Introduction to Mathematical Reason	Test # 3 MATH 300	April 8, 2008
Name: ________________	R. Hammack	Score: ______

Directions: Please answer the questions in the space provided. To get full credit you must show all of your work. Use of calculators and other computing or communication devices is not allowed on this test.

1. Suppose $a, b \in \mathbb{Z}$. Prove that $(a-3)b^2$ is even if and only if a is odd or b is even.

   Proof. First we will prove that if $(a-3)b^2$ is even, then a is odd or b is even. For this we use contrapositive proof. Suppose it is not the case that a is odd or b is even. Then by DeMorgan's law, a is even and b is odd. Thus there are integers m and n for which $a = 2m$ and $b = 2n+1$. Now observe $(a-3)b^2 = (2m-3)(2n+1)^2 = (2m-3)(4n^2+4n+1)\ 2mn^2+8mn+2m-6n-3 = 2mn^2+8mn+2m-6n-4+1 = 2(mn^2+4mn+m-3n-2)+1$. This shows $(a-3)b^2$ is odd, so it's not even.

   Conversely, we need to show that if a is odd or b is even, then $(a-3)b^2$ is even. For this we use direct proof, with cases. **Case 1.** Suppose a is odd. Then $a = 2m+1$ for some integer m. Thus $(a-3)b^2 = (2m+1-3)b^2 = (2m-2)b^2 = 2(m-1)b^2$. Thus in this case $(a-3)b^2$ is even.
   Case 2. Suppose b is even. Then $b = 2n$ for some integer n. Thus $(a-3)b^2 = (a-3)(2n)^2 = (a-3)4n^2 = 2(a-3)2n^2 =$. Thus in this case $(a-3)b^2$ is even.

   Therefore, in any event, $(a-3)b^2$ is even. ■

2. Suppose A and B are sets. Prove that $\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B)$.

   Proof. To prove $\mathscr{P}(A) \cap \mathscr{P}(B) = \mathscr{P}(A \cap B)$, we must prove $\mathscr{P}(A) \cap \mathscr{P}(B) \subseteq \mathscr{P}(A \cap B)$ and $\mathscr{P}(A \cap B) \subseteq \mathscr{P}(A) \cap \mathscr{P}(B)$.

   First we will show that $\mathscr{P}(A) \cap \mathscr{P}(B) \subseteq \mathscr{P}(A \cap B)$.
   Suppose $X \in \mathscr{P}(A) \cap \mathscr{P}(B)$.
   By definition of intersection, this means $X \in \mathscr{P}(A)$ and $X \in \mathscr{P}(B)$.
   By the definition of power sets, this means $X \subseteq A$ and $X \subseteq B$.
   Thus, any element $x \in X$ is in both A and B, so $x \in A \cap B$. Hence $X \subseteq A \cap B$, which means $X \in \mathscr{P}(A \cap B)$.
   We have seen that $X \in \mathscr{P}(A) \cap \mathscr{P}(B)$ implies $X \in \mathscr{P}(A \cap B)$, so $\mathscr{P}(A) \cap \mathscr{P}(B) \subseteq \mathscr{P}(A \cap B)$.

   Next we will show that $\mathscr{P}(A \cap B) \subseteq \mathscr{P}(A) \cap \mathscr{P}(B)$.
   Suppose $X \in \mathscr{P}(A \cap B)$.
   By definition of the power set, this means $X \subseteq A \cap B$.
   Thus any element $x \in X$ is in $A \cap B$, so $x \in A$ and $x \in B$. Hence $X \subseteq A$ and $X \subseteq B$.
   Thus $X \in \mathscr{P}(A)$ and $X \in \mathscr{P}(B)$, so $X \in \mathscr{P}(A) \cap \mathscr{P}(B)$, by definition of intersection.
   We have seen that $X \in \mathscr{P}(A \cap B)$ implies $X \in \mathscr{P}(A) \cap \mathscr{P}(B)$, so $\mathscr{P}(A \cap B) \subseteq \mathscr{P}(A) \cap \mathscr{P}(B)$.

   The previous two paragraphs imply $\mathscr{P}(A) \cap \mathscr{P}(B) = \mathscr{P}(A \cap B)$. ■

3. Suppose A, B and C are sets. If $B \subseteq C$, then $A \times B \subseteq A \times C$.

   Proof. This is a conditional statement, and we'll prove it with direct proof. Suppose $B \subseteq C$. (Now we need to prove $A \times B \subseteq A \times C$.)

   Suppose $(a, b) \in A \times B$. Then by definition of the Cartesian product we have $a \in A$ and $b \in B$. But since $b \in B$ and $B \subseteq C$, we have $b \in C$. Since $a \in A$ and $b \in C$, it follows that $(a, b) \in A \times C$. Now we've shown $(a, b) \in A \times B$ implies $(a, b) \in A \times C$, so $A \times B \subseteq A \times C$.

   In summary, we've shown that if $B \subseteq C$, then $A \times B \subseteq A \times C$. This completes the proof. ■

4. Prove by induction: If $n \in \mathbb{N}$, then $6|(n^3 - n)$.

   Proof. The proof is by mathematical induction.

   (a) When $n = 1$, the statement is $6|(1^3 - 1)$, or $6|0$, which is true.

   (b) Now assume the statement is true for some integer $n = k \geq 0$, that is assume $6|(k^3 - k)$. This means $k^3 - k = 6a$ for some integer a. We need to show that $6|((k + 1)^3 - (k + 1))$. Observe that

   $$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= (k^3 - k) + 3k^2 + 3k \\ &= 6a + 3k^2 + 3k \\ &= 6a + 3k(k+1)\end{aligned}$$

   Thus we have deduced $(k+1)^3-(k+1) = 6a+3k(k+1)$. Since one of k or $(k+1)$ must be even, it follows that $k(k+1)$ is even, so $k(k+1) = 2b$ for some integer b. Consequently $(k+1)^3-(k+1) = 6a+3k(k+1) = 6a + 3(2b) = 6(a + b)$. Since $(k + 1)^3 - (k + 1) = 6(a + b)$ it follows that $6|((k + 1)^3 - (k + 1))$.

   Thus the result follows by mathematical induction. ■

FOR THE PROBLEMS ON THIS PAGE:
Decide if the statement is true or false. If it is true, prove it; if it is false, give a counterexample.

5. Let A and B be sets. If $A - B = B - A$, then $A - B = \emptyset$.

   This is TRUE.

   Proof. Suppose for the sake of contradiction that $A - B = B - A$ but $A - B \neq \emptyset$.
   Now since $A - B \neq \emptyset$, then there must be some $a \in A - B$.
   And since $a \in A - B = \{x \in A : x \notin B\}$, it follows that $a \in A$ but $a \notin B$.
   But also $a \in A - B = B - A = \{x \in B : x \notin A\}$, which means $x \notin A$.
   Thus $a \in A$ and $a \notin A$, which is a contradiction. ■

6. For every two sets A and B, $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

   This is FALSE. Here is a counterexample.

   Let $A = \{1\}$ and $B = \{2\}$.
   Then $\mathcal{P}(A \cup B) = \mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.
   Also $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}\} \cup \{\emptyset, \{2\}\} = \{\emptyset, \{1\}, \{2\}\}$.
   Thus we see that $\mathcal{P}(A \cup B) \neq \mathcal{P}(A) \cup \mathcal{P}(B)$.

7. Suppose A, B, C and D are sets. If $A \times B \subseteq C \times D$, then $A \subseteq C$ and $B \subseteq D$.

   This is FALSE. Here is a counterexample.

   Suppose $A = \{1\}, B = \emptyset, C = \{2\}$ and $D = \{3\}$.
   Then $A \times B = \emptyset \subseteq C \times D$, but $A \not\subseteq C$, so it is not true that $A \subseteq C$ and $B \subseteq D$.