

Increase-Decrease

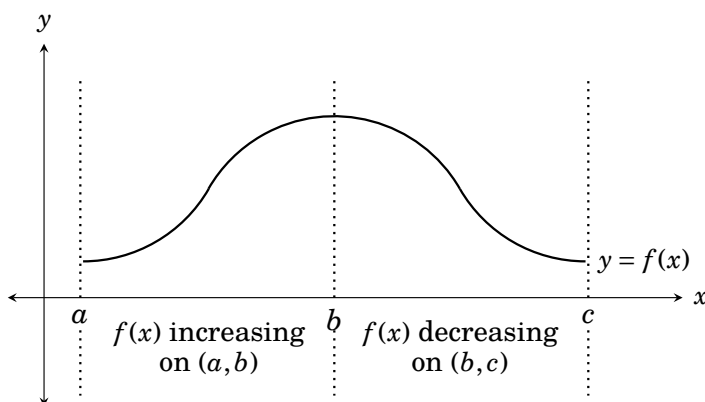
Part 3 of this course dealt with derivatives of functions: what they are, how to compute them, what they mean and how to work with them. In Part 4 the focus now shifts to how derivatives are useful. In Chapters 30 through 34 the theme is what the derivative f' tells us about the function f . Here the primary interest will be the behavior of some function f , and the derivative is a *tool* that gives information about f .

In this chapter we examine one of the most immediate things f' tells us about f : where f increases and where f decreases.

Definition 30.1 Suppose $f(x)$ is a function defined on some interval I .

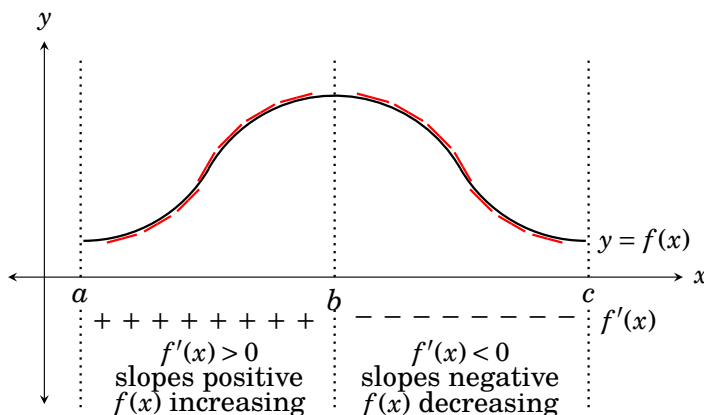
- $f(x)$ **increases** on I if x moving to the right on I causes $f(x)$ to increase. (That is, if x and x' are on I and $x' > x$, then $f(x') > f(x)$.)
- $f(x)$ **decreases** on I if x moving to the right on I causes $f(x)$ to decrease. (That is, if x and x' are on I and $x' > x$, then $f(x') < f(x)$.)

For example, the function f below increases on the interval (a, b) and it decreases on the interval (b, c) .



For another example, your familiarity with the parabola $f(x) = x^2$ tells you that this function decreases on $(-\infty, 0)$ and increases on $(0, \infty)$.

A function's derivative tells where the function increases and where it decreases. Consider the function $f(x)$ graphed on the previous page, shown again below. Notice that, as $f(x)$ increases on (a, b) , the slopes of its tangent lines are *positive*. And as $f(x)$ decreases on (b, c) , the slopes of its tangent lines are *negative*. (We have called attention to this by putting a row of +++ or --- to show where $f'(x)$ is positive or negative.)



So positive derivative means the function increases; negative derivative means the function decreases. Let's record this very useful, far-reaching (and obvious!) fact.


Fact 30.1 Suppose $f(x)$ is a function defined on some interval I .

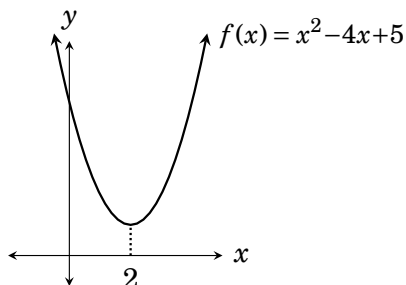
- $f(x)$ increases on I if $f'(x) > 0$ for all x in I .
- $f(x)$ decreases on I if $f'(x) < 0$ for all x in I .

Example 30.1 Find the intervals on which the function $f(x) = x^2 - 4x + 5$ increases/decreases.

Solution Fact 30.1 says that we can find an answer by looking at the derivative, which is $f'(x) = 2x - 4 = 2(x - 2)$. By inspection, $f'(x) = 2(x - 2)$ is negative when $x < 2$, and it is positive when $x > 2$. This means $f'(x)$ is positive on $(2, \infty)$, and negative on $(-\infty, 2)$.

Answer: The function $f(x) = x^2 - 4x + 5$ decreases on the interval $(-\infty, 2)$ and increases on the interval $(2, \infty)$.

We got this answer from looking at the derivative alone, not a graph. To underscore that our answer is correct the graph shown on the right. 



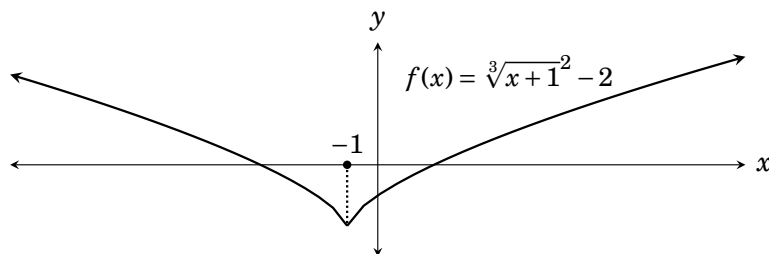
Example 30.2 Find the intervals on which the function $f(x) = \sqrt[3]{x+1}^2 - 2$ increases/decreases.

Solution Fact 30.1 says that we can get an answer by looking at the sign of the derivative. Since, $f(x) = (x+1)^{2/3} - 2$, the generalized power rule gives


$$f'(x) = \frac{2}{3}(x+1)^{-1/3} \frac{d}{dx}[x+1] = \frac{2}{3\sqrt[3]{x+1}}.$$

The sign of $f'(x)$ is controlled by the cube root $\sqrt[3]{x+1}$ in the denominator. Notice that $\sqrt[3]{x+1}$ is negative when $x+1 < 0$, and it is positive when $x+1 > 0$. In other words, $\sqrt[3]{x+1}$ is negative when $x < -1$, and it is positive when $x > -1$. Therefore $f'(x) = \frac{2}{3\sqrt[3]{x+1}}$ is negative when $x < -1$, and positive when $x > -1$.

Answer: The function $f(x) = \sqrt[3]{x+1}^2 - 2$ decreases on the interval $(-\infty, -1)$ (where $f'(x)$ is negative) and it increases on $(-1, \infty)$ (where $f'(x)$ is positive).



To check this answer let's draw a quick sketch of the graph of $f(x) = \sqrt[3]{x+1}^2 - 2$. It is the graph of $y = \sqrt[3]{x^2}$ moved 1 unit left and 2 units down. (See above.) Indeed this graph decreases on $(-\infty, -1)$ and increases on $(-1, \infty)$.

Notice that the graph of $f(x)$ has a cusp at -1 . This makes sense because $f'(-1) = -\frac{2}{3\sqrt[3]{-1+1}} = -\frac{2}{0}$ does not exist, so $f(x)$ has no tangent at $x = -1$. 

Examples 30.1 and 30.2 illuminate a very significant fact about what happens at the point that a function switches from decreasing to increasing (or increasing to decreasing).

In Example 30.1, $f(x)$ stopped decreasing and started increasing at $x = 2$, and $f'(2) = 0$. The function “bottomed out” at 2 with a horizontal tangent.

In Example 30.2, $f(x)$ stopped decreasing and started increasing at $x = -1$, and $f'(-1)$ was not defined. The function “hits bottom with a kink” at -1 .

These two examples illustrate the two possibilities that signal a switch in increase/decrease. Draw the graph of any continuous $f(x)$, like the one in Figure 30. It will be the case that whenever $f(x)$ switches increase/decrease at some number c , then either $f'(c) = 0$ or $f'(c)$ does not exist.

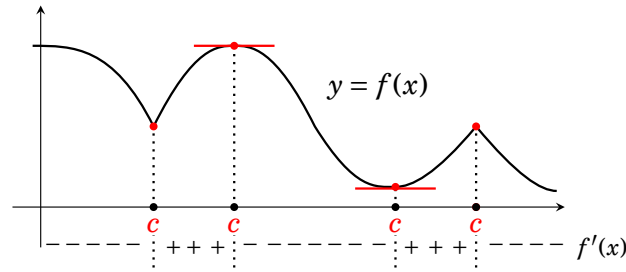


Figure 30.1. If a function $f(x)$ changes from decreasing to increasing (or increasing to decreasing) at a number $x=c$, then either $f'(c)=0$ or $f'(c)$ is not defined. Such a number c is called a **critical point** for $f(x)$.

The reason for this should be intuitively clear: Suppose that $f(x)$ switches increase/decrease at $x=c$. If it happened that $f'(c)$ were positive, then $f(x)$ would continue rising through c . If $f'(c)$ were negative, then $f(x)$ would continue falling through c . Because neither of these two alternatives holds, we conclude that $f'(c)$ is neither positive nor negative. There are only two ways this can happen: either $f'(c)=0$ or $f'(c)$ simply doesn't exist.

So the values $x=c$ that make a function's derivative zero or undefined are going to play an important role. They are called *critical points*.

Definition 30.2 A number c in the domain of a function f is called a **critical point** for f if either $f'(c)=0$ or $f'(c)$ is not defined.

We summarize our observations as the following fact.

Fact 30.2 If a function $f(x)$ switches from increasing to decreasing (or decreasing to increasing) at a number $x=c$, then c is a critical point for $f(x)$.

With this we have a simple procedure to find the intervals on which a function increase or decreases. (We will assume that any function f under discussion here is differentiable on its domain, except possibly at a discrete set of points at which its derivative is not defined.)

To find the intervals on which a function f increases or decreases

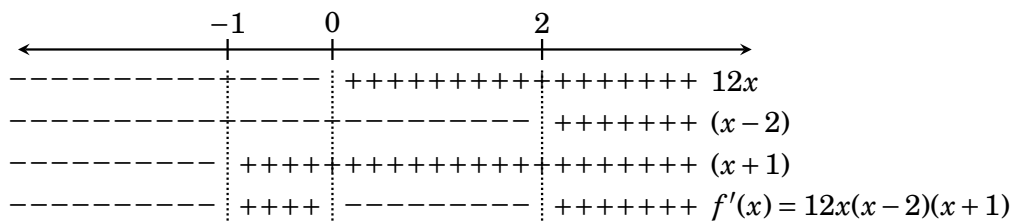
1. Find all critical points of the function.
2. The critical points divide the function's domain into a set of intervals.
3. For each interval, check if $f'(x) > 0$. If so, f increases on this interval. Otherwise, if $f'(x) < 0$, then f decreases on this interval.

Example 30.3 Find the intervals on which $f(x) = 3x^4 - 4x^3 - 12x^2 + 24$ is increasing/decreasing.

Solution The first step is to find the critical points, the values of x that make the derivative zero or undefined. To find them we must examine the derivative, $f'(x) = 12x^3 - 12x^2 - 24x$. This polynomial is defined for all real x , so there are no critical points that make $f'(x)$ undefined. To find the critical points that make $f'(x)$ zero, we solve the equation $f'(x) = 0$:

$$\begin{aligned} 12x^3 - 12x^2 - 24x &= 0 \\ 12x(x^2 - x - 2) &= 0 \\ 12x(x - 2)(x + 1) &= 0. \end{aligned}$$

So the derivative factors as $f'(x) = 12x(x - 2)(x + 1)$, and we can see that the critical points are $x = 0$, $x = 2$ and $x = -1$. They divide the number line into four intervals, as shown in the diagram below.

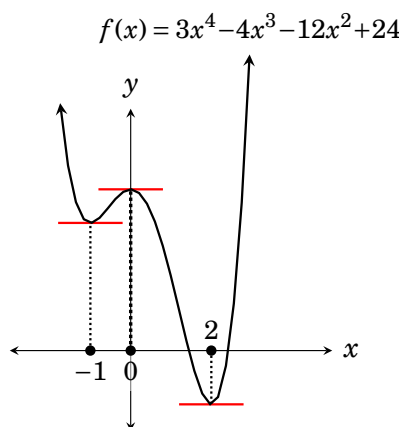


For each factor of the derivative, we indicate the intervals on which it is negative (-) or positive (+). For example, the factor $12x$ is negative on the interval $(-\infty, 0)$ and positive on $(0, \infty)$. Once this is done for all factors, we can read off the sign of $f'(x)$ for each of the four intervals. For example, on $(-\infty, -1)$, $f'(x)$ is a product of three negatives, so it is negative (-). From this chart we can read off our answer.

Answer: $f(x) = 3x^4 - 4x^3 - 12x^2 + 24$ increases on the intervals $(-1, 0)$ and $(2, \infty)$. It decreases on the intervals $(-\infty, -1)$ and $(0, 2)$.

Note: our final answer does not involve $f'(x)$ at all. The derivative was just a tool used to get the answer

The function is sketched on the right. Notice the zero slope at the critical points.

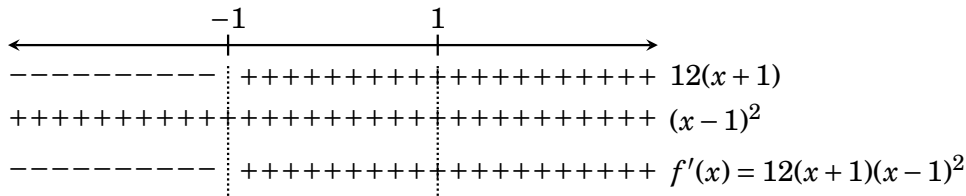


Example 30.4 Find the intervals on which $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 18$ is increasing/decreasing.

Solution The first step is to find the critical points, and to find them we must examine the derivative, $f'(x) = 12x^3 - 12x^2 - 12x + 12$. This polynomial is defined for all real x , so there are no critical points that make $f'(x)$ undefined. To find the critical points that make $f'(x)$ zero, we solve the equation $f'(x) = 0$:


$$\begin{aligned} 12x^3 - 12x^2 - 12x + 12 &= 0 \\ 12(x^3 - x^2 - x + 1) &= 0 \\ 12(x^2(x-1) - (x-1)) &= 0 \\ 12(x^2 - 1)(x-1) &= 0 \\ 12(x+1)(x-1)(x-1) &= 0 \\ 12(x+1)(x-1)^2 &= 0. \end{aligned}$$

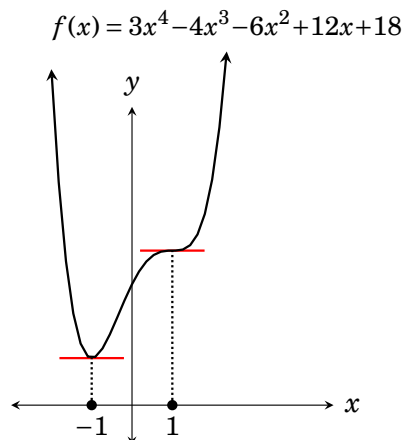
So the derivative factors as $f'(x) = 12(x+1)(x-1)^2$, and the critical points are $x = -1$, and $x = 1$. They divide the number line into three intervals, as shown in the diagram below.



As indicated, the factor $12(x+1)$ is negative for $x < -1$ and positive for $x > -1$. But the factor $(x-1)^2$ is *never negative*, because it is squared. Therefore, the derivative $f'(x) = 12(x+1)(x-1)^2$ is negative when $x < -1$, and it is positive when $x > -1$.

Answer: $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 18$ decreases the interval $(-\infty, -1)$ and increases on $(-1, 1)$ and $(1, \infty)$.

Notice how the derivative does not change signs at $x = 1$, even though $f'(1) = 0$. The function $f(x)$ (graphed on the right) rises before getting to $x = 1$, then levels out at $x = 1$, then continues rising. Given this, it is allowable to say that $f(x)$ increases on the interval $(-1, \infty)$. 



Example 30.5 Find the intervals on which the function $f(x) = e^{(3\sqrt[3]{x^2} - 4x)}$ is increasing/decreasing.

Solution The first step is to find all critical points, and this involves examining $f'(x)$. By the chain rule (or generalized exponential rule),

$$f'(x) = \frac{d}{dx} \left[e^{(3\sqrt[3]{x^2} - 4x)} \right] = e^{(3\sqrt[3]{x^2} - 4x)} \frac{d}{dx} \left[3\sqrt[3]{x^2} - 4x \right] = e^{(3\sqrt[3]{x^2} - 4x)} \left(\frac{2}{\sqrt[3]{x}} - 4 \right).$$

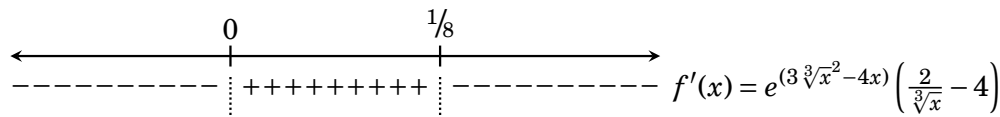
From this we can see that $x = 0$ is a critical point, for $f'(0)$ involves division by zero, so $f'(0)$ is not defined. But $f'(x)$ is defined for all other x , so $x = 0$ is the only critical point of $f(x)$ that makes $f'(x)$ undefined. Any other critical point will make $f'(x)$ zero, so to find them we solve the equation $f'(x) = 0$:

$$e^{(3\sqrt[3]{x^2} - 4x)} \left(\frac{2}{\sqrt[3]{x}} - 4 \right) = 0$$

Since e to any power is positive, we can divide both sides of this equation by the nonzero expression $2e^{(3\sqrt[3]{x^2} - 4x)}$, getting

$$\begin{aligned} \frac{1}{\sqrt[3]{x}} - 2 &= 0 \\ \frac{1}{\sqrt[3]{x}} &= 2 \\ \frac{1}{2} &= \sqrt[3]{x} \\ x &= \frac{1}{8} \end{aligned}$$

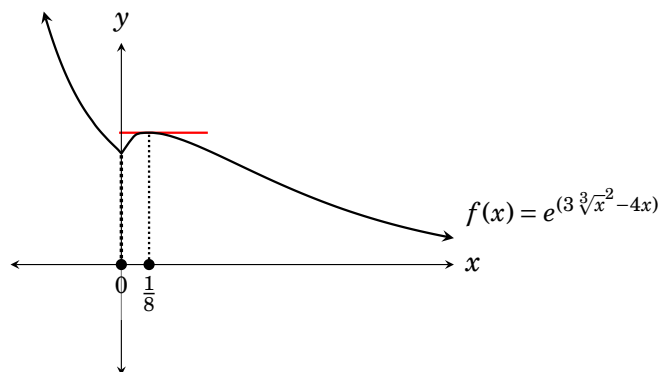
Thus we have just two critical points $x = 0$ and $x = \frac{1}{8}$. These divide the domain of f into three intervals, $(-\infty, 0)$, $(0, \frac{1}{8})$ and $(\frac{1}{8}, \infty)$.




An alternative approach to finding the sign of $f'(x)$ on these intervals is to select a “test point” in each interval and plug it into $f'(x)$. For example:

- 1 is in $(\frac{1}{8}, \infty)$, and $f'(1) = e^{(3 - 4)} \left(\frac{2}{\sqrt[3]{1}} - 4 \right) < 0$, so $f'(x)$ is negative on $(\frac{1}{8}, \infty)$.
- -1 is in $(-\infty, 0)$, and $f'(-1) = e^{(-3 + 4)} \left(\frac{2}{\sqrt[3]{-1}} - 4 \right) < 0$, so $f'(x)$ is negative on $(-\infty, 0)$.
- $\frac{1}{27}$ is in $(0, \frac{1}{8})$, and $f'(\frac{1}{27}) > 0$, so $f'(x)$ is positive on $(0, \frac{1}{8})$.

Answer: The function $f(x) = e^{(3\sqrt[3]{x^2} - 4x)}$ increases on $(0, \frac{1}{8})$, and decreases on $(-\infty, 0)$ and $(\frac{1}{8}, \infty)$.



The function $f(x) = e^{(3\sqrt[3]{x^2} - 4x)}$ has been sketched with a graphing utility above. Notice that there is a cusp at the critical point 0, where $f'(0)$ is not defined. And the slope is zero at the critical point $1/8$, where $f'(1/8) = 0$. 

In all of this chapter's examples the domain of the function has been all real numbers, $(-\infty, \infty)$, and the critical points split $(-\infty, \infty)$ into smaller intervals. By contrast, the function $f(x) = \frac{1}{x} + x$ from Example 5 below has domain $(-\infty, 0) \cup (0, \infty)$, and its critical points will further split these two intervals into smaller intervals. Test your understanding by working this exercise.

Exercises for Chapter 30

- Find the intervals on which $y = x^4 - 8x^2 + 16$ increases/decreases.
- Find the intervals on which $y = x^3 - 27x + 36$ increases/decreases.
- Find the intervals on which $f(x) = (x - 2)e^x$ increases/decreases.
- Find the intervals on which $y = \sqrt{x} - x$ increases/decreases.
- Find the intervals on which $y = \frac{1}{x} + x$ increases/decreases.
- Find the intervals on which $y = e^x - x$ increases/decreases.
- Find the intervals on which $y = \ln(x^2 + 10x + 26)$ increases/decreases.
- Find the intervals on which $y = \tan^{-1}(x^2 + 10x + 24)$ increases/decreases.
- Find the intervals on which $y = \tan^{-1}(\sqrt[3]{x^2} + 3)$ increases/decreases.
- Find the intervals on which $y = x \ln|x|$ increases/decreases.

Exercises Solutions for Chapter 30

1. Find the intervals on which $y = x^4 - 8x^2 + 16$ increases/decreases.

The derivative is $f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x - 2)(x + 2)$. From this we can see that there are three critical points, 0, -2 and 2. These divide the domain $(-\infty, \infty)$ of f into four intervals, $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$ and $(2, \infty)$.

Let's pick a test point a in each interval to determine the sign of f' on that interval. This is tabulated in the table below.

Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
Test point a	-3	-1	1	3
$f'(a)$	$f'(-3) = -60$	$f'(-1) = 12$	$f'(1) = -12$	$f'(3) = 60$
Sign of $f'(a)$	-	+	-	+
f is	decreasing	increasing	decreasing	increasing

Answer: f increases on $(-2, 0)$ and $(2, \infty)$, and decreases on $(-\infty, -2)$ and $(0, 2)$.

3. Find the intervals on which $f(x) = (x - 2)e^x$ increases/decreases.

By the product rule, the derivative is $f'(x) = 1 \cdot e^x + (x - 2)e^x = e^x(1 + x - 2) = e^x(x - 1)$. Since e^x is positive for any x , we can just look at this and see that there is only one critical point, $x = 1$. This critical point divides the domain $(-\infty, \infty)$ of f into two intervals $(-\infty, 1)$ and $(1, \infty)$. By inspection, $f'(x)$ is negative on $(-\infty, 1)$, and positive on $(1, \infty)$.

Answer: f decreases on $(-\infty, 1)$ and increases on $(1, \infty)$.

5. Find the intervals on which $y = \frac{1}{x} + x$ increases/decreases.

Observe that the domain of this function is $(-\infty, 0) \cup (0, \infty)$. Its derivative is $\frac{dy}{dx} = -\frac{1}{x^2} + 1$, and this is zero if $x = \pm 1$. The critical points $x = \pm 1$ divide the domain into intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, \infty)$. Let's pick a test point a in each interval to determine the sign of f' on that interval. This is tabulated in the table below.

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Test point a	-2	-1/2	1/2	2
$f'(a)$	$f'(-2) = 3/4$	$f'(-1/2) = -3$	$f'(1/2) = -3$	$f'(2) = 3/4$
Sign of $f'(a)$	+	-	-	+
f is	increasing	decreasing	decreasing	increasing

Answer: f decreases on $(-1, 0)$ and $(1, \infty)$, and increases on $(-\infty, -1)$ and $(0, 1)$.

7. Find the intervals on which $y = \ln(x^2 + 10x + 26)$ increases/decreases.

Notice that $x^2 + 10x + 26 = (x^2 + 10x + 25) + 1 = (x + 5)^2 + 1 > 0$, so $\ln(x^2 + 10x + 26)$ is defined for all x . Hence the domain of this function is $(-\infty, \infty)$. The derivative is $\frac{dy}{dx} = \frac{2x+10}{x^2+10x+26} = \frac{2(x+5)}{x^2+10x+26}$, and the only critical point is $x = -5$. This splits the domain into two intervals $(-\infty, -5)$ and $(-5, \infty)$.

Interval	$(-\infty, -5)$	$(-5, \infty)$
Test point a	-6	0
$f'(a)$	$f'(-6) = \frac{-2}{2} < 0$	$f'(0) = \frac{10}{26}$
Sign of $f'(a)$	$-$	$+$
f is	decreasing	increasing

Thus the function decreases on $(-\infty, -5)$ and increases on $(-5, \infty)$.

9. Find the intervals on which $y = \tan^{-1}(\sqrt[3]{x^2} + 3)$ increases/decreases.

The derivative is $\frac{dy}{dx} = \frac{1}{1 + (\sqrt[3]{x^2} + 3)^2} \cdot \frac{2}{3\sqrt[3]{x}}$. This is never zero, but it is undefined

for $x = 0$. Thus $x = 0$ is the only critical point, splitting the domain into two intervals $(-\infty, 0)$ and $(0, \infty)$. The derivative is negative on the first interval and positive on the second. Therefore the function decreases on $(-\infty, 0)$ and increases on $(0, \infty)$.