Introduction to Mathematical Reasoning	MATH 300 R. Hammack	April 11, 2007 Test #3

Name: ______________________ Score: ________

Directions: This is a take-home test. It is due at the beginning of class on Monday, April 16. Please answer all questions in the space provided. Consider working the problems on scratch paper, then rewriting them neatly on the test. Additional copies of this test can be downloaded from my web page if needed.

- Do not discuss this test with anyone other than the instructor. Ask me if you have any questions.
- You may consult your text and notes, but **no** other source.
- To get full credit you must show and explain all of your work.
- Each problem is worth 10 points.

1. Suppose that R is a symmetric and transitive relation on a set A, and there is an element $a \in A$ with the property that aRx for every $x \in A$. Prove that R is reflexive.

   Proof. Take an arbitrary element $b \in A$. We need to show bRb.
   The hypothesis states that aRb, and since R is symmetric, it also follows that bRa.
   Then we have bRa and aRb, and from this transitivity implies bRb.
   We have now shown that bRb is true for all $b \in A$, and this means R is reflexive. ■

2. Let $B = \{1, 2, 3, 4\}$. Define a relation R on $\mathcal{P}(B)$ as $X \, R \, Y$ if $|X| = |Y|$.

   (a) Show that R is an equivalence relation.

   i. Any $X \in \mathcal{P}(B)$ satisfies $|X| = |X|$, so $X \, R \, X$. Thus R is reflexive.
   ii. Suppose $X, Y \in \mathcal{P}(B)$ and $X \, R \, Y$. This means $|X| = |Y|$, so $|Y| = |X|$, and therefore $Y \, R \, X$. Consequently R is symmetric.
   iii. Suppose $X R Y$ and $Y R Z$. This means $|X| = |Y|$ and $|Y| = |Z|$, so it follows that $|X| = |Z|$, which means $X R Z$. Therefore R is transitive.

   Since R has been shown to be reflexive, symmetric and transitive, it follows that R is an equivalence relation.

   (b) List the equivalence classes of R.

   By definition of R, two elements of $\mathcal{P}(B)$ are in the same equivalence class if and only if they have the same cardinality, so there are 5 equivalence classes, as follows.

   i. $\{\emptyset\}$
   ii. $\{\{1\}, \{2\}, \{3\}, \{4\}\}$
   iii. $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$
   iv. $\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$
   v. $\{\{1, 2, 3, 4\}\}$

3. Let $\mathbb{Q}^+$ denote the set of positive rational numbers. That is, $\mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x > 0\}$. Consider the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$ defined as $f((x, y)) = \frac{x}{y}$.

   (a) Is f is injective? Explain.

   No. Observe $(1, 2) \neq (2, 4)$, yet $f((1, 2)) = \frac{1}{2} = \frac{2}{4} = f((2, 4))$.

   (b) Is f is surjective? Explain.

   Yes. If $b \in \mathbb{Q}^+$, then $b = \frac{c}{d}$ for some positive integers c and d.
   Therefore $(c, d) \in \mathbb{N} \times \mathbb{N}$ and $f((c, d)) = \frac{c}{d} = b$.

4. Consider the function $g : \mathbb{N} \to \mathbb{Q}^+$ defined as $g(x) = \frac{1}{x}$.

   (a) Is g is injective? Explain.

   Yes. If $g(x) = g(y)$, then $\frac{1}{x} = \frac{1}{y}$, from which it follows $x = y$.

   (b) Is g is surjective? Explain.

   No. Observe that $2 \in \mathbb{Q}^+$.
   However, for any $x \in \mathbb{N}$ we have $g(x) = \frac{1}{x} \leq 1$, so $g(x) \neq 2$ for all x in the domain of g.

5. Write the addition table for $\mathbb{Z}_7$.

+	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[0]	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[1]	[1]	[2]	[3]	[4]	[5]	[6]	[0]
[2]	[2]	[3]	[4]	[5]	[6]	[0]	[1]
[3]	[3]	[4]	[5]	[6]	[0]	[1]	[2]
[4]	[4]	[5]	[6]	[0]	[1]	[2]	[3]
[5]	[5]	[6]	[0]	[1]	[2]	[3]	[4]
[6]	[6]	[0]	[1]	[2]	[3]	[4]	[5]

6. Consider the function $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{2\}$ defined as $f(x) = \sqrt[3]{\dfrac{8x}{x-1}}$.

(a) Show that f is injective.

Suppose $f(x) = f(y)$ for some $x, y \in \mathbb{R} - \{1\}$. This means $\sqrt[3]{\dfrac{8x}{x-1}} = \sqrt[3]{\dfrac{8y}{y-1}}$.

Cubing both sides produces $\dfrac{8x}{x-1} = \dfrac{8y}{y-1}$.
Cross-multiplying gives $8x(y-1) = 8y(x-1)$,
so $8xy - 8x = 8xy - 8y$.
Subtracting $8xy$ from both sides gives $-8x = -8y$,
and finally dividing by -8 results in $x = y$.

Since $f(x) = f(y)$ implies $x = y$, it follows that f is injective.

(b) Show that f is surjective.

Suppose $b \in \mathbb{R} - \{2\}$. Then since $b \neq 2$, it follows that $b^3 \neq 8$.
Therefore $b^3 - 8 \neq 0$ so the number $x = \dfrac{b^3}{b^3-8}$ is defined.
Moreover, $x \neq 1$ (since the numerator and denominator are not equal) so x is in the domain $\mathbb{R} - \{1\}$ of f.

Notice that $f(x) = f\left(\dfrac{b^3}{b^3-8}\right) = \sqrt[3]{\dfrac{8\frac{b^3}{b^3-8}}{\frac{b^3}{b^3-8} - 1}} = \sqrt[3]{\dfrac{8\frac{b^3}{b^3-8}}{\frac{8}{b^3-8}}} = \sqrt[3]{b^3} = b.$

This shows that for any b the codomain $\mathbb{R} - \{2\}$, there is an x in the domain $\mathbb{R} - \{1\}$ for which $f(x) = b$.
Therefore f is surjective.

(c) Find a formula for f^{-1}.

Observe that $f(f^{-1}(x)) = x$, and using the expression for f, this is $\sqrt[3]{\dfrac{8f^{-1}(x)}{f^{-1}(x)-1}} = x$.

Solving for $f^{-1}(x)$, we get:

$$\begin{aligned}
\frac{8f^{-1}(x)}{f^{-1}(x)-1} &= x^3 \\
8f^{-1}(x) &= x^3(f^{-1}(x) - 1) \\
8f^{-1}(x) &= x^3 f^{-1}(x) - x^3 \\
8f^{-1}(x) - x^3 f^{-1}(x) &= -x^3 \\
f^{-1}(x)(8 - x^3) &= -x^3 \\
f^{-1}(x) &= \frac{-x^3}{8 - x^3}
\end{aligned}$$

Therefore $\boxed{f^{-1}(x) = \dfrac{x^3}{x^3-8}}$

7. Use mathematical induction to prove that $1+3+5+7+\cdots+(2n+1)=(n+1)^2$ for every integer $n \geq 0$.

Proof. If $n=0$, then this statement is $(2\cdot 0+1)=(0+1)^2$, which is true.

Assume that $1+3+5+7+\cdots+(2k+1)=(k+1)^2$ for some $k \geq 0$.
We need to show that this implies $1+3+5+7+\cdots+(2k+1)+(2(k+1)+1)=((k+1)+1)^2$.
Observe that

$$\begin{aligned}
1+3+5+7+\cdots+(2k+1)+(2(k+1)+1) &= [1+3+5+7+\cdots+(2k+1)]+(2(k+1)+1) \\
&= [(k+1)^2]+(2(k+1)+1) \\
&= [k^2+2k+1]+2k+2+1) \\
&= k^2+4k+4 \\
&= (k+2)^2 \\
&= ((k+1)+1)^2
\end{aligned}$$

Therefore $1+3+5+7+\cdots+(2k+1)+(2(k+1)+1)=((k+1)+1)^2$, and the result follows by the principle of Mathematical Induction. ∎

8. Use mathematical induction to prove that $n! > 2^n$ for every integer $n > 4$.

Proof. In $n=5$, then $n! = 120$ and $2^n = 32$. So $n! > 2^n$ if $n=5$.

Next, assume $k! > 2^k$ for some integer $k \geq 5$. We need to show this implies $(k+1)! > 2^{k+1}$.
Begin by multiplying both sides of the inequality $k! > 2^k$ by the positive number $k+1$.
We get $(k+1)k! > (k+1)2^k$ which simplifies to

$$(k+1)! > (k+1)2^k.$$

Now, since $k > 4$, it follows that $k+1 > 2$, and so $(k+1)2^k > 2\cdot 2^k$. Thus we have

$$(k+1)! > (k+1)2^k > 2\cdot 2^k = 2^{k+1}.$$

Therefore $(k+1)! > 2^{k+1}$.

In summary, we have shown that $5! > 2^5$, and $k! > 2^k$ implies $(k+1)! > 2^{k+1}$.
Thus it follows by the principle of mathematical induction that $n! > 2^n$ for every integer $n > 4$. ∎

The problems on this page concern the Fibonacci Sequence $F_1, F_2, F_3, \ldots$.
Recall that this sequence is defined as $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for any $n > 2$.

9. Use mathematical induction to prove that $\displaystyle\sum_{i=1}^{n} F_i = F_{n+2} - 1$ for every $n \in \mathbb{N}$.

   Proof. (Induction) If $n = 1$, the statement is $\displaystyle\sum_{i=1}^{1} F_i = F_{1+2} - 1$.

   The left-hand side equals $\displaystyle\sum_{i=1}^{1} F_i = F_1 = 1$, and the right-hand side equals $F_{1+2} - 1 = F_3 - 1 = 2 - 1 = 1$.
   Since the two sides are equal in this case, the statement is true for $n = 1$.

   Now suppose the statement is true for some positive integer k, that is $\displaystyle\sum_{i=1}^{k} F_i = F_{k+2} - 1$.

   We must show that the statement is true for $k+1$, that is, we must show $\displaystyle\sum_{i=1}^{k+1} F_i = F_{(k+1)+2} - 1$.

   Observe that

$$\begin{aligned}
\sum_{i=1}^{k+1} F_i &= \left(\sum_{i=1}^{k} F_i\right) + F_{k+1} && \text{(split the sequence)} \\
&= (F_{k+2} - 1) + F_{k+1} && \text{(inductive hypothesis)} \\
&= F_{k+2} + F_{k+1} - 1 && \text{(regroup)} \\
&= F_{k+3} - 1 && \text{(use } F_n = F_{n-1} + F_{n-2}\text{)} \\
&= F_{(k+1)+2} - 1 && \text{(regroup)}
\end{aligned}$$

   This shows $\displaystyle\sum_{i=1}^{k+1} F_i = F_{(k+1)+2} - 1$, so the induction proof is complete. ■

10. Prove that $3|F_{4n}$ for each $n \in \mathbb{N}$.

    Proof. The Fibonacci Sequence begins as $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$,
    so it follows that if $n = 1$, then $F_{4\cdot 1} = F_4 = 3$, and so $3|F_{4\cdot 1}$.
    Thus the statement is true if $n = 1$.

    Now assume that $3|F_{4k}$ for some integer $k \geq 1$, which means $F_{4k} = 3a$ for some integer a.

    We need to show $3|F_{4(k+1)}$. By using the equation $F_n = F_{n-1} + F_{n-2}$ (from the definition of the Fibonacci Sequence) multiple times, we get

$$\begin{aligned}
F_{4(k+1)} &= F_{4k+4} \\
&= F_{4k+3} + F_{4k+2} \\
&= (F_{4k+2} + F_{4k+1}) + (F_{4k+1} + F_{4k}) \\
&= ((F_{4k+1} + F_{4k}) + F_{4k+1}) + (F_{4k+1} + F_{4k}) \\
&= 3F_{4k+1} + 2F_{4k} \\
&= 3F_{4k+1} + 2 \cdot 3a \\
&= 3(F_{4k+1} + 2a)
\end{aligned}$$

    This shows that $F_{4(k+1)} = 3b$, where b is the integer $b = F_{4k+1} + 2a$. Therefore $3|F_{4(k+1)}$.

    In summary, we have shown that $3|F_{4\cdot 1}$, and that $3|F_{4k}$ implies $3|F_{4(k+1)}$.
    Thus by mathematical induction $3|F_{4n}$ for each $n \in \mathbb{N}$. ■