Introduction to Mathematical Reason	Midterm MATH 300	March 17, 2008
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Directions: Please answer the questions in the space provided. To get full credit you must show all of your work. Use of calculators and other computing or communication devices is not allowed on this test.

1. Write each of the following sets by listing its elements or describing it with a familiar symbol or symbols.

   (a) $\{n \in \mathbb{Z} : |n| \leq 2\} = \boxed{\{-2, -1, 0, 1, 2\}}$

   (b) $\{x \in \mathbb{R} \,:\, \cos(\pi x) = -1\} = \boxed{\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}} = \boxed{\{2n + 1 : n \in \mathbb{Z}\}}$

   (c) $\{X \in \mathscr{P}(\mathbb{N}) \,:\, X \cap \{1, 2\} = X\} = \boxed{\{\{\}, \{1\}, \{2\}, \{1, 2\}\}}$

   (d) $\displaystyle\bigcap_{n \in \mathbb{N}} \left[1, 2 + \tfrac{1}{n}\right] = \boxed{[1, 2]} = \boxed{\{x \in \mathbb{R} : 1 \leq x \leq 2\}}$

   (e) $\mathscr{P}(\{1\}) \times \mathscr{P}(\{2\}) = \boxed{\{\ (\emptyset, \emptyset),\ (\emptyset, \{2\}),\ (\{1\}, \emptyset),\ (\{1\}, \{2\})\ \}}$

2. Write a truth table to decide if $(\sim P) \Rightarrow Q$ and $(P \wedge Q) \Rightarrow P$ are logically equivalent.

P	Q	$\sim P$	$P \wedge Q$	$(\sim P) \Rightarrow Q$	$(P \wedge Q) \Rightarrow P$
T	T	F	T	**T**	**T**
T	F	F	F	**T**	**T**
F	T	T	F	**T**	**T**
F	F	T	F	**F**	**T**

Statements $(\sim P) \Rightarrow Q$ and $(P \wedge Q) \Rightarrow P$ are **NOT** logically equivalent because their truth tables do not match up.

3. This problem concerns the following statement.
P : For every subset X of $\mathbb{N}$, there is an integer m for which $|X| = m$.

(a) Is the statement P true or false? Explain.

P says that no matter which set $X \subseteq \mathbb{N}$ you might pick,
there will always be an integer m (depending on X) for which $|X| = m$.

The statement is **FALSE.**
Notice that the set $X = \{2, 4, 6, 8, \ldots\}$ of even numbers is a subset of $\mathbb{N}$,
but there is no integer m for which $|X| = m$, because X is infinite.

Therefore it is untrue that for every subset X of $\mathbb{N}$, there is an integer m for which $|X| = m$.

(b) Form the negation $\sim P$. Write your answer as an English sentence.

Symbolically, the statement P is $\forall X \subseteq \mathbb{N}, \exists m \in \mathbb{Z}, |X| = m$
Its negation is

$$\begin{aligned}
\sim P &= \sim (\forall X \subseteq \mathbb{N}, \exists m \in \mathbb{Z}, |X| = m) \\
&= \exists X \subseteq \mathbb{N}, \sim (\exists m \in \mathbb{Z}, |X| = m) \\
&= \exists X \subseteq \mathbb{N}, \forall m \in \mathbb{Z}, \sim (|X| = m) \\
&= \exists X \subseteq \mathbb{N}, \forall m \in \mathbb{Z}, |X| \neq m
\end{aligned}$$

Translating back into words, the negation is:
There is a subset X of $\mathbb{N}$ for which for every integer m we have $|X| \neq m$.

Putting this into a more natural form, the translation is
$\boxed{\text{There is a subset } X \text{ of } \mathbb{N} \text{ for which } |X| \neq m \text{ for every integer } m.}$

4. Suppose that $(R \Rightarrow S) \vee \sim (P \wedge Q)$ is **false**.
Is there enough information to determine the truth values of P, Q, R and S? If so, what are they?
(This is most easily done without a truth table.)

For this to be false, both $R \Rightarrow S$ and $\sim (P \wedge Q)$ must be false.
The only way that $R \Rightarrow S$ can be false is if $\boxed{R \text{ is true and } S \text{ is false.}}$
The only way that $\sim (P \wedge Q)$ can be false is if $P \wedge Q$ is true.
The only way that $P \wedge Q$ can be true is if $\boxed{P \text{ and } Q \text{ are both true}}$.

Therefore $\boxed{R, P \text{ and } Q \text{ are all true, and } S \text{ is false.}}$

5. Suppose $a, b, c, d \in \mathbb{Z}$ and $n \in \mathbb{N}$. Prove that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$. (Suggestion: Try direct proof.)

   Proof. (Direct) Suppose $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$.
   By definition of congruence modulo n, this means $n|(a-b)$ and $n|(c-d)$.
   By definition of divisibility, $a - b = nk$ and $c - d = n\ell$ for some $k, \ell \in \mathbb{Z}$.
   Therefore we have $a = b + nk$ and $c = d + n\ell$. Consequently,

$$\begin{aligned} ac &= (b+nk)(d+n\ell) \\ ac &= bd + bn\ell + nkd + n^2k\ell \\ ac - bd &= bn\ell + nkd + n^2k\ell \\ ac - bd &= n(b\ell + kd + nk\ell). \end{aligned}$$

   Since $b\ell + kd + nk\ell \in \mathbb{Z}$, it follows from the above equation that $n|(ac - bd)$.
   This means that $ac \equiv bd \pmod{n}$. ∎

6. Suppose $a, b \in \mathbb{Z}$. If $a^2(b^2 - 2b)$ is odd, then both a and b are odd. (Suggestion: Try contrapositive proof.)

   Proof. (Contrapositive) Suppose it is not the case that a and b are odd.
   Then, by DeMorgan's Law, a is even or b is even. Let us look at these cases separately.

   Case 1. Suppose a is even. Then $a = 2c$ for some integer c.
   Thus $a^2(b^2 - 2b) = (2c)^2(b^2 - 2b) = 2(2c^2(b^2 - 2b))$, which is even.

   Case 2. Suppose b is even. Then $b = 2c$ for some integer c.
   Thus $a^2(b^2 - 2b) = a^2((2c)^2 - 2(2c)) = 2(a^2(2c^2 - 2c))$, which is even.

   Thus in either case $a^2(b^2 - 2b)$ is even, so it is not odd. ∎

   (NOTE: A third case where both a and b are even is not necessary.
   In that case a is even, a scenario addressed in Case 1.)

7. Prove: If $a, b \in \mathbb{Z}$, then $a^2 - 4b - 2 \neq 0$.
(Suggestion: Contradiction may be easiest.)

Proof. Suppose for the sake of contradiction that $a, b \in \mathbb{Z}$ but $a^2 - 4b - 2 = 0$.
Then we have $a^2 = 4b + 2 = 2(2b + 1)$, which means a^2 is even.
Therefore a is even also, so $a = 2c$ for some integer c. Plugging this back into $a^2 - 4b - 3 = 0$ gives us

$$\begin{aligned} (2c)^2 - 4b - 2 &= 0 \\ 4c^2 - 4b - 2 &= 0 \\ 4c^2 - 4b &= 2 \\ 2c^2 - 2b &= 1 \\ 2(c^2 - b) &= 1 \end{aligned}$$

From this last equation, we conclude that 1 is an even number, a contradiction. ■

8. Suppose $x \in \mathbb{Z}$. Prove that x is even if and only if $3x + 5$ is odd.

Proof. We first use direct proof to show that if x is even, then $3x + 5$ is odd. Suppose x is even. Then $x = 2n$ for some integer n. Thus $3x + 5 = 3(2n) + 5 = 6n + 5 = 6n + 4 + 1 = 2(3n + 2) + 1$. Thus $3x + 5$ is odd because it has form $2k + 1$, where $k = 3n + 2 \in \mathbb{Z}$.

Conversely, we need to show that if $3x + 5$ is odd, then x is even. We will prove this using contrapositive proof. Suppose x is *not* even. Then x is odd, so $x = 2n + 1$ for some integer n. Thus $3x + 5 = 3(2n + 1) + 5 = 6n + 8 = 2(3n + 4)$. This means says $3x + 5$ is twice the integer $3n + 4$, so $3x + 5$ is odd, not even. ■

9. Prove that $\{9^n : n \in \mathbb{Q}\} = \{3^n : n \in \mathbb{Q}\}$.

Proof. First we will show $\{9^n : n \in \mathbb{Q}\} \subseteq \{3^n : n \in \mathbb{Q}\}$.
Suppose $a \in \{9^n : n \in \mathbb{Q}\}$. This means that $a = 9^n$ for some $n \in \mathbb{Q}$.
By definition of a rational number, we have $n = \frac{b}{c}$, where b and c integers.
Then $a = 9^n = 9^{\frac{b}{c}} = (3^2)^{\frac{b}{c}} = 3^{\frac{2a}{b}}$. Now we have $a = 3^{\frac{2b}{c}}$, so a is a rational power of 3.
Thus $a \in \{3^n : n \in \mathbb{Q}\}$. This completes the demonstration that $\{9^n : n \in \mathbb{Q}\} \subseteq \{3^n : n \in \mathbb{Q}\}$.

Next we will show $\{3^n : n \in \mathbb{Q}\} \subseteq \{9^n : n \in \mathbb{Q}\}$.
Suppose $a \in \{3^n : n \in \mathbb{Q}\}$. This means that $a = 3^n$ for some $n \in \mathbb{Q}$.
As before, $n = \frac{b}{c}$ where b and c are integers.
Then $a = 3^n = 3^{\frac{b}{c}} = (9^{\frac{1}{2}})^{\frac{b}{c}} = 3^{\frac{b}{2c}}$. Now we have $a = 9^{\frac{b}{2c}}$, so a is a rational power of 9.
Thus $a \in \{9^n : n \in \mathbb{Q}\}$. This completes the demonstration that $\{3^n : n \in \mathbb{Q}\} \subseteq \{9^n : n \in \mathbb{Q}\}$.

Now that we've shown $\{9^n : n \in \mathbb{Q}\} \subseteq \{3^n : n \in \mathbb{Q}\}$ and $\{3^n : n \in \mathbb{Q}\} \subseteq \{9^n : n \in \mathbb{Q}\}$,
it follows that $\{9^n : n \in \mathbb{Q}\} = \{3^n : n \in \mathbb{Q}\}$. ■

10. Suppose A and B are sets. Prove that if $A \times A \subseteq A \times B$, then $A \subseteq B$.

Proof. (Direct) Suppose $A \times A \subseteq A \times B$.

We now need to show that $A \subseteq B$.
To do this, suppose $x \in A$.
Then $(x, x) \in A \times A$, but since $A \times A \subseteq A \times B$, it follows that $(x, x) \in A \times B$.
But this means that $x \in A$ and $x \in B$. In particular, $x \in B$.
Now we've shown that $x \in A$ implies $x \in B$, so it follows from this that $A \subseteq B$. ■