

## Chapter 11

**Ex. 11.1** Suppose that we may write the power series  $1 + a_1u + a_2u^2 + \cdots$  as the quotient of two polynomials  $P(u)/Q(u)$ . Show that we may assume that  $P(0) = Q(0) = 1$ .

*Proof.* Here  $f(u) = 1 + a_1u + a_2u^2 + \cdots \in \mathbb{C}[[u]]$  is a formal series in the variable  $u$ .

We suppose that  $f(u) = P(u)/Q(u)$ , where we may assume, after simplification, that the two polynomials are relatively prime. Then  $P(1)/Q(1) = 1$ . Write  $c = P(1) = Q(1) \in F$ .

If  $c = 0$ , then  $u \mid P(u)$  and  $u \mid Q(u)$ . This is impossible since  $P \wedge Q = 1$ . So  $c \neq 0$ .

Define  $P_1(u) = (1/c)P(u)$ ,  $Q_1(u) = (1/c)Q(u)$ . Then  $f(u) = P_1(u)/Q_1(u)$  and  $P_1(0) = Q_1(0) = 1$ . If we replace  $P, Q$  by  $P_1, Q_1$ , then the pair  $(P_1, Q_1)$  has the required properties.  $\square$

**Ex. 11.2** Prove the converse to Proposition 11.1.1.

*Proof.* If  $N_s = \sum_{j=1}^e \beta_j^s - \sum_{i=1}^d \alpha_i^s$ , where  $\alpha_i, \beta_j$  are complex numbers, then

$$\begin{aligned} \sum_{s=1}^{\infty} \frac{N_s u^s}{s} &= \sum_{j=1}^e \left( \sum_{s=1}^{\infty} \frac{(\beta_j u)^s}{s} \right) - \sum_{i=1}^d \left( \sum_{s=1}^{\infty} \frac{(\alpha_i u)^s}{s} \right) \\ &= - \sum_{j=1}^e \ln(1 - \beta_j u) + \sum_{i=1}^d \ln(1 - \alpha_i u). \end{aligned}$$

Here  $u$  is a variable, and both members are formal polynomials in  $\mathbb{C}[[u]]$ , so we don't study convergence. Nevertheless, the left member has a radius of convergence at least  $q^{-n}$ , and the right member  $\min_{i,j} (1/|\beta_j|, 1/|\alpha_i|)$ .

Therefore,

$$Z_f(u) = \exp \left( \sum_{s=1}^{\infty} \frac{N_s u^s}{s} \right) = \prod_{j=1}^e (1 - \beta_j u)^{-1} \prod_{i=1}^d (1 - \alpha_i u) = \frac{\prod_{i=1}^d (1 - \alpha_i u)}{\prod_{j=1}^e (1 - \beta_j u)}$$

is a rational fraction.  $\square$

**Ex. 11.3** Give the details of the proof that  $N_s$  is independent of the field  $F_s$  (see the concluding paragraph to section 1).

*Proof.* Suppose that  $E$  and  $E'$  are two fields containing  $F$  both with  $q^s$  elements. We first show that there is a isomorphism  $\sigma : E \rightarrow E'$  which fixes the elements of  $F$ , by showing that both  $E$  and  $E'$  are isomorphic over  $F$  to  $F[x]/(f(x))$  for some irreducible polynomial  $f(x) \in F(x)$ .

There is a primitive element  $\alpha' \in E'$ , i.e. such that  $E' = F(\alpha')$ . For example, take  $\alpha'$  to be a primitive  $q^s - 1$  root of unity : since  $\alpha$  is a generator of  $E'^*$ , every element  $\gamma \in E'^*$  is equal to  $\alpha'^k$  for some integer  $k$ , thus  $\gamma \in F(\alpha')$  (and  $0 \in F(\alpha')$ ). This proves  $E' \subset F(\alpha')$ , and since  $\alpha' \in E'$  and  $F \subset E'$ ,  $F(\alpha') \subset E'$ , so  $E' = F(\alpha')$ .

Let  $f(x) \in F[x]$  be the minimal polynomial of  $\alpha'$  over  $F$ . Then

$$E' = F(\alpha') \simeq F(x)/(f(x)),$$

where the isomorphism  $\sigma_1 : F(\alpha') \rightarrow F(x)/(f(x))$  maps  $\alpha'$  to  $\bar{x} = x + (f(x))$ , and maps  $a \in F$  on  $\bar{a} = a + (f(x))$ . Since  $\alpha'$  is a root of  $x^{q^s} - x$ ,  $f(x) \mid x^{q^s} - x$ .

$E$  is a field with  $q^s$  elements, so we have  $x^{q^s} - x = \prod_{\alpha \in E} (x - \alpha)$ . Thus  $f(x) \mid \prod_{\alpha \in E} (x - \alpha)$ , where  $\deg(f(x)) = s \geq 1$ , so  $f(\alpha) = 0$  for some  $\alpha \in E$ . The polynomial  $f$  being irreducible over  $F$ ,  $f$  is the minimal polynomial of  $\alpha$  over  $F$ , thus  $F(\alpha) \simeq F[x]/(f(x))$  is a field with  $q^s$  elements. Since  $F(\alpha) \subset E$ , and  $|F(\alpha)| = |E|$ , we conclude  $E = F(\alpha)$ , therefore

$$E = F(\alpha) \simeq F(x)/(f(x)),$$

where the isomorphism  $\sigma_2 : F(\alpha) \rightarrow F(x)/(f(x))$  maps  $\alpha$  to  $\bar{x} = x + (f(x))$ , and maps  $a \in F$  on  $\bar{a} = a + (f(x))$ .

Then  $\sigma = \sigma_1^{-1} \circ \sigma_2 : E \rightarrow E'$  is an isomorphism, and  $\sigma(a) = a$  for all  $a \in F$ .

We can now use the isomorphism  $\sigma$  to induce a map

$$\bar{\sigma} \begin{cases} P^n(E) & \rightarrow P^n(E') \\ [\alpha_0, \dots, \alpha_n] & \mapsto [\sigma(\alpha_0), \dots, \sigma(\alpha_n)]. \end{cases}$$

Then  $\bar{\sigma}$  is injective: if  $[\sigma(\alpha_0), \dots, \sigma(\alpha_n)] = [\sigma(\beta_0), \dots, \sigma(\beta_n)]$ , then there is  $\lambda \in F^*$  such that  $\beta_i = \lambda \sigma(\alpha_i) = \sigma(\lambda) \sigma(\alpha_i) = \sigma(\lambda \alpha_i)$ ,  $i = 0, \dots, n$ , thus  $\beta_i = \lambda \alpha_i$ , which proves  $[\alpha_0, \dots, \alpha_n] = [\beta_0, \dots, \beta_n]$ .

If  $[\gamma_0, \dots, \gamma_n]$  is any projective point of  $P^n(E')$ , then

$$[\gamma_0, \dots, \gamma_n] = \bar{\sigma}([\sigma^{-1}(\gamma_0), \dots, \sigma^{-1}(\gamma_n)]).$$

This proves that  $\bar{\sigma}$  is surjective. So  $\bar{\sigma}$  is a bijection.

Now take  $f(y_0, \dots, y_n) \in F[y_0, \dots, y_n]$  an homogeneous polynomial,  $\bar{H}_f(E)$  the corresponding projective hypersurface in  $P^n(E)$ , and  $\bar{H}_f(E')$  the corresponding projective hypersurface in  $P^n(E')$ . We show that  $\bar{\sigma}(\bar{H}_f(E)) = \bar{H}_f(E')$ .

Since  $\sigma$  is a  $F$ -isomorphism,  $\sigma(f(\alpha_0, \dots, \alpha_n)) = f(\sigma(\alpha_0), \dots, \sigma(\alpha_n))$  ( $\alpha_i \in E$ ), and similarly  $\sigma^{-1}(f(\beta_0, \dots, \beta_n)) = f(\sigma^{-1}(\beta_0), \dots, \sigma^{-1}(\beta_n))$  ( $\beta_i \in E'$ ), thus

$$\begin{aligned} [\alpha_0, \dots, \alpha_n] \in \bar{H}_f(E) &\Rightarrow f(\alpha_0, \dots, \alpha_n) = 0 \\ &\Rightarrow \sigma(f(\alpha_0, \dots, \alpha_n)) = \sigma(0) = 0 \\ &\Rightarrow f(\sigma(\alpha_0), \dots, \sigma(\alpha_n)) = 0 \\ &\Rightarrow \bar{\sigma}([\alpha_0, \dots, \alpha_n]) = [\sigma(\alpha_0), \dots, \sigma(\alpha_n)] \in \bar{H}_f(E'). \end{aligned}$$

This shows  $\bar{\sigma}(\bar{H}_f(E)) \subset \bar{H}_f(E')$ .

Conversely,

$$\begin{aligned} [\beta_0, \dots, \beta_n] \in \bar{H}_f(E') &\Rightarrow f(\beta_0, \dots, \beta_n) = 0 \\ &\Rightarrow \sigma^{-1}(f(\beta_0, \dots, \beta_n)) = \sigma^{-1}(0) = 0 \\ &\Rightarrow f(\sigma^{-1}(\beta_0), \dots, \sigma^{-1}(\beta_n)) = 0 \\ &\Rightarrow \bar{\sigma}^{-1}([\beta_0, \dots, \beta_n]) = [\sigma^{-1}(\beta_0), \dots, \sigma^{-1}(\beta_n)] \in \bar{H}_f(E). \end{aligned}$$

If we define  $\alpha_i = \sigma^{-1}(\beta_i)$ ,  $i = 0, \dots, n$ , then  $[\alpha_0, \dots, \alpha_n] \in \bar{H}_f(E)$ , and  $[\beta_0, \dots, \beta_n] = \bar{\sigma}([\alpha_0, \dots, \alpha_n]) \in \bar{\sigma}(\bar{H}_f(E))$ . This shows  $\bar{H}_f(E') \subset \bar{\sigma}(\bar{H}_f(E))$ , and so

$$\bar{\sigma}(\bar{H}_f(E)) = \bar{H}_f(E').$$

Since  $\bar{\sigma}$  is a bijection,

$$N_s = |\bar{H}_f(E)| = |\bar{H}_f(E')| = N'_s.$$

So  $N_s$  is independent of the choice of the extension  $F_s = \mathbb{F}_{q^s}$  of  $F = \mathbb{F}_q$ . □

**Ex. 11.4** Calculate the zeta function of  $x_0x_1 - x_2x_3 = 0$  over  $\mathbb{F}_p$ .

*Proof.* Here  $F = \mathbb{F}_p$ , and  $F_s = \mathbb{F}_{p^s}$ .

To calculate  $N_s$ , we calculate the number of points at infinity (such that  $x_0 = 0$ ), and the numbers of affine points of the curve  $\overline{H}_f(\mathbb{F}_{p^s})$  associate to

$$f(x_0, x_1, x_2, x_3) = x_0x_1 - x_2x_3.$$

- To estimate the number of points at infinity, we calculate first the cardinality of the set

$$U = \{(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in F_s^4 \mid \alpha_0\alpha_1 - \alpha_2\alpha_3 = 0, \alpha_0 = 0\}.$$

Then  $\alpha_1$  takes an arbitrary value  $a \in F_s$ . Write

$$U_a = \{(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in U \mid \alpha_1 = a\}.$$

Then  $U_a = \{(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in F_s^4 \mid \alpha_0 = 0, \alpha_1 = a, \alpha_2\alpha_3 = 0\}$ , thus  $U_a = A \cup B$ , where

$$A = \{(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in U_a \mid \alpha_2 = 0\},$$

$$B = \{(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in U_a \mid \alpha_3 = 0\}.$$

Since  $\alpha_0, \alpha_1, \alpha_3$  are fixed in  $A$ , the map  $A \rightarrow F_s$  defined by  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \mapsto \alpha_2$  is a bijection, therefore  $|A| = p^s$ , and similarly  $|B| = p^s$ . But  $A \cap B = \{(0, 0, 0, 0)\}$ , thus

$$|U_a| = |A| + |B| - |A \cap B| = 2p^s - 1.$$

Since  $U$  is the disjoint union of the  $U_a$ , thus

$$|U| = \sum_{a \in F_s} |U_a| = \sum_{a \in F_s} (2p^s - 1) = 2p^{2s} - p^s.$$

Therefore the number of projective points  $[\alpha_0, \alpha_1, \alpha_2, \alpha_3] \in P^3(F_s)$  at infinity (such that  $\alpha_0 = 0$ ) is

$$N_\infty = \frac{|U| - 1}{p^s - 1} = \frac{2p^{2s} - p^s - 1}{p^s - 1} = 2p^s + 1.$$

- Now we calculate the number of points of the affine surface  $H_f(\mathbb{F}_s)$  associate to the equation  $y_1 = y_2y_3$  (where  $y_i = x_i/x_0$ ).

The maps

$$u \left\{ \begin{array}{ccc} F_s^2 & \rightarrow & H_f(F_s) \\ (\beta, \gamma) & \mapsto & (\beta\gamma, \beta, \gamma) \end{array} \right. \quad \left\{ \begin{array}{ccc} H_f(F_s) & \rightarrow & F_s^2 \\ (\alpha, \beta, \gamma) & \mapsto & (\beta, \gamma) \end{array} \right.$$

satisfy  $u \circ v = \text{id}, v \circ u = \text{id}$ , so  $u$  is a bijection. With more informal words, the arbitrary choice of  $\beta, \gamma \in F_s$  gives the affine point  $(\alpha, \beta, \gamma)$ , where  $\alpha = \beta\gamma$ .

This gives  $|H_f(F_s)| = p^{2s}$ .

Therefore

$$N_s = |\overline{H}_f(F_s)| = p^{2s} + 2p^s + 1.$$

We obtain in  $\mathbb{C}[[u]]$

$$\begin{aligned}\sum_{s=1}^{\infty} \frac{N_s u^s}{s} &= \sum_{s=1}^{\infty} \frac{(p^2 u)^s}{s} + 2 \sum_{s=1}^{\infty} \frac{(pu)^s}{s} + \sum_{s=1}^{\infty} \frac{u^s}{s} \\ &= -\ln(1 - p^2 u) - 2 \ln(1 - pu) - \ln(1 - u).\end{aligned}$$

This gives

$$Z_f(u) = (1 - p^2 u)^{-1} (1 - pu)^{-2} (1 - u)^{-1}.$$

Note: The result for  $N_s$  is verified with the naive and very slow following code in Sage:

```
def N(p,s):
    Fs = GF(p^s)
    counter = 0
    for x in Fs:
        for y in Fs:
            for z in Fs:
                for t in Fs:
                    if x*y == z*t:
                        counter += 1
    return (counter - 1)/(p^s - 1)

p, s = 5, 3
print N(p,s), p^(2*s) + 2*p^s + 1
```

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There is a misprint in the “Selected Hints for the Exercises” in Ireland-Rosen p.371.  $\square$

**Ex. 11.5** Calculate as explicitly as possible the zeta function of  $a_0 x_0^2 + a_1 x_1^2 + \cdots + a_n x_n^2$  over  $\mathbb{F}_q$ , where  $q$  is odd. The answer will depend on whether  $n$  is odd or even and whether  $q \equiv 1 \pmod{4}$  or  $q \equiv 3 \pmod{4}$ .

*Proof.* Since  $q$  is odd, there is a unique character  $\chi$  of order 2 over  $F = \mathbb{F}_q$ , and a unique character of order 2 over  $F_s = \mathbb{F}_{q^s}$ . We first compute the number in  $\mathbb{F}_q^{n+1}$  of solutions of the equation  $f(x_0, \dots, x_n) = 0$ , where  $f(x_0, \dots, x_n) = a_0 x_0^2 + \cdots + a_n x_n^2 \in F[x_0, \dots, x_n]$ .

$$\begin{aligned}N(a_0 x_0^2 + \cdots + a_n x_n^2 = 0) &= \sum_{a_0 u_0 + \cdots + a_n u_n = 0} N(x_0^2 = u_0) \cdots N(x_n^2 = u_n) \\ &= \sum_{a_0 u_0 + \cdots + a_n u_n = 0} (1 + \chi(u_0)) \cdots (1 + \chi(u_n)) \\ &= \sum_{v_0 + \cdots + v_n = 0} (1 + \chi(a_0^{-1}) \chi(v_0)) \cdots (1 + \chi(a_n^{-1}) \chi(v_n)) \quad (v_i = a_i u_i) \\ &= q^n + \chi(a_0^{-1}) \cdots \chi(a_n^{-1}) J_0(\chi, \chi, \dots, \chi),\end{aligned}$$

Indeed  $J_0(\varepsilon, \dots, \varepsilon) = q^{l-1}$ , and  $J_0(\chi_0, \dots, \chi_n) = 0$  if some but not all of the  $\chi_i$  are trivial (generalization of Proposition 8.5.1).

We estimate  $J_0(\chi, \dots, \chi)$ , where there are  $n + 1$  entries of  $\chi$ .

- If  $n$  is even, then  $\chi^{n+1} = \chi \neq \varepsilon$ , thus  $J_0(\chi, \dots, \chi) = 0$  (Proposition 8.5.1(d)), and so

$$N(a_0x_0^2 + \dots + a_nx_n^2 = 0) = q^n,$$

and the number of projective points on the hypersurface is given by

$$N_1 = \frac{q^n - 1}{q - 1} = q^{n-1} + \dots + q + 1.$$

- If  $n$  is odd, then  $\chi^{n+1} = \varepsilon$ , thus  $J_0(\chi, \dots, \chi) = \chi(-1)(q-1)J(\chi, \dots, \chi)$ , with  $n$  entries of  $\chi$  (same Proposition).

By Theorem 3 of chapter 8,

$$J(\chi, \dots, \chi) = \frac{g(\chi)^n}{g(\chi)} = g(\chi)^{n-1}.$$

Since  $g(\chi)^2 = g(\chi)g(\chi)^{-1} = \chi(-1)q$  (Exercise 10.22),

$$\begin{aligned} \frac{1}{q-1} J_0(\chi, \dots, \chi) &= \chi(-1)g(\chi)^{n-1} \\ &= \chi(-1)g(\chi)^{n-1} \\ &= \frac{\chi(-1)g(\chi)^{n+1}}{g(\chi)^2} \\ &= \frac{1}{q} g(\chi)^{n+1}. \end{aligned}$$

Therefore

$$N(a_0x_0^2 + \dots + a_nx_n^2 = 0) = q^n + \chi(a_0)^{-1} \dots \chi(a_n)^{-1} \frac{q-1}{q} g(\chi)^{n_1},$$

and

$$N_1 = q^{n-1} + \dots + q + 1 + \frac{1}{q} \chi(a_0)^{-1} \dots \chi(a_n)^{-1} g(\chi)^{n+1}.$$

To conclude this first part,

$$\begin{aligned} N_1 &= q^{n-1} + \dots + q + 1 && \text{if } n \text{ is even,} \\ N_1 &= q^{n-1} + \dots + q + 1 + \frac{1}{q} \chi(a_0)^{-1} \dots \chi(a_n)^{-1} g(\chi)^{n+1} && \text{if } n \text{ is odd.} \end{aligned}$$

To compute  $N_s$ , we must replace  $q$  by  $q^s$  and  $\chi$  by  $\chi_s$ , the character of order 2 on  $F_s$ . Then

$$\begin{aligned} N_s &= q^{s(n-1)} + \dots + q^s + 1 && \text{if } n \text{ is even,} \\ N_s &= q^{s(n-1)} + \dots + q^s + 1 + \frac{1}{q^s} \chi_s(a_0)^{-1} \dots \chi_s(a_n)^{-1} g(\chi_s)^{n+1} && \text{if } n \text{ is odd.} \end{aligned}$$

(These two results can also be obtained by using the equations (1) and (2) in Theorem 2 of Chapter 10.)

It remains to study  $\chi_s$  in the odd case.

Since  $\chi_s^2 = \varepsilon$ , for all  $\alpha \in F_s$ ,  $\chi_s(\alpha)^{-1} = \chi_s(\alpha)$ , and  $\chi_s(\alpha) = -1 \in \mathbb{C}$  if  $\alpha^{\frac{q^s-1}{2}} = -1 \in F_s$ ,  $\chi_s(\alpha) = 1$  otherwise.

If  $a \in F$ ,  $a^{\frac{q-1}{2}} = \pm 1 = \varepsilon$ . Since  $q$  is odd,  $1 + q + \cdots + q^{s-1} \equiv s \pmod{2}$ , thus

$$a^{\frac{q^s-1}{2}} = a^{\frac{q-1}{2}(1+q+\cdots+q^{s-1})} = \varepsilon^{1+q+\cdots+q^{s-1}} = \varepsilon^s,$$

so

$$\chi_s(a) = \chi(a)^s \quad (a \in F).$$

We know that  $g(\chi_s)^2 = \chi_s(-1)q^s$  (Ex. 10.22), thus, as  $n$  is odd,

$$\begin{aligned} g(\chi_s)^{n+1} &= [g(\chi_s)^2]^{\frac{n+1}{2}} \\ &= \chi_s(-1)^{\frac{n+1}{2}} q^{s\frac{n+1}{2}}. \end{aligned}$$

If  $q \equiv 1 \pmod{4}$ , then  $(-1)^{\frac{q-1}{2}} = 1$ , so  $-1$  is a square in  $\mathbb{F}_q$ . In this case,  $-1$  is a square in  $\mathbb{F}_{q^s}$ , and  $\chi_s(-1) = 1$  for all  $s \geq 1$ . In this case, using  $a_i \in F$ ,

$$\begin{aligned} N_s &= q^{s(n-1)} + \cdots + q^s + 1 + \chi_s(a_0) \cdots \chi_s(a_n) q^{s\frac{n-1}{2}} \\ &= q^{s(n-1)} + \cdots + q^s + 1 + [\chi(a_0) \cdots \chi(a_n)]^s q^{s\frac{n-1}{2}} \end{aligned}$$

If  $q \equiv -1 \pmod{4}$ , then  $\chi(-1) = (-1)^{\frac{q-1}{2}} = -1$ , and

$$\chi_s(-1) = \chi(-1)^s = (-1)^s,$$

thus

$$\frac{1}{q^s} g(\chi_s)^{n+1} = (-1)^{s\frac{n+1}{2}} q^{s\frac{n-1}{2}}.$$

This gives for odd integers  $n$ , and  $q \equiv -1 \pmod{4}$ ,

$$\begin{aligned} N_s &= q^{s(n-1)} + \cdots + q^s + 1 + (-1)^{s\frac{n+1}{2}} \chi_s(a_0) \cdots \chi_s(a_n) q^{s\frac{n-1}{2}} \\ &= q^{s(n-1)} + \cdots + q^s + 1 + [(-1)^{\frac{n+1}{2}} \chi(a_0) \cdots \chi(a_n)]^s q^{s\frac{n-1}{2}}. \end{aligned}$$

To collect all these cases, we have proved

$$\begin{aligned} N_s &= q^{s(n-1)} + \cdots + q^s + 1 && \text{if } n \equiv 0 \pmod{2}, \\ N_s &= q^{s(n-1)} + \cdots + q^s + 1 + [\chi(a_0) \cdots \chi(a_n)]^s q^{s\frac{n-1}{2}} && \text{if } n \equiv 1 \pmod{2}, q \equiv +1 \pmod{4}, \\ N_s &= q^{s(n-1)} + \cdots + q^s + 1 + [(-1)^{\frac{n+1}{2}} \chi(a_0) \cdots \chi(a_n)]^s q^{s\frac{n-1}{2}} && \text{if } n \equiv 1 \pmod{2}, q \equiv -1 \pmod{4}. \end{aligned} \quad (4)$$

If  $n$  is even this gives, as in paragraph 1,

$$Z_f(u) = (1 - q^{n-1}u)^{-1} \cdots (1 - qu)^{-1} (1 - u)^{-1}.$$

In the case  $n \equiv 1 \pmod{2}, q \equiv +1 \pmod{4}$ , we write for simplicity  $\varepsilon = \chi(a_0) \cdots \chi(a_n) = \pm 1$ . Then

$$\begin{aligned} \sum_{s=1}^{\infty} \frac{N_s u^s}{s} &= \sum_{m=0}^{n-1} \left( \sum_{s=1}^{\infty} \frac{(q^m u)^s}{s} \right) + \sum_{s=1}^{\infty} \frac{(\varepsilon q^{\frac{n-1}{2}} u)^s}{s} \\ &= - \sum_{m=0}^{n-1} \ln(1 - q^m u) - \ln(1 - \varepsilon q^{\frac{n-1}{2}} u). \end{aligned}$$

Therefore

$$Z_f(u) = \left[ \prod_{m=0}^{n-1} (1 - q^m u)^{-1} \right] (1 - \chi(a_0) \cdots \chi(a_n) q^{\frac{n-1}{2}} u)^{-1}.$$

(Same calculation in the last case, with  $\varepsilon = (-1)^{\frac{n+1}{2}} \chi(a_0) \cdots \chi(a_n)$ .)

We obtain

$$\begin{aligned} Z_f(u) &= P(u) && \text{if } n \equiv 0 \pmod{2}, \\ Z_f(u) &= P(u)(1 - \chi(a_0) \cdots \chi(a_n) q^{\frac{n-1}{2}} u)^{-1} && \text{if } n \equiv 1 \pmod{2}, q \equiv +1 \pmod{4}, \\ Z_f(u) &= P(u)(1 - (-1)^{\frac{n+1}{2}} \chi(a_0) \cdots \chi(a_n) q^{\frac{n-1}{2}} u)^{-1} && \text{if } n \equiv 1 \pmod{2}, q \equiv -1 \pmod{4}, \end{aligned}$$

where  $P(u) = (1 - q^{n-1}u)^{-1} \cdots (1 - qu)^{-1}(1 - u)^{-1}$ .

(These results are consistent with the example  $N_s = q^{2s} + q^s + 1 + \chi_s(-1)q^s$  given in paragraph 1 for the surface defined by  $-y_0^2 + y_1^2 + y_2^2 + y_3^2 = 0$ , where  $n = 3$  is odd.

$$\begin{aligned} Z_f(u) &= (1 - q^2u)^{-1}(1 - qu)^{-1}(1 - u)^{-1}(1 - \chi(-1)qu)^{-1} \\ &= \begin{cases} (1 - q^2u)^{-1}(1 - qu)^{-2}(1 - u)^{-1} & \text{if } q \equiv 1 \pmod{4}, \\ (1 - q^2u)^{-1}(1 - qu)^{-1}(1 - u)^{-1}(1 + qu)^{-1} & \text{if } q \equiv -1 \pmod{4}. \end{cases} \end{aligned}$$

□

**Ex. 11.6** Consider  $x_0^3 + x_1^3 + x_2^3 = 0$  as an equation over  $F_4$ , the field with four elements. Show that there are nine points on the curve in  $P^2(F_4)$ . Calculate the zeta function. [Answer:  $(1 + 2u)^2 / ((1 - u)(1 - 4u))$ .]

*Proof.* Since  $q = 4 \equiv 1 \pmod{3}$ , we can apply Theorem 2 of Chapter 10. Let  $\chi$  be a character of order 3 over  $F = \mathbb{F}_4$ . The only other character of order 3 is then  $\chi^2$ . Thus

$$N_1 = q + 1 + \frac{1}{q-1} \sum_{i,j,k} J_0(\chi^i, \chi^j, \chi^k),$$

where the sum is over all  $(i, j, k) \in \{1, 2\}^3$  such that  $i + j + k \equiv 0 \pmod{3}$ , that is  $(1, 1, 1)$  and  $(2, 2, 2)$ . Thus

$$N_1 = q + 1 + \frac{1}{q-1} (J_0(\chi, \chi, \chi) + J_0(\chi^2, \chi^2, \chi^2)).$$

Using  $\frac{1}{q-1} J_0(\chi^k, \chi^k, \chi^k) = \frac{1}{q} g(\chi^k)^3$  for  $k = 1, 2$ , we obtain

$$N_1 = q + 1 + \frac{1}{q} (g(\chi)^3 + g(\chi^2)^3).$$

Consider  $\mathbb{F}_4 = \mathbb{F}_2[x]/(x^2 + x + 1)$ , where  $a = \bar{x} = x + (x^2 + x + 1)$  is a generator of  $\mathbb{F}_4^*$ . Then  $\mathbb{F}_4 = \{0, 1, a, a^2 = a + 1\}$ . We compute  $g(\chi)$  for the character  $\chi$  of order 3 defined by

$$\frac{t}{\chi(t)} \mid \begin{array}{cccc} 0 & 1 & a & a^2 \\ 0 & 1 & \omega & \omega^2 \end{array}$$

where  $\omega = e^{\frac{2i\pi}{3}}$ .

for each  $t \in \mathbb{F}_4$ ,  $\text{tr}(a) = a + a^2 \in \mathbb{F}_2$ , so the traces are  $\text{tr}(1) = 1 + 1 = 0$ ,  $\text{tr}(a) = a + a^2 = 1$ ,  $\text{tr}(a^2) = a^2 + a^4 = a^2 + a = 1$ . Therefore

$$\begin{aligned} g(\chi) &= \sum_{t \in \mathbb{F}_4} \chi(t) \zeta_2^{\text{tr}(t)} \\ &= \sum_{t \in \mathbb{F}_4} \chi(t) (-1)^{\text{tr}(t)} \\ &= 1 - \omega - \omega^2 \\ &= 2. \end{aligned}$$

(This is in accordance with  $|g(\chi)| = q^{1/2} = 2$ .) Then  $g(\chi^2) = g(\chi^{-1}) = \chi(-1)\overline{g(\chi)} = g(\chi) = 2$ . Therefore

$$\begin{aligned} N_1 &= q + 1 + \frac{1}{q}g(\chi)^3 + \frac{1}{q}g(\chi^2)^3 \\ &= 5 + \frac{1}{4}(8 + 8) \\ &= 9. \end{aligned}$$

There are nine points on the curve with equation  $x_0^3 + x_1^3 + x_2^3 = 0$  in  $P^2(F_4)$  (this is verified with a naive program in Sage).

Now we compute  $N_s$ . We must replace  $q = 4$  by  $q^s = 4^s$ , and  $\chi$  by  $\chi_s$ , a character with order 3 on  $F_s = \mathbb{F}_{4^s}$ .

We obtain

$$N_s = q^s + 1 + \frac{1}{q^s} (g(\chi_s)^3 + g(\chi_s^2)^3).$$

Now we compute  $g(\chi_s)^3$ . By the generalization of Corollary of Proposition 8.3.3.,

$$g(\chi_s)^3 = q^s J(\chi_s, \chi_s),$$

thus

$$N_s = q^s + 1 + J(\chi_s, \chi_s) + J(\chi_s^2, \chi_s^2).$$

We know that  $|J(\chi_s, \chi_s)|^2 = q^s = 4^s$  (generalization of Corollary of Theorem 1). Writing  $J(\chi_s, \chi_s) = a + b\omega$ ,  $a, b \in \mathbb{Z}$ , we search the solutions of

$$|a + b\omega|^2 = a^2 - ab + b^2 = 4^s.$$

Since  $\mathbb{Z}[\omega]$  is a PID, the factorization in primes is unique. Here 2 is a prime element of  $\mathbb{Z}[\omega]$ , and  $(a + b\omega)(a + b\omega^2) = 2^{2s}$ , therefore  $a + b\omega = \varepsilon 2^k$ ,  $a + b\omega^2 = \zeta 2^l$ , where  $l, k \in \mathbb{N}$  and  $\varepsilon, \zeta$  are units. Moreover  $2^k = |a + b\omega| = |a + b\omega^2| = 2^l$ , so  $k = l = s$ . This shows that every solution  $a + b\omega$  of  $|a + b\omega|^2 = 4^s$  is associated to  $2^s$ :

$$|a + b\omega|^2 = 4^s \iff a + b\omega \in \{-2^s, -1 - 2^s\omega, -2^s\omega, 2^s, 1 + 2^s\omega, 2^s\omega\}.$$

Moreover, we know that  $a \equiv -1 \pmod{3}$ ,  $b \equiv 0 \pmod{3}$  (generalization of Proposition 8.3.4.). Therefore

$$J(\chi_s, \chi_s) = a + b\omega = -(-2)^s,$$

and similarly  $J(\chi_s^2, \chi_s^2) = -(-2)^s$ . This gives

$$N_s = 4^s + 1 - 2(-2)^s.$$

For  $s = 1$ , we find anew  $N_1 = 9$ .

Then

$$\begin{aligned} \sum_{s=1}^{\infty} \frac{N_s u^s}{s} &= \sum_{s=1}^{\infty} \frac{(4u)^s}{s} + \sum_{s=1}^{\infty} \frac{u^s}{s} - 2 \sum_{s=1}^{\infty} \frac{(-2u)^s}{s} \\ &= -\ln(1 - 4u) - \ln(1 - u) + 2\ln(1 + 2u). \end{aligned}$$

This gives

$$Z_f(u) = \frac{(1 + 2u)^2}{(1 - 4u)(1 - u)}.$$

This is the first example where  $Z_f$  has a zero, which satisfies the Riemann hypothesis for curves.  $\square$



**Ex. 11.7** Try this exercise if you know a little projective geometry. Let  $N_s$  be the number of lines in  $P_n(F_{p^s})$ . Find  $N_s$  and calculate  $\sum_{s=1}^{\infty} N_s u^s / s$ . (The set of lines in projective space form an algebraic variety called a Grassmannian variety. So do the set of planes three-dimensional linear subspaces, etc.)

*Proof.* Write  $q = p^s$ . The set of lines in  $P_n(F_q)$  is in bijective correspondence with the set of planes of the vector space  $F_q^{n+1}$ . To count these planes, consider the set  $A$  of linearly independent pairs  $(u, v)$  of the space  $F_q^{n+1}$ , and  $B$  the set of planes of  $F_q^{n+1}$ , and

$$f \left\{ \begin{array}{ll} A & \rightarrow B \\ (u, v) & \mapsto \langle u, v \rangle. \end{array} \right.$$

The set of pre-images of a fixed plane  $P$  in  $B$  is the set of basis of this plane  $P$ . Thus, to obtain  $N_s$ , we divide the number of linearly independent pairs  $(u, v)$  of the space by the number of basis of a fixed plane. To build such a pair, we choose first a nonzero vector  $u$ , and then a vector  $v$  not on the line generated by  $u$ . Therefore

$$\begin{aligned} N_s &= \frac{(q^{n+1} - 1)(q^{n+1} - q)}{(q^2 - 1)(q^2 - q)} \\ &= \frac{(q^{n+1} - 1)(q^n - 1)}{(q^2 - 1)(q - 1)}. \end{aligned}$$

□

• If  $n = 2m + 1$  is odd, then

$$\begin{aligned} N_s &= \frac{q^{2m+2} - 1}{q^2 - 1} \cdot \frac{q^{2m+1} - 1}{q - 1} \\ &= \sum_{k=0}^m q^{2k} \sum_{l=0}^2 q^l \\ &= \sum_{k=0}^m \sum_{l=0}^{2m} q^{2k+l} \\ &= \sum_{r=0}^{4m} a_r q^r \quad (r = 2k + l), \end{aligned}$$

where  $a_r$  is the cardinality of the set

$$A_r = \{(k, l) \in \llbracket 0, m \rrbracket \times \llbracket 0, 2m \rrbracket \mid 2k + l = r\}.$$

We note that  $0 \leq l = r - 2k \leq 2m$  gives

$$\left\{ \begin{array}{lll} \frac{r}{2} - m \leq k & \leq \frac{r}{2}, \\ 0 \leq k & \leq m, \end{array} \right.$$

that is

$$\max\left(0, \frac{r}{2} - m\right) \leq k \leq \min\left(\frac{r}{2}, m\right), \quad (1)$$

and each such  $k$  gives a unique pair  $(k, l) = (k, r - 2k)$  in  $A_r$ .

– If  $0 \leq r \leq 2m$ , then (1)  $\iff 0 \leq k \leq \frac{r}{2}$ , thus  $a_r = \lfloor \frac{r}{2} \rfloor + 1$ .

– If  $2m < r \leq 4m$ , then (1)  $\iff \frac{r}{2} - m \leq k \leq m$ , thus  $a_r = 2m - \left\lceil \frac{r}{2} \right\rceil + 1$ .

If  $n$  is odd, we have proved that

$$\begin{aligned} N_s &= \sum_{r=0}^{2m} \left( \left\lfloor \frac{r}{2} \right\rfloor + 1 \right) q^r + \sum_{r=2m+1}^{4m} \left( 2m + 1 - \left\lceil \frac{r}{2} \right\rceil \right) q^r \\ &= \sum_{r=0}^{n-1} \left( \left\lfloor \frac{r}{2} \right\rfloor + 1 \right) p^{sr} + \sum_{r=n}^{2n-2} \left( n - \left\lceil \frac{r}{2} \right\rceil \right) p^{sr}. \end{aligned}$$

• If  $n = 2m$  is even, then

$$\begin{aligned} N_s &= \frac{q^{2m} - 1}{q^2 - 1} \cdot \frac{q^{2m+1} - 1}{q - 1} \\ &= \sum_{k=0}^{m-1} q^{2k} \sum_{l=0}^{2m} q^l \\ &= \sum_{k=0}^{m-1} \sum_{l=0}^{2m} q^{2k+l} \\ &= \sum_{r=0}^{4m-2} b_r q^r \quad (r = 2k + l), \end{aligned}$$

where  $b_r$  is the cardinality of the set

$$B_r = \{(k, l) \in \llbracket 0, m-1 \rrbracket \times \llbracket 0, 2m \rrbracket \mid 2k + l = r\}.$$

Here  $0 \leq l = r - 2k \leq 2m$  gives

$$\begin{cases} \frac{r}{2} - m \leq k \leq \frac{r}{2}, \\ 0 \leq k \leq m-1, \end{cases}$$

that is

$$\max\left(0, \frac{r}{2} - m\right) \leq k \leq \min\left(\frac{r}{2}, m-1\right), \quad (2)$$

and each such  $k$  gives a unique pair  $(k, l) = (k, r - 2k)$  in  $B_r$ .

– If  $0 \leq r \leq 2m-1$ , then (2)  $\iff 0 \leq k \leq \frac{r}{2}$ , thus  $b_r = \left\lfloor \frac{r}{2} \right\rfloor + 1$ .

– If  $2m \leq r \leq 4m-2$ , then (2)  $\iff \frac{r}{2} - m \leq k \leq m-1$ , thus  $b_r = 2m - \left\lceil \frac{r}{2} \right\rceil$ .

If  $n$  is odd, we have proved that

$$\begin{aligned} N_s &= \sum_{r=0}^{2m-1} \left( \left\lfloor \frac{r}{2} \right\rfloor + 1 \right) q^r + \sum_{r=2m}^{4m-2} \left( 2m - \left\lceil \frac{r}{2} \right\rceil \right) q^r \\ &= \sum_{r=0}^{n-1} \left( \left\lfloor \frac{r}{2} \right\rfloor + 1 \right) p^{sr} + \sum_{r=n}^{2n-2} \left( n - \left\lceil \frac{r}{2} \right\rceil \right) p^{sr}. \end{aligned}$$

This is the same formula as in the odd case ! To conclude, for all dimension  $n$ ,

$$N_s = \sum_{r=0}^{n-1} \left( \left\lfloor \frac{r}{2} \right\rfloor + 1 \right) p^{sr} + \sum_{r=n}^{2n-2} \left( n - \left\lceil \frac{r}{2} \right\rceil \right) p^{sr},$$

therefore

$$\sum_{s=1}^{\infty} \frac{N_s u^s}{s} = - \sum_{r=0}^{n-1} \left( \left\lfloor \frac{r}{2} \right\rfloor + 1 \right) \ln(1 - p^r u) - \sum_{r=n}^{2n-2} \left( n - \left\lceil \frac{r}{2} \right\rceil \right) \ln(1 - p^r u)$$

This gives the order of the poles  $p^{-r}$  of  $Z(u) = \exp \left( \sum_{s=1}^{\infty} \frac{N_s u^s}{s} \right)$ .

To verify the equality between the two formulas giving  $N_s$ , we test this equality with a Sage program.

```
def N(n,p,s):
    q = p^s
    num = (q^(n+1) - 1)*(q^(n+1) - q)
    den = (q^2 - 1)*(q^2-q)
    return num // den

def M(n,p,s):
    q = p^s
    a = sum((floor(r/2) + 1)*q^r for r in range(n))
    b = sum((n - ceil(r/2))*q^r for r in range(n,2*n-1))
    return a+b
```

`N(4,5,3),M(4,5,3)`

`(3845707062626, 3845707062626)`

**Ex. 11.8** If  $f$  is a nonhomogeneous polynomial, we can consider the zeta function of the projective closure of the hypersurface defined by  $f$  (see Chapter 10). One way to calculate this is to count the number of points on  $H_f(F_q)$  and then add to it the number of points at infinity. For example, consider  $y^2 = x^3$  over  $F_{p^s}$ . Show that there is one point at infinity. The origin  $(0,0)$  is clearly on this curve. If  $x \neq 0$ , write  $(y/x)^2 = x$  and show that there are  $p^s$  more points on this curve. Altogether we have  $p^s$  points and the zeta function over  $F_p$  is  $(1 - pu)^{-1}$ .

*Proof.* Consider the polynomial  $f(x, y) = y^2 - x^3$  and  $g(x, z) = y^2 - x$ , and

$$\begin{aligned} \Gamma &= H_f(F_q) = \{(x, y) \in F_p^2 \mid y^2 = x^3\}, \\ \Gamma_1 &= H_g(F_q) = \{(x, y) \in F_q^2 \mid y^2 = x\}. \end{aligned}$$

Then

$$\varphi \begin{cases} \Gamma \setminus \{(0,0)\} & \rightarrow \Gamma_1 \setminus \{(0,0)\} \\ (x, y) & \mapsto (x, \frac{y}{x}) \end{cases}$$

is defined, since  $(\frac{y}{x})^2 = x$  for  $(x, y) \in \Gamma \setminus \{(0,0)\}$ , thus  $(x, \frac{y}{x}) \in \Gamma_1$ . Moreover

$$\psi \begin{cases} \Gamma_1 \setminus \{(0,0)\} & \rightarrow \Gamma \setminus \{(0,0)\} \\ (x, y) & \mapsto (x, xy) \end{cases}$$

is correctly defined, since for each  $(x, y) \in \Gamma_1 \setminus \{(0,0)\}$ ,  $y^2 = x$ , then  $x \neq 0$ , thus  $(xy)^2 = x^3$ , and  $(x, xy) \in \Gamma$ , where  $(x, xy) \neq (0,0)$ .

Moreover  $\psi$  satisfies  $\psi \circ \varphi = \text{id}$ ,  $\varphi \circ \psi = \text{id}$ :

$$\begin{aligned} (\psi \circ \varphi)(x, y) &= \psi\left(x, \frac{y}{x}\right) = \left(x, x \frac{y}{x}\right) = (x, y) & ((x, y) \in \Gamma \setminus \{(0, 0)\}), \\ (\varphi \circ \psi)(x, y) &= \varphi(x, xy) = \left(x, \frac{xy}{x}\right) = (x, y) & ((x, y) \in \Gamma_1 \setminus \{(0, 0)\}). \end{aligned}$$

So  $\varphi$  is a bijection. This shows that  $|\Gamma \setminus \{(0, 0)\}| = |\Gamma_1 \setminus \{(0, 0)\}|$ , where  $(0, 0) \in \Gamma$  and  $(0, 0) \in \Gamma_1$ , thus

$$|\Gamma_1| = |\Gamma|.$$

To count the points on  $\Gamma_1$ , we consider

$$\lambda \begin{cases} F_q & \rightarrow \Gamma_1 \\ y & \mapsto (y^2, y). \end{cases}$$

Then  $\lambda$  is bijective, with inverse  $\mu : (x, y) \mapsto y$ . This show that

$$|\Gamma| = |\Gamma_1| = q = p^s.$$

Therefore the zeta function of the affine curve  $y^2 = x^3$  over  $F_p$  is

$$Z_f(u) = (1 - pu)^{-1}.$$

But the projective completion  $H_{\bar{f}}(F_q)$  of this curve has  $p^s + 1$  points, with only one point at infinity, since  $ty^2 = x^3$  has only one point  $[t, x, y]$  satisfying  $t = 0$ , the point  $[0, 0, 1]$ .

The zeta function of the curve with homogeneous equation  $\bar{f}(t, x, y) = ty^2 - x^3$  over  $F_p$  is

$$Z_{\bar{f}}(u) = (1 - u)^{-1}(1 - pu)^{-1}.$$

□

**Ex. 11.9** Calculate the zeta function of  $y^2 = x^3 + x^2$  over  $F_p$ .

*Proof.* The curve  $\Gamma$  defined by the equation  $y^2 = x^3 + x^2$  has a singularity at the origine, as in the previous exercise. The same method applies here: if we use  $z = y/x$ , then  $z^2 = x + 1$ .

Watch out! Here there are two points  $(x, z) \in \Gamma_1$  such that  $x = 0$ , the points  $(0, 1)$  and  $(0, -1)$  (here we assume that  $p \neq 2$ ). The curve  $\Gamma_1$  defined by the equation  $z^2 = x + 1$  is such that

$$\varphi \begin{cases} \Gamma \setminus \{(0, 0)\} & \rightarrow \Gamma_1 \setminus \{(0, 1), (0, -1)\} \\ (x, y) & \mapsto \left(x, \frac{y}{x}\right) \end{cases}$$

is bijective, thus  $|\Gamma| = |\Gamma_1| - 1$ . Since each point of  $\Gamma_1$  is determined by its coordinate  $z$ ,  $|\Gamma_1| = q = p^s$ , and  $|\Gamma| = p^s - 1$ .

Therefore the zeta function of the affine curve  $y^2 = x^3 + x^2$  over  $F_p$  is

$$Z_f(u) = (1 - u)(1 - pu)^{-1},$$

There is only one point  $p$  at infinity, given by  $y^2t = x^3 + x^2t, t = 0$ , i.e.  $p = [0, 0, 1]$ . Thus  $N_s = p^s$ , and the zeta function of the projective completion of  $\Gamma$  is

$$Z_{\bar{f}}(u) = (1 - pu)^{-1}.$$

□

The results of Ex.8 and Ex. 9 concern only singular cubics.

**Ex. 11.10** If  $A \neq 0$  in  $F_q$  and  $q \equiv 1 \pmod{3}$ , show that the zeta function of  $y^2 = x^3 + A$  over  $F_q$  has the form  $Z(u) = (1 + au + qu^2)/((1-u)(1-qu))$ , where  $a \in \mathbb{Z}$  and  $|a| \leq 2q^{1/2}$ .

*Proof.* Here we compute the zeta function of the projective completion  $\overline{H}_f(F_q)$ , with equation  $f(x, y, t) = y^2t = x^3 + At^3$ . If  $t = 0$ , then  $x = 0$ , thus there is only one point  $[0, 1, 0]$  at infinity (over  $F_q$  or over  $F_{q^s}$ ).

We assume that the characteristic is not 2. Then  $q$  is odd, and so  $q \equiv 1 \pmod{6}$ . Therefore, there are characters of order 2 and 3 on  $F_q$ . Write  $\rho$  the unique character of order 2, and write  $\chi$  a character of order 3. As  $\chi$  is a character of order 3, the characters whose order divides 3 are  $\varepsilon, \chi, \chi^2$ .

We compute first  $N_1$ . We write  $N(y^2 = x^3 + A)$  for the number of points of the affine cubic over  $F_q$ , and  $N_1$  for the number of points of the projective cubic, so that  $N_1 = N(y^2 = x^3 + A) + 1$ . We recall the results obtained in Ex. 8.15.

The map  $x \mapsto -x$  is a bijection between the set of roots of  $x^3 = b$  and the set of roots of  $(-x)^3 = b$ , so  $N(x^3 = b) = N((-x)^3 = b) = N(x^3 = -b)$ .

Using Prop. 8.1.5, we obtain, since  $A \neq 0$ ,

$$\begin{aligned} N(y^2 = x^3 + A) &= \sum_{a+b=A} N(y^2 = a)N(x^3 = -b) \\ &= \sum_{a+b=A} N(y^2 = a)N(x^3 = b) \\ &= \sum_{a+b=A} (1 + \rho(a))(1 + \chi(b) + \chi^2(b)) \\ &= \sum_{i=0}^1 \sum_{j=0}^2 \sum_{a+b=A} \rho^i(a) \chi^j(b) \\ &= \sum_{i=0}^1 \sum_{j=0}^2 \rho(A)^i \chi(A)^j \sum_{a'+b'=1} \rho^i(a') \chi^j(b') \quad (a = Aa', b = Ab') \\ &= \sum_{i=0}^1 \sum_{j=0}^2 \rho(A)^i \chi(A)^j J(\chi^j, \rho^i). \end{aligned}$$

We know (generalization of Theorem 1, Chapter 8) that  $J(\chi, \varepsilon) = J(\chi^2, \varepsilon) = J(\varepsilon, \rho) = 0$ , and  $J(\varepsilon, \varepsilon) = q$ , so

$$N(y^2 = x^3 + A) = q + \rho(A)\chi(A)J(\chi, \rho) + \rho(A)\chi^2(A)J(\chi^2, \rho).$$

As  $\chi^2(A) = \chi^{-1}(A) = \overline{\chi(A)}$ , and as  $\overline{\rho(A)} = \rho(A)$ , then  $J(\chi^2, \rho) = J(\overline{\chi}, \overline{\rho}) = \overline{J(\chi, \rho)}$ , and

$$N(y^2 = x^3 + A) = q + \pi + \bar{\pi}, \text{ where } \pi = \rho(A)\chi(A)J(\chi, \rho),$$

therefore

$$N_1 = q + 1 + \pi + \bar{\pi}, \text{ where } \pi = \rho(A)\chi(A)J(\chi, \rho).$$

Since the orders of  $\chi, \rho$ , and  $\chi\rho$  are 3, 2 and 6,  $\chi \neq \varepsilon, \rho \neq \varepsilon, \chi\rho \neq \varepsilon$ , thus Theorem 1 of Chapter 6 gives

$$J(\chi, \rho) = \frac{g(\chi)g(\rho)}{g(\chi\rho)}, \quad \pi = \rho(A)\chi(A) \frac{g(\chi)g(\rho)}{g(\chi\rho)}.$$

Write  $\chi' = \chi \circ N_{F_{q^s}/F_q}$ ,  $\rho' = \rho \circ N_{F_{q^s}/F_q}$ . Then  $\chi', \rho'$  are characters on  $F_{q^s}$ , and the orders of  $\chi', \rho'$  are 3 and 2 (by properties (a), (b) of §3). The same reasoning in  $F_{q^s}$  gives

$$N_s = q^s + 1 + \pi' + \overline{\pi'}, \quad \pi' = \rho'(A)\chi'(A)\frac{g(\chi')g(\rho')}{g(\chi'\rho')}.$$

Since  $A \in F_q$ , the property (c) of §3 gives  $\chi'(A) = \chi(A)^s$ ,  $\rho'(A) = \rho(A)^s$ . Using the Hasse-Davenport Relation, and  $(\chi\rho)' = \chi'\rho'$ , we obtain

$$\begin{aligned} \pi' &= \rho'(A)\chi'(A)\frac{g(\chi')g(\rho')}{g(\chi'\rho')} \\ &= -\rho(A)^s\chi(A)^s\frac{(-g(\chi))^s(-g(\rho))^s}{(-g(\chi\rho))^s} \\ &= (-1)^{s+1}\rho(A)^s\chi(A)^s\left[\frac{g(\chi)g(\rho)}{g(\chi\rho)}\right]^s \\ &= -\left[-\rho(A)\chi(A)\frac{g(\chi)g(\rho)}{g(\chi\rho)}\right]^s \\ &= -(-\pi)^s. \end{aligned}$$

This gives  $N_s$  in the appropriate form:

$$N_s = q^s + 1 - (-\pi)^s - (-\overline{\pi})^s, \quad \pi = \rho(A)\chi(A)J(\chi, \rho) = \rho(A)\chi(A)\frac{g(\chi)g(\rho)}{g(\chi\rho)}.$$

Using the converse to Proposition 11.1.1 given in Exercise 2, we obtain

$$Z_f(u) = \frac{(1 + \pi u)(1 + \overline{\pi} u)}{(1 - u)(1 - qu)}.$$

Note that  $\pi\overline{\pi} = |\pi|^2 = q$  (by Exercise 10.22). Expanding the numerator, this gives

$$Z_f(u) = \frac{1 + au + qu^2}{(1 - u)(1 - qu)},$$

where  $a = \pi + \overline{\pi}$ .

For all  $t \in F_q^*$ ,  $\chi^3(t) = 1$ , thus  $\chi(t) \in \{1, \omega, \omega^2\} \subset \mathbb{Z}[\omega]$ , and  $\rho(t) = \pm 1$ , therefore  $\pi = \rho(A)\chi(A)\sum_{t \in F_q^*} \chi(t)\rho(t) \in \mathbb{Z}[\omega]$ . Writing  $\pi = u + v\omega$ ,  $u, v \in \mathbb{Z}$ , we obtain  $a = \pi + \overline{\pi} = 2u - v \in \mathbb{Z}$ .

Moreover,

$$|a| \leq |\pi| + |\overline{\pi}| = 2|\pi| = 2q^{1/2}.$$

To conclude,

$$Z_f(u) = \frac{1 + au + qu^2}{(1 - u)(1 - qu)}, \quad a \in \mathbb{Z}, |a| \leq 2q^{1/2}.$$

□

**Ex. 11.11** Consider the curve  $y^2 = x^3 - Dx$  over  $F_p$ , where  $D \neq 0$ . Call this curve  $C_1$ . Show that the substitution  $x = \frac{1}{2}(u + v^2)$  and  $y = \frac{1}{2}v(u + v^2)$  transforms  $C_1$  into the curve  $C_2$  given by  $u^2 - v^4 = 4D$ . Show that in any given finite field the number of finite points on  $C_1$  is one more than the number of finite points on  $C_2$ .

*Proof.* Let  $F$  be a finite field such that the characteristic of  $F$  is not 2. Here

$$C_1 = \{(x, y) \in F^2 \mid y^2 = x^3 - Dx\},$$

$$C_2 = \{(u, v) \in F^2 \mid u^2 - v^4 = 4D\}.$$

Consider the maps

$$\varphi \begin{cases} C_1 \setminus \{(0, 0)\} & \rightarrow C_2 \\ (x, y) & \mapsto \left(2x - \left(\frac{y}{x}\right)^2, \frac{y}{x}\right), \end{cases} \quad \psi \begin{cases} C_2 & \rightarrow C_1 \setminus \{(0, 0)\} \\ (u, v) & \mapsto \left(\frac{1}{2}(u + v^2), \frac{1}{2}v(u + v^2)\right). \end{cases}$$

• The map  $\varphi$  is well defined: If  $(x, y) \in C_1 \setminus \{(0, 0)\}$ , then  $y^2 = x^3 - Dx$ , and  $x \neq 0$ , otherwise  $y^2 = x^3 - Dx = 0$ , and then  $(x, y) = (0, 0)$ .

Write  $(u, v) = \left(2x - \left(\frac{y}{x}\right)^2, \frac{y}{x}\right)$ , then  $x = \frac{1}{2}(u + v^2)$  and  $y = \frac{1}{2}v(u + v^2)$ . The equality  $y^2 = x^3 - Dx$  gives

$$\begin{aligned} \frac{1}{2}v^2(u + v^2) &= \frac{1}{4}(u + v^2)^2 - D, \\ 4D &= (u + v^2)^2 - 2v^2(u + v^2), \\ 4D &= u^2 - v^4, \end{aligned}$$

so that  $(u, v) = \left(2x - \left(\frac{y}{x}\right)^2, \frac{y}{x}\right) \in C_2$ .

• The map  $\psi$  is well defined: if  $(u, v) \in C_2$ , then  $u^2 - v^4 = 4D$ . Then  $u + v^2 \neq 0$ , otherwise  $4D = u^2 - v^4 = (u - v^2)(u + v^2) = 0$ , where  $4D \neq 0$  ( $D \neq 0$ ), and the characteristic is not 2 by hypothesis).

Write  $(x, y) = \left(\frac{1}{2}(u + v^2), \frac{1}{2}v(u + v^2)\right)$ . Then  $x = \frac{1}{2}(u + v^2) \neq 0$ , and  $(u, v) = \left(2x - \left(\frac{y}{x}\right)^2, \frac{y}{x}\right)$ . The equality  $u^2 - v^4 = 4D$  gives

$$\begin{aligned} \left(2x - \left(\frac{y}{x}\right)^2\right)^2 - \left(\frac{y}{x}\right)^4 &= 4D, \\ 4x^2 - 4\frac{y^2}{x} &= 4D, \\ x^3 - Dx &= y^2, \end{aligned}$$

so that  $(x, y) = \left(\frac{1}{2}(u + v^2), \frac{1}{2}v(u + v^2)\right) \in C_1$ , and  $(x, y) \neq (0, 0)$ .

Take any point  $(x, y) \in C_1 \setminus \{(0, 0)\}$ , then  $x \neq 0$ . Write  $(u, v) = \varphi(x, y) = \left(2x - \left(\frac{y}{x}\right)^2, \frac{y}{x}\right)$ . Then  $(x, y) = \left(\frac{1}{2}(u + v^2), \frac{1}{2}v(u + v^2)\right) = \psi(u, v) = (\psi \circ \varphi)(x, y)$ . Thus  $\psi \circ \varphi = 1_{C_1 \setminus \{(0, 0)\}}$ . Similarly, take any point  $(u, v) \in C_2$ . Write  $(x, y) = \psi(u, v) = \left(\frac{1}{2}(u + v^2), \frac{1}{2}v(u + v^2)\right)$ . Then  $(u, v) = \left(2x - \left(\frac{y}{x}\right)^2, \frac{y}{x}\right) = \varphi(x, y) = (\varphi \circ \psi)(u, v)$ . Thus  $\varphi \circ \psi = 1_{C_2}$ .

This proves that  $\varphi$  and  $\psi$  are bijections.

Therefore  $|C_2| = |C_1 \setminus \{(0, 0)\}| = |C_1| - 1$ , and  $|C_1| = |C_2| + 1$ .

To conclude, in any given finite field whose characteristic is not 2, the number of finite points on  $C_1$  is one more than the number of finite points on  $C_2$ .  $\square$

**Ex. 11.12** (*continuation*)

If  $p \equiv 3 \pmod{4}$ , show that the number of projective points on  $C_1$  is just  $p + 1$ .

If  $p \equiv 1 \pmod{4}$ , show that the answer is  $p + 1 + \overline{\chi(D)}J(\chi, \chi^2) + \chi(D)J(\chi, \chi^2)$ , where  $\chi$  is a character of order 4 on  $F_p$ .

Note: There is an obvious misprint. We must read  $p + 1 + \overline{\chi(D)}J(\chi, \chi^2) + \chi(D)\overline{J(\chi, \chi^2)}$

*Proof.* • Assume first that  $p \equiv 3 \pmod{4}$ . First, we count the number of affine points on  $C_2$ .

In this case, there is no character of order 4, and the only characters whose order divides 4 are  $\varepsilon$  and  $\rho$ , where  $\rho$  is the Legendre's character. Then Exercises 8.1, 8.2, with  $d = 4 \wedge (p - 1) = 2$ , and Proposition 8.1.5 show that  $N(x^4 = a) = N(y^2 = a) = 1 + \rho(a)$ . Therefore

$$\begin{aligned}
N(u^2 - v^4 = 4D) &= \sum_{a-b=4D} N(u^2 = a)N(v^4 = b) \\
&= \sum_{a-b=4D} (1 + \rho(a))(1 + \rho(b)) \\
&= \sum_{a \in F} (1 + \rho(a))(1 + \rho(a - 4D)) \\
&= \sum_{a \in F} 1 + \sum_{a \in F} \rho(a) + \sum_{a \in F} \rho(a - 4D) + \sum_{a \in F} \rho(a)\rho(a - 4D) \\
&= p + \sum_{a \in F} \rho(a)\rho(a - 4D).
\end{aligned}$$

We compute this last sum.

$$\begin{aligned}
\sum_{a \in F} \rho(a)\rho(a - 4D) &= \rho(-1) \sum_{a \in F} \rho(a)\rho(c) \\
&= \rho(-1) \sum_{a+c=4D} \rho(a)\rho(c) \\
&= \rho(-1) \sum_{a'+c'=1} \rho(4D)^2 \rho(a')\rho(b') \quad (a = 4Da', c = 4Db') \\
&= \rho(-1)J(\rho, \rho).
\end{aligned}$$

Moreover, by Theorem 1(c), Chapter 8, since  $\rho^2 = \varepsilon$ ,

$$J(\rho, \rho) = J(\rho, \rho^{-1}) = -\rho(-1).$$

Putting all together, we obtain

$$N(u^2 - v^4 = 4D) = p - 1.$$

Then Exercise 11 gives

$$N(y^2 = x^3 - Dx) = p.$$

The projective completion of  $C_1$  has equation  $y^2t = x^3 - Dxt^2$ . For  $t = 0$ ,  $x = 0$ , thus  $[0, 1, 0]$  is the only point at infinity. The number of projective points on  $C_1$  is

$$N_1 = p + 1.$$



- Now we assume that  $p \equiv 1 \pmod{4}$ . Then there is a character  $\chi$  of order 4 on  $F_p$ .

$$\begin{aligned}
N(u^2 - v^4 = 4D) &= \sum_{a-b=4D} N(u^2 = a)N(v^4 = b) \\
&= \sum_{a-b=4D} (1 + \rho(a))(1 + \chi(b) + \chi^2(b) + \chi^3(b)) \\
&= \sum_{i=0}^1 \sum_{j=0}^3 \sum_{a-b=4D} \rho^i(a) \chi^j(b).
\end{aligned}$$

The inner sum for each fixed pair  $(i, j)$  is

$$\begin{aligned}
\sum_{a-b=4D} \rho^i(a) \chi^j(b) &= \sum_{a \in F_p} \rho^i(a) \chi^j(a - 4D) \\
&= \chi^j(-1) \sum_{a \in F_p} \rho^i(a) \chi^j(4D - a) \\
&= \chi^j(-1) \sum_{a+c=4D} \rho^i(a) \chi^j(c) \\
&= \chi^j(-1) \sum_{a'+c'=1} \rho^i(a') \chi^j(c') \quad (a = 4Da', c = 4Db') \\
&= \chi^j(-1) \rho^i(4D) \chi^j(4D) J(\rho^i, \chi^j).
\end{aligned}$$

Since  $\chi^2$  is of order 2,  $\rho = \chi^2$ , thus

$$\sum_{a-b=4D} \rho^i(a) \chi^j(b) = \chi^j(-1) \chi^{2i+j}(4D) J(\chi^{2i}, \chi^j),$$

and, using  $J(\varepsilon, \varepsilon) = p$ ,  $J(\varepsilon, \chi^j) = 0$  if  $j \neq 0$ ,

$$\begin{aligned}
N(u^2 - v^4 = 4D) &= \sum_{i=0}^1 \sum_{j=0}^3 \chi^j(-1) \chi^{2i+j}(4D) J(\chi^{2i}, \chi^j) \\
&= p + \chi(-1) \chi^3(4D) J(\chi^2, \chi) \\
&\quad + \chi^2(-1) \chi^4(4D) J(\chi^2, \chi^2) \\
&\quad + \chi^3(-1) \chi^5(4D) J(\chi^2, \chi^3).
\end{aligned}$$

Since  $J(\chi^2, \chi^2) = J(\chi^2, \chi^{-2}) = -\chi^2(-1) = -1$ , and  $\chi^3 = \bar{\chi}$ , we obtain

$$N(u^2 - v^4 = 4D) = p - 1 + \chi(-1) [\overline{\chi(4D)} J(\chi, \chi^2) + \chi(4D) \overline{J(\chi, \chi^2)}].$$

Comme  $\chi(4)^2 = \chi(2^4) = \chi^4(2) = 1$ ,  $\chi(4) = \pm 1$  is real. Therefore

$$N(u^2 - v^4 = 4D) = p - 1 + \chi(-4) \left[ \overline{\chi(D)} J(\chi, \chi^2) + \chi(D) \overline{J(\chi, \chi^2)} \right].$$

We must add one to obtain the number of affine points of  $C_1$ , and one more to the point at infinity. Thus the number of projective points on  $C_1$  is

$$N_1 = p + 1 + \chi(-4) [\overline{\chi(D)} J(\chi, \chi^2) + \chi(D) \overline{J(\chi, \chi^2)}].$$

But  $\chi(-1) = (-1)^{\frac{p-1}{4}}$ . To prove this equality, take  $g$  a generator of  $F_p^*$  such that  $\chi(g) = i$  (such a generator exists, since  $\chi(g) = \pm i$ : if  $\chi(g) = -i$ , replace  $g$  by  $g^{-1}$ ). Since  $g^{p-1} = 1$ , and  $g^{(p-1)/2} \neq 1$ , we obtain  $g^{(p-1)/2} = -1$ , thus  $\chi(-1) = \chi(g)^{(p-1)/2} = i^{(p-1)/2} = (-1)^{(p-1)/4}$ . Moreover  $\chi(4) = \chi^2(2) = \rho(2) = (-1)^{(p^2-1)/8}$ . Thus, for  $p = 4k + 1$ ,

$$\chi(-4) = \chi(-1)\chi(4) = (-1)^{\frac{p-1}{4}}(-1)^{\frac{p^2-1}{8}} = (-1)^k(-1)^{2k^2+k} = 1.$$

Alleluia! We conclude

$$N_1 = p + 1 + \overline{\chi(D)}J(\chi, \chi^2) + \chi(D)\overline{J(\chi, \chi^2)}.$$

□

**Ex. 11.13** (continuation) If  $p \equiv 1 \pmod{4}$ , calculate the zeta function of  $y^2 = x^3 - Dx$  over  $F$  in terms of  $\pi$  and  $\chi(D)$ , where  $\pi = -J(\chi, \chi^2)$ . This calculation in somewhat sharpened form is contained in [23]. The result has played a key role in recent empirical work of B.J.Birch and H.P.F. Swinnerton-Dyer on elliptic curves.

*Proof.* Here  $p \equiv 1 \pmod{4}$ , thus  $p^s \equiv 1 \pmod{4}$ . We consider here the two fields  $F = \mathbb{F}_p$  and  $F_s = \mathbb{F}_{p^s}$ , where  $|F| = p$  and  $F_s = p^s$ .

Let  $\rho' = \rho \circ N_{F_s/F}$ , and  $\chi' = \chi \circ N_{F_s/F}$ . The results of §3 show that the map  $\xi \mapsto \xi' = \xi \circ N_{F_s/F}$  induces a group isomorphism between the group cyclic  $C_n$  of characters on  $F$  whose order divides  $n$  on the group cyclic  $C'_n$  of characters on  $F_s$  whose order divides  $n$ . Thus the order of  $\rho'$  is 2 and the order of  $\chi'$  is 4, and  $\chi'^2 = \rho'$ .

Replacing  $\chi, \rho$  by  $\chi', \rho'$ , and  $p$  by  $p^s$ , we obtain by the same reasoning that the number of projective point of  $C_1$  in  $\overline{H}_f(F_s)$  is

$$N_s = p^s + 1 + \chi'(-4) \left[ \overline{\chi'(D)}J(\chi', \chi'^2) + \chi'(D)\overline{J(\chi', \chi'^2)} \right].$$

To compute  $\chi'(-4)$  and  $\chi'(D)$  we use the property (c) of §3. Since  $-4$  and  $D$  are in  $F$ ,

$$\chi'(-4) = \chi(-4)^s = 1, \quad \chi'(D) = \chi(D)^s.$$

Therefore

$$N_s = p^s + 1 + \overline{\chi(D)}^s J(\chi', \chi'^2) + \chi(D)^s \overline{J(\chi', \chi'^2)}.$$

It remains to compute  $J(\chi', \chi'^2)$ . Since  $\chi' \neq \varepsilon, \chi'^2 \neq \varepsilon, \chi'^3 \neq \varepsilon$ ,

$$J(\chi', \chi'^2) = \frac{g(\chi')g(\chi'^2)}{g(\chi'^3)}.$$

The Hasse-Davenport relation gives  $g(\chi'^k) = -(-g(\chi^k))^s$ , thus

$$\begin{aligned} J(\chi', \chi'^2) &= - \left[ -\frac{g(\chi)g(\chi^2)}{g(\chi^3)} \right]^s \\ &= -(-J(\chi, \chi^2))^s \\ &= -\pi^s, \end{aligned}$$

where  $\pi = -J(\chi, \chi^2) \in \mathbb{Z}[i]$ . To conclude,

$$N_s = p^s + 1 - \overline{\chi(D)}^s \pi^s - \chi(D)^s \overline{\pi}^s, \quad \pi = -J(\chi, \chi^2).$$

Then Exercise 2 gives

$$Z_f(u) = \frac{(1 - \overline{\chi(D)}\pi u)(1 - \chi(D)\overline{\pi}u)}{(1 - u)(1 - pu)}, \quad \pi = -J(\chi, \chi^2).$$

Since  $|\pi|^2 = |J(\chi, \chi^2)|^2 = p$  (corollary of Theorem 1, chapter 8), expanding the numerator, we obtain

$$Z_f(u) = \frac{1 + au + pu^2}{(1 - u)(1 - pu)}, \quad a = -\text{tr} \left( \overline{\chi(D)} \pi \right) \in \mathbb{Z}, \quad \pi = -J(\chi, \chi^2) \in \mathbb{Z}[i].$$

Note: Since  $Z_f(u) = \exp(N_1 u + \cdots) = 1 + N_1 u + \cdots$ , and

$$\begin{aligned} Z_f(u) &= (1 + au + pu^2)(1 + u + u^2 + \cdots)(1 + pu + p^2 u^2 + \cdots) \\ &= 1 + (a + p + 1)u + \cdots, \end{aligned}$$

the comparison of the coefficient of  $u$  in the two power series gives

$$a = N_1 - p - 1, \quad \text{where } N_1 = p + 1 - \overline{\chi(D)}\pi - \chi(D)\overline{\pi}, \quad \pi = -J(\chi, \chi^2).$$

This gives anew  $a = -\text{tr} \left( \overline{\chi(D)} \right)$ . □