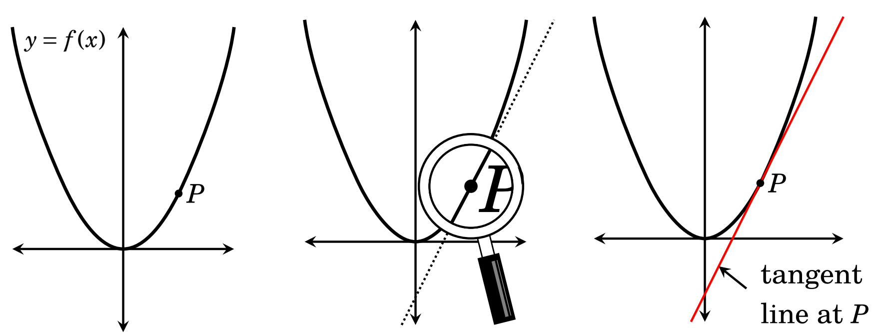
CHAPTER 7

Limits: The Way to Tangent Lines

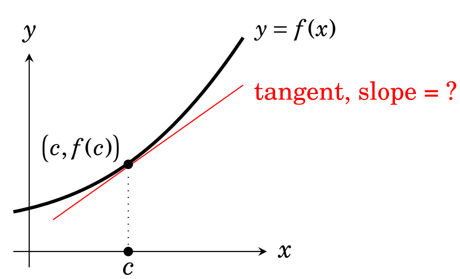
Now that we have reviewed the fundamental idea of a function, we turn to the central problem of calculus: finding slopes of tangent lines.

Recall that, up close, non-linear functions tend to look linear. Take a point P on the graph of a function $f(x)$ and magnify the graph at P. The graph looks like a straight line there because the curve hasn't had much space to bend. The higher the magnification, the straighter the curve looks.



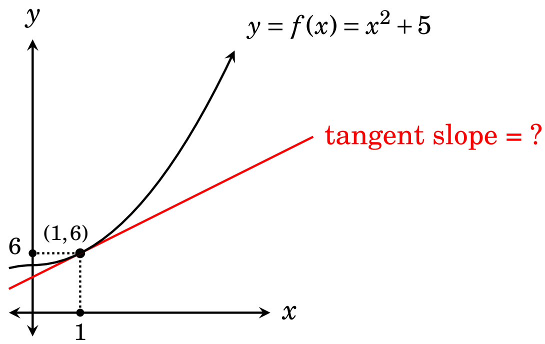
If we were to extend this apparent straight line through P we'd get what is called the *tangent line* to the curve at P. Near P, it touches the graph only at P. Think of it as the best straight line "fit" to the curve at P. At P it has the same direction as the curve.

The **fundamental problem of calculus** is to compute the slope of a tangent line to the graph of a function $y = f(x)$ at a point $(c, f(c))$.



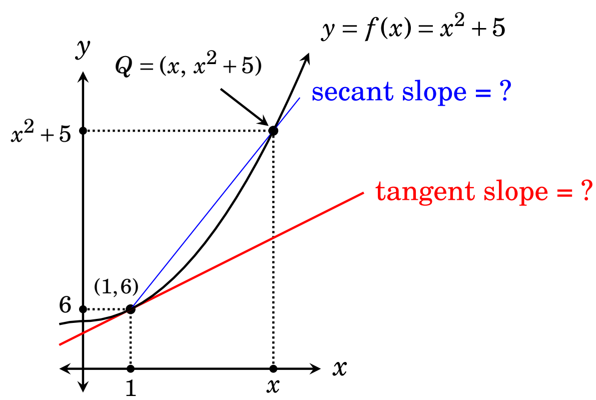
In what follows we will solve this problem for a specific point on a specific function. The solution involves a new mathematical idea called a *limit*. The rest of Part II will be a careful study of this important concept.

Our motivational example concerns the function $f(x) = x^2 + 5$. Its graph is the parabola $y = x^2$ moved up 5 units. On it is the point $(1, f(1)) = (1, 6)$. Our goal is to find the slope of the tangent to the graph at $(1, 6)$.



To compute its slope we need two points on the line so we can work out rise over run. Unfortunately we know of only one definite point on the tangent line, namely $(1, 6)$. We need a second.

We will do the best we can do with the information given. The second point will be not on the tangent line to the curve, but on the curve itself. Take a value of x that is near 1. To x there corresponds a point $Q = (x, f(x)) = (x, x^2 + 5)$ *on the curve*. Draw a line through $(1, 6)$ and Q. This new line is not our tangent line because it crosses the graph twice. A line—like this one—crossing a curve at two (or more) points is called a **secant line** to the curve.

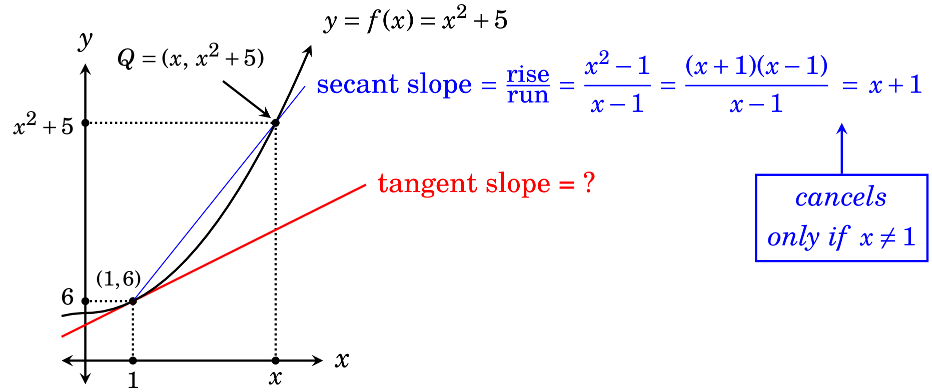


If x is fairly close to 1, the secant line has roughly the same slope as the tangent line. But (unlike the tangent line) we can compute the slope of the secant from the two points $(1, 6)$ and $Q = (x, x^2 + 5)$. We get

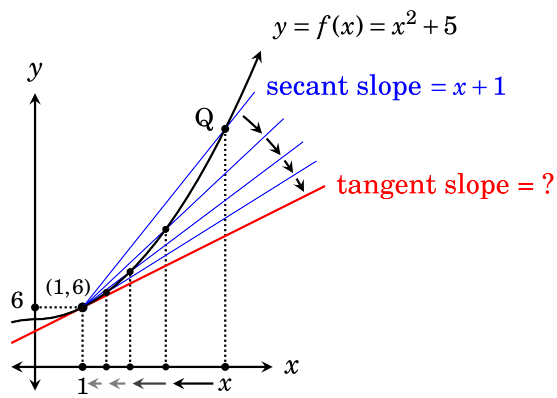
$$\text{secant slope} = \frac{\text{rise}}{\text{run}} = \frac{(x^2 + 5) - 6}{x - 1} = \frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x - 1} = x + 1.$$

Notice that this secant slope depends on x because the location of Q depends on x. Notice also that our set-up relies on $x \neq 1$ because if $x = 1$ the point Q coincides with $(1,6)$ and we no longer have two points to form the secant.

Also, in simplifying the expression for secant slope, the cancellation $\frac{(x+1)(x-1)}{x-1} = x+1$ works only when $x \neq 1$. The reason is that in canceling we are relying on $\frac{x-1}{x-1} = 1$. If $x = 1$ then this fraction is $\frac{0}{0}$, which is not defined.



Of course the secant line, whose slope is $x+1$, is not the tangent line. But if x is very close to 1, the secant line is a reasonable approximation of the tangent line, and the closer x is to 1, the better the approximation.



Imagine x getting closer and closer to the number 1. This makes the point Q move down the curve, getting closer and closer to the point $(1,6)$. The secant line pivots on the point $(1,6)$, rotating toward the tangent line. Thus as x approaches 1, the secant slope $x+1$ approaches the tangent slope. So as x approaches 1, the secant slope $x+1$ approaches $1+1 = 2$, which must be the tangent slope. Now we have our answer.

Conclusion: The tangent to $f(x) = x^2 + 5$ at $(1,6)$ has slope **2**.

In summary, the tangent slope is the value that the secant slope $\frac{x^2-1}{x-1}$ approaches as x approaches 1. To underscore this point we tally the information into the table below. The first column contains values of x getting closer and closer to 1. The second column shows the corresponding secant slope $\frac{x^2-1}{x-1}$. We can see that as x approaches 1, the corresponding value $\frac{x^2-1}{x-1}$ approaches 2.

x	secant slope $= \dfrac{x^2-1}{x-1} = x+1$
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999
0.9999	1.9999
↓	↓
1	**2**
↑	↑
1.0001	2.0001
1.001	2.001
1.01	2.01
1.1	2.1
1.5	2.5
2	3

Mathematics has a special notation for the kind of *limiting process* expressed by the table, where x approaching 1 forces $\frac{x^2-1}{x-1}$ to approach 2. We write it as

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2,$$

where the symbol $\lim\limits_{x\to 1}$ means that x is getting closer and closer to 1. We call this expression a **limit**. From this discussion we see that in general the problem of finding the slope of a tangent line to the graph of a function can be solved by a limit of the form

$$\lim_{x\to c} g(x) = L.$$

The picture that emerges from our discussion is this: Finding slopes of tangent lines involves understanding limits. Consequently all of Part 2 of this book is devoted to limits. Once we have a thorough understanding of limits we will return to slopes of tangent lines in Part 3.