbb{F}_p^*\}$  the set of squares in  $\mathbb{F}_p^*$ ,  $\overline{C}$  its complementary in  $\mathbb{F}_p^*$  :

$$\mathbb{F}_p = \{0\} \cup C \cup \overline{C}.$$

Then

$$\begin{aligned} J(\chi, \rho) &= \sum_{a+b=1} \chi(a)N(x^2 = b) \\ &= \sum_{a+b=1, b=0} \chi(a)N(x^2 = b) + \sum_{a+b=1, b \in C} \chi(a)N(x^2 = b) + \sum_{a+b=1, b \in \overline{C}} \chi(a)N(x^2 = b) \\ &= \chi(1) + 2 \sum_{b \in C} \chi(1 - b) \end{aligned}$$

(because  $N(x^2 = b) = 0$  if  $x \in \overline{C}$ , and  $N(x^2 = b) = 2$  if  $x \in C$ ). As each  $b \in C$  has two roots, and as the set of roots of two distinct  $b$  are disjointed,

$$J(\chi, \rho) = \chi(1) + \sum_{t \in \mathbb{F}_p^*} \chi(1 - t^2) = \sum_{t \in \mathbb{F}_p} \chi(1 - t^2).$$

Conclusion : if  $\chi$  is a non trivial multiplicative character of  $\mathbb{F}_p$  and  $\rho$  the character of order 2,

$$J(\chi, \rho) = \sum_{t \in \mathbb{F}_p} \chi(1 - t^2).$$

□

**Ex. 8.4** Show, if  $k \in \mathbb{F}_p, k \neq 0$ , that  $\sum_t \chi(t(k - t)) = \chi(k^2/2^2)J(\chi, \rho)$ .

*Proof.* We know from Ex. 8.3 that  $J(\chi, \rho) = \sum_t \chi(1 - t^2)$ , so

$$\begin{aligned}
&\leq J(\chi, \rho) = \sum_{t \in \mathbb{F}_p} \chi(1-t)\chi(1+t) \\
&= \sum_{u \in \mathbb{F}_p} \chi(u)\chi(2-u) \quad (u = 1-t) \\
&= \chi(2^2) \sum_{u \in \mathbb{F}_p} \chi\left(\frac{u}{2}\right) \chi\left(1 - \frac{u}{2}\right) \\
&= \chi(2^2) \sum_{v \in \mathbb{F}_p} \chi(v)\chi(1-v) \quad (u = 2v) \\
&= \chi(2^2)\chi(k^{-2}) \sum_{w \in \mathbb{F}_p} \chi(kw)\chi(k-kw) \\
&= \chi(2^2/k^2) \sum_{t \in \mathbb{F}_p} \chi(t)\chi(k-t) \quad (t = kw).
\end{aligned}$$

Conclusion : if  $k \in \mathbb{F}^*$ , and  $\chi$  is a non trivial character,  $\rho$  the character of order 2,

$$\sum_{t \in \mathbb{F}_p} \chi(t(k-t)) = \chi(k^2/2^2)J(\chi, \rho).$$

□

**Ex. 8.5** If  $\chi^2 \neq \varepsilon$ , show that  $g(\chi)^2 = \chi(2)^{-2}J(\chi, \rho)g(\chi^2)$ . [Hint: Write out  $g(\chi)^2$  explicitly and use Exercise 4.]

*Proof.* Let  $\zeta = e^{2i\pi/p}$ . Using the result of Ex. 8.4, we obtain

$$\begin{aligned}
g(\chi)^2 &= \left( \sum_t \chi(t)\zeta^t \right) \left( \sum_s \chi(s)\zeta^s \right) \\
&= \sum_{s,t} \chi(t)\chi(s)\zeta^{t+s} \\
&= \sum_k \left( \sum_{s+t=k} \chi(t)\chi(s) \right) \zeta^k \\
&= \sum_k \left( \sum_t \chi(t(k-t)) \right) \zeta^k \\
&= \chi(-1) \sum_t \chi(t^2) + \sum_{k \neq 0} \chi(k^2/2^2)J(\chi, \rho)\zeta^k \\
&= \chi(-1) \sum_t \chi^2(t) + \chi(2)^{-2}J(\chi, \rho) \sum_{k \neq 0} \chi^2(k)\zeta^k
\end{aligned}$$

If  $\chi^2 \neq \varepsilon$ ,  $\sum_t \chi^2(t) = 0$ , so

$$g(\chi)^2 = \chi(2)^{-2}J(\chi, \rho)g(\chi^2).$$

□

**Ex. 8.6** (continuation) Show that  $J(\chi, \chi) = \chi(2)^{-2}J(\chi, \rho)$ .

*Proof.* As  $\chi^2 \neq \rho$ , Theorem 1 Chapter 8 gives  $J(\chi, \chi) = g(\chi)^2/g(\chi^2)$ , and Exercise 8.5 gives  $g(\chi)^2/g(\chi^2) = \chi(2)^{-2}J(\chi, \rho)$ , so

$$J(\chi, \chi) = \chi(2)^{-2}J(\chi, \rho).$$

□

**Ex. 8.7** Suppose that  $p \equiv 1 \pmod{4}$  and that  $\chi$  is a character of order 4. Then  $\chi^2 = \rho$  and  $J(\chi, \chi) = \chi(-1)J(\chi, \rho)$ . [Hint: Evaluate  $g(\chi)^4$  in two ways.]

*Proof.* As  $\chi$  is a character of order 4,  $\chi^2$  is a character of order 2, and  $\rho$  (Legendre's character) is the unique character of order 2, so  $\chi^2 = \rho$ .

From Prop. 8.3.3 we have

$$g(\chi)^4 = \chi(-1)pJ(\chi, \chi)J(\chi, \chi^2) = \chi(-1)pJ(\chi, \chi)J(\chi, \rho).$$

Squaring the result of Ex. 8.5, we obtain

$$g(\chi)^4 = \chi(2)^{-4}J(\chi, \rho)^2 [g(\chi^2)]^2.$$

Moreover  $\chi(2^4) = \chi^4(2) = \varepsilon(2) = 1$ , and  $g(\chi^2) = g(\rho) = g$ , so  $[g(\chi^2)]^2 = g^2 = (-1)^{(p-1)/2}p = p$  (From Prop. 6.3.2 and  $p \equiv 1 \pmod{4}$ ).

Equating these two result, we obtain

$$\chi(-1)pJ(\chi, \chi)J(\chi, \rho) = J(\chi, \rho)^2p.$$

As  $g(\chi)^4 \neq 0$  since  $|g(\chi)|^2 = p$ , we have  $J(\chi, \rho) \neq 0$ , so

$$\chi(-1)J(\chi, \chi) = J(\chi, \rho).$$

$[\chi(-1)]^2 = \chi((-1)^2) = \chi(1) = 1$ , so  $\chi(-1) = \pm 1$ , and  $\chi(-1)^{-1} = \chi(-1)$ , thus

$$J(\chi, \chi) = \chi(-1)J(\chi, \rho).$$

□

**Ex. 8.8** Generalize Exercise 3 in the following way. Suppose that  $p$  is a prime,  $\sum_t \chi(1 - t^m) = \sum_\lambda J(\chi, \lambda)$ , where  $\lambda$  varies over all characters such that  $\lambda^m = \varepsilon$ . Conclude that  $|\sum_t \chi(1 - t^m)| \leq (m-1)p^{1/2}$ .

*Proof.* For all  $y \in \mathbb{F}_p$ , write  $A_y = \{x \in \mathbb{F}_p \mid x^m = y\}$ . Then  $|A_y| = N(x^m = y)$ .

$\mathbb{F}_p = \coprod_{y \in \mathbb{F}_p} A_y$  is the disjoint union of the  $A_y$ , so

$$\sum_{t \in \mathbb{F}_p} \chi(1 - t^m) = \sum_{y \in \mathbb{F}_p} \sum_{t \in A_y} \chi(1 - t^m) = \sum_{y \in \mathbb{F}_p} |A_y| \chi(1 - y) = \sum_{y \in \mathbb{F}_p} N(x^m = y) \chi(1 - y).$$



Moreover,  $N(x^m = y) = \sum_{\lambda^m = \varepsilon} \lambda(y)$  (Prop. 8.1.5), so

$$\begin{aligned} \sum_{t \in \mathbb{F}_p} \chi(1 - t^m) &= \sum_{y \in \mathbb{F}_p} \sum_{\lambda^m = \varepsilon} \lambda(y) \chi(1 - y) \\ &= \sum_{\lambda^m = \varepsilon} \sum_{x+y=1} \chi(x) \lambda(y) \\ &= \sum_{\lambda^m = \varepsilon} J(\chi, \lambda) \end{aligned}$$

Conclusion :

$$\sum_{t \in \mathbb{F}_p} \chi(1 - t^m) = \sum_{\lambda^m = \varepsilon} J(\chi, \lambda).$$

We know that there exist  $m$  character whose order divides  $m$ . As  $\chi \neq \varepsilon$ ,  $J(\chi, \varepsilon) = 0$ , and  $|J(\chi, \lambda)| = \sqrt{p}$  for every  $\lambda \neq \varepsilon$ ,

$$\left| \sum_{t \in \mathbb{F}_p} \chi(1 - t^m) \right| \leq \sum_{\lambda^m = \varepsilon, \lambda \neq \varepsilon} |J(\chi, \lambda)| = (m-1)\sqrt{p}.$$

□

**Ex. 8.9** Suppose that  $p \equiv 1 \pmod{3}$  and that  $\chi$  is a character of order 3. Prove (using Exercise 5) that  $g(\chi)^3 = p\pi$ , where  $\pi = \chi(2)J(\chi, \rho)$ .

*Proof.* As  $\chi$  is a character of order 3,  $\chi^2 \neq \varepsilon$ . From Exercise 5, we know that

$$g(\chi)^2 = \chi(2)^{-2} J(\chi, \rho) g(\chi^2).$$

So

$$g(\chi)^3 = \chi(2)^{-2} J(\chi, \rho) g(\chi^2) g(\chi).$$

Recall (§8.2) that

$$\overline{g(\chi)} = \sum_t \overline{\chi(t)} \zeta^{-t} = \chi(-1) \sum_t \overline{\chi(-t)} \zeta(-t) = \chi(-1) g(\chi),$$

Here  $\chi(-1) = 1$ , because  $\chi(-1) = \chi((-1)^3) = \chi^3(-1) = \varepsilon(-1) = 1$ . Hence

$$g(\chi^2) g(\chi) = g(\bar{\chi}) g(\chi) = \overline{g(\chi)} g(\chi) = |g(\chi)|^2 = p.$$

Moreover  $\chi(2)^3 = \chi^3(2) = 1$ , so  $\chi(2)^{-2} = \chi(2)$ .

Conclusion : if  $\chi$  is a character of order 3,

$$g(\chi)^3 = p\pi, \text{ where } \pi = \chi(2)J(\chi, \rho).$$

□

**Ex. 8.10** (continuation) Show that  $\chi\rho$  is a character of order 6 and that

$$g(\chi\rho)^6 = (-1)^{(p-1)/2} p\bar{\pi}^4$$

*Proof.*  $(\chi\rho)^6 = \chi^6\rho^6 = \varepsilon$ ,  $(\chi\rho)^2 = \chi^2 \neq \varepsilon$ ,  $(\chi\rho)^3 = \rho^3 = \rho \neq \varepsilon$ , so  $\chi\rho$  is of order 6.  
 $J(\chi, \rho)g(\chi\rho) = g(\chi)g(\rho)$  since  $\chi, \rho, \chi\rho$  are non trivial characters. So

$$g(\chi\rho)^6 = \frac{g(\chi)^6 g(\rho)^6}{J(\chi, \rho)^6}.$$

From Exercise 8.9,  $g(\chi)^6 = p^2\pi^2$ . Proposition 6.3.2 gives  $g(\rho)^2 = (-1)^{(p-1)/2}p$ , so  $g(\rho)^6 = (-1)^{(p-1)/2}p^3$ . As  $\pi = \chi(2)J(\chi, \rho)$ ,  $J(\chi, \rho)^6 = \chi(2)^{-6}\pi^6 = \pi^6$ , since  $\chi(2)^3 = 1$ . Therefore

$$g(\chi\rho)^6 = \frac{p^2\pi^2(-1)^{(p-1)/2}p^3}{\pi^6} = (-1)^{(p-1)/2}p^5\pi^{-4}.$$

Moreover,  $\pi\bar{\pi} = \chi(2)\overline{\chi(2)}J(\chi, \rho)\overline{J(\chi, \rho)} = |J(\chi, \rho)|^2 = p$  (Theorem 8.1, Corollary), so  $\pi^{-1} = \bar{\pi}/p$ . In conclusion,

$$g(\chi\rho)^6 = (-1)^{(p-1)/2} p\bar{\pi}^4.$$

□

**Ex. 8.11** Use Gauss' theorem to find the number of solutions to  $x^3 + y^3 = 1$  in  $\mathbb{F}_p$  for  $p = 13, 19, 37$ , and 97.

*Proof.* •  $p = 13$ .

$4 \times 13 = 52 = (-5)^2 + 27 \times 1^2$ , where  $-5 \equiv 1 \pmod{3}$ , so  $A = -5$ .

If  $p = 13$ ,  $N(x^3 + y^3 = 1) = p - 2 + A = 13 - 2 - 5 = 6$ : the solutions are only the trivial solutions.

•  $p = 19$ .

$4 \times 19 = 76 = 7^2 + 27 \times 1^2$ , where  $7 \equiv 1 \pmod{3}$ , so  $A = 7$ .

If  $p = 19$ ,  $N(x^3 + y^3 = 1) = 19 - 2 + 7 = 24$ .

•  $p = 37$ .

$4 \times 37 = 148 = (-11)^2 + 27 \times 1^2$ , where  $-11 \equiv 1 \pmod{3}$ , so  $A = -11$ .

If  $p = 37$ ,  $N(x^3 + y^3 = 1) = 37 - 2 - 11 = 24$ .

•  $p = 97$ .

$4 \times 97 = 388 = 19^2 + 27 \times 1^2$ , where  $19 \equiv 1 \pmod{3}$ , so  $A = 19$ .

If  $p = 97$ ,  $N(x^3 + y^3 = 1) = 97 - 2 + 19 = 114$ .

(These results were verified on pari/gp.)

□

**Ex. 8.12** If  $p \equiv 1 \pmod{4}$ , then we have seen that  $p = a^2 + b^2$  with  $a, b \in \mathbb{Z}$ . If we require that  $a$  and  $b$  are positive, that  $a$  be odd, and that  $b$  is even, show that  $a$  and  $b$  are uniquely determined. (Hint: Use the fact that unique factorization holds in  $\mathbb{Z}[i]$  and that if  $p = a^2 + b^2$  then  $a + bi$  is a prime in  $\mathbb{Z}[i]$ .)

*Proof.* Suppose that  $p$  is prime,  $p \equiv 1 \pmod{4}$ , and  $p = a^2 + b^2 = c^2 + d^2$ , where  $a, b, c, d$  are positive integers,  $a, c$  odd,  $b, d$  even. We will show that  $a = c, b = d$ .

As  $p = N(a + bi)$ ,  $\pi = a + bi$  is irreducible in  $\mathbb{Z}[i]$ : indeed  $\pi = uv$  implies that  $p = N(\pi) = N(u)N(v)$ , so  $N(u) = 1$  or  $N(v) = 1$ , and  $u$  or  $v$  is an unit.

Since  $\mathbb{Z}[i]$  is a principal ideal domain,  $\pi$  is a prime in  $\mathbb{Z}[i]$ .

$(a + bi)(a - bi) = (c + di)(c - di)$ , so the prime  $\pi$  divides  $c + di$ , or it divides  $c - di$ .

As  $N(\pi) = N(c + di) = N(c - di)$ , the quotient is a unit. Therefore  $\pi$  is an associate of  $c + di$  or  $c - di$ . Since the units in  $\mathbb{Z}[i]$  are  $1, -1, i, -i$ ,

$$a + bi = \pm(c + di), \text{ or } a + bi = \pm i(c + di), \text{ or } a + bi = \pm(c - di), \text{ or } a + bi = \pm i(c - di).$$

In all cases,  $a = \pm c, b = \pm d$ , or  $a = \pm d, b = \pm c$ . Since  $a, b, c, d$  are positive,  $a = c, b = d$ , or  $a = d, b = c$ . As  $ac$  are odds, and  $b, d$  even,  $a = c, b = d$ : the unicity of the decomposition is proved.  $\square$

**Ex. 8.13** If  $p \equiv 1 \pmod{3}$ , we have seen that  $4p = A^2 + 27B^2$ , with  $A, B \in \mathbb{Z}$ . If we require that  $A \equiv 1 \pmod{3}$ , show that  $A$  is uniquely determined. (Hint: Use the fact that unique factorization holds in  $\mathbb{Z}[\omega]$ . This proof is a little trickier than that for Exercise 12.)

*Proof.* Suppose that  $4p = A^2 + 27B^2 = C^2 + 27D^2$ , where  $A \equiv C \equiv 1 \pmod{3}$ . We will show that  $A = C$ .

Let  $\omega = e^{2i\pi/3} = -1/2 + i\sqrt{3}/2$ . Then  $i\sqrt{3} = 2\omega + 1$ , and for all  $x, y$ ,  $x^3 + 3y^2 = (x + i\sqrt{3}y)(x - i\sqrt{3}y) = (x + (2\omega + 1)y)(x - (2\omega + 1)y)$ ,

$$x^2 + 3y^2 = (x + y + 2jy)(x - y - 2jy).$$

With  $x = A, y = 3B$ , we obtain

$$4p = A^2 + 27B^2 = (A + 3B + 6\omega B)(A - 3B - 6\omega B).$$

Note that  $A, B$  are of same parity, since  $4p = A^2 + 27B^2$ .

So we can write  $p = ((A + 3B)/2 + 3\omega B)((A - 3B)/2 - 6\omega B)$ :

$$p = \pi\bar{\pi}, \text{ where } \pi = \frac{A + 3B}{2} + 3\omega B \in \mathbb{Z}[\omega].$$

$\pi$  is a prime in  $\mathbb{Z}[\omega]$ : indeed  $\pi = uv$ ,  $u, v \in \mathbb{Z}[\omega]$  implies  $p = N(\pi) = N(u)N(v)$ , then  $N(u) = 1$  or  $N(v) = 1$ ,  $u$  or  $v$  is a unit, so  $\pi$  is irreducible in the principal ideal domain  $\mathbb{Z}[\omega]$ , thus  $\pi$  is a prime in  $\mathbb{Z}[\omega]$ .

$$\pi\bar{\pi} = \left(\frac{A + 3B}{2} + 3\omega B\right) \left(\frac{A - 3B}{2} - 3\omega B\right) = \left(\frac{C + 3D}{2} + 3\omega D\right) \left(\frac{C - 3D}{2} - 3\omega D\right).$$

As  $\pi$  is a prime, it divides  $\frac{C + 3D}{2} + 3\omega D$  or its conjugate. Since they have the same norm

$p$ , they are associated. The units of  $\mathbb{Z}[\omega]$  are  $\pm 1, \pm j, \pm j^2$ , so there exists 12 cases :

$$\begin{aligned}\frac{A+3B}{2} + 3\omega B &= \pm \left( \frac{C+3D}{2} + 3\omega D \right) \\ \frac{A+3B}{2} + 3\omega B &= \pm \omega \left( \frac{C+3D}{2} + 3\omega D \right) \\ \frac{A+3B}{2} + 3\omega B &= \pm \omega^2 \left( \frac{C+3D}{2} + 3\omega D \right) \\ \frac{A+3B}{2} + 3\omega B &= \pm \left( \frac{C-3D}{2} - 3\omega D \right) \\ \frac{A+3B}{2} + 3\omega B &= \pm \omega \left( \frac{C-3D}{2} - 3\omega D \right) \\ \frac{A+3B}{2} + 3\omega B &= \pm \omega^2 \left( \frac{C-3D}{2} - 3\omega D \right)\end{aligned}$$

If we replace  $D$  by  $-D$ , we obtain the 6 last cases from the 6 first cases, so it is sufficient to examine the first 6 cases. Recall that  $(1, \omega)$  is a  $\mathbb{Z}$ -base of  $\mathbb{Z}[\omega]$ .

1)  $A + 3B + 6\omega B = C + 3D + 6\omega D$ .

Then  $B = D$  and  $A + 3B = C + 3D$ , so  $A = C$ , which is the expected result. The five other cases are impossible :

2)  $A + 3B + 6\omega B = -C - 3D - 6\omega D$ .

Then  $B = -D, A = -C$ . As  $A \equiv C \equiv 1 \pmod{3}$ , this is impossible.

3)  $A + 3B + 6\omega B = \omega(C + 3D + 6\omega D) = \omega(C + 3D) + (-1 - \omega)6D = -6D + \omega(C - 3D)$ .

Then  $A + 3B = -6D, A \equiv 0 \pmod{3}$ , this is impossible.

4)  $A + 3B + 6\omega B = -\omega(C + 3D + 6\omega D) = -\omega(C + 3D) + (1 + \omega)6D = 6D + \omega(-C + 3D)$ .

Then  $A + 3B = -6D, A \equiv 0 \pmod{3}$ , this is impossible.

5)  $A + 3B + 6\omega B = \omega^2(C + D + 6\omega D) = (-1 - \omega)(C + 3D) + 6D = -C + 3D + \omega(-C - 3D)$ . Then  $A + 3B = -C + 3D, A \equiv -C \pmod{3}$ , this is impossible.

6)  $A + 3B + 6\omega B = -\omega^2(C + 3D + 6\omega D) = (1 + \omega)(C + 3D) - 6D = (C - 3D) + \omega(C + 3D)$ .

Then  $6B = C + 3D, C \equiv 0 \pmod{3}$ , this is impossible.

In conclusion  $A = C$ . □

**Ex. 8.14** Suppose that  $p \equiv 1 \pmod{n}$  and that  $\chi$  is a character of order  $n$ . Show that  $g(\chi^n) \in \mathbb{Z}[\zeta]$ , where  $\zeta = e^{2\pi i/n}$ .

*Proof.* From Proposition 8.3.3 we know that

$$g(\chi)^n = \chi(-1)pJ(\chi, \chi)J(\chi, \chi^2) \cdots J(\chi, \chi^{n-2}).$$

Let  $\mathbb{U}_n = \{x \in \mathbb{C} \mid x^n = 1\} = \{1, \zeta, \dots, \zeta^{n-1}\}$ , with  $\zeta = e^{2\pi i/n}$ , the group of  $n$ -th roots of unity. As the order of  $\chi$  is  $n$ , for all  $x \in \mathbb{F}_p^*$ ,  $(\chi(x))^n = \chi^n(x) = \varepsilon(x) = 1$ , so  $\chi(x) \in \mathbb{U}_n$ , and also  $\chi^k(x) = (\chi(x))^k$ .

Therefore  $J(\chi, \chi^k) = \sum_{x+y=1} \chi(x)\chi^k(x) \in \mathbb{Z}[\zeta]$ . Moreover  $\chi(-1) = \pm 1$ , so  $\chi(-1)$  and  $p$  are in  $\mathbb{Z}[\zeta]$ . In conclusion  $g(\chi^n) \in \mathbb{Z}[\zeta]$ . □

**Ex. 8.15** Suppose that  $p \equiv 1 \pmod{6}$  and let  $\chi$  and  $\rho$  be characters of order 3 and 2, respectively. Show that the number of solutions to  $y^2 = x^3 + D$  in  $\mathbb{F}_p$  is  $p + \pi + \bar{\pi}$ , where  $\pi = \chi\rho(D)J(\chi\rho)$ . If  $\chi(2) = 1$ , show that the number of solutions to  $y^2 = x^3 + 1$  is  $p + A$ , where  $4p = A^2 + 27B^2$  and  $A \equiv 1 \pmod{3}$ . Verify this result numerically when  $p = 31$ .

*Proof.*  $x \mapsto -x$  is a bijection between the set of roots of  $x^3 = b$  and the set of roots of  $(-x)^3 = b$ , so  $N(x^3 = b) = N((-x)^3 = b) = N(x^3 = -b)$ .

As  $\chi$  is a character of order 3, the characters whose order divides 3 are  $\varepsilon, \chi, \chi^2$ . Using Prop. 8.1.5, we obtain

$$\begin{aligned}
N(y^2 = x^3 + D) &= \sum_{a+b=D} N(y^2 = a)N((-x)^3 = b) \\
&= \sum_{a+b=D} N(y^2 = a)N(x^3 = b) \\
&= \sum_{a+b=D} (1 + \rho(a))(1 + \chi(b) + \chi^2(b)) \\
&= \sum_{i=0}^1 \sum_{j=0}^2 \sum_{a+b=D} \rho^i(a)\chi^j(b) \\
&= \sum_{i=0}^1 \sum_{j=0}^2 \rho(D)^i \chi(D)^j \sum_{a'+b'=1} \rho^i(a')\chi^j(b') \quad (a = Da', b = Db') \\
&= \sum_{i=0}^1 \sum_{j=0}^2 \rho(D)^i \chi(D)^j J(\chi^j, \rho^i)
\end{aligned}$$

We know (Theorem 1) that  $J(\chi, \varepsilon) = J(\chi^2, \varepsilon) = J(\varepsilon, \rho) = 0$ ,  $J(\varepsilon, \varepsilon) = p$ , so

$$N(y^2 = x^3 + D) = p + \rho(D)\chi(D)J(\chi, \rho) + \rho(D)\chi^2(D)J(\chi^2, \rho).$$

As  $\chi^2(D) = \chi^{-1}(D) = \overline{\chi(D)}$ , and as  $\overline{\rho(D)} = \rho(D)$ , then  $J(\chi^2, \rho) = J(\overline{\chi}, \bar{\rho}) = \overline{J(\chi, \rho)}$ , and

$$N(y^2 = x^3 + D) = p + \pi + \bar{\pi}, \text{ where } \pi = (\rho\chi)(D)J(\chi, \rho).$$

If  $\chi(2) = 1$ , then from Exercise 8.6 we have

$$J(\chi, \chi) = \chi(2)^{-2}J(\chi, \rho) = J(\chi, \rho).$$

With  $D = 1$  (if  $\chi(2) = 1$ ), we obtain

$$N(y^2 = x^3 + 1) = p + \pi + \bar{\pi}, \pi = J(\chi, \rho) = J(\chi, \chi).$$

From Prop. 8.3.4 we know that  $J(\chi, \chi) = a + b\omega$ ,  $b \equiv 0 \pmod{3}$ ,  $a \equiv -1 \pmod{3}$ .

$\pi + \bar{\pi} = 2 \operatorname{Re} J(\chi, \chi) = 2a - b \equiv 1 \pmod{3}$ , and  $p = N(J(\chi, \rho)) = a^2 - ab + b^2$ , so  $4p = (2a - b)^2 + 3b^2$ .

Writing  $A = 2a - b$ ,  $B = b/3$ , we obtain  $4p = A^2 + 27B^2$ ,  $A \equiv 1 \pmod{3}$  (the unicity of  $A$  if proved in Exercise 8.13).

Conclusion :  $N(y^2 = x^3 + 1) = p + A$ , where  $4p = A^2 + 27B^2$ ,  $A \equiv 1 \pmod{3}$ .

If  $p = 31$ , 3 is a primitive element, and  $2 = 3^{24} = (3^8)^3$  in  $\mathbb{F}_{31}$ , therefore  $\chi(2) = 1$ .

$31 = 4 + 27$ ,  $4 \times 31 = 124 = 4^2 + 27 \times 2^2$ , and  $4 \equiv 1 \pmod{3}$ , so

if  $p = 31$ ,  $N(y^2 = x^3 + 1) = 35$ . □

**Ex. 8.16** Suppose that  $p \equiv 1 \pmod{4}$  and that  $\chi$  is a character of order 4. Let  $N$  be the number of solutions to  $x^4 + y^4 = 1$  in  $\mathbb{F}_p$ . Show that  $N = p + 1 - \delta_4(-1)4 + 2 \operatorname{Re} J(\chi, \chi) + 4 \operatorname{Re} J(\chi, \rho)$ .

*Proof.* Let  $\chi$  a character of order 4 : such a character exists since  $p \equiv 1 \pmod{4}$ . Then

$$\begin{aligned}
N(x^4 + y^4 = 1) &= \sum_{a+b=1} N(x^4 = a)N(y^4 = b) \\
&= \sum_{a+b=1} \sum_{i=0}^3 \chi^i(a) \sum_{j=0}^3 \chi^j(b) \\
&= \sum_{i=0}^3 \sum_{j=0}^3 \sum_{a+b=1} \chi^i(a) \chi^j(b) \\
&= \sum_{i=0}^3 \sum_{j=0}^3 J(\chi^i, \chi^j) \\
&= p - \chi(-1) - \chi^2(-1) - \chi^3(-1) \\
&\quad + J(\chi, \chi) + J(\chi, \chi^2) + J(\chi^2, \chi) \\
&\quad + J(\chi^2, \chi^3) + J(\chi^3, \chi^2) + J(\chi^3, \chi^3),
\end{aligned}$$

since from Theorem 1, we have  $J(\varepsilon, \varepsilon) = p$ ,  $J(\varepsilon, \chi^j) = 0$  for  $j = 1, 2, 3$ , and  $J(\chi^i, \chi^{4-i}) = -\chi^i(-1)$ .

Moreover

$$-[\chi(-1) + \chi^2(-1) + \chi^3(-1)] = 1 - [1 + \chi(-1) + \chi^2(-1) + \chi^3(-1)],$$

and

$$\begin{cases} 1 + \chi(-1) + \chi^2(-1) + \chi^3(-1) = \frac{1-\chi^4(-1)}{1-\chi(-1)} & = 0 \quad \text{if } \chi(-1) \neq 1 \\ & = 4 \quad \text{if } \chi(-1) = 1. \end{cases}$$

Let  $g$  a generator of  $\mathbb{F}_p^*$ . Recall that  $\chi(g) = e^{qi\pi/2}$  with  $q$  odd, so  $\chi : a = g^k \mapsto e^{iqk\pi/2} = i^{qk}$ , thus

$$\chi(a) = 1 \iff \chi(g^k) = 1 \iff i^{qk} = 1 \iff 4 \mid k \iff a = b^4, b \in \mathbb{F}^*.$$

$\delta_4$  is defined by  $\delta_4(a) = 1$  if  $a$  is a fourth power, 0 if not. Then

$$-[\chi(-1) + \chi^2(-1) + \chi^3(-1)] = 1 - \delta_4(-1)4.$$

Moreover  $J(\chi, \chi) + J(\chi^3, \chi^3) = 2 \operatorname{Re} (J(\chi, \chi))$ , and

$$J(\chi, \chi^2) + J(\chi^3, \chi^2) + J(\chi^2, \chi) + J(\chi^2, \chi^3) = 2 \operatorname{Re} (J(\chi, \chi^2)) + 2 \operatorname{Re} (J(\chi^2, \chi)) = 4 \operatorname{Re} (J(\chi, \chi^2)).$$

$\chi$  is of order 4, so  $\rho = \chi^2$  is the unique character of order 2, the Legendre's character.

In conclusion,

$$N(x^4 + y^4 = 1) = p + 1 - \delta_4(-1)4 + 2 \operatorname{Re} (J(\chi, \chi)) + 4 \operatorname{Re} (J(\chi, \rho)).$$

□

**Ex. 8.17** (continuation) By Exercise 8.7,  $J(\chi, \chi) = \chi(-1)J(\chi, \rho)$ . Let  $\pi = -J(\chi, \rho)$ . Show that

(a)  $N = p - 3 - 6 \operatorname{Re} \pi$  if  $p \equiv 1 \pmod{8}$ .

(b)  $N = p + 1 - 2 \operatorname{Re} \pi$  if  $p \equiv 5 \pmod{8}$ .

*Proof.* Let  $g$  a generator in  $\mathbb{F}_p^*$ . As  $(g^{(p-1)/2})^2 = 1$  and  $g^{(p-1)/2} \neq 1$ , then  $g^{(p-1)/2} = -1$ . As in Exercise 8.16, write  $\chi(g) = e^{qi\pi/2}$ , with  $q$  odd.

Then  $-1$  is a fourth power in  $\mathbb{F}_p^*$  iff (see Exercise 8.16)

$$\begin{aligned} \delta_4(-1) = 1 &\iff \chi(-1) = 1 \\ &\iff \chi(g^{(p-1)/2}) = 1 \\ &\iff e^{q((p-1)/2)i\pi/2} = 1 \\ &\iff 4 \mid q(p-1)/2 \\ &\iff 4 \mid (p-1)/2 \\ &\iff p \equiv 1 \pmod{8}. \end{aligned}$$

By Exercise 8.7, as  $\chi$  is a character of order 4,

$$J(\chi, \chi) = \chi(-1)J(\chi, \rho).$$

- If  $p \equiv 1[8]$ ,  
 $\chi(-1) = 1$ , so  $J(\chi, \chi) = J(\chi, \rho)$ , and  $\delta_4(-1) = 1$ .

$$\begin{aligned} N &= p + 1 - \delta_4(-1)4 + 2 \operatorname{Re} J(\chi, \chi) + 4 \operatorname{Re} J(\chi, \rho) \\ &= p - 3 + 6 \operatorname{Re} J(\chi, \rho) \\ &= p - 3 - 6 \operatorname{Re} \pi, \quad \text{where } \pi = -J(\chi, \rho). \end{aligned}$$

- If  $p \equiv 5[8]$ ,  
 $\chi(-1) = -1$ , donc  $J(\chi, \chi) = -J(\chi, \rho)$ , et  $\delta_4(-1) = 0$

$$\begin{aligned} N &= p + 1 - \delta_4(-1)4 + 2 \operatorname{Re} J(\chi, \chi) + 4 \operatorname{Re} J(\chi, \rho) \\ &= p + 1 + 2 \operatorname{Re} J(\chi, \rho) \\ &= p + 1 - 2 \operatorname{Re} \pi. \end{aligned}$$

□

**Ex. 8.18** (continuation) Let  $\pi = a + bi$ . One can show (see Chapter 11, Section 5) that  $a$  is odd,  $b$  is even, and  $a \equiv 1 \pmod{4}$  if  $4 \mid b$  and  $a \equiv -1 \pmod{4}$  if  $4 \nmid b$ . Let  $p = A^2 + B^2$  and fix  $A$  by requiring that  $A \equiv 1 \pmod{4}$ . Then show that

(a)  $N = p - 3 - 6A$  if  $p \equiv 1 \pmod{8}$ ,

(b)  $N = p + 1 + 2A$  if  $p \equiv 5 \pmod{8}$ .

*Proof.* Recall that  $\pi = -J(\chi, \rho) \in \mathbb{Z}[i]$ , so  $\pi = a + bi$ ,  $a, b \in \mathbb{Z}$ .

- 1) We begin by proving that  $\pi \equiv 1 \pmod{2+2i}$  ( see Chapter 11, Section 5).

For all  $t \in \mathbb{F}_p^*$ ,  $\rho(t) = \pm 1$ , so  $\rho(t) - 1 \equiv 0 \pmod{2}$ .

Let's verify that  $\chi(t) - 1 \equiv 0 \pmod{1+i}$ .  $\chi(t) \in \{1, -1, i, -i\}$ , so  $\chi(t) - 1 \in \{0, -2, i-1, -i-1\}$ . As  $2 = (1-i)(1+i)$  and  $i-1 = i(1+i)$ , we obtain

$$\forall t \in \mathbb{F}_p^*, 1+i \mid \chi(t) - 1.$$

Thus

$$\forall s \in \mathbb{F}_p^*, \forall t \in \mathbb{F}_p^*, (\rho(s) - 1)(\chi(t) - 1) \equiv 0 \pmod{2+2i}.$$

Moreover, if  $s = 0, t = 1$ , then  $\chi(b) = 1$ , and if  $s = 1, t = 0$ , then  $\rho(s) = 1$ , so

$$\sum_{s+t=1} (\rho(s) - 1)(\chi(b) - 1) \equiv 0 \pmod{2+2i}.$$

This gives, when developing this expression, :

$$-\pi - \sum_{b \in \mathbb{F}_p} \chi(b) - \sum_{a \in \mathbb{F}_p} \rho(a) + p \equiv 0 \pmod{2+2i}.$$

As  $\sum_b \chi(b) = \sum_a \rho(a) = 0$ , we obtain

$$\pi \equiv p \pmod{2+2i}.$$

Finally,  $p \equiv 1 \pmod{4}$ , and  $2+2i \mid 4$  since  $4 = (1-i)(2+2i)$ , so  $p \equiv 1 \pmod{2+2i}$ , so

$$\pi \equiv 1 \pmod{2+2i}.$$

2) By Corollary of Theorem 1,  $N(\pi) = N(J(\chi, \rho)) = p = a^2 + b^2$ .

We know that  $p \equiv 1 \pmod{4}$ ,  $p = a^2 + b^2$  and  $a + ib \equiv 1 \pmod{2+2i}$ . Then we prove that  $a$  is odd,  $b$  is even, and  $a \equiv 1 \pmod{4}$  if  $4 \mid b$  and  $a \equiv -1 \pmod{4}$  if  $4 \nmid b$ .

$a + bi \equiv 1 \pmod{2+2i}$ , so  $a + bi \equiv 1 \pmod{2}$ , so  $a$  is odd, and  $b$  is even.

• If  $4 \mid b$ , then  $2+2i \mid b$ .

$a \equiv 1 \pmod{2+2i}$ , and by complex conjugation,  $a \equiv 1 \pmod{2-2i}$ , so  $52 + 2i)(2-2i) = 8 \mid (a-1)^2$ , thus  $4 \mid a-1$ .

• If  $4 \nmid b$ , then  $b = 4k + 2, k \in \mathbb{Z}$ .

Therefore,  $1 \equiv a + bi \equiv a + 2i \pmod{2+2i}$ . As  $2i \equiv -2 \pmod{2+2i}$ ,  $a \equiv 3 \equiv -1 \pmod{2+2i}$ . By conjugation,  $a \equiv -1 \pmod{2-2i}$ . Multiplying these congruences, we obtain  $8 \mid (a+1)^2$ , so  $a \equiv -1 \pmod{4}$ .

3)  $\pi = -J(\chi, \rho) = a + bi$  is such that  $a^2 + b^2 = p$ ,  $a$  odd,  $b$  even and also

$$(4 \mid b \text{ and } a \equiv 1 \pmod{4}) \text{ or } (4 \nmid b \text{ and } a \equiv -1 \pmod{4}).$$

If  $p = A^2 + B^2$ ,  $A$  odd and  $B$  even, then also  $p = (-A)^2 + B^2$ , and  $A \equiv 1 \pmod{4}$  or  $-A \equiv 1 \pmod{4}$ . So there exists a decomposition  $p = A^2 + B^2$  such that  $A \equiv 1 \pmod{4}$ . Such a decomposition is unique. Let's verify that  $4 \mid b$  if  $p \equiv 1 \pmod{8}$ ,  $4 \nmid b$  if  $p \equiv 5 \pmod{8}$ .

$$p = a^2 + b^2, a = 2a' + 1, b = 2b', \text{ so } p = 4a'^2 + 4a' + 1 + 4b'^2 = 8\frac{a'(a'+1)}{2} + 1 + 4b'^2.$$

$$\text{Hence } 4 \mid b \iff 2 \mid b' \iff 8 \mid p - 1.$$

Therefore if  $p \equiv 1 \pmod{8}$ ,  $\text{Re } \pi = a = A$ , and if  $p \equiv 5 \pmod{8}$ ,  $\text{Re } \pi = a = -A$ .

In conclusion, by Exercise 8.17 :

if  $p = A^2 + B^2, A \equiv 1 \pmod{4}$ , and  $N = N(x^4 + y^4 = 1)$  in  $\mathbb{F}_p$ ,



- (a)  $N = p - 3 - 6A$  if  $p \equiv 1 \pmod{8}$ ,  
(b)  $N = p + 1 + 2A$  if  $p \equiv 5 \pmod{8}$ .

Note : if  $p \equiv -1 \pmod{4}$ , then there is no character of order 4 on  $\mathbb{F}_p^*$ , and  $d = 4 \wedge (p-1) = 4 \wedge (4k+2) = 2$ , so

$$N(x^4 = a) = \sum_{\chi_d=1} \chi(a) = 1 + \rho(a) = N(x^2 = a).$$

$$\begin{aligned} N(x^4 + y^4 = 1) &= \sum_{a+b=1} N(x^4 = a)N(y^4 = b) \\ &= \sum_{a+b=1} a + b = 1N(x^2 = a)N(y^2 = b) \\ &= N(x^2 + y^2 = 1) = 1 \end{aligned}$$

Using Chapter 8, Section 3, we obtain

$$N(x^4 + y^4 = 1) = p + 1 \text{ if } p \equiv -1 \pmod{4}.$$

□

**Ex. 8.19** Find a formula for the number of solutions to  $x_1^2 + x_2^2 + \cdots + x_r^2 = 0$  in  $\mathbb{F}_p$ .

*Proof.* Let  $\chi$  be the Legendre character. Then

$$\begin{aligned} N(x_1^2 + x_2^2 + \cdots + x_r^2 = 0) &= \sum_{a_1+a_2+\cdots+a_r=0} N(x_1^2 = a_1)N(x_2^2 = a_2) \cdots N(x_r^2 = a_r) \\ &= \sum_{a_1+a_2+\cdots+a_r=0} (1 + \chi(a_1))(1 + \chi(a_2)) \cdots (1 + \chi(a_r)) \\ &= p^{r-1} + J_0(\chi, \chi, \cdots, \chi) \end{aligned}$$

(We used Proposition 8.5.1) For all  $k$ ,  $\chi^{2k} = \varepsilon$ ,  $\chi^{2k+1} = \chi$ .

- If  $r$  is odd,  $\chi^r \neq \varepsilon$ , so  $J_0(\chi, \chi, \cdots, \chi) = 0$  (Proposition 8.5.1).

$$N(x_1^2 + x_2^2 + \cdots + x_r^2 = 0) = p^{r-1}.$$

- If  $r$  is even,  $\chi^r = \varepsilon$ , so  $J_0(\chi, \chi, \cdots, \chi) = \chi(-1)(p-1)J(\chi, \chi, \cdots, \chi)$ , where there are  $r-1$  components in the Jacobi sum (Proposition 8.5.1).

By Theorem 3,  $J(\chi, \chi, \cdots, \chi)g(\chi^{r-1}) = g(\chi)^{r-1}$ , and  $g(\chi^{r-1}) = g(\chi)$ , so

$$J(\chi, \chi, \cdots, \chi) = g(\chi)^{r-2}.$$

$g(\chi)^2 = \chi(-1)p$ , therefore  $\chi^{r-2} = \chi(-1)^{(r/2)-1}p^{(r/2)-1} = (-1)^{((p-1)/2)(r/2-1)}p^{(r/2)-1}$ .  
So

$$N(x_1^2 + x_2^2 + \cdots + x_r^2 = 0) = p^{r-1} + (-1)^{\frac{p-1}{2} \frac{r}{2}} (p-1)p^{\frac{r}{2}-1}.$$

(Verified in C++ with small values of  $p$  and  $r$ .)

Conclusion :

$$\begin{cases} N(x_1^2 + x_2^2 + \cdots + x_r^2 = 0) &= p^{r-1} & \text{if } r \text{ is odd} \\ &= p^{r-1} + (-1)^{\frac{p-1}{2} \frac{r}{2}} (p-1)p^{\frac{r}{2}-1} & \text{if } r \text{ is even.} \end{cases}$$

□

**Ex. 8.20** Generalize Proposition 8.6.1 by finding an explicit formula for the number of solutions to  $a_1x_1^2 + a_2x_2^2 + \cdots + a_rx_r^2 = 1$  in  $\mathbb{F}_p$ .

*Proof.* Write  $\chi$  the Legendre character.

$$\begin{aligned} N(a_1x_1^2 + \cdots + a_rx_r^2 = 1) &= \sum_{a_1u_1 + \cdots + a_ru_r = 1} N(x_1^2 = u_1) \cdots N(x_r^2 = u_r) \\ &= \sum_{a_1u_1 + \cdots + a_ru_r = 1} (1 + \chi(u_1)) \cdots (1 + \chi(u_r)) \quad (v_i = a_iu_i) \\ &= \sum_{v_1 + \cdots + v_r = 1} (1 + \chi(a_1)^{-1}\chi(v_1)) \cdots (1 + \chi(a_r)^{-1}\chi(v_r)) \\ &= p^{r-1} + \chi(a_1^{-1}) \cdots \chi(a_r^{-1}) J(\chi, \chi, \dots, \chi) \end{aligned}$$

$$\chi(a_i^{-1}) = \overline{\chi(a_i)} = \chi(a_i) = \left(\frac{a_i}{p}\right)$$

$J(\chi, \chi, \dots, \chi)$  is computed in Chapter 5 Section 6. We obtain

$$\begin{cases} N(a_1x_1^2 + \cdots + a_rx_r^2 = 1) &= p^{r-1} + \left(\frac{a_1}{p}\right) \cdots \left(\frac{a_r}{p}\right) (-1)^{\frac{r-1}{2} \frac{p-1}{2}} p^{\frac{r-1}{2}} & \text{if } r \text{ is odd} \\ &= p^{r-1} - \left(\frac{a_1}{p}\right) \cdots \left(\frac{a_r}{p}\right) (-1)^{\frac{r}{2} \frac{p-1}{2}} p^{\frac{r}{2}-1} & \text{if } r \text{ is even.} \end{cases}$$

□

**Ex. 8.21** Suppose that  $p \equiv 1 \pmod{d}$ ,  $\zeta = e^{2\pi i/p}$ , and consider  $\sum_x \zeta^{ax^d}$ . Show that  $\sum_x \zeta^{ax^d} = \sum_r m(r) \zeta^{ar}$ , where  $m(r) = N(x^d = r)$ .

*Proof.* Let  $A_r = \{x \in \mathbb{F}_p \mid x^d = r\}$

Then  $\mathbb{F}_p = \coprod_r A_r$ , thus

$$\sum_{x \in \mathbb{F}_p} \zeta^{ax^d} = \sum_{r \in \mathbb{F}_p} \sum_{x \in A_r} \zeta^{ax^d} = \sum_{r \in \mathbb{F}_p} |A_r| \zeta^{ar} = \sum_{r \in \mathbb{F}_p} m(r) \zeta^{ar},$$

where  $m(r) = |A_r| = N(x^d = r)$

□

**Ex. 8.22** (continuation) Prove that  $\sum_x \zeta^{ax^d} = \sum_{\chi} g_a(\chi)$ , where the sum is over all  $\chi$  such that  $\chi^d = \varepsilon$ ,  $\chi \neq \varepsilon$ . Assume that  $p \nmid a$ .

*Proof.* By Exercise 8.21,

$$S = \sum_{x \in \mathbb{F}_p} \zeta^{ax^d} = \sum_{r \in \mathbb{F}_p} m(r) \zeta^{ar}.$$

As  $d \mid p-1$ , by Proposition 8.1.5,

$$m(r) = N(x^d = r) = \sum_{\chi^d = \varepsilon} \chi(r).$$

Therefore

$$S = \sum_{r \in \mathbb{F}_p} \sum_{\chi^d = \varepsilon} \chi(r) \zeta^{ar} = \sum_{\chi^d = \varepsilon} \sum_{r \in \mathbb{F}_p} \chi(r) \zeta^{ar}.$$

If  $\chi = \varepsilon$ ,  $\sum_{r \in \mathbb{F}_p} \chi(r) \zeta^{ar} = \sum_{r \in \mathbb{F}_p} \zeta^{ar} = 0$ , since  $a \not\equiv 0 \pmod{p}$ .

By definition  $g_a(\chi) = \sum_r \chi(r) \zeta^{ar}$ , so, if  $d \mid p-1$ ,  $p \nmid a$ ,

$$\sum_{x \in \mathbb{F}_p} \zeta^{ax^d} = \sum_{\chi^d = \varepsilon, \chi \neq \varepsilon} g_a(\chi)$$

□

**Ex. 8.23** Let  $f(x_1, x_2, \dots, x_n) \in \mathbb{F}_p[x_1, x_2, \dots, x_n]$ . Let  $N$  be the number of zeros of  $f$  in  $\mathbb{F}_p$ . Show that  $N = p^{n-1} + p^{-1} \sum_{a \neq 0} (\sum_{x_1, \dots, x_n} \zeta^{af(x_1, \dots, x_n)})$ .

*Proof.* Let  $A_r = \{(x_1, x_2, \dots, x_n) \in \mathbb{F}_p^n \mid f(x_1, x_2, \dots, x_n) = r\}$ . Then  $\mathbb{F}_p^n = \coprod_{r \in \mathbb{F}_p} A_r$ , so, for all  $a \in \mathbb{F}_p$ ,

$$\begin{aligned} \sum_{(x_1, x_2, \dots, x_n) \in \mathbb{F}_p^n} \zeta^{af(x_1, x_2, \dots, x_n)} &= \sum_{r \in \mathbb{F}_p} \sum_{(x_1, x_2, \dots, x_n) \in A_r} \zeta^{ar} \\ &= \sum_{r \in \mathbb{F}_p} |A_r| \zeta^{ar} \end{aligned}$$

Let  $m(r) = |A_r| = N(f(x_1, x_2, \dots, x_n) = r)$ . Then

$$\begin{aligned} \sum_{a \in \mathbb{F}_p} \sum_{(x_1, x_2, \dots, x_n) \in \mathbb{F}_p^n} \zeta^{af(x_1, x_2, \dots, x_n)} &= \sum_{a \in \mathbb{F}_p} \sum_{r \in \mathbb{F}_p} m(r) \zeta^{ar} \\ &= \sum_{r \in \mathbb{F}_p} m(r) \sum_{a \in \mathbb{F}_p} \zeta^{ar} \end{aligned}$$

As  $\sum_{a \in \mathbb{F}_p} \zeta^{ar} = 0$  if  $r \neq 0$ , and  $\sum_{a \in \mathbb{F}_p} \zeta^{ar} = p$  if  $r = 0$ , we obtain

$$\sum_{a \in \mathbb{F}_p} \sum_{(x_1, x_2, \dots, x_n) \in \mathbb{F}_p^n} \zeta^{af(x_1, x_2, \dots, x_n)} = m(0)p = pN.$$

Moreover

$$\sum_{a \in \mathbb{F}_p} \sum_{(x_1, x_2, \dots, x_n) \in \mathbb{F}_p^n} \zeta^{af(x_1, x_2, \dots, x_n)} = p^n + \sum_{a \in \mathbb{F}_p^*} \sum_{(x_1, x_2, \dots, x_n) \in \mathbb{F}_p^n} \zeta^{af(x_1, x_2, \dots, x_n)},$$

so

$$pN = p^n + \sum_{a \in \mathbb{F}_p^*} \sum_{(x_1, x_2, \dots, x_n) \in \mathbb{F}_p^n} \zeta^{af(x_1, x_2, \dots, x_n)}.$$

In conclusion,

$$N = p^{n-1} + p^{-1} \sum_{a \in \mathbb{F}_p^*} \sum_{(x_1, x_2, \dots, x_n) \in \mathbb{F}_p^n} \zeta^{af(x_1, x_2, \dots, x_n)}.$$

□

**Ex. 8.24** (continuation) Let  $f(x_1, x_2, \dots, x_n) = a_1 x_1^{m_1} + a_2 x_2^{m_2} + \dots + a_n x_n^{m_n}$ . Let  $d_i = (m_i, p-1)$ . Show that  $N = p^{n-1} + p^{-1} \sum_{a \neq 0} \prod_{i=1}^n \sum_{\chi_i} g_{aa_i}(\chi_i)$  where  $\chi_i$  runs over all characters such that  $\chi_i^{d_i} = \varepsilon$  and  $\chi_i \neq \varepsilon$ .

*Proof.* By Exercise 8.2,

$$N = N(a_1 x_1^{m_1} + \dots + a_n x_n^{m_n} = 0) = N(a_1 x_1^{d_1} + \dots + a_n x_n^{d_n} = 0),$$

where  $d_i = m_i \wedge (p-1)$  divides  $p-1$ .

By Exercise 8.23,

$$N = p^{n-1} + p^{-1} \sum_{a \in \mathbb{F}_p^*} \sum_{(x_1, x_2, \dots, x_n) \in \mathbb{F}_p^n} \zeta^{a(a_1 x_1^{d_1} + \dots + a_n x_n^{d_n})}$$

By Exercise 8.22, since  $p \nmid a, p \nmid a_i$ ,

$$\begin{aligned} \sum_{(x_1, x_2, \dots, x_n) \in \mathbb{F}_p^n} \zeta^{a(a_1 x_1^{d_1} + \dots + a_n x_n^{d_n})} &= \left( \sum_{x_1 \in \mathbb{F}_p} \zeta^{a a_1 x_1^{d_1}} \right) \cdots \left( \sum_{x_n \in \mathbb{F}_p} \zeta^{a a_n x_n^{d_n}} \right) \\ &= \left( \sum_{\chi_1^{d_1} = \varepsilon, \chi_1 \neq \varepsilon} g_{aa_1}(\chi_1) \right) \cdots \left( \sum_{\chi_n^{d_n} = \varepsilon, \chi_n \neq \varepsilon} g_{aa_n}(\chi_n) \right) \\ &= \prod_{i=1}^n \sum_{\chi_i^{d_i} = \varepsilon, \chi_i \neq \varepsilon} g_{aa_i}(\chi_i) \end{aligned}$$

In conclusion,

$$N = p^{n-1} + p^{-1} \sum_{a \in \mathbb{F}_p^*} \prod_{i=1}^n \sum_{\chi_i^{d_i} = \varepsilon, \chi_i \neq \varepsilon} g_{aa_i}(\chi_i)$$

□

**Ex. 8.25** Deduce from Exercise 8.24 that  $|N - p^{n-1}| \leq (p-1)(d_1-1) \cdots (d_n-1)p^{(n/2)-1}$ .

*Proof.* As  $|g_{aa_i}(\chi_i)| = \sqrt{p}$ ,

$$\left| \sum_{\chi_i^{d_i} = \varepsilon, \chi_i \neq \varepsilon} g_{aa_i}(\chi_i) \right| \leq \sqrt{p} n_i,$$

where  $n_i = \text{Card} \{ \chi_i \neq \varepsilon \mid \chi_i^{d_i} = \varepsilon \}$ .

As  $d_i \mid p-1$ , there exists exactly  $d_i$  characters of order dividing  $d_i$ , so  $n_i = d_i - 1$  :

$$\left| \sum_{\chi_i^{d_i} = \varepsilon, \chi_i \neq \varepsilon} g_{aa_i}(\chi_i) \right| \leq \sqrt{p}(d_i - 1).$$

By Exercise 8.24,

$$N = p^{n-1} + p^{-1} \sum_{a \in \mathbb{F}_p^*} \prod_{i=1}^n \sum_{\chi_i^{d_i} = \varepsilon, \chi_i \neq \varepsilon} g_{aa_i}(\chi_i)$$

so

$$|N - p^{n-1}| \leq p^{-1}(p-1)\sqrt{p}(d_1-1) \cdots \sqrt{p}(d_n-1),$$

that is

$$|N - p^{n-1}| \leq (p-1)(d_1-1) \cdots (d_n-1)p^{\frac{n}{2}-1}$$

□

**Ex. 8.26** Let  $p$  be a prime,  $p \equiv 1 \pmod{4}$ ,  $\chi$  a multiplicative character of order 4 on  $\mathbb{F}_p$ , and  $\rho$  the Legendre symbol. Put  $J(\chi, \rho) = a + bi$ . Show

(a)  $N(y^2 + x^4 = 1) = p - 1 + 2a$ .

(b)  $N(y^2 = 1 - x^4) = p + \sum \rho(1 - x^4)$ .

(c)  $2a \equiv -(-1)^{(p-1)/4} \binom{2m}{m} \pmod{p}$  where  $m = (p-1)/4$ .

(d) Verify (c) for  $p = 13, 17, 29$ .

*Proof.* (a) By Proposition 8.1.5,

$$\begin{aligned} N(y^2 + x^4 = 1) &= \sum_{a+b=1} N(y^2 = a)N(x^4 = b) \\ &= \sum_{a+b=1} (1 + \rho(a))(1 + \chi(b) + \chi^2(b) + \chi^3(b)) \\ &= \sum_{i=0}^1 \sum_{j=0}^3 \sum_{a+b=1} \rho^i(a) \chi^j(b) \\ &= \sum_{i=0}^1 \sum_{j=0}^3 J(\rho^i, \chi^j) \end{aligned}$$

As  $J(\varepsilon, \varepsilon) = p$ , and  $0 = J(\varepsilon, \chi) = J(\varepsilon, \chi^2) = J(\varepsilon, \chi^3) = J(\rho, \varepsilon)$ , we obtain

$$N(y^2 + x^4 = 1) = p + J(\rho, \chi) + J(\rho, \chi^2) + J(\rho, \chi^3).$$

As  $J(\rho, \chi^3) = J(\bar{\rho}, \bar{\chi}) = \overline{J(\rho, \chi)}$ , and  $J(\rho, \chi^2) = J(\rho, \rho) = J(\rho, \rho^{-1}) = -\rho(-1) = -(-1)^{(p-1)/2} = -1$  (since  $p \equiv 1 \pmod{4}$ ). Moreover, by Exercise 8.7,

$$J(\chi, \rho) = \chi(-1)J(\chi, \chi) = \pm J(\chi, \chi) \in \mathbb{Z}[i] : J(\chi, \rho) = a + bi, (a, b) \in \mathbb{Z}^2.$$

Thus  $N(y^2 + x^4 = 1) = p + 2 \operatorname{Re} J(\chi, \rho) + J(\rho, \rho) = p - 1 + 2a$

In conclusion,

$$N(y^2 + x^4 = 1) = p - 1 + 2a, \text{ where } J(\chi, \rho) = a + bi.$$

(b) By Exercise 8.8,

$$\begin{aligned} \sum_{t \in \mathbb{F}_p} \rho(1 - t^4) &= \sum_{\lambda^4 = \varepsilon} J(\rho, \lambda) \\ &= J(\rho, \varepsilon) + J(\rho, \chi) + J(\rho, \chi^2) + J(\rho, \chi^3) \\ &= J(\rho, \rho) + J(\rho, \chi) + J(\rho, \bar{\chi}) \\ &= -1 + 2 \operatorname{Re} J(\chi, \rho) \\ &= -1 + 2a \end{aligned}$$

So  $N(y^2 = 1 - x^4) = p - 1 + 2a = p + \sum_{t \in \mathbb{F}_p} \rho(1 - t^4)$

(c) Reducing modulo  $p$ , we obtain :

$$\begin{aligned}
2\bar{a} &= 1 + \sum_{t \in \mathbb{F}_p} \rho(1 - t^4) \\
&= 1 + \sum_{t \in \mathbb{F}_p} (1 - t^4)^{\frac{p-1}{2}} \\
&= 1 + \sum_{t \in \mathbb{F}_p} \sum_{k=0}^{(p-1)/2} \binom{\frac{p-1}{2}}{k} (-1)^k t^{4k} \\
&= 1 + \sum_{k=0}^{(p-1)/2} (-1)^k \binom{\frac{p-1}{2}}{k} \sum_{t \in \mathbb{F}_p} t^{4k} \\
&= 1 + \sum_{k=1}^{(p-1)/2} (-1)^k \binom{\frac{p-1}{2}}{k} \sum_{t \in \mathbb{F}_p^*} t^{4k}
\end{aligned}$$

Let  $S_k = \sum_{t \in \mathbb{F}_p^*} t^{4k}$ ,  $k > 0$ , and  $g$  a generator of  $\mathbb{F}_p^*$  :  $t = g^i, 0 \leq i \leq p-2$ .

$$S_k = \sum_{i=0}^{p-2} (g^i)^{4k} = \sum_{i=0}^{p-2} (g^{4k})^i.$$

If  $g^{4k} \neq 1$ ,  $S_k \equiv \frac{g^{4k(p-1)} - 1}{g^{4k} - 1} = 0$ , if not  $S_k = p - 1 = -1$ .

For every  $k$ ,  $1 \leq k \leq (p-1)/2$ ,

$$g^{4k} = 1 \iff p-1 \mid 4k \iff \frac{p-1}{4} \mid k \iff k = \frac{p-1}{4} \text{ or } k = \frac{p-1}{2},$$

therefore

$$2a \equiv 1 - (-1)^{\frac{p-1}{4}} \binom{\frac{p-1}{2}}{\frac{p-1}{4}} - (-1)^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{\frac{p-1}{2}} \pmod{p},$$

$$2a \equiv -(-1)^{\frac{p-1}{4}} \binom{\frac{p-1}{2}}{\frac{p-1}{4}} \pmod{p}$$

(d) • If  $p = 13$ , I choose the primitive root  $g = \bar{2}$ , and  $\chi$  the character of order 4 defined by  $\chi(g) = i$  (the only other character of order 4 is  $\bar{\chi}$ ).

Then  $J(\chi, \rho) = -3 + 2i, a = -3$ .

$$N = p - 1 + 2a = 6.$$

$$2a = -6, -(-1)^{\frac{13-1}{4}} \binom{6}{3} = 20 \text{ et } -6 \equiv 20 \pmod{13}.$$

• If  $p = 17$ ,  $g = 3$ ,  $\chi(g) = i$ ,  $J(\chi, \rho) = -1 + 4i, a = -1$ .

$$N = p - 1 + 2a = 14.$$

$$2a = -2, -(-1)^{\frac{17-1}{4}} \binom{8}{4} = -70 \equiv -2 \pmod{17}.$$

- Si  $p = 29$ ,  $g = 3$ ,  $\chi(g) = i$ ,  $J(\chi, \rho) = 5 + 2i$ ,  $a = 5$ .

$$N = p - 1 + 2a = 38.$$

$$2a = 10, -(-1)^{\frac{29-1}{4}} \binom{14}{7} = 3432 \equiv 10 \pmod{29} \quad (3422 = 118 \times 29).$$

□

*Note :* By Prop. 8.3.1 (and Ex. 8.7),  $p = |J(\chi, \rho)|^2 = |J(\chi, \chi)|$ , so  $a^2 + b^2 = p$  (where  $p = 4m + 1$ ).

As  $\binom{2m}{m} = 2 \binom{2m-1}{m-1}$  is even, and as  $p$  is an odd prime  $a \equiv \pm \frac{1}{2} \binom{2m}{m} \pmod{p}$ .

Since  $p \geq 5$ ,  $p = a^2 + b^2$  implies  $|a| < \sqrt{p} < p/2$ , thus the least remainder of  $\frac{1}{2} \binom{2m}{m}$  is  $\pm a$ .

Moreover, from Wilson theorem, we obtain a square root of  $-1$  in  $\mathbb{F}_p$  ( $p = 4m + 1$ ) :

$$-1 \equiv (p-1)! \equiv \left[ (-1)^{(p-1)/4} \left( \frac{p-1}{2} \right)! \right]^2 = [(2m)!]^2.$$

Since  $\bar{b}a^{-1} = -\bar{1}$  in  $\mathbb{F}_p$ , we obtain  $b \equiv (2m)!a \pmod{p}$ . The conclusion is the proposition of Gauss, which gives an explicit formula for the solution of  $p = a^2 + b^2$  :

**Proposition** Let  $p$  a prime of the form  $p = 4m + 1$ .

If

$$\begin{aligned} a &\equiv \frac{1}{2} \binom{2m}{m} \pmod{p}, & -\frac{p}{2} < a < \frac{p}{2}, \\ b &\equiv (2m)!a \pmod{p}, & -\frac{p}{2} < b < \frac{p}{2}, \end{aligned}$$

then  $p = a^2 + b^2$ .

**Ex. 8.27** Let  $p \equiv 1 \pmod{3}$ ,  $\chi$  a character of order 3,  $\rho$  the Legendre symbol. Show

$$(a) \quad N(y^2 = 1 - x^3) = p + \sum \rho(1 - x^3).$$

$$(b) \quad N(y^2 + x^3 = 1) = p + 2 \operatorname{Re} J(\chi, \rho).$$

$$(c) \quad 2a - b \equiv -\binom{(p-1)/2}{(p-1)/3} \pmod{p} \text{ where } J(\chi, \rho) = a + b\omega.$$

*Proof.* (b) By Exercise 8.15,

$$N(y^2 = x^3 + D) = p + 2 \operatorname{Re} (\pi), \text{ where } \pi = (\rho\chi)(D)J(\chi, \rho).$$

Moreover, with  $D = 1$ , we obtain

$$\begin{aligned} N(y^2 + x^3 = 1) &= N(y^2 + (-x)^3 = 1) \\ &= p + 2 \operatorname{Re} J(\chi, \rho). \end{aligned}$$

$$J(\chi, \rho) = \sum_{a+b=1} \chi(a)\rho(b), \text{ where } \rho(b) \in \{-1, 1\}, \chi(a) \in \{1, \omega, \omega^2\}, \text{ so } J(\chi, \rho) \in \mathbb{Z}[\omega].$$

$$J(\chi, \rho) = a + b\omega, a \in \mathbb{Z}, b \in \mathbb{Z} \text{ and } 2 \operatorname{Re} (J(\chi, \rho)) = 2a - b.$$

(a) By Exercise 8.8,

$$\begin{aligned}\sum_x \rho(1-x^3) &= \sum_{\lambda^3=\varepsilon} J(\rho, \lambda) \\ &= J(\rho, \varepsilon) + J(\rho, \chi) + J(\rho, \chi^2) \\ &= 2 \operatorname{Re} J(\chi, \rho)\end{aligned}$$

so

$$\begin{aligned}N(y^2 = 1 - x^3) &= p + \sum_x \rho(1 - x^3) \\ &= p + 2a - b\end{aligned}$$

(c) Reducing modulo  $p$ , we obtain in  $\mathbb{F}_p$  :

$$\begin{aligned}2\bar{a} - \bar{b} &= \sum_x \rho(1 - x^3) \\ &= \sum_{t \in \mathbb{F}_p} (1 - t^3)^{\frac{p-1}{2}} \\ &= \sum_{t \in \mathbb{F}_p} \sum_{k=0}^{(p-1)/2} \binom{\frac{p-1}{2}}{k} (-1)^k t^{3k} \\ &= \sum_{k=0}^{(p-1)/2} (-1)^k \binom{\frac{p-1}{2}}{k} \sum_{t \in \mathbb{F}_p} t^{3k} \\ &= \sum_{k=1}^{(p-1)/2} (-1)^k \binom{\frac{p-1}{2}}{k} \sum_{t \in \mathbb{F}_p} t^{3k} \pmod{p}\end{aligned}$$

Let  $S_k = \sum_{t \in \mathbb{F}_p} t^{3k}$  ( $0 < k \leq \frac{p-1}{2}$ ), and  $g$  a primitive root in  $\mathbb{F}_p$  :  $t = g^i, 0 \leq i \leq p-2$ .

$$S_k = \sum_{i=0}^{p-2} (g^i)^{3k} = \sum_{i=0}^{p-2} (g^{3k})^i$$

If  $g^{3k} \neq 1$ ,  $S_k = \frac{g^{3k(p-1)} - 1}{g^{3k} - 1} = 0$ , if not  $S_k = p - 1 = -1$ .

$$g^{3k} = 1 \iff p-1 \mid 3k \iff \frac{p-1}{3} \mid k \iff k = \frac{p-1}{3}$$

and  $\frac{p-1}{3}$  even, so

$$2a - b \equiv -\binom{\frac{p-1}{2}}{\frac{p-1}{3}} \pmod{p} \quad (\text{with } J(\chi, \rho) = a + b\omega)$$

□

**Ex. 8.28** Let  $p \equiv 3 \pmod{4}$  and  $\chi$  the quadratic character defined on  $\mathbb{Z}/p\mathbb{Z}$ . Show

$$(a) \sum_{x=1}^{p-1} x\chi(x) = 2 \sum_{x=1}^{(p-1)/2} x\chi(x) - p \sum_{x=1}^{(p-1)/2} \chi(x).$$

$$(b) \sum_{x=1}^{p-1} x\chi(x) = 4\chi(2) \sum_{x=1}^{(p-1)/2} x\chi(x) - p\chi(2) \sum_{x=1}^{(p-1)/2} \chi(x).$$



(c) If  $p \equiv 3 \pmod{8}$  then  $\sum_{x=1}^{p-1} x\chi(x)/p = -\frac{1}{3} \sum_{x=1}^{(p-1)/2} \chi(x)$ .

(d) If  $p \equiv 7 \pmod{8}$  then  $\sum_{x=1}^{p-1} x\chi(x)/p = -\sum_{x=1}^{(p-1)/2} \chi(x)$ .

*Note : I added two minus signs in (c) and (d) to write a true sentence. See the verification below.*

*Proof.* (a)  $\chi(-1) = (-1)^{(p-1)/2} = -1$ , thus

$$\begin{aligned}
\sum_{x=1}^{p-1} x\chi(x) &= \sum_{x=1}^{(p-1)/2} x\chi(x) + \sum_{x=(p-1)/2+1}^{p-1} x\chi(x) \\
&= \sum_{x=1}^{(p-1)/2} x\chi(x) + \sum_{y=1}^{(p-1)/2} (p-y)\chi(p-y) \quad (x = p-y) \\
&= \sum_{x=1}^{(p-1)/2} x\chi(x) - \left[ p \sum_{x=1}^{(p-1)/2} \chi(x) - \sum_{x=1}^{(p-1)/2} x\chi(x) \right] \\
&= 2 \sum_{x=1}^{(p-1)/2} x\chi(x) - p \sum_{x=1}^{(p-1)/2} \chi(x)
\end{aligned}$$

(b) If we separate even and odd indices, we obtain, as  $p$  is odd :

$$\begin{aligned}
\sum_{x=1}^{p-1} x\chi(x) &= \sum_{k=1}^{(p-1)/2} 2k\chi(2k) + \sum_{k=0}^{(p-1)/2-1} (2k+1)\chi(2k+1) \\
&= \sum_{x=1}^{(p-1)/2} 2x\chi(2x) + \sum_{x=1}^{(p-1)/2} (p-2x)\chi(p-2x) \quad (2k+1 = p-2x) \\
&= 2\chi(2) \sum_{x=1}^{(p-1)/2} x\chi(x) - \chi(2) \sum_{x=1}^{(p-1)/2} (p-2x)\chi(x) \\
&= 4\chi(2) \sum_{x=1}^{(p-1)/2} x\chi(x) - p\chi(2) \sum_{x=1}^{(p-1)/2} \chi(x)
\end{aligned}$$

(c) Let  $S = \sum_{x=1}^{(p-1)/2} \chi(x)$ ,  $T = \sum_{x=1}^{(p-1)/2} x\chi(x)$ . Then

$$(a) \sum_{x=1}^{p-1} x\chi(x) = 2T - pS$$

$$(b) \sum_{x=1}^{p-1} x\chi(x) = 4\chi(2)T - p\chi(2)S$$

Subtracting theses equalities , we obtain

$$\begin{aligned}
(4\chi(2) - 2)T &= p(\chi(2) - 1)S \\
\frac{T}{p} &= \frac{\chi(2) - 1}{4\chi(2) - 2} S
\end{aligned}$$

$\chi(2) = (-1)^{(p^2-1)/8} = -1$  if  $p \equiv 3 \pmod{8}$ , so  $\frac{\chi(2)-1}{4\chi(2)-2} = \frac{1}{3}$ , and  $T/p = (1/3)S$ .

$$\sum_{x=1}^{p-1} x\chi(x)/p = 2T/p - S = -(1/3)S,$$

$$\sum_{x=1}^{p-1} x\chi(x)/p = -\frac{1}{3} \sum_{x=1}^{(p-1)/2} \chi(x).$$

(d)  $\chi(2) = (-1)^{(p^2-1)/8} = 1$  if  $p \equiv 7 \pmod{8}$  :  $\frac{\chi(2)-1}{4\chi(2)-2} = 0$ .

$$\sum_{x=1}^{p-1} x\chi(x)/p = -\sum_{x=1}^{(p-1)/2} \chi(x).$$

Verification : with  $p = 7$ , the squares are 1, 4, 2 = 9, so

$$\begin{aligned} \sum_{x=1}^{p-1} x\chi(x)/p &= \frac{1}{7} \left[ \left(\frac{1}{7}\right) + 2\left(\frac{2}{7}\right) + 3\left(\frac{3}{7}\right) + 4\left(\frac{4}{7}\right) + 5\left(\frac{5}{7}\right) + 6\left(\frac{6}{7}\right) \right] \\ &= \frac{1}{7} (1 + 2 - 3 + 4 - 5 - 6) = -1 \\ \sum_{x=1}^{(p-1)/2} \chi(x) &= \left(\frac{1}{7}\right) + \left(\frac{2}{7}\right) + \left(\frac{3}{7}\right) \\ &= 1 + 1 - 1 = 1. \end{aligned}$$

With  $p = 3$ ,

$$\begin{aligned} \sum_{x=1}^{p-1} x\chi(x)/p &= \frac{1}{3} \left[ \left(\frac{1}{3}\right) + 2\left(\frac{2}{3}\right) \right] \\ &= \frac{1}{3} (1 - 2) = -\frac{1}{3} \\ \sum_{x=1}^{(p-1)/2} \chi(x) &= \left(\frac{1}{3}\right) \\ &= 1 \end{aligned}$$

This confirms the misprints in the initial sentence. □

## Chapter 9

**Ex. 9.1** If  $\alpha \in \mathbb{Z}[\omega]$ , show that  $\alpha$  is congruent to either 0, 1, or  $-1$  modulo  $1 - \omega$ .

*Proof.* Let  $\lambda = 1 - \omega$ , and  $z = a + b\omega \in D = \mathbb{Z}[\omega]$ ,  $a, b \in \mathbb{Z}$ .

$\omega \equiv 1 \pmod{\lambda}$ , so  $z \equiv a + b \pmod{\lambda}$ , with  $c = a + b \in \mathbb{Z}$ .

$c \equiv 0, 1, -1 \pmod{3}$ , and since  $\lambda \mid 3$ ,  $\lambda \equiv 0, 1, -1 \pmod{\lambda}$ .

Conclusion : every  $z \in D$  is congruent to either 0, 1, or  $-1$  modulo  $\lambda = 1 - \omega$ .

Note :  $1 \not\equiv -1 \pmod{\lambda}$ , if not  $\lambda \mid 2$ , so  $2 = \lambda\lambda'$ ,  $N(2) = N(\lambda)N(\lambda')$ , thus  $4 = 3N(\lambda')$ , so  $3 \mid 4$  : this is absurd.

$\pm 1 \equiv 0 \pmod{\lambda}$  implies  $\lambda \mid 1$ , so  $\lambda$  would be an unit, in contradiction with  $\lambda$  prime.

So there exist exactly three classes modulo  $\lambda$  in  $D$  :  $|D/\lambda D| = 3 = N(\lambda)$ . □

**Ex. 9.2** From now on we shall set  $D = \mathbb{Z}[\omega]$  and  $\lambda = 1 - \omega$ . For  $\mu$  in  $D$  show that we can write  $\mu = (-1)^a \omega^b \lambda^c \pi_1^{a_1} \pi_2^{a_2} \cdots \pi_t^{a_t}$ , where  $a, b, c$ , and the  $a_i$  are nonnegative integers and the  $\pi_i$  are primary primes.

*Proof.* Let  $S$  the set containing  $\lambda = 1 - \omega$  and all primary primes. By Proposition 9.3.5,

- (a) Every prime in  $D$  is associate to a prime in  $S$ .
- (b) No two primes in  $S$  are associate.

By Theorem 3, Chapter 1, as  $D = \mathbb{Z}[\omega]$  is a principal ideal domain, every  $\mu \in D$  is of the form

$$\mu = u \prod_{\lambda \in S} \lambda^{e(\lambda)},$$

where  $u$  is a unit, so  $u = (-1)^a \omega^b$ . Thus

$$\mu = (-1)^a \omega^b \lambda^c \pi_1^{a_1} \pi_2^{a_2} \cdots \pi_t^{a_t},$$

where the  $\pi$  are primary primes, and  $a, b, c$  and the  $a_i$  are nonnegative integers.  $\square$

**Ex. 9.3** Let  $\gamma$  a primary prime. To evaluate  $\chi_\gamma(\mu)$  we see, by Exercise 2, that it is enough to evaluate  $\chi_\gamma(-1)$ ,  $\chi_\gamma(\omega)$ ,  $\chi_\gamma(\lambda)$ , and  $\chi_\gamma(\pi)$ , where  $\pi$  is a primary prime. Since  $-1 = (-1)^3$  we have  $\chi_\gamma(-1) = 1$ . We now consider  $\chi_\gamma(\omega)$ . Let  $\gamma = a + b\omega$  and set  $a = 3m - 1$  and  $b = 3n$ . Show that  $\chi_\gamma(\omega) = \omega^{m+n}$ .

*Proof.* Let  $\gamma = a + b\omega = 3m - 1 + 3n\omega$ . Then  $\chi_\gamma(\omega) = \omega^{\frac{N(\gamma)-1}{3}}$ .

$$\begin{aligned} N(\gamma) - 1 &= (3m - 1)^2 + (3n)^2 - 3n(3m - 1) - 1 \\ &= 9m^2 - 6m + 9n^2 - 9nm + 3n \\ \frac{N(\gamma) - 1}{3} &= 3m^2 - 2m + 3n^2 - 3nm + n \equiv n + m \pmod{3} \end{aligned}$$

Thus, for  $\gamma = a + b\omega = 3m - 1 + 3n\omega$ ,

$$\chi_\gamma(\omega) = \omega^{\frac{N(\gamma)-1}{3}} = \omega^{n+m}$$

$\square$

**Ex. 9.4** (continuation) Show that  $\chi_\gamma(\omega) = 1, \omega$ , or  $\omega^2$  according to whether  $\gamma$  is congruent to 8, 2, or 5 modulo  $3\lambda$ . In particular, if  $q$  is a rational prime,  $q \equiv 2 \pmod{3}$ , then  $\chi_q(\omega) = 1, \omega$ , or  $\omega^2$  according to whether  $q \equiv 8, 2$ , or  $5 \pmod{9}$ . [Hint :  $\gamma = a + b\omega = -1 + 3(m + n\omega)$ , and so  $\gamma \equiv -1 + 3(m + n) \pmod{3\lambda}$ .]

*Proof.*  $\lambda = 1 - \omega$ , so  $\omega \equiv 1 \pmod{\lambda}$ . Thus

$$\begin{aligned} m + n\omega &\equiv m + n \pmod{\lambda} \\ 3(m + n\omega) &\equiv 3(m + n) \pmod{3\lambda} \\ \gamma = -1 + 3(m + n\omega) &\equiv -1 + 3(m + n) \pmod{3\lambda} \end{aligned}$$

Moreover  $9 = 3\lambda\bar{\lambda} \equiv 0 \pmod{3\lambda}$ , thus  $\gamma$  is congruent modulo  $3\lambda$  to an integer between 0 and 8 of the form  $3k - 1 : \gamma \equiv 8, 2$  or  $5 \pmod{3\lambda}$ .

By Ex. 9.3,  $\chi_\gamma(\omega) = 1 \iff m + n \equiv 0 [3]$ , and  $m + n \equiv 0 [3]$  implies  $m + n = 3k, k \in \mathbb{Z}$ , so  $\gamma \equiv -1 + 9k \equiv -1 \equiv 8 [3\lambda]$ .

Reciprocally, if  $\gamma \equiv 8 \equiv -1 [3\lambda]$ , then  $3\lambda \mid 3(m+n)$ , so  $\lambda \mid m+n$ , and  $N(\lambda) \mid N(m+n)$ ,  $3 \mid (m+n)^2$ , thus  $3 \mid m+n$ ,  $m+n \equiv 0 [3]$ , and so  $\chi_\gamma(\omega) = 1$ . As the two other cases are similar, we obtain

$$\begin{aligned}\chi_\gamma(\omega) = 1 &\iff m + n \equiv 0 [3] \iff \gamma \equiv 8 [3\lambda] \\ \chi_\gamma(\omega) = \omega &\iff m + n \equiv 1 [3] \iff \gamma \equiv 2 [3\lambda] \\ \chi_\gamma(\omega) = \omega^2 &\iff m + n \equiv 2 [3] \iff \gamma \equiv 5 [3\lambda]\end{aligned}$$

If  $\gamma = q$  is a rational prime,  $q \equiv 8 [9]$  implies  $q \equiv 8 [3\lambda]$ , since  $3\lambda \mid 9 = 3\lambda\bar{\lambda}$ , thus  $\chi_q(\omega) = 1$ .

Reciprocally, if  $\chi_q(\omega) = 1$ , then  $q \equiv 8 [3\lambda]$ ,  $q - 8 = \mu(3\lambda), \mu \in D$ , therefore  $(q-8)^2 = N(\mu)3^3, 3^3 \mid (q-8)^2$ , thus  $3^2 \mid q-8$  and so  $q \equiv 8 [9]$ . The two other cases are similar.

$$\begin{aligned}\chi_q(\omega) = 1 &\iff q \equiv 8 [9] \\ \chi_q(\omega) = \omega &\iff q \equiv 2 [9] \\ \chi_q(\omega) = \omega^2 &\iff q \equiv 5 [9]\end{aligned}$$

□

**Ex. 9.5** In the text we stated Eisenstein's result  $\chi_\gamma(\lambda) = \omega^{2m}$ . Show that  $\chi_\gamma(3) = \omega^{2n}$ .

*Proof.*  $(1 - \omega)^2 = -3\omega$ , thus  $\chi_\gamma((1 - \omega)^2) = \chi_\gamma(-1)\chi_\gamma(3)\chi_\gamma(\omega)$ .

$$\chi_\gamma((1 - \omega)^2) = \chi_\gamma(\lambda^2) = \omega^{4m} = \omega^m$$

As  $-1 = (-1)^3, \chi_\gamma(-1) = 1$ . Finally  $\chi_\gamma(\omega) = \omega^{m+n}$  by Exercise 9.3. Thus

$$\omega^m = \chi_\gamma(3)\omega^{m+n}, \quad \chi_\gamma(3) = \omega^{-n} = \omega^{2n}.$$

Conclusion :

$$\chi_\gamma(3) = \omega^{2n}$$

□

**Ex. 9.6** Prove that

(a)  $\chi_\gamma(\lambda) = 1$  for  $\gamma \equiv 8, 8 + 3\omega, 8 + 6\omega [9]$ .

(b)  $\chi_\gamma(\lambda) = \omega$  for  $\gamma \equiv 5, 5 + 3\omega, 5 + 6\omega [9]$ .

(c)  $\chi_\gamma(\lambda) = \omega^2$  for  $\gamma \equiv 2, 2 + 3\omega, 2 + 6\omega [9]$ .

*Proof.*  $\gamma = -1 + 3(m + n\omega)$ , et  $\chi_\gamma(\lambda) = \omega^{2m}$ .

$$\begin{aligned}\chi_\gamma(\lambda) = 1 &\iff m \equiv 0 [3] \Rightarrow \gamma \equiv 8 + 3n\omega [9] \Rightarrow \gamma \equiv 8, 8 + 3\omega, 8 + 6\omega [9] \\ \chi_\gamma(\lambda) = \omega &\iff m \equiv 2 [3] \Rightarrow \gamma \equiv 5 + 3n\omega [9] \Rightarrow \gamma \equiv 5, 5 + 3\omega, 5 + 6\omega [9] \\ \chi_\gamma(\lambda) = \omega^2 &\iff m \equiv 1 [3] \Rightarrow \gamma \equiv 2 + 3n\omega [9] \Rightarrow \gamma \equiv 2, 2 + 3\omega, 2 + 6\omega [9]\end{aligned}$$

As  $\chi_\gamma(\lambda) \in \{1, \omega, \omega^2\}$ , these 9 cases are the only possibilities. Moreover these 9 cases are mutually exclusive, since 9 doesn't divide any difference. Thus the reciprocals are true.

$$\begin{aligned}\chi_\gamma(\lambda) = 1 &\iff \gamma \equiv 8, 8 + 3\omega, 8 + 6\omega \text{ [9]} \\ \chi_\gamma(\lambda) = \omega &\iff \gamma \equiv 5, 5 + 3\omega, 5 + 6\omega \text{ [9]} \\ \chi_\gamma(\lambda) = \omega^2 &\iff \gamma \equiv 2, 2 + 3\omega, 2 + 6\omega \text{ [9]}\end{aligned}$$

□

**Ex. 9.7** Find primary primes associate to  $1 - 2\omega$ ,  $-7 - 3\omega$ , and  $3 - \omega$ .

*Proof.* :

- $(1 - 2\omega)\omega = 2 + 3\omega \equiv 2 \pmod{3}$ , so  $2 + 3\omega$  is primary, and associate to  $1 - 2\omega$ .  
 $N(2 + 3\omega) = 7$  and 7 is a rational prime, thus  $2 + 3\omega$  is a primary prime.
- $-7 - 3\omega \equiv 2 \pmod{3}$ .  
 $N(-7 - 3\omega) = 37$  and 37 is a rational prime, thus  $-7 - 3\omega$  is a primary prime.
- $(3 - \omega)\omega^2 = -4 - 3\omega \equiv 2 \pmod{3}$ , so  $-4 - 3\omega$  is primary, and associate to  $3 - \omega$ .  
 $N(-4 - 3\omega) = 13$  and 13 is a rational prime, thus  $-4 - 3\omega$  is a primary prime.

□

**Ex. 9.8** Factor the following numbers into primes in  $D$  : 7, 21, 45, 22, and 143.

*Proof.*  $7 = N(2 + 3\omega)$ , thus  $7 = (2 + 3\omega)(2 + 3\omega^2) = (2 + 3\omega)(-1 - 3\omega)$ .

$$21 = 3 \times 7 = -\omega^2 \lambda^2 (2 + 3\omega)(-1 - 3\omega) \text{ since } 3 = -\omega^2(1 - \omega)^2.$$

$$45 = 3^2 \times 5 = \omega \lambda^4 5$$

$$22 = 2 \times 11 \text{ (2 and 11 are primes in } D\text{)}$$

$$143 = 11 \times 13 = 11(-4 - 3\omega)(-4 - 3\omega^2) = 11(-4 - 3\omega)(-1 + 3\omega)$$

□

**Ex. 9.9** Show that  $\bar{\alpha} \neq 0$ , the residue class of  $\alpha$ , is a cube in the field  $D/\pi D$  iff  $\alpha^{(N\pi-1)/3} \equiv 1 \pmod{\pi}$ . Conclude that there are  $(N\pi - 1)/3$  cubes in  $(D/\pi D)^*$ .

*Solution 1 :*

*Proof.* Let  $\pi$  a prime in  $D$ ,  $N\pi \neq 3$ , and  $\alpha \in D, \pi \nmid \alpha$ .

$\bar{\alpha}$  is a cube in  $(D/\pi D)^*$

$$\iff x^3 \equiv \alpha \pmod{\pi} \text{ has a solution}$$

$$\iff \chi_\pi(\alpha) = 1 \text{ (by Prop. 9.3.3(a))}$$

$$\iff \alpha^{\frac{N\pi-1}{3}} \equiv 1 \pmod{\pi}$$

$$\iff \bar{\alpha}^{\frac{N\pi-1}{3}} = \bar{1}.$$

The cubes in  $(D/\pi D)^*$  are then the roots of the polynomial  $f(x) = x^{\frac{N\pi-1}{3}} - \bar{1}$  in  $D/\pi D$ .

As  $d = |D/\pi D| = N\pi$ ,  $(N\pi - 1)/3 \mid q - 1$ ,  $f(x) \mid x^{q-1} - 1 \mid x^q - x$ . By Corollary 2 of Proposition 8.1.1,  $f$  has  $\deg(f) = \frac{N\pi-1}{3}$  roots.

Conclusion : there exist exactly  $\frac{N\pi-1}{3}$  cubes in  $(D/\pi D)^*$ .

□

*Solution 2 :*

*Proof.* Let  $\varphi : (D/\pi D)^* \rightarrow (D/\pi D)^*$  the group homomorphism defined by  $\varphi(x) = x^3$ .

Then  $\text{im}(\varphi)$  is the set of cubes in  $(D/\pi D)^*$ .

The equation  $x^3 = \bar{1}$  has three distinct solutions  $\bar{1}, \bar{\omega}, \bar{\omega}^2$  in  $D/\pi D$  if  $N\pi \neq 3$  (see the demonstration of Proposition 9.3.1).

So  $\ker(\varphi) = \{\bar{1}, \bar{\omega}, \bar{\omega}^2\}$  and  $|\ker(\varphi)| = 3$ . Thus  $|\text{im}\varphi| = |(D/\pi D)^*| / |\ker(\varphi)| = (N\pi - 1)/3$ . There exist exactly  $\frac{N\pi-1}{3}$  cubes in  $(D/\pi D)^*$ .  $\square$

*Note :* if  $N\pi = 3$ , that is to say if  $\pi$  is associate to  $1 - \omega$ ,  $D/\pi D = \{\bar{0}, \bar{1}, \bar{2}\}$ . As  $\bar{1}^3 = \bar{1}, \bar{2}^3 = \bar{2}$ , all the elements of  $(D/\pi D)^*$  are cubes.

**Ex. 9.10** What is the factorisation of  $x^{24} - 1$  in  $D/5D$ .

*Proof.*  $|(D/5D)^*| = N(5) - 1 = 24$ , thus  $x^{24} - 1 = \prod_{\alpha \in (D/5D)^*} (x - \alpha)$ .

(  $\alpha = a + b\bar{\omega}$ ,  $0 \leq a < 5, 0 \leq b < 5$  ).  $\square$

**Ex. 9.11** How many cubes are there in  $D/5D$  ?

*Proof.* By Exercise 9.9, there exist  $(N(5) - 1)/3 = 8$  cubes in  $D/5D$ .  $\square$

**Ex. 9.12** Show that  $\omega\lambda$  has order 8 in  $D/5D$  and that  $\omega^2\lambda$  has order 24. [Hint : Show first that  $(\omega\lambda)^2$  has order 4.]

*Proof.*  $\alpha = (\omega\lambda)^2 = \omega^2(1 - \omega)^2 = \omega^2(1 + \omega^2 - 2\omega) = 3\omega^3 = -3$ .

$\alpha^2 = 9 \equiv -1 \pmod{5}$ ,  $\alpha^4 \equiv 1 \pmod{5}$ , thus  $\alpha = (\omega\lambda)^2$  is of order 4 in  $D/5D$ , and  $\omega\lambda$  of order 8.

Let  $\beta = \omega^2\lambda$ .  $|(D/5D)^*| = 24$ , thus  $\bar{\beta}^{24} = 1$ .

To verify that  $\bar{\beta}$  has order 24, it is sufficient to verify  $\bar{\beta}^8 \neq 1, \bar{\beta}^{12} \neq 1$  :

$\beta^8 = \omega^{16}\lambda^8 = \omega\lambda^8 = (\omega\lambda)^8\omega^2 \equiv \omega^2 \not\equiv 1 \pmod{5}$ .

$\beta^{12} = (\omega^2\lambda)^{12} = \lambda^{12} = (\omega\lambda)^{12} \equiv (\omega\lambda)^4 \equiv -1 \pmod{5}$  (since  $(\omega\lambda)$  has order 8 in  $D/5D$ ).

Conclusion :  $\omega\lambda$  has order 8,  $\omega\lambda^2$  has order 24.  $\square$