Introduction to Mathematical Reason	Sample Midterm MATH 300	March 7, 2007
Name: __________________	R. Hammack	Score: ______

Directions: Please answer the questions in the space provided. To get full credit you must show all of your work. Use of calculators and other computing or communication devices is not allowed on this test.

1. Write each of the following sets by listing its elements or describing it with a familiar symbol.

   (a) $\{x \in \mathbb{Z} : |x| \leq 3\} = \boxed{\{-3, -2, -1, 0, 1, 2, 3\}}$

   (b) $\{X \in \mathscr{P}(\mathbb{N}) : |X \cup \{1,2,3\}| \leq 3\} = \boxed{\{\, \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\} \,\}}$

   (c) $\{(x, y) \in \mathbb{N} \times \mathbb{R} : x^2 = 4, y^2 = 2\} = \boxed{\{(2, \sqrt{2}), (2, -\sqrt{2})\}}$

   (d) $\{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2\} \cap \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x\} = \boxed{\{(0,0), (1,1)\}}$

   (e) $\mathbb{R} - \mathscr{P}(\mathbb{R}) = \boxed{\mathbb{R}}$ (Reason: $\mathbb{R}$ is a set of numbers and $\mathscr{P}(\mathbb{R})$ is a set of sets of numbers; no element of $\mathscr{P}(\mathbb{R})$ is an element of $\mathbb{R}$.)

2. Write each of the following sets by listing its elements or describing it with a familiar symbol.

   (a) $\mathscr{P}(\mathscr{P}(\{\emptyset\})) = \boxed{\{\, \emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\} \,\}}$

   (b) $\{\emptyset\} \times \{\emptyset\} = \boxed{\{(\emptyset, \emptyset)\}}$

   (c) $\emptyset \times \mathbb{N} = \boxed{\emptyset}$

   (d) $(\mathbb{R} - \mathbb{Z}) \cap \mathbb{N} = \boxed{\emptyset}$

   (e) $\displaystyle\bigcup_{X \in \mathcal{P}(\mathbb{N})} \overline{X} = \boxed{\mathbb{N}}$ (Reason: $X \in \mathscr{P}(\mathbb{N})$ means $X \subseteq \mathbb{N}$, so $\overline{X} = \mathbb{N} - X \subseteq \mathbb{N}$. Thus the union of all such $\overline{X}$ is a subset of $\mathbb{N}$. But if $X = \emptyset$, we have $\overline{X} = \mathbb{N}$, so the union is all of $\mathbb{N}$.)

3. (a) Suppose you know that P is false, and that the statement $(R \Rightarrow S) \Leftrightarrow (P \wedge Q)$ is true.
   Can the true/false values of R and S be determined? Explain. (This can be done without a truth table.)

   Since P is false, it must be the case that $P \wedge Q$ is also false.
   Given this and the fact that $(R \Rightarrow S) \Leftrightarrow (P \wedge Q)$ is true, the truth table for $\Leftrightarrow$ shows that $R \Rightarrow S$ is false.
   But from the truth table for $\Rightarrow$, the only way that $R \Rightarrow S$ can be false is if $\boxed{\text{R is TRUE and S is FALSE.}}$

   (b) Write an expression that is logically equivalent to $(\sim P) \vee (\sim Q)$ and contains only one $\sim$.

   By DeMorgan's Law, $(\sim P) \vee (\sim Q)$ is logically equivalent to $\sim (P \wedge Q)$.

4. Write out a truth table to decide if $(\sim P) \wedge (P \Rightarrow Q)$ and $\sim (Q \Rightarrow P)$ are logically equivalent.

P	Q	$\sim P$	$P \Rightarrow Q$	$Q \Rightarrow P$	$(\sim P) \wedge (P \Rightarrow Q)$	$\sim (Q \Rightarrow P)$
T	T	F	T	T	**F**	**F**
T	F	F	F	T	**F**	**F**
F	T	T	T	F	**T**	**T**
F	F	T	T	T	**T**	**F**

Since the final two columns are not the same, the two expressions $(\sim P) \wedge (P \Rightarrow Q)$ and $\sim (Q \Rightarrow P)$ are **NOT logically equivalent.**

5. Suppose $a, b, c \in \mathbb{Z}$, and $a \neq 0$. Prove the following statement: If $a \nmid bc$, then $a \nmid b$ and $a \nmid c$. [Suggestion: Contrapositive may be easiest.]

   Proof. (Contrapositive) Assume that it is not true that $a \nmid b$ and $a \nmid c$.
   Then $a \mid b$ or $a \mid c$. Thus $b = ak$ or $c = ak$ for some $k \in \mathbb{Z}$. Consider these cases separately.
   Case 1. If $b = ak$, then multiply both sides by c to get $bc = a(kc)$, which means $a \mid bc$.
   Case 2. If $c = ak$, then multiply both sides by b to get $bc = a(kb)$, which means $a \mid bc$.
   Thus, in either case $a \mid bc$, so it is not true that $a \nmid b$. ■

6. Prove that $\sqrt{2}$ is irrational. [Suggestion: proof by contradiction is probably easiest.]

   Proof. Suppose to the contrary that $\sqrt{2}$ is rational. Then there exists $a, b \in \mathbb{N}$ for which $\sqrt{2} = \frac{a}{b}$.
   We may assume that the fraction $\frac{a}{b}$ is reduced, **so a and b are not both even.**

   From $\sqrt{2} = \frac{a}{b}$, we get $2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$, so $a^2 = 2b^2$, so a^2 is even, and hence a is even.
   Since a and b are not both even, it follows that b **is odd**.

   Since a is even, $a = 2k$ for some integer k. The equation $a^2 = 2b^2$ then yields $(2k)^2 = 2b^2$ or $4k^2 = 2b^2$, which implies $2k^2 = b^2$. Consequently b^2 is even, so **so b is even**.

   We have therefore deduced that b is even and b is odd. This contradiction proves the theorem. ■

7. Suppose A and B are sets. Prove that $\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B)$.

   Proof. To prove $\mathscr{P}(A) \cap \mathscr{P}(B) = \mathscr{P}(A \cap B)$, we must prove $\mathscr{P}(A) \cap \mathscr{P}(B) \subseteq \mathscr{P}(A \cap B)$ and $\mathscr{P}(A \cap B) \subseteq \mathscr{P}(A) \cap \mathscr{P}(B)$.

   First we will show that $\mathscr{P}(A \cap B) \subseteq \mathscr{P}(A) \cap \mathscr{P}(B)$.
   Suppose $X \in \mathscr{P}(A) \cap \mathscr{P}(B)$.
   By definition of intersection, this means $X \in \mathscr{P}(A)$ and $X \in \mathscr{P}(B)$.
   By the definition of power sets, this means $X \subseteq A$ and $X \subseteq B$.
   Thus, any element $x \in X$ is in both A and B, so $x \in A \cap B$. Hence $X \subseteq A \cap B$, which means $X \in \mathscr{P}(A \cap B)$.
   We have seen that $X \in \mathscr{P}(A) \cap \mathscr{P}(B)$ implies $X \in \mathscr{P}(A \cap B)$, so $\mathscr{P}(A) \cap \mathscr{P}(B) \subseteq \mathscr{P}(A \cap B)$.

   Next we will show that $\mathscr{P}(A) \cap \mathscr{P}(B) \subseteq \mathscr{P}(A \cap B)$.
   Suppose $X \in \mathscr{P}(A \cap B)$.
   By definition of the power set, this means $X \subseteq A \cap B$.
   Thus any element $x \in X$ is in $A \cap B$, so $x \in A$ and $x \in B$. Hence $X \subseteq A$ and $X \subseteq B$.
   Thus $X \in \mathscr{P}(A)$ and $X \in \mathscr{P}(B)$, so $X \in \mathscr{P}(A) \cap \mathscr{P}(B)$, by definition of intersection.
   We have seen that $X \in \mathscr{P}(A \cap B)$ implies $X \in \mathscr{P}(A) \cap \mathscr{P}(B)$, so $\mathscr{P}(A \cap B) \subseteq \mathscr{P}(A) \cap \mathscr{P}(B)$.

   The previous two paragraphs imply $\mathscr{P}(A) \cap \mathscr{P}(B) = \mathscr{P}(A \cap B)$. ■

8. Prove that if $n \in \mathbb{Z}$, then $n^2 - 3n + 9$ is odd.

   Proof. Suppose $n \in \mathbb{Z}$. Then n is either even or odd. Consider these two cases separately.

   Case 1. Suppose n is even. Then there is an integer k for which $n = 2k$.
   Then $n^2 - 3n + 9 = (2k)^2 - 3(2k) + 9 = 4k^2 - 6k + 9 = 4k^2 - 6k + 8 + 1 = 2(2k^2 - 3k + 4) + 1$.
   Thus $n^2 - 3n + 9 = 2(2k^2 - 3k + 4) + 1$. Letting $m = 2k^2 - 3k + 4$, this becomes $n^2 - 3n + 9 = 2m + 1$, which means $n^2 - 3n + 9$ is odd.

   Case 2. Suppose n is odd. Then there is an integer k for which $n = 2k + 1$.
   Then $n^2 - 3n + 9 = (2k + 1)^2 - 3(2k + 1) + 9 = 4k^2 + 4k + 1 - 6k - 3 + 9 = 4k^2 - 2k + 7 = 4k^2 - 2k + 6 + 1 = 2(2k^2 - k + 3) + 1$.
   Letting $m = 2k^2 - k + 3$, this becomes $n^2 - 3n + 9 = 2m + 1$, which means $n^2 - 3n + 9$ is odd.

   The two cases above show that no matter what parity n has, $n^2 - 3n + 9$ is odd. ■

9. Suppose $a, b \in \mathbb{Z}$. Prove that $(a-3)b^2$ is even if and only if a is odd or b is even.

   Proof. First we will prove that if $(a-3)b^2$ is even, then a is odd or b is even. For this we use contrapositive proof. Suppose it is not the case that a is odd or b is even. Then by DeMorgan's law, a is even and b is odd. Thus there are integers m and n for which $a = 2m$ and $b = 2n+1$. Now observe $(a-3)b^2 = (2m-3)(2n+1)^2 = (2m-3)(4n^2+4n+1)\ 2mn^2+8mn+2m-6n-3 = 2mn^2+8mn+2m-6n-4+1 = 2(mn^2+4mn+m-3n-2)+1$. This shows $(a-3)b^2$ is odd, so it's not even.

   Conversely, we need to show that if a is odd or b is even, then $(a-3)b^2$ is even. For this we use direct proof, with cases. **Case 1.** Suppose a is odd. Then $a = 2m+1$ for some integer m. Thus $(a-3)b^2 = (2m+1-3)b^2 = (2m-2)b^2 = 2(m-1)b^2$. Thus in this case $(a-3)b^2$ is even.
   Case 2. Suppose b is even. Then $b = 2n$ for some integer n. Thus $(a-3)b^2 = (a-3)(2n)^2 = (a-3)4n^2 = 2(a-3)2n^2 =$. Thus in this case $(a-3)b^2$ is even.

   Therefore, in any event, $(a-3)b^2$ is even. ∎

10. Suppose A, B and C are sets. If $B \subseteq C$, then $A \times B \subseteq A \times C$.

    Proof. This is a conditional statement, and we'll prove it with direct proof. Suppose $B \subseteq C$. (Now we need to prove $A \times B \subseteq A \times C$.)

    Suppose $(a, b) \in A \times B$. Then by definition of the Cartesian product we have $a \in A$ and $b \in B$. But since $b \in B$ and $B \subseteq C$, we have $b \in C$. Since $a \in A$ and $b \in C$, it follows that $(a, b) \in A \times C$. Now we've shown $(a, b) \in A \times B$ implies $(a, b) \in A \times C$, so $A \times B \subseteq A \times C$.

    In summary, we've shown that if $B \subseteq C$, then $A \times B \subseteq A \times C$. This completes the proof. ∎