MATH 656

Extra Credit Problems

Proof of Mac Lane's Theorem

Recall that a **2-basis** for a graph G is a basis $\mathscr{B} = \{C_1, C_2, \ldots, C_n\}$ for the cycle space $\mathscr{C}(G)$ having the property that no edge of G belongs to more than 2 elements of $\mathscr{B}$.

In the case that G has no cycles (that is, it is a forest), then $\mathscr{C}(G) = \{\emptyset\}$ is zero-dimensional. In this case the only basis is $\mathscr{B} = \emptyset$. Observe that this is vacuously a 2-basis because certainly no edge of G belongs to more than two elements of $\mathscr{B}$.

Observe that if $\mathscr{B} = \{C_1, C_2, \ldots, C_n\}$ is a 2-basis for G, then G must have some edges that belong to only one C_i, because otherwise every edge of G would be on exactly zero or two elements of $\mathscr{B}$, and this would yield $\sum_{i=1}^{n} C_i = \emptyset$, violating linear independence of $\mathscr{B}$.

Theorem (Saunders Mac Lane, 1937): A graph is planar if and only if it has a 2-basis.

Exercise: The following optional exercises culminate with a proof of Mac Lane's theorem (using Kuratowski's theorem).

1. Prove that $K_{3,3}$ has no 2-basis.
2. Prove that K_5 has no 2-basis.
3. Prove that if a graph G has a 2-basis, then any subgraph H of G has a 2-basis.

   Suggestion: Use induction on the number of edges in G.
4. Prove that if a subdivision of a graph K has a 2-basis, then K has a 2-basis.
5. Use the above facts (and Kuratowski's theorem) to prove Mac Lane's theorem.