556F18 COURSE OVERVIEW

BUSHAW

"Begin at the beginning," the King said, very gravely, "and go on till you come to the end: then stop."
–Lewis Carroll–

Your primary study guide should be our several months of classes, and your notes from these sessions. No guarantees are made that I haven't missed things here! Theorems preceded by a $\star$ are of an acceptable level for me to ask you about their proof.

Basics

Definitions. Sec 1.1: graph, vertices, edges, incidence function, $V(G)$, $E(G)$, $|G|$, $||G||$, simple graph, incident, adjacent, order, size.
Sec 1.2: identical graphs, isomorphic, complete graph K_n, empty graph, bipartite, complete bipartite graph $K_{m,n}$
Sec 1.3: Incidence matrix, adjacency matrix
Sec 1.4 subgraph, spanning subgraph, induced subgraph, $G[S]$ for $S \subseteq V(G)$, $G - S$ for $S \subseteq V(G)$, $G - v$ for $v \in V(G)$, $G - F$ for $F \subseteq E(G)$, $G - e$ for $e \in E(G)$, $G_1 \cup G_2$, $G_1 \cap G_2$.
Sec 1.5 degree, minimum degree, maximum degree, $\delta(G)$, $\Delta(G)$, k-regular
Sec 1.6 walk, internal vertices, W^{-1}, concatenation of walks, (x, v)-section of a walk, trail, path, connected, components, disconnected, distance, $d(u, v)$
Sec 1.7 closed walk, cycle, C_k
Sec 1.4 Shortest path

Theorems / Algorithms.

- $\star$ The Handshake Lemma
- $\star$ Every graph has an even number of vertices of odd degree.
- $\star$ If G is a bipartite k-regular graph with bipartition (X, Y), then $|X| = |Y|$.
- $\star$ "x and y are connected" is an equivalence relation on $V(G)$.
- A graph is bipartite if and only if it contains no odd cycle.
- $\star$ If G containsa. closed odd walk, it contains a closed odd cycle.
- $\star$ A graph is bipartite if all its components are bipartite.
- $\star$ Every graph G with $\delta(G) \geq 2$ contains a cycle.
- Dijkstra's algorithm

Trees

Definitions. Sec 2.1 acyclic, tree, leaf
Sec 2.2 cut edge, edge cut, bond
Sec 2.3 cut vertex

Theorems / Algorithms.

- $\star$ In a tree, every pair of vertices is connected by a unique path.
- $\star$ Every tree T satisfies $||T|| = |T| - 1$.
- $\star$ Every (nontrivial) tree contains at least two leaves.
- TFAE: (1) G is a tree (2) G is connected and has $||G|| = |G| - 1$ (3) $||G|| = |G| - 1$ and G is acyclic (4) G has no loops and exactly one path between each pair of vertices.
- An edge of a graph is a cut edge iff it is contained in no cycle.
- $\star$ A connected graph is a tree if and only if every edge is a cut edge
- $\star$ Every connected graph has a spanning tree.
- $\star$ Every connected graph G satisfies $||G|| \geq |G| - 1$.
- $\star$ A vertex v in a tree is a cut vertex iff $d(v) > 1$.
- $\star$ Every non-trivial connected graph has at least two vertices which are **not** cut vertices.
- $\star$ Find the Prüfer code of a tree.

- ⋆ Reconstruct a tree based on its Prüfer code.
- The number of trees on vertex set $\{1, 2, \ldots, n\}$ is n^{n-2}.
- Kruskal's Algorithm

Connectivity

Definitions. Separating set, k-connected, connectivity of a graph, $\kappa(G)$, k-edge-connected, edge-connectivity of a graph, $\kappa'(G)$, H-pathblock, block graph, internally disjoint paths

Theorems / Algorithms.

- ⋆ For every non-trivial graph, $\kappa'(G) \leq \delta(G)$.
- For every non-trivial graph, $\kappa(G) \leq \kappa'(G)$.
- G is 2-connected if and only if it can be constructed from a cycle by successively adding H-paths to graphs H already constructed. ($\Leftarrow$ is $\star$, $\Rightarrow$ is harder)
- ⋆ The cycles of G are the cycles of its blocks.
- ⋆ The bonds of G are the bonds of its blocks.
- Whitney's Theorem: G is 2-connected if and only if any two vertices of G are connected by at least two internally disjoint paths. ($\Leftarrow$ is $\star$-ish, $\Rightarrow$ is hard)
- Distinct edges e, f lie in the same block iff they lie in the same cycle ($\star$ if you're allowed to use Whitney's Theorem as a tool, hard to prove directly)
- The block graph of a connected graph is a tree.

Eulerian and Hamiltonian Graphs

Definitions. Euler Tour; Hamilton path; Hamilton Cycle; hamiltonian graph; closure of a graph, degree sequence

Theorems.

- ⋆ A connected graph G has an Euler tour if and only if all degrees are even.

- $\star$ If G is hamiltonian then for every nonempty proper $S \subseteq V(G)$, we have $C(G - S) \leq |S|$. (Where $C(H)$ is the number of components in H)
- $\star$ Dirac's Theorem
- $\star$ Bondy/Chvatal: Let G be a graph with $uv \notin E(G)$ and $d(u) + d(v) \geq |G|$. Then G is Hamiltonian iff $G + uv$ is Hamiltonian. (Proof is the same as Dirac)
- $\star$ A graph is Hamiltonian if and only if its closure is Hamiltonian (Proof via Bondy-Chvatal)
- $\star$ If the close of a graph is complete, then the graph is Hamiltonian.
- Let G be a graph with degree sequence $(d_1, \ldots, d_n)$, written in order with $d_1 \leq d_2 \leq \ldots \leq d_n$ and $n \geq 3$. Suppose there is no $m < \frac{n}{2}$ for which both $d_m \leq m$ and $d_{n-m} < n - m$. Then G is Hamiltonian.

Matchings

Definitions. matching; maximal matching; maximum matching; M-saturated vertex; perfect matching; M-alternating path; M-augmenting path; neighborhood of a set $N(S)$; vertex cover; $o(H)$ the number of odd components in H; k-factor; unstable pair; stable matching

Theorems / Algorithms.

- $\star$ If a matching M in G is maximum, then G has no M-augmenting path.
- If G has no M-augmenting path, then M is a maximum matching.
- Hall's Theorem
- König's Theorem
- $\star$ If G is a k-regular bipartite graph with $k \geq 1$, then G has a perfect matching. (Proof using Hall's Theorem)
- Tutte's Theorem ($\star$: If G has a 1-factor, then for every set $S \subseteq V(G)$ we must have $o(G - S) \leq |S|$; the other direction is hard.)

- $\star$ Gayle-Shapley Stable Matching Algorithm (Proposal Algorithm)

Flows

Definitions. Kirchoff's Law / Conservation Law; flow; oriented edges; $\overrightarrow{E}$; $\overrightarrow{F}(X,Y)$; $\vec{e}$; $\overleftarrow{e}$; given a function $f : \overrightarrow{E} \to H$ and $X, Y \subseteq V$, $f(X,Y)$; circulation; network; flow on network $N = (G, s, t, c)$; integral flow; cut; capacity of a cut; total value of a flow $|f|$

Theorems / Algorithms.

- $\star$ In any circulation, the flow across every cut is 0.
- $\star$ In any circulation, f(X,X).
- $\star$ For a flow f and a cut $(S, \overline{S})$, $f(S, \overline{S}) = f(s, V)$.
- (Max flow min cut) In every network, the maximum total value of any flow is equal to the minimum capacity of any cut.
- Vertex capacity version of max-flow min cut. (derive from edge version, but not prove)
- Ford-Fulkerson Augmenting Path Algorithm (use, but not prove)
- Menger's Theorem

Coloring

Definitions. k-vertex coloring; proper coloring; k-colorable; chromatic number $\chi(G)$; k-critical graph; S-components; colorings which agree on S; type I and type II components in a critical graph; $G \cdot uv$; Mycielski Construction; k-edge coloring; proper edge coloring; k-edge coloring; chromatic index $\chi'(G)$

Theorems / Algorithms.

- $\star$ If G is k-critical, then $\delta(G) \geq k - 1$.
- $\star$ Every k-chromatic graph has at least k vertices of degree at least $k - 1$.
- $\star$ (Cor) For any graph G, $\chi(G) \leq \Delta + 1$.
- $\star$ In a k-critical graph, no vertex cut is a clique.
- $\star$ (Cor) No k-critical graph has a cut-vertex.
- $\star$ (Cor) If G is k-critical with vertex cut $\{u, v\}$, then $uv \notin E(G)$.

- $\star$ Every critical graph is a block.
- Let k be k-critical with vertex cut $\{u, v\}$. Then (i) $G = G_1 \cup G_2$, where each G_i is a $\{u, v\}$-component of type i, and (ii) $G_1 + uv$ and $G_2 \cdot uv$ are both k-critical.
- (Cor) Let G be a k-critical graph with vertex cut $\{u, v\}$. Then $d(u) + d(v) \geq 3k - 5$.
- Brooks' Theorem
- $\star$ Mycielski Theorem: if G is triangle free and chromatic number k, then the graph G' built by the Mycielski construction is triangle free and has chromatic number $k + 1$.
- $\star$ For every graph G, we have $\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1$.
- $\star$ König's Theorem
- $\star$ Vizing's Theorem

Coloring Planar Graphs

Definitions. drawing; crossing; planar embedding; planar graph; faces.

Theorems / Algorithms.

- $\star$ If a connected plane graph has exactly n vertices, e edges, and f faces, then $n - e + f = 2$.
- $\star$ If G is a simple planar graph with at leas three vertices, then $||G|| \leq 3|G| - 6$. If G is also triangle-free, then $||G|| \leq 2|G| - 4$.
- Every planar graph has chromatic number at most five.