MATH 1853 Tutorial

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Tutorial 1: Matrix Operation

Tips: The following exercises are provided **for reference only** and as a supplement to the tutorial material.

1. Please calculate $\begin{bmatrix} 1 & 3 \\ 5 & 2 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 1 \end{bmatrix}$.

   Hints: $\begin{bmatrix} 1 & 3 \\ 5 & 2 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 2 \\ -1 & -1 \end{bmatrix}$.

2. Please calculate $3 \cdot \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 1 \end{bmatrix}$.

   Hints: $3 \cdot \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 9 & 0 \\ 0 & 3 \end{bmatrix}$.

3. Assume $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & -1 \\ -1 & 0 & -1 & 0 \end{bmatrix}$. Please calculate $\mathbf{AB}$.

   Hints: $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & -1 \\ -1 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 3 & 1 & 4 & 1 \end{bmatrix}$

4. Assume $\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Please calculate $\mathbf{AB}$ and $\mathbf{BA}$.

   Hints: $\mathbf{AB} = \begin{bmatrix} 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1$. $\mathbf{BA} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

5. Assume $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Please calculate $\mathbf{AB}$ and $\mathbf{BA}$.

   Hints: $\mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. $\mathbf{BA} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

6. Assume $\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$. Please calculate $\mathbf{A}^n$.

   Hints: $\mathbf{A^2} = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} =$ $4\mathbf{I}$. Therefore, if n is a even number, then $\mathbf{A}^n = (\mathbf{A}^2)^{\frac{n}{2}} = 4^{\frac{n}{2}}\mathbf{I}$, if n is a odd number, then $\mathbf{A}^n = 4^{\frac{n-1}{2}}\mathbf{A}$.

Tutorial 2: Solving Linear Equations

Tips: The following exercises are provided **for reference only** and as a supplement to the tutorial material.

1. Please solving the linear equations $\begin{cases} 2x_1 - x_2 - x_3 + x_4 = 2 \\ x_1 + x_2 - 2x_3 + x_4 = 4 \\ 4x_1 - 6x_2 + 2x_3 - 2x_4 = 4 \\ 3x_1 + 6x_2 - 9x_3 + 7x_4 = 9 \end{cases}$.

   Hints: $\begin{bmatrix} 2 & -1 & -1 & 1 & 2 \\ 1 & 1 & -2 & 1 & 4 \\ 4 & -6 & 2 & -2 & 4 \\ 3 & 6 & -9 & 7 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 2 & -1 & -1 & 1 & 2 \\ 2 & -3 & 1 & -1 & 2 \\ 3 & 6 & -9 & 7 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 0 & 2 & -2 & 2 & 0 \\ 0 & -5 & 5 & -3 & -6 \\ 0 & 3 & -3 & 4 & -3 \end{bmatrix} \to$
   $\begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.
   Therefore, $\begin{cases} x_1 = x_3 + 4 \\ x_2 = x_3 + 3 \\ x_4 = -3 \end{cases}$.

2. Please solving the linear equations $\begin{cases} x_1 + 2x_2 + 2x_3 + x_4 = 0 \\ 2x_1 + x_2 - 2x_3 - 2x_4 = 0 \\ x_1 - x_2 - 4x_3 - 3x_4 = 0 \end{cases}$.

   Hints: $\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & -2 & -2 \\ 1 & -1 & -4 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -3 & -6 & -4 \\ 0 & -3 & -6 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & \frac{4}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$
   $\begin{bmatrix} 1 & 0 & -2 & -\frac{5}{3} \\ 0 & 1 & 2 & \frac{4}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Therefore, $\begin{cases} x_1 = 2x_3 + \frac{5}{3}x_4 \\ x_2 = -2x_3 - \frac{4}{3}x_4 \end{cases}$.

3. Please solving the linear equations $\begin{cases} x_1 - 2x_2 + 3x_3 - x_4 = 1 \\ 3x_1 - x_2 + 5x_3 - 3x_4 = 2 \\ 2x_1 + x_2 + 2x_3 - 2x_4 = 3 \end{cases}$.

   Hints: $\begin{bmatrix} 1 & -2 & 3 & -1 & 1 \\ 3 & -1 & 5 & -3 & 2 \\ 2 & 1 & 2 & -2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & -1 & 1 \\ 0 & 5 & -4 & 0 & -1 \\ 0 & 5 & -4 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & -1 & 1 \\ 0 & 5 & -4 & 0 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$.
   Therefore, there is no solution for original linear equations.

4. Please solving the linear equations $\begin{cases} x_1 + x_2 - 3x_3 - x_4 = 1 \\ 3x_1 - x_2 - 3x_3 + 4x_4 = 4 \\ x_1 + 5x_2 - 9x_3 - 8x_4 = 0 \end{cases}$.

Hints: $\begin{bmatrix} 1 & 1 & -3 & -1 & 1 \\ 3 & -1 & -3 & 4 & 4 \\ 1 & 5 & -9 & -8 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -3 & -1 & 1 \\ 0 & -4 & 6 & 7 & 1 \\ 0 & 4 & -6 & -7 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -3 & -1 & 1 \\ 0 & 1 & -\frac{3}{2} & -\frac{7}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to$
$\begin{bmatrix} 1 & 0 & -\frac{3}{2} & \frac{3}{4} & \frac{5}{4} \\ 0 & 1 & -\frac{3}{2} & -\frac{7}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Therefore, $\begin{cases} x_1 = \frac{3}{2}x_3 - \frac{3}{4}x_4 + \frac{5}{4} \\ x_2 = \frac{3}{2}x_3 + \frac{7}{4}x_4 - \frac{1}{4} \end{cases}$.

5. Please solving the linear equations $\begin{cases} x_1 - x_2 - x_3 = 2 \\ 2x_1 - x_2 - 3x_3 = 1 \\ 3x_1 + 2x_2 - 5x_3 = 0 \end{cases}$.

Hints: $\begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -1 & -3 & 1 \\ 3 & 2 & -5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 5 & -2 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to$
$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$. Therefore, $x_1 = 5, x_2 = 0, x_3 = 3$.

6. Please solving the equation: $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A} = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 2 & -2 \\ -1 & 3 & 2 \end{bmatrix}$ and
$\mathbf{B} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ -2 & 5 \end{bmatrix}$.

Hints: $\mathbf{A}, \mathbf{B} = \begin{bmatrix} 2 & 1 & -3 & 1 & -1 \\ 1 & 2 & -2 & 2 & 0 \\ -1 & 3 & 2 & -2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 & 2 & 0 \\ 0 & -3 & 1 & -3 & -1 \\ 0 & 5 & 0 & 0 & 5 \end{bmatrix} \to$
$\begin{bmatrix} 1 & 0 & 0 & -4 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -3 & 2 \end{bmatrix}$. Therefore, $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} -4 & 2 \\ 0 & 1 \\ -3 & 2 \end{bmatrix}$ is the solution.

Tutorial 3: Vector Spaces and Linear Spaces

Tips: The following exercises are provided **for reference only** and as a supplement to the tutorial material.

1. For what values of a are the following vectors linearly dependent?

$$\mathbf{a}_1 = \begin{pmatrix} a \\ 1 \\ 1 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 1 \\ a \\ -1 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} 1 \\ -1 \\ a \end{pmatrix}$$

Hints: Vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are linearly dependent if and only if $R(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) < 3$, that is, $|\mathbf{A}| = 0$, where $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$.

$$|\mathbf{A}| = \begin{vmatrix} a & 1 & 1 \\ 1 & a & -1 \\ 1 & -1 & a \end{vmatrix} = \begin{vmatrix} 0 & a+1 & 1-a^2 \\ 0 & a+1 & -1-a \\ 1 & -1 & a \end{vmatrix} = (a+1)^2(a-2).$$

Therefore, the vectors are linearly dependent when $a = -1$ or $a = 2$.

2. Given that vectors $\mathbf{a}_1$, $\mathbf{a}_2$ are linearly independent, and $\mathbf{a}_1 + \mathbf{b}$, $\mathbf{a}_2 + \mathbf{b}$ are linearly dependent, find the linear representation of vector $\mathbf{b}$ using $\mathbf{a}_1$, $\mathbf{a}_2$.

Hints: Let $\mathbf{b} = k_1\mathbf{a}_1 + k_2\mathbf{a}_2$. Since $\mathbf{a}_1 + \mathbf{b}$, $\mathbf{a}_2 + \mathbf{b}$ are linearly dependent, there exist non-zero x_1, x_2 such that

$$x_1(\mathbf{a}_1 + \mathbf{b}) + x_2(\mathbf{a}_2 + \mathbf{b}) = \mathbf{0}.$$

Substituting $\mathbf{b} = k_1\mathbf{a}_1 + k_2\mathbf{a}_2$, we get

$$[x_1(k_1 + 1) + x_2k_1]\mathbf{a}_1 + [x_1k_2 + x_2(k_2 + 1)]\mathbf{a}_2 = \mathbf{0}. \tag{2.1}$$

Since $\mathbf{a}_1$, $\mathbf{a}_2$ are linearly independent, we have

$$\begin{cases} x_1(k_1 + 1) + x_2k_1 = 0, \\ x_1k_2 + x_2(k_2 + 1) = 0. \end{cases} \tag{2.2}$$

For non-zero x_1, x_2, the coefficient matrix of system (4.2) must have zero determinant, thus

$$\begin{vmatrix} k_1 + 1 & k_1 \\ k_2 & k_2 + 1 \end{vmatrix} = k_1 + k_2 + 1 = 0.$$

Therefore,

$$\mathbf{b} = k_1\mathbf{a}_1 + k_2\mathbf{a}_2 = k_1\mathbf{a}_1 - (1 + k_1)\mathbf{a}_2,$$

or equivalently

$$\mathbf{b} = c\mathbf{a}_1 - (1+c)\mathbf{a}_2, \quad (c \in \mathbb{R}).$$

3. Let $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ be a group of n-dimensional vectors. Given that n unit coordinate vectors $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ can be linearly represented by them, prove that $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ are linearly independent.

   Hints: If vector group $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ can be linearly represented by vector group $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$, then

$$R(\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n) \leq R(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n). \tag{1}$$

   And since

$$R(\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n) = n, \text{ and } R(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n) \leq n,$$

   therefore

$$R(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n) = n.$$

   Thus $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ are linearly independent.

4. Let $\mathbf{b}_1 = \mathbf{a}_1$, $\mathbf{b}_2 = \mathbf{a}_1 + \mathbf{a}_2$, $\cdots$, $\mathbf{b}_r = \mathbf{a}_1 + \mathbf{a}_2 + \cdots + \mathbf{a}_r$, and vector group $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_r$ is linearly independent. Prove that vector group $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_r$ is linearly independent.

   Hints: Let

$$k_1\mathbf{b}_1 + k_2\mathbf{b}_2 + \cdots + k_r\mathbf{b}_r = \mathbf{0}, \tag{4.1}$$

   then

$$(k_1 + \cdots + k_r)\mathbf{a}_1 + (k_2 + \cdots + k_r)\mathbf{a}_2 + \cdots + (k_i + \cdots + k_r)\mathbf{a}_i + \cdots + k_r\mathbf{a}_r = \mathbf{0}. \tag{4.2}$$

   Since vector group $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_r$ is linearly independent, we have

$$\begin{cases} k_1 + k_2 + \cdots + k_r = 0, \\ k_2 + \cdots + k_r = 0, \\ \cdots\cdots \\ k_r = 0. \end{cases} \tag{4.3}$$

   By back substitution, we can directly solve to get $k_1 = k_2 = \cdots = k_r = 0$. Therefore, equation (4.1) holds if and only if $k_1 = k_2 = \cdots = k_r = 0$. Thus $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_r$ are linearly independent.

5. Let vector group $B : \mathbf{b}_1, \cdots, \mathbf{b}_r$ be linearly represented by vector group $A : \mathbf{a}_1, \cdots, \mathbf{a}_s$ as

$$(\mathbf{b}_1, \cdots, \mathbf{b}_r) = (\mathbf{a}_1, \cdots, \mathbf{a}_s)\mathbf{K}$$

where $\mathbf{K}$ is an $s \times r$ matrix, and vector group A is linearly independent. Please prove that the necessary and sufficient condition for vector group B to be linearly independent is that the rank $R(\mathbf{K}) = r$.

Hints: (Necessity) Suppose vector group B is linearly independent. Let $\mathbf{B} = (\mathbf{b}_1, \cdots, \mathbf{b}_r)$, $\mathbf{A} = (\mathbf{a}_1, \cdots, \mathbf{a}_s)$ then we have

$$\mathbf{B} = \mathbf{AK}. \tag{5.1}$$

From matrix properties, we know

$$R(\mathbf{B}) = R(\mathbf{AK}) \leq R(\mathbf{K}). \tag{5.2}$$

Since vector group B is linearly independent, we know $R(\mathbf{B}) = r$, thus $R(\mathbf{K}) \geq r$.

Also, $\mathbf{K}$ is an $r \times s$ matrix, so $R(\mathbf{K}) \leq \min\{r, s\} \leq r$.

Therefore, we know $R(\mathbf{K}) = r$.

(Sufficiency) If $R(\mathbf{K}) = r$. Let

$$x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_r\mathbf{b}_r = \mathbf{0}. \tag{5.3}$$

To prove equation (5.3) has only zero solution. Let equation (5.3) be written as

$$\mathbf{Bx} = \mathbf{0}. \tag{5.4}$$

Substituting (5.1) gives

$$\mathbf{AKx} = \mathbf{0}. \tag{5.5}$$

Since vector group $A : \mathbf{a}_1, \cdots, \mathbf{a}_s$ is linearly independent, we have $R(\mathbf{A}) = r$. Therefore, equation (5.5) has only zero solution:

$$\mathbf{Kx} = \mathbf{0}. \tag{5.6}$$

Also, since $R(\mathbf{K}) = r$, equation (4.13) has only zero solution:

$$\mathbf{x} = \mathbf{0}$$

Therefore, $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_r$ are linearly independent.

6. Given a 3rd order matrix $\mathbf{A}$ and a 3-dimensional vector $\mathbf{x}$ satisfying $\mathbf{A}^3\mathbf{x} = 3\mathbf{A}\mathbf{x} - \mathbf{A}^2\mathbf{x}$, and vectors $\mathbf{x}$, $\mathbf{A}\mathbf{x}$, $\mathbf{A}^2\mathbf{x}$ are linearly independent. Let $\mathbf{P} = (\mathbf{x}, \mathbf{A}\mathbf{x}, \mathbf{A}^2\mathbf{x})$, find a 3rd order matrix $\mathbf{B}$ such that $\mathbf{AP} = \mathbf{PB}$ and find $|\mathbf{A}|$.

   Hints: (1) From $\mathbf{P} = (\mathbf{x}, \mathbf{A}\mathbf{x}, \mathbf{A}^2\mathbf{x})$, we have

$$\begin{aligned}\mathbf{AP} &= \mathbf{A}(\mathbf{x}, \mathbf{A}\mathbf{x}, \mathbf{A}^2\mathbf{x}) \\ &= (\mathbf{A}\mathbf{x}, \mathbf{A}^2\mathbf{x}, \mathbf{A}^3\mathbf{x}) \\ &= (\mathbf{x}, \mathbf{A}\mathbf{x}, \mathbf{A}^2\mathbf{x}) \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix} \qquad (\text{since } \mathbf{A}^3\mathbf{x} = 3\mathbf{A}\mathbf{x} - \mathbf{A}^2\mathbf{x}) \\ &= \mathbf{P} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}.\end{aligned}$$

   Note that matrix $\mathbf{P}$ is a 3rd order matrix, and since vectors $\mathbf{x}$, $\mathbf{A}\mathbf{x}$, $\mathbf{A}^2\mathbf{x}$ are linearly independent, matrix $\mathbf{P}$ is invertible. Therefore,

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP} = \mathbf{P}^{-1}\mathbf{P} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}.$$

   (2) From $\mathbf{A} = \mathbf{PBP}^{-1}$, taking determinants of both sides gives

$$|\mathbf{A}| = |\mathbf{B}| = 0.$$

Tutorial 4: Determinant

Tips: The following exercises are provided **for reference only** and as a supplement to the tutorial material.

1. Please calculate $\begin{vmatrix} 2 & 0 & 1 \\ 1 & -4 & -1 \\ -1 & 8 & 3 \end{vmatrix}$.

   Hints:

   $$\begin{vmatrix} 2 & 0 & 1 \\ 1 & -4 & -1 \\ -1 & 8 & 3 \end{vmatrix} = 2\times(-4)\times3+0\times(-1)\times1+1\times8-0\times1\times3-2\times(-1)\times8-1\times(-4)\times(-1)$$

   $$= -24 + 8 + 16 - 4 = -4.$$

   .

2. Please calculate $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

   Hints:

   $$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = acb + bac + cba - bbb - aaa - ccc = 3abc - a^3 - b^3 - c^3.$$

   .

3. Please calculate $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$.

   Hints:

   $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = bc^2 + ca^2 + ab^2 - ac^2 - ba^2 - cb^2 = (a-b)(b-c)(c-a).$$

   .

4. Please calculate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$.

   Hints:

   $$\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} = x(x+y)y+yx(x+y)+(x+y)yx-y^3-(x+y)^3-x^3$$

   $$= 3xy(x+y) - y^3 - 3x^2y - 3y^2x - x^3 - y^3 = -2(x^3+y^3).$$

   .

5. Please calculate $\begin{vmatrix} a^2 & ab & b^2 \\ 2a & a+b & 2b \\ 1 & 1 & 1 \end{vmatrix}$.

Hints:

$$\begin{vmatrix} a^2 & ab & b^2 \\ 2a & a+b & 2b \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a^2 & ab-a^2 & b^2-a^2 \\ 2a & b-a & 2b-2a \\ 1 & 0 & 0 \end{vmatrix} = (-1)^{3+1} \begin{vmatrix} ab-a^2 & b^2-a^2 \\ b-a & 2b-2a \end{vmatrix}$$

$$= (b-a)(b-a) \begin{vmatrix} a & b+a \\ 1 & 2 \end{vmatrix} = (a-b)^3.$$

6. Please calculate $\begin{vmatrix} ax+by & ay+bz & az+bx \\ ay+bz & az+bx & ax+by \\ az+bx & ax+by & ay+bz \end{vmatrix}$.

Hints:

$$\begin{vmatrix} ax+by & ay+bz & az+bx \\ ay+bz & az+bx & ax+by \\ az+bx & ax+by & ay+bz \end{vmatrix} = a \begin{vmatrix} x & ay+bz & az+bx \\ y & az+bx & ax+by \\ z & ax+by & ay+bz \end{vmatrix} + b \begin{vmatrix} y & ay+bz & az+bx \\ z & az+bx & ax+by \\ x & ax+by & ay+bz \end{vmatrix}.$$

$$= a^2 \begin{vmatrix} x & ay+bz & z \\ y & az+bx & x \\ z & ax+by & y \end{vmatrix} + b^2 \begin{vmatrix} y & z & az+bx \\ z & x & ax+by \\ x & y & ay+bx \end{vmatrix} = a^3 \begin{vmatrix} y & z & x \\ z & x & y \\ x & y & z \end{vmatrix} + b^3 \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}.$$

$$= (a^3+b^3) \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}.$$

Tutorial 5: Eigenvalue and Eigenvector

Tips: The following exercises are provided **for reference only** and as a supplement to the tutorial material.

1. Please find the eigenvalues and eigenvectors of $\begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}$.

   Hints: By

   $|\mathbf{A} - \lambda\mathbf{E}| = \begin{vmatrix} 2-\lambda & -1 & 2 \\ 5 & -3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{vmatrix} = -(\lambda+1)^3,$

   Therefore, the eigenvalues of $\mathbf{A}$ are $\lambda_1 = \lambda_2 = \lambda_3 = -1$.

   When $\lambda_1 = \lambda_2 = \lambda_3 = -1$, solve $(\mathbf{A}+\mathbf{E})\mathbf{x} = \mathbf{0}$. By

   $(\mathbf{A}+\mathbf{E}) = \begin{pmatrix} 3 & -1 & 2 \\ 5 & -2 & 3 \\ -1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & -1 & -1 \\ 0 & -2 & -2 \\ -1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$

   We get the basic solution $\mathbf{p} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$. Therefore, $k\mathbf{p}$ $(k \neq 0)$ is the general eigenvector corresponding to $\lambda_1 = \lambda_2 = \lambda_3 = -1$..

2. Let $\mathbf{A}$, $\mathbf{B}$ be n-th order matrices satisfying $R(\mathbf{A}) + R(\mathbf{B}) < n$. Prove that $\mathbf{A}$ and $\mathbf{B}$ have common eigenvalues and common eigenvectors.

   Hints: Since $R(\mathbf{A}) + R(\mathbf{B}) < n$, we have $R(\mathbf{A}) < n$, and

   $$R(\mathbf{A}) < n \Leftrightarrow |\mathbf{A}| = 0 \Leftrightarrow |\mathbf{A} - 0\mathbf{E}| = 0 \Leftrightarrow 0 \text{ is an eigenvalue of } \mathbf{A}.$$

   Similarly, 0 is also an eigenvalue of $\mathbf{B}$. Therefore, $\mathbf{A}$ and $\mathbf{B}$ have a common eigenvalue 0.

   Now we prove $\mathbf{A}$ and $\mathbf{B}$ have common eigenvectors corresponding to $\lambda = 0$.

   $\mathbf{A}$ and $\mathbf{B}$ have common eigenvectors corresponding to $\lambda = 0$

   $\Leftrightarrow$ there exists a non-zero vector $\mathbf{p}$ simultaneously satisfying $\mathbf{Ap} = \mathbf{0p}, \mathbf{Bp} = \mathbf{0p}$

   $\Leftrightarrow$ system of equations $\begin{cases} \mathbf{Ax} = \mathbf{0} \\ \mathbf{Bx} = \mathbf{0} \end{cases}$ has non-zero solutions

   $\Leftrightarrow$ system of equations $\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} \mathbf{x} = \mathbf{0}$ has non-zero solutions

   And

$$R\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} \leq R(\mathbf{A}) + R(\mathbf{B}) < n.$$

Therefore, $\mathbf{A}$ and $\mathbf{B}$ have common eigenvectors.

3. Let $\mathbf{A}$ be an n-th order matrix. Prove that $\mathbf{A}^{\mathrm{T}}$ and $\mathbf{A}$ have the same eigenvalues.

   Hints: It suffices to prove that they have the same characteristic equation (or characteristic polynomial). By the property $|\mathbf{A}^{\mathrm{T}}| = |\mathbf{A}|$, we know

   $$|\mathbf{A}^{\mathrm{T}} - \lambda\mathbf{E}| = |(\mathbf{A} - \lambda\mathbf{E})^{\mathrm{T}}| = |\mathbf{A} - \lambda\mathbf{E}|. \tag{2}$$

   Therefore, $\mathbf{A}^{\mathrm{T}}$ and $\mathbf{A}$ have the same eigenvalues.

4. Let $\lambda \neq 0$ be an eigenvalue of the $m \times n$ matrix $\mathbf{A}_{m\times n}\mathbf{B}_{n\times m}$. Please prove that λ is also an eigenvalue of the $n \times n$ matrix $\mathbf{BA}$.

   Hints: Let $\mathbf{p}$ be an eigenvector of $\mathbf{A}_{m\times n}\mathbf{B}_{n\times m}$ corresponding to eigenvalue λ, then

   $$(\mathbf{AB})\mathbf{p} = \lambda\mathbf{p}$$

   Multiply both sides of the equation by $\mathbf{B}$ to get $\mathbf{B}(\mathbf{AB})\mathbf{p} = \mathbf{B}\lambda\mathbf{p}$, i.e.,

   $$(\mathbf{BA})(\mathbf{Bp}) = \lambda(\mathbf{Bp})$$

   Now we prove that $\mathbf{Bp}$ is non-zero. Because, if $\mathbf{Bp} = \mathbf{0}$, then from the first equation, the left side $(\mathbf{AB})\mathbf{p} = \mathbf{A}(\mathbf{Bp}) = \mathbf{0}$; however, since $\lambda \neq 0$ and eigenvector $\mathbf{p}$ is non-zero, we have $\lambda\mathbf{p} \neq \mathbf{0}$. This leads to a contradiction.

   Therefore, λ is also an eigenvalue of the $n \times n$ matrix $\mathbf{BA}$.

5. Let $\mathbf{A}$ be an orthogonal matrix, and $|\mathbf{A}| = -1$. Please prove that $\lambda = -1$ is an eigenvalue of $\mathbf{A}$.

   Hints: We need to prove that $\lambda = -1$ satisfies the characteristic equation $|\mathbf{A} - \lambda\mathbf{E}| = 0$, i.e., $|\mathbf{A} + \mathbf{E}| = 0$. Because

   $$\begin{aligned}
   |\mathbf{A} + \mathbf{E}| &= |\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{A}| && (\mathbf{A} \text{ is orthogonal})\\
   &= |\mathbf{E} + \mathbf{A}^{\mathrm{T}}||\mathbf{A}| \\
   &= -|\mathbf{A}^{\mathrm{T}} + \mathbf{E}| && (|\mathbf{A}| = -1)\\
   &= -|(\mathbf{A} + \mathbf{E})^{\mathrm{T}}| \\
   &= -|\mathbf{A} + \mathbf{E}|
   \end{aligned}$$

   Therefore $2|\mathbf{A} + \mathbf{E}| = 0$, i.e., $|\mathbf{A} + \mathbf{E}| = 0$. This proves that $\lambda = -1$ is an eigenvalue of $\mathbf{A}$.

6. Let $\mathbf{A}^2 - 3\mathbf{A} + 2\mathbf{E} = \mathbf{O}$. Please prove that the eigenvalues of $\mathbf{A}$ can only be 1 or 2.

   Hints: Let λ be an eigenvalue of $\mathbf{A}$, then $\lambda^2 - 3\lambda + 2$ is an eigenvalue of $\mathbf{A}^2 - 3\mathbf{A} + 2\mathbf{E}$. Therefore, there exists a non-zero vector $\mathbf{p}$ such that

   $$(\mathbf{A}^2 - 3\mathbf{A} + 2\mathbf{E})\mathbf{p} = (\lambda^2 - 3\lambda + 2)\mathbf{p}$$

   Also, since $\mathbf{A}^2 - 3\mathbf{A} + 2\mathbf{E} = \mathbf{O}$, substituting into the above equation yields

   $$(\lambda^2 - 3\lambda + 2)\mathbf{p} = \mathbf{0}$$

   Since the eigenvector $\mathbf{p} \neq \mathbf{0}$, we must have

   $$\lambda^2 - 3\lambda + 2 = 0$$

   Solving this equation gives $\lambda = 1$ or 2.

   Therefore, the eigenvalues of $\mathbf{A}$ can only be 1 or 2.