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MIDTERM EXAM ♣

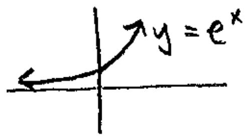
MATH 200
October 26, 2022

1. (35 pts.) Evaluate the following limits. Show steps, as appropriate.

(a) $\lim_{x\to 0} \frac{1-\cos(x)}{x} = \boxed{0}$ (Standard fact)

(b) $\lim_{x\to\infty} \sin^{-1}\left(\frac{1}{2}+\frac{1}{x}\right) = \sin^{-1}\left(\lim_{x\to\infty}\left(\frac{1}{2}+\frac{1}{x}\right)\right) = \sin^{-1}\left(\frac{1}{2}+0\right) = \sin^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{6}}$

(c) $\lim_{x\to -\infty} e^x = \boxed{0}$



(d) $\lim_{x\to\infty} \frac{x^2-4x+3}{x^2+4x-5} = \lim_{x\to\infty} \frac{x^2-4x+3}{x^2+4x-5}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1-\frac{4}{x}+\frac{3}{x^2}}{1+\frac{4}{x}-\frac{5}{x^2}} = \frac{1-0+0}{1+0-0} = \boxed{1}$

(e) $\lim_{x\to 1} \frac{x^2-4x+3}{x^2+4x-5} = \lim_{x\to 1}\frac{\cancel{(x-1)}(x-3)}{\cancel{(x-1)}(x+5)} = \lim_{x\to 1}\frac{x-3}{x+5} = \frac{1-3}{1+5} = \frac{-2}{6} = \boxed{\frac{-1}{3}}$

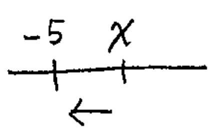
$\frac{0}{0}$

(f) $\lim_{x\to -5^+} \frac{x^2-4x+3}{x^2+4x-5} = \lim_{x\to -5^+}\frac{x-3}{x+5} = \boxed{-\infty}$

approaching -8

approaching 0, positive

factor and cancel as above



(g) $\lim_{x\to 16} \frac{\sqrt{x}-4}{x-16} = \lim_{x\to 16}\frac{\sqrt{x}-4}{x-16}\cdot\frac{\sqrt{x}+4}{\sqrt{x}+4} = \lim_{x\to 16}\frac{x+4\sqrt{x}-4\sqrt{x}-16}{(x-16)(\sqrt{x}+4)}$

$\frac{0}{0}$

$= \lim_{x\to 16}\frac{\cancel{x-16}}{\cancel{(x-16)}(\sqrt{x}+4)} = \lim_{x\to 16}\frac{1}{\sqrt{x}+4} = \frac{1}{\sqrt{16}+4} = \boxed{\frac{1}{8}}$

2. (5 pts.) Use a limit definition of a derivative to find the derivative of $f(x) = 2x^2 - 3$.

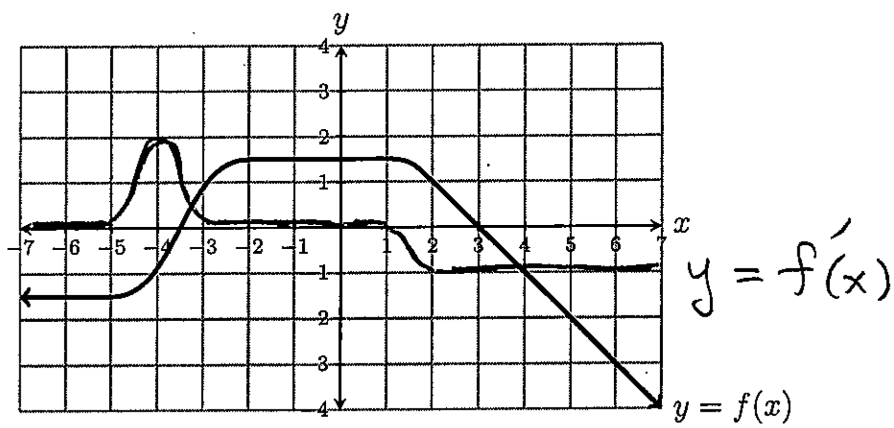
$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{(2z^2 - 3) - (2x^2 - 3)}{z - x}$$

$$= \lim_{z \to x} \frac{2z^2 - 3 - 2x^2 + 3}{z - x} = \lim_{z \to x} \frac{2z^2 - 2x^2}{z - x}$$

$$= \lim_{z \to x} \frac{2(z^2 - x^2)}{z - x} = \lim_{z \to x} \frac{2(z+x)\cancel{(z-x)}}{\cancel{z-x}}$$

$$= \lim_{z \to x} 2(z + x) = 2(x + x) = 2(2x) = \boxed{4x}$$

3. (5 pts.) The graph of a function $f(x)$ is shown. Using the same grid, sketch the graph of $f'(x)$.



4. (5 pts.) Find all points (x, y) on the graph of $y = x + \frac{1}{x-3}$ where the tangent line is horizontal.

Solve $y' = 0$

$$1 - \frac{1}{(x-3)^2} = 0$$

$$(x-3)^2\left(1 - \frac{1}{(x-3)^2}\right) = 0(x-3)^2$$

$$(x-3)^2 - 1 = 0$$

$$x^2 - 6x + 9 - 1 = 0$$

$$x^2 - 6x + 8 = 0$$

$$(x-2)(x-4) = 0$$

$\boxed{x = 2}$ $\boxed{x = 4}$

Points:

$$\left(2, 2 + \frac{1}{2-3}\right) = \left(2, 2 + \frac{1}{-1}\right) = \boxed{(2, 1)}$$

$$\left(4, 4 + \frac{1}{4-3}\right) = \left(4, 4 + \frac{1}{1}\right) = \boxed{(4, 5)}$$

5. (30 pts.) Find the indicated derivatives.

(a) $f(\theta) = \sqrt{\theta^3} + \ln(\theta)$ $= \theta^{3/2} + \ln(\theta)$

$f'(\theta) =$ $\frac{3}{2}\theta^{\frac{1}{2}} + \frac{1}{\theta} = \boxed{\frac{3}{2}\sqrt{\theta} + \frac{1}{\theta}}$

$f''(\theta) =$ $\frac{3}{4}\theta^{-1/2} - \frac{1}{\theta^2} = \boxed{\frac{3}{4\sqrt{\theta}} - \frac{1}{\theta^2}}$

(b) $D_x\left[\frac{x^3+x^2+1}{x}\right] =$ $\frac{(3x^2+2x)x - (x^3+x^2+1)\cdot 1}{x^2}$

$= \frac{3x^3+2x^2-x^3-x^2-1}{x^2} = \boxed{\frac{2x^3+x^2-1}{x^2}} = \boxed{2x+1-\frac{1}{x^2}}$

(c) $D_x\left[4xe^{\sqrt{3x+1}}\right] =$ $D_x[4x]e^{\sqrt{3x+1}} + 4x D_x\left[e^{\sqrt{3x+1}}\right]$

$= 4e^{\sqrt{3x+1}} + 4xe^{\sqrt{3x+1}} D_x\left[(3x+1)^{\frac{1}{2}}\right] = 4e^{\sqrt{3x+1}} + 4xe^{\sqrt{3x+1}}\frac{3}{2\sqrt{3x+1}}$

$= \boxed{4e^{\sqrt{3x+1}} + \frac{6xe^{\sqrt{3x+1}}}{\sqrt{3x+1}}}$

(d) $D_x\left[\left(\sec\left(\ln(x)\right)\right)^3\right] =$ $3\left(\sec(\ln(x))\right)^2 D_x\left[\sec(\ln(x))\right]$

$= \boxed{3\left(\sec(\ln(x))\right)^2 \sec(\ln(x))\tan(\ln(x))\frac{1}{x}}$

$= \boxed{3\sec^3(\ln(x))\tan(\ln(x))\frac{1}{x}}$

(e) $D_x\left[\sin^{-1}(\pi x)\right] =$ $\frac{1}{\sqrt{1-(\pi x)^2}} D_x[\pi x] = \boxed{\frac{\pi}{\sqrt{1-\pi^2x^2}}}$

6. (5 pts.) Consider the equation $y\sin(x) = y^3$. Use implicit differentiation to find $\frac{dy}{dx}$.

$$D_x\left[y\sin(x)\right] = D_x\left[y^3\right]$$

$$\frac{dy}{dx}\sin(x) + y\cos(x) = 3y^2\frac{dy}{dx}$$

$$\frac{dy}{dx}\sin(x) - 3y^2\frac{dy}{dx} = -y\cos(x)$$

$$\frac{dy}{dx}\left(\sin(x) - 3y^2\right) = -y\cos(x)$$

$$\boxed{\frac{dy}{dx} = \frac{-y\cos(x)}{\sin(x) - 3y^2}}$$

7. (5 pts.) Use logarithmic differentiation to find the derivative of $f(x) = x^{1+2x}$.

$$y = x^{1+2x}$$

$$\ln|y| = \ln\left|x^{1+2x}\right|$$

$$\ln|y| = (1+2x)\ln|x|$$

$$D_x\left[\ln|y|\right] = D_x\left[(1+2x)\ln|x|\right]$$

$$\frac{y'}{y} = 2\ln|x| + (1+2x)\frac{1}{x}$$

$$y' = y\left(2\ln|x| + \frac{1+2x}{x}\right)$$

$$\boxed{f'(x) = y' = x^{1+2x}\left(2\ln|x| + \frac{1+2x}{x}\right)}$$

8. (10 pts.) A rock is thrown from a tower at time $t = 0$. At time t (in seconds) it has a height of $s(t) = 48 + 32t - 16t^2$ feet. Please show your work in answering the following questions.

(a) When does the rock hit the ground?

When $s(t) = 0$

$$48 + 32t - 16t^2 = 0$$
$$-16(t^2 - 2t - 3) = 0$$
$$-16(t+1)(t-3) = 0$$

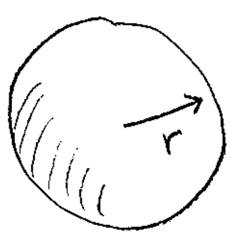
Ignore negative time $\rightarrow t = -1$ $\qquad \boxed{t = 3}$

$\boxed{\text{Hits ground at } t = 3 \text{ seconds,}}$

(b) What is its velocity when it hits the ground?

$$v(t) = s'(t) = 32 - 32t$$
$$v(3) = 32 - 32 \cdot 3 = -32 \cdot 2 = \boxed{-64 \text{ ft/sec}}$$

9. (**Bonus:** 5 pts.) A spherical balloon is inflated and its volume increases at a rate of 15 cubic inches per minute. What is the rate of change of its radius when the radius is 10 inches?



V = Volume
r = radius

Know: $\frac{dV}{dt} = 15$

Want: $\frac{dr}{dt}$ when $r = 10$

$$V = \frac{4}{3}\pi r^3$$
$$D_t[V] = D_t\left[\frac{4}{3}\pi r^3\right]$$
$$\frac{dV}{dt} = \frac{4}{3}\pi 3 r^2 \frac{dr}{dt}$$
$$15 = 4\pi r^2 \frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{15}{4\pi r^2}$$

Ans $\left.\frac{dr}{dt}\right|_{r=10} = \frac{15}{4\pi 10^2} = \frac{3}{4\pi 20} = \boxed{\frac{3}{80\pi} \text{ in/min}}$

Sphere formulas: Volume $= \frac{4}{3}\pi r^3$ Area $= \frac{1}{3}\pi r^2$