Introduction to Mathematical Reason	Test #2 MATH 300	October 22, 2007
Name: ____________	R. Hammack	Score: ______

Directions: Please answer the questions in the space provided. To get full credit you must show all of your work. Use of calculators and other computing or communication devices is not allowed on this test.

1. Consider the sets $A = \{n \in \mathbb{Z} : 24|n\}$, $B = \{n \in \mathbb{Z} : 3|n\}$ and $C = \{n \in \mathbb{Z} : 4|n\}$. Prove that $A \subseteq B \cap C$.

   Proof.
   Suppose $x \in A$. By definition of A, this means $24|x$.
   Thus $x = 24k$ for some $k \in \mathbb{Z}$, by definition of divisibility.
   From $x = 24k$ we get $x = 3 \cdot (8k)$, so $3|x$, which means $x \in B$, by definition of B.
   Also from $x = 24k$ we get $x = 4 \cdot (6k)$, so $4|x$, which means $x \in C$, by definition of C.
   Since $x \in B$ and $x \in C$, it follows that $x \in B \cap C$, by definition of intersection.

   We've shown that $x \in A$ implies $x \in B \cap C$, so $A \subseteq B \cap C$.

   ■

2. Suppose x and y are real numbers. Prove the following statement.
   If $x^2 - 4x = y^2 - 4y$ and $x \neq y$, then $x + y = 4$.

   Proof. Suppose that $x^2 - 4x = y^2 - 4y$ and $x \neq y$.
   Then $x^2 - y^2 = 4x - 4y$.
   Thus $(x + y)(x - y) = 4(x - y)$.
   Since $x \neq y$, it follows that $x - y \neq 0$.
   Thus we may divide both sides of the above equation by $x - y$.
   Doing this produces $x + y = 4$. ■

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3. Suppose A, B, and C are sets, and $C \neq \emptyset$. Prove the following statement.
If $B \times C = A \times C$, then $A = B$.

Proof. (Direct) Assume that $B \times C = A \times C$.

First we will show that $A \subseteq B$.
Suppose $a \in A$.
Since $C \neq \emptyset$, there is an element $c \in C$.
Then $(a, c) \in A \times C$, by definition of the Cartesian product.
Thus $(a, c) \in B \times C$, because $B \times C = A \times C$.
Consequently $a \in B$, by definition of the Cartesian product.
This shows $a \in A$ implies $a \in B$, so $A \subseteq B$.

Next we will show that $B \subseteq A$.
Suppose $b \in B$.
Since $C \neq \emptyset$, there is an element $c \in C$.
Then $(b, c) \in B \times C$, by definition of the Cartesian product.
Thus $(b, c) \in A \times C$, because $B \times C = A \times C$.
Consequently $b \in A$, by definition of the Cartesian product.
This shows $b \in B$ implies $b \in A$, so $B \subseteq A$.

We've shown $A \subseteq B$ and $B \subseteq A$, so it follows that $A = B$. ■

4. Suppose x and y are nonzero real numbers. Use proof by contradiction to prove the following result.
If x is rational and y is irrational, then xy is irrational.

Proof. Assume for the sake of contradiction that x is rational and y is irrational, but xy is **not** irrational.

Since x is rational, there exists $a, b \in \mathbb{Z}$ (with $b \neq 0$) for which $x = \dfrac{a}{b}$. Since x is nonzero, we have $a \neq 0$.
Since xy is not irrational, it is rational, so there exists $c, d \in \mathbb{Z}$ (with $d \neq 0$) for which $xy = \dfrac{c}{d}$.

From $x = \dfrac{a}{b}$ and $xy = \dfrac{c}{d}$, we get $\dfrac{a}{b}y = \dfrac{c}{d}$, from which it follows that $\boxed{y = \dfrac{bc}{ad}.}$
Observe that bc and ad are integers, and as noted above $a \neq 0$ and $d \neq 0$, so $ad \neq 0$.
Thus the boxed equation shows y is a quotient of two integers (with nonzero denominator) so y is rational.
Thus we see that y is both rational and irrational, a contradiction. ■

FOR THE PROBLEMS ON THIS PAGE:
Decide if the statement is true or false. If it is true, prove it. If it is false, disprove it.

5. If a and b are integers, then $(a+b)^3 \equiv a^3 + b^3 \pmod{3}$.

   This is **True**.

   Proof. Observe that $(a+b)^3 - (a^3+b^3) = a^3 + 3a^2b + 3ab^2 + b^3 - a^3 - b^3 = 3(a^2b + ab^2)$.
   Let k be the integer $a^2b + ab^2$.
   Then we have $(a+b)^3 - (a^3+b^3) = 3k$, so $3|((a+b)^3 - (a^3+b^3))$ by definition of divisibility.
   This means $(a+b)^3 \equiv a^3 + b^3 \pmod{3}$, by definition of congruence modulo 3. ■

6. Suppose a, b and c are integers. If ab, ac and bc are all even, then a, b and c are all even.

   This is **False**, as the following counterexample shows.
   Let $a = 2$, $b = 4$ and $c = 3$.
   Then $ab = 8$, $ac = 6$ and $bc = 12$ are all even, but a, b and c are **not** all even.

7. For all integers a and b, if $a \equiv b \pmod{56}$, then $a \equiv b \pmod{8}$.

   This is **True**.

   Proof. Suppose $a \equiv b \pmod{56}$.
   This means $56|(a-b)$, so $a - b = 56k$ for some $k \in \mathbb{Z}$.
   Therefore $a - b = 8 \cdot (7k)$, so $8|(a-b)$, which means $a \equiv b \pmod{8}$. ■