Fall 2018 Research

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1 Theorems

Lemma 1.1. *Suppose G or H is even palindromic, then $G \times H$ is even palindromic.*

Proof. Suppose, without loss of generality, that G is even palindromic. Let $\alpha : G \to G$ defined by $x \mapsto \alpha(x)$ be even palindromic. Then $\varphi : G \times H \to G \times H$ defined by $(x, y) \mapsto (\alpha(x), y)$ is an even palindromic involution. Since α has no fixed points, $G \times H$ is even palindromic. □

Lemma 1.2. *Suppose G and H are odd palindromic, then $G \times H$ is odd palindromic.*

Proof. Suppose G and H are odd palindromic. Then G has an involution α with exactly one fixed vertex x_0, so $\alpha(x_0) = x_0$. Likewise, H has an involution β with exactly one fixed vertex y_0, so $\beta(y_0) = y_0$. Then $\varphi : G \times H \to G \times H$ defined by $(x, y) \mapsto (\alpha(x), \beta(y))$ is an involution of $G \times H$ with exactly one fixed vertex (x_0, y_0), so $G \times H$ is odd palindromic. □

Theorem 1.3. *(Handbook of Product Graphs Theorem 8.18)*

Suppose φ is an automorphism of a connected nonbipartite R-thin graph G that has a prime factorization $G = G_1 \times G_2 \times ... \times G_k$. Then there exists a permutation of π of $\{1, 2, ..., k\}$, together with isomorphisms $\varphi_i : G_{\pi(i)} \to G_i$ such that $\varphi(x_1, x_2, ..., x_k) = (\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), ..., \varphi_k(x_{\pi(k)}))$. Thus $Aut(G)$ is generated by the automorphisms of the prime factors and transpositions of isomorphic factors. Consequently, $Aut(G)$ is isomorphic to the automorphism group of the disjoint union of the prime factors of G.

Lemma 1.4. *Suppose G and H are both connected, nonbipartite, R-thin, and prime. Then:*

1. *$G \times H$ is even palindromic if and only if G or H is even palindromic*
2. *$G \times H$ is odd palindromic if and only if G and H are odd palindromic*

Proof. Suppose $G \times H$ is palindromic. Then there exists an involution $\varphi : G \times H \to G \times H$ with at most one fixed point. Suppose G and H are both connected, nonbipartite, R-thin, and prime.

Case 1: φ operates such that $\varphi(x, y) = (\alpha(x), \beta(y))$ where $\alpha : G \to G$ and $\beta : H \to H$. Observe that $\varphi(x, y) = (\alpha(x), \beta(y))$ and $\varphi^2(x, y) = (\alpha^2(x), \beta^2(y)) = (x, y)$, since $G \times H$ is palindromic. Thus, $\alpha^2(x) = x$ and $\beta^2(y) = y$ which means α and β are involutions.

Now, suppose φ has no fixed points. Then $\varphi(x, y) \neq (x, y)$ for all $(x, y) \in V(G \times H)$. Thus, $\alpha(x) \neq x$ for all $x \in G$ or $\beta(y) \neq y$ for all $y \in H$. So α or β has no fixed points. Therefore, one of G or H is even palindromic.

Now, suppose φ has one fixed point, so $\varphi(x, y) = (x, y)$. Then $\alpha(x) = x$ and $\beta(y) = y$. Since φ may have at most one fixed point, both α and β have exactly one fixed point. If one of α or β has more than one fixed point, say α, then $\alpha(x) = x$ and $\alpha(x') = x'$ for some $x \neq x'$. Thus, $\varphi(x, y) = (x, y)$ and $\varphi(x', y) = (x', y)$. Therefore, φ would have more than one fixed point. Since that cannot be the case, both G and H are odd palindromic.

Case 2: φ operates such that $\varphi(x, y) = (\alpha(y), \beta(x))$ where $\alpha : H \to G$ and $\beta : G \to H$ such that $G \cong H$. We will show Case 2 cannot happen.

Since φ is an involution, $\varphi^2(x, y) = (x, y)$. Thus, if $\varphi(x, y) = (\alpha(y), \beta(x))$, then $\varphi^2(x, y) = \varphi(\alpha(y), \beta(x)) = (\alpha(\beta(x)), \beta(\alpha(y)) = (x, y)$. So $\alpha(\beta(x)) = x$ and $\beta(\alpha(y)) = y$. Therefore, $\beta(x) = \alpha^{-1}(x)$ and $\alpha(y) = \beta^{-1}(y)$. We can rewrite φ as $\varphi(x, y) = (\alpha(y), \alpha^{-1}(x))$.

Suppose φ is an involution, and φ has no fixed points. Then $\varphi(x, y) \neq (x, y)$, so fix an (x, y) such that $\varphi(x, y) = (x', y')$ for some $(x', y') \in G \times H$ where $(x', y') \neq (x, y)$. Note that we will never have a situation such as $\varphi(x, y) = (\alpha(y), \alpha^{-1}(x)) = (x', y)$ where one coordinate element is fixed, since this implies that $\alpha(y) = x'$ and $\alpha^{-1}(x) = y$ which means α^{-1} cannot be the inverse of α. Now, when $\varphi(x,y) = (x',y')$, this means $\alpha(y) = x'$ and $\alpha^{-1}(x) = y'$. Since φ is an involution, we have $\varphi(x',y') = (x,y)$. So $\alpha(y') = x$ and $\alpha^{-1}(x') = y$. Now, note that $\varphi(x,y') = (\alpha(y'), \alpha^{-1}(x)) = (x,y')$. So (x,y') is a fixed point of φ. This contradicts our original assumption that φ has no fixed points.

Thus, φ must have a fixed point. So, suppose φ is an involution and has exactly one fixed point. Observe from our previous construction that our fixed point was at (x,y'). However, observe that $\varphi(x',y) = (\alpha(y), \alpha^{-1}(x')) = (x',y)$. Thus, (x',y) is also a fixed point of φ. Therefore, φ has more than one fixed point.

Thus, case 2 cannot happen. Therefore, G or H is even palindromic, or G and H are both odd palindromic. □

Lemma 1.5. *Suppose $G \times H$ is palindromic, and G and H are both connected, nonbipartite, and R-thin. Then G or H is even palindromic, or G and H are both odd palindromic.*

Proof. Suppose $G \times H$ is palindromic with palindromic involution φ. Consider the connected, nonbipartite, R-thin, prime factorings $G = G_1 \times \cdots \times G_j$ and $H = G_{j+1} \times \cdots \times G_k$, so we have an involution φ of $G \times H = (G_1 \times \cdots \times G_j) \times (G_{j+1} \times \cdots \times G_k)$.

Utilizing Theorem 1.3, the involution φ permutes the prime factors of this product such that the permutation π satisfies $\pi^2 = \mathrm{id}$. Using commutativity of $\times$, we can group together the prime factors G_i of G for which $1 < \pi(i) \leq j$, and call their product A. Note that $A = K_1^*$, where K_1^* is a single vertex with a loop, if no such factors G_i exist. The same applies for the graphs B and D defined below. Let B be the product of the remaining factors G_i of G. Also group together the prime factors G_i of H for which $j+1 < \pi(i) \leq k$, and call their product D. The direct product of the remaining factors of H is then a graph isomorphic to B. The structure of φ under this scheme is as indicated below, where the arrows represent isomorphisms $\varphi_i : G_{\pi(i)} \to G_i$ between factors.

$$\begin{tikzcd}[column sep=small]
G\times H \arrow[d, "\varphi"'] & = & (G_1 \times \arrow[dr] & G_2 \times \arrow[dl] & G_3 \times \arrow[d] & G_4 \times \arrow[drr] & G_5) \times \arrow[drr] & (G_6 \times \arrow[dll] & G_7 \times \arrow[dll] & G_8 \times \arrow[d] & G_9 \times \arrow[dr] & G_{10} \times \arrow[dl] & G_{11}) \arrow[d] \\
G\times H & = & (G_1 \times & G_2 \times & G_3 \times & G_4 \times & G_5) \times & (G_6 \times & G_7 \times & G_8 \times & G_9 \times & G_{10} \times & G_{11})
\end{tikzcd}$$

$$G \times H = (\overbrace{\overbrace{G_1\times G_2\times G_3}^{A}\times\overbrace{G_4\times G_5}^{B}}^{G})\times(\overbrace{\overbrace{G_6\times G_7}^{B}\times\overbrace{G_8\times G_9\times G_{10}\times G_{11}}^{D}}^{H})$$

$$G \times H = (\underbrace{G_1\times G_2\times G_3}_{A}\times\underbrace{G_4\times G_5}_{B})\times(\underbrace{G_6\times G_7}_{B}\times\underbrace{G_8\times G_9\times G_{10}\times G_{11}}_{D})$$

We have now coordinatized G and H as $G = A\times B$ and $H = B\times D$, and φ is an involution of $G\times H = (A\times B)\times(B\times D)$ for which $\varphi\big((a,b),(b',d)\big) = \big((\alpha(a),\beta(b')),(\gamma(b),\delta(d))\big)$, for automorphisms $\alpha : A \to A$, $\beta,\gamma : B \to B$ and $\delta : D \to D$. But because φ^2 is the identity, it must be that $\alpha^2 = \mathrm{id}$, $\gamma = \beta^{-1}$ and $\delta^2 = \mathrm{id}$. Thus we have involutions α and δ of A and D, respectively, and

$$\varphi\big((a,b),(b',d)\big) = \big((\alpha(a),\beta(b')),(\beta^{-1}(b),\delta(d))\big), \tag{1}$$

From (1) it is evident that the fixed points of φ (if any) are precisely

$$\big((a_0,\beta(b)),(b,d_0)\big) \quad \text{with } \alpha(a_0) = a_0,\ \delta(d_0) = d_0, \text{ and } b \in V(B). \tag{2}$$

Thus φ has a fixed point if and only if both α and δ have fixed points. Further, if φ has a fixed point, then it has exactly $|V(B)|$ of them.

Now suppose $G \times H$ is even palindromic. Let φ be an even palindromic involution of $G \times H$ (having no fixed point). From (2), at least one of α or δ has no fixed point, so suppose it is α. Then α is an even palindromic involution of A, so A is even palindromic. By the first part of the theorem, $G = A \times B$ is even palindromic. Similarly H is even palindromic if δ has no fixed points.

Suppose $G\times H$ is odd palindromic. Let φ be an odd palindromic involution whose sole fixed point is $\big((a_0,\beta(b_0)),(b_0,d_0)\big)$. The remark following (2) implies φ has at least $|V(B)|$ fixed points, so $B = K_1$. Thus we can drop B from our discussion, so $G = A$, $H = D$ and $\varphi(a,d) = \big(\alpha(a),\delta(d)\big)$. We now have involutions $\alpha : G \to G$ and $\delta : H \to H$ with fixed points a_0 and d_0, respectively. Also (a_0,d_0) is a fixed point of φ. If the involution

α of G had a second fixed point a_1, then (a_0, d_0) and (a_1, d_0) would be two distinct fixed points of φ. Thus a_0 is the only fixed point of α, so α (hence also G) is odd palindromic. By the same reasoning H is odd palindromic. $\square$

We say that vertices x and y of a graph are in relation R, written xRy, provided that each has the same open neighborhood, that is, $N_G(x) = N_G(y)$. It is straightforward to check that R is an equivalence relation on G.

Proposition 1.6. *R is an equivalence relation on G.*

Proof. To see that R is reflexive, let $x \in V(G)$. Then $N(x) = N(x)$, so $xRx\ \forall x \in V(G)$.

To see that R is symmetric, let $x, y \in V(G)$. Suppose xRy, that is, $N(x) = N(y)$, so $N(y) = N(x)$, hence yRx. Thus, xRy implies that $yRx\ \forall x, y \in V(G)$.

To see that R is transitive, suppose xRy and yRz. Then $N(x) = N(y)$ and $N(y) = N(z)$. Thus, $N(x) = N(z)$, so xRz. $\square$

We will refer to an R-equivalence class of $V(G)$ as an **R-class**.

Proposition 1.7. *The subgraph induced on an R-class is either completely disconnected or is the complete graph with loops at each vertex.*

Proof. Let $x_1, ..., x_n$ be vertices in the R-class $[x]$. Thus, $N(x_1) = N(x_2) = ... = N(x_n)$.

Case 1: $x_1 \notin N(x_1)$

If x_1 is not in its own neighborhood (i.e. there is no loop at x_1), then $x_1 \notin N(x_2) = ... = N(x_n)$. So $x_1x_i \notin E(G)$ for all $1 \leq i \leq n$. Then $N(x_1) \cap \{x_1, x_2, ..., x_n\} = \emptyset$. Therefore, $N(x_j) \cap \{x_1, ..., x_n\} = \emptyset$. Thus, $x_ix_j \notin E(G)$, so the subgraph induced on $[x]$ has no edges in it, thus it is completely disconnected.

Case 2: $x_1 \in N(x_1)$

If $x_1 \in N(x_1)$, then $x_1 \in N(x_2) = N(x_3) = ... = N(x_n)$. So, x_1 has a loop, and since $x_1 \in N(x_1)$, we have that $x_1 \in N(x_2)$ and $x_2 \in N(x_1)$, thus $x_2 \in N(x_2)$. So, x_2 also has a loop, and it must be that $x_1x_2 \in E(G)$. Thus, if $i \neq 1$, then $x_i \in N(x_i)$, so $x_1x_i \in E(G)$. Hence $N(x_1) = [x] = N(x_2) = N(x_3) = ... = N(x_n)$. Thus any vertex of $[x]$ is adjacent to all other vertices of $[x]$. $\square$

Let G/R be the graph whose vertices are the R-classes of a graph G in Γ_0 (where Γ_0 is the set of finite graphs in which loops are admitted), and there is an edge joining R-classes X and Y if G has an edge from X to Y.

Proposition 1.8. *If $X, Y \in G/R$, then either every vertex of X is adjacent in G to every vertex of Y or no vertex of X is adjacent to any vertex of Y.*

Proof. Let $X, Y \in G/R$. Let $x_1, ..., x_n \in X$ and $y_1, ..., y_n \in Y$. Then $N(x_1) = N(x_2) = ... = N(x_n)$ and $N(y_1) = N(y_2) = ... = N(y_n)$.

Case 1: X and Y are adjacent.

Without loss of generality, select an arbitrary $x_n \in X$. Since X and Y are adjacent in G/R, we have that x_n is adjacent to some y_n since G has an edge from X to Y. So, $x_n \in N(y_n)$, so $x_n \in N(y_1) = N(y_2) = ... = N(y_n)$. Hence, x_n is adjacent to $y_1, ..., y_n$, and since $N(x_1) = ... = N(x_n)$, we have that all $x_1, ..., x_n$ are adjacent to all $y_1, ..., y_n$.

Case 2: X and Y are not adjacent.

Without loss of generality, select an arbitrary $x_n \in X$. Since X and Y are not adjacent in G/R, we have that $y_1, ..., y_n \notin N(x_n)$. So, x_n is not adjacent to any $y_1, ..., y_n$, and since $N(x_1) = ... = N(x_n)$, we have that none of $x_1, ..., x_n$ are adjacent to any $y_1, ..., y_n$. □

Let $[x] = \{y \in V(G) \mid N_G(x) = N_G(y)\}$.

Proposition 1.9. *For $G \in \Gamma_0$, $xy \in E(G) \iff [x][y] \in E(G/R)$.*

Proof. This follows from the proof of Proposition 1.8. □

A graph is called **R-thin** if all of its R-classes contain just one vertex.

Proposition 1.10. *G/R is always R-thin.*

Proof. Suppose not. Then for two distinct $[x], [y] \in V(G/R)$, we must have that $N([x]) = N([y])$. Let $[g] \in N([x])$, meaning that $[x][g] \in E(G/R)$. By Proposition 1.8, since $[x]$ and $[g]$ are adjacent, we have that all $x_1, .., x_n \in [x]$ are adjacent to all $g_1, ..., g_n \in [g]$ in G. Since $N([x]) = N([y])$, we have that if $[g] \in N([x])$, then $[g] \in N([y])$. Again, by Proposition 1.8, we have that all $g_1, ..., g_n \in [g]$ are adjacent to all $y_1, ..., y_n \in [y]$ in G. So, for all $[g] \in [x]$ (and therefore, for all $[g] \in [y]$), we have that $g_i \in N(x_1) = ... = N(x_n)$ for all $i = 1, ..., n$ and $g_i \in N(y_1) = ... = N(y_n)$. Thus, $N(x_1) = N(x_2) = ... = N(x_n) = N(y_1) = ... = N(y_n)$, meaning that $[x] = [y]$ which contradicts our assumption. Therefore, G/R is always R-thin. □

Because R is defined in terms of the adjacency structure of a graph, given an isomorphism $\varphi : G \to H$, we have $xR_G y$ if and only if $\varphi(x)R_H\varphi(y)$. So φ maps equivalence classes of R_G to equivalence classes of R_H, and, in particular, $\varphi([x]) = [\varphi(x)]$.

Proposition 1.11. *For any isomorphism $\varphi : G \to H$, there is a corresponding isomorphism $\tilde{\varphi} : G/R \to H/R$ such that $\tilde{\varphi}([x]) = [\varphi(x)]$.*

Proof. Observe that $\tilde{\varphi}$ is well-defined and bijective because φ is an isomorphism that maps R-classes onto R-classes. By Proposition 1.9, we have that

$$\begin{aligned}
[x][y] \in E(G/R) &\iff xy \in E(G) \\
&\iff \varphi(x)\varphi(y) \in E(H) \\
&\iff [\varphi(x)][\varphi(y)] \in E(H/R) \\
&\iff \tilde{\varphi}([x])\tilde{\varphi}([y]) \in E(H/R).
\end{aligned}$$

So, $\tilde{\varphi}$ is a corresponding isomorphism. □

However, the existence of the isomorphism $\tilde{\varphi} : G/R \to H/R$ does not necessarily mean there is an isomorphism $\varphi : G \to H$. But if $|X| = |\tilde{\varphi}(X)|$ for each $X \in V(G/R)$, then we can lift $\tilde{\varphi}$ to an isomorphism $\varphi : G \to H$ by declaring φ to restrict to a bijection $X \to \tilde{\varphi}(X)$ for each X.

Proposition 1.12. *If $|X| = |\tilde{\varphi}(X)|$ for each $X \in V(G/R)$, then we can lift $\tilde{\varphi}$ to an isomorphism $\varphi : G \to H$ by declaring φ to restrict to a bijection $X \to \tilde{\varphi}(X)$ for each X.*

Proof. Let $\tilde{\varphi}(X) : G/R \to H/R$ such that $|X| = |\tilde{\varphi}(X)|$ for all $X \in V(G/R)$, and let $\varphi : G \to H$ be any map such that $\varphi : X \to \tilde{\varphi}(X)$. Since φ maps R-classes to R-classes, and by Proposition 1.11, we have that $\varphi([x]) = \tilde{\varphi}([x])$, making φ a bijection. To see that φ is an isomorphism, we have that

$$\begin{aligned} xy \in E(G) &\iff [x][y] \in E(G/R) \\ &\iff \tilde{\varphi}([x])\tilde{\varphi}([y]) \in E(H/R) \\ &\iff \varphi([x])\varphi([y]) \in E(H/R) \\ &\iff \varphi(x)\varphi(y) \in E(H) \end{aligned}$$

. Thus we can lift $\tilde{\varphi}$ to an isomorphism φ provided we restrict φ to a bijection. □

We will make use of the following propositions:

Proposition 1.13. $N_{G\times H}((x,y)) = N_G(x) \times N_H(y)$

Proof. For $x \in V(G)$, consider $N_G(x) = \{u \mid ux \in E(G)\}$, and for $y \in V(H)$, consider $N_H(y) = \{v \mid vy \in E(H)\}$. Then:

$$\begin{aligned} (u,v) \in N_G(x) \times N_H(y) &\iff u \in N_G(x), v \in N_H(y) \\ &\iff xu \in E(G), yv \in E(H) \\ &\iff (x,y)(u,v) \in E(G \times H) \\ &\iff (u,v) \in N_{G\times H}((x,y)) \end{aligned}$$

Thus, $N_{G\times H}((x,y)) = N_G(x) \times N_N(y)$. □

Proposition 1.14. *If G and H in Γ_0 has no isolated vertices, then $V((G \times H)/R) = \{X \times Y \mid X \in V(G/R),\ Y \in V(H/R)\}$. In particular, $[(x,y)] = [x] \times [y]$.*

Proof. For some arbitrary $[(x,y)] \in (G \times H)/R$, observe that $(x',y') \in [(x,y)] \iff N_{G\times H}((x',y') = N_{G\times H}((x,y))$. Then, by Proposition 1.13,

$$\begin{aligned} (x',y') \in [(x,y)] &\iff N_G(x') \times N_H(y') = N_G(x) \times N_H(y) \\ &\iff N_G(x') = N_G(x), N_H(y') = N_H(y), \end{aligned}$$

since there are no isolated vertices (this is due to notions in set theory: if $X \times Y = Z \times W$, then $X = Z$ and $Y = W$ provided that $X = Y = W = Z \neq \emptyset$). Then, $(x', y') \in [(x, y)] \iff x' \in [x]$ and $y' \in [y]$, so $(x', y') \in [x] \times [y]$. Therefore, $[(x, y)] = [x] \times [y]$. $\square$

Proposition 1.15. *$(G \times H)/R \cong G/R \times H/R$ with isomorphism $[(x, y)] \mapsto ([x], [y])$.*

Proof. To see that $[(x, y)] \mapsto ([x], [y])$ is bijective, first let $([x], [y]) = ([z], [w])$, so $[x] = [z]$ and $[y] = [w]$ meaning that $[x] \times [y] = [z] \times [w]$, and by Proposition 1.14, $[(x, y)] = [(z, w)]$. Hence, this map is injective. The map is surjective since given some $([x], [y])$ in the codomain, there certainly exists an $[(x, y)]$ in the domain provided we have $x \in G$ and $y \in H$.

Consider $[(x, y)][(x', y')] \in E(G \times H)/R$. By Proposition 1.8, $[(x, y)][x', y'] \in E(G \times H)/R) \iff (x, y)(x', y') \in E(G \times H) \iff xx' \in E(G), yy' \in E(H) \iff [x][x'] \in E(G/R), [y][y'] \in E(H/R) \iff ([x], [y])([x'], [y']) \in E(G/R \times H/R)$. Thus the map $[(x, y)] \mapsto ([x], [y])$ is an isomorphism. So $(G \times H)/R \cong G/R \times H/R$. $\square$

Theorem 1.16. *Suppose G and H are connected and non-bipartite. Then:*

1. *G or H is even palindromic if and only if $G \times H$ is even palindromic*

2. *G and H are odd palindromic if and only if $G \times H$ is odd palindromic*

Proof. If G or H (say G) is even palindromic, then there exists an even palindromic involution α of G, so $(x, y) \mapsto (\alpha(x), y)$ is an even palindromic involution of $G \times H$. Now, suppose G and H are odd palindromic. Then G has an odd palindromic involution α with fixed point x_0, and H has an odd palindromic involution β with fixed point y_0. Then $(x, y) \mapsto (\alpha(x), \beta(y))$ is an odd palindromic involution of $G \times H$ who sole fixed point is (x_0, y_0).

Part I (Involution Structure) Let $\varphi : G \times H \to G \times H$ be an involution. By the remarks preceding this theorem, φ induces an automorphism $\tilde{\varphi}$ of the R-thin graph $(G \times H)/R \cong G/R \times H/R$. Because φ is an involution, we have $\tilde{\varphi}^2 = id$.

Take prime factorings $G/R = G_1 \times ... \times G_j$ and $H/R = G_{j+1} \times ... \times G_k$. Then $\tilde{\varphi}$ is an automorphism (of order 2, or possibly of order 1, if φ fixes each R-class) of the graph

$G/R \times H/R = (G_1 \times ... \times G_j) \times (G_{j+1} \times G_k)$. Now, $\tilde{\varphi}$ permutes the prime factors of this product in the sense of Theorem 1.3, where the permutation π satisfies $\pi^2 = id$. As in the proof of Lemma 1.5 $\square$