MATH 501, Section 6 Solutions

(18)
Consider the cyclic subgroup of $\mathbb{Z}_{42}$ generated by 30.
This contains the elements $0 \cdot 30,\ \ 1 \cdot 30,\ \ 2 \cdot 30,\ \ 4 \cdot 30, \ldots$, and so on.
Working this out, we get the list $0, 30, 18, 6, 36, 24, 12, 0, 30, 18, 6, 36, 24, 12, 0, \ldots$
This is the subgroup $\langle 6 \rangle = \{0, 6, 12, 18, 24, 30, 36\}$ and its order is 7.

(18)
Notice that $\dfrac{1+i}{\sqrt{2}} = \cos\dfrac{2\pi}{8} + i \sin\dfrac{2\pi}{8}$ is the generator ζ of the subgroup U_8 of $\mathbb{C}^*$. The order of U_8 is 8.

(24)

$$\begin{tikzcd}[column sep=small, row sep=small]
\mathbb{Z}_8 \arrow[dr, dash] & & & & & \mathbb{Z}_8 \arrow[dr, dash] & & & \\
& \langle 2 \rangle \arrow[dr, dash] & & & & & \{0,2,4,6\} \arrow[dr, dash] & & \\
& & \langle 4 \rangle \arrow[dr, dash] & \text{or} & & & & \{0,4\} \arrow[dr, dash] & \\
& & & \langle 0 \rangle & & & & & \{0\}
\end{tikzcd}$$

(46)
Suppose a and b are elements of a group G. If ab has order n, then ba has order n also.

Proof. Recall the definition of the order of an element of a group.
The order of ab is the cardinality of the cyclic subgroup $\langle ab \rangle = \{(ab)^k | k \in \mathbb{Z}\}$.
The order of ba is the cardinality of the cyclic subgroup $\langle ba \rangle = \{(ba)^k | k \in \mathbb{Z}\}$.
(Notice these could be different subgroups, since it's possible $ab \neq ba$.)
To show the orders are the same, we will show that the cardinalities of $\langle ab \rangle$ and $\langle ba \rangle$ are the same.
This will be done by constructing a one-to-one and onto function $\varphi : \langle ab \rangle \to \langle ba \rangle$.

Define $\varphi : \langle ab \rangle \to \langle ba \rangle$ by the rule $\varphi(x) = a^{-1}xa$.
Notice $\varphi((ab)^k) = a^{-1}(ab)^k a = a^{-1}(ab)(ab)\cdots(ab)a = (ba)(ba)\cdots(ba) = (ba)^k$ so φ does send elements of $\langle ab \rangle$ to elements of $\langle ba \rangle$.

To see that φ is one-to-one, suppose $\varphi(x) = \varphi(y)$. This means $a^{-1}xa = a^{-1}ya$. Left-multiplying both sides by a produces the equation $xa = ya$. Now right-multiplying both sides by a^{-1} gives $x = y$, so φ is one-to-one.

To see that φ is onto, consider an arbitrary element $y = (ba)^k$ of $\langle ba \rangle$. Then $(ab)^k \in \langle ab \rangle$, and $\varphi((ab)^k) = a^{-1}(ab)^k a = a^{-1}(ab)(ab)\cdots(ab)a = (ba)(ba)\cdots(ba) = (ba)^k = y$. Thus φ is onto.

Since we have a one-to-one an onto function from $\langle ab \rangle$ to $\langle ba \rangle$, it follows that ab and ba have the same order.

(49)
It's not true that if every proper subgroup of a group is cyclic then the group is cyclic.
For a counterexample, consider the Klein 4-group V. It is not cyclic, but its proper subgroups are $\{e, a\} = \langle a \rangle$, $\{e, b\} = \langle b \rangle$, $\{e, c\} = \langle c \rangle$ and $\{e\} = \langle e \rangle$, which are all cyclic.