Introduction to Mathematical Reason	Test 1 MATH 300	February 21, 2008

Name: ____________________ R. Hammack Score: ______

Directions: Please answer the questions in the space provided. To get full credit you must show all of your work. Use of calculators and other computing or communication devices is not allowed on this test.

1. Short Answer. Write each of the following sets by listing its elements or describing it with a familiar symbol.

(a) $\{x \in \mathbb{Z} : |x| \leq 3\} = \boxed{\{-3,-2,-1,0,1,2,3\}}$

(b) $\{(x,y) \in \mathbb{N} \times \mathbb{R} : x^2 = 4, y^2 = 2\} = \boxed{\{(2,\sqrt{2}),(2,-\sqrt{2})\}}$

(c) $\{(x,y) \in \mathbb{R} \times \mathbb{R} : y = x^2\} \cap \{(x,y) \in \mathbb{R} \times \mathbb{R} : y = x\} = \boxed{\{(0,0),(1,1)\}}$

(d) $\mathbb{R} - \mathscr{P}(\mathbb{R}) = \boxed{\mathbb{R}}$ (Reason: $\mathbb{R}$ is a set of numbers and $\mathscr{P}(\mathbb{R})$ is a set of sets of numbers; no element of $\mathscr{P}(\mathbb{R})$ is an element of $\mathbb{R}$.)

(e) $\{x \in \mathbb{R} : \cos(\pi x) = -1\} = \boxed{\{\ldots,-5,-3,-1,1,3,5,\ldots\}} = \boxed{\{2n+1 : n \in \mathbb{Z}\}}$

(f) $\{X \in \mathscr{P}(\mathbb{N}) : X \cap \{1,2\} = X\} = \boxed{\{\{\},\{1\},\{2\},\{1,2\}\}}$

(g) $\mathscr{P}(\{1\}) \times \mathscr{P}(\{2\}) = \boxed{\{\ (\emptyset,\emptyset),\ (\emptyset,\{2\}),\ (\{1\},\emptyset),\ (\{1\},\{2\})\ \}}$

(h) $\mathscr{P}(\mathscr{P}(\{\emptyset\})) = \boxed{\{\ \emptyset,\{\emptyset\},\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\ \}}$

(i) $\{\emptyset\} \times \{\emptyset\} = \boxed{\{(\emptyset,\emptyset)\}}$

(j) $\emptyset \times \mathbb{N} = \boxed{\emptyset}$

(k) $(\mathbb{R} - \mathbb{Z}) \cap \mathbb{N} = \boxed{\emptyset}$

(l) $\displaystyle\bigcup_{X \in \mathcal{P}(\mathbb{N})} \overline{X} = \boxed{\mathbb{N}}$ (Reason: $X \in \mathscr{P}(\mathbb{N})$ means $X \subseteq \mathbb{N}$, so $\overline{X} = \mathbb{N} - X \subseteq \mathbb{N}$. Thus the union of all such $\overline{X}$ is a subset of $\mathbb{N}$. But if $X = \emptyset$, we have $\overline{X} = \mathbb{N}$, so the union is all of $\mathbb{N}$.)

2. This problem concerns the following statement.
   $P:$ For every subset X of $\mathbb{N}$, there is an integer m for which $|X| = m$.

   (a) Is the statement P true or false? Explain.

   P says that no matter which set $X \subseteq \mathbb{N}$ you might pick,
   there will always be an integer m (depending on X) for which $|X| = m$.

   The statement is **FALSE.**
   Notice that the set $X = \{2, 4, 6, 8, \ldots\}$ of even numbers is a subset of $\mathbb{N}$,
   but there is no integer m for which $|X| = m$, because X is infinite.

   Therefore it is untrue that for every subset X of $\mathbb{N}$, there is an integer m for which $|X| = m$.

   (b) Form the negation $\sim P$. Write your answer as an English sentence.

   Symbolically, the statement P is $\forall X \subseteq \mathbb{N}, \exists m \in \mathbb{Z}, |X| = m$
   Its negation is

$$\begin{aligned} \sim P &= \sim (\forall X \subseteq \mathbb{N}, \exists m \in \mathbb{Z}, |X| = m) \\ &= \exists X \subseteq \mathbb{N}, \sim (\exists m \in \mathbb{Z}, |X| = m) \\ &= \exists X \subseteq \mathbb{N}, \forall m \in \mathbb{Z}, \sim (|X| = m) \\ &= \exists X \subseteq \mathbb{N}, \forall m \in \mathbb{Z}, |X| \neq m \end{aligned}$$

   Translating back into words, the negation is:
   There is a subset X of $\mathbb{N}$ for which for every integer m we have $|X| \neq m$.

   Putting this into a more natural form, the translation is
   $\boxed{\text{There is a subset } X \text{ of } \mathbb{N} \text{ for which } |X| \neq m \text{ for every integer } m.}$

3. Suppose that $(R \Rightarrow S) \vee \sim (P \wedge Q)$ is **false**.
   Is there enough information to determine the truth values of P, Q, R and S? If so, what are they?
   (This is most easily done without a truth table.)

   For this to be false, both $R \Rightarrow S$ and $\sim (P \wedge Q)$ must be false.
   The only way that $R \Rightarrow S$ can be false is if $\boxed{R \text{ is true and } S \text{ is false.}}$
   The only way that $\sim (P \wedge Q)$ can be false is if $P \wedge Q$ is true.
   The only way that $P \wedge Q$ can be true is if $\boxed{P \text{ and } Q \text{ are both true}}$.

   Therefore $\boxed{R, P \text{ and } Q \text{ are all true, and } S \text{ is false.}}$

4. Write out a truth table to decide if $(\sim P) \wedge (P \Rightarrow Q)$ and $\sim (Q \Rightarrow P)$ are logically equivalent.

P	Q	$\sim P$	$P \Rightarrow Q$	$Q \Rightarrow P$	$(\sim P) \wedge (P \Rightarrow Q)$	$\sim (Q \Rightarrow P)$
T	T	F	T	T	**F**	**F**
T	F	F	F	T	**F**	**F**
F	T	T	T	F	**T**	**T**
F	F	T	T	T	**T**	**F**

Since the final two columns are not the same, the two expressions $(\sim P) \wedge (P \Rightarrow Q)$ and $\sim (Q \Rightarrow P)$ are **NOT logically equivalent.**

5. How many 10-digit integers contain no 0's and exactly three 6's?

Answer: Make such a number as follows: Start with 10 blank spaces and choose three of these spaces for the 6's. There are $\binom{10}{3} = 120$ ways of doing this. For each of these 120 choices we can fill in the remaining seven blanks with choices from the digits 1,2,3,4,5,7,8,9, and there are 8^7 to do this. Thus the answer to the question is $\binom{10}{3} \cdot 8^7 = \mathbf{251658240}$.