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TEST 2 ♡
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MATH 201
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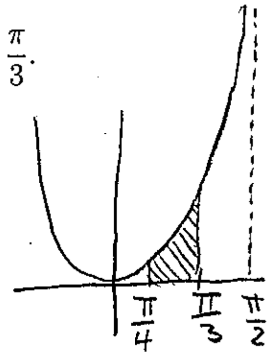
1. Find the area of the region bounded by $y = \tan^2(x)$, $y = 0$, $x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$.



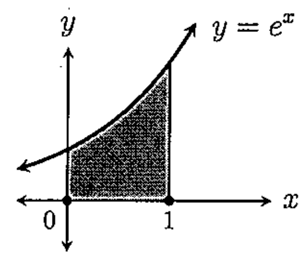
$$A = \int_{\pi/4}^{\pi/3} \tan^2(x)\,dx = \Big[\tan(x) - x\Big]_{\pi/4}^{\pi/3}$$

$$= \left(\tan\left(\frac{\pi}{3}\right) - \frac{\pi}{3}\right) - \left(\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\right)$$

$$= \frac{\sqrt{3}/2}{1/2} - \frac{\pi}{3} - \frac{\sqrt{2}/2}{\sqrt{2}/2} + \frac{\pi}{4} = \sqrt{3} - 1 - \frac{\pi}{3} + \frac{\pi}{4}$$

$$= \sqrt{3} - 1 - \frac{4\pi}{12} + \frac{3\pi}{12} = \boxed{\sqrt{3} - 1 - \frac{\pi}{12} \text{ square units}}$$

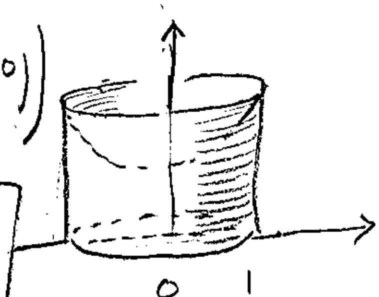
2. The shaded region below is rotated around the y-axis. Find the volume of the resulting solid.



Volume by shells:

$$V = \int_0^1 2\pi x e^x\,dx = 2\pi \int_0^1 x e^x\,dx$$

$$= 2\pi\Big[x e^x - e^x\Big]_0^1 = 2\pi\left(1 \cdot e^1 - e^1 - (0e^0 - e^0)\right)$$



$$= 2\pi(e - e + 1) = \boxed{2\pi \text{ cubic units}}$$

$$\int x e^x\,dx = x e^x - \int e^x\,dx = x e^x - e^x$$

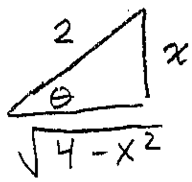
$$\boxed{\begin{array}{ll} u = x & dv = e^x dx \\ du = dx & v = e^x \end{array}}$$

3. $\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2\sqrt{4-(2\sin\theta)^2}} = \int \frac{2\cos\theta\, d\theta}{4\sin^2\theta\sqrt{4-4\sin^2\theta}}$

$x = 2\sin(\theta)$

$dx = 2\cos(\theta)d\theta$

$\sin\theta = \frac{x}{2}$



$= \int \frac{2\cos\theta\, d\theta}{4\sin^2\theta \; 2\cos\theta} = \frac{1}{4}\int \frac{1}{\sin^2\theta}\, d\theta$

$= \frac{1}{4}\int \csc^2(\theta)\, d\theta = \frac{-1}{4}\cot(\theta) + C$

$= -\frac{1}{4}\frac{\text{ADJ}}{\text{OPP}} + C = \boxed{-\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C}$

4. $\int \tan^5(x)\sec^4(x)\, dx = \int \tan^5(x)\sec^2(x)\sec^2(x)\, dx$

$= \int \tan^5(x)\left(1+\tan^2(x)\right)\sec^2(x)\, dx$

$u = \tan x$

$du = \sec^2 x\, dx$

$= \int u^5(1+u^2)\, du = \int u^5 + u^7\, du$

$= \frac{u^6}{6} + \frac{u^8}{8} + C = \boxed{\frac{\tan^6 x}{6} + \frac{\tan^8 x}{8} + C}$

5. $\int \frac{4x^2+6x+1}{x^2+x}\, dx = \int 4 + \frac{2x+1}{x^2+x}\, dx = \boxed{4x + \ln|x^2+x| + C}$

$$\begin{array}{r|l} & \quad 4 \\ x^2+x & 4x^2+6x+1 \\ & 4x^2+4x \\ \hline & \quad 2x+1 \end{array}$$

6. $\int x^3 e^{x^2}\,dx = \int x^2 e^{x^2} x\,dx = x^2 \frac{1}{2} e^{x^2} - \int \frac{1}{2} e^{x^2} 2x\,dx.$

$u = x^2 \quad dv = e^{x^2} x\,dx$

$du = 2x\,dx \quad v = \frac{1}{2} e^{x^2}$

$$= \frac{x^2 e^{x^2}}{2} - \int e^{x^2} x\,dx$$

$$= \frac{x^2 e^{x^2}}{2} - \frac{1}{2} e^{x^2} + C$$

$$= \boxed{\frac{e^{x^2}}{2}(x^2 - 1) + C}$$

7. $\int \frac{x}{x^2 - 2x + 1}\,dx = \int \frac{x}{(x-1)^2}\,dx = \int \frac{1}{x-1} + \frac{1}{(x-1)^2}\,dx$

$$= \boxed{\ln|x-1| - \frac{1}{x-1} + C}$$

$\frac{x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$

$x = A(x-1) + B \rightarrow x = Ax - A + B$

$\boxed{A = 1}, \boxed{B = 1}$

8. $\int \frac{x-1}{x^2+3}\,dx = \int \frac{x}{x^2+3} - \frac{1}{x^2+3}\,dx$

$$= \frac{1}{2}\int \frac{2x}{x^2+3}\,dx - \int \frac{1}{x^2 + \sqrt{3}^2}\,dx$$

$$= \boxed{\frac{1}{2}\ln|x^2+3| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C}$$

9. $\int_5^\infty \frac{4}{x^3}\,dx = \lim_{b\to\infty} \int_5^b \frac{4}{x^3}\,dx = \lim_{b\to\infty} \left[\frac{-2}{x^2}\right]_5^b$

$= \lim_{b\to\infty} \left(-\frac{2}{b^2} - \frac{-2}{5^2}\right) = -0 + \frac{2}{25} = \boxed{\frac{2}{25}}$

10. $\int_0^1 \ln(x)\,dx = \lim_{a\to 0^+} \int_a^1 \ln(x)\,dx$ (Note that $\ln(x)$ is not continuous on $[0,1]$!)

Not continuous at 0!

$= \lim_{a\to 0^+} \left[x\ln(x) - x\right]_a^1$

$= \lim_{a\to 0^+} \left(\left(1\ln(1) - 1\right) - \left(a\ln a - a\right)\right)$

$= \lim_{a\to 0^+} \left(1\cdot 0 - 1 - a\ln a + a\right)$

$= -1 - \lim_{a\to 0^+} a\ln a + \lim_{a\to 0^+} a$

$= -1 - \lim_{a\to 0^+} \frac{\ln a}{\frac{1}{a}} + 0 = -1 - \lim_{a\to 0^+} \frac{\frac{1}{a}}{-\frac{1}{a^2}}$

form $\frac{\infty}{\infty}$

$= -1 - \lim_{a\to 0^+}(-a)$

$= -1 - 0 = \boxed{-1}$