Introduction to Mathematical Reasoning	MATH 300 R. Hammack	February 7, 2007 Test #1

Name: ________________________ Score: ________

Directions: This is a take-home test. It is due at the beginning of class on Monday, February 12. Please answer all questions in the space provided. Consider working the problems on scratch paper, then rewriting them neatly on the test. Additional copies of this test can be downloaded from my web page if needed.

- Do not discuss this test with anyone other than the instructor. Ask me if you have any questions.
- You may consult your text and notes, but **no** other source.
- To get full credit on problems 4–10, you must show all of your work.
- Each problem is worth 10 points.

1. Write each of the following sets by listing its elements or describing it with a familiar symbol.

   (a) $\{x \in \mathbb{Z} : |x| \leq 3\} = \boxed{\{-3,-2,-1,0,1,2,3\}}$

   (b) $\{X \in \mathcal{P}(\mathbb{N}) : |X \cup \{1,2,3\}| \leq 3\} = \boxed{\{\ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\ \}}$

   (c) $\{(x,y) \in \mathbb{N} \times \mathbb{R} : x^2 = 4, y^2 = 2\} = \boxed{\{(2,\sqrt{2}), (2,-\sqrt{2})\}}$

   (d) $\{(x,y) \in \mathbb{R} \times \mathbb{R} : y = x^2\} \cap \{(x,y) \in \mathbb{R} \times \mathbb{R} : y = x\} = \boxed{\{(0,0),(1,1)\}}$

   (e) $\mathbb{R} - \mathcal{P}(\mathbb{R}) = \boxed{\mathbb{R}}$ (Reason: $\mathbb{R}$ is a set of numbers and $\mathcal{P}(\mathbb{R})$ is a set of sets of numbers; no element of $\mathcal{P}(\mathbb{R})$ is an element of $\mathbb{R}$.)

2. Write each of the following sets by listing its elements or describing it with a familiar symbol.

   (a) $\mathcal{P}(\mathcal{P}(\{\emptyset\})) = \boxed{\{\ \emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\ \}}$

   (b) $\{\emptyset\} \times \{\emptyset\} = \boxed{\{(\emptyset,\emptyset)\}}$

   (c) $\emptyset \times \mathbb{N} = \boxed{\emptyset}$

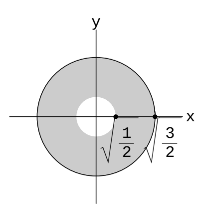
   (d) $(\mathbb{R} - \mathbb{Z}) \cap \mathbb{N} = \boxed{\emptyset}$

   (e) $\displaystyle\bigcup_{X \in \mathcal{P}(\mathbb{N})} \overline{X} = \boxed{\mathbb{N}}$ (Reason: $X \in \mathcal{P}(\mathbb{N})$ means $X \subseteq \mathbb{N}$, so $\overline{X} = \mathbb{N} - X \subseteq \mathbb{N}$. Thus the union of all such $\overline{X}$ is a subset of $\mathbb{N}$. But if $X = \emptyset$, we have $\overline{X} = \mathbb{N}$, so the union is all of $\mathbb{N}$.)

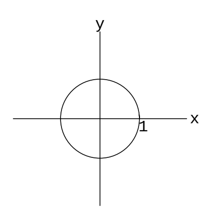
3. For each $n \in \mathbb{N}$, let $A_n = \{(x,y) \in \mathbb{R} \times \mathbb{R} : 1 - \frac{1}{n} < x^2 + y^2 \leq 1 + \frac{1}{n}\}$.

   (a) Describe the set A_2. A carefully drawn picture will suffice.
   Notice that for each n, $A_n = \{(x,y) \in \mathbb{R} \times \mathbb{R} : 1 - \frac{1}{n} < x^2 + y^2 \leq 1 + \frac{1}{n}\}$
   $= \{(x,y) \in \mathbb{R} \times \mathbb{R} : \sqrt{1 - \frac{1}{n}} < \sqrt{x^2+y^2} \leq \sqrt{1 + \frac{1}{n}}\}$. This is the set of all points (x,y) on the plane whose distance from the origin is between $\sqrt{1 - \frac{1}{n}}$ and $\sqrt{1 + \frac{1}{n}}$.
   Thus A_2 is the set of all points on the plane whose distance from the origin is between $\sqrt{1/2}$ and $\sqrt{3/2}$.

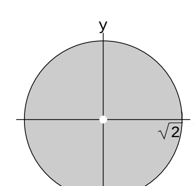
   Here it is drawn shaded on the plane $\mathbb{R} \times \mathbb{R}$.



   (b) $\displaystyle\bigcap_{n=1}^{\infty} A_n = \boxed{\{(x,y) : x^2+y^2 = 1\} \quad \text{(That is, the unit circle.)}}$



   (c) $\displaystyle\bigcup_{n=1}^{\infty} A_n = \boxed{\{(x,y) : 0 < x^2+y^2 \leq 2\} \quad \text{(i.e. solid disk of radius } \sqrt{2}\text{, minus origin)}}$



4. Write an expression that is logically equivalent to $\sim(\ \forall x, P(x) \vee \sim Q(x)\ )$ and contains only one $\sim$.

$$\begin{aligned} \sim(\ \forall x, P(x) \vee \sim Q(x)\ ) &= \exists x, \sim (P(x) \vee \sim Q(x)) \\ &= \exists x, (\sim P(x)) \wedge (\sim\sim Q(x)) \quad \text{(DeMorgan's Rule)} \\ &= \exists x, (\sim P(x)) \wedge Q(x) \quad \text{(Double negative)} \end{aligned}$$

5. Use a truth table to show $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$ are logically equivalent.

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Since the columns for $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$ are the same, these two statements are are logically equivalent.

6. Write out truth tables for the statements $P \Rightarrow Q$ and $(P \wedge \sim Q) \Rightarrow (Q \wedge \sim Q)$. How do these two statements compare?

P	Q	$\sim Q$	$P \wedge \sim Q$	$Q \vee \sim Q$	$P \Rightarrow Q$	$(P \wedge \sim Q) \Rightarrow (Q \wedge \sim Q)$
T	T	F	F	F	T	T
T	F	T	T	F	F	F
F	T	F	F	F	T	T
F	F	T	F	F	T	T

Since the columns for $P \Rightarrow Q$ and $(P \wedge \sim Q) \Rightarrow (Q \wedge \sim Q)$ agree on each line, it follows that the two statements are logically equivalent.

7. Let $x \in \mathbb{Z}$. Prove that $5x - 11$ is even if and only if x odd.

   Proof. Since this is a biconditional (if and only if) statement, we need to prove two things.
   First ($\Rightarrow$) we need to prove that if $5x - 11$ is even, then x odd.
   Then ($\Leftarrow$) we need to prove that if x is odd, then $5x - 11$ is even.

   ($\Rightarrow$) (Contrapositive) We will assume x is not odd and prove that $5x - 11$ is not even.
   Thus suppose x is not odd, so it is even, and thus there is an integer k for which $x = 2k$.
   Then $5x - 11 = 5(2k) - 11 = 10k - 11 = 10k - 12 + 1 = 2(5k - 6) + 1$, which means $5x - 11$ is odd (since it has form $2m + 1$). Since $5x - 11$ is odd, it follows that $5x - 11$ is not even.

   ($\Leftarrow$) We need to prove that if x is odd, then $5x - 11$ is even. Direct proof is appropriate here.
   Suppose x is odd, which means $x = 2k + 1$ for some integer k.
   Then $5x - 11 = 5(2k + 1) - 11 = 10k + 5 - 11 = 10k - 6 = 2(5k - 3)$.
   Hence $5x - 11$ is twice an integer, so $5x - 11$ is even. ■

8. Prove that if $n \in \mathbb{Z}$, then $n^2 - 3n + 9$ is odd.

   Proof. Suppose $n \in \mathbb{Z}$. Then n is either even or odd. Consider these two cases separately.

   Case 1. Suppose n is even. Then there is an integer k for which $n = 2k$.
   Then $n^2 - 3n + 9 = (2k)^2 - 3(2k) + 9 = 4k^2 - 6k + 9 = 4k^2 - 6k + 8 + 1 = 2(2k^2 - 3k + 4) + 1$.
   Thus $n^2 - 3n + 9 = 2(2k^2 - 3k + 4) + 1$. Letting $m = 2k^2 - 3k + 4$, this becomes $n^2 - 3n + 9 = 2m + 1$, which means $n^2 - 3n + 9$ is odd.

   Case 2. Suppose n is odd. Then there is an integer k for which $n = 2k + 1$.
   Then $n^2 - 3n + 9 = (2k + 1)^2 - 3(2k + 1) + 9 = 4k^2 + 4k + 1 - 6k - 3 + 9 = 4k^2 - 2k + 7 = 4k^2 - 2k + 6 + 1 = 2(2k^2 - k + 3) + 1$.
   Letting $m = 2k^2 - k + 3$, this becomes $n^2 - 3n + 9 = 2m + 1$, which means $n^2 - 3n + 9$ is odd.

   The two cases above show that no matter what parity n has, $n^2 - 3n + 9$ is odd. ■

9. Suppose $a, b \in \mathbb{Z}$. Prove that if ab is odd, then a and b are both odd.

   Proof. (Contrapositive) Suppose that it is not the case that a and b are both odd.
   This means that one or both of a are b is even.

   Case 1. Suppose a is even. Then there is an integer k for which $a = 2k$.
   Then $ab = (2k)b = 2(kb)$ and hence ab is even, and therefore not odd.

   Case 2. Suppose b is even. Then there is an integer k for which $b = 2k$.
   Then $ab = a(2k) = 2(ak)$ and hence ab is even, and therefore not odd.

   Either way, we see that ab is not odd, so the proof is complete. ■

10. Recall that if a and b are integers, we say that a *divides* b, written $a|b$, if there is an integer n for which $b = an$.
    Prove that if $a|b$ and $a|c$, then $a|(b + c)$.

    Proof. Suppose $a|b$ and $a|c$. This means there are integers m and n for which $b = am$ and $c = an$.
    Then $b + c = am + an = a(m + n)$. Therefore we have $b + c = ak$ for $k = m + n \in \mathbb{Z}$.
    By definition of divisible, this means $a|(b + c)$. ■