CHAPTER 25

Derivatives of Inverse Trig Functions

Our goal is simple, and the answers will come quickly. We will derive six new derivative formulas for the six inverse trigonometric functions:

$$\frac{d}{dx}\Big[\sin^{-1}(x)\Big] \qquad \frac{d}{dx}\Big[\tan^{-1}(x)\Big] \qquad \frac{d}{dx}\Big[\sec^{-1}(x)\Big]$$

$$\frac{d}{dx}\Big[\cos^{-1}(x)\Big] \qquad \frac{d}{dx}\Big[\cot^{-1}(x)\Big] \qquad \frac{d}{dx}\Big[\csc^{-1}(x)\Big]$$

These formulas will flow from the inverse rule from Chapter 24 (page 280):

$$\frac{d}{dx}\Big[f^{-1}(x)\Big] = \frac{1}{f'\big(f^{-1}(x)\big)}. \tag{25.1}$$

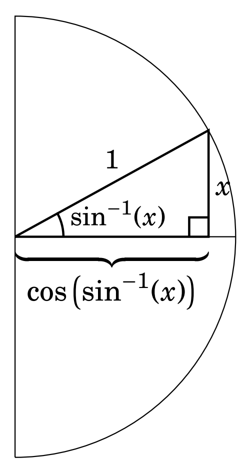
25.1 Derivatives of Inverse Sine and Cosine

Applying the inverse rule (25.1) with $f(x) = \sin(x)$ yields

$$\frac{d}{dx}\Big[\sin^{-1}(x)\Big] = \frac{1}{\cos\big(\sin^{-1}(x)\big)}. \tag{25.2}$$

We are almost there. We just have to simplify the $\cos\big(\sin^{-1}(x)\big)$ in the denominator. To do this recall

$$\sin^{-1}(x) = \left(\begin{array}{l} \text{the angle } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ \text{for which } \sin(\theta) = x \end{array} \right).$$



Thus $\sin^{-1}(x)$ is the angle (between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$) of a the triangle on the unit circle whose opposite side is x. (Because sin of this angle equals x.) Then $\cos\big(\sin^{-1}(x)\big)$ is the length of the adjacent side. By the Pythagorean theorem this side length is $\sqrt{1-x^2}$. Putting $\cos\big(\sin^{-1}(x)\big) = \sqrt{1-x^2}$ into the above Equation (25.2), we get our latest rule:

Rule 20 $\quad \dfrac{d}{dx}\Big[\sin^{-1}(x)\Big] = \dfrac{1}{\sqrt{1-x^2}}$

We reviewed $\sin^{-1}(x)$ In Section 6.1 and presented its graph on page 101. Figure 25.1 repeats the graph, along with the derivative from Rule 20.

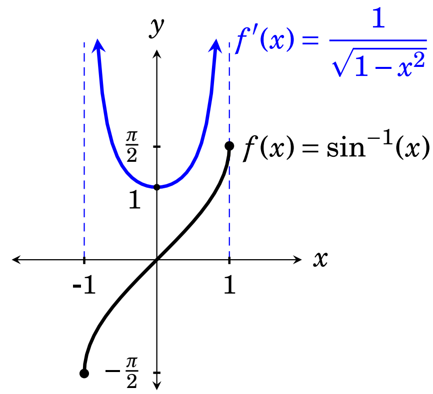


Figure 25.1. The function $\sin^{-1}(x)$ and its derivative. The derivative is always positive, reflecting the fact that the tangents to $\sin^{-1}(x)$ have positive slope. The derivative has vertical asymptotes at $x = \pm 1$, as the tangents to $\sin^{-1}(x)$ become increasingly steep as x approaches ± 1.

Now consider $\cos^{-1}(x)$. The tangents to its graph (Figure 25.2 below) have *negative* slope, and the geometry suggests that its derivative is *negative* the derivative of $\sin^{-1}(x)$. Indeed this turns out to be exactly the case. This chapter's Exercise 1 asks you to prove our next rule:

$$\boxed{\textbf{Rule 21} \quad \frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}}$$

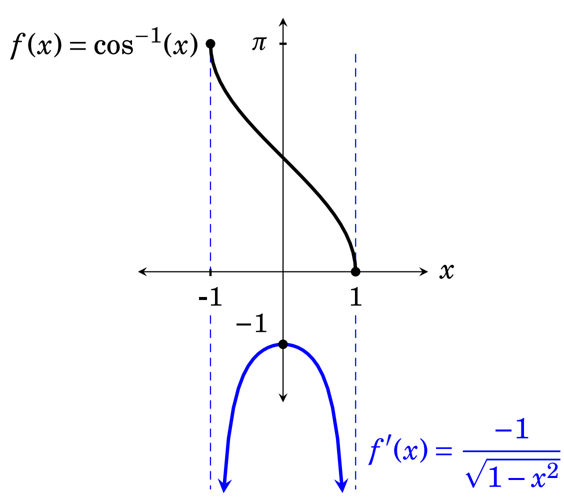


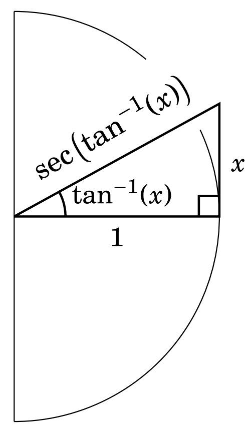
Figure 25.2. The function $\cos^{-1}(x)$ and its derivative.

25.2 Derivatives of Inverse Tangent and Cotangent

Now let's find the derivative of $\tan^{-1}(x)$. Putting $f(x) = \tan(x)$ into the inverse rule (25.1), we have $f^{-1}(x) = \tan^{-1}(x)$ and $f'(x) = \sec^2(x)$, and we get

$$\frac{d}{dx}\Big[\tan^{-1}(x)\Big] = \frac{1}{\sec^2\big(\tan^{-1}(x)\big)} = \frac{1}{\Big(\sec\big(\tan^{-1}(x)\big)\Big)^2}. \tag{25.3}$$

The expression $\sec\big(\tan^{-1}(x)\big)$ in the denominator is the length of the hypotenuse of the triangle to the right. (See example 6.3 in Chapter 6, page 114.) By the Pythagorean theorem, the length is $\sec\big(\tan^{-1}(x)\big) = \sqrt{1+x^2}$. Inserting this into the above Equation (25.4) yields



$$\frac{d}{dx}\Big[\tan^{-1}(x)\Big] = \frac{1}{\Big(\sec\big(\tan^{-1}(x)\big)\Big)^2} = \frac{1}{\big(\sqrt{1+x^2}\big)^2} = \frac{1}{1+x^2}.$$

We now have: $\boxed{\textbf{Rule 22} \quad \frac{d}{dx}\Big[\tan^{-1}(x)\Big] = \frac{1}{1+x^2}}$

We discussed $\tan^{-1}(x)$ in Chapter 6, and its graph is in Figure 6.3. Below Figure 25.3 repeats the graph, along with the derivative $\frac{1}{x^2+1}$.

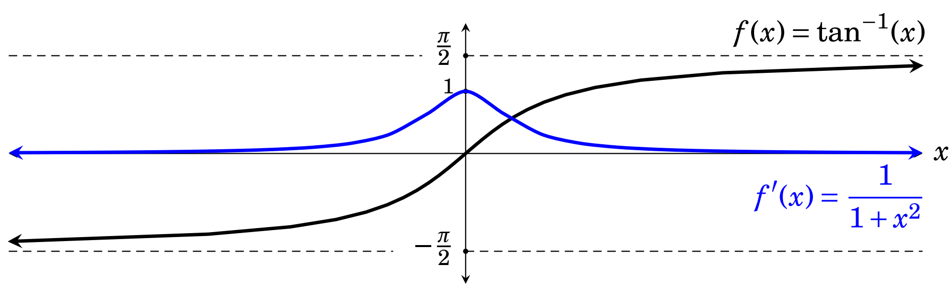


Figure 25.3. The function $\tan^{-1}(x)$ and its derivative $\frac{1}{1+x^2}$. Note $\lim\limits_{x\to\infty} \frac{1}{1+x^2} = 0$ and $\lim\limits_{x\to-\infty} \frac{1}{1+x^2} = 0$, reflecting the fact that the tangent lines to $y = \tan^{-1}(x)$ become closer and closer to horizontal as $x \to \pm\infty$. The derivative bumps up to 1 at $x = 0$, where the tangent to $y = \tan^{-1}(x)$ is steepest, with slope 1

Exercise 3 below asks you to mirror the above arguments to deduce:

$$\boxed{\textbf{Rule 23} \quad \frac{d}{dx}\Big[\cot^{-1}(x)\Big] = \frac{-1}{1+x^2}}$$

25.3 Derivatives of Inverse Secant and Cosecant

We reviewed $\sec^{-1}(x)$ in Section 6.3. For its derivative, put $f(x) = \sec(x)$ into the inverse rule (25.1), with $f^{-1}(x) = \sec^{-1}(x)$ and $f'(x) = \sec(x)\tan(x)$. We get

$$\begin{aligned}\frac{d}{dx}\Big[\sec^{-1}(x)\Big] &= \frac{1}{f'\big(f^{-1}(x)\big)}\\ &= \frac{1}{\sec\big(f^{-1}(x)\big)\cdot\tan\big(f^{-1}(x)\big)}\\ &= \frac{1}{\sec\big(\sec^{-1}(x)\big)\cdot\tan\big(\sec^{-1}(x)\big)}.\end{aligned}$$

Because $\sec\big(\sec^{-1}(x)\big) = x$, the above becomes

$$\frac{d}{dx}\Big[\sec^{-1}(x)\Big] = \frac{1}{x\cdot\tan\big(\sec^{-1}(x)\big)}. \tag{25.4}$$

In Example 6.5 we showed that $\tan\big(\sec^{-1}(x)\big) = \begin{cases}\sqrt{x^2-1} & \text{if } x \text{ is positive}\\ -\sqrt{x^2-1} & \text{if } x \text{ is negative}\end{cases}$
With this, Equation 25.4 above becomes

$$\frac{d}{dx}\Big[\sec^{-1}(x)\Big] = \begin{cases}\dfrac{1}{x\sqrt{x^2-1}} & \text{if } x \text{ is positive}\\[2ex] \dfrac{1}{-x\sqrt{x^2-1}} & \text{if } x \text{ is negative.}\end{cases}$$

But if x is negative, then $-x$ is positive, and the above consolidates to

$$\boxed{\textbf{Rule 24} \quad \frac{d}{dx}\Big[\sec^{-1}(x)\Big] = \frac{1}{|x|\sqrt{x^2-1}}}$$

This graph of $\sec^{-1}(x)$ and its derivative is shown in Figure 25.3.

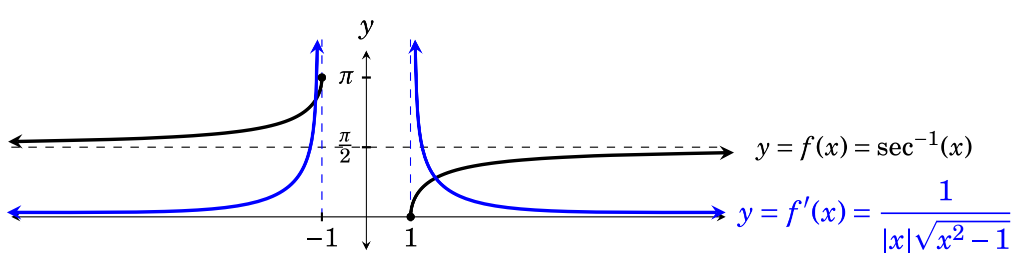


Figure 25.4. The graph of $\sec^{-1}(x)$ and its derivative. The domain of both functions is $(-\infty,-1]\cup[1,\infty)$. Note that the derivative has vertical asymptotes at $x = \pm 1$, where the tangent line to $y = \sec^{-1}(x)$ is vertical.

This chapter's Exercise 2 asks you to use reasoning similar to the above to deduce our final rule.

Rule 25 $\dfrac{d}{dx}\Big[\csc^{-1}(x)\Big] = \dfrac{-1}{|x|\sqrt{x^2-1}}$

Each of our new rules has a chain rule generalization. For example, Rule 25 generalizes as

$$\frac{d}{dx}\Big[\csc^{-1}\big(g(x)\big)\Big] = \frac{-1}{\big|g(x)\big|\sqrt{(g(x))^2-1}}g'(x) = \frac{-g'(x)}{\big|g(x)\big|\sqrt{(g(x))^2-1}}.$$

Here is a summary of this Chapter's new rules, along with their chain rule generalizations.

$$\begin{array}{lcllcl}
\dfrac{d}{dx}\Big[\sin^{-1}(x)\Big] & = & \dfrac{1}{\sqrt{1-x^2}} & \dfrac{d}{dx}\Big[\sin^{-1}\big(g(x)\big)\Big] & = & \dfrac{1}{\sqrt{1-(g(x))^2}}g'(x) \\
\dfrac{d}{dx}\Big[\cos^{-1}(x)\Big] & = & \dfrac{-1}{\sqrt{1-x^2}} & \dfrac{d}{dx}\Big[\cos^{-1}\big(g(x)\big)\Big] & = & \dfrac{-1}{\sqrt{1-(g(x))^2}}g'(x) \\
\dfrac{d}{dx}\Big[\tan^{-1}(x)\Big] & = & \dfrac{1}{1+x^2} & \dfrac{d}{dx}\Big[\tan^{-1}\big(g(x)\big)\Big] & = & \dfrac{1}{1+(g(x))^2}g'(x) \\
\dfrac{d}{dx}\Big[\cot^{-1}(x)\Big] & = & \dfrac{-1}{1+x^2} & \dfrac{d}{dx}\Big[\cot^{-1}\big(g(x)\big)\Big] & = & \dfrac{-1}{1+(g(x))^2}g'(x) \\
\dfrac{d}{dx}\Big[\sec^{-1}(x)\Big] & = & \dfrac{1}{|x|\sqrt{x^2-1}} & \dfrac{d}{dx}\Big[\sec^{-1}\big(g(x)\big)\Big] & = & \dfrac{g'(x)}{\big|g(x)\big|\sqrt{(g(x))^2-1}} \\
\dfrac{d}{dx}\Big[\csc^{-1}(x)\Big] & = & \dfrac{-1}{|x|\sqrt{x^2-1}} & \dfrac{d}{dx}\Big[\csc^{-1}\big(g(x)\big)\Big] & = & \dfrac{-g'(x)}{\big|g(x)\big|\sqrt{(g(x))^2-1}}
\end{array}$$

Example 25.1 $\dfrac{d}{dx}\Big[\sqrt{\cos^{-1}(x)}\Big] = \dfrac{d}{dx}\Big[\big(\cos^{-1}(x)\big)^{\frac{1}{2}}\Big] = \dfrac{1}{2}\big(\cos^{-1}(x)\big)^{-\frac{1}{2}}\dfrac{d}{dx}\Big[\cos^{-1}(x)\Big]$

$$= \frac{1}{2}\big(\cos^{-1}(x)\big)^{-\frac{1}{2}}\frac{-1}{\sqrt{1-x^2}} = \boxed{\frac{-1}{2\sqrt{\cos^{-1}(x)}\sqrt{1-x^2}}.}$$ ✍

Example 25.2 $\dfrac{d}{dx}\Big[e^{\tan^{-1}(x)}\Big] = e^{\tan^{-1}(x)}\dfrac{d}{dx}\Big[\tan^{-1}(x)\Big] = e^{\tan^{-1}(x)}\dfrac{1}{1+x^2} = \boxed{\dfrac{e^{\tan^{-1}(x)}}{1+x^2}.}$

Example 25.3 $\dfrac{d}{dx}\Big[\tan^{-1}\big(e^x\big)\Big] = \dfrac{1}{1+(e^x)^2}\dfrac{d}{dx}\Big[e^x\Big] = \dfrac{1}{1+e^{2x}}e^x = \boxed{\dfrac{e^x}{1+e^{2x}}.}$ ✍

25.4 Summary of Derivative Rules

We have reached the end of our derivative rules. In summary, we have the following rules for specific functions. The corresponding chain rule generalizations are shown to the right.

	Rule	**Chain Rule Generalization**
Constant Rule	$\frac{d}{dx}[c] = 0$	
Identity Rule	$\frac{d}{dx}[x] = 1$	
Power Rule	$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$	$\frac{d}{dx}\left[\big(g(x)\big)^n\right] = n\big(g(x)\big)^{n-1}g'(x)$
Exp Rules	$\frac{d}{dx}\left[e^x\right] = e^x$ $\frac{d}{dx}\left[a^x\right] = \ln(a)a^x$	$\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)}g'(x)$ $\frac{d}{dx}\left[a^{g(x)}\right] = \ln(a)a^{g(x)}g'(x)$
Log Rules	$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}$ $\frac{d}{dx}\left[\log_a(x)\right] = \frac{1}{x\ln(a)}$	$\frac{d}{dx}\left[\ln\big(g(x)\big)\right] = \frac{1}{g(x)}g'(x)$ $\frac{d}{dx}\left[\log_a\big(g(x)\big)\right] = \frac{1}{g(x)\ln(a)}g'(x)$
Trig Rules	$\frac{d}{dx}\left[\sin(x)\right] = \cos(x)$ $\frac{d}{dx}\left[\cos(x)\right] = -\sin(x)$ $\frac{d}{dx}\left[\tan(x)\right] = \sec^2(x)$ $\frac{d}{dx}\left[\cot(x)\right] = -\csc^2(x)$ $\frac{d}{dx}\left[\sec(x)\right] = \sec(x)\tan(x)$ $\frac{d}{dx}\left[\csc(x)\right] = -\csc(x)\cot(x)$	$\frac{d}{dx}\left[\sin\big(g(x)\big)\right] = \cos\big(g(x)\big)g'(x)$ $\frac{d}{dx}\left[\cos\big(g(x)\big)\right] = -\sin\big(g(x)\big)g'(x)$ $\frac{d}{dx}\left[\tan\big(g(x)\big)\right] = \sec^2\big(g(x)\big)g'(x)$ $\frac{d}{dx}\left[\cot\big(g(x)\big)\right] = -\csc^2\big(g(x)\big)g'(x)$ $\frac{d}{dx}\left[\sec\big(g(x)\big)\right] = \sec\big(g(x)\big)\tan\big(g(x)\big)g'(x)$ $\frac{d}{dx}\left[\csc\big(g(x)\big)\right] = -\csc\big(g(x)\big)\cot\big(g(x)\big)g'(x)$
Inverse Trig Rules	$\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2}$ $\frac{d}{dx}\left[\cot^{-1}(x)\right] = \frac{-1}{1+x^2}$ $\frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$ $\frac{d}{dx}\left[\csc^{-1}(x)\right] = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}\left[\sin^{-1}\big(g(x)\big)\right] = \frac{1}{\sqrt{1-(g(x))^2}}g'(x)$ $\frac{d}{dx}\left[\cos^{-1}\big(g(x)\big)\right] = \frac{-1}{\sqrt{1-(g(x))^2}}g'(x)$ $\frac{d}{dx}\left[\tan^{-1}\big(g(x)\big)\right] = \frac{1}{1+(g(x))^2}g'(x)$ $\frac{d}{dx}\left[\cot^{-1}\big(g(x)\big)\right] = \frac{-1}{1+(g(x))^2}g'(x)$ $\frac{d}{dx}\left[\sec^{-1}\big(g(x)\big)\right] = \frac{1}{\lvert g(x)\rvert\sqrt{(g(x))^2-1}}g'(x)$ $\frac{d}{dx}\left[\csc^{-1}\big(g(x)\big)\right] = \frac{-1}{\lvert g(x)\rvert\sqrt{(g(x))^2-1}}g'(x)$

In addition we have the following general rules for the derivatives of combinations of functions.

Constant Multiple Rule:	$\frac{d}{dx}\left[\, cf(x)\,\right]$	$=$	$cf'(x)$
Sum/Difference Rule:	$\frac{d}{dx}\left[f(x) \pm g(x)\right]$	$=$	$f'(x) \pm g'(x)$
Product Rule:	$\frac{d}{dx}\left[f(x)g(x)\right]$	$=$	$f'(x)g(x) + f(x)g'(x)$
Quotient Rule:	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$	$=$	$\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
Chain Rule:	$\frac{d}{dx}\left[\, f\big(g(x)\big)\,\right]$	$=$	$f'(g(x))\,g'(x)$
Inverse Rule:	$\frac{d}{dx}\left[\, f^{-1}(x)\,\right]$	$=$	$\frac{1}{f'\big(f^{-1}(x)\big)}$

We used this last rule, the inverse rule, to find the derivatives of $\ln(x)$ and the inverse trig functions. After it has served these purposes it is mostly retired for the remainder of Calculus I, except for the stray exercise or quiz or test question.

This looks like a lot of rules to remember, and it is. But through practice and usage you will reach the point of using them automatically, with hardly a thought. Be sure to get enough practice!

Exercises for Chapter 25

1. Show that $\frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}$.

2. Show that $\frac{d}{dx}\left[\csc^{-1}(x)\right] = \frac{-1}{|x|\sqrt{x^2-1}}$.

3. Show that $\frac{d}{dx}\left[\cot^{-1}(x)\right] = \frac{-1}{1+x^2}$.

Find the derivatives of the given functions.

4. $\sin^{-1}\left(\sqrt{2x}\right)$ **5.** $\ln\left(\tan^{-1}(x)\right)$ **6.** $e^x \tan^{-1}(x)$

7. $\tan^{-1}\left(\pi x\right)$ **8.** $\sec^{-1}\left(\pi x\right)$ **9.** $\ln\left(\sin^{-1}(x)\right)$

10. $\cos^{-1}\left(\pi x\right)$ **11.** $\sec^{-1}\left(x^5\right)$ **12.** $e^{\tan^{-1}(\pi x)}$

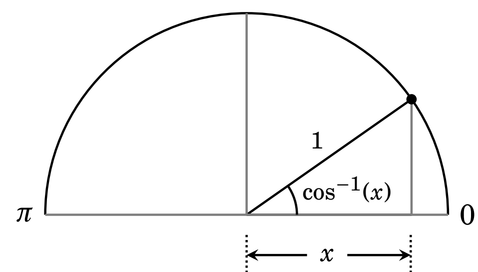
13. $\tan^{-1}\big(\ln(x)\big) + \pi$ **14.** $\tan^{-1}\big(x\sin(x)\big)$ **15.** $x\sin^{-1}\big(\ln(x)\big)$

Exercise Solutions for Chapter 25

1. Show that $\frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}$.

By the inverse rule, $\frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{1}{-\sin\left(\cos^{-1}(x)\right)}$.

Now we simplify the denominator. From the standard diagram for $\cos^{-1}(x)$ we get $\sin\left(\cos^{-1}(x)\right) = \frac{\text{OPP}}{\text{HYP}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$. With this, the above becomes $\frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}$.



3. Show that $\frac{d}{dx}\left[\cot^{-1}(x)\right] = \frac{-1}{1+x^2}$.

Suggestion: Verify the identity $\cot^{-1}(x) = \frac{\pi}{2} - \tan^{-1}(x)$. Then differentiate both sides of this.

5. $\frac{d}{dx}\left[\ln\left(\tan^{-1}(x)\right)\right] = \frac{1}{\tan^{-1}(x)}\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{\tan^{-1}(x)}\frac{1}{1+x^2} = \frac{1}{\tan^{-1}(x)\left(1+x^2\right)}$

7. $\frac{d}{dx}\left[\tan^{-1}(\pi x)\right] = \frac{\pi}{1+(\pi x)^2} = \frac{\pi}{1+\pi^2 x^2}$

9. $\frac{d}{dx}\left[\ln\left(\sin^{-1}(x)\right)\right] = \frac{1}{\sin^{-1}(x)}\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sin^{-1}(x)}\frac{1}{\sqrt{1-x^2}} = \frac{1}{\sin^{-1}(x)\sqrt{1-x^2}}$

11. $\frac{d}{dx}\left[\sec^{-1}\left(x^5\right)\right] = \frac{1}{|x^5|\sqrt{\left(x^5\right)^2-1}}5x^4 = \frac{5x^4}{|x^5|\sqrt{x^{10}-1}} = \frac{5}{|x|\sqrt{x^{10}-1}}$

13. $\frac{d}{dx}\left[\tan^{-1}\left(\ln(x)\right)+\pi\right] = \frac{1}{1+\left(\ln(x)\right)^2}\frac{1}{x} = \frac{1}{x+x\left(\ln(x)\right)^2}$

15. $\frac{d}{dx}\left[x\sin^{-1}\left(\ln(x)\right)\right] = 1\cdot\sin^{-1}\left(\ln(x)\right) + x\frac{d}{dx}\left[\sin^{-1}\left(\ln(x)\right)\right]$

$$= \sin^{-1}\left(\ln(x)\right) + x\frac{1}{\sqrt{1-\left(\ln(x)\right)^2}}\frac{1}{x} = \sin^{-1}\left(\ln(x)\right) + \frac{1}{\sqrt{1-\left(\ln(x)\right)^2}}$$