

# MATH 201

## Integration by Substitution

You learned the **Substitution Rule** in Calculus I, but it is so important in Calculus II that we are going to spend some time reviewing it.

The **substitution rule** is an integration rule that is the chain rule in reverse. Think of it as the *chain rule for integration*. To apply it, you must first have command of the basic integration formulas.

### Basic Integration Formulas

$\int c \, du = cu + C$	$\int \sec^2(u) \, du = \tan(u) + C$
$\int u^n \, du = \frac{u^{n+1}}{n+1} + C$	$\int \csc^2(u) \, du = -\cot(u) + C$
$\int \frac{1}{u} \, du = \ln u  + C$	$\int \sec(u) \tan(u) \, du = \sec(u) + C$
$\int e^u \, du = e^u + C$	$\int \csc(u) \cot(u) \, du = -\csc(u) + C$
$\int b^u \, du = \frac{1}{\ln(b)} b^u + C$	$\int \frac{1}{\sqrt{1-u^2}} \, du = \sin^{-1}(u) + C$
$\int \sin(u) \, du = -\cos(u) + C$	$\int \frac{1}{1+u^2} \, du = \tan^{-1}(u) + C$
$\int \cos(u) \, du = \sin(u) + C$	$\int \frac{1}{u\sqrt{u^2-1}} \, du = \sec^{-1} u  + C$

### Substitution Rule

$$\text{If } u = g(x), \text{ then } \int f(g(x)) g'(x) \, dx = \int f(u) \, du.$$

### Substitution Rule for Definite Integrals

$$\text{If } u = g(x), \text{ then } \int_a^b f(g(x)) g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du.$$

# EXAMPLES

$$1. \int \sin(x^2) 2x \, dx = \int \sin(u) \, du = -\cos(u) + C = \boxed{-\cos(x^2) + C}$$

$$\begin{cases} u = x^2 \\ \frac{du}{dx} = 2x \\ du = 2x \, dx \end{cases}$$

$$2. \int x \sin(x^2) \, dx = \int \sin(x^2) \cancel{x} \, dx = \int \sin(u) \frac{1}{2} \, du$$

$$\begin{cases} u = x^2 \\ \frac{du}{dx} = 2x \\ du = 2x \, dx \rightarrow \frac{1}{2} du = x \, dx \end{cases}$$

$$= \frac{1}{2} \int \sin(u) \, du$$

$$= \frac{1}{2} (-\cos(u)) + C$$

$$= \boxed{-\frac{1}{2} \cos(x^2) + C}$$

$$3. \int \frac{\sec(\ln(x)) \tan(\ln(x))}{x} \, dx =$$

$$\begin{cases} u = \ln(x) \\ \frac{du}{dx} = \frac{1}{x} \\ du = \frac{1}{x} \, dx \end{cases}$$

$$= \int \sec(\ln(x)) \tan(\ln(x)) \frac{1}{x} \, dx$$

$$= \int \sec(u) \tan(u) \, du$$

$$= \sec(u) + C = \boxed{\sec(\ln(x)) + C}$$

$$4. \int \sqrt{e^x} e^x \, dx = \int \sqrt{u} \, du$$

$$\begin{cases} u = e^x \\ \frac{du}{dx} = e^x \\ du = e^x \, dx \end{cases}$$

$$= \int u^{\frac{1}{2}} \, du = \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{2}{3} \sqrt{u^3} + C = \boxed{\frac{2}{3} \sqrt{e^{3x}} + C}$$

$$5. \int \frac{3x^2 + 8x - 1}{x^3 + 4x^2 - x + 1} \, dx = \int \frac{1}{x^3 + 4x^2 - x + 1} (3x^2 + 8x - 1) \, dx$$

$$u = x^3 + 4x^2 - x + 1$$

$$\frac{du}{dx} = 3x^2 + 8x - 1$$

$$du = (3x^2 + 8x - 1) \, dx$$

$$= \int \frac{1}{u} \, du = \ln|u| + C$$

$$= \boxed{\ln|x^3 + 4x^2 - x + 1| + C}$$

$$6. \int \frac{5x+2}{25x^2+20x+4} dx = \int \frac{1}{25x^2+20x+4} (5x+2) dx = \int \frac{1}{u} \frac{1}{10} du$$

$$u = 25x^2 + 20x + 4$$

$$\frac{du}{dx} = 50x + 20$$

$$du = (50x + 20) dx \Rightarrow \frac{1}{10} du = (5x + 2) dx$$

$$= \frac{1}{10} \ln|u| + C = \boxed{\frac{1}{10} \ln|25x^2 + 20x + 4| + C}$$

$$7. \int \frac{1}{25x^2+20x+4} dx = \int \frac{1}{(5x+2)(5x+2)} dx = \int \frac{1}{(5x+2)^2} dx$$

$$u = 5x + 2$$

$$\frac{du}{dx} = 5$$

$$du = 5 dx \rightarrow dx = \frac{1}{5} du$$

$$= \int \frac{1}{u^2} \frac{1}{5} du = \frac{1}{5} \int u^{-2} du = \frac{1}{5} \frac{u^{-2+1}}{-2+1} + C$$

$$= -\frac{1}{5} u^{-1} + C = \frac{-1}{5u} + C = \frac{-1}{5(5x+2)} + C$$

$$8. \int \frac{1}{25x^2+20x+5} dx = \int \frac{1}{25x^2+20x+4+1} dx = \boxed{\frac{-1}{25x+10} + C}$$

$$u = 5x + 2$$

$$du = 5 dx$$

$$\frac{1}{5} du = dx$$

$$= \int \frac{1}{(5x+2)^2 + 1} dx = \int \frac{1}{u^2 + 1} \frac{1}{5} du$$

$$= \frac{1}{5} \tan^{-1}(u) + C = \boxed{\frac{1}{5} \tan^{-1}(5x+2) + C}$$

$$9. \int_1^2 \frac{5}{(5x-1)^2} dx = \int_1^2 (5x-1)^{-2} 5 dx$$

$$u = 5x - 1$$

$$\frac{du}{dx} = 5$$

$$du = 5 dx$$

$$= \int_{5 \cdot 1 - 1}^{5 \cdot 2 - 1} u^{-2} du = \left[ \frac{u^{-2+1}}{-2+1} \right]_4^9 = \left[ -\frac{1}{u} \right]_4^9 = -\frac{1}{9} - \left(-\frac{1}{4}\right)$$

$$= -\frac{1}{9} + \frac{1}{4} = -\frac{4}{36} + \frac{9}{36} = \boxed{\frac{5}{36}}$$

$$10. \int_0^1 \frac{x+1}{x^2+2x+2} dx =$$

$$u = x^2 + 2x + 2$$

$$\frac{du}{dx} = 2x + 2$$

$$du = (2x + 2) dx$$

$$\frac{1}{2} du = (x + 1) dx$$

$$\int_0^1 \frac{1}{x^2+2x+2} (x+1) dx = \int_{0^2+2 \cdot 0+2}^{1^2+2 \cdot 1+2} \frac{1}{u} du$$

$$= \int_2^5 \frac{1}{u} du = [\ln|u|] = \ln|5| - \ln|2|$$

$$= \boxed{\ln\left(\frac{5}{2}\right)}$$

## Some Useful Integration Rules

Next we are going to derive some integration formulas that will be useful in Calculus II. In these exercises,  $a \neq 0$  is a constant. Use the substitution  $u = ax$  to get the answers on the right.

In each case your substitution will work like this:

$u = ax$	$\frac{du}{dx} = a$	$du = a \, dx$	$dx = \frac{1}{a} du$
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$$1. \int e^{ax} dx = \int e^u \frac{1}{a} du = \frac{1}{a} \int e^u du = \frac{1}{a} e^u + C = \frac{1}{a} e^{ax} + C$$

$$2. \int \sin(ax) dx = \int \sin(u) \frac{1}{a} du = \frac{1}{a} \int \sin(u) du = \frac{1}{a} (-\cos(u) + C) = -\frac{1}{a} \cos(ax) + C$$

$$3. \int \cos(ax) dx = \int \cos(u) \frac{1}{a} du = \frac{1}{a} \int \cos(u) du = \frac{1}{a} \sin(u) + C = \frac{1}{a} \sin(ax) + C$$

$$4. \int \sec^2(ax) dx = \int \sec^2(u) \frac{1}{a} du = \frac{1}{a} \int \sec^2(u) du = \frac{1}{a} \tan(u) + C = \frac{1}{a} \tan(ax) + C$$

$$5. \int \csc^2(ax) dx = \int \csc^2(u) \frac{1}{a} du = \frac{1}{a} \int \csc^2(u) du = \frac{1}{a} (-\cot(u) + C) = -\frac{1}{a} \cot(ax) + C$$

$$6. \int \sec(ax) \tan(ax) dx = \int \sec(u) \tan(u) \frac{1}{a} du = \frac{1}{a} \sec(u) + C = \frac{1}{a} \sec(ax) + C$$

$$7. \int \csc(ax) \cot(ax) dx = \int \csc(u) \cot(u) \frac{1}{a} du = \frac{1}{a} (-\csc(u) + C) = -\frac{1}{a} \csc(ax) + C$$

For the examples on this page, use the substitution  $u = \frac{1}{a}x$ , as follows.

$u = \frac{1}{a}x$ (so $x = au$ )	$\frac{du}{dx} = \frac{1}{a}$	$dx = a du$
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$$8. \int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{dx}{\sqrt{a^2 - (au)^2}} = \int \frac{dx}{\sqrt{a^2 - a^2 u^2}} = \int \frac{1}{\sqrt{a^2(1-u^2)}} dx$$

$$= \int \frac{1}{|a| \sqrt{1-u^2}} a du = \frac{a}{|a|} \int \frac{du}{\sqrt{1-u^2}} = \frac{a}{|a|} \sin^{-1}(u) + C = \boxed{\sin^{-1}\left(\frac{x}{a}\right) + C}$$

$\left\{ \frac{a}{|a|} = 1 \text{ if } a > 0 \right\}$ 
If  $a > 0$

$$9. \int \frac{1}{a^2 + x^2} dx = \int \frac{1}{a^2 + (au)^2} a du = \int \frac{1}{a^2 + a^2 u^2} a du$$

$$= \int \frac{1}{(1+u^2)a^2} a du = \frac{1}{a} \int \frac{1}{1+u^2} du = \frac{1}{a} \tan^{-1}(u) + C = \boxed{\frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C}$$

$$10. \int \frac{1}{x\sqrt{x^2 - a^2}} dx = \int \frac{1}{au\sqrt{(au)^2 - a^2}} a du = \int \frac{1}{u\sqrt{a^2(u^2 - 1)}} du$$

$$= \int \frac{1}{|a| u \sqrt{u^2 - 1}} du = \frac{1}{|a|} \int \frac{1}{u \sqrt{u^2 - 1}} du = \frac{1}{|a|} \sec^{-1}\left|\frac{x}{a}\right| + C = \boxed{\frac{1}{a} \sec^{-1}\left|\frac{x}{a}\right| + C}$$

If  $a > 0$

You should regard the examples on this page as giving **new formulas** for integrals that you will see often. For instance, we will apply #9 above in the next example.

Example:  $\int \frac{1}{7+x^2} dx = \int \frac{1}{\sqrt{7^2+x^2}} dx = \boxed{\frac{1}{\sqrt{7}} \tan^{-1}\left(\frac{x}{\sqrt{7}}\right) + C}$  (using #9 above)

Example:  $\int \frac{1}{\sqrt{16-x^2}} dx = \int \frac{1}{\sqrt{4^2-x^2}} dx = \boxed{\sin^{-1}\left(\frac{x}{4}\right) + C}$  (using #8 above)

# TAKEAWAY

Remember these general integration formulas. Use them in applying the Substitution Rule.

## General Integration Formulas

$\int c \, du = cu + C$	$\int \sec^2(u) \, du = \tan(u) + C$
$\int u^n \, du = \frac{u^{n+1}}{n+1} + C$	$\int \csc^2(u) \, du = -\cot(u) + C$
$\int \frac{1}{u} \, du = \ln u  + C$	$\int \sec(u) \tan(u) \, du = \sec(u) + C$
$\int e^u \, du = e^u + C$	$\int \csc(u) \cot(u) \, du = -\csc(u) + C$
$\int b^u \, du = \frac{1}{\ln(b)} b^u + C$	$\int \frac{1}{\sqrt{a^2 - u^2}} \, du = \sin^{-1}\left(\frac{u}{a}\right) + C \quad (a > 0)$
$\int \sin(u) \, du = -\cos(au) + C$	$\int \frac{1}{a^2 + u^2} \, du = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C$
$\int \cos(u) \, du = \sin(au) + C$	$\int \frac{1}{u\sqrt{u^2 - a^2}} \, du = \frac{1}{a} \sec^{-1}\left \frac{u}{a}\right  + C \quad (a > 0)$

## Substitution Rule

$$\text{If } u = g(x), \text{ then } \int f(g(x)) g'(x) \, dx = \int f(u) \, du.$$

## Substitution Rule for Definite Integrals

$$\text{If } u = g(x), \text{ then } \int_a^b f(g(x)) g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du.$$