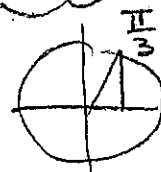


1. (35 pts.) Evaluate the following limits. Show steps, as appropriate.

$$(a) \lim_{x \rightarrow 0} \frac{\pi \sin(x)}{3x} = \frac{\pi}{3} \lim_{x \rightarrow 0} \frac{\sin(x)}{x} = \frac{\pi}{3} \cdot 1 = \boxed{\frac{\pi}{3}}$$

$$(b) \lim_{x \rightarrow \infty} \frac{\sin(x)}{x} = \boxed{0} \leftarrow \text{The denominator goes to } \infty, \text{ but numerator oscillates between } -1 \text{ and } 1.$$

$$(c) \lim_{x \rightarrow \pi/3} \frac{\sin(x)}{x} = \frac{\sin(\pi/3)}{\pi/3} = \frac{\frac{\sqrt{3}}{2}}{\frac{\pi}{3}} = \boxed{\frac{3\sqrt{3}}{2\pi}}$$



$$(d) \lim_{x \rightarrow -\infty} \frac{x^2 - 3x - 10}{x^2 - 8x + 15} = \lim_{x \rightarrow -\infty} \frac{x^2 - 3x - 10}{x^2 - 8x + 15} \cdot \frac{1/x^2}{1/x^2}$$

$$= \lim_{x \rightarrow -\infty} \frac{1 - 3/x - 10/x^2}{1 - 8/x - 15/x^2} = \frac{1 - 0 - 0}{1 - 0 - 0} = \frac{1}{1} = \boxed{1}$$

$$(e) \lim_{x \rightarrow 5} \frac{x^2 - 3x - 10}{x^2 - 8x + 15} = \lim_{x \rightarrow 5} \frac{(x-5)(x+2)}{(x-5)(x+3)} = \lim_{x \rightarrow 5} \frac{x+2}{x+3} = \frac{5+2}{5+3} = \boxed{\frac{7}{8}}$$

getting  $\frac{0}{0}$

$$(f) \lim_{x \rightarrow 0} \frac{(x-3)\sin(x)}{2x^2 - 6x} = \lim_{x \rightarrow 0} \frac{(x-3)\sin(x)}{2x(x-3)} = \lim_{x \rightarrow 0} \frac{\sin(x)}{2x}$$

$$= \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin(x)}{x} = \frac{1}{2} \cdot 1 = \boxed{\frac{1}{2}}$$

getting  $\frac{0}{0}$

$$(g) \lim_{h \rightarrow 0} \frac{\sqrt{6+h} - \sqrt{6}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{6+h} - \sqrt{6}}{h} \cdot \frac{\sqrt{6+h} + \sqrt{6}}{\sqrt{6+h} + \sqrt{6}}$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{6+h}^2 - \sqrt{6}^2}{h(\sqrt{6+h} + \sqrt{6})} = \lim_{h \rightarrow 0} \frac{6+h-6}{h(\sqrt{6+h} + \sqrt{6})} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{6+h} + \sqrt{6})}$$

$$= \lim_{h \rightarrow 0} \frac{1}{\sqrt{6+h} + \sqrt{6}} = \frac{1}{\sqrt{6+0} + \sqrt{6}} = \frac{1}{\sqrt{6} + \sqrt{6}} = \boxed{\frac{1}{2\sqrt{6}}}$$

2. (5 pts) Sketch the graph of **one** function with domain  $[-8, 8]$  that meets all of the following criteria.

(a)  $\lim_{x \rightarrow \infty} f(x) = 1$

(b)  $\lim_{x \rightarrow -\infty} f(x) = 2$

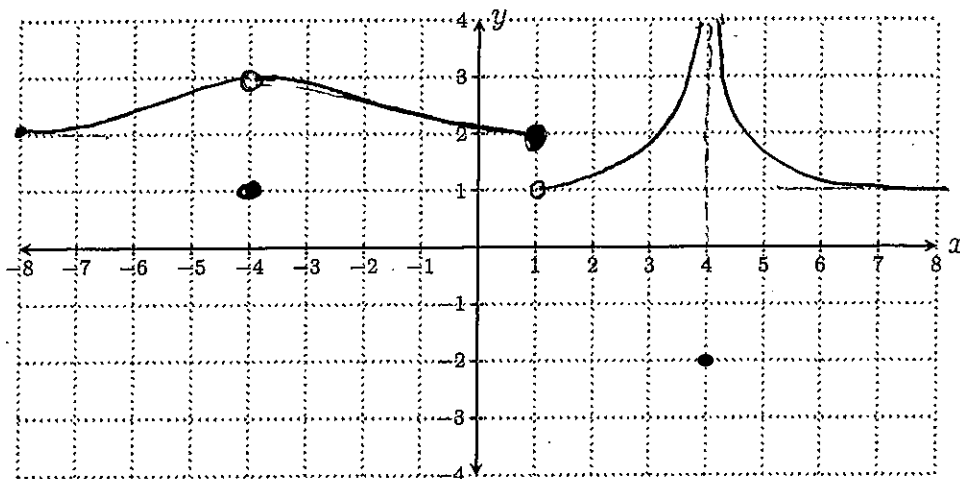
(c)  $\lim_{x \rightarrow 4} f(x) = \infty$

(d)  $\lim_{x \rightarrow 1^+} f(x) = 1$

(e)  $\lim_{x \rightarrow 1^-} f(x) = 2$

(f)  $\lim_{x \rightarrow -4} f(x) = 3$

(g)  $f$  is not continuous at  $x = -4$ .



3. (5 pts.) Find the following limit. Explain your reasoning.

$$\lim_{z \rightarrow 5} \frac{\ln(z) - \ln(5)}{z - 5} = f'(5) = \boxed{\frac{1}{5}}$$

Let  $f(x) = \ln(x)$ . Then  $f'(x) = \frac{1}{x}$

Also  $f'(x) = \lim_{z \rightarrow x} \frac{\ln(z) - \ln(x)}{z - x}$ , so  $f'(5) = \lim_{z \rightarrow 5} \frac{\ln(z) - \ln(5)}{z - 5}$

Thus the above limit equals  $f'(5) = \frac{1}{5}$

4. (5 pts.) Suppose  $f(x) = \frac{6}{x}$ . Use a limit definition of the derivative to find  $f'(x)$ .

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} = \lim_{z \rightarrow x} \frac{\frac{6}{z} - \frac{6}{x}}{z - x}$$

$$= \lim_{z \rightarrow x} \frac{\frac{6}{z} - \frac{6}{x}}{z - x} \cdot \frac{zx}{zx} = \lim_{z \rightarrow x} \frac{6x - 6z}{(z - x)zx}$$

$$= \lim_{z \rightarrow x} \frac{-6(z - x)}{(z - x)zx} = \lim_{z \rightarrow x} \frac{-6}{zx} = \frac{-6}{x \cdot x} = \boxed{\frac{-6}{x^2}}$$

Hence  $\boxed{f'(x) = \frac{-6}{x^2}}$

5. (30 points) Find the derivatives.

$$(a) \frac{d}{dx} [\sin^{-1}(x)] = \boxed{\frac{1}{\sqrt{1-x^2}}}$$

$$(b) \frac{d}{dx} [\sqrt{x^4+x^2+1}] = \frac{d}{dx} [(x^4+x^2+1)^{\frac{1}{2}}] = \frac{1}{2}(x^4+x^2+1)^{-\frac{1}{2}}(4x^3+2x)$$

$$= \frac{4x^3+2x}{2\sqrt{x^4+x^2+1}} = \boxed{\frac{2x^3+x}{\sqrt{x^4+x^2+1}}}$$

$$(c) \frac{d}{dx} [x^2 \cos(x^2)] = \frac{d}{dx} [x^2] \cos(x^2) + x^2 \frac{d}{dx} [\cos(x^2)]$$

$$= 2x \cos(x^2) + x^2 (-\sin(x^2) 2x)$$

$$= \boxed{2x \cos(x^2) - 2x^3 \sin(x^2)}$$

$$(d) \frac{d}{dx} \left[ \frac{e^x}{x} \right] = \frac{D_x[e^x]x - e^x D_x[x]}{x^2} = \frac{e^x x - e^x \cdot 1}{x^2}$$

$$= \boxed{\frac{e^x(x-1)}{x^2}}$$

$$(e) \frac{d}{dx} \left[ \frac{1}{\sqrt{3x+1}} \right] = \frac{d}{dx} [(3x+1)^{-\frac{1}{2}}] = -\frac{1}{2}(3x+1)^{-\frac{1}{2}-1} (3)$$

$$= -\frac{3}{2} (3x+1)^{-\frac{3}{2}} = \boxed{\frac{-3}{2\sqrt{3x+1}^3}}$$

$$(f) \frac{d}{dx} [\ln(\sec(e^x))] = \frac{D_x[\sec(e^x)]}{\sec(e^x)} = \frac{\sec(e^x) \tan(e^x) e^x}{\sec(e^x)}$$

$$= \boxed{e^x \tan(e^x)}$$

6. (5 pts.) Consider the equation  $x^5 + 4xy^3 - 3y^5 = 2$ . Use implicit differentiation to find  $\frac{dy}{dx}$ .

$$D_x [x^5 + 4xy^3 - 3y^5] = D_x [2] \quad \{y = f(x)\}$$

$$5x^4 + 4y^3 + 4x \cdot 3y^2 \frac{dy}{dx} - 15y^4 \frac{dy}{dx} = 0$$

$$12xy^2 \frac{dy}{dx} - 15y^4 \frac{dy}{dx} = -5x^4 - 4y^3$$

$$\frac{dy}{dx} (12xy^2 - 15y^4) = -5x^4 - 4y^3$$

$$\boxed{\frac{dy}{dx} = \frac{-5x^4 - 4y^3}{12xy^2 - 15y^4}}$$

7. (5 pts.) Use logarithmic differentiation to find the derivative of  $f(x) = \left(\frac{1}{x}\right)^x$ .

$$y = \left(\frac{1}{x}\right)^x$$

$$\ln(y) = \ln\left(\left(\frac{1}{x}\right)^x\right)$$

$$\ln(y) = x \ln\left(\frac{1}{x}\right)$$

$$\ln(y) = -x \ln(x)$$

$$D_x [\ln(y)] = D_x [-x \ln(x)]$$

$$\frac{y'}{y} = -1 \cdot \ln(x) - x \frac{1}{x}$$

$$\frac{y'}{y} = -\ln(x) - 1$$

$$y' = y(-\ln(x) - 1)$$

$$\boxed{y' = -\left(\frac{1}{x}\right)^x (\ln(x) + 1)}$$

8. (10 pts.) An object is propelled straight down from atop a 160-foot-high tower at time  $t = 0$  seconds. At time  $t$  seconds its height is  $s(t) = 160 - 32t - 16t^2$  feet.

(a) Find the object's height when its velocity is  $-96$  feet per second.

Know  $v(t) = s'(t) = -32 - 32t$

To find when velocity is  $-96$  ft/sec, solve

$$v(t) = -96$$

$$-32 - 32t = -96$$

$$-32t = -64$$

$$t = -64 / -32 = \boxed{2 \text{ sec.}}$$
 so at  $t = 2$  sec, the

velocity is  $-96$  ft/sec. At this time height is

$$s(2) = 160 - 32 \cdot 2 - 16 \cdot 2^2 = 160 - 64 - 64 = \boxed{32 \text{ ft}}$$

(b) What is object's acceleration when its velocity is  $-96$  feet per second.

$$a(t) = v'(t) = \frac{d}{dt}[-32 - 32t] = -32 \text{ ft/sec/sec}$$

So object's acceleration is always  $-32 \text{ ft/sec/sec}$

9. (Bonus: 5 pts.) A plane is taxiing down a runway that is one mile from a tower, as shown below. When the plane is  $5/3$  miles from the tower, the distance  $y$  between tower and plane is increasing at a rate of 100 mph. What is the plane's velocity at this point in time?

Let  $x$  be as shown

Know  $\frac{dy}{dt} = 100$  (when  $y = \frac{5}{3}$ )

Want  $\frac{dx}{dt}$  (when  $y = \frac{5}{3}$ )

By pythagorean theorem

$$1 + x^2 = y^2$$

$$D_t[1 + x^2] = D_t[y^2]$$

$$0 + 2x \frac{dx}{dt} = 2y \frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{2y}{2x} \frac{dy}{dt} = \frac{y}{x} 100$$

$$\boxed{\frac{dx}{dt} = \frac{100y}{x}}$$

But when  $y = 5/3$ ,

$$x = \sqrt{\left(\frac{5}{3}\right)^2 - 1^2}$$

$$= \sqrt{\frac{25}{9} - \frac{9}{9}} = \sqrt{\frac{16}{9}} = \frac{4}{3}$$

$$\boxed{\text{So } \frac{dx}{dt} = \frac{100 \cdot 5/3}{4/3} = 125 \text{ mph}}$$

