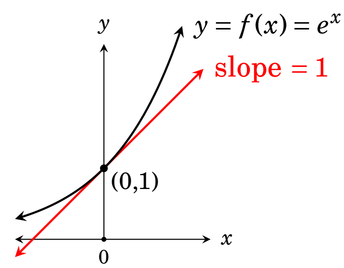
CHAPTER 19

The Derivative of e^x

This chapter's goal is to find a derivative rule for the natural exponential function. We ask: If $f(x) = e^x$, what is $f'(x)$? We will answer this by working out the limit $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

Actually, we already know a little about this from Chapter 5. In Section 5.6 we found that the tangent to the graph of $f(x) = e^x$ at the point $(0,1)$ has a slope of 1. (Fact 5.2 on page 93). This fact is illustrated on the right. It tells us that



$$\begin{aligned} 1 = f'(0) &= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0} \frac{e^{0+h}-e^0}{h} = \lim_{h\to 0} \frac{e^h-1}{h}. \end{aligned}$$

We will need this fact shortly. Note that it gives the value of a certain limit:

$$\lim_{h\to 0} \frac{e^h-1}{h} = 1. \tag{19.1}$$

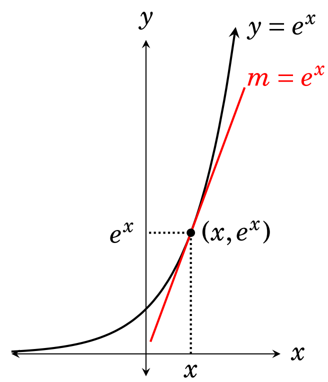
Now let's find the derivative of $f(x) = e^x$ using the limit definition of $f'(x)$.

$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} && \text{(definition of } f'(x)\text{)} \\ &= \lim_{h\to 0} \frac{e^x e^h - e^x}{h} && \text{(using } e^{x+h} = e^x e^h\text{)} \\ &= \lim_{h\to 0} \frac{e^x\left(e^h-1\right)}{h} && \text{(factor out } e^x\text{)} \\ &= \lim_{h\to 0} e^x \cdot \lim_{h\to 0} \frac{e^h-1}{h} && \text{(limit law)} \\ &= e^x \cdot 1 = e^x && \text{(Equation 19.1)} \end{aligned}$$

We have just found that if $f(x) = e^x$, then $f'(x) = e^x$. In other words, e^x is its own derivative! This is our latest derivative rule.

Rule 6 $D_x[e^x] = e^x$.

Geometrically, this new rule tells us that the tangent to the graph of $y = e^x$ at the point (x, e^x) has slope e^x. (See the diagram on the right.) The slope at the point (x, e^x) is as big as the point is high.



The fact that e^x is its own derivative is yet another indication of how special the natural exponential function e^x is, and why we place more importance on it than on other exponential functions a^x. The derivative of e^x is e^x. (As we will see, the derivative of, say, 2^x is **not** 2^x.)

You will often use this new rule in conjunction with other rules. For example, suppose we need to find the derivative of $x^5 - 3e^x + 1$. The answer comes from combining Rule 6 with rules 1–5:

$$\begin{aligned} D_x[x^5 - 3e^x + 1] &= D_x[x^5] - D_x[3e^x] + D_x[1] \\ &= 5x^4 - 3D_x[e^x] + 0 \\ &= 5x^4 - 3e^x \end{aligned}$$

Of course you will typically skip steps and get the answer immediately.

Be careful not to apply Rule 6 blindly. Notice that, for instance, $D_x[e^3] = 0$ because $e^3 \approx 2.71828^3 = 20.08555$ is a constant, and the derivative of a constant is zero. (Some students mistakenly write $D_x[e^3] = e^3$, or, even worse, $D_x[e^3] = 3e^2$. These are **wrong**. The first is a misapplication of Rule 6. The second is a misapplication of the power rule.)

Exercises for Chapter 19

Find the derivatives of the following functions in problems 1-6.

1. $f(x) = \sqrt{2}\,e^x + \sqrt{x}$

2. $f(x) = \dfrac{1}{x} - e^x + 3$

3. $w = z + e^2$

4. $y = e^{5+x}$ Hint: $e^{a+b} = e^a e^b$.

5. $f(x) = 6x^3 + e^x - 4$

6. $f(x) = \dfrac{3}{x^4} + \dfrac{e^x}{3}$

7. Find the equation of the tangent line to $y = 3e^x$ at the point $(2, 3e^2)$.

8. For what x is the tangent to $y = e^x - x$ at $(x, e^x - x)$ horizontal?

Exercise Solutions for Chapter 19

1. $D_x\left[\sqrt{2}\,e^x+\sqrt{x}\right] \;=\; D_x\left[\sqrt{2}\,e^x\right]+D_x\left[\sqrt{x}\right] \;=\; \sqrt{2}D_x\left[e^x\right]+D_x\left[x^{1/2}\right]$

$$= \sqrt{2}e^x+\frac{1}{2}x^{1/2-1} \;=\; \sqrt{2}e^x+\frac{1}{2}x^{-1/2} \;=\; \sqrt{2}e^x+\frac{1}{2x^{1/2}} \;=\; \boxed{\sqrt{2}e^x+\frac{1}{2\sqrt{x}}}$$

3. $\dfrac{d}{dz}\left[z+e^2\right] \;=\; 1+0 \;=\; \boxed{1}$

5. $f'(x)=18x^2+e^x$

7. Find the equation of the tangent line to $y=3e^x$ at the point $(2,3e^2)$.

The slope of the tangent to $y=3e^x$ at $(x,3e^x)$ is $\frac{dy}{dx}=3e^x$. We are interested in the tangent line at $(2,3e^2)$, and its slope is $\frac{dy}{dx}\big|_{x=2}=3e^2$. So its slope is $m=3e^2$ and it passes through $(2,3e^2)$. We can get its equation with the point-slope formula.

$$\begin{aligned} y-y_0 &= m(x-x_0)\\ y-3e^2 &= 3e^2(x-2)\\ y &= 3e^2x-3e^2\cdot 2+3e^2\\ y &= 3e^2x-3e^2 \end{aligned}$$

Answer: The tangent line has equation $y=3e^2x-3e^2$.