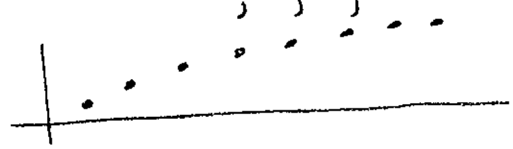
§10.2 Continued Monotonic Sequences

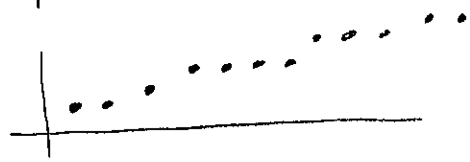
Definitions: A sequence $\{a_n\} = a_1, a_2, a_3, \ldots$ is

1. **Increasing** if $a_{n+1} > a_n$

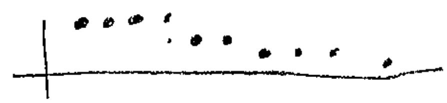


2. **Decreasing** if $a_{n+1} < a_n$

3. **Non decreasing** if $a_{n+1} \geq a_n$



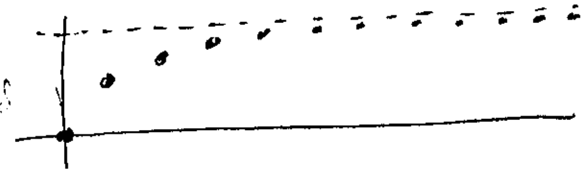
4. **Non increasing** if $a_{n+1} \leq a_n$



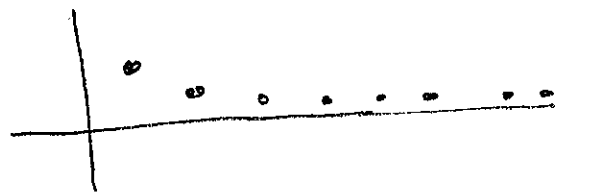
5. **Monotonic** if its non increasing or non decreasing
(i.e. $a_{n+1} \geq a_n$ or $a_{n+1} \leq a_n$ for all n.)

Examples

$\left\{ \dfrac{n}{n+1} \right\}_{n=0}^{\infty}$ increasing, nondecreasing, monotonic



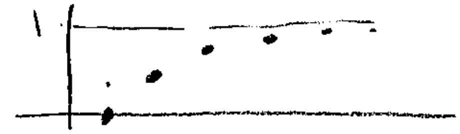
$\left\{ \dfrac{1}{n} \right\}_{n=1}^{\infty}$ decreasing, non increasing, monotonic



Testing a sequence $\{a_n\} = \{f(n)\}$ for monotinicity

non decreasing	non increasing	
$a_{n+1} \geq a_n$	$a_{n+1} \leq a_n$	
$a_{n+1} - a_n \geq 0$	$a_{n+1} - a_n \leq 0$	
$\dfrac{a_{n+1}}{a_n} \geq 1$	$\dfrac{a_{n+1}}{a_n} \leq 1$	only if $a_n > 0$
$f'(n) \geq 0$	$f'(n) \leq 0$	

Ex $\left\{1-\frac{1}{n}\right\}_{n=1}^{\infty}$

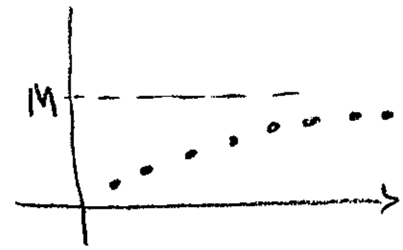


(A) $a_{n+1}-a_n = \left(1-\frac{1}{n+1}\right)-\left(1-\frac{1}{n}\right) = \frac{1}{n}-\frac{1}{n+1} > 0$ Increasing

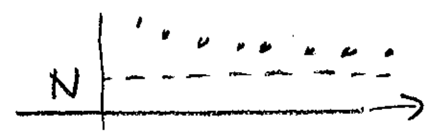
(B) $f'(n) = \frac{1}{n^2} > 0$ Increasing

Definitions A sequence $\{a_n\}$ is

(1) Bounded above if there is a number M for which $a_n \le M$



(2) Bounded below if there is a number N for which $N \le a_n$



(3) Bounded if it's bounded above and below

$\boxed{\text{Theorem 10.5 Any bounded monotonic sequence converges}}$

Ex Converge or diverge? $\left\{\frac{n!}{n^n}\right\}_{n=1}^{\infty}$

[The factorial makes the limit hard to work out!]

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{(n+1)!}{(n+1)^{n+1}}\,\frac{n^n}{n!} = \frac{n+1}{(n+1)^{n+1}}\,n^n = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n < 1$$

Thus the sequence decreases and is bounded below by $N=0$. (And automatically bounded above by $M = a_1 = 1$ because its decreasing.

So the sequence converges!