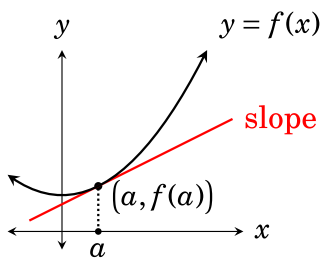
CHAPTER 16

The Derivative of a Function

This chapter introduces the primary concept in calculus – the idea of the *derivative* of a function. Derivatives are closely related to slopes of tangent lines, so let's begin by recalling the last chapter's Theorem 15.1. It gave two limits for the slope of the tangent line to a function's graph.

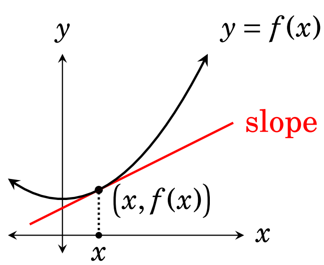
$$\left(\begin{array}{l}\text{Slope of tangent to}\\ y=f(x)\text{ at point }(a,f(a))\end{array}\right) = \lim_{z\to a}\frac{f(z)-f(a)}{z-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$



$$\text{slope} = \lim_{z\to a}\frac{f(z)-f(a)}{z-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Let's make a very minor adjustment to this. The number $x=a$ could be any point in the domain of f, so let's just call it x instead of a. In other words, replace all the a's in the above box with x's. We get the following.

$$\left(\begin{array}{l}\text{Slope of tangent to}\\ y=f(x)\text{ at point }(x,f(x))\end{array}\right) = \lim_{z\to x}\frac{f(z)-f(x)}{z-x} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$



$$\text{slope} = \lim_{z\to x}\frac{f(z)-f(x)}{z-x} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

This gives the slope of the tangent line to $y=f(x)$ at *any* point $(x,f(x))$.

So if you want to find the slope of the tangent to the graph of $y = f(x)$ at the point $(x, f(x))$, just work out one of the limits $\lim\limits_{z\to x} \frac{f(z)-f(x)}{z-x}$ or $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$. As we will see, we can usually do these limits even though we don't have a specific value for x. But the answer will depend on x, because different values of x yield different points $(x, f(x))$, hence different tangent lines, hence different slopes. Our first example illustrates this.

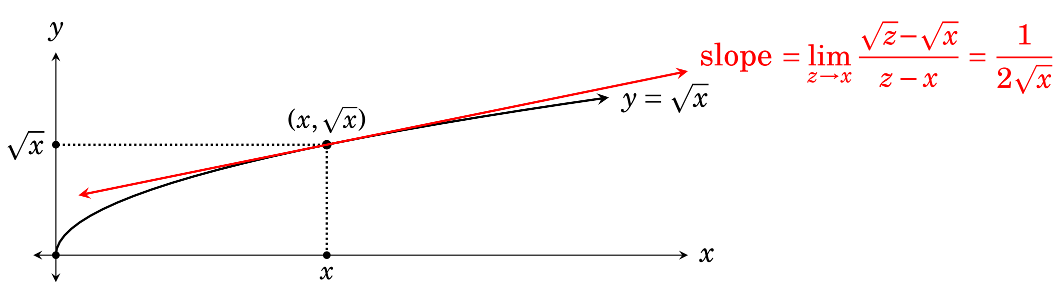
Example 16.1 Find the slope of the tangent line to the graph of $f(x) = \sqrt{x}$ at the point $(x, f(x)) = (x, \sqrt{x})$.

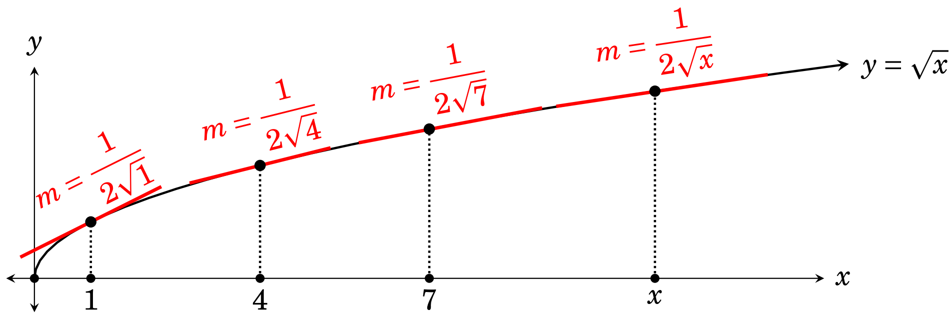
Let's try this with the first limit, so the slope in question is $\lim\limits_{z\to x} \dfrac{f(z)-f(x)}{z-x}$. In this limit the variable z is approaching a number x. In working the limit, think of x as a specific (but unspecified) number.

$$
\begin{aligned}
\text{Slope} = \lim_{z\to x} \frac{f(z)-f(x)}{z-x} &= \lim_{z\to x} \frac{\sqrt{z}-\sqrt{x}}{z-x} \\
&= \lim_{z\to x} \frac{\sqrt{z}-\sqrt{x}}{z-x} \cdot \frac{\sqrt{z}+\sqrt{x}}{\sqrt{z}+\sqrt{x}} && \text{(multiply by conjugate)} \\
&= \lim_{z\to x} \frac{z-x}{(z-x)\left(\sqrt{z}+\sqrt{x}\right)} && \text{(FOIL top)} \\
&= \lim_{z\to x} \frac{1}{\sqrt{z}+\sqrt{x}} && \text{(cancel)} \\
&= \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}} && \text{(apply limit law \& simplify)}
\end{aligned}
$$

Answer: The tangent to $y = \sqrt{x}$ at the point $(x, f(x))$ has slope $\dfrac{1}{2\sqrt{x}}$.

So once we have worked out the limit, we have a tidy formula $\dfrac{1}{2\sqrt{x}}$ that gives *any tangent slope at any point*, as illustrated below.





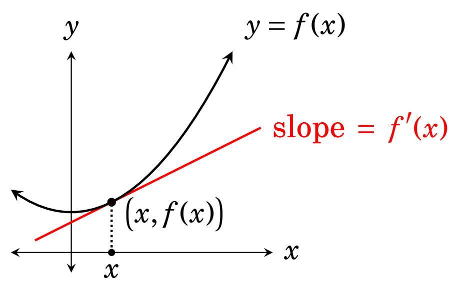
Thus, for instance, as shown above, the tangent slope at $(1, \sqrt{1})$ is $\frac{1}{2\sqrt{1}} = \frac{1}{2}$. The tangent slope at $(4, \sqrt{4})$ is $\frac{1}{2\sqrt{4}} = \frac{1}{4}$. At $(7, \sqrt{7})$ the slope is $\frac{1}{2\sqrt{7}}$. ✍

Example 16.1 above motivates the main definition in calculus. In it we have a function $f(x) = \sqrt{x}$ and *another* function $\frac{1}{2\sqrt{x}}$ that gives the slope of the tangent line to $f(x)$ at $(x, f(x))$. We are going to call this other function the **derivative** of $f(x)$ and denote it as $f'(x)$. Here is the exact definition.

Definition 16.1

The **derivative** of a function $f(x)$ is another function $f'(x)$ defined as

$$\begin{aligned} f'(x) &= \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \left(\begin{array}{l} \text{Slope of tangent to} \\ \text{graph of } y = f(x) \\ \text{at point } (x, f(x)) \end{array} \right). \end{aligned}$$



This is the most important definition in calculus. Given a function f, the definition defines its derivative f' in two ways—as two limits—and assigns to it a meaning: slope of the tangent to $y = f(x)$ at $(x, f(x))$.

In Example 16.1 we found that the tangent to the graph of $y = f(x) = \sqrt{x}$ at $(x, f(x))$ has slope $\dfrac{1}{2\sqrt{x}}$. Thus the derivative of $f(x) = \sqrt{x}$ is $f'(x) = \dfrac{1}{2\sqrt{x}}$. Still, our next example reworks this, finding the derivative using the language of Definition 16.1.

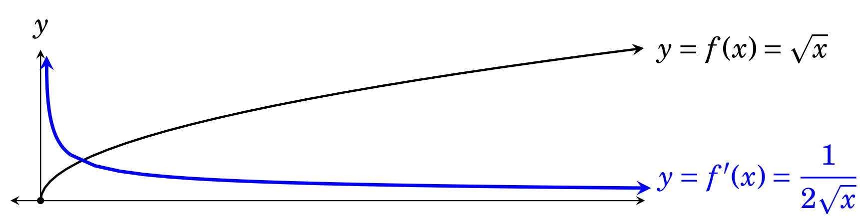
Example 16.2 Find the derivative of the function $f(x)=\sqrt{x}$.

This time let's try the second limit in Definition 16.1 (though the first would work just as well). The derivative is

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} && \text{(Definition 16.1)}\\
&= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} && (f(x)=\sqrt{x})\\
&= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} && \text{(multiply by conjugate)}\\
&= \lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} && \text{(FOIL top)}\\
&= \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\
&= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} && \text{(cancel)}\\
&= \frac{1}{\sqrt{x+0}+\sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}} && \text{(limit law \& simplify).}
\end{aligned}$$

Answer: The derivative of the function $f(x)=\sqrt{x}$ is the function $f'(x)=\dfrac{1}{2\sqrt{x}}$.

The graphs of $f(x)$ and its derivative $f'(x)$ are shown below. Comparing the graphs reinforces the fact that $f'(x)$ gives the slope of the tangent line to $y=f(x)$ at x. As x approaches 0, the tangent to $y=f(x)$ gets steeper and steeper, approaching vertical, and indeed $f'(x)$ becomes increasingly large. (In fact, $f'(x)$ has a vertical asymptote at $x=0$.)



And $f'(x)$ gets small as x gets big because the tangents to $f(x)$ get closer to horizontal (slope 0). (In fact, $y=0$ is a horizontal asymptote for $f'(x)$.) ✍

In doing the limit $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, we must work out $f(x+h)$. This involves inserting an $x+h$ in every occurrence of x in the expression for $f(x)$. For instance, our next example concerns the function $f(x)=x^2-x+2$. In this case, $f(x+h)=(x+h)^2-(x+h)+2$.

Example 16.3 Find the derivative $f'(x)$ of the function $f(x) = x^2 - x + 2$.

Definition 16.1 says we can find the $f'(x)$ using either one of two limits. For the sake of illustration, we will do it both ways. The first limit gives

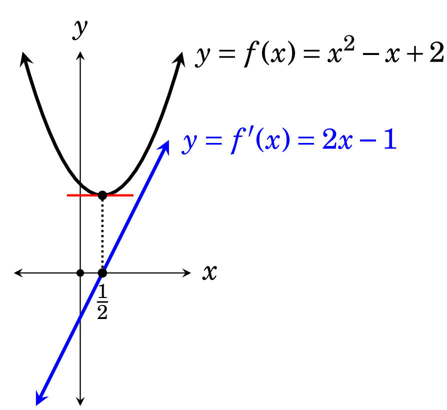
$$\begin{aligned}
f'(x) &= \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \\
&= \lim_{z \to x} \frac{\big(z^2 - z + 2\big) - \big(x^2 - x + 2\big)}{z - x} \\
&= \lim_{z \to x} \frac{z^2 - x^2 - z + x}{z - x} \\
&= \lim_{z \to x} \frac{(z - x)(z + x) - (z - x)}{z - x} \\
&= \lim_{z \to x} \frac{(z - x)\big((z + x) - 1\big)}{z - x} \\
&= \lim_{z \to x} \big(z + x - 1\big) \;=\; x + x - 1 \;=\; \boxed{2x - 1}
\end{aligned}$$

Therefore the derivative is the function $\boxed{f'(x) = 2x - 1}$. We are done, but let's now also compute this same answer using the second limit in Definition 16.1.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\big((x + h)^2 - (x + h) + 2\big) - \big(x^2 - x + 2\big)}{h} \\
&= \lim_{h \to 0} \frac{\big(x^2 + 2xh + h^2 - (x + h) + 2\big) - \big(x^2 - x + 2\big)}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2 - h}{h} \\
&= \lim_{h \to 0} \frac{h(2x + h - 1}{h} \\
&= \lim_{h \to 0} \big(2x + h - 1\big) \;=\; 2x + 0 - 1 \;=\; \boxed{2x - 1}
\end{aligned}$$

We have now found the derivative $f'(x) = 2x - 1$ in two ways.

The graphs of $f(x)$ and $f'(x)$ are shown on the right. Notice that the derivative equals 0 at $x = \frac{1}{2}$, which is precisely where the tangent to $f(x)$ has slope 0. The derivative is positive where the tangents to $f(x)$ have positive slope and negative where the tangents have negative slope. ✍



Example 16.4 Find the derivative of the function $f(x) = x^3$.

For this, let's use the limit $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ from Definition 16.1. Notice that this is going to involve the expression $f(x+h) = (x+h)^3$. Let's work this out before beginning the limit:

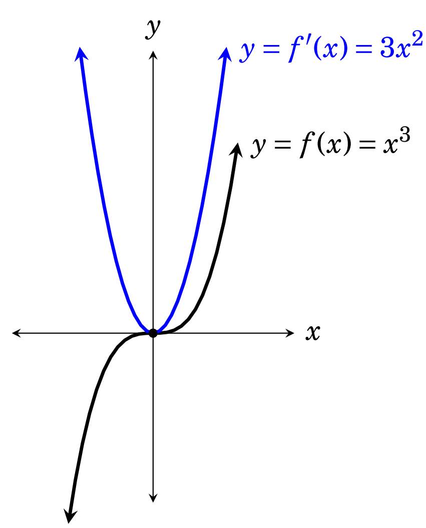
$$\begin{aligned}
(x+h)^3 &= (x+h)(x+h)^2 \\
&= (x+h)\left(x^2+2xh+h^2\right) \\
&= x^3+2x^2h+xh^2 + hx^2+2xh^2+h^3 \\
&= x^3+3x^2h+3xh^2+h^3.
\end{aligned}$$

Now let's do the limit:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3-x^3}{h} \\
&= \lim_{h\to 0} \frac{\left(x^3+3x^2h+3xh^2+h^3\right)-x^3}{h} \\
&= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} \\
&= \lim_{h\to 0} \frac{h\left(3x^2+3xh+h^2\right)}{h} \\
&= \lim_{h\to 0} \left(3x^2+3xh+h^2\right) \\
&= 3x^2+3x\cdot 0+0^2 \\
&= 3x^2.
\end{aligned}$$

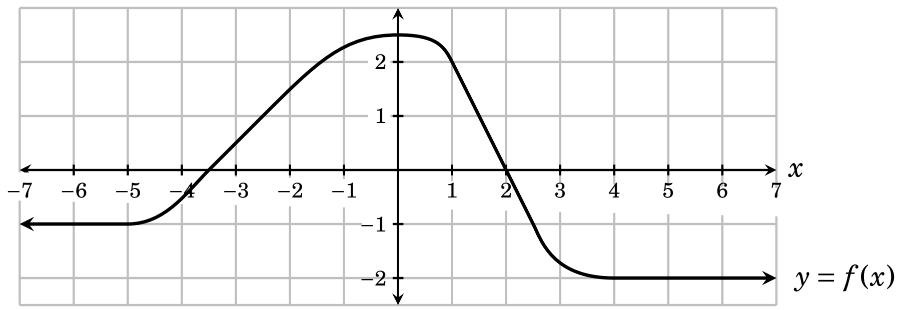
Therefore the derivative of the function $f(x) = x^3$ is the function $f'(x) = 3x^2$.

The graph of $f(x)$ and its derivative are shown on the right. This reinforces the *meaning* of the derivative as giving the slopes of tangent lines to $f(x)$. The tangent lines to $y = x^3$ all have positive slope, and indeed $f'(x) = 3x^2$ is always positive. The one exception is that the tangent to $y = x^3$ is horizontal at $x = 0$, and indeed $f'(0) = 0$. ✍

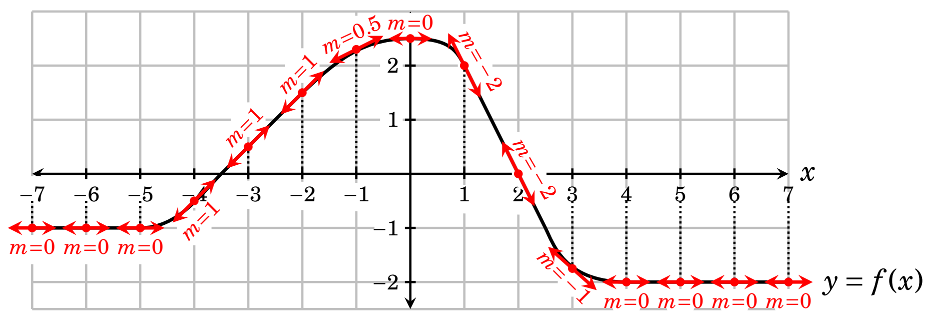


In our final example, we are given the graph of a function and asked for the graph of its derivative. Since we don't have an algebraic expression for the function, we can't get the derivative as a limit.

Example 16.5 The graph of a function $f(x)$ is shown. Sketch the graph of its derivative $f'(x)$.

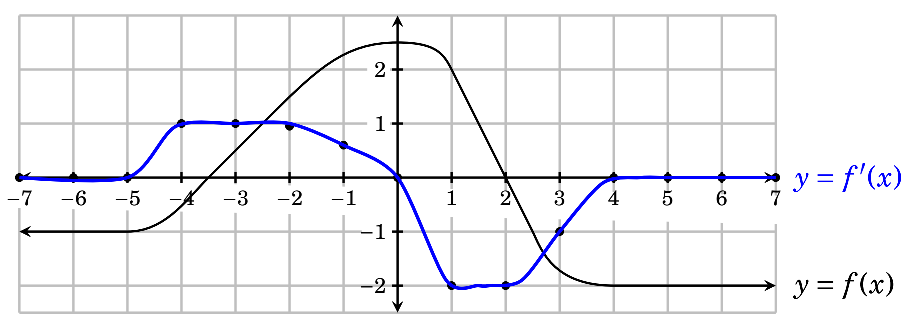


Our strategy is to plot points $(x, f'(x))$ on the graph of $f'(x)$ and connect dots. Pick some sample x values, say the grid points $-7. -6, -5, \ldots, 7$. For each such x, we know $f'(x)$ is the slope of the tangent line to $f(x)$ at $(x, f(x))$. So sketch in the tangent lines and estimate their slopes, as done below.



Here is a table listing the pairs $(x, f'(x))$ on the graph of the derivative $f'(x)$. Under each x we enter the tangent slope at x from the drawing above.

x	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6.	7
$f'(x)$	0	0	0	1	1	1	0.5	0	-2	-2	-1	0	0	0.	0



Plotting these points and connecting dots, we get a sketch of $y = f'(x)$. ✍

Exercises for Chapter 16

In problems 1–8 use Definition 16.1 to find the derivative of the given function.

1. $f(x) = x^2 + 1$

2. $f(x) = \dfrac{1}{x^2}$

3. $f(x) = 4 - 3x^2$

4. $f(x) = \sqrt{6x}$

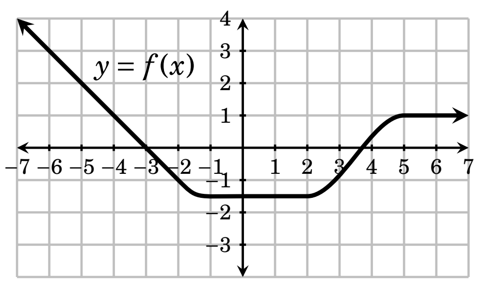
5. $f(x) = \sqrt{3x}$

6. $f(x) = \dfrac{1}{7x}$

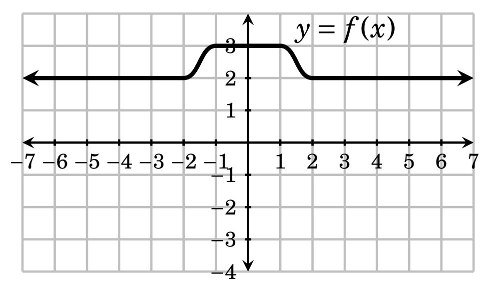
7. $f(x) = \dfrac{3}{5x^2}$

8. $f(x) = 3x - x^2$

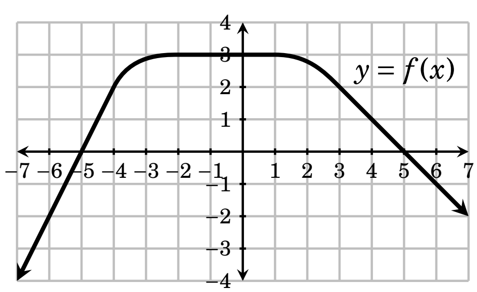
9. A function $f(x)$ is graphed below. Sketch the graph of $f'(x)$.



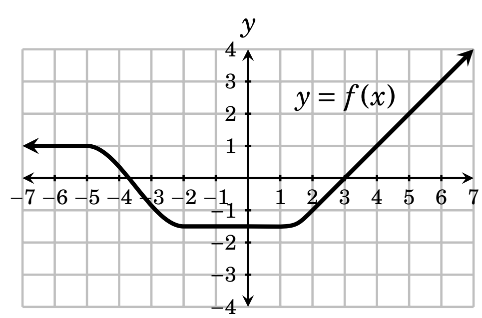
10. A function $f(x)$ is graphed below. Sketch the graph of $f'(x)$.



11. A function $f(x)$ is graphed below. Sketch the graph of $f'(x)$.



12. A function $f(x)$ is graphed below. Sketch the graph of $f'(x)$.



Exercises Solutions for Chapter 16

1. Find the derivative of $f(x) = x^2 + 1$.

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{(z^2+1) - (x^2+1)}{z - x} = \lim_{z \to x} \frac{z^2 - x^2}{z - x} = \lim_{z \to x} \frac{(z+x)(z-x)}{z - x}$$
$$= \lim_{z \to x} (z + x) = 2x$$

$$\boxed{\text{Therefore } f'(x) = 2x.}$$

3. Find the derivative of $f(x) = 4 - 3x^2$.

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{(4 - 3z^2) - (4 - 3x^2)}{z - x} = \lim_{z \to x} \frac{-3z^2 + 3x^2}{z - x}$$
$$= \lim_{z \to x} \frac{-3\left(z^2 - x^2\right)}{z - x} = \lim_{z \to x} \frac{-3((z-x)(z+x))}{z - x} = \lim_{z \to x} -3(z + x) = -3(x + x) = -6x$$

$$\boxed{\text{Therefore } f'(x) = -6x.}$$

5. Find the derivative of $f(x) = \sqrt{3x}$.

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{\sqrt{3z} - \sqrt{3x}}{z - x} = \lim_{z \to x} \frac{\sqrt{3z} - \sqrt{3x}}{z - x} \cdot \frac{\sqrt{3z} + \sqrt{3x}}{\sqrt{3z} + \sqrt{3x}}$$
$$= \lim_{z \to x} \frac{3z - 3x}{(z-x)\left(\sqrt{3z}+\sqrt{3x}\right)} = \lim_{z \to x} \frac{3(z - x)}{(z-x)\left(\sqrt{3z}+\sqrt{3x}\right)} = \lim_{z \to x} \frac{3}{\sqrt{3z}+\sqrt{3x}} = \frac{3}{2\sqrt{3x}}$$

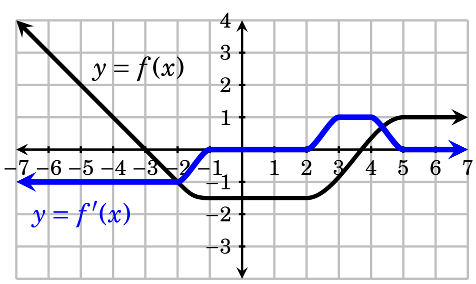
$$\boxed{\text{Therefore } f'(x) = \frac{3}{2\sqrt{3x}}.}$$

7. Find the derivative of $f(x) = \dfrac{3}{5x^2}$.

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{\frac{3}{5z^2} - \frac{3}{5x^2}}{z - x} = \lim_{z \to x} \frac{\frac{3}{5z^2} - \frac{3}{5x^2}}{z - x} \cdot \frac{5z^2x^2}{5z^2x^2} = \lim_{z \to x} \frac{3x^2 - 3z^2}{(z - x)5z^2x^2}$$
$$= \lim_{z \to x} \frac{3(x^2 - z^2)}{(z - x)5z^2x^2} = \lim_{z \to x} \frac{3(x - z)(x + z)}{(z - x)5z^2x^2} = \lim_{z \to x} \frac{-3(z + x)}{5z^2x^2} = \frac{-3(x + x)}{5x^2x^2} = \frac{-6x}{5x^4} = -\frac{6}{5x^3}$$

$$\boxed{\text{Therefore } f'(x) = -\frac{6}{5x^3}.}$$

9. A function $f(x)$ is graphed below. Sketch the graph of $f'(x)$.



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