Introduction to Mathematical Reason	Test #2 MATH 300	March 7, 2007
Name: ______________	R. Hammack	Score: ________

Directions: Please answer the questions in the space provided. To get full credit you must show all of your work. Use of calculators and other computing or communication devices is not allowed on this test.

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1. Complete the following definitions.

   (a) Suppose $a, b \in \mathbb{Z}$. Then $a|b$ if $b = ac$ *for some* $c \in \mathbb{Z}$.

   (b) Suppose $a, b, n \in \mathbb{Z}$. Then $a \equiv b \pmod{n}$ if $n|(a-b)$.

   (c) A number r is rational if $r = \frac{a}{b}$ *for some* $a, b \in \mathbb{Z}$ *with* $b \neq 0$.

   (d) A number r is irrational if *it is not rational. (That is if* $r \neq \frac{a}{b}$ *for all* $a, b \in \mathbb{Z}$.*)*

   (e) If X and Y are sets, then $X - Y = \{x \in X \ : \ x \notin Y\}$.

2. Suppose $a, b, c \in \mathbb{Z}$, and $a \neq 0$. Prove the following statement: If $a \nmid bc$, then $a \nmid b$ and $a \nmid c$.
   [Suggestion: Contrapositive may be easiest.]

   Proof. (Contrapositive) Assume that it is not true that $a \nmid b$ and $a \nmid c$.
   Then $a|\, b$ or $a|\, c$. Thus $b = ak$ or $c = ak$ for some $k \in \mathbb{Z}$. Consider these cases separately.
   Case 1. If $b = ak$, then multiply both sides by c to get $bc = a(kc)$, which means $a \mid bc$.
   Case 2. If $c = ak$, then multiply both sides by b to get $bc = a(kb)$, which means $a \mid bc$.
   Thus, in either case $a \mid bc$, so it is not true that $a \nmid b$. ■

3. Suppose $a, b, c, d, n \in \mathbb{Z}$, and $n \geq 2$. Prove the following statement.
If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a + c \equiv b + d \pmod{n}$.

Proof. (Direct) Assume that $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$.
This means $n|(a-b)$ and $n|(c-d)$.
In turn, *this* means there are integers p and q for which $a - b = np$ and $c - d = nq$.
Adding, $(a-b)+(c-d) = np + nq$, which becomes $(a+c)-(b+d) = n(p+q)$.
From this it follows that $n|((a+c)-(b+d))$, and therefore $a + c \equiv b + d \pmod{n}$. ■

4. Let $x \in \mathbb{R}$. Prove the following statement: If $3x^4 + 1 \leq x^7 + x^3$, then $x \geq 0$.

Proof. (Contrapositive) Suppose $x \geq 0$ is not true, so $x < 0$. (That is, x is negative.)
Then $3x^4 + 1 > 0$ because the fourth power of a negative number is not negative.
Also $0 > x^7 + x^3$ because a negative number to an odd power is negative.
Therefore $3x^4 + 1 > x^7 + x^3$. (Because the left-hand side is positive and the right-hand side is negative.)
Therefore $3x^4 + 1 \leq x^7 + x^3$ is not true. ■

5. Prove that $\sqrt{2}$ is irrational. [Suggestion: proof by contradiction is probably easiest.]

Proof. Suppose to the contrary that $\sqrt{2}$ is rational. Then there exists $a, b \in \mathbb{N}$ for which $\sqrt{2} = \dfrac{a}{b}$.

We may assume that the fraction $\dfrac{a}{b}$ is reduced, **so a and b are not both even.**

From $\sqrt{2} = \dfrac{a}{b}$, we get $2 = \left(\dfrac{a}{b}\right)^2 = \dfrac{a^2}{b^2}$, so $a^2 = 2b^2$, so a^2 is even, and hence a is even.
Since a and b are not both even, it follows that b **is odd**.

Since a is even, $a = 2k$ for some integer k. The equation $a^2 = 2b^2$ then yields $(2k)^2 = 2b^2$ or $4k^2 = 2b^2$, which implies $2k^2 = b^2$. Consequently b^2 is even, so **so b is even**.

We have therefore deduced that b is even and b is odd. This contradiction proves the theorem. ■

6. Suppose A, B, C and D are sets. Prove the following statement.
If $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

Proof. (Direct) Suppose $A \subseteq C$ and $B \subseteq D$.

Let $(x, y) \in A \times B$. By definition of the Cartesian product it follows that $x \in A$ and $y \in B$.
Since $A \subseteq C$ and $B \subseteq D$, it further follows that $x \in C$ and $y \in D$.
From $x \in C$ and $y \in D$, it follows that $(x, y) \in C \times D$. (By definition of the Cartesian product.)

We have seen that $(x, y) \in A \times B$ implies $(x, y) \in C \times D$, and this means $A \times B \subseteq C \times D$. ■

FOR THE PROBLEMS ON THIS PAGE:
Decide if the statement is true or false. If it is true, prove it; if it is false, give a counterexample.

7. Let A and B be sets. If $A - B = B - A$, then $A - B = \emptyset$.

   This is TRUE.

   Proof. Suppose for the sake of contradiction that $A - B = B - A$ but $A - B \neq \emptyset$.
   Now since $A - B \neq \emptyset$, then there must be some $a \in A - B$.
   And since $a \in A - B = \{x \in A : x \notin B\}$, it follows that $a \in A$ but $a \notin B$.
   But also $a \in A - B = B - A = \{x \in B : x \notin A\}$, which means $x \notin A$.
   Thus $a \in A$ and $a \notin A$, which is a contradiction. ■

8. If $x, y \in \mathbb{R}$ and $x^2 < y^2$, then $x < y$.

   This is FALSE. Here is a counterexample.
   Let $x = 1$ and $y = -2$.
   Then $x^2 < y^2$ is $1 < 2$, which is true, but $x < y$ is false.

9. For every two sets A and B, $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

   This is FALSE. Here is a counterexample.

   Let $A = \{1\}$ and $B = \{2\}$.
   Then $\mathcal{P}(A \cup B) = \mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.
   Also $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}\} \cup \{\emptyset, \{2\}\} = \{\emptyset, \{1\}, \{2\}\}$.
   Thus we see that $\mathcal{P}(A \cup B) \neq \mathcal{P}(A) \cup \mathcal{P}(B)$.

10. Suppose A, B, C and D are sets. If $A \times B \subseteq C \times D$, then $A \subseteq C$ and $B \subseteq D$.

    This is FALSE. Here is a counterexample.

    Suppose $A = \{1\}, B = \emptyset, C = \{2\}$ and $D = \{3\}$.
    Then $A \times B = \emptyset \subseteq C \times D$, but $A \not\subseteq C$, so it is not true that $A \subseteq C$ and $B \subseteq D$.