

# Solutions to Ireland, Rosen “A Classical Introduction to Modern Number Theory”

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## Chapter 4

**Ex. 4.1** *Show that 2 is a primitive root modulo 29.*

*Proof.* Let  $p = 29 : p - 1 = 2^2 \times 7$ .

$$2^4 = 16 \not\equiv 1 [29]$$

$$2^{14} = 4^7 = 4 \times 16^3 = 64 \times 256 \equiv 6 \times (-34) = -204 \equiv 86 = 3 \times 29 - 1 \equiv -1 [29]$$

$$2^{28} \equiv 1 [29] \text{ and } 2^d \not\equiv 1 \text{ if } d \mid 28, d < 28, \text{ hence 2 is a primitive element modulo 29. } \square$$

**Ex. 4.2** *Compute all primitive roots for  $p = 11, 13, 17$ , and 19.*

*Proof.* •  $p = 11$ . Then  $p - 1 = 10 = 2 \times 5$ .

$2^2 = 4 \not\equiv 1 \pmod{11}$ , and  $2^5 = 32 \equiv -1 \not\equiv 1 \pmod{11}$ , so 2 is a primitive element modulo 11.

The other primitive elements modulo 11 are congruent to the powers  $2^i, i \wedge 10 = 1, 1 \leq i < 10$ , namely  $2, 2^3, 2^7, 2^9$ .

$$2^7 \equiv 7 \pmod{11}, 2^9 \equiv 6 \pmod{11}, \text{ so}$$

$$\{\bar{2}, \bar{8}, \bar{7}, \bar{6}\} \text{ is the set of the generators of } U(\mathbb{Z}/11\mathbb{Z}).$$

Similarly :

$$\bullet p = 13 : \{2, 6, 11, 7\} \text{ is the set of the generators of } U(\mathbb{Z}/13\mathbb{Z}).$$

$$\bullet p = 17 : \{3, 10, 5, 11, 14, 7, 12, 6\} \text{ is the set of the generators of } U(\mathbb{Z}/17\mathbb{Z}).$$

$$\bullet p = 19 : \{2, 13, 14, 15, 3, 10\} \text{ is the set of the generators of } U(\mathbb{Z}/19\mathbb{Z}).$$

I obtain these results with the direct orders in S.A.G.E. :

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p = 19; Fp = GF(p); a = Fp.multiplicative_generator()
print([a^k for k in range(1,p) if gcd(k,p-1) == 1])
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□

**Ex. 4.3** *Suppose that  $a$  is a primitive root modulo  $p^n$ ,  $p$  an odd prime. Show that  $a$  is a primitive root modulo  $p$ .*

*Proof.* Suppose that  $a$  is a primitive root modulo  $p^n$  : then  $\bar{a}$  is a generator of  $U(\mathbb{Z}/p^n\mathbb{Z})$ .

If  $a$  was not a primitive root modulo  $p$ ,  $\bar{a}$  is not a generator of  $U(\mathbb{Z}/p\mathbb{Z})$ , so there exists  $b \in \mathbb{Z}, b \wedge p = 1$  such that  $a^k \not\equiv b \pmod{p}$  for all  $k \in \mathbb{Z}$ . A fortiori  $a^k \not\equiv b \pmod{p^n}$ , and  $b \wedge p^n = 1$ , so  $\bar{b} \in U(\mathbb{Z}/p^n\mathbb{Z})$  and  $\bar{b} \notin \langle \bar{a} \rangle$  in  $U(\mathbb{Z}/p^n\mathbb{Z})$ , in contradiction with the hypothesis. So  $a$  is a primitive root modulo  $p$ .

(the reasoning on the orders of  $a$ , modulo  $p$  and modulo  $p^n$ , is possible, but not so easy.) □

**Ex. 4.4** Consider a prime  $p$  of the form  $4t + 1$ . Show that  $a$  is a primitive root modulo  $p$  iff  $-a$  is a primitive root modulo  $p$ .

*Proof.* Solution 1.

As  $p - 1$  is even,  $(-a)^{p-1} = a^{p-1} \equiv 1 \pmod{p}$ .

If  $(-a)^n \equiv 1 \pmod{p}$ , with  $n \in \mathbb{N}$ , then  $a^n \equiv (-1)^n \pmod{p}$ .

If  $n$  is odd, then  $a^n \equiv -1, a^{2n} \equiv 1 \pmod{p}$ . As  $a$  is a primitive root modulo  $p$ ,  $p - 1 \mid 2n$ ,  $2t \mid n$ , so  $n$  is even : this is a contradiction.

Consequently,  $n$  is even, and  $a^n \equiv 1 \pmod{p}$ , so  $p - 1 \mid n$ , so the least  $n \in \mathbb{N}^*$  such that  $a^n \equiv 1 \pmod{p}$  is  $p - 1$  : the order of  $a$  modulo  $p$  is  $p - 1$ ,  $a$  is a primitive root modulo  $p$ .

Reciprocally, if  $-a$  is a primitive root modulo  $p$ , we apply the previous result at  $-a$  to obtain that  $-(-a) = a$  is a primitive root.

Solution 2.

Let  $p - 1 = 2^{a_0} p_1^{a_1} \cdots p_k^{a_k}$  the decomposition of  $p - 1$  in prime factors.

As  $p_i$  is odd for  $i = 1, 2, \dots, k$ ,  $(p - 1)/p_i$  is even, and  $a$  is primitive, so

$$\begin{aligned} (-a)^{(p-1)/p_i} &= a^{(p-1)/p_i} \not\equiv 1 \pmod{p}, \\ (-a)^{(p-1)/2} &= (-a)^{2k} = a^{2k} = a^{(p-1)/2} \not\equiv 1 \pmod{p}. \end{aligned}$$

So the order of  $a$  is  $p - 1$  modulo  $p$  (see Ex. 4.8) :  $a$  is a primitive element modulo  $p$ .  $\square$

**Ex. 4.5** Consider a prime  $p$  of the form  $4t + 3$ . Show that  $a$  is a primitive root modulo  $p$  iff  $-a$  has order  $(p - 1)/2$ .

*Proof.* Let  $a$  a primitive root modulo  $p$ .

As  $a^{p-1} \equiv 1 \pmod{p}$ ,  $p \mid (a^{(p-1)/2} - 1)(a^{(p-1)/2} + 1)$ , so  $p \mid a^{(p-1)/2} - 1$  or  $p \mid a^{(p-1)/2} + 1$ . As  $a$  is a primitive root modulo  $p$ ,  $a^{(p-1)/2} \not\equiv 1 \pmod{p}$ , so

$$a^{(p-1)/2} \equiv -1 \pmod{p}.$$

Hence  $(-a)^{(p-1)/2} = (-1)^{2t+1} a^{(p-1)/2} \equiv (-1) \times (-1) = 1 \pmod{p}$ .

Suppose that  $(-a)^n \equiv 1 \pmod{p}$ , with  $n \in \mathbb{N}$ .

Then  $a^{2n} = (-a)^{2n} \equiv 1 \pmod{p}$ , so  $p - 1 \mid 2n$ ,  $\frac{p-1}{2} \mid n$ .

So  $-a$  has order  $(p - 1)/2$  modulo  $p$ .

Reciprocally, suppose that  $-a$  has order  $(p - 1)/2 = 2t + 1$  modulo  $p$ . Let  $2, p_1, \dots, p_k$  the prime factors of  $p - 1$ , where  $p_i$  are odd.

$a^{(p-1)/2} = a^{2t+1} = -(-a)^{2t+1} = -(-a)^{(p-1)/2} \equiv -1$ , so  $a^{(p-1)/2} \not\equiv 1 \pmod{p}$ .

As  $p - 1$  is even,  $(p - 1)/p_i$  is even, so

$a^{(p-1)/p_i} = (-a)^{(p-1)/p_i} \not\equiv 1 \pmod{p}$  (since  $-a$  has order  $p - 1$ ).

So the order of  $a$  is  $p - 1$  (see Ex. 4.8) :  $a$  is a primitive root modulo  $p$ .  $\square$

**Ex. 4.6** If  $p = 2^{2^n} + 1$  is a Fermat prime, show that 3 is a primitive root modulo  $p$ .

*Proof.* Solution 1 (with quadratic reciprocity).

Write  $p = 2^k + 1$ , with  $k = 2^n$ .

We suppose that  $n > 0$ , so  $k \geq 2, p \geq 5$ . As  $p$  is prime,  $3^{p-1} \equiv 1 \pmod{p}$ .

In other words,  $3^{2^k} \equiv 1 \pmod{p}$  : the order of 3 is a divisor of  $2^k$ , a power of 2.

3 has order  $2^k$  modulo  $p$  iff  $3^{2^{k-1}} \not\equiv 1 \pmod{p}$ . As  $(3^{2^{k-1}})^2 \equiv 1 \pmod{p}$ , where  $p$  is prime, this is equivalent to  $3^{2^{k-1}} \equiv -1 \pmod{p}$ , which remains to prove.

$$3^{2^{k-1}} = 3^{(p-1)/2} \equiv \left(\frac{3}{p}\right) \pmod{p}.$$

As the result is true for  $p = 5$ , we can suppose  $n \geq 2$ . From the law of quadratic reciprocity :

$$\left(\frac{3}{p}\right)\left(\frac{p}{3}\right) = (-1)^{(p-1)/2} = (-1)^{2^{k-1}} = 1.$$

$$\text{So } \left(\frac{3}{p}\right) = \left(\frac{p}{3}\right)$$

$$\begin{aligned} p = 2^{2^n} + 1 &\equiv (-1)^{2^n} + 1 \pmod{3} \\ &\equiv 2 \equiv -1 \pmod{3}, \end{aligned}$$

so  $\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right) = -1$ , that is to say

$$3^{2^{k-1}} \equiv -1 \pmod{p}.$$

The order of 3 modulo  $p = 2^{2^n} + 1$  is  $p - 1 = 2^{2^n} : 3$  is a primitive root modulo  $p$ .  
(On the other hand, if 3 is of order  $p - 1$  modulo  $p$ , then  $p$  is prime, so

$$F_n = 2^{2^n} + 1 \text{ is prime} \iff 3^{(F_n-1)/2} = 3^{2^{2^n}-1} \equiv -1 \pmod{F_n}.)$$

Solution 2 (without quadratic reciprocity, with the hint of chapter 4).

As above, if we suppose that 3 is not a primitive root modulo  $p$ , then  $3^{2^{n-1}} \equiv 1 \pmod{p}$ , so  $n \geq 2$ , and  $(-3)^{(p-1)/2} = 3^{2^{n-1}} \equiv 1 \pmod{p}$ , so  $-3$  is a square modulo  $p$  : there exists  $a \in \mathbb{Z}$  such that  $-3 \equiv a^2 \pmod{p}$ .

As  $2 \wedge p = 1$ , there exists  $u \in \mathbb{Z}$  such that  $2u \equiv -1 + a \pmod{p}$  ( $\bar{u}$  is similar to  $\omega = \frac{-1+i\sqrt{3}}{2} \in \mathbb{C}$ ). Then

$$\begin{aligned} 8u^3 &\equiv (-1 + a)^3 \\ &\equiv -1 + 3a - 3a^2 + a^3 \\ &\equiv -1 + 3a + 9 - 3a \\ &\equiv 8 \pmod{p} \end{aligned}$$

As  $p \wedge 2 = p \wedge 8 = 1$ ,  $u^3 \equiv 1 \pmod{p}$ . Moreover, if  $u \equiv 1 \pmod{3}$ , then  $a \equiv 3 \pmod{p}$ ,  $-3 \equiv 9 \pmod{p}$ ,  $p \mid 12$ , so  $p = 2$  or  $p = 3$ , in contradiction with  $p \geq 5$ . So the order of  $u$  modulo  $p$  is 3 :  $(\mathbb{Z}/p\mathbb{Z})^*$  contains an element  $\bar{u}$  of order 3. So  $3 \mid p - 1$ ,  $p \equiv 1 \pmod{3}$ , but  $p \equiv (-1)^{2^n} + 1 \equiv 2 \equiv -1 \pmod{3}$  : this is a contradiction, so 3 is a primitive root modulo  $p = 2^{2^n} + 1$ .  $\square$

**Ex. 4.7** Suppose that  $p$  is a prime of the form  $8t + 3$  and that  $q = (p - 1)/2$  is also a prime. Show that 2 is a primitive root modulo  $p$ .

*Proof.* The first examples of such couples  $(q, p)$  are  $(5, 11)$ ,  $(29, 59)$ ,  $(41, 83)$ ,  $(53, 107)$ ,  $(89, 179)$ .  
 $p = 2q + 1 = 8t + 3$  and  $p, q$  are prime numbers.

From Fermat's little theorem,  $2^{p-1} \equiv 1 \pmod{p}$ , so  $2^{2q} \equiv 1 \pmod{p}$ .

The order of 2 modulo  $p$  divides  $2q$  : to prove that the order of 2 is  $2q = p - 1$ , it is sufficient to prove

$$2^2 \not\equiv 1 \pmod{p}, \quad 2^q \not\equiv 1 \pmod{p}.$$

If  $2^2 \equiv 1 \pmod{p}$ , then  $p \mid 3$ ,  $p = 3$  and  $q = 1$  :  $q$  is not a prime, so  $2^2 \not\equiv 1 \pmod{p}$ .

If  $2^q = 2^{(p-1)/2} \equiv 1 \pmod{p}$ , then 2 is a square modulo  $p$  (prop. 4.2.1) : there exists  $a \in \mathbb{Z}$  such that  $2 \equiv a^2 \pmod{p}$ .

From the complementary case of law of quadratic reciprocity (see next chapter, prop. 5.1.3), 2 is a square modulo  $p$  iff

$$1 = \left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}.$$

Yet  $p \equiv 3 \pmod{8}$ , so  $p^2 \equiv 1 \pmod{16}$ ,  $\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} = -1$ , so 2 is not a square modulo  $p$ . This is a contradiction, so  $2^q \not\equiv 1 \pmod{p}$  : 2 is a primitive root modulo  $p$ .  $\square$

**Ex. 4.8** Let  $p$  be an odd prime. Show that  $a$  is a primitive root modulo  $p$  iff  $a^{(p-1)/q} \not\equiv 1 \pmod{p}$  for all prime divisors  $q$  of  $p-1$ .

*Proof.* • If  $a$  is a primitive root, then  $a^k \not\equiv 1$  for all  $k$ ,  $1 \leq k < p-1$ , so  $a^{(p-1)/q} \not\equiv 1 \pmod{p}$  for all prime divisors  $q$  of  $p-1$ .

• In the other direction, suppose  $a^{(p-1)/q} \not\equiv 1 \pmod{p}$  for all prime divisors  $q$  of  $p-1$ .

Let  $\delta$  the order of  $a$ , and  $p-1 = q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k}$  the decomposition of  $p-1$  in prime factors. As  $\delta \mid p-1$ ,  $\delta = q_1^{b_1} q_2^{b_2} \cdots q_k^{b_k}$ , with  $b_i \leq a_i$ ,  $i = 1, 2, \dots, k$ . If  $b_i < a_i$  for some index  $i$ , then  $\delta \mid (p-1)/q_i$ , so  $a^{(p-1)/q_i} \equiv 1 \pmod{p}$ , which is in contradiction with the hypothesis. Thus  $b_i = a_i$  for all  $i$ , and  $\delta = q-1$  :  $a$  is a primitive root modulo  $p$ .  $\square$

**Ex. 4.9** Show that the product of all the primitive roots modulo  $p$  is congruent to  $(-1)^{\phi(p-1)}$  modulo  $p$ .

*Proof.* Here we suppose  $p$  prime,  $p > 2$ . Let  $g$  a primitive root modulo  $p$ .  $U(\mathbb{Z}/p\mathbb{Z})$  is cyclic, generated by  $\bar{g}$ :

$$U(\mathbb{Z}/p\mathbb{Z}) = \{\bar{1}, \bar{g}, \bar{g}^2, \dots, \bar{g}^{p-2}\}, \quad \bar{g}^{p-1} = \bar{1}.$$

$\bar{g}^k$  is a primitive element iff  $k \wedge (p-1) = 1$ , so the product of primitive elements in  $U(\mathbb{Z}/p\mathbb{Z})$  is

$$\bar{P} = \prod_{\substack{k \wedge (p-1) = 1 \\ 1 \leq k < p-1}} \bar{g}^k.$$

so  $\bar{P} = \bar{g}^S$ , where  $S = \sum_{\substack{k \wedge (p-1) = 1 \\ 1 \leq k < p-1}} k$ .

From Ex. 2.22, we know that for  $n \geq 2$ ,

$$\sum_{\substack{k \wedge n = 1 \\ 1 \leq k < n}} k = \frac{1}{2} n \phi(n).$$

So  $S = \sum_{\substack{k \wedge (p-1) = 1 \\ 1 \leq k < p-1}} k = \frac{1}{2} (p-1) \phi(p-1)$ .

As  $p > 2$ ,  $p-1$  is even.  $(\bar{g}^{(p-1)/2})^2 = \bar{g}^{p-1} = \bar{1}$ , and  $\bar{g}^{(p-1)/2} \neq \bar{1}$ . As  $\mathbb{Z}/p\mathbb{Z}$  is a field,  $\bar{g}^{(p-1)/2} = -\bar{1}$ .

Thus  $\bar{P} = (-\bar{1})^{\phi(p-1)}$  : so the product  $P$  of all the primitive roots modulo  $p$  is such that

$$P \equiv (-1)^{\phi(p-1)} \pmod{p}.$$

$\square$

**Ex. 4.10** Show that the sum of all the primitive roots modulo  $p$  is congruent to  $\mu(p-1)$  modulo  $p$ .

*Proof.* Notation :  $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$  is the field with  $p$  elements,  $|x|$  the multiplicative order of an element  $x \in \mathbb{F}_p^*$ ,  $\mathbb{N}^* = \{1, 2, 3, \dots\}$ .

Let

$$\psi : \begin{cases} \mathbb{N}^* & \rightarrow \\ n & \mapsto \psi(n) = \sum_{d \in \mathbb{F}_p^*, |d|=n} d \end{cases}$$

$\psi(n)$  is the sum of the elements with order  $n$  in  $\mathbb{F}_p^*$ . So  $\psi(n) = 0$  if  $n \nmid p-1$ , and  $S = \psi(p-1)$  is the sought sum of all the primitive roots modulo  $p$ .

We compute for all  $n \in \mathbb{N}^*$

$$f(n) = \sum_{d|n} \psi(d).$$

$f(n)$  is the sum of elements whose order divides  $n$ , in other words the sum of the roots of  $x^n - 1$ . This sum is, up to the sign, the coefficient of  $x^{n-1}$ , so is null, except in the case  $n = 1$ , where the sum of the unique root 1 of  $x - 1$  is 1. So

$$f(1) = 1, \quad \forall n > 1, f(n) = 0,$$

( $f = \chi_{\{1\}}$  is the characteristic function of  $\{1\}$ ).

From the Möbius inversion formula, for all  $n \in \mathbb{N}^*$ ,  $\psi(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d)$ , so

$$\psi(p-1) = \sum_{d|p-1} \mu\left(\frac{p-1}{d}\right) f(d) = \mu(p-1).$$

Conclusion :

$$S = \sum_{d \in \mathbb{F}_p^*, |d|=p-1} d = \mu(p-1) :$$

the sum of all the primitive roots modulo  $p$  is congruent to  $\mu(p-1)$  modulo  $p$ .  $\square$

**Ex. 4.11** Prove that  $1^k + 2^k + \dots + (p-1)^k \equiv 0 \pmod{p}$  if  $p-1 \nmid k$ , and  $-1 \pmod{p}$  if  $p-1 \mid k$ .

*Proof.* Let  $S_k = 1^k + 2^k + \dots + (p-1)^k$ .

Let  $g$  a primitive root modulo  $p$  :  $\bar{g}$  a generator of  $\mathbb{F}_p^*$ .

As  $(\bar{1}, \bar{g}, \bar{g}^2, \dots, \bar{g}^{p-2})$  is a permutation of  $(\bar{1}, \bar{2}, \dots, \overline{p-1})$ ,

$$\begin{aligned} \overline{S_k} &= \bar{1}^k + \bar{2}^k + \dots + \overline{p-1}^k \\ &= \sum_{i=0}^{p-2} \bar{g}^{ki} = \begin{cases} \overline{p-1} = -\bar{1} & \text{if } p-1 \mid k \\ \frac{\bar{g}^{(p-1)k} - 1}{\bar{g}^k - 1} = \bar{0} & \text{if } p-1 \nmid k \end{cases} \end{aligned}$$

since  $p-1 \mid k \iff \bar{g}^k = \bar{1}$ .

Conclusion :

$$\begin{aligned} 1^k + 2^k + \dots + (p-1)^k &\equiv 0 \pmod{p} \text{ if } p-1 \nmid k \\ 1^k + 2^k + \dots + (p-1)^k &\equiv -1 \pmod{p} \text{ if } p-1 \mid k \end{aligned}$$

$\square$

**Ex. 4.12** Use the existence of a primitive root to give another proof of Wilson's theorem  $(p-1)! \equiv -1 \pmod{p}$ .

*Proof.* As the result is trivial if  $p = 2$ , we suppose that  $p$  is an odd prime.

Let  $g$  a primitive root modulo  $p$  :  $\bar{g}$  a generator of  $\mathbb{F}_p^*$ .

As  $(\bar{g}^{(p-1)/2})^2 = \bar{g}^{p-1} = \bar{1}$ , and  $\bar{g}^{(p-1)/2} \neq 1$  in the field  $\mathbb{F}_p^*$ , then  $\bar{g}^{(p-1)/2} = -1$ , and  $(\bar{1}, \bar{g}, \bar{g}^2, \dots, \bar{g}^{p-2})$  is a permutation of  $(\bar{1}, \bar{2}, \dots, \overline{p-1})$ , so

$$\begin{aligned} \overline{(p-1)!} &= \prod_{k=0}^{p-2} \bar{g}^k \\ &= \bar{g}^{\sum_{k=0}^{p-2} k} \\ &= \bar{g}^{(p-2)(p-1)/2} \\ &= \left(\bar{g}^{(p-1)/2}\right)^{p-2} \\ &= (-\bar{1})^{p-2} \\ &= -1. \end{aligned}$$

Hence  $(p-1)! \equiv -1 \pmod{p}$  for each prime  $p$ . □

**Ex. 4.13** Let  $G$  be a finite cyclic group and  $g \in G$  a generator. Show that all the other generators are of the form  $g^k$ , where  $(k, n) = 1$ ,  $n$  being the order of  $G$ .

*Proof.* Suppose  $G = \langle g \rangle$ , with  $\text{Card } G = n$ , so the order of  $g$  is  $n$ .

Let  $x$  another generator of  $G$ , then  $x = g^k$ , and  $g = x^l$ ,  $k, l \in \mathbb{Z}$ , so  $g = g^{kl}$ ,  $g^{kl-1} = e$  :  $n \mid kl - 1$ , then  $kl - 1 = qn$ ,  $q \in \mathbb{Z}$ , so  $n \wedge k = 1$ .

Reciprocally, if  $u \wedge k = 1$ , there exist  $u, v \in \mathbb{Z}$  such that  $un + vk = 1$ , so  $g = g^{un+vk} = (g^n)^u (g^k)^v = x^v \in \langle x \rangle$ , so  $G \subset \langle x \rangle$ ,  $G = \langle x \rangle$  :  $x$  is a generator of  $G$ .

Conclusion : if  $g$  is a generator of  $G$ , all the other generators are the elements  $g^k$ , where  $k \wedge n = 1$ ,  $n = |G|$ . □

**Ex. 4.14** Let  $A$  be a finite abelian group and  $a, b \in A$  elements of order  $m$  and  $n$ , respectively. If  $(m, n) = 1$ , prove that  $ab$  has order  $mn$ .

*Proof.* Suppose  $|a| = m$ ,  $|b| = n$ ,  $m \wedge n = 1$ .

• If  $(ab)^k = e$ , then  $a^k = b^{-k}$ , so  $a^{kn} = b^{-kn} = (b^n)^{-k} = e$ , so  $m \mid kn$ , with  $m \wedge n = 1$ , so  $m \mid k$ .

Similarly,  $b^{km} = a^{-km} = (a^m)^{-k} = e$ , so  $n \mid km$ ,  $n \wedge m = 1$  :  $n \mid k$ .

As  $n \mid k$ ,  $m \mid k$ ,  $n \wedge m = 1$ ,  $nm \mid k$ .

• Reciprocally, if  $nm \mid k$ ,  $nm = qnm$ ,  $q \in \mathbb{Z}$ , so  $(ab)^k = a^k b^k = (a^m)^{qn} (b^n)^{qm} = e$ .

$$\forall k \in \mathbb{Z}, (ab)^k = e \iff nm \mid k.$$

So  $|ab| = nm$ . □

**Ex. 4.15** Let  $K$  be a field and  $G \subset K^*$  a finite subgroup of the multiplicative group of  $K$ . Extend the arguments used in the proof of Theorem 4.1 to show that  $G$  is cyclic.

**Solution 1.**

*Proof.* Let  $n = |G|$ . From Lagrange's theorem,  $a^n = 1$  for all  $a \in G$ , so the polynomial  $x^n - 1 \in K[x]$  has exactly  $n$  roots in  $G$ , and so

$$\forall x \in K, x \in G \iff x^n = 1.$$

If  $d \mid n$ , the polynomial  $x^d - 1 \in K[x]$  has exactly  $d$  roots in  $K$  otherwise  $x^n - 1 = (x^d - 1)g(x)$ ,  $g(x) \in K[x]$ , and  $\deg(g) = n - d$  has at most  $n - d$  roots, so  $x^n - 1$  would have less than  $n$  roots in  $K$ . As  $x_0^d = 1 \Rightarrow x_0^n = 1$ , all these roots are in  $G$ :  $x^d - 1$  has  $d$  roots in  $G$ .

Let  $\psi(d)$  the number of elements in  $G$  of order  $d$  ( $\psi(d) = 0$  if  $d \nmid n$ ). Then  $\sum_{c \mid d} \psi(c) = d$ . Applying the Möbius inversion theorem,  $\psi(d) = \sum_{c \mid d} \mu(c) d/c = \Phi(d)$  (Prop. 2.2.5), in particular,  $\psi(n) = \phi(n) > 1$  if  $n > 2$ . Since a group of order 2 is cyclic, we have shown in all cases the existence of an element of order  $n$  in  $G$ , so  $G$  is cyclic.

(variation :  $\psi(d) = 0$  if there exists no element of order  $d$ , and  $\psi(d) = \phi(d)$  otherwise : see Ex.4.13. So  $\psi(d) \leq \phi(d)$  for all  $d \mid n$ . As  $\sum_{d \mid n} \psi(d) = \sum_{d \mid n} \phi(d) = n$ ,  $\psi(d) = \phi(d)$  for all  $d \mid n$ . So there exists in  $G$  an element of order  $n$ , and  $G$  is cyclic.)  $\square$

### Solution 2.

*Proof.* Let  $n = |G| = p_1^{a_1} \cdots p_k^{a_k}$ . From Lagrange's theorem,  $y^n = 1$  for all  $y \in G$ .

$p(x) = x^{n/p_1} - 1 \in K[x]$  has at most  $n/p_1 < n$  roots in  $K^*$ , a fortiori in  $G$ , so there exists  $a \in G$  such that  $a^{n/p_1} \neq 1$ .

Let  $c_1 = a^{n/p_1^{a_1}} = a^{p_2^{a_2} \cdots p_k^{a_k}}$ . Then  $c_1^{p_1^{a_1}} = 1$  and  $c_1^{p_1^{a_1-1}} = a^{n/p_1} \neq 1$ , so  $|c_1| = p_1^{a_1}$ .

Similarly, there exist  $c_2, \dots, c_k$  with respective orders  $|c_i| = p_i^{a_i}$ .

From exercise 4.14, we obtain by induction that  $c = c_1 \cdots c_k$  has order  $p_1^{a_1} \cdots p_k^{a_k} = n$ , so  $G$  is cyclic.  $\square$

**Ex. 4.16** Calculate the solutions to  $x^3 \equiv 1 \pmod{19}$  and  $x^4 \equiv 1 \pmod{17}$ .

*Proof.* Here we note  $a$  the class of  $a$  in  $\mathbb{Z}/p\mathbb{Z}$ .

Let  $x \in \mathbb{F}_{19}$ .  $x^3 - 1 = 0 \iff x - 1 = 0$  or  $x^2 + x + 1 = 0$ .

$$\begin{aligned} x^2 + x + 1 = 0 &\iff (x + 10) - 99 = 0 \\ &\iff (x + 10)^2 - 4 = 0 \\ &\iff (x + 8)(x + 12) = 0 \end{aligned}$$

So, for all  $x \in \mathbb{Z}$ ,

$$x^3 \equiv 1 \pmod{19} \iff x \equiv 1, 7, 11 \pmod{19}.$$

Let  $x \in \mathbb{F}_{17}$ .

$$\begin{aligned} x^4 = 1 &\iff x^2 = 1 \text{ or } x^2 = -1 = 4^2 \\ &\iff x = \pm 1 \text{ or } x = \pm 4 \end{aligned}$$

So, for all  $x \in \mathbb{Z}$ ,

$$x^4 \equiv 1 \pmod{17} \iff x \equiv -1, 1, -4, 4 \pmod{17}.$$

Alternatively, we can take primitives roots modulo 19 and 17.

2 is a primitive root modulo 19, Let  $x = 2^k \in \mathbb{F}_{19}$ .

$$\begin{aligned} x^3 = 1 &\iff 2^{3k} = 1 \\ &\iff 18 \mid 3k \\ &\iff 6 \mid k \\ &\iff x = 1, 2^6 = 7, 2^{12} = 11 \end{aligned}$$

3 is a primitive root modulo 17. Let  $x = 3^k \in \mathbb{F}_{17}$ .

$$\begin{aligned} x^4 = 1 &\iff 3^{4k} = 1 \\ &\iff 16 \mid 4k \\ &\iff 4 \mid k \\ &\iff x = 1, 3^4 = -4, 3^8 = -1, 3^{12} = 4 \end{aligned}$$

□

**Ex. 4.17** Use the fact that 2 is a primitive root modulo 29 to find the seven solutions to  $x^7 \equiv 1 \pmod{29}$ .

*Proof.* Let  $x \in \mathbb{Z}$ , then  $x \equiv 2^k \pmod{29}, k \in \mathbb{N}$ .

$$\begin{aligned} x^7 \equiv 1 \pmod{29} &\iff 2^{7k} \equiv 1 \pmod{29} \\ &\iff 28 \mid 7k \\ &\iff 4 \mid k \end{aligned}$$

So the group cyclic  $S$  of the roots of  $x^7 - 1$  in  $\mathbb{F}_{29}$  are

$$\begin{aligned} S &= \{1, 2^4, 2^8, 2^{12}, 2^{16}, 2^{20}, 2^{24}\}, \\ S &= \{1, 16, 24, 7, 25, 23, 20\}. \end{aligned}$$

□

**Ex. 4.18** Solve the congruence  $1 + x + \cdots + x^6 \equiv 0 \pmod{29}$ .

*Proof.* As  $(1 + x + \cdots + x^6)(1 - x) = 1 - x^7$ ,

$$1 + x + \cdots + x^6 \equiv 0 \pmod{29} \iff \begin{cases} x^7 \equiv 1 \pmod{29} \\ x \not\equiv 1 \pmod{29} \end{cases}$$

From Ex. 4.17, the solutions are congruent to  $2^4, 2^8, 2^{12}, 2^{16}, 2^{20}, 2^{24}$  modulo 29. □

**Ex. 4.19** Determine the numbers  $a$  such that  $x^3 \equiv a \pmod{p}$  is solvable for  $p = 7, 11, 13$ .

*Proof.* (a) If  $p = 7$ , then  $3 \mid p - 1, d = 3 \wedge (p - 1) = 3$ . From Prop. 4.2.1,

$$\exists x \in \mathbb{Z}, a \equiv x^3 \pmod{7} \iff a \equiv 0 \pmod{7} \text{ or } a^{(p-1)/3} = a^2 \equiv 1 \pmod{7}.$$

So the numbers  $a$  such that  $x^3 \equiv a \pmod{7}$  is solvable are congruent at  $0, 1, -1$  modulo 7.



(b) If  $p = 11$ , then  $d = 3 \wedge (p - 1) = 1$ . With the same proposition,

$$\exists x \in \mathbb{Z}, a \equiv x^3 \pmod{11} \iff a \equiv 0 \pmod{11} \text{ or } a^{p-1} = a^6 \equiv 1 \pmod{11}.$$

So all integers  $a$  are cube modulo 11, in only one way.

For an alternative proof, the application

$$f : \begin{cases} \mathbb{F}_{11}^* & \rightarrow \mathbb{F}_{11}^* \\ x & \mapsto x^3 \end{cases}$$

$f$  is a bijection. Indeed,

- $f$  is a group homomorphism,
- $x^3 = 1 \Rightarrow (x^3)^7 = 1 \Rightarrow x = 1$  so  $\ker(f) = \{1\}$ ,
- $f : \mathbb{F}_{11}^* \rightarrow \mathbb{F}_{11}^*$  is injective and  $\mathbb{F}_{11}^*$  is finite, so  $f$  is bijective.

In  $\mathbb{F}_{11}$ ,  $0 = 0^3, 1 = 1^3, 2 = 7^3, 3 = 9^3, 4 = 5^3, 5 = 3^3, 6 = 8^3, 7 = 6^3, 8 = 2^3, 9 = 4^3, 10 = 10^3$ .

(c) If  $p = 13$ , then  $3 \mid p - 1, 3 \wedge (p - 1) = 3$ , so

$$\begin{aligned} \exists x \in \mathbb{Z}, a \equiv x^3 \pmod{13} &\iff a \equiv 0 \pmod{13} \text{ or } a^{(p-1)/3} = a^4 \equiv 1 \pmod{13} \\ &\iff a \equiv 0, 1, -1, 5, -5 \pmod{13} \end{aligned}$$

$$(5 \equiv 8^3 \pmod{13}).)$$

□

**Ex. 4.20** Let  $p$  be a prime, and  $d$  a divisor of  $p - 1$ . Show that  $d$ th powers form a subgroup of  $U(\mathbb{Z}/p\mathbb{Z})$  of order  $(p - 1)/d$ . Calculate this subgroup for  $p = 11, d = 5$ , for  $p = 17, d = 4$ , and for  $p = 19, d = 6$ .

*Proof.* Here  $p$  is a prime number, and  $d \mid p - 1$ . Let

$$f : \begin{cases} \mathbb{F}_p^* & \rightarrow \mathbb{F}_p^* \\ x & \rightarrow x^d \end{cases}$$

Then  $f$  is a group homomorphism, and  $\text{im}(f)$  is the set of  $d$ th powers, and consequently is a subgroup of  $U(\mathbb{F}_p) = \mathbb{F}_p^*$ .  $\ker(f)$  is the group of the roots of  $x^d - 1$ . As  $d \mid p - 1$ , the polynomial  $x^d - 1$  has exactly  $d$  roots (Prop. 4.1.2), so  $|\ker(f)| = d$ .

As  $\text{im}(f) \simeq \mathbb{F}_p^* / \ker(f)$ ,

$$|\text{im}(f)| = |\mathbb{F}_p^*| / |\ker(f)| = (p - 1)/d.$$

So there exist exactly  $(p - 1)/d$   $d$ th powers in  $(\mathbb{Z}/p\mathbb{Z})^*$ .

From Prop. 4.2.1, as  $d \mid p - 1, d \wedge p - 1$ , for all  $x \in \mathbb{F}_p^*$ ,

$$x \in \text{im}(f) \iff x^{(p-1)/d} = 1.$$

So the group of  $d$ th powers is the group of the roots of  $x^{(p-1)/d} - 1$ .

- If  $p = 11, d = 5$ ,  $\text{im}(f) = \{1, -1\}$ .
- If  $p = 17, d = 4$ ,  $x \in \text{im}(f) \iff x^4 = 1 : \text{im}(f) = \{1, -1, 4, -4\}$ .
- If  $p = 19, d = 6$ ,  $x \in \text{im}(f) \iff x^3 = 1 : \text{im}(f) = \{1, 7, 7^2 = 11\}$ , where  $7 \equiv 2^6 \pmod{19}$ .

□

**Ex. 4.21** If  $g$  is a primitive root modulo  $p$ , and  $d \mid p-1$ , show that  $g^{(p-1)/d}$  has order  $d$ . Show also that  $a$  is a  $d$ th power iff  $a \equiv g^{kd} \pmod{p}$  for some  $k$ . Do Exercises 16-20 making use of those observations.

*Proof.* Let  $x = \bar{g}^{(p-1)/d} \in \mathbb{F}_p^*$ , where  $g$  is a primitive root modulo  $p$ . For all  $k \in \mathbb{Z}$ ,

$$\begin{aligned} x^k = 1 &\iff g^{k \frac{p-1}{d}} = 1 \\ &\iff p-1 \mid k \frac{p-1}{d} \\ &\iff d \mid k \end{aligned}$$

So the order of  $\bar{g}^{(p-1)/d}$  is  $d$ .

- If  $\bar{a} = \bar{g}^{kd}$ , then  $\bar{a} = x^k$ , where  $x = \bar{g}^{(p-1)/d}$ , so  $\bar{a}$  is a  $d$ th power.
- If  $\bar{a} \neq \bar{0}$  is a  $d$ th power,  $\bar{a} = x^k, x \in \langle \bar{g} \rangle$ ,  $x = \bar{g}^{(p-1)/d}$ , so  $\bar{a} = \bar{g}^{kd}$ .

So, if  $a \not\equiv 0 \pmod{p}$ ,  $a$  is a  $d$ th power iff  $a \equiv g^{kd} \pmod{p}$  for some  $k$ .

By example (Ex. 4.20), 2 is a primitive root modulo 19, so the 6th powers modulo 19 are  $2^0 = 1, 2^6 = 7, 2^{12} = 11$ .  $\square$

**Ex. 4.22** If  $a$  has order 3 modulo  $p$ , show that  $1+a$  has order 6.

*Proof.* If  $a$  has order 3 modulo  $p$ , then  $0 \equiv a^3 - 1 = (a-1)(a^2 + a + 1) \pmod{p}$ , with  $a \not\equiv 1 \pmod{p}$ , so  $a^2 + a + 1 \equiv 0 \pmod{p}$ . Thus

$$\begin{aligned} (1+a)^3 &\equiv 1 + 3a + 3a^2 + a^3 \\ &\equiv 1 + 3a + 3(-1-a) + 1 \\ &\equiv -1 \pmod{p} \end{aligned}$$

So  $(1+a)^6 \equiv 1 \pmod{p}$ .

$$(1+a)^2 \equiv 1 + 2a + a^2 = 1 + 2a + (-1-a) \equiv a \not\equiv 1 \pmod{p}.$$

So  $(1+a)^6 \equiv 1, (1+a)^2 \not\equiv 1, (1+a)^3 \not\equiv 1 \pmod{p}$ , so the order of  $1+a$  divides 6, but doesn't divide 2 or 3, so  $1+a$  has order 6 modulo  $p$ .  $\square$

**Ex. 4.23** Show that  $x^2 \equiv -1 \pmod{p}$  has a solution iff  $p \equiv 1 \pmod{4}$ , and that  $x^4 \equiv -1 \pmod{p}$  has a solution iff  $p \equiv 1 \pmod{8}$ .

*Proof.* If  $x^2 \equiv -1 \pmod{p}$ , then  $\bar{x}$  has order 4 in  $\mathbb{F}_p^*$ , hence from Lagrange's theorem,  $4 \mid p-1$ .

Reciprocally, suppose  $4 \mid p-1$ , so  $p = 4k+1, k \in \mathbb{N}^*$ . From proposition 4.2.1, as  $2 \mid p-1$ ,  $-1$  is a square modulo  $p$  iff  $(-1)^{(p-1)/2} \equiv 1 \pmod{p}$ , which is true because  $(-1)^{(p-1)/2} = (-1)^{2k} = 1$ .

If  $x^4 \equiv -1 \pmod{p}$ , then  $\bar{x}^8 = 1 \in \mathbb{F}_p^*$ , and  $\bar{x}^4 \neq 1$ , so  $x$  has order 8 in  $\mathbb{F}_p^*$ , so  $8 \mid p-1$ .

Reciprocally, if  $p \equiv 1 \pmod{8}$ ,  $p = 8K+1, K \in \mathbb{N}^*$ . From Prop.4.2.1, as  $4 \mid p-1$ , there exists  $x \in \mathbb{Z}$  such that  $-1 = x^4$  iff  $(-1)^{(p-1)/4} \equiv 1 \pmod{8}$ , which is true because  $(-1)^{(p-1)/4} = (-1)^{2K} = 1$ .

Conclusion :

$$\exists x \in \mathbb{Z}, x^4 \equiv -1 \pmod{p} \iff p \equiv 1 \pmod{8}.$$

$\square$

**Ex. 4.24** Show that  $ax^m + by^n \equiv c \pmod{p}$  has the same number of solutions as  $ax^{m'} + by^{n'} \equiv c \pmod{p}$ , where  $m' = (m, p-1)$  and  $n' = (n, p-1)$ .

*Proof.* If  $a \wedge b \nmid c$ , the two equations have no solution. So we can suppose  $a \wedge b \mid c$ , and after division by  $\delta = a \wedge b$ , we obtain an equation  $a'x^m + b'y^n = c'$ ,  $a' = a/\delta, b' = b\delta, c' = c\delta$ , and  $a' \wedge b' = 1$ . So it remains to prove that  $ax^m + by^n \equiv c \pmod{p}$  has the same number of solutions as  $ax^{m'} + by^{n'} \equiv c \pmod{p}$  when  $a \wedge b = 1$ .

In this case the equation  $au + bv = c$  has solutions. Let  $N$  the number of solutions  $(\bar{x}, \bar{y})$  of the equation  $\bar{a}\bar{x}^m + \bar{b}\bar{y}^n = \bar{c}$ ,  $N'$  the number of solutions  $(\bar{x}, \bar{y})$  of the equation  $\bar{a}\bar{x}^{m'} + \bar{b}\bar{y}^{n'} = \bar{c}$ . Then

$$\begin{aligned} N &= \text{Card}\{(\bar{x}, \bar{y}) \in \mathbb{F}_p \times \mathbb{F}_p \mid \bar{a}\bar{x}^m + \bar{b}\bar{y}^n = \bar{c}\} \\ &= \sum_{\bar{a}\bar{u} + \bar{b}\bar{v} = \bar{c}} \text{Card}\{(\bar{x}, \bar{y}) \in \mathbb{F}_p \times \mathbb{F}_p \mid \bar{x}^m = \bar{u}, \bar{y}^n = \bar{v}\} \\ &= \sum_{\bar{a}\bar{u} + \bar{b}\bar{v} = \bar{c}} \text{Card}\{\bar{x} \in \mathbb{F}_p \mid \bar{x}^m = \bar{u}\} \times \text{Card}\{\bar{y} \in \mathbb{F}_p \mid \bar{y}^n = \bar{v}\}. \end{aligned}$$

The same is true for  $N'$ , so it is sufficient to prove that

$$\text{Card}\{\bar{x} \in \mathbb{F}_p \mid \bar{x}^m = \bar{u}\} = \text{Card}\{\bar{x} \in \mathbb{F}_p \mid \bar{x}^{m'} = \bar{u}\},$$

where  $m' = m \wedge (p-1)$ , and a similar equality for the equation  $\bar{y}^n = \bar{v}$ .

Let  $\bar{g}$  a generator of  $\mathbb{F}_p^*$ . Write  $\bar{u} = \bar{g}^r, r \in \mathbb{N}$ .

$$\begin{aligned} \exists \bar{x} \in \mathbb{F}_p, \bar{x}^m = \bar{u} &\iff \exists k \in \mathbb{Z}, \bar{g}^{mk} = \bar{g}^r \\ &\iff \exists k \in \mathbb{Z}, p-1 \mid mk - r \\ &\iff \exists k \in \mathbb{Z}, \exists l \in \mathbb{Z}, r = mk + l(p-1) \\ &\iff m \wedge (p-1) \mid r \end{aligned}$$

So

$$\{\bar{x} \in \mathbb{F}_p \mid \bar{x}^m = \bar{u}\} \neq \emptyset \iff m \wedge (p-1) \mid r,$$

and similarly

$$\{\bar{x} \in \mathbb{F}_p \mid \bar{x}^{m'} = \bar{u}\} \neq \emptyset \iff m' \wedge (p-1) \mid r.$$

Since  $m' \wedge (p-1) = (m \wedge (p-1)) \wedge (p-1) = m \wedge (p-1)$ , these two conditions are equivalent, so these two sets are empty for the same values of  $\bar{u}$ .

Let  $\bar{u}$  is such that  $\{\bar{x} \in \mathbb{F}_p \mid \bar{x}^m = \bar{u}\} \neq \emptyset$ , and  $x_0$  a fixed solution of  $\bar{x}^m = \bar{u}$ .

Write  $\bar{x} = \bar{g}^k, \bar{x}_0 = \bar{g}^{k_0}$ . Let  $d = m \wedge (p-1) (= m')$ .

$$\begin{aligned} \bar{x}^m = u &\iff \bar{x}^m = \bar{x}_0^m \\ &\iff \bar{g}^{mk} = \bar{g}^{mk_0} \\ &\iff p-1 \mid m(k - k_0) \\ &\iff \frac{p-1}{d} \mid \frac{m}{d}(k - k_0) \\ &\iff \frac{p-1}{d} \mid k - k_0 \\ &\iff \exists j \in \mathbb{Z}, k = k_0 + j \frac{p-1}{d} \end{aligned}$$

As  $g$  is a primitive root modulo  $p$ , the distinct solutions are  $x_0, x_0g^{\frac{p-1}{d}}, \dots, x_0g^{k\frac{p-1}{d}}, \dots, x_0g^{(d-1)\frac{p-1}{d}}$ , so in this case

$$\text{Card}\{\bar{x} \in \mathbb{F}_p \mid \bar{x}^m = \bar{u}\} = d = m \wedge (p-1).$$

As  $m' \wedge (p-1) = m \wedge (p-1)$ ,

$$\text{Card}\{\bar{x} \in \mathbb{F}_p \mid \bar{x}^m = \bar{u}\} = \text{Card}\{\bar{x} \in \mathbb{F}_p \mid \bar{x}^{m'} = \bar{u}\}.$$

So  $N = N' : ax^m + by^n \equiv c \pmod{p}$  has the same number of solutions as  $ax^{m'} + by^{n'} \equiv c \pmod{p}$ , where  $m' = (m, p-1)$  and  $n' = (n, p-1)$ .  $\square$

**Ex. 4.25** Prove Propositions 4.2.2 and 4.2.4.

**Proposition 4.2.2.** Suppose that  $a$  is odd,  $e \geq 3$ , and consider the congruence  $x^n \equiv a \pmod{2^e}$ . If  $n$  is odd, a solution always exists and it is unique.

If  $n$  is even, a solution exists iff  $a \equiv 1 \pmod{4}$ ,  $a^{2^{e-2}/d} \equiv 1 \pmod{2^e}$ , where  $d = (n, 2^{e-2})$ . When a solution exists there are exactly  $2d$  solutions.

*Proof.* We suppose that  $a$  is odd and  $e \geq 3$ .

From Theorem 2', we know that  $\{(-1)^a 5^b \mid 0 \leq a \leq 1, 0 \leq b \leq 2^{e-2}\}$  constitutes a reduced residue system modulo  $2^e$ , so we can write

$$\begin{aligned} a &\equiv (-1)^s 5^t \pmod{2^e}, 0 \leq s \leq 1, 0 \leq t \leq 2^{e-2}, \\ x &\equiv (-1)^y 5^z \pmod{2^e}, 0 \leq y \leq 1, 0 \leq z \leq 2^{e-2}. \end{aligned}$$

For all  $x \in \mathbb{Z}$ ,

$$x^n \equiv a \pmod{2^e} \iff (-1)^{ny} 5^{nz} \equiv (-1)^s 5^t \pmod{2^e}$$

Then  $(-1)^{ny} \equiv (-1)^s \pmod{4}$ ,  $ny \equiv s \pmod{2}$ ,  $(-1)^{ny} = (-1)^s$ , so  $5^{nz} \equiv 5^t \pmod{2^e}$ .

Reciprocally, if  $ny \equiv s \pmod{2}$  and  $5^{nz} \equiv 5^t \pmod{2^e}$ , then  $x^n \equiv a \pmod{2^e}$ , so

$$x^n \equiv a \pmod{2^e} \iff \begin{cases} ny \equiv s \pmod{2} \\ 5^{nz} \equiv 5^t \pmod{2^e} \end{cases} \iff \begin{cases} ny \equiv s \pmod{2} \\ nz \equiv t \pmod{2^{e-2}} \end{cases}$$

since the order of 5 modulo  $2^e$  is  $2^{e-2}$ .

• Suppose that  $n$  is an odd integer. Then

$$\begin{cases} ny \equiv s \pmod{2} \\ nz \equiv t \pmod{2^{e-2}} \end{cases} \iff \begin{cases} y \equiv s \pmod{2} \\ z \equiv n't \pmod{2^{e-2}} \end{cases}$$

where  $n'$  is an inverse of  $n$  modulo  $2^{e-2}$ :  $nn' \equiv 1 \pmod{2^{e-2}}$ .

So  $x^n \equiv a \pmod{2^e}$  has an unique solution modulo  $2^e$ .

• Suppose that  $n$  is an even integer.

Then  $\begin{cases} ny \equiv s \pmod{2} \\ nz \equiv t \pmod{2^{e-2}} \end{cases}$  implies  $s \equiv 0 \pmod{2}$  and  $d = n \wedge 2^{e-2} \mid t$ .

Then  $a \equiv (-1)^s 5^t \equiv 5^t \pmod{2^e}$ , so  $a \equiv 1 \pmod{4}$ .

Hence  $a^{\frac{2^{e-2}}{d}} \equiv \left(5^{2^{e-2}}\right)^{\frac{t}{d}} \equiv 1 \pmod{2^e}$ , since 5 has order  $2^{e-2}$ , and  $d \mid t$ .

So, if  $n$  is even, and  $d = n \wedge 2^{e-2}$ ,

$$\exists x \in \mathbb{Z}, x^n \equiv a \pmod{2^e} \Rightarrow \begin{cases} a \equiv 1 \pmod{4} \\ a^{\frac{2^{e-2}}{d}} \equiv 1 \pmod{2^e} \end{cases}$$

Reciprocally, suppose that  $\begin{cases} a \equiv 1 \pmod{4} \\ a^{\frac{2^{e-2}}{d}} \equiv 1 \pmod{2^e} \end{cases}$ . Then  $a \equiv (-1)^s 5^t \pmod{2^e}$  implies  $a \equiv (-1)^s \pmod{4}$ , so  $s$  is even, and  $a \equiv 5^t \pmod{2^e}$ .

Therefore  $5^{t\frac{2^{e-2}}{d}} \equiv 1 \pmod{2^e}$ , which implies  $2^{e-2} \mid t\frac{2^{e-2}}{d}$ , so  $d \mid t$ .

$$\begin{aligned} \exists x \in \mathbb{Z}, x^n \equiv a \pmod{2^e} &\iff \exists y \in \mathbb{Z}, \exists z \in \mathbb{Z}, \begin{cases} ny \equiv s \pmod{2} \\ nz \equiv t \pmod{2^{e-2}} \end{cases} \\ &\iff \exists z \in \mathbb{Z}, nz \equiv t \pmod{2^{e-2}} \quad (\text{since } n, s \text{ even}) \\ &\iff \exists z \in \mathbb{Z}, 2^{e-2} \mid nz - t \\ &\iff \exists z \in \mathbb{Z}, \frac{2^{e-2}}{d} \mid \frac{n}{d}z - \frac{t}{d} \\ &\iff \exists z \in \mathbb{Z}, \exists q \in \mathbb{Z}, q\frac{2^{e-2}}{d} + z\frac{n}{d} = \frac{t}{d} \end{aligned}$$

As  $\frac{2^{e-2}}{d} \wedge \frac{n}{d} = 1$ , there exists a solution  $(q, z_0)$  of this last equation, where  $0 \leq z_0 < \frac{2^{e-2}}{d}$ , and so  $x_0 = 5^{z_0}$  is a particular solution of  $x^n \equiv a \pmod{2^e}$ , therefore

$$\exists x \in \mathbb{Z}, x^n \equiv a \pmod{2^e} \iff \begin{cases} a \equiv 1 \pmod{4} \\ a^{\frac{2^{e-2}}{d}} \equiv 1 \pmod{2^e} \end{cases}$$

If there exists a particular solution  $x_0 \equiv (-1)^{y_0} 5^{z_0}$ , then

$$\begin{aligned} x^n \equiv a \pmod{2^e} &\iff x^n \equiv x_0^n \pmod{2^e} \\ &\iff \begin{cases} ny \equiv ny_0 \pmod{2} \\ nz \equiv nz_0 \pmod{2^{e-2}} \end{cases} \\ &\iff n(z - z_0) \equiv 0 \pmod{2^{e-2}} \quad (\text{since } n \text{ even}) \\ &\iff \frac{2^{e-2}}{d} \mid \frac{n}{d}(z - z_0) \\ &\iff \frac{2^{e-2}}{d} \mid z - z_0, \quad (\text{since } \frac{2^{e-2}}{d} \wedge \frac{n}{d} = 1) \\ &\iff \exists k \in \mathbb{Z}, z = z_0 + k\frac{2^{e-2}}{d} \end{aligned}$$

As the order of 5 modulo  $2^e$  is  $2^{e-2}$ , the solutions of  $x^n \equiv a \pmod{2^e}$  are

$$x_k = (-1)^{y_0} 5^{z_0 + k\frac{2^{e-2}}{d}}, \quad 0 \leq y_0 < 2, \quad 0 \leq k < d,$$

so there are exactly  $2d$  solutions modulo  $2^e$ . □

**Proposition 4.2.4.** *Let  $2^l$  be the highest power of 2 dividing  $n$ . Suppose that  $a$  is odd and that  $x^n \equiv a \pmod{2^{2l+1}}$  is solvable. Then  $x^n \equiv a \pmod{2^e}$  is solvable for all  $e \geq 2l + 1$ , and consequently for all  $e \geq 1$ . Moreover, all these congruences have the same number of solutions.*

*Proof.* We suppose that  $a$  is odd, and that  $x^n \equiv a \pmod{2^{2l+1}}$  is solvable.  $l$  is such that  $n = 2^l n'$ , where  $n'$  is an odd integer.

Let the induction hypothesis be, for a fixed integer  $m \geq 2l + 1$ ,

$$\exists x_0 \in \mathbb{Z}, x_0^n \equiv a \pmod{2^m}.$$

Let  $x_1 = x_0 + b2^{m-l}$  : we show that for an appropriate choice of  $b \in \{0, 1\}$ ,  $x_1^n \equiv a \pmod{2^{m+1}}$ .

$$x_1^n = x_0^n + nb2^{m-l}x_0^{n-1} + 2^{2m-2l}A, \quad A \in \mathbb{Z}.$$

Since  $m \geq 2l + 1$ ,  $2m - 2l \geq m + 1$ , so

$$x_1^n \equiv x_0^n + nb2^{m-l}x_0^{n-1} \pmod{2^{m+1}}.$$

$$\begin{aligned} x_1^n \equiv a \pmod{2^{m+1}} &\iff (x_0^n - a) + n'bx_0^{n-1}2^m \equiv 0 \pmod{2^{n+1}} \\ &\iff \frac{x_0^n - a}{2^m} + n'bx_0^{n-1} \equiv 0 \pmod{2} \end{aligned}$$

As  $a$  is odd, and  $x_0^n \equiv a \pmod{2^m}$ ,  $m \geq 1$ ,  $x_0$  is odd, and  $n'$  is odd, so there exists a unique  $b \in \{0, 1\}$  such that  $\frac{x_0^n - a}{2^m} + n'bx_0^{n-1} \equiv 0 \pmod{2}$ . So there exists  $x_1 \in \mathbb{Z}$  such that  $x_1^n \equiv a \pmod{2^{m+1}}$ , and the induction is completed. Therefore,  $x^n \equiv a \pmod{2^e}$  is solvable for all  $e \geq 2l + 1$ , and consequently for all  $e \geq 1$ .

From the Proposition 4.2.2., with the hypothesis  $e \geq 3$ , we know that the number of solutions of the solvable equation  $x^n \equiv a \pmod{2^e}$ ,  $e \geq 2l + 1$ , is 1 if  $n$  is odd,  $2(n \wedge 2^{e-2})$  if  $n$  is even.

If  $n$  is even,  $l \geq 1$ ,  $e \geq 2l + 1 \geq 3$ . Since  $e \geq 2l + 1$ , and  $n = 2^l n'$  for an odd  $n'$ ,  $l \leq \frac{e-1}{2} \leq e - 2$ , so  $n \wedge 2^{e-2} = n'2^l \wedge 2^{e-2} = 2^l$ , and the number of solutions is  $2^{l+1}$ , independent of  $e \geq 2l + 1$ .

Conclusion : under the hypothesis  $x^n \equiv a \pmod{2^{2l+1}}$ , where  $l = \text{ord}_2(n)$ , then  $x^n \equiv a \pmod{2^e}$  is solvable for all  $e \geq 1$ , and all these congruences have the same number of solutions for  $e \geq 2l + 1$ ,  $e \geq 3$ .  $\square$

## Chapter 5

**Ex. 5.1** Use Gauss' lemma to determine  $\left(\frac{5}{7}\right)$ ,  $\left(\frac{3}{11}\right)$ ,  $\left(\frac{6}{13}\right)$ ,  $\left(\frac{-1}{p}\right)$ .

*Proof.* •  $a = 5, p = 7$ .

The array of values of the least residues modulo  $p = 7$ , for  $1 \leq k \leq (p-1)/2$ .

$k \pmod{7}$	1	2	3
$5k \pmod{7}$	-2	3	1

So the number of negative least residues is  $\mu = 1$ , and  $\left(\frac{5}{7}\right) = (-1)^\mu = -1$ .

•  $a = 3, p = 11$ .

$k \pmod{11}$	1	2	3	4	5
$3k \pmod{11}$	3	-5	-2	1	4

So  $\mu = 2$ ,  $\left(\frac{3}{11}\right) = (-1)^\mu = 1$ .

•  $a = 6, p = 13$ .

$k \pmod{13}$	1	2	3	4	5	6
$6k \pmod{13}$	6	-1	5	-2	4	-3

So  $\mu = 3$ ,  $\left(\frac{6}{13}\right) = (-1)^\mu = -1$ .

• If  $a = -1$ , and  $p$  an odd prime, the values of the least residues of  $-k$  modulo  $p$  for  $k = 1, 2, \dots, (p-1)/2$  are  $-k$ , all negative. So the number of negative least residues is  $\mu = (p-1)/2$ , and  $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$ .  $\square$

**Ex. 5.2** Show that the number of solutions to  $x^2 \equiv a \pmod{p}$  is equal to  $1 + (a/p)$ .

*Proof.* Let  $N$  the number of solutions of  $x^2 \equiv a \pmod{p}$ .

- If  $(\frac{a}{p}) = 0$ , then  $p \mid a$ ,  $a \equiv 0 \pmod{p}$ , so the unique solution of  $x^2 \equiv a = 0$  is  $x \equiv 0 \pmod{p}$ , so  $N = 1 = 1 + (\frac{a}{p})$ .
- If  $(\frac{a}{p}) = -1$ , then  $N = 0 = 1 + (\frac{a}{p})$ .
- If  $(\frac{a}{p}) = 1$ , then  $x^2 \equiv a \pmod{p}$  has a solution  $x_0$ , and  $x^2 \equiv a \pmod{p} \iff x^2 \equiv x_0^2 \pmod{p} \iff p \mid (x - x_0)(x + x_0) \iff x \equiv \pm x_0 \pmod{p}$ , so  $N = 2 = 1 + (\frac{a}{p})$ .  $\square$

**Ex. 5.3** Suppose  $p \nmid a$ . Show that the number of solutions to  $ax^2 + bx + c \equiv 0 \pmod{p}$  is equal to  $1 + ((b^2 - 4ac)/p)$ .

*Proof.* Here  $p$  is an odd prime number, and  $p \nmid a$ . Let  $N$  be the number of solutions of  $ax^2 + bx + c \equiv 0 \pmod{p}$

For  $\bar{x} \in \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ ,

$$\begin{aligned} \overline{a}\bar{x}^2 + \overline{b}\bar{x} + \overline{c} &= \overline{a} \left( \bar{x}^2 + \frac{\overline{b}}{\overline{a}} \bar{x} + \frac{\overline{c}}{\overline{a}} \right) \\ &= \overline{a} \left( \left( \bar{x} + \frac{\overline{b}}{2\overline{a}} \right)^2 - \frac{\overline{b}^2 - 4\overline{a}\overline{c}}{4\overline{a}^2} \right) \end{aligned}$$

Let  $\Delta = b^2 - 4ac$ . Then  $N$  is the number of solutions of  $\left( \bar{x} + \frac{\overline{b}}{2\overline{a}} \right)^2 - \frac{\overline{\Delta}}{4\overline{a}^2} = \overline{0}$  in  $\mathbb{F}_p$ . As in Ex.5.2,  $N = 1$  if  $\overline{\Delta} = \overline{0}$ ,  $N = 0$  if  $\overline{\Delta}$  is not a square in  $\mathbb{F}_p^*$ , otherwise  $\overline{\Delta} = \delta^2$ ,  $\delta \in \mathbb{F}_p^*$ , and the solutions are  $\bar{x} = (-\overline{b} \pm \delta)/2\overline{a}$ , so  $N = 2$ . In the three cases,  $N = 1 + (\frac{\Delta}{p})$ .  $\square$

**Ex. 5.4** Prove that  $\sum_{a=1}^{p-1} (a/p) = 0$ .

*Proof.* Here  $p$  is an odd prime (the result is false if  $p = 2$ ). In the interval  $[1, p-1]$ , there exist  $(p-1)/2$  residues, and  $(p-1)/2$  nonresidues (Prop. 5.1.2., Corollary 1), so  $\sum_{a=1}^{p-1} (a/p) = 0$ .  $\square$

*Proof.* As an alternative proof, let  $S = \sum_{a=1}^{p-1} (\frac{a}{p})$ , and  $b$  a nonresidue modulo  $p$  :  $(\frac{b}{p}) = -1$  (such a  $b$  exists if  $p \neq 2$ ). As  $a \mapsto ab$  is a bijection from  $\mathbb{F}_p^*$  to itself,

$$\left( \frac{b}{p} \right) S = \sum_{a=1}^{p-1} \left( \frac{ab}{p} \right) = \sum_{c=1}^{p-1} \left( \frac{c}{p} \right) = S,$$

so  $-S = S$ ,  $S = 0$ .  $\square$

**Ex. 5.5** Prove that  $\sum_{x=1}^{p-1} ((ax+b)/p) = 0$  provided that  $p \nmid a$ .

There is a mistake in the sentence : we must read

Prove that  $\sum_{x=0}^{p-1} ((ax+b)/p) = 0$  provided that  $p \nmid a$ .

By example,

$$\sum_{x=1}^{5-1} \left( \frac{x+1}{5} \right) = \left( \frac{2}{5} \right) + \left( \frac{3}{5} \right) + \left( \frac{4}{5} \right) = -1 \neq 0.$$

*Proof.* From exercise 5.3, as  $\left(\frac{0}{p}\right) = 0$ , we know that

$$\sum_{\bar{x} \in \mathbb{F}_p} \left(\frac{x}{p}\right) = \sum_{x=0}^{p-1} \left(\frac{x}{p}\right) = \sum_{x=1}^{p-1} \left(\frac{x}{p}\right) = 0.$$

(This sum is well defined, since  $\left(\frac{x}{p}\right)$  depends only of  $\bar{x} : x \equiv x' \pmod{p} \Rightarrow \left(\frac{x}{p}\right) = \left(\frac{x'}{p}\right)$ .)

As  $\bar{a} \neq \bar{0}$  in  $\mathbb{F}_p$ ,  $f : \begin{cases} \mathbb{F}_p & \rightarrow \mathbb{F}_p \\ x & \mapsto \bar{a}x + \bar{b} \end{cases}$  is a bijection. Thus

$$\begin{aligned} \sum_{x=0}^{p-1} \left(\frac{ax+b}{p}\right) &= \sum_{x \in \mathbb{F}_p} \left(\frac{f(x)}{p}\right) \\ &= \sum_{y \in \mathbb{F}_p} \left(\frac{y}{p}\right) \quad (y = f(x)) \\ &= 0 \end{aligned}$$

□

**Ex. 5.6** Show that the number of solutions to  $x^2 - y^2 \equiv a \pmod{p}$  is given by:

$$\sum_{y=0}^{p-1} \left(1 + \left(\frac{y^2 + a}{p}\right)\right).$$

*Proof.* Let  $S = \{(\bar{x}, \bar{y}) \in \mathbb{F}_p^2 \mid \bar{x}^2 - \bar{y}^2 = \bar{a}\}$ . From Ex.5.2,

$$\begin{aligned} |S| &= \sum_{\bar{y} \in \mathbb{F}_p} \text{Card} \{\bar{x} \in \mathbb{F}_p \mid \bar{x}^2 = \bar{y}^2 + \bar{a}\} \\ &= \sum_{y=0}^{p-1} \left(1 + \left(\frac{y^2 + a}{p}\right)\right). \end{aligned}$$

□

**Ex. 5.7** By calculating directly show that the number of solutions to  $x^2 - y^2 \equiv a \pmod{p}$  is  $p-1$  if  $p \nmid a$ , and  $2p-1$  if  $p \mid a$ . (Hint. Use the change of variables  $u = x + y, v = x - y$ .)

*Proof.* Let  $S = \{(\bar{x}, \bar{y}) \in \mathbb{F}_p^2 \mid \bar{x}^2 - \bar{y}^2 = \bar{a}\}$ , and  $T = \{(\bar{u}, \bar{v}) \in \mathbb{F}_p^2 \mid \bar{u}\bar{v} = \bar{a}\}$ . Then  $f : \begin{cases} S & \rightarrow T \\ (\bar{x}, \bar{y}) & \mapsto (\bar{x} + \bar{y}, \bar{x} - \bar{y}) \end{cases}$  is well defined (if  $(\bar{x}, \bar{y}) \in S$ ,  $(\bar{x} - \bar{y})(\bar{x} + \bar{y}) = a$ , so  $(\bar{x} + \bar{y}, \bar{x} - \bar{y}) \in T$ ). Moreover  $f$  is a bijection, with inverse  $(\bar{u}, \bar{v}) \mapsto ((\bar{u} + \bar{v})/2, (\bar{u} - \bar{v})/2)$ , so  $|S| = |T|$ .

We compute  $|T|$ .

- Suppose  $p \nmid a$ , so  $\bar{a} \neq \bar{0}$ . For  $\bar{v} \neq 0$ , there is no solution, and for each  $\bar{v} \neq 0$ , we obtain the unique solution  $(\bar{a}\bar{v}^{-1}, \bar{v})$ , so there exist  $p-1$  solutions.

- Suppose  $p \mid a$ . The solutions of  $\bar{u}\bar{v} = \bar{0}$  are  $(\bar{0}, \bar{0})$ ,  $(\bar{0}, \bar{v})$  for each  $\bar{v} \neq \bar{0}$ ,  $(\bar{u}, \bar{0})$  for each  $\bar{u} \neq \bar{0}$ , that is to say  $N = 1 + (p-1) + (p-1) = 2p-1$  solutions.



Conclusion :

$$\begin{aligned} \text{Card } \{(\bar{x}, \bar{y}) \in \mathbb{F}_p^2 \mid \bar{x}^2 - \bar{y}^2 = \bar{a}\} &= p - 1 \quad \text{if } p \nmid a \\ &= 2p - 1 \quad \text{if } p \mid a \end{aligned}$$

□

**Ex. 5.8** Combining the results of Ex. 5.6 and 5.7 show that:

$$\sum_{y=0}^{p-1} \left( \frac{y^2 + a}{p} \right) = \begin{cases} -1 & \text{if } p \nmid a \\ p - 1 & \text{if } p \mid a \end{cases}$$

*Proof.* Let  $S = \{(\bar{x}, \bar{y}) \in \mathbb{F}_p^2 \mid \bar{x}^2 - \bar{y}^2 = \bar{a}\}$ .

We obtain in Ex 5.6,  $|S| = \sum_{y=0}^{p-1} \left( 1 + \left( \frac{y^2 + a}{p} \right) \right)$ , and in Ex. 5.7. ,  $|S| = p - 1$  if  $p \nmid a$ ,  
 $|S| = 2p - 1$  if  $p \mid a$ .

So

$$S - p = \sum_{y=0}^{p-1} \left( \frac{y^2 + a}{p} \right) = \begin{cases} -1 & \text{if } p \nmid a \\ p - 1 & \text{if } p \mid a \end{cases}$$

□

**Ex. 5.9** Prove that  $1^2 3^2 \dots (p-2)^2 \equiv (-1)^{(p+1)/2} \pmod{p}$  using Wilson's theorem.

*Proof.* Here  $p$  is an odd prime.

From Wilson's theorem, as  $k(p-k) \equiv -k^2 \pmod{p}$  for  $k = 1, 2, \dots, p-1$ ,

$$\begin{aligned} -1 &\equiv (p-1)! \\ &\equiv \left[ 1 \times 2 \times \dots \times k \times \dots \times \left( \frac{p-1}{2} \right) \right] \times \left[ \left( \frac{p+1}{2} \right) \times \dots \times (p-k) \dots \times (p-2) \times (p-1) \right] \\ &\equiv \prod_{k=1}^{(p-1)/2} k(p-k) \\ &\equiv (-1)^{(p-1)/2} \prod_{k=1}^{(p-1)/2} k^2 \\ &\equiv (-1)^{(p-1)/2} \left[ \left( \frac{p-1}{2} \right)! \right]^2 \pmod{p} \end{aligned}$$

So

$$\left[ \left( \frac{p-1}{2} \right)! \right]^2 \equiv (-1)^{(p+1)/2} \pmod{p}.$$

Moreover, from Wilson' theorem and Fermat's little theorem,

$$\begin{aligned} 1^2 2^2 3^2 \dots (p-1)^2 &= [(p-1)!]^2 \equiv 1 \pmod{p} \\ 2^2 4^2 \dots (p-1)^2 &= (2^{p-1})^2 \left[ \left( \frac{p-1}{2} \right)! \right]^2 \equiv \left[ \left( \frac{p-1}{2} \right)! \right]^2 \pmod{p} \end{aligned}$$

Thus

$$1^2 3^2 \dots (p-2)^2 \left[ \left( \frac{p-1}{2} \right)! \right]^2 \equiv 1 \pmod{p}.$$

which gives

$$1^2 3^2 \dots (p-2)^2 \equiv (-1)^{(p+1)/2} \pmod{p}.$$

□

**Ex. 5.10** Let  $r_1, r_2, \dots, r_{(p-1)/2}$  be the quadratic residues between 1 and  $p$ . Show that their product is congruent to 1 (mod  $p$ ) if  $p \equiv 3 \pmod{4}$ , and to  $-1$  if  $p \equiv 1 \pmod{4}$ .

*Proof.* We proved in Ex. 5.9 that

$$\left[ \left( \frac{p-1}{2} \right)! \right]^2 \equiv (-1)^{(p+1)/2} \pmod{p}.$$

The application  $f : \left\{ \begin{array}{ccc} \{\bar{1}, \bar{2}, \dots, \overline{(p-1)/2}\} & \mapsto & \{\bar{r}_1, \bar{r}_2, \dots, \overline{r_{(p-1)/2}}\} \\ x & \mapsto & x^2 \end{array} \right.$  is a bijection, so

$$\prod_{i=1}^{(p-1)/2} r_i \equiv \left[ \left( \frac{p-1}{2} \right)! \right]^2 \pmod{p},$$

so

$$\prod_{i=1}^{(p-1)/2} r_i \equiv (-1)^{(p+1)/2} \pmod{p}.$$

That is to say, the product of the quadratic residues between 1 and  $p$  is congruent to 1 (mod  $p$ ) if  $p \equiv 3 \pmod{4}$ , and to  $-1$  if  $p \equiv 1 \pmod{4}$ . □

**Ex. 5.11** Suppose that  $p \equiv 3 \pmod{4}$ , and that  $q = 2p + 1$  is also prime. Prove that  $2^p - 1$  is not prime. (Hint : Use the quadratic character of 2 to show that  $q \mid 2^p - 1$ ) One must assume that  $p > 3$ .

*Proof.* The result is false if  $p = 3$ , so we must suppose  $p > 3$ .

$p = 4k + 3$  for an integer  $k$ , so  $q = 2p + 1 = 8k + 7 \equiv -1 \pmod{8}$ . Thus

$$\left( \frac{2}{q} \right) = (-1)^{(q^2-1)/8} = 1.$$

So  $2^{(q-1)/2} \equiv 1 \pmod{q}$ ,  $2^p \equiv 1 \pmod{q}$ , so  $q \mid 2^p - 1$ .

Moreover, as  $p > 3$ ,  $q = 2p + 1 < 2^p - 1$

$$(2p + 1 < 2^p - 1 \iff 2p < 2^p - 2 \iff p + 1 < 2^{p-1}.$$

$4 + 1 < 2^{4-1}$  and for all  $k \geq 4$ ,  $k + 1 < 2^{k-1}$  implies  $k + 2 < 2^{k-1} + 1 \leq 2^k$ , and  $4 + 1 < 2^{4-1}$ , so by induction  $k + 1 < 2^{k-1}$  for all  $k > 3$ .

So  $q \mid 2^p - 1$  with  $1 < q < 2^p - 1$  :  $2^p - 1$  is composite.

Conclusion : if  $p \equiv 3 \pmod{4}$ ,  $p > 3$  is prime, and  $q = 2p + 1$  is also prime, then  $2^p - 1$  is not a prime.

For instance, le Mersenne's number  $2^{11} - 1 = 2047$  is not a prime :  $2047 = 23 \times 89$ . □

**Ex. 5.12** Let  $f(x) \in \mathbb{Z}[x]$ . We say that a prime  $p$  divides  $f(x)$  if there's an integer  $n$  such that  $p \mid f(n)$ . Describe the prime divisors of  $x^2 + 1$  and  $x^2 - 2$ .

*Proof.*  $p$  divides  $x^2 + 1$  iff there exists  $a \in \mathbb{Z}$  such that  $-1 \equiv a^2 \pmod{p}$ , iff  $p = 2$  or  $\left(\frac{-1}{p}\right) = 1$  iff  $p = 2$  or  $p \equiv 1 \pmod{4}$ .

$p$  divides  $x^2 - 2$  iff there exists  $a \in \mathbb{Z}$  such that  $2 \equiv a^2 \pmod{p}$ , iff  $p = 2$  or  $\left(\frac{2}{p}\right) = 1$  iff  $p = 2$  or  $p \equiv \pm 1 \pmod{8}$ .  $\square$

**Ex. 5.13** Show that any prime divisor of  $x^4 - x^2 + 1$  is congruent to 1 modulo 12.

*Proof.* • As  $a^6 + 1 = (a^2 + 1)(a^4 - a^2 + 1)$ ,  $p \mid a^4 - a^2 + 1$  implies  $p \mid a^6 + 1$ , so  $\left(\frac{-1}{p}\right) = 1$  and  $p \equiv 1 \pmod{4}$ .

•  $p \mid 4a^4 - 4a^2 + 4 = (2a - 1)^2 + 3$ , so  $\left(\frac{-3}{p}\right) = 1$ .

As  $-3 \equiv 1 \pmod{4}$ ,  $\left(\frac{-3}{p}\right) = \left(\frac{p}{3}\right)$ , so  $\left(\frac{p}{3}\right) = 1$ , thus  $p \equiv 1 \pmod{3}$ .

$4 \mid p - 1$  and  $3 \mid p - 1$ , thus  $12 \mid p - 1$ :

$$p \equiv 1 \pmod{12}.$$

$\square$

**Ex. 5.14** Use the fact that  $U(\mathbb{Z}/p\mathbb{Z})$  is cyclic to give a direct proof that  $(-3/p) = 1$  when  $p \equiv 1 \pmod{3}$ . [Hint : There is a  $\rho$  in  $U(\mathbb{Z}/p\mathbb{Z})$  of order 3. Show that  $(2\rho + 1)^2 = -3$ .]

*Proof.* Suppose that  $p \equiv 1 \pmod{3}$ . Let  $g$  a generator of  $\mathbb{F}_p^*$ . Then  $g$  has order  $p - 1$ , thus  $\rho = g^{(p-1)/3}$  has order 3. As  $\rho^3 = 1, \rho \neq 1$ , then  $\rho^2 + \rho + 1 = 0$ .

$$\begin{aligned} (2\rho + 1)^2 &= 4\rho^2 + 4\rho + 1 \\ &= 4(\rho^2 + \rho + 1) - 3 \\ &= -3. \end{aligned}$$

Thus  $\left(\frac{-3}{p}\right) = 1$ .  $\square$

The inverse form of this proposition is also true for an odd prime  $p$  : if  $\left(\frac{-3}{p}\right) = 1$ , then there exists  $a \in \mathbb{F}_p^*$  such that  $-\bar{3} = a^2$ .  $\rho = \frac{-1+a}{2}$  has order 3. Indeed  $\rho^2 = \frac{1+a^2-2a}{4} = \frac{-2-2a}{4} = \frac{-1-a}{2}$ , so

$$\begin{aligned} 1 + \rho + \rho^2 &= 1 + \frac{-1+a}{2} + \frac{-1-a}{2} \\ &= 0 \end{aligned}$$

so  $\rho \neq 1, \rho^3 = 1$ . The group  $\mathbb{F}_p^*$  contains an element of order 3, thus from Lagrange's theorem  $3 \mid p - 1$  :  $p \equiv 1 \pmod{3}$ .

**Ex. 5.15** If  $p \equiv 1 \pmod{5}$ , show directly that  $(5/p) = 1$  by the method of Ex. 5.14. [Hint : Let  $\rho$  be an element of  $U(\mathbb{Z}/p\mathbb{Z})$  of order 5. Show that  $(\rho + \rho^4)^2 + (\rho + \rho^4) - \bar{1} = \bar{0}$ , etc.]

*Proof.* Let  $g$  a generator of  $\mathbb{F}_p^*$ .  $g$  has order  $p-1$ , thus  $\rho = g^{(p-1)/5}$  has order 5.

Let

$$\begin{cases} \alpha &= \rho + \rho^4 \\ \beta &= \rho^2 + \rho^3 \end{cases}$$

As  $0 = \rho^5 - 1 = (\rho - 1)(1 + \rho + \rho^2 + \rho^3 + \rho^4)$  and  $\rho \neq 1$ , then  $1 + \rho + \rho^2 + \rho^3 + \rho^4 = 0$ , thus

$$\begin{aligned} \alpha + \beta &= -1 \\ \alpha\beta &= \rho^3 + \rho^4 + \rho + \rho^2 = -1 \end{aligned}$$

So  $\alpha, \beta$  are the roots in  $\mathbb{F}_p$  of  $x^2 + x - 1 : \alpha^2 + \alpha - 1 = 0$ .

Thus  $4\alpha^2 + 4\alpha - 4 = (2\alpha + 1)^2 - 5 = 0 : 5$  is a square in  $\mathbb{F}_p^*$  and  $\left(\frac{5}{p}\right) = 1$ .  $\square$

**Ex. 5.16** Using quadratic reciprocity find the primes for which 7 is quadratic residue. Do the same for 15.

*Proof.* 7 is a quadratic residue for 2 and for the odd primes such that  $\left(\frac{7}{p}\right) = 1$ .

From the law of quadratic reciprocity,

$$\left(\frac{7}{p}\right) = 1 \iff (-1)^{(p-1)/2} \left(\frac{p}{7}\right) = 1$$

iff either  $p \equiv 1 \pmod{4}$  and  $\left(\frac{p}{7}\right) = 1$ , or  $p \equiv -1 \pmod{4}$  and  $\left(\frac{p}{7}\right) = -1$ .

In the first case,  $p \equiv 1 \pmod{4}, p \equiv 1, 4, 2 \pmod{7}$ , which gives  $p \equiv 1, -3, 9 \pmod{28}$ .

In the second case,  $p \equiv -1 \pmod{4}, p \equiv -1, -4, -2 \pmod{7}$ , which gives  $p \equiv -1, 3, -9 \pmod{28}$ .

Conclusion : the primes for which 7 is a quadratic residue are 2 and the odd primes  $p$  such that

$$\left(\frac{7}{p}\right) = 1 \iff p \equiv \pm 1, \pm 3, \pm 9 \pmod{28}.$$

$\square$

15 is a quadratic residue for 2 and for the odd primes such that  $\left(\frac{15}{p}\right) = 1$ .

$$\left(\frac{15}{p}\right) = 1 \iff \left(\frac{3}{p}\right) = \left(\frac{5}{p}\right) = 1 \text{ or } \left(\frac{3}{p}\right) = \left(\frac{5}{p}\right) = -1$$

From the examples of theorem 2, we know that

$$\left(\frac{3}{p}\right) = 1 \iff p \equiv 1, -1 \pmod{12}, \quad \left(\frac{3}{p}\right) = -1 \iff p \equiv 5, -5 \pmod{12},$$

$$\left(\frac{5}{p}\right) = 1 \iff p \equiv 1, -1 \pmod{5}, \quad \left(\frac{5}{p}\right) = -1 \iff p \equiv 2, -2 \pmod{5}.$$

As  $5 \wedge 12 = 1$ , there exist 8 cases, all possible, which give

$$\left(\frac{15}{p}\right) = 1 \iff p \equiv \pm 1, \pm 7, \pm 11, \pm 17 \pmod{60}.$$

For instance, the primes 2, 7, 11, 17, 43, 53, 59, 61, 67, 137, ... are suitable.

**Ex. 5.17** Supply the details to the proof of Proposition 5.2.1 and to the corollary to the lemma following it.

**Proposition 5.2.1**

(a)  $(a_1/b) = (a_2/b)$  if  $a_1 \equiv a_2 \pmod{b}$ .

(b)  $(a_1 a_2/b) = (a_1/b)(a_2/b)$ .

(c)  $(a/b_1 b_2) = (a/b_1)(a/b_2)$ .

*Proof.* (a) Let  $b = p_1 p_2 \cdots p_m$ , where the  $p_i$  are not necessarily distinct primes. For each prime  $p_i$ ,  $(a_1, p_i) = (a_2, p_i)$  (Prop. 5.1.2 (c)), so  $\prod_i (a_1, p_i) = \prod_i (a_2, p_i)$ , thus  $(a_1/b) = (a_2/b)$ .

(b) From Prop. 5.1.2(b),

$$(a_1 a_2/b) = \prod_i (a_1 a_2/p_i) = \prod_i (a_1/p_i)(a_2/p_i) = \prod_i (a_1/p_i) \prod_i (a_2/p_i) = (a_1/b)(a_2/b).$$

(c) Let  $b_1 = p_1 p_2 \cdots p_m$ ,  $b_2 = q_1 q_2 \cdots q_l$ . Then  $b_1 b_2 = p_1 p_2 \cdots p_m q_1 q_2 \cdots q_l = \prod_{i=1}^{m+l} r_i$ , where  $r_i = p_i$  for  $i = 1, \dots, m$ ,  $r_i = q_{i-m}$  for  $i = m+1, \dots, m+l$ . Then

$$(a/b_1 b_2) = \prod_{i=1}^{m+l} (a/r_i) = \prod_{i=1}^m (a/p_i) \prod_{j=1}^l (a/q_j) = (a/b_1)(a/b_2).$$

□

**Lemma.** Let  $r$  and  $s$  be odd integers. Then

(a)  $(rs - 1)/2 \equiv ((r - 1)/2) + ((s - 1)/2) \pmod{2}$ .

(b)  $(r^2 s^2 - 1)/8 \equiv ((r^2 - 1)/8) + ((s^2 - 1)/8) \pmod{2}$ .

(Proof in the book.)

**Corollary.** Let  $r_1, r_2, \dots, r_m$  be odd integers. Then

(a)  $\sum_{i=1}^m (r_i - 1)/2 \equiv (r_1 r_2 \cdots r_m - 1)/2 \pmod{2}$ .

(b)  $\sum_{i=1}^m (r_i^2 - 1)/8 \equiv (r_1^2 r_2^2 \cdots r_m^2 - 1)/8 \pmod{2}$ .

*Proof.* Let  $\mathcal{P}(m)$  the proposition defined by

$$\mathcal{P}(m) \iff \sum_{i=1}^m (r_i - 1)/2 \equiv (r_1 r_2 \cdots r_m - 1)/2 \pmod{2}.$$

Then  $\mathcal{P}(1) \iff (r_1 - 1)/2 \equiv (r_1 - 1)/2 \pmod{2}$  is true, and  $\mathcal{P}(2)$  is part (a) of the lemma. If we make the induction hypothesis  $\mathcal{P}(m)$ , then

$$\begin{aligned} \sum_{i=1}^{m+1} (r_i - 1)/2 &= \sum_{i=1}^m (r_i - 1)/2 + (r_{m+1} - 1)/2 \\ &\equiv (r_1 r_2 \cdots r_m - 1)/2 + (r_{m+1} - 1)/2 \pmod{2} \\ &\equiv (r_1 r_2 \cdots r_m r_{m+1} - 1)/2 \pmod{2}, \end{aligned}$$

where the last congruence is a consequence of the part (a) of the Lemma : the induction is completed, and  $\mathcal{P}(m)$  is true for all  $m \geq 1$ .

The proof of part (b) is similar. □

**Ex. 5.18** Let  $D$  be a square-free integer that is also odd and positive. Show that there's an integer  $b$  prime to  $D$  such that  $(b/D) = -1$ .

*Proof.* Let  $D = p_1 p_2 \cdots p_k$ , where the  $p_i$  are distinct odd primes.

Let  $s$  a nonresidue modulo  $p_k$ . From Chinese remainder theorem, as  $p_i \wedge p_j = 1$  if  $i \neq j$ , there exists an integer  $b$  such that

$$b \equiv 1 \pmod{p_1}, b \equiv 1 \pmod{p_2}, \dots, b \equiv 1 \pmod{p_{k-1}}, b \equiv s \pmod{p_k}.$$

Then  $(b/p_i) = 1$ ,  $i = 1, 2, \dots, k-1$ ,  $(b/p_k) = -1$ , so  $b \wedge p_i = 1$  for all  $i = 1, 2, \dots, k$ . Then  $b \wedge D = b \wedge p_1 \cdots p_k = 1$ , and

$$\left(\frac{b}{D}\right) = \prod_{i=1}^k \left(\frac{b}{p_i}\right) = \left(\frac{b}{p_k}\right) = -1.$$

□

**Ex. 5.19** Let  $D$  be as in Exercise 18. Show that  $\sum(a/D) = 0$ , where the sum is over a reduced residue system modulo  $D$ . Conclude that exactly one half of the elements in  $U(\mathbb{Z}/D\mathbb{Z})$  satisfy  $(a/D) = 1$ .

*Proof.* Let  $b$  such that  $(b/D) = -1$ : the existence of  $b$  comes from Ex 5.18.

Let  $S = \sum_{a \in A} (a/D)$ , where  $A$  is reduced residue system modulo  $D$ . As two reduced system modulo  $D$  represent the same elements in  $U(\mathbb{Z}/D\mathbb{Z})$ , the sum is independent of the reduced residue system  $A$ : we can write

$$S = \sum_{\bar{a} \in U(\mathbb{Z}/D\mathbb{Z})} (a/D).$$

As  $b \wedge D = 1$ , we know from Ex. 3.6 that  $B = bA = \{ba \mid a \in A\}$  is also a reduced system modulo  $D$ . In other words, the application  $U(\mathbb{Z}/D\mathbb{Z}) \rightarrow U(\mathbb{Z}/D\mathbb{Z}), \bar{a} \mapsto \bar{a}\bar{b}$  is a bijection, so

$$\left(\frac{b}{D}\right) S = \sum_{\bar{a} \in U(\mathbb{Z}/D\mathbb{Z})} \left(\frac{b}{D}\right) \left(\frac{a}{D}\right) = \sum_{\bar{a} \in U(\mathbb{Z}/D\mathbb{Z})} \left(\frac{ba}{D}\right) = \sum_{\bar{c} \in U(\mathbb{Z}/D\mathbb{Z})} \left(\frac{c}{D}\right) = S \quad (\bar{c} = \bar{a}\bar{b}).$$

As  $(b/D) = -1$ ,  $-S = S$ , so  $S = 0$ .

Since  $(a/D) = \pm 1$ , one half of the elements in  $U(\mathbb{Z}/D\mathbb{Z})$  satisfy  $(a/D) = 1$ , and one half of the elements in  $U(\mathbb{Z}/D\mathbb{Z})$  satisfy  $(a/D) = -1$ . □

**Ex. 5.20** (continuation) Let  $a_1, a_2, \dots, a_{\phi(D)/2}$  be integers between 1 and  $D$  such that  $(a_i, D) = 1$  and  $(a_i/D) = 1$ . Prove that  $D$  is a quadratic residue modulo a prime  $p \nmid D$ ,  $p \equiv 1 \pmod{4}$  iff  $p \equiv a_i \pmod{D}$  for some  $i$ .

*Proof.* From Ex. 5.19 we know that there exist exactly  $\phi(D)/2$  integers  $a_i$  between 1 and  $D$  such that  $a_i \wedge D = 1$  and  $(a_i/D) = 1$ . So  $\{\bar{a}_1, \dots, \bar{a}_{\phi(D)/2}\}$  is the set of all  $\bar{a} \in U(\mathbb{Z}/D\mathbb{Z})$  such that  $(a/D) = 1$ .

Let  $D = p_1 p_2 \cdots p_k$ , with distinct  $p_i$ , and  $p$  a prime number,  $p \equiv 1 \pmod{4}$ ,  $p \notin \{p_1, \dots, p_k\}$  (so  $p = 4k + 1, k \in \mathbb{N}$ ).

( $\Leftarrow$ ) Suppose that  $p \equiv a_i$  for some  $i$ ,  $1 \leq i \leq \phi(D)/2$ , then  $(p/D) = (a_i/D) = 1$ , so (Prop. 5.2.2)

$$\left(\frac{D}{p}\right) = (-1)^{\frac{p-1}{2} \frac{D-1}{2}} \left(\frac{p}{D}\right) = (-1)^{2k(\frac{D-1}{2})} \left(\frac{p}{D}\right) = \left(\frac{p}{D}\right) = 1.$$

( $\Rightarrow$ ) Suppose that  $D$  is a quadratic residue modulo  $p$ . Then  $(D/p) = 1$ , so

$$\left(\frac{p}{D}\right) = (-1)^{\frac{p-1}{2} \frac{D-1}{2}} \left(\frac{D}{p}\right) = 1.$$

Thus  $\bar{p} \in \{\bar{a}_1, \dots, \bar{a}_{\phi(D)/2}\}$  since  $\{\bar{a}_1, \dots, \bar{a}_{\phi(D)/2}\}$  is the set of all  $\bar{a} \in U(\mathbb{Z}/D\mathbb{Z})$  such that  $(a/D) = 1$ . Consequently  $p \equiv a_i \pmod{D}$  for some  $i$ .  $\square$

**Ex. 5.21** Apply the method of Ex. 5.19 and 5.20 to find those primes for which 21 is a quadratic residue.

*Proof.* Let  $D = 21 = 3 \times 7$  ( $D$  is positive, odd and square-free). We first search the  $\phi(D)/2 = 6$  integers  $a$ ,  $1 \leq a \leq 21$ , such that  $(a/D) = 1$ .

$$\left(\frac{a}{21}\right) = 1 \iff \left(\frac{a}{3}\right) = \left(\frac{a}{7}\right) = 1 \text{ or } \left(\frac{a}{3}\right) = \left(\frac{a}{7}\right) = -1.$$

The first case is equivalent to  $a \equiv 1 \pmod{3}$ ,  $a \equiv 1, 2, 4 \pmod{7}$ , that is  $a \equiv 1, 16, 4 \pmod{21}$ .

The second case gives  $a \equiv -1 \pmod{3}$ ,  $a \equiv -1, -2, -4 \pmod{7}$ , that is  $a \equiv -1, -16, -4 \pmod{21}$ , or equivalently  $a \equiv 20, 5, 17 \pmod{21}$ .

So  $A = \{1, 4, 5, 16, 17, 20\}$  is the set of the integers  $a$  such that  $1 \leq a \leq 21$ ,  $(a/D) = 1$ .

As  $(21/3) = (21/7) = 0$ , 21 is not a quadratic residue modulo 3 or 7.

•  $p \equiv 1 \pmod{4}$ .

From Ex.5.20, we know that  $D = 21$  is a quadratic residue modulo an odd prime  $p$ ,  $p \neq 3, p \neq 7$ ,  $p \equiv 1 \pmod{4}$ , iff  $p \equiv a \pmod{D}$  for some  $a \in A$ .

•  $p \equiv -1 \pmod{4}$ .

As  $D = 21 \equiv 1 \pmod{4}$ ,  $\left(\frac{D}{p}\right) \left(\frac{p}{D}\right) = (-1)^{\frac{p-1}{2} \frac{D-1}{2}} = 1$ , so the same reasoning as in Ex. 5.20 show that  $D$  is a quadratic residue modulo 21 iff  $p \equiv a, a \in A$ .

Conclusion : 21 is a quadratic residue for 2, and for the primes  $p$  such that

$$p \equiv 1, 4, 5, 16, 17, 20 \pmod{21}.$$

$\square$

**Ex. 5.22** Use the Jacobi symbol to determine  $(113/997)$ ,  $(215/761)$ ,  $(514/1093)$ , and  $(401/757)$ .

*Proof.*  $(113/997) = (997/113) = (93/11) = (113/93) = (20/93) = (2^2/93)(5/93) = (5/93) = (93/5) = (3/5) = (5/3) = (2/3) = -1$ .

$(215/761) = (761/215) = (116/215) = (2^2/215)(29/215) = (29/215) = (215/29) = (12/29) = (2^2/29)(3/29) = (3/29) = (29/3) = (2/3) = -1$ .

$(514/1093) = (2/1093)(257/1093) = -(257/1093) = -(1093/57) = -(65/257) = -(257/65) = -(62/65) = -(2/65)(31/65) = -(31/65) = -(65/31) = -(3/31) = (31/3) = (1/3) = 1$ .

$(401/757) = (757/401) = (356/401) = (401/89) = (45/89) = (89/45) = (44/45) = (2^2/45)(11/45) = (11/45) = (45/11) = (1/11) = 1$ .  $\square$

**Ex. 5.23** Suppose that  $p \equiv 1 \pmod{4}$ . Show that there exist integers  $s$  and  $t$  such that  $pt = 1 + s^2$ . Conclude that  $p$  is not a prime in  $\mathbb{Z}[i]$ . Remember that  $\mathbb{Z}[i]$  has unique factorization.

*Proof.* As  $p \equiv 1 \pmod{4}$ , then  $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = 1$  :  $-1$  is a square modulo  $p$ .

So  $-1 \equiv s^2 \pmod{p}$ ,  $s \in \mathbb{Z}$  : there exist  $s \in \mathbb{Z}, t \in \mathbb{Z}$  such that  $pt = 1 + s^2$ .

In  $\mathbb{Z}[i]$ ,  $p \mid (s+i)(s-i)$ .

If  $p$  was a prime in  $\mathbb{Z}[i]$ , then  $p \mid s+i$  ou  $p \mid s-i$ .

This implies  $s \pm i = (a+bi)p$ ,  $(a,b) \in \mathbb{Z}^2$ , thus  $\pm 1 = bp$ ,  $p \mid 1$  : it's impossible.

Conclusion : if  $p \equiv 1 \pmod{4}$ ,  $p$  is not a prime in  $\mathbb{Z}[i]$ .  $\square$

**Ex. 5.24** If  $p \equiv 1 \pmod{4}$ , show that  $p$  is a sum of two squares, i.e.  $p = a^2 + b^2$  with  $a, b \in \mathbb{Z}$ . (Hint :  $p = \alpha\beta$ , with  $\alpha$  and  $\beta$  being non units in  $\mathbb{Z}[i]$ . Remember that  $\mathbb{Z}[i]$  has unique factorisation.)

*Proof.*  $\mathbb{Z}[i]$  is a principal ideal domain, thus  $p$  prime is in  $\mathbb{Z}[i]$  iff  $p$  is irreducible in  $\mathbb{Z}[i]$ .

If  $p \equiv 1 \pmod{4}$ ,  $p$  is not a prime from Ex.5.23, so it is not irreducible :

$p = \alpha\beta$ ,  $\alpha, \beta \in \mathbb{Z}[i]$ ,  $N(\alpha) > 1$ ,  $N(\beta) > 1$  (where  $N(a+bi) = a^2 + b^2$  is the complex norm).

$N(p) = p^2 = N(u)N(v)$ ,  $1 < N(u) < p^2$

Thus  $N(u) = p$ , that is  $p = a^2 + b^2$ , where  $u = a+bi$ .

Conclusion : if  $p$  is prime in  $\mathbb{N}$ ,  $p \equiv 1 \pmod{4}$ , then  $p = a^2 + b^2$ ,  $a, b \in \mathbb{Z}$ ,  $p$  is a sum of two squares.  $\square$

**Ex. 5.25** An integer is called a biquadratic residue modulo  $p$  if it is congruent to a fourth power. Using the identity  $x^4 + 4 = ((x+1)^2 + 1)((x-1)^2 + 1)$  show that  $-4$  is a biquadratic residue modulo  $p$  iff  $p \equiv 1 \pmod{4}$ .

*Proof.*  $x^4 + 4 = (x^4 + 4x^2 + 4) - 4x^2 = (x^2 + 2)^2 - 4x^2 = (x^2 + 2 - 2x)(x^2 + 2 + 2x)$ , so

$$x^4 + 4 = ((x-1)^2 + 1)((x+1)^2 + 1).$$

If  $-4 \equiv x^4 \pmod{p}$ , then  $p \mid (x+1)^2 + 1$  or  $p \mid (x-1)^2 + 1$

In the two cases,  $-1$  is a quadratic residue modulo  $p$ , thus  $\left(\frac{-1}{p}\right) = 1$  :  $p \equiv 1 \pmod{4}$ .

Reciprocally, if  $p \equiv 1 \pmod{4}$ ,  $\left(\frac{-1}{p}\right) = 1$ , then it exists an integer  $a$  such that  $-1 \equiv a^2 \pmod{p}$ .

Let  $x = a - 1$ . Then  $p \mid (x+1)^2 + 1$ , thus  $p \mid x^4 + 4$  :  $-4$  is a biquadratic residue modulo  $p$ .

Conclusion :

$$\exists x \in \mathbb{Z}, x^4 \equiv -4 \pmod{p} \iff p \equiv 1 \pmod{4}.$$

$\square$

**Ex. 5.26** This exercise and Ex. 5.27 and 5.28 give Dirichlet's beautiful proof that 2 is a biquadratic residue modulo  $p$  iff  $p$  can be written in the form  $A^2 + 64B^2$ , where  $A, B \in \mathbb{Z}$ . Suppose that  $p \equiv 1 \pmod{4}$ . Then  $p = a^2 + b^2$  by Ex. 5.24. Take  $a$  to be odd. Prove the following statements:

(a)  $(a/p) = 1$ .

(b)  $((a+b)/p) = (-1)^{((a+b)^2-1)/8}$ .



$$(c) \ (a+b)^2 \equiv 2ab \pmod{p}$$

$$(d) \ (a+b)^{(p-1)/2} \equiv (2ab)^{(p-1)/4} \pmod{p}.$$

*Proof.* Let  $p$  a prime number,  $p \equiv 1 \pmod{4}$  :  $p = 4k + 1, k \in \mathbb{N}^*$ .

Then  $p = a^2 + b^2$  (Ex. 5.24).

As  $a, b$  are not of the same parity, up to exchange  $a$  and  $b$ , we will suppose that  $a$  is odd (then  $b$  is even).

(a)

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} = a^{2k} \pmod{p}.$$

Using the law of quadratic reciprocity for Jacobi's symbol (Proposition 5.2.2), where  $a, p$  are odd numbers :

$$\left(\frac{a}{p}\right) = \left(\frac{p}{a}\right) (-1)^{\frac{p-1}{2} \frac{a-1}{2}} = \left(\frac{p}{a}\right),$$

since  $p \equiv 1 \pmod{4}$ .

If  $a = p_1 p_2 \cdots p_l$  is the decomposition of  $a$  in prime factors, with not necessarily distinct primes , then

$$\left(\frac{p}{a}\right) = \left(\frac{p}{p_1}\right) \left(\frac{p}{p_2}\right) \cdots \left(\frac{p}{p_l}\right).$$

Since  $p = a^2 + b^2$ ,  $p \equiv b^2 \pmod{p_i}$ , thus  $\left(\frac{p}{p_i}\right) = 1$  for all  $i$ .

$$\left(\frac{a}{p}\right) = \left(\frac{p}{a}\right) = 1.$$

(b)  $a + b$  is odd, and  $p \equiv 1 \pmod{4}$ , thus

$$\left(\frac{a+b}{p}\right) = \left(\frac{p}{a+b}\right) = \left(\frac{2^2 p}{a+b}\right) = \left(\frac{2}{a+b}\right) \left(\frac{2p}{a+b}\right).$$

If  $a + b = q_1 q_2 \cdots q_l$ , as  $2p = (a+b)^2 + (a-b)^2$ ,  $2p \equiv (a-b)^2 \pmod{q_i}$ , thus  $\left(\frac{2p}{q_i}\right) = 1$ .

$$\left(\frac{2p}{a+b}\right) = \left(\frac{2p}{q_1}\right) \cdots \left(\frac{2p}{q_l}\right) = 1.$$

Moreover  $\left(\frac{2}{a+b}\right) = (-1)^{\frac{(a+b)^2-1}{8}}$ , so

$$\left(\frac{a+b}{p}\right) = (-1)^{\frac{(a+b)^2-1}{8}}.$$

$$(c) \ (a+b)^2 = a^2 + b^2 + 2ab = p + 2ab \equiv 2ab \pmod{p}$$

$$(d) [(a+b)^2]^{\frac{p-1}{4}} \equiv (2ab)^{\frac{p-1}{4}} \pmod{p}, \text{ thus}$$

$$(a+b)^{\frac{p-1}{2}} \equiv (2ab)^{\frac{p-1}{4}} \pmod{p}.$$

□

**Ex. 5.27** Suppose that  $f$  is such that  $b \equiv af \pmod{p}$ . Show that  $f^2 \equiv -1 \pmod{p}$ , and that  $2^{(p-1)/4} \equiv f^{ab/2} \pmod{p}$ .

*Proof.* Let  $f$  such as  $b \equiv af \pmod{p}$ .

This is equivalent to  $\bar{f} = \bar{b}\bar{a}^{-1}$  dans  $\mathbb{F}_p^*$ .

As  $\bar{a}^2 = -\bar{b}^2$ ,  $\bar{f}^2 = -1$  :  $f^2 \equiv -1 \pmod{p}$ .

We deduce from Ex. 5.26 (d) and (b) that

$$\begin{aligned} (2ab)^{\frac{p-1}{4}} &\equiv (a+b)^{\frac{p-1}{2}} = \left(\frac{a+b}{p}\right) \\ &\equiv (-1)^{\frac{(a+b)^2-1}{8}} \\ &\equiv (f^2)^{\frac{(a+b)^2-1}{8}} \\ &\equiv f^{\frac{(a+b)^2-1}{4}} = f^{\frac{a^2+b^2-1+2ab}{4}} \\ &\equiv f^{\frac{p-1}{4}} f^{\frac{ab}{2}} \pmod{p} \end{aligned}$$

Since  $a^{\frac{p-1}{2}} = \left(\frac{a}{p}\right) = 1$  from Ex. 5.26(a)), then

$$(ab)^{\frac{p-1}{4}} \equiv (a^2 f)^{\frac{p-1}{4}} \equiv a^{\frac{p-1}{2}} f^{\frac{p-1}{4}} \equiv f^{\frac{p-1}{4}} \pmod{p},$$

so

$$2^{\frac{p-1}{4}} f^{\frac{p-1}{4}} \equiv f^{\frac{ab}{2}} f^{\frac{p-1}{4}} \pmod{p}.$$

As  $f^{\frac{p-1}{4}} \not\equiv 0 \pmod{p}$ ,

$$2^{\frac{p-1}{4}} \equiv f^{\frac{ab}{2}} \pmod{p}.$$

□

**Ex. 5.28** Show that  $x^4 \equiv 2 \pmod{p}$  has a solution for  $p \equiv 1 \pmod{4}$  iff  $p$  is of the form  $A^2 + 64B^2$ .

*Proof.* If  $p \equiv 1 \pmod{4}$  and if there exists  $x \in \mathbb{Z}$  such that  $x^4 \equiv 2 \pmod{p}$ , then

$$2^{\frac{p-1}{4}} \equiv x^{p-1} \equiv 1 \pmod{p}.$$

From Ex. 5.27, where  $p = a^2 + b^2$ ,  $a$  odd, we know that

$$f^{\frac{ab}{2}} \equiv 2^{\frac{p-1}{4}} \equiv 1 \pmod{p}.$$

Since  $f^2 \equiv -1 \pmod{p}$ , the order of  $f$  modulo  $p$  is 4, thus  $4 \mid \frac{ab}{2}$ , so  $8 \mid ab$ .

As  $a$  is odd,  $8 \mid b$ , then  $p = A^2 + 64B^2$  (with  $A = a$ ,  $B = b/8$ ).

Reciprocally, if  $p = A^2 + 64B^2$ , then  $p \equiv 1 \pmod{4}$ .

Let  $a = A$ ,  $b = 8B$ . Then

$$2^{\frac{p-1}{4}} \equiv f^{\frac{ab}{2}} \equiv f^{4AB} \equiv (-1)^{2AB} \equiv 1 \pmod{p}.$$

As  $2^{\frac{p-1}{4}} \equiv 1 \pmod{p}$ ,  $x^4 \equiv 2 \pmod{p}$  has a solution in  $\mathbb{Z}$  (Prop. 4.2.1) : 2 is a biquadratic residue modulo  $p$ .

Conclusion :

$$\exists A \in \mathbb{Z}, \exists B \in \mathbb{Z}, p = A^2 + 64B^2 \iff (p \equiv 1 \pmod{4} \text{ and } \exists x \in \mathbb{Z}, x^4 \equiv 2 \pmod{p}).$$

Remark : the equation  $x^4 \equiv 2 \pmod{p}$  has also solutions if  $p \equiv -1 \pmod{8}$ .

Indeed, the equation  $x^4 \equiv 2 \pmod{p}$  has a solution in  $\mathbb{Z}$  iff  $2^{\frac{p-1}{d}} \equiv 1$ , where  $d = 4 \wedge (p-1) = 2$ , thus iff  $2^{\frac{p-1}{2}} \equiv 1 \pmod{p}$ , which is true as  $\left(\frac{2}{p}\right) = 1$ .

For instance,  $8^4 \equiv 2 \pmod{23}$ , with  $23 \equiv -1 \pmod{8}$ . □

**Ex. 5.29** Let  $(RR)$  be the number of pairs  $(n, n+1)$  in the set  $1, 2, 3, \dots, p-1$  such that  $n$  and  $n+1$  are both quadratic residues modulo  $p$ . Let  $(NR)$  be the number of pairs  $(n, n+1)$  in the set  $1, 2, 3, \dots, p-1$  such that  $n$  is a quadratic nonresidue and  $n+1$  is a quadratic residue. Similarly, define  $(RN)$  and  $(NN)$ . Determine the sums  $(RR) + (RN)$ ,  $(NR) + (NN)$ ,  $(RR) + (NR)$ , and  $(RN) + (NN)$ .

*Proof.* Let  $E$  the set of pairs  $(n, n+1) \in \mathbb{N}^2, 1 \leq n \leq p-2 : |E| = p-2$ .

Write  $RR$  the set of pairs  $(n, n+1)$  such that  $n$  and  $n+1$  are both a quadratic residues, and  $(RR) = |RR|$  its cardinality, and similar definitions for  $RN, NR, NN$ .

As  $E = RR \cup RN \cup NR \cup NN$  (disjoint union),

$$(RR) + (RN) + (NR) + (NN) = |E| = p-2.$$

•  $RR \cup RN$  is the set of pairs  $(n, n+1)$  in  $E$  such that  $n$  is a residue. Its cardinality is the number of residues in  $[1, p-2]$ , so the number of residues in  $[1, p-1]$ , minus  $s$ , where  $s = 1$  if  $p-1$  is a residue,  $s = 0$  otherwise. In both cases  $p \equiv 1, 3 \pmod{4}$ ,  $s = \frac{1+(-1)^{\frac{p-1}{2}}}{2}$ , and the total number of residues is  $(p-1)/2$ , so

$$(RR) + (RN) = \frac{p-1}{2} - s = \frac{p-1}{2} - \frac{1+(-1)^{\frac{p-1}{2}}}{2} = \frac{1}{2}(p-2 - (-1)^{\frac{p-1}{2}}).$$

• Similarly,  $(NR) + (NN)$  is the number of nonresidues in  $[1, p-1]$ , minus  $t$ , where  $t = 1$  if  $p-1$  is a nonresidue,  $t = 0$  otherwise :  $t = \frac{1-(-1)^{\frac{p-1}{2}}}{2}$ , so

$$(NR) + (NN) = \frac{1}{2}(p-2 + (-1)^{\frac{p-1}{2}})$$

(the sum of these two results is indeed  $p-2 = |E|$ ).

• As 1 is a residue,  $(RR) + (NR)$  is the number of residues in  $[1, p-1]$ , minus 1 :

$$(RR) + (NR) = \frac{p-1}{2} - 1.$$

•  $(RN) + (NN)$  is the number of nonresidues in  $[2, p-1]$ , equal to the number of residues in  $[1, p-1]$  :

$$(RN) + (NN) = \frac{p-1}{2}.$$

□

**Ex. 5.30** Show that  $(RR) + (NN) - (RN) - (NR) = \sum_{n=1}^{p-1} (n(n+1)/p)$ . Evaluate this sum and show that it is equal to  $-1$ . (Hint : The result of Exercise 8 is useful.)

*Proof.* Let  $\chi$  be the characteristic function of  $RR \cup NN$  : if  $1 \leq n \leq p-1$ ,  $\chi(n) = 1$  if  $n, n+1$  are both residues, or if  $n, n+1$  are both non residues. Then

$$\chi(n) = \frac{1}{2} \left( 1 + \left( \frac{n}{p} \right) \left( \frac{n+1}{p} \right) \right)$$

(if  $\chi(n) = 1$ ,  $\left( \frac{n}{p} \right) \left( \frac{n+1}{p} \right) = 1$ , and  $\left( \frac{n}{p} \right) \left( \frac{n+1}{p} \right) = -1$  otherwise.)

Similarly, let  $\chi'$  the characteristic function of  $RN \cup NR$  :  $\chi(n) = 1$  if exactly one of the integer  $n, n+1$  is a residue, 0 otherwise. Then

$$\chi'(n) = \frac{1}{2} \left( 1 - \left( \frac{n}{p} \right) \left( \frac{n+1}{p} \right) \right).$$

As each integer  $n$  between 1 and  $p-1$  brings the contribution 1 if  $n \in RR \cup NN$ , and  $-1$  if  $n \in RN \cup NR$ , then

$$\begin{aligned} (RR) + (NN) - (RN) - (NR) &= \sum_{n=1}^{p-1} (\chi(n) - \chi'(n)) \\ &= \frac{1}{2} \sum_{n=1}^{p-1} \left( 1 + \left( \frac{n(n+1)}{p} \right) \right) - \left( 1 - \left( \frac{n(n+1)}{p} \right) \right) \\ &= \sum_{n=1}^{p-1} \left( \frac{n(n+1)}{p} \right) \end{aligned}$$

To evaluate this sum  $S$ , note that  $4n(n+1) = (2n+1)^2 - 1$ , so

$$S = \sum_{n=1}^{p-1} \left( \frac{n(n+1)}{p} \right) = \sum_{n=1}^{p-1} \left( \frac{4n(n+1)}{4p} \right) = \sum_{n=1}^{p-1} \left( \frac{(2n+1)^2 - 1}{4p} \right).$$

This sum can be written  $S = \sum_{\bar{n} \in \mathbb{F}_p^*} ((2\bar{n}+1)^2 - 1)/p = \sum_{\bar{n} \in \mathbb{F}_p} ((2\bar{n}+1)^2 - 1)/p$ , since  $(0/p) = 0$ . As  $f : \mathbb{F}_p \rightarrow \mathbb{F}_p, \bar{n} \mapsto (2\bar{n}+1)$  is a bijection (2 is invertible in  $\mathbb{F}_p^*$ ),

$$\sum_{\bar{n} \in \mathbb{F}_p} \left( \frac{(2\bar{n}+1)^2 - 1}{p} \right) = \sum_{\bar{y} \in \mathbb{F}_p} \left( \frac{y^2 - 1}{p} \right) \quad (y = 2\bar{n}+1).$$

As  $p \nmid 1$ , the evaluation of this last sum is given in Exercise 5.8 :  $S = -1$ , so

$$(RR) + (NN) - (RN) - (NR) = \sum_{n=1}^{p-1} \left( \frac{n(n+1)}{p} \right) = -1.$$

□

**Ex. 5.31** Use the results of Exercises 29 and 30 to show that  $(RR) = \frac{1}{4}(p-4-\varepsilon)$ , where  $\varepsilon = (-1)^{(p-1)/2}$

*Proof.* To summarize the results of the Ex. 5.29 and 5.30,

$$\begin{aligned}(a)(RR) + (RN) + (NR) + (NN) &= p - 2 \\ (b)(RR) + (NN) - (RN) - (NR) &= -1\end{aligned}$$

and

$$\begin{aligned}(c)(RR) + (RN) &= \frac{1}{2} \left( p - 2 - (-1)^{\frac{p-1}{2}} \right) \\ (d)(RR) + (NR) &= \frac{p-1}{2} - 1\end{aligned}$$

The sum of (a) and (b) gives

$$(e)(RR) + (NN) = \frac{p-3}{2}.$$

The sum of (c),(d),(e) gives (using (a))

$$2(RR) + p - 2 = \frac{p-2}{2} + \frac{p-1}{2} + \frac{p-3}{2} - 1 - \frac{(-1)^{\frac{p-1}{2}}}{2},$$

so

$$\begin{aligned}2(RR) &= \frac{p-1}{2} + \frac{p-3}{2} - \frac{p-2}{2} - 1 - \frac{(-1)^{\frac{p-1}{2}}}{2} = \frac{p}{2} - 2 - \frac{(-1)^{\frac{p-1}{2}}}{2}, \\ (RR) &= \frac{1}{4}(p - 4 - \varepsilon), \text{ where } \varepsilon = (-1)^{\frac{p-1}{2}}.\end{aligned}$$

□

**Ex. 5.32** If  $p$  is an odd prime, show that  $(2/p) = \prod_{j=1}^{(p-1)/2} 2 \cos(2\pi j/p)$ . Use this to give another proof to Proposition 5.1.3.

*Proof.* Let  $p$  an odd prime number, and  $\zeta = e^{2i\pi/p}$  : then  $\zeta^p = 1$ .

Let

$$P = \prod_{j=0}^{p-1} (\zeta^j + \zeta^{-j}) = \prod_{j=0}^{p-1} 2 \cos(2\pi j/p).$$

$$\begin{aligned}P &= \zeta^0 \zeta^{-1} \dots \zeta^{-(p-1)} \prod_{j=0}^{p-1} (\zeta^{2j} + 1) \\ &= (\zeta^p)^{-(p-1)/2} \prod_{j=0}^{p-1} (\zeta^{2j} + 1) \\ &= \prod_{j=0}^{p-1} (\zeta^{2j} + 1)\end{aligned}$$

As  $\zeta^j$  depends only of the class  $\bar{j} \in \mathbb{F}_p$ , this product can be written

$$P = \prod_{\bar{j} \in \mathbb{F}_p} (\zeta^{2j} + 1) = \prod_{\bar{k} \in \mathbb{F}_p} (\zeta^k + 1) \quad (k = 2j),$$

since  $f : \mathbb{F}_p \rightarrow \mathbb{F}_p, x \mapsto 2x$  is a bijection. So

$$P = \prod_{k=0}^{p-1} (\zeta^k + 1).$$

Since  $\zeta^0 = 1, \zeta, \dots, \zeta^{p-1}$  are the roots of the polynomial  $f(x) = x^p - 1$ , then  $1 + \zeta^0, \dots, 1 + \zeta^{p-1}$  are the roots of  $g(x) = (x - 1)^p - 1 = f(x - 1)$ , so  $g(x) = \prod_{k=0}^{p-1} (x - (1 + \zeta^k))$ .

As  $g(0) = (-1)^p - 1 = -2 = (-1 - \zeta^0) \cdots (-1 - \zeta^{p-1}) = -\prod_{k=0}^{p-1} (\zeta^k + 1)$ , we obtain

$$P = \prod_{j=0}^{p-1} 2 \cos(2\pi j/p) = \prod_{k=0}^{p-1} (\zeta^k + 1) = 2,$$

so

$$\prod_{j=1}^{p-1} 2 \cos(2\pi j/p) = 1.$$

$$\begin{aligned} 1 &= \prod_{j=1}^{p-1} 2 \cos(2\pi j/p) \\ &= \prod_{j=1}^{(p-1)/2} 2 \cos(2\pi j/p) \prod_{j=(p+1)/2}^{p-1} 2 \cos(2\pi j/p) \\ &= \prod_{j=1}^{(p-1)/2} 2 \cos(2\pi j/p) \prod_{k=1}^{(p-1)/2} 2 \cos(2\pi - 2\pi k/p) \quad (k = p - j) \end{aligned}$$

As  $\cos(2\pi - \alpha) = \cos(\alpha)$ ,

$$1 = \left( \prod_{j=1}^{(p-1)/2} 2 \cos(2\pi j/p) \right)^2, \text{ so } \prod_{j=1}^{(p-1)/2} 2 \cos(2\pi j/p) = \pm 1$$

Case 1 : if  $1 \leq j \leq p/4, 0 \leq 2\pi j/p < \pi/2$ , so  $\cos(2\pi j/p) > 0$ ,

case 2 : if  $p/4 < j \leq (p-1)/2, \pi/2 < 2\pi j/p < \pi$ , so  $\cos(2\pi j/p) < 0$ .

In the first case,  $2 \leq 2j \leq (p-1)/2$  : the least residue of  $2j$  is positive. In the second case  $p/2 < 2j \leq p-1$  : the least residue of  $2j$  is negative.

Let  $\mu$  the number of negative least residues of the integer  $2j$ ,  $1 \leq j \leq (p-1)/2$  : we know from Gauss' Lemma that  $(2/p) = (-1)^\mu$ . As  $\mu$  is also the number of  $j$ ,  $1 \leq j \leq (p-1)/2$  such that  $\cos(2\pi j/p) > 0$ ,

$$\prod_{j=1}^{(p-1)/2} 2 \cos(2\pi j/p) = (-1)^\mu = \left(\frac{2}{p}\right).$$

If  $p \equiv 1 \pmod{8}$ ,  $p = 8q + 1, q \in \mathbb{N}$ . For  $1 \leq j \leq (p-1)/2$ ,

$$\cos(2\pi j/p) < 0 \iff p/4 \leq j \leq (p-1)/2 \iff 2q + 1 \leq j \leq 4q,$$

so  $\mu = 2q$  and  $(2/p) = (-1)^\mu = 1$ .

If  $p \equiv -1 \pmod{8}$ ,  $p = 8q - 1$ ,  $q \in \mathbb{N}^*$ .

$$\cos(2\pi j/p) < 0 \iff p/4 \leq j \leq (p-1)/2 \iff 2q \leq j \leq 4q-1,$$

so  $\mu = 2q$  and  $(2/p) = (-1)^\mu = 1$ .

If  $p \equiv 3 \pmod{8}$ ,  $p = 8q + 3$ ,  $q \in \mathbb{N}$ .

$$\cos(2\pi j/p) < 0 \iff p/4 \leq j \leq (p-1)/2 \iff 2q+1 \leq j \leq 4q+1,$$

so  $\mu = 2q+1$  and  $(2/p) = (-1)^\mu = 1$ .

If  $p \equiv -3 \pmod{8}$ ,  $p = 8q - 3$ ,  $q \in \mathbb{N}^*$ ,

$$\cos(2\pi j/p) < 0 \iff p/4 \leq j \leq (p-1)/2 \iff 2q \leq j \leq 4q-2,$$

so  $\mu = 2q-1$  and  $(2/p) = (-1)^\mu = 1$ . □

**Ex. 5.33** Use Proposition 5.3.2 to derive the quadratic character of  $-1$ .

*Proof.* Let  $f(z) = e^{2\pi iz} - e^{-2\pi iz}$ . If  $p$  is an odd prime,  $a \in \mathbb{Z}$ , and  $p \nmid a$ , we know from Prop. 5.3.2 that

$$\prod_{l=1}^{(p-1)/2} f\left(\frac{la}{p}\right) = \left(\frac{a}{p}\right) \prod_{l=1}^{(p-1)/2} f\left(\frac{l}{p}\right).$$

For  $a = -1$ , as  $f(-z) = -f(z)$ ,

$$\begin{aligned} \left(\frac{-1}{p}\right) \prod_{l=1}^{(p-1)/2} f\left(\frac{l}{p}\right) &= \prod_{l=1}^{(p-1)/2} f\left(\frac{-l}{p}\right) \\ &= (-1)^{(p-1)/2} \prod_{l=1}^{(p-1)/2} f\left(\frac{l}{p}\right) \end{aligned}$$

Moreover  $f(z) = 0 \iff e^{4\pi iz} = 1 \iff 4\pi iz = 2ki\pi, k \in \mathbb{Z} \iff z = k/2, k \in \mathbb{Z}$ , so, if  $l \in \mathbb{Z}$ ,  $f\left(\frac{l}{p}\right) = 0 \iff l/p = k/2, k \in \mathbb{Z} \iff p \mid 2l \iff p \mid l$ . For  $1 \leq l < p$ , this is impossible, so  $\prod_{l=1}^{(p-1)/2} f\left(\frac{l}{p}\right) \neq 0$ . Consequently,

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}.$$

□

**Ex. 5.34** If  $p$  is an odd prime distinct from 3, show that

$$\left(\frac{3}{p}\right) = \prod_{j=1}^{(p-1)/2} \left(3 - 4\sin^2\left(\frac{2\pi j}{p}\right)\right).$$

*Proof.* Let  $p$  an odd prime number,  $p \neq 3$  and  $\zeta = e^{2i\pi/p}$ .

$$\begin{aligned} 3 - 4 \sin^2 \left( \frac{2\pi j}{p} \right) &= 3 - 4 \left( \frac{\zeta^j - \zeta^{-j}}{2i} \right)^2 \\ &= 3 + \zeta^{2j} + \zeta^{-2j} - 21 \\ &= 1 + \zeta^{2j} + \zeta^{-2j} \\ &= 1 + 2 \cos \left( \frac{4\pi j}{p} \right) \end{aligned}$$

(Or  $\cos(2\alpha) = 1 - 2 \sin^2 \alpha$ , so  $3 - 4 \sin^2 \alpha = 1 + 2 \cos \alpha$ .)

Let

$$P = \prod_{j=1}^{p-1} \left( 3 - 4 \sin^2 \left( \frac{2\pi j}{p} \right) \right) = \prod_{\bar{j} \in \mathbb{F}_p^*} \left( 3 - 4 \sin^2 \left( \frac{2\pi j}{p} \right) \right).$$

Then

$$\begin{aligned} P &= \prod_{\bar{j} \in \mathbb{F}_p^*} (1 + \zeta^{2j} + \zeta^{-2j}) \\ &= \prod_{\bar{k} \in \mathbb{F}_p^*} (1 + \zeta^k + \zeta^{-k}) \quad (k = 2j) \end{aligned}$$

since  $f : \mathbb{F}_p \rightarrow \mathbb{F}_p, \bar{j} \mapsto 2\bar{j}$  is a bijection. So

$$\begin{aligned} P &= \prod_{k=0}^{p-1} \zeta^{-k} (1 + \zeta^k + \zeta^{2k}) \\ &= 3 \prod_{k=1}^{p-1} \zeta^{-k} (1 + \zeta^k + \zeta^{2k}) \\ &= 3 \prod_{k=0}^{p-1} \zeta^{-k} \frac{\prod_{k=1}^{p-1} (1 - \zeta^{3k})}{\prod_{k=1}^{p-1} (1 - \zeta^k)} \end{aligned}$$

$\prod_{k=0}^{p-1} \zeta^{-k} = (\zeta^p)^{-(p-1)/2} = 1$ . Moreover,  $\prod_{k=1}^{p-1} (1 - \zeta^{3k}) = \prod_{k=1}^{p-1} (1 - \zeta^k)$ , since  $\bar{k} \mapsto 3\bar{k}$  is a bijection in  $\mathbb{F}_p^*$ , so  $P = 3$ , and consequently

$$\begin{aligned} 1 &= \prod_{j=1}^{p-1} \left( 3 - 4 \sin^2 \left( \frac{2\pi j}{p} \right) \right) \\ &= \prod_{j=1}^{(p-1)/2} \left( 3 - 4 \sin^2 \left( \frac{2\pi j}{p} \right) \right) \prod_{j=(p+1)/2}^{p-1} \left( 3 - 4 \sin^2 \left( \frac{2\pi j}{p} \right) \right) \\ &= \prod_{j=1}^{(p-1)/2} \left( 3 - 4 \sin^2 \left( \frac{2\pi j}{p} \right) \right) \prod_{k=1}^{(p-1)/2} \left( 3 - 4 \sin^2 \left( \frac{2\pi(k-j)}{p} \right) \right) \quad (k = p-j) \\ &= \left[ \prod_{j=1}^{(p-1)/2} \left( 3 - 4 \sin^2 \left( \frac{2\pi j}{p} \right) \right) \right]^2 \end{aligned}$$

So  $\prod_{j=1}^{(p-1)/2} \left( 3 - 4 \sin^2 \left( \frac{2\pi j}{p} \right) \right) = \pm 1$ .



Let  $\nu$  the number of negative factors in this product.

If  $1 \leq j \leq (p-1)/2$ , then  $0 < 4\pi j/p < 2\pi$ .

$$\begin{aligned}
1 + 2 \cos \frac{4\pi j}{p} < 0 &\iff \cos \frac{4\pi j}{p} < \cos \frac{2\pi}{3} \\
&\iff \frac{2\pi}{3} < \frac{4\pi j}{p} < \frac{4\pi}{3} \\
&\iff \frac{p}{6} < j < \frac{p}{3} \\
&\iff \frac{p}{2} < 3j < p
\end{aligned}$$

Let  $\mu$  the number of integers  $j, 1 \leq j \leq (p-1)/2$  such their least remainder is negative. Since  $3 \leq 3j \leq 3(p-1)/2$  and  $3j \neq p/2$ , these  $j$  are such that  $\frac{p}{2} < 3j < p$ , so  $\mu = \nu$ . Therefore

$$\prod_{j=1}^{(p-1)/2} \left( 3 - 4 \sin^2 \left( \frac{2\pi j}{p} \right) \right) = (-1)^\nu = (-1)^\mu = \left( \frac{3}{p} \right).$$

□

**Ex. 5.35** Use the preceding exercise to show that 3 is a square modulo  $p$  iff  $p$  is congruent to 1 or  $-1$  modulo 12.

*Proof.* We know from Ex. 5.34 that  $\nu = \text{Card} \{j \in [1, (p-1)/2] \mid p/2 \leq 3j < p\} = \mu$ .  $\nu$  is the number of  $j$  such that  $p/6 \leq j < p/3$ , so  $\nu = \lfloor p/3 \rfloor - \lfloor p/6 \rfloor$ .

If  $p = 12k + 1$ ,  $\nu = \lfloor p/3 \rfloor - \lfloor p/6 \rfloor = 4k - 2k = 2k : (3/p) = (-1)^\nu = 1$

If  $p = 12k + 5$ ,  $\nu = \lfloor p/3 \rfloor - \lfloor p/6 \rfloor = 4k + 1 - 2k = 2k + 1 : (3/p) = (-1)^\nu = -1$

If  $p = 12k - 5$ ,  $\nu = \lfloor p/3 \rfloor - \lfloor p/6 \rfloor = 4k - 2 - (2k - 1) = 2k - 1 : (3/p) = (-1)^\nu = -1$

If  $p = 12k - 1$ ,  $\nu = \lfloor p/3 \rfloor - \lfloor p/6 \rfloor = 4k - 1 - (2k - 1) = 2k : (3/p) = (-1)^\nu = 1$

3 is a square modulo  $p$  ( $p \neq 2, p \neq 3$ ) iff  $p$  is congruent to 1 or  $-1$  modulo 12. □

**Ex. 5.36** Show that part (c) of Proposition 5.2.2 is true if  $a$  is negative and  $b$  is positive (both still odd).

As said by Adam Michalik, the Jacobi symbol  $\left(\frac{a}{b}\right)$  only defined for positive  $b$ , so the question, which concerns  $\left(\frac{a}{b}\right)\left(\frac{b}{a}\right)$ ,  $a < 0$  makes no sense.

To give sense to this question, we must substitute the Kronecker symbol to the Jacobi symbol. The Kronecker symbol (not defined in Ireland-Rosen) is the usual extension of Jacobi symbol (see for instance [Henri Cohen] A course in computational algebraic number theory, [Henri Cohen] Number theory (vol. 1), or [Harvey Cohn] Advanced number theory).

We define Kronecker (or Kronecker-Jacobi) symbol  $\left(\frac{a}{b}\right)$  for any  $a$  and  $b$  in  $\mathbb{Z}$  in the following way.

(1) If  $b = 0$ , then  $\left(\frac{a}{0}\right) = 1$  if  $a = \pm 1$ , and is equal to 0 otherwise.

(2) For  $b \neq 0$ , write  $b = \prod p$ , where the  $p$  are not necessarily distinct primes (including 2), or  $p = -1$  to take care of the sign. Then we set

$$\left(\frac{a}{b}\right) = \prod \left(\frac{a}{p}\right),$$

where  $\left(\frac{a}{p}\right)$  is the Legendre symbol defined above for  $p > 2$ , and where we define

$$\left(\frac{a}{2}\right) = \begin{cases} 0 & \text{if } a \text{ is even} \\ (-1)^{(a^2-1)/8} & \text{if } a \text{ is odd,} \end{cases}$$

and also

$$\left(\frac{a}{-1}\right) = \begin{cases} 1 & \text{if } a \geq 0 \\ -1 & \text{if } a < 0 \end{cases}$$

*Proof.* Suppose  $a < 0, b > 0$ , both odd. Let  $a = -A, A > 0, A = p_1 p_2 \cdots p_k$ , where the  $p_i$  are not necessarily distinct primes. Then

$$\begin{aligned} \left(\frac{a}{b}\right) \left(\frac{b}{a}\right) &= \left(\frac{-A}{b}\right) \left(\frac{b}{-A}\right) \\ \left(\frac{-A}{b}\right) &= \left(\frac{-1}{b}\right) \left(\frac{A}{b}\right) = (-1)^{(b-1)/2} \left(\frac{A}{b}\right) \\ \left(\frac{b}{-A}\right) &= \left(\frac{b}{-1}\right) \left(\frac{b}{p_1}\right) \cdots \left(\frac{b}{p_k}\right) = \left(\frac{b}{A}\right) \end{aligned}$$

so, from Prop. 5.2.2, as  $A, b$  are odd and positive,

$$\begin{aligned} \left(\frac{a}{b}\right) \left(\frac{b}{a}\right) &= (-1)^{\frac{b-1}{2}} \left(\frac{A}{b}\right) \left(\frac{b}{A}\right) \\ &= (-1)^{\frac{b-1}{2}} (-1)^{\frac{A-1}{2} \frac{b-1}{2}} \\ &= (-1)^{\frac{b-1}{2} [1 + \frac{A-1}{2}]} \\ &= (-1)^{\frac{b-1}{2} \frac{1+A}{2}} \\ &= (-1)^{\frac{b-1}{2} \frac{a-1}{2}} \end{aligned}$$

So the law of quadratic reciprocity remains valid for the Kronecker symbol when  $a$  is negative ( $b > 0, a, b$  both odd).  $\square$

**Ex. 5.37** Show that if  $a$  is negative, then  $p \equiv q \pmod{4a}$ ,  $p \nmid a$  implies  $(a/p) = (a/q)$ .

*Proof.* Write  $a = -A, A > 0$ . As  $p \equiv q \pmod{4a}$ , we know from Prop. 5.3.3. (b) that  $(A/p) = (A/q)$ .

Moreover,

$$\begin{aligned} \left(\frac{a}{p}\right) &= \left(\frac{-A}{p}\right) = (-1)^{(p-1)/2} \left(\frac{A}{p}\right) \\ \left(\frac{a}{q}\right) &= \left(\frac{-A}{q}\right) = (-1)^{(q-1)/2} \left(\frac{A}{q}\right) \end{aligned}$$

As  $p \equiv q \pmod{4a}$ ,  $p = q + 4ak, k \in \mathbb{Z}$ , so

$$(-1)^{(p-1)/2} = (-1)^{(q+4ak-1)/2} = (-1)^{(q-1)/2},$$

so  $(a/p) = (a/q)$ .  $\square$

**Ex. 5.38** Let  $p$  be an odd prime. Derive the quadratic character of 2 modulo  $p$  by verifying the following steps, involving the Jacobi symbol:

$$\left(\frac{2}{p}\right) = \left(\frac{8-p}{p}\right) = \left(\frac{p}{p-8}\right) = \left(\frac{8}{p-8}\right) = \left(\frac{2}{p-8}\right).$$

Generalize the argument to show that

$$\left(\frac{a}{p}\right) = \left(\frac{a}{p-4a}\right), \quad a > 0, p \nmid a.$$

(As in Ex. 5.36, since  $8-p$  or  $p-8$  is negative, we interpret  $(a/b)$  as the Kronecker symbol : see definition in Ex. 5.36.)

*Proof.* As  $(2^2/p) = 1$  and  $8-p \equiv 8 \pmod{p}$ ,

$$\left(\frac{2}{p}\right) = \left(\frac{2^2}{p}\right) \left(\frac{2}{p}\right) = \left(\frac{8}{p}\right) = \left(\frac{8-p}{p}\right).$$

As  $p$  and  $8-p$  are odd numbers and  $p > 0$ , from the extension of the law of quadratic reciprocity to  $a < 0$  proved in Ex. 5.36, we obtain

$$\left(\frac{8-p}{p}\right) = (-1)^{\frac{7-p}{2} \frac{p-1}{2}} \left(\frac{p}{8-p}\right).$$

Moreover

$$(7-p)(p-1) \equiv (-1-p)(p-1) = 1-p^2 \pmod{8}$$

As  $p = 2k+1$  is odd,  $p^2 = 4k^2 + 4k + 1 = 8 \frac{k(k+1)}{2} + 1 \equiv 1 \pmod{8}$ , so  $(7-p)(p-1) \equiv 0 \pmod{8}$  and  $\frac{7-p}{2} \frac{p-1}{2}$  is even, so

$$\left(\frac{8-p}{p}\right) = \left(\frac{p}{8-p}\right).$$

As  $p > 0$ ,  $\left(\frac{p}{-1}\right) = 1$ , thus  $\left(\frac{p}{8-p}\right) = \left(\frac{p}{-1}\right) \left(\frac{p}{p-8}\right) = \left(\frac{p}{p-8}\right)$  (with the same argument, this is also true for the 3 odd primes such that  $8-p > 0$ ), so

$$\left(\frac{8-p}{p}\right) = \left(\frac{p}{p-8}\right).$$

□

As  $p \equiv 8 \pmod{p-8}$ ,  $\left(\frac{p}{p-8}\right) = \left(\frac{8}{p-8}\right)$ , and since  $8 = 2^2 \times 2$ ,  $\left(\frac{8}{p-8}\right) = \left(\frac{2}{p-8}\right)$ . We have proved for all odd primes  $p$  that

$$\left(\frac{2}{p}\right) = \left(\frac{8-p}{p}\right) = \left(\frac{p}{p-8}\right) = \left(\frac{8}{p-8}\right) = \left(\frac{2}{p-8}\right).$$

The preceding arguments remain valid if we replace the odd prime  $p$  by any odd positive integer. So with an immediate induction, we see that for all  $k \in \mathbb{N}$ ,

$$\left(\frac{2}{p}\right) = \left(\frac{2}{p-8k}\right).$$

So the quadratic character of 2 modulo  $p$  depends only of the class of  $p$  modulo 8.

If  $p \equiv 1 \pmod{8}$ ,  $\left(\frac{2}{p}\right) = \left(\frac{2}{1}\right) = 1$ .

If  $p \equiv -1 \pmod{8}$ ,  $\left(\frac{2}{p}\right) = \left(\frac{2}{-1}\right) = 1$ .

If  $p \equiv \pm 3 \pmod{8}$ ,  $\left(\frac{2}{p}\right) = \left(\frac{2}{\pm 3}\right) = -1$ .

Generalization : let  $a > 0$  and  $p$  an odd positive integer such that  $p \wedge a = 1$  (not necessarily prime).

$$\left(\frac{a}{p}\right) = \left(\frac{4ap}{p}\right) = \left(\frac{4a-p}{p}\right) = (-1)^{\frac{4a-p-1}{2} \frac{p-1}{2}} \left(\frac{p}{4a-p}\right).$$

$(4a-p-1)(p-1) = 4a(p-1) + 1 - p^2 \equiv 0 \pmod{8}$ , so

$$\left(\frac{a}{p}\right) = \left(\frac{p}{4a-p}\right).$$

As  $\left(\frac{p}{-1}\right) = 1$ ,

$$\left(\frac{p}{4a-p}\right) = \left(\frac{p}{p-4a}\right).$$

Since  $p \equiv 4a \pmod{p} - 4a$ , and 4 is a square,

$$\left(\frac{p}{p-4a}\right) \equiv \left(\frac{4a}{p-4a}\right) = \left(\frac{a}{p-4a}\right).$$

We have proved

$$\left(\frac{a}{p}\right) = \left(\frac{4a-p}{p}\right) = \left(\frac{p}{p-4a}\right) = \left(\frac{4a}{p-4a}\right) = \left(\frac{a}{p-4a}\right).$$

By induction, for all  $k \geq 0$ ,  $\left(\frac{a}{p}\right) = \left(\frac{a}{p-4a}\right)$ , so  $\left(\frac{a}{p}\right)$  depends only of the class of  $p$  modulo  $4a$ .

## Chapter 6

**Ex. 6.1** Show that  $\sqrt{2} + \sqrt{3}$  is an algebraic integer.

*Proof.* Let  $x = \sqrt{2} + \sqrt{3}$ . Then  $x^2 = 5 + 2\sqrt{6}$ .

$(x^2 - 5)^2 = (2\sqrt{6})^2 = 24$ , so  $x^4 - 10x^2 + 1 = 0$  :  $x$  is an algebraic integer.  $\square$

**Ex. 6.2** Let  $\alpha$  be an algebraic number. Show that there's an integer  $n$  such that  $n\alpha$  is an algebraic integer.

(0 is a valid answer to this sentence ! More seriously, we search a *positive* integer  $n$ .)

*Proof.* Let  $\alpha$  an algebraic number. By definition, there exist  $a_0, a_1, \dots, a_n \in \mathbb{Z}, a_n \neq 0$ , such that

$$a_n \alpha^n + a_{n-1} \alpha^{n-1} + \dots + a_k \alpha^k + \dots + a_0 = 0.$$

(Up to multiply this equation by  $-1$ , we can suppose that  $a_n > 0$ ).

Multiplying by  $a_n^{n-1}$ , we obtain

$$a_n^n \alpha^n + a_n^{n-1} a_{n-1} \alpha^{n-1} + \dots + a_n^{n-1} a_k \alpha^k + \dots + a_n^{n-1} a_0 = 0.$$

So

$$(a_n \alpha)^n + a_{n-1} (a_n \alpha)^{n-1} + \cdots + a_n^{n-k-1} a_k (a_n \alpha)^k + \cdots + a_n^{n-1} a_0 = 0.$$

Soit  $p(x) = x^n + \sum_{k=0}^{n-1} a_n^{n-k-1} a_k x^k$ . Then  $p(x) \in \mathbb{Z}[x]$ ,  $p(x)$  is monic, and  $p(a_n x) = 0$ .

So  $a_n x$  is an algebraic integer, with  $m = a_n \in \mathbb{N}^*$ .

Conclusion : if  $\alpha$  is an algebraic number, there exists an integer  $m > 0$  such that  $m\alpha$  is an algebraic integer.  $\square$

**Ex. 6.3** If  $\alpha$  and  $\beta$  are algebraic integers, prove that any solution to  $f(x) = x^2 + \alpha x + \beta = 0$  is an algebraic integer. Generalize this result.

*Proof.* Let  $\gamma$  a root of  $x^2 + \alpha x + \beta$ , where  $\alpha, \beta$  verify :

$$\alpha^n + r_1 \alpha^{n-1} + \cdots + r_n = 0, \quad r_i \in \mathbb{Z},$$

$$\beta^m + s_1 \beta^{m-1} + \cdots + s_m = 0, \quad s_j \in \mathbb{Z}.$$

Let  $V$  the set of linear combinations with integer coefficients of

$$\alpha^i \beta^j \gamma^k, 0 \leq i < n, 0 \leq j < m, 0 \leq k < 2.$$

Then  $V$  is a finitely generated  $\mathbb{Z}$ -module.

Moreover, for all  $\delta \in V, \gamma \delta \in V$ . Indeed, every  $\delta \in V$  is a linear combination with coefficients in  $\mathbb{Z}$  of  $\alpha^i \beta^j, \alpha^i \beta^j \gamma$ , and

$$\gamma(\alpha^i \beta^j) = \alpha^i \beta^j \gamma \in V$$

$$\gamma(\alpha^i \beta^j \gamma) = \alpha^i \beta^j \gamma^2 = \alpha^i \beta^j (-\alpha \gamma - \beta) = -\alpha^{i+1} \beta^j \gamma - \alpha^i \beta^{j+1} \in V.$$

(if  $i+1 = n$ , we replace  $\alpha^{i+1} = \alpha^n$  by  $-\sum_{k=1}^{n-1} r_k \alpha^{n-k}$ , and a similar replacement if  $j+1 = m$ .)

As for each  $\gamma \in V$ , where  $V$  is a finitely generated  $\mathbb{Z}$ -module,  $x\gamma \in V$ , so  $\gamma$  is an algebraic integer (Proposition 6.1.4).

More generally, if  $\gamma^n + \alpha_1 \gamma^{n-1} + \cdots + \alpha_n = 0$ , where the  $\alpha_i$  are algebraic integers, then  $x$  is an algebraic integer.  $\square$

**Ex. 6.4** A polynomial  $f(x) \in \mathbb{Z}[x]$  is said to be primitive if the greatest common divisor of its coefficients is 1. Prove that product of primitive polynomials is also primitive.

### Solution 1

*Proof.* Let  $p(x) = \sum_{i=0}^n a_i x^i, q(x) = \sum_{j=0}^m b_j x^j$  two primitive polynomials, and  $p$  a prime number. There exist a coefficient of  $p(x)$  (and of  $q(x)$ ) not divisible by  $p$ . Let

$$i_0 = \min\{i \in [0, n] \mid a_i \not\equiv 0 \pmod{p}\}$$

$$j_0 = \min\{j \in [0, m] \mid b_j \not\equiv 0 \pmod{p}\}$$

Let  $p(x)q(x) = \sum_{k=0}^{n+m} c_k x^k$ . Then  $c_k = \sum_{i+j=k} a_i b_j, k = 0, \dots, n+m$ . Then

$$c_{i_0+j_0} = \sum_{i+j=i_0+j_0} a_i b_j.$$

- If  $i < i_0$ , then  $a_i \equiv 0 \pmod{p}$ .
- If  $i > i_0$ , then  $j < j_0$  and  $b_j \equiv 0 \pmod{p}$ .

In the two cases  $a_i b_j \equiv 0 \pmod{p}$ , so  $c_{i_0+j_0} \equiv a_{i_0} b_{j_0} \pmod{p}$ , so  $c_{j_0} \not\equiv 0 \pmod{p}$  : as it's true for all primes  $p$ , the polynomial  $p(x)q(x)$  is primitive.  $\square$

## Solution 2

*Proof.* Let

$$\varphi : \begin{cases} \mathbb{Z}[x] & \rightarrow \mathbb{F}_p[x] \\ p(x) = a_0 + \cdots + a_n x^n & \mapsto \bar{p}(x) = \bar{a}_0 + \cdots + \bar{a}_n x^n, \end{cases}$$

where  $\bar{a}_i$  is the class of  $a_i$  in  $\mathbb{F}_p$ .  $\varphi$  is a ring homomorphism.

As  $\mathbb{F}_p[x]$  is an integrity domain, if  $p(x), q(x)$  are both primitive,

$$\overline{p(x)} \neq 0, \overline{q(x)} \neq 0 \Rightarrow \overline{p(x)q(x)} = \overline{p(x)} \overline{q(x)} \neq 0.$$

As  $\overline{p(x)q(x)} \neq 0$  in all fields  $\mathbb{F}_p$ ,  $p(x)q(x)$  is a primitive polynomial.  $\square$

**Ex. 6.5** Let  $\alpha$  be an algebraic integer and  $f(x) \in \mathbb{Q}[x]$  be the monic polynomial of least degree such that  $f(\alpha) = 0$ . Use Exercise 6.4 to show that  $f(x) \in \mathbb{Z}[x]$ .

*Proof.* As  $\alpha$  is an algebraic integer, there exists a monic polynomial  $h(x) \in \mathbb{Z}[x]$  such that  $h(\alpha) = 0$ . As  $f(x) \in \mathbb{Q}[x]$  is the minimal polynomial of  $\alpha$ , and  $h(\alpha) = 0$ ,  $f(x)$  divides  $h(x)$  in  $\mathbb{Q}[x]$ .

(quick reminder :  $h(x) = q(x)f(x) + r(x)$ ,  $q(x), r(x) \in \mathbb{Q}[x]$ ,  $\deg(r(x)) < \deg(f(x))$  or  $r(x) = 0$ . As  $r(\alpha) = 0$  and  $f(x) \in \mathbb{Q}[x]$  is the monic polynomial of least degree such that  $f(\alpha) = 0$ ,  $r = 0$  so  $f(x) \mid h(x)$ ).

So there exists  $g(x) \in \mathbb{Q}[x]$  such that  $h(x) = f(x)g(x)$ . As  $h(x), f(x)$  are both monic,  $g(x)$  is also monic.

Let  $d \in \mathbb{Z}, d \neq 0$  such that  $df(x) = \sum_{i=0}^m a_i x^i \in \mathbb{Z}[x]$ , and  $c = a_1 \wedge a_2 \wedge \cdots \wedge a_m$ ,  $a_i = cb_i$ , with  $b_1 \wedge b_2 \wedge \cdots \wedge b_m = 1$ , so  $f(x) = \frac{c}{d} f_1(x)$ , with  $f_1$  is primitive. Similarly  $g(x) = \frac{s}{t} g_1(x)$ ,  $s, t \in \mathbb{Z}$ ,  $g_1(x)$  primitive.

So  $h(x) = \frac{cs}{dt} f_1(x)f_2(x) = \frac{u}{v} f_1(x)f_2(x)$ , where  $u \wedge v = 1$ . The polynomial  $f_1(x)f_2(x) = \sum_{k=0}^r c_k x^k$  is primitive (Ex. 6.4). As  $vh(x)(x) = uf_1(x)f_2(x)$ ,  $c \mid uc_k$ , and  $u \wedge v = 1$ , thus  $v \mid c_k, k = 0, 1, \dots, r$ . As  $c_1 \wedge \cdots \wedge c_r = 1$ ,  $v \mid 1$ , so  $v = \pm 1$ .  $h(x) = uf_1(x)f_2(x)$  is monic, thus  $u = 1$ , and  $f_1, f_2$  are monic. From  $f(x) = \frac{c}{d} f_1(x)$  we deduce  $\frac{c}{d} = 1$  and  $f(x) = f_1(x) \in \mathbb{Z}[x]$ .

Conclusion : if  $f(x)$  is the minimal polynomial of an algebraic integer  $\alpha$ ,  $f \in \mathbb{Z}[x]$ .  $\square$

**Ex. 6.6** Let  $x^2 + mx + n \in \mathbb{Z}[x]$  be irreducible, and  $\alpha$  be a root. Show that  $\mathbb{Q}[\alpha] = \{r + s\alpha : r, s \in \mathbb{Q}\}$  is a ring (in fact, it is a field). Let  $m^2 - 4n = D_0^2 D$ , where  $D$  is square-free. Show that  $\mathbb{Q}[\alpha] = \mathbb{Q}[\sqrt{D}]$ .

*Proof.* By definition, for all  $z \in \mathbb{C}, z \in \mathbb{Q}[\alpha] \iff \exists P \in \mathbb{Q}[x], z = P(\alpha)$ .

The Euclidean division gives  $P = Q_1(x^2 + mx + n) + R(x)$ ,  $Q_1, R \in \mathbb{Q}[x]$ ,  $\deg(R) < 2$ , so  $R = rx + s$ ,  $r, s \in \mathbb{Q}$ . So  $z = Q_1(\alpha)(\alpha^2 + m\alpha + n) + r\alpha + s = r\alpha + s$  :

$$\mathbb{Q}(\alpha) = \{z \in \mathbb{C} \mid \exists r \in \mathbb{Q}, \exists s \in \mathbb{Q}, z = r + s\alpha\}.$$

•  $\mathbb{Q}[\alpha] \subset \mathbb{C}$ , where  $(\mathbb{C}, +, \times)$  is a field.  $1 \in \mathbb{Q}[\alpha]$  ( $1 = P_0(\alpha)$ , where  $P_0$  is the constant polynomial 1).

• Let  $\beta, \gamma \in \mathbb{Q}[\alpha] : \beta = P(\alpha), \gamma = Q(\alpha)$ , where  $P, Q$  are in  $\mathbb{Q}[x]$ . Then  $\alpha - \beta = P(\alpha) - Q(\alpha) = R(\alpha)$ , where  $R = P - Q \in \mathbb{Q}[x]$ , and  $\alpha\beta = P(\alpha)Q(\alpha) = S(\alpha)$ , where  $S = PQ \in \mathbb{Q}[x]$ . So  $\alpha - \beta \in \mathbb{Q}[\alpha], \alpha\beta \in \mathbb{Q}[\alpha]$ . So  $\mathbb{Q}[\alpha]$  is a subring of  $(\mathbb{C}, +, \times)$ .

• Let  $\beta = P(\alpha) \in \mathbb{Q}[\alpha], P \in \mathbb{Q}[x]$  and  $\beta \neq 0$ . As  $\beta \neq 0, Q = x^2 + mx + n \nmid P$ .

Let  $D \in \mathbb{Q}[x]$  such that  $D \mid P, D \mid Q$ . As  $Q$  is irreducible by hypothesis,  $D = \lambda$  or  $D = \lambda Q, \lambda \in \mathbb{C}^*$  ( $D$  is an associate of 1 or  $Q$ ). If  $D = \lambda Q$ , then  $Q \mid D$ , and  $D \mid P$ , so  $Q \mid P$ . Since  $Q(\alpha) = 0$ , this implies  $\beta = P(\alpha) = 0$ , in contradiction with the definition of  $\beta$ . So  $D = \lambda \mid 1$ . Therefore  $P \wedge Q = 1$ .

From Bézout's theorem, there exist polynomials  $U, V \in \mathbb{Q}[x]$  such that  $UP + VQ = 1$ . As  $\mathbb{Q}(\alpha) = 0, U(\alpha)P(\alpha) = 1$  and  $\gamma = U(\alpha) \in \mathbb{Q}[\alpha]$  is such that  $\gamma\beta = 1$ . Therefore  $\mathbb{Q}[\alpha]$  is a subfield of  $(\mathbb{C}, +, \times)$  (and  $\mathbb{Q}(\alpha) = \mathbb{Q}[\alpha]$ ).

As  $x^2 + mx + n$  is irreducible,  $\Delta = m^2 - 4n \neq 0$  (if not,  $x^2 + mx + n = (x + m/2)^2 - (m^2 - 4n)/4 = (x + m/2)^2$  is not irreducible). So  $\Delta \in \mathbb{Z} \setminus \{0\}$  can be written  $\Delta = m^2 - 4n = D_0^2 D$ , where  $D$  is square-free (positive or negative),  $D \neq 0, D_0 \neq 0$ .

$\alpha = -\frac{m}{2} + \varepsilon \frac{\sqrt{\Delta}}{2}, \varepsilon = \pm 1$ , so  $\alpha = -\frac{m}{2} + \varepsilon D_0 \frac{\sqrt{D}}{2}$ , thus  $\alpha \in \mathbb{Q}[\sqrt{D}]$  and  $\mathbb{Q}[\alpha] \subset \mathbb{Q}[\sqrt{D}]$ .

As  $D_0 \neq 0, \sqrt{D} = \varepsilon \frac{2\alpha + m}{D_0} \in \mathbb{Q}[\alpha]$ , so  $\mathbb{Q}[\sqrt{D}] \subset \mathbb{Q}[\alpha]$  :

$$\mathbb{Q}[\alpha] = \mathbb{Q}[\sqrt{D}].$$

□

**Ex. 6.7** (continuation) If  $D \equiv 2, 3 \pmod{4}$ , show that all the algebraic integers in  $\mathbb{Q}[\sqrt{D}]$  have the form  $a + b\sqrt{D}$ , where  $a, b \in \mathbb{Z}$ . If  $D \equiv 1 \pmod{4}$ , show that all the algebraic integers in  $\mathbb{Q}[\sqrt{D}]$  have the form  $a + b((-1 + \sqrt{D})/2)$ , where  $a, b \in \mathbb{Z}$ .

*Proof.* (We write  $\overline{\mathbb{Z}}$  the ring of algebraic integers in  $\mathbb{C}$ , and  $\mathcal{O}_K$  (or  $\mathbb{Z}_K$ ) the ring of algebraic integers in the field  $K$ .)

If  $D = 1, \mathbb{Q}[\sqrt{D}] = \mathbb{Q}$ . If  $D \neq 1$ , as  $D$  is square-free,  $D$  is not a square, so  $\sqrt{D}$  is irrational.

Let  $\gamma = r + s\sqrt{D} \in \mathbb{Q}[\sqrt{D}]$  ( $r, s \in \mathbb{Q}$ ) an algebraic integer of  $\mathbb{Q}[\sqrt{D}]$  ( $D \in \mathbb{Z}, D$  square-free).  $(\gamma - r)^2 = s^2 D$ , so  $\gamma^2 - 2r\gamma + r^2 - Ds^2 = 0$ .  $\gamma$  is a root of

$$p(x) = x^2 - 2rx + r^2 - Ds^2.$$

If  $s = 0$ , then the minimal polynomial of  $\gamma$  is  $x - r$ . As  $r = \gamma$  is an algebraic integer and  $r \in \mathbb{Q}$ , then  $r \in \mathbb{Z}$ . In this case  $r \in \mathbb{Z}$  and  $s = 0$ .

If  $s \neq 0, \gamma \notin \mathbb{Q}$ , so no polynomial of degree  $d \leq 1$  has the root  $\gamma$ . Thus the minimal polynomial of  $\gamma$  is  $p(x)$ . From Exercise 6.5,  $p(x) \in \mathbb{Z}[x]$ , so (in the two cases  $s = 0, s \neq 0$ )

$$2r \in \mathbb{Z}, r^2 - Ds^2 \in \mathbb{Z}.$$

Reciprocally, if  $2r \in \mathbb{Z}, r^2 - Ds^2 \in \mathbb{Z}$ , then  $p(x) \in \mathbb{Z}[x]$  and  $p(\gamma) = 0$ , thus  $\gamma$  is an algebraic integer.

If  $r, s \in \mathbb{Q}, D \neq 1$  square-free,

$$r + s\sqrt{D} \in \overline{\mathbb{Z}} \iff 2r \in \mathbb{Z}, r^2 - Ds^2 \in \mathbb{Z}.$$

Let  $\gamma = r + s\sqrt{D} \in \overline{\mathbb{Z}}$ . We can write

$$r = \frac{a}{d}, s = \frac{b}{d}, \quad a, b, d \in \mathbb{Z}, d \geq 1, d \wedge a \wedge b = 1.$$

Then

$$n = \frac{2a}{d} \in \mathbb{Z}, \quad m = \frac{a^2 - Db^2}{d^2} \in \mathbb{Z}.$$

As  $D$  is square-free,  $D \not\equiv 0 \pmod{4}$ .

- Case 1 :  $D \equiv 2, 3 \pmod{4}$ .

$$n^2 - 4m = \frac{4Db^2}{d^2}, \text{ so } d \mid 2a, d^2 \mid 4Db^2.$$

If  $2 \mid d$ ,  $4 \mid a^2 - Db^2$ ,  $a^2 \equiv Db^2 \pmod{4}$ . As  $d \wedge a \wedge b = 1$ , and  $2 \mid d$ ,  $a$  or  $b$  is odd, and  $a^2 \equiv Db^2 \pmod{4}$ ,  $D \not\equiv 0 \pmod{4}$ , implies that  $a$  and  $b$  are both odd. Then  $a^2 \equiv b^2 \equiv 1 \pmod{4}$ , so  $D \equiv 1 \pmod{4}$  : this is in contradiction with the hypothesis  $D \equiv 2, 3 \pmod{4}$ . So  $d$  is an odd number.

Consequently,  $d \mid p, d^2 \mid Dq^2$ . If  $p \in \mathbb{N}$  is a prime factor of  $d$ ,  $p \mid d, p \mid a$ , and  $d \wedge a \wedge b = 1$ , so  $p \nmid b$ , and since  $p^2 \mid Db^2$ ,  $p^2 \mid D$ , in contradiction with  $D$  square-free. So  $d \geq 1$  has no prime factor :  $d = 1$  and  $r = a, s = b \in \mathbb{Z}$ . Reciprocally, any  $\gamma = a + b\sqrt{D}, a, b \in \mathbb{Z}$  is an algebraic integer, so

$$\mathcal{O}_{\mathbb{Q}[\sqrt{D}]} = \overline{\mathbb{Z}} \cap \mathbb{Q}[\sqrt{D}] = \{a + b\sqrt{D} \mid a, b \in \mathbb{Z}\}.$$

- Case 2 :  $D \equiv 1 \pmod{4}$ .

Then  $r = \frac{n}{2}, n \in \mathbb{Z}$ . Write  $s = \frac{u}{v}, u \wedge v = 1, v \geq 1$ .

$m = r^2 - Ds^2 = \frac{n^2}{4} - D\frac{u^2}{v^2} \in \mathbb{Z}$ ,  $4D\frac{u^2}{v^2} = n^2 - 4m \in \mathbb{Z}$ , so  $v^2 \mid 4Du^2$ . Since  $u \wedge v = 1, u^2 \wedge v^2 = 1$ , so  $v^2 \mid 4D$ . As  $D$  is square-free,  $v$  has no odd prime factor, so  $v = 2^k$ . Since  $D$  is odd,  $k \leq 1$  and  $v = 1$  or  $v = 2$ . So  $r, s$  are both half-integers:  $r = n/2, s = n'/2, n, n' \in \mathbb{Z}$ .

$4m = n^2 - Dn'^2$ , thus  $n^2 \equiv n'^2 \pmod{4}$ , so  $n, n'$  have the same parity. Let  $a = \frac{n+n'}{2} \in \mathbb{Z}, b = n' \in \mathbb{Z}$ . Then  $n = 2a - b, n' = b$  and  $\gamma = \frac{n}{2} + \frac{n'}{2}\sqrt{D} = a - \frac{b}{2} + \frac{b}{2}\sqrt{D} = a + b\left(\frac{-1+\sqrt{D}}{2}\right)$ .

Reciprocally,  $\frac{-1+\sqrt{D}}{2}$  is a root of  $x^2 + x + \frac{1-D}{4} \in \mathbb{Z}[x]$ , so every  $a + b\left(\frac{-1+\sqrt{D}}{2}\right)$  is an algebraic integer.

$$\mathcal{O}_{\mathbb{Q}[\sqrt{D}]} = \overline{\mathbb{Z}} \cap \mathbb{Q}[\sqrt{D}] = \left\{a + b\left(\frac{-1+\sqrt{D}}{2}\right) \mid a, b \in \mathbb{Z}\right\}.$$

□

**Ex. 6.8** Let  $\omega = e^{2\pi i/3}$ ,  $\omega$  satisfies  $x^3 - 1 = 0$ . Show that  $(2\omega + 1)^2 = -3$ , and use this to determine  $(-3/p)$  by the method of section 2.

*Proof.* As  $\omega^2 + \omega + 1 = 0, (2\omega + 1)^2 = 4\omega^2 + 4\omega + 1 = -4 + 1 = -3$ . Let  $\alpha = 2\omega + 1$ , so  $\alpha^2 = -3$

$$\left(\frac{-3}{p}\right) \equiv (-3)^{(p-1)/2} \pmod{p}$$

$$\equiv \alpha^{p-1} \pmod{p}$$

$$\alpha^p = \left(\frac{-3}{p}\right)\alpha.$$



From Prop. 6.1.6,

$$\begin{aligned}\alpha^p &= (2\omega + 1)^p \\ &\equiv 2^p \omega^p + 1 \pmod{p} \\ &\equiv 2\omega^p + 1 \pmod{p}\end{aligned}$$

- If  $p \equiv 0 \pmod{3}$ ,  $\left(\frac{-3}{p}\right) = 0$ .
- If  $p \equiv 1 \pmod{3}$ ,  $\omega^p = \omega$ , so  $\alpha^p \equiv \alpha \pmod{p}$ .  
 $\left(\frac{-3}{p}\right)\alpha \equiv \alpha \pmod{p}$ , thus  $\left(\frac{-3}{p}\right)\alpha^2 \equiv \alpha^2 \pmod{p}$ ,  $\left(\frac{-3}{p}\right)3 \equiv 3 \pmod{p}$ . As  $p \wedge 3 = 1$ ,  
 $\left(\frac{-3}{p}\right) \equiv 1 \pmod{p}$ . Since  $\left(\frac{-3}{p}\right) = \pm 1$ ,  $\left(\frac{-3}{p}\right) = 1$ .
- If  $p \equiv -1 \pmod{3}$ ,

$$\begin{aligned}\alpha^p &\equiv 2\omega^p + 1 \pmod{p} \\ &\equiv 2\omega^2 + 2 = 2(-1 - \omega) + 1 = -2\omega - 1 = -\alpha \pmod{p}.\end{aligned}$$

$\left(\frac{-3}{p}\right)\alpha \equiv -\alpha \pmod{p}$ , thus  $\left(\frac{-3}{p}\right)\alpha^2 \equiv -\alpha^2 \pmod{p}$ ,  $\left(\frac{-3}{p}\right)3 \equiv -3 \pmod{p}$ . As  
 $p \wedge 3 = 1$ ,  $\left(\frac{-3}{p}\right) \equiv -1 \pmod{p}$ . Since  $\left(\frac{-3}{p}\right) = \pm 1$ ,  $\left(\frac{-3}{p}\right) = -1$ .

Conclusion :

$$\begin{aligned}p \equiv 0[3] &\iff \left(\frac{-3}{p}\right) = 0 \\ p \equiv 1[3] &\iff \left(\frac{-3}{p}\right) = 1 \\ p \equiv -1[3] &\iff \left(\frac{-3}{p}\right) = -1\end{aligned}$$

In other words,  $\left(\frac{-3}{p}\right) = \left(\frac{p}{3}\right)$ .

Remark :  $\alpha = 2\omega + 1 = \omega - \omega^2 = g$ , the quadratic Gauss sum for  $p = 3$ .  $\square$

**Ex. 6.9** Verify Proposition 6.3.2 explicitly for  $p = 3, 5$ , i.e., write out the Gauss sum longhand and square.

*Proof.* •  $p=3$ . Let  $\omega = e^{2i\pi/3}$ . Let  $g = \sum_{t=0}^2 (t/3)\omega^t$  the quadratic Gauss sum. Then  
 $g = \omega - \omega^2$ .

As  $1 + \omega + \omega^2 = 0$ ,  $g^2 = (\omega - \omega^2)^2 = \omega^2 - 2\omega^3 + \omega^4 = \omega^2 - 2 + \omega = -3$  :

$$g^2 = -3.$$

- $p=5$ . Let  $\zeta = e^{2i\pi/5}$ .  
 $g = \sum_{t=0}^4 (t/5)\zeta^t = \zeta - \zeta^2 - \zeta^3 + \zeta^4$ .  
Then  $g = \alpha - \beta$ , where  $\alpha = \zeta + \zeta^4, \beta = \zeta^2 + \zeta^3$ .  
 $\alpha + \beta = \zeta + \zeta^4 + \zeta^2 + \zeta^3 = -1$ .  
 $\alpha\beta = \zeta^3 + \zeta^4 + \zeta^6 + \zeta^7 = \zeta^3 + \zeta^4 + \zeta + \zeta^2 = -1$

So  $\alpha, \beta$  are the two roots of  $x^2 + x - 1$ .

$$\begin{aligned} g^2 &= (\alpha - \beta)^2 \\ &= \alpha^2 + \beta^2 - 2\alpha\beta \\ &= (\alpha + \beta)^2 - 4\alpha\beta \\ &= (-1)^2 - 4(-1) \\ &= 5. \end{aligned}$$

Remark : here we know explicitly  $g$  :

if  $p = 3$ ,  $g = \omega - \omega^2 = i\sqrt{3}$ .

If  $p = 5$ ,  $g = \alpha - \beta = (-1 + \sqrt{5})/2 - (-1 - \sqrt{5})/2 = \sqrt{5}$ .

□

**Ex. 6.10** What is  $\sum_{a=1}^{p-1} g_a$ ?

*Proof.* From Prop. 6.3.1 and Lemma 2,

$$\sum_{a=1}^{p-1} g_a = g_1 \sum_{a=1}^{p-1} \left( \frac{a}{p} \right) = 0.$$

□

**Ex. 6.11** By evaluating  $\sum_t (1 + (t/p))\zeta^t$  in two ways, prove that  $g = \sum_t \zeta^{t^2}$ .

*Proof.* For  $a \in \mathbb{F}_p$ , Write  $N[x^2 = a]$  the number of solutions of the equation  $x^2 = a$  in  $\mathbb{F}_p$ . We know from Ex. 5.2 that  $N[x^2 = a] = 1 + (a/p)$ . So

$$\begin{aligned} \sum_{t=0}^{p-1} \zeta^{t^2} &= \sum_{\bar{t} \in \mathbb{F}_p} \zeta^{t^2} \\ &= \sum_{\bar{t} \in \mathbb{F}_p} N[x^2 = t] \zeta^t \\ &= \sum_{\bar{t} \in \mathbb{F}_p} \left( 1 + \left( \frac{t}{p} \right) \right) \zeta^t \\ &= \sum_{\bar{t} \in \mathbb{F}_p} \zeta^t + \sum_{\bar{t} \in \mathbb{F}_p} \left( \frac{t}{p} \right) \zeta^t \\ &= \sum_{t=0}^{p-1} \left( \frac{t}{p} \right) \zeta^t \\ &= g \end{aligned}$$

□

**Ex. 6.12** Write  $\psi_a(t) = \zeta^{at}$ . Show that

$$(a) \quad \overline{\psi_a(t)} = \psi_a(-t) = \psi_{-a}(t)$$

$$(b) \quad (1/p) \sum_a \psi_a(t-s) = \delta(t, s)$$

*Proof.* (a) Let  $a \in \mathbb{Z}$ . As  $\bar{\zeta} = \zeta^{-1}$ ,

$$\begin{aligned} \overline{\psi_a(t)} &= \overline{\zeta^{at}} = \zeta^{-at} \\ &= \zeta^{a(-t)} = \zeta^{(-a)t} \\ &= \psi_a(-t) = \psi_{-a}(t) \end{aligned}$$

$$\overline{\psi_a(t)} = \psi_a(-t) = \psi_{-a}(t)$$

(b) From Corollary of Lemma 1 :

$$\begin{aligned} \frac{1}{p} \sum_{a=0}^{p-1} \psi_a(t-s) &= \frac{1}{p} \sum_{a=0}^{p-1} \zeta^{a(t-s)} = \delta(t, s) \\ \frac{1}{p} \sum_a \psi_a(t-s) &= \delta(t, s). \end{aligned}$$

□

**Ex. 6.13** Let  $f$  be a function from  $\mathbb{Z}$  to the complex numbers. Suppose that  $p$  is a prime and that  $f(n+p) = f(n)$  for all  $n \in \mathbb{Z}$ . Let  $\hat{f}(a) = p^{-1} \sum_t f(t) \psi_{-a}(t)$ . Prove that  $f(t) = \sum_a \hat{f}(a) \psi_a(t)$ . This result is directly analogous to a result in the theory of Fourier series.

*Proof.* Let  $\hat{f}(a) = p^{-1} \sum_t f(t) \psi_{-a}(t)$ . Then

$$\begin{aligned} \sum_{a=0}^{p-1} \hat{f}(a) \psi_a(t) &= \sum_{a=0}^{p-1} p^{-1} \sum_{s=0}^{p-1} f(s) \psi_{-a}(s) \psi_a(t) \\ &= p^{-1} \sum_{s=0}^{p-1} f(s) \sum_{a=0}^{p-1} \psi_{-a}(s) \psi_a(t) \\ &= p^{-1} \sum_{s=0}^{p-1} f(s) \sum_{a=0}^{p-1} \psi_a(t-s) \\ &= \sum_{s=0}^{p-1} f(s) \delta(s, t) \\ &= f(t) \end{aligned}$$

□

**Ex. 6.14** In Ex. 13 take  $f$  to be the Legendre symbol and show that  $\hat{f}(a) = p^{-1} g_{-a}$ .

*Proof.* Here  $f(a) = \left(\frac{a}{p}\right)$ . Then  $\hat{f}(a) = p^{-1} \sum_{t=0}^{p-1} \left(\frac{t}{p}\right) \zeta^{-at} = p^{-1} g_{-a}$ . □

**Ex. 6.15** Show that

$$\left| \sum_{t=n}^m \binom{t}{p} \right| < \sqrt{p} \log p.$$

The inequality holds for the sum over any range.

**Lemma.** If  $0 \leq x \leq \frac{\pi}{2}$ ,  $\sin x \geq \frac{2}{\pi}x$ .

*Proof.* As  $-\sin$  is a convex function on  $[0, \pi/2]$ , the graph of  $\sin$  is above any chord, and the chord between the points  $(0, 0)$  and  $(\pi/2, 1)$  has equation  $y = (2/\pi)x$ , we conclude that  $\sin x \geq \frac{2}{\pi}x$  for  $0 \leq x \leq \pi/2$ .  $\square$

*Proof.* Let  $S = \sum_{t=n}^m \binom{t}{p} g$  with  $n \leq m$ . Then  $|S| = \sqrt{p} \left| \sum_{t=n}^m \binom{t}{p} \right|$ . As  $(t/p)g = g_t$ ,

$$\begin{aligned} S &= \sum_{t=m}^n g_t \\ &= \sum_{t=m}^n \sum_{s=0}^{p-1} \binom{s}{p} \zeta^{ts} \\ &= \sum_{s=0}^{p-1} \binom{s}{p} \zeta^{ms} \sum_{t=m}^n \zeta^{(t-m)s} \\ &= \sum_{s=0}^{p-1} \binom{s}{p} \zeta^{ms} \sum_{u=0}^{n-m} \zeta^{us} \quad (u = t - m) \\ &= \sum_{s=1}^{p-1} \binom{s}{p} \zeta^{ms} \frac{\zeta^{(n-m+1)s} - 1}{\zeta^s - 1} \end{aligned}$$

(since for  $s = 0$ , the sum  $\sum_{u=0}^{n-m} \zeta^{us} = n - m + 1$  and  $\sum_{s=0}^{p-1} \binom{s}{p} = 0$ ). So

$$\begin{aligned} S &= \sum_{s=1}^{p-1} \binom{s}{p} \frac{\zeta^{(n+1)s} - \zeta^{ms}}{\zeta^s - 1} \\ &= \sum_{s=1}^{p-1} \binom{s}{p} \frac{\zeta^{\frac{n+m+1}{2}s}}{\zeta^{\frac{s}{2}}} \frac{\zeta^{\frac{n-m+1}{2}s} - \zeta^{\frac{-n+m-1}{2}s}}{\zeta^{\frac{s}{2}} - \zeta^{\frac{-s}{2}}} \\ &= \sum_{s=1}^{p-1} \binom{s}{p} \zeta^{\frac{n+m}{2}s} \frac{\sin\left((n-m+1)s\frac{\pi}{p}\right)}{\sin\left(s\frac{\pi}{p}\right)} \end{aligned}$$

As  $\sin(x) \geq \frac{2}{\pi}x$  for  $x \in [0, \frac{\pi}{2}]$ , for all  $s$ ,  $1 \leq s < \frac{p}{2}$ ,  $0 \leq \frac{s\pi}{p} \leq \frac{\pi}{2}$ , so

$$\left| \frac{\sin\left((n-m+1)s\frac{\pi}{p}\right)}{\sin\left(s\frac{\pi}{p}\right)} \right| \leq \frac{1}{\frac{2}{\pi}\left(s\frac{\pi}{p}\right)} = \frac{p}{2s} \quad (s = 1, 2, \dots, (p-1)/2).$$

Since  $\binom{s}{p} \zeta^{ts}$  depends only of the class of  $s$ , we can replace in the preceding calculation the values  $s = 1, 2, \dots, p-1$  by  $s = -(p-1)/2, \dots, -1, 1, \dots, (p-1)/2$ , so

$$S = \sum_{s=1}^{(p-1)/2} \binom{s}{p} \zeta^{\frac{n+m}{2}s} \frac{\sin\left((n-m+1)s\frac{\pi}{p}\right)}{\sin\left(s\frac{\pi}{p}\right)} + \sum_{s=-(p-1)/2}^{-1} \binom{s}{p} \zeta^{\frac{n+m}{2}s} \frac{\sin\left((n-m+1)s\frac{\pi}{p}\right)}{\sin\left(s\frac{\pi}{p}\right)}.$$

As  $\sin$  is an odd function,

$$S = \sum_{s=1}^{(p-1)/2} \left(\frac{s}{p}\right) \zeta^{\frac{n+m}{2}s} \frac{\sin\left((n-m+1)s\frac{\pi}{p}\right)}{\sin\left(s\frac{\pi}{p}\right)} + \sum_{s=1}^{(p-1)/2} \left(\frac{-s}{p}\right) \zeta^{-\frac{n+m}{2}s} \frac{\sin\left((n-m+1)s\frac{\pi}{p}\right)}{\sin\left(s\frac{\pi}{p}\right)}.$$

Thus

$$|S| \leq 2 \sum_{s=1}^{(p-1)/2} \frac{p}{2s} = p \sum_{s=1}^{(p-1)/2} \frac{1}{s}.$$

As  $S = \sum_{t=n}^m \left(\frac{t}{p}\right)g$  and  $|g| = \sqrt{p}$ ,

$$\left| \sum_{t=n}^m \left(\frac{t}{p}\right) \right| \leq \sqrt{p} \sum_{s=1}^{(p-1)/2} \frac{1}{s}.$$

It remains to do a sufficient estimation of the harmonic sum. We prove by induction that for all  $n \geq 1$ ,

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq \log(2n+1).$$

As  $1 \leq \log(3)$ , this proposition is true for  $n = 1$ . Suppose that it is true for  $n - 1$  :

$$1 + \frac{1}{2} + \cdots + \frac{1}{n-1} \leq \log(2n-1).$$

Then

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq \frac{1}{n} + \log(2n-1).$$

If we prove that  $\frac{1}{n} + \log(2n-1) \leq \log(2n+1)$ , the induction is done.

Let  $u(x) = \log(2x+1) - \log(2x-1) - \frac{1}{x}$ ,  $x > \frac{1}{2}$ .

$$\begin{aligned} u'(x) &= \frac{2}{2x+1} - \frac{2}{2x-1} - \frac{1}{x^2} \\ &= \frac{-4}{4x^2-1} + \frac{1}{x^2} \\ &= \frac{-1}{(4x^2-1)x^2} < 0 \end{aligned}$$

As  $u(x) = \log\left(\frac{2x+1}{2x-1}\right) - \frac{1}{x}$ ,  $\lim_{x \rightarrow +\infty} u(x) = 0$ . Moreover  $u$  is a decreasing function, so for all  $x > 1/2$ ,  $u(x) > 0$ , and for all  $n \in \mathbb{N}$ ,  $n \geq 1$ ,

$$\frac{1}{n} + \log(2n-1) \leq \log(2n+1).$$

We have proved by induction that for all  $n \geq 1$ ,

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq \log(2n+1).$$

If  $n = (p-1)/2$ , where  $p$  is an odd prime ( $p \geq 3$ ),

$$\sum_{s=1}^{(p-1)/2} \frac{1}{s} \leq \log p.$$

Conclusion :

$$\left| \sum_{t=n}^m \left(\frac{t}{p}\right) \right| < \sqrt{p} \log p.$$

□

**Ex. 6.16** Let  $\alpha$  be an algebraic number with minimal polynomial  $f(x)$ . Show that  $f(x)$  does not have repeated roots in  $\mathbb{C}$ .

*Proof.* Let  $\gamma$  a repeated root of  $f(x)$ . Then  $f(\gamma) = f'(\gamma) = 0$ , so  $x - \gamma$  is a common factor of  $f$  and  $f'$ . Thus  $f \wedge f' \neq 1$  ( $\deg(f \wedge f') \geq 1$ ). Since  $f \wedge f' \mid f$  and  $f$  is irreducible (with  $f, f \wedge f'$  monic), we conclude  $f \wedge f' = f$ , so  $f \mid f'$ . In  $\mathbb{C}$ , this is impossible since  $\deg(f) \geq 1$ , so  $f' \neq 0$ , and  $\deg(f') < \deg(f)$ .  $f(x)$  does not have repeated roots in  $\mathbb{C}$ .  $\square$

**Ex. 6.17** Show that the minimal polynomial for  $\sqrt[3]{2}$  is  $x^3 - 2$ .

*Proof.* Let  $f(x) = x^3 - 2$ . Then  $f(\sqrt[3]{2}) = 0$ . If  $f(x)$  was not irreducible, then  $f(x) = u(x)v(x)$ , with  $1 \leq \deg(u) \leq \deg(v) \leq 2$ ,  $\deg(u) + \deg(v) = \deg(f) = 3$ , so  $\deg(u) = 1$ ,  $\deg(v) = 2$ .

Then  $f(x) = (ax + b)(cx^2 + dx + e)$ ,  $a, b, c, d, e \in \mathbb{Q}$ . Let  $w = -b/a$ . Then  $f(w) = w^3 - 2 = 0$  and  $w \in \mathbb{Q}$ , so there exist  $p, q \in \mathbb{Z}$ , such that  $w = p/q$ ,  $p \wedge q = 1$ .

Thus  $p^3 = 2q^3$ , so  $p^3$  is even, therefore  $p$  is even :  $p = 2p', p' \in \mathbb{Z}$ .

$8p'^3 = 2q^3$ ,  $4p'^3 = q^3$ , so  $q^3$  is even, which implies that  $q$  is even. Then  $2 \mid p \wedge q = 1$  : this is a contradiction.

So  $f(\sqrt[3]{2}) = 0$ , and  $f$  is monic, irreducible :  $f$  is the minimal polynomial of  $\sqrt[3]{2}$  on  $\mathbb{Q}$ .  $\square$

**Ex. 6.18** Show that there exist algebraic numbers of arbitrarily high degree.

*Proof.* As  $1 + x + \dots + x^{p-1}$  is irreducible on  $\mathbb{Q}[x]$  (Prop. 6.4.1), the numbers  $\zeta_p = e^{2i\pi/p}$ , with  $p$  prime number, are algebraic numbers of arbitrary large degree.  $\square$

**Ex. 6.19** Find the conjugates of  $\cos(2\pi/5)$ .

*Proof.* Let  $\gamma = \cos(2\pi/5)$ ,  $\zeta = e^{2i\pi/5}$  and  $\alpha = \zeta + \zeta^4, \beta = \zeta^2 + \zeta^3$ .

Then  $\gamma = \frac{\zeta + \zeta^{-1}}{2} = \frac{\zeta + \zeta^4}{2} = \frac{\alpha}{2}$ .

$\alpha + \beta = \zeta + \zeta^4 + \zeta^2 + \zeta^3 = -1$ .

$\alpha\beta = \zeta^3 + \zeta^4 + \zeta^6 + \zeta^7 = \zeta^3 + \zeta^4 + \zeta + \zeta^2 = -1$

So  $\alpha, \beta$  are the two roots of  $x^2 + x - 1$  :

$\alpha^2 + \alpha - 1 = 0$ , so  $4(\alpha/2)^2 + 2(\alpha/2) - 1 = 0$  :  $\gamma = \alpha/2$  is a root of

$$f(x) = 4x^2 + 2x - 1.$$

As  $\Delta = 4 \times 5$ , the two roots of  $f$  are irrational.  $\deg(f) = 2$  and  $f$  has no root in  $\mathbb{Q}$ , so  $f(x)$  is irreducible in  $\mathbb{Q}[x]$ . So the minimal polynomial of  $\gamma = \cos(2\pi/5)$  is  $f(x) = 4x^2 + 2x - 1$ . The other root of  $f$  is  $\beta/2 = (\zeta^2 + \zeta^3)/2 = \cos(4\pi/5)$ .

Conclusion : the conjugates of  $\gamma = \cos(2\pi/5)$  are  $\gamma = \cos(2\pi/5)$  and  $\cos(4\pi/5)$ .  $\square$

**Ex. 6.20** Let  $F$  be a subfield of  $\mathbb{C}$  which is a finite-dimensional vector space over  $\mathbb{Q}$  of degree  $n$ . Show that every element of  $F$  is algebraic of degree at most  $n$ .

*Proof.* Let  $\alpha \in F$ , with  $\dim_{\mathbb{Q}} F = n$ . Any subset of  $n + 1$  vectors in  $F$  is linearly dependent, so  $\{1, \alpha, \alpha^2, \dots, \alpha^n\}$  is linearly dependent.

Thus there exists  $(a_0, \dots, a_n) \in \mathbb{Q}^{n+1}$ ,  $(a_0, \dots, a_n) \neq (0, 0, \dots, 0)$  such that  $a_0 + a_1\alpha + \dots + a_n\alpha^n = 0$ .

Let  $f(x) = a_0 + a_1x + \dots + a_nx^n$ . Then  $f(x) \in \mathbb{Q}[x]$ ,  $f(x) \neq 0$  and  $f(\alpha) = 0$ ,  $\deg(f(x)) \leq n$ . So every element of  $F$  is algebraic of degree at most  $n$ .  $\square$

**Ex. 6.21** Let  $f(x) = \sum_{n=0}^{\infty} a_n x^n / n!$  and  $g(x) = \sum_{n=0}^{\infty} b_n x^n / n!$  be power series with  $a_n$  and  $b_n$  integers. If  $p$  is a prime such that  $p \nmid a_i$  for  $i = 0, \dots, p-1$ , show that each coefficient  $c_t$  of the product  $f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n$  for  $t = 0, \dots, p-1$  may be written in the form  $p(A/B)$ ,  $p \nmid B$ .

*Proof.* Let  $k \in \mathbb{N}, 0 \leq k \leq p-1$ .

$$\begin{aligned} c_k &= \sum_{i+j=k} \frac{a_i}{i!} \frac{b_j}{j!} \\ &= \sum_{i=0}^k \frac{a_i}{i!} \frac{b_{k-i}}{(k-i)!} \\ &= \frac{1}{k!} \sum_{i=0}^k \binom{k}{i} a_i b_{k-i} \end{aligned}$$

As  $k! \wedge p = 1$ , and  $\sum_{i=0}^k \binom{k}{i} a_i b_{k-i} \equiv 0 \pmod{p}$  for  $k = 0, 1, \dots, p-1$ ,  
 $c_k = p(A/B), p \wedge B = 1$ . □

**Ex. 6.22** Show that the relation  $\varepsilon \equiv 1 \pmod{p}$  in Proposition 6.4.4 can also be achieved by replacing  $x$  by  $1+t$  instead of  $e^z$ .

*Proof.* We know from Prop. 6.4.3 that

$$g(\zeta) = \varepsilon \prod_{k=1}^{(p-1)/2} (\zeta^{2k-1} - \zeta^{-(2k-1)}),$$

where  $\varepsilon = \pm 1$ . Let

$$f(x) = \sum_{j=1}^{p-1} \chi(j) x^j - \varepsilon \prod_{k=1}^{(p-1)/2} (x^{2k-1} - x^{p-(2k-1)}).$$

Then  $f(0) = 0$  and  $f(\zeta) = 0$ , so  $(x^p - 1)$  divides  $f(x)$ . As  $f(x) \in \mathbb{Z}[x]$  and  $x^p - 1 \in \mathbb{Z}[x]$  is monic,  $f(x) = (x^p - 1)h(x)$ ,  $h(x) \in \mathbb{Z}[x]$ . If we replace  $x$  by  $1+t$ , we obtain

$$f(1+t) = \sum_{j=1}^{p-1} \chi(j)(1+t)^j - \varepsilon \prod_{k=1}^{(p-1)/2} \left( (1+t)^{2k-1} - (1+t)^{p-(2k-1)} \right).$$

We compute the coefficient of  $t^{(p-1)/2}$  in the polynomial  $f(1+t)$  :

$$\begin{aligned} \sum_{j=1}^{p-1} \chi(j)(1+t)^j &= \sum_{j=1}^{p-1} \chi(j) \sum_{i=1}^j \binom{j}{i} t^i \\ &= \sum_{i=1}^{p-1} \sum_{j=i}^{p-1} \chi(j) \binom{j}{i} t^i \end{aligned}$$

So the coefficient of  $t^{(p-1)/2}$  in  $\sum_{j=1}^{p-1} \chi(j)(1+t)^j$  is  $\sum_{j=(p-1)/2}^{p-1} \chi(j) \binom{j}{(p-1)/2}$ .

$$\begin{aligned} \prod_{k=1}^{(p-1)/2} ((1+t)^{2k-1} - (1+t)^{p-(2k-1)}) &= \prod_{k=1}^{(p-1)/2} ((1+(2k-1)t) - (1+(p-(2k-1))t + t^2 u(t))) \\ &= \prod_{k=1}^{(p-1)/2} ((4k-2-p)t + t^2 v(t)) \\ &= t^{(p-1)/2} \prod_{k=1}^{(p-1)/2} (4k-2-p) + t^{(p-1)/2} w(t), \end{aligned}$$

where  $u(t), v(t), w(t)$  are polynomials. So the coefficient of  $t^{(p-1)/2}$  in  $f(1+t)$  is

$$c_{(p-1)/2} = \sum_{j=(p-1)/2}^{p-1} \chi(j) \binom{j}{(p-1)/2} - \varepsilon \prod_{k=1}^{(p-1)/2} (4k-2-p).$$

Furthermore,

$$\begin{aligned} f(1+t) &= ((1+t)^p - 1) h(1+t) \\ &= \left[ \sum_{i=1}^p \binom{p}{i} t^i \right] h(1+t) \\ &= \left[ \sum_{i=1}^p i! \binom{p}{i} \frac{t^i}{i!} \right] h(1+t) \\ &= \left[ \sum_{i=0}^p a_i \frac{t^i}{i!} \right] h(1+t), \end{aligned}$$

where  $a_0 = 0, a_i = i! \binom{p}{i} = \frac{p!}{(p-i)!}$ , so  $p \mid a_i, i = 0, \dots, p-1$ : the conditions of Ex.21 are verified, so  $f(1+t) = \sum_{i=0}^{p-1} c_i t^i$  is such that  $c_{(p-1)/2} = p(A/B)$ ,  $p \nmid B$ . Equating these two evaluations of  $c_{(p-1)/2}$ , we obtain

$$\sum_{j=(p-1)/2}^{p-1} \chi(j) \binom{j}{(p-1)/2} - \varepsilon \prod_{k=1}^{(p-1)/2} (4k-2-p) = p \frac{A}{B}, \quad p \nmid B.$$

Multiplying by  $B(p-1)!/2$ , we obtain, as  $p \nmid B$ ,

$$\begin{aligned} \frac{(p-1)!}{2} \sum_{j=(p-1)/2}^{p-1} \chi(j) \binom{j}{(p-1)/2} &\equiv \varepsilon \frac{(p-1)!}{2} \prod_{k=1}^{(p-1)/2} (4k-2) \\ &\equiv \varepsilon (2 \cdot 4 \cdot 6 \cdots (p-1)) \prod_{k=1}^{(p-1)/2} (2k-1) \equiv \varepsilon (p-1)! \\ &\equiv -\varepsilon \pmod{p} \end{aligned}$$



To prove that  $\varepsilon = +1$ , it remains to prove

$$\left(\frac{(p-1)}{2}\right)! \sum_{j=(p-1)/2}^{p-1} \chi(j) \binom{j}{(p-1)/2} \equiv -1 \pmod{p}$$

The factor of  $((p-1)/2)!$  cancels the denominator of  $\binom{j}{(p-1)/2}$ , which leaves

$$\begin{aligned} & \sum_{j=(p-1)/2}^{p-1} \chi(j) \cdot j(j-1) \cdots \left(j - \frac{p-1}{2} + 1\right) \\ &= \sum_{j=1}^{p-1} \chi(j) \cdot j(j-1) \cdots \left(j - \frac{p-1}{2} + 1\right). \end{aligned}$$

The equality is justified because all terms for  $j < \frac{p-1}{2}$  are zero. Collecting powers of  $j$ , this is

$$\sum_{j=1}^{p-1} \sum_{k=0}^{(p-1)/2} \chi(j) a_k j^k \equiv \sum_{k=0}^{(p-1)/2} a_k \sum_{j=1}^{p-1} j^{k+\frac{p-1}{2}} \pmod{p}$$

for some integers  $a_k$ . It's important to note that  $a_{(p-1)/2} = 1$ .

Now, we compute  $\sum_{j=1}^{p-1} j^n \pmod{p}$ . Let  $S$  denote this sum and let  $g$  be a generator of  $\mathbb{Z}/p^\times$ . Then

$$g^n S = \sum_{j=1}^{p-1} (gj)^n \equiv \sum_{j=1}^{p-1} (j)^n = S \pmod{p}$$

because  $gj$  also runs over a complete system of nonzero residues mod  $p$ . If  $g^n \neq 1$ , then  $S \equiv 0$ . If  $g^n \equiv 1$ , then also  $j^n \equiv 1$  for all  $j$ , hence  $S \equiv p-1 \equiv -1$ .

Returning to the previous sum, the only nonzero term modulo  $p$  is  $k = \frac{p-1}{2}$ , so

$$\equiv \sum_{k=0}^{(p-1)/2} a_k \sum_{j=1}^{p-1} j^{k+\frac{p-1}{2}} \equiv a_{(p-1)/2} \cdot (-1) = -1 \pmod{p}$$

as desired.  $\square$

**Ex. 6.23** If  $f(x) = x^n + a_1 x^{n-1} + \dots + a_n$ ,  $a_i \in \mathbb{Z}$ , and  $p$  is prime such that  $p \mid a_i$  for  $i = 1, \dots, n$ , and  $p^2 \nmid a_n$ , show that  $f(x)$  is irreducible over  $\mathbb{Q}$  (Eisenstein's irreducibility criterion).

**Lemma.** If  $f \in \mathbb{Z}[x]$ ,  $\deg(f) \geq 1$ , is not irreducible in  $\mathbb{Q}[x]$ , then there exist  $g, h \in \mathbb{Z}[x]$ ,  $\deg(g) \geq 1$ ,  $\deg(h) \geq 1$  such that  $f = gh$ .

*Proof.* (lemma) Suppose that  $f(x) = \sum_{k=0}^n a_k x^k$ ,  $a_k \in \mathbb{Z}$ , is not irreducible in  $\mathbb{Q}[x]$ .

Then  $f(x) = f_1(x)f_2(x)$ , with  $f_1, f_2 \in \mathbb{Q}[X]$ , and  $\deg(f_1) \geq 1$ ,  $\deg(f_2) \geq 1$ . As in Ex. 6.5, we can write  $f_1(x) = \lambda p(x)$ ,  $f_2(x) = \mu q(x)$  where  $\lambda, \mu \in \mathbb{Q}$ , and  $p, q \in \mathbb{Z}[X]$  are primitive. Let  $\nu = \lambda\mu \in \mathbb{Q}$ : write  $\nu = u/v$ ,  $u \wedge v = 1$ ,  $v \geq 1$ . Then  $r(x) = p(x)q(x) = \sum_{k=0}^n c_k x^k$  is primitive (Ex. 6.4), and  $f(x) = \frac{u}{v} r(x) = \frac{u}{v} p(x)q(x)$ .

As  $vf(x) = ur(x)$ ,  $v \mid uc_i$ ,  $i = 0, 1, \dots, n$ , with  $u \wedge v = 1$ , so  $u \mid c_i$  for all  $i$ . The polynomial  $r$  being primitive,  $v \mid 1$ , so  $v = \varepsilon = \pm 1$ .

Let  $g(x) = \varepsilon up(x)$ ,  $h(x) = q(x)$ . Then  $g, h \in \mathbb{Z}[x]$ ,  $\deg(g) \geq 1$ ,  $\deg(h) \geq 1$ , and  $f = gh$  is the product of two non constant polynomials in  $\mathbb{Z}[x]$ . □

*Proof.* (Ex. 6.23)

Let

$$\varphi: \begin{cases} \mathbb{Z}[x] & \rightarrow \mathbb{F}_p[x] \\ p(x) = a_0 + \cdots + a_n x^n & \mapsto \bar{p}(x) = \bar{a}_0 + \cdots + \bar{a}_n x^n, \end{cases}$$

where  $\bar{a}_i$  is the class of  $a_i$  in  $\mathbb{F}_p$ .  $\varphi$  is a ring homomorphism.

We show that  $f(x) = g(x)h(x)$ ,  $g, h \in \mathbb{Z}[x]$ ,  $\deg(g) \geq 1$ ,  $\deg(h) \geq 1$  is impossible. Indeed in such a situation,

$$\bar{f}(x) = x^n = \bar{g}(x)\bar{h}(x).$$

As the only irreducible factor of  $x^n$  is  $x$ , the unicity of the decomposition of a polynomial in irreducible factors in  $\mathbb{F}_p[x]$  gives

$$\bar{g}(x) = \lambda x^i, \bar{h}(x) = \mu x^j, \lambda, \mu \in \mathbb{F}_p, i, j \in \mathbb{N}.$$

As  $\deg(\bar{g}) \leq \deg(g)$ ,  $\deg(\bar{h}) \leq \deg(h)$  and  $\deg(\bar{g}) + \deg(\bar{h}) = n = \deg(f) + \deg(g)$ , this implies that  $i = \deg(\bar{f}) = \deg(f)$ ,  $j = \deg(\bar{g}) = \deg(g)$ , so  $i \geq 1, j \geq 1$ . Therefore  $p \mid g(0), p \mid h(0)$ , so  $p^2 \mid a_n = g(0)h(0)$ , which is in contradiction with the hypothesis.

From the lemma we deduce that  $f(x)$  is irreducible in  $\mathbb{Q}[x]$ . □