MATH 501, Section 22 Solutions

4. (a) $(2x^3 + 4x^2 + 3x + 2)(3x^4 + 2x + 4) = \boxed{3x^4 + 2x^3 + 4x^2 + 1}$

   (b) $(2x^3 + 4x^2 + 3x + 2)(3x^4 + 2x + 4) = \boxed{x^7 + 2x^6 + 4x^5 + x^3 + 2x^2 + x + 3}$

6. How many polynomials of degree 2 or less are there in $\mathbb{Z}_5[x]$?

   Such a polynomial will have form $a + bx + cx^2$. Since a, b, c are all in $\mathbb{Z}_5$, there are five possibilities for each of them. This gives a total of $5 \cdot 5 \cdot 5 = \boxed{\text{125 polynomials.}}$

8. $\varphi_i(2x^3 - x^2 + 3x + 2) = 2(i)^3 - (i)^2 + 3i + 2 = \boxed{3 + i}$

14. Find all the zeros of $f(x) = x^5 + 3x^3 + x^2 + 2x$ in $\mathbb{Z}_5$.

    Let's just plug in all the elements and see what we get:
    $f(0) = 0^5 + 3 \cdot 0^3 + 0^2 + 2 \cdot 0 = 0$
    $f(1) = 1^5 + 3 \cdot 1^3 + 1^2 + 2 \cdot 1 = 2$
    $f(2) = 2^5 + 3 \cdot 2^3 + 2^2 + 2 \cdot 2 = 4$
    $f(3) = 3^5 + 3 \cdot 3^3 + 3^2 + 2 \cdot 3 = 4$
    $f(4) = 4^5 + 3 \cdot 4^3 + 4^2 + 2 \cdot 4 = 0$

    Thus $\boxed{\text{the zeros are 0 and 4.}}$

22. Find a polynomial of positive degree in $\mathbb{Z}_4[x]$ that is a unit.

    Notice that $\boxed{2x + 1 \text{ is a unit}}$ because it is its own inverse: $(2x + 1)(2x + 1) = 4x^2 + 4x + 1 = 1$.

    Editiorial Comment: That would NEVER happen in a ring of polynomials over an integral domain.

25. (a) Let D be an integral domain. Find the units in $D[x]$.

    Suppose we have two polynonials $f(x)$ and $g(x)$ in $D[x]$. Notice that the highest degree term in the product $f(x)g(x)$ is just the product of the two highest degree terms of $f(x)$ and $g(x)$, respectively. The coefficient of that term is the product of the coefficients of the two highest terms of$f(x)$ and $g(x)$, and cannot be 0 since D has no zero divisors. The degree of that highest term is the sum of the degrees of $f(x)$ and $g(x)$. What all this means is that the fact that D is an integral domain forces the following equation to hold:

    $$\text{degree}(\ f(x)g(x)\ ) = \text{degree}(\ f(x)\ ) + \text{degree}(\ g(x)\ )$$

    It follows that the only way $f(x)g(x)$ can be the degree 0 polynomial 1 is if $f(x)$ and $g(x)$ both have degree 0, that is if they are both elements of D itself. (i.e. they must be polynomials of the form $a + 0x$) Thus, the units in $D[x]$ are the units in D.

    (b) By the above, the units in $\mathbb{Z}[x]$ are 1 and -1.

    (c) By the above, the units in $\mathbb{Z}_7[x]$ are $1, 2, 3, 4, 5, 6$.