Introduction to Mathematical Reason	Test #3 MATH 300	November 28, 2007

Name: ____________________ R. Hammack Score: ________

Directions: Please answer the questions in the space provided. To get full credit you must show all of your work. Use of calculators and other computing or communication devices is not allowed on this test.

1. (10 points) Let $A = \{a, b, c, d\}$. Suppose R is an equivalence relation on A for which $(a, b) \in R$, $(c, b) \in R$, but $(a, d) \notin R$. List out the set R as a subset of $A \times A$.

   $R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (c, b), (b, c), (a, c), (c, a)\}$

2. (10 points) The relation $\{(x, y) : x, y \in \mathbb{Z},\ 2|(x + y)\}$ is an equivalence relation on $\mathbb{Z}$. Describe its equivalence classes.

   This relation is xRy if $2|(x + y)$.

   $[0] = \{y \in \mathbb{Z} : 0Ry\} = \{y \in \mathbb{Z} : 2|(0 + y)\} = \{y \in \mathbb{Z} : 2|y\} = \{y : y = 2c, c \in \mathbb{Z}\} = \{2c : c \in \mathbb{Z}\}$
   (set of even integers)

   $[1] = \{y \in \mathbb{Z} : 1Ry\} = \{y \in \mathbb{Z} : 2|(1 + y)\} = \{y : 1 + y = 2c, c \in \mathbb{Z}\} = \{y : y = 2c - 1, c \in \mathbb{Z}\}$
   (set of odd integers)

   Thus there are just two equivalence classes. One is the set of even integers. The other is the set of odd integers.

3. (10 points) Do the following operations in $\mathbb{Z}_5$. In each case your answer should be one of [0], [1], [2], [3], or [4].

   (a) $[4] \cdot [4] = [4 \cdot 4] = [16] = \boxed{[1]}$

   (b) $[3] + [4] = [3 + 4] = [7] = \boxed{[2]}$

   (c) $([2] + [4]) \cdot [3] = ([2 + 4]) \cdot [3] = [6] \cdot [3] = [1] \cdot [3] = \boxed{[3]}$

4. (15 points) Suppose R and R' are two symmetric relations on a set A. Show that the relation $S = R \cap R'$ is also symmetric.

   Proof. We want to show that for any $x, y \in A$, if xSy, then ySx.

   Suppose $x, y \in A$ and xSy.
   Recall that xSy means $(x, y) \in S$.
   Thus $(x, y) \in R \cap R'$ because $S = R \cap R'$.
   Therefore $(x, y) \in R$ and $(x, y) \in R'$, by definition of intersection.
   Since $(x, y) \in R$, it follows that $(y, x) \in R$, because R is symmetric.
   Also, since $(x, y) \in R'$, it follows that $(y, x) \in R'$, because R' is symmetric.
   Then since $(y, x) \in R$ and $(y, x) \in R'$, it follows that $(y, x) \in R \cap R'$, by definition of intersection.
   Consequently, $(y, x) \in S$, because $S = R \cap R'$.
   Finally, $(y, x) \in S$ means ySx.

   The above paragraph has shown that if xSy, then ySx. Thus S is a symmetric relation. ■

5. (10 points) Consider the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined as $f((x, y)) = xy + 5$.

(a) Is f injective? Explain.
No. For notice that $(1, 0) \neq (0, 1)$ but $f((1, 0)) = 1 \cdot 0 + 5 = 5 = 0 \cdot 1 + 5 = f((0, 1))$.

(b) Is f surjective? Explain.
Yes. Suppose $b \in \mathbb{Z}$. Then $(b - 5, 1) \in \mathbb{Z} \times \mathbb{Z}$ and $f((b - 5, 1)) = (b - 5) \cdot 1 + 5 = b$.

6. (15 points) Consider the function $f : \mathbb{R} \to \mathbb{R}^+$ defined as $f(x) = e^{3x+4}$.

(a) Show that f is injective.

Suppose $f(x) = f(y)$ for some $x, y \in \mathbb{R}$.
This means $e^{3x+4} = e^{3y+4}$.
Since any real power of e is positive, we can take ln of both sides.
Then $\ln(e^{3x+4}) = \ln(e^{3y+4})$.
Thus $3x + 4 = 3y + 4$, so $3x = 3y$, and thus $x = y$.

Since $f(x) = f(y)$ implies $x = y$, it follows that f is injective.

(b) Show that f is surjective.

Take an arbitrary element $b \in \mathbb{R}^+$.
Since b is a positive real number, it follows that $\ln(b)$ exists.
Let $x = \dfrac{\ln(b) - 4}{3}$.
Then $f(x) = e^{3\frac{\ln(b)-4}{3}+4} = e^{\ln(b)-4+4} = e^{\ln(b)} = b$
This shows f is surjective.

(c) Find a formula for f^{-1}.

Notice that $f(f^{-1}(x)) = x$.
This means $e^{3f^{-1}(x)+4} = x$.
So $\ln(e^{3f^{-1}(x)+4}) = \ln(x)$.
Thus $3f^{-1}(x) + 4 = \ln(x)$.
So $3f^{-1}(x) = \ln(x) - 4$.
Therefore $\boxed{f^{-1}(x) = \dfrac{\ln(x) - 4}{3}.}$

7. (15 points) Prove that $3|(4^n - 1)$ for every $n \in \mathbb{N}$.

Proof. (Induction) For the basis step, notice that if $n = 1$, then $4^n - 1 = 4^1 - 1 = 3$, so $3|(4^n - 1)$ when $n = 1$.

Now assume that $3|(4^k - 1)$ for some integer $k \geq 1$.
This means $4^k - 1 = 3a$ for some integer a, so $\boxed{4^k = 3a + 1.}$

Observe that $4^{k+1} - 1 = 4 \cdot 4^k - 1 = 4(3a + 1) - 1 = 12a + 4 - 1 = 12a + 3 = 3(4a + 1)$.
Since $4^{k+1} - 1 = 3(4a + 1)$, it follows that $3|(4^{k+1} - 1)$.

This shows that $3|(4^k - 1)$ implies that $3|(4^{k+1} - 1)$, so the proof by induction is complete. ■

8. (15 points) Prove that $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$ for every natural number n.

Proof. (Induction)
For the basis step, notice that when $n = 1$ the statement is $1^3 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$, which is true.

Now assume the statement is true for some integer $n = k \geq 1$, that is assume
$1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 = \dfrac{k^2(k+1)^2}{4}$.

Observe that this implies the statement is true for $n = k + 1$, as follows:

$$\begin{aligned}
1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 + (k+1)^3 &= (1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3) + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1)^1)}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \frac{(k+1)^2((k+1)+1)^2}{4}
\end{aligned}$$

Therefore $1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 + (k+1)^3 = \dfrac{(k+1)^2((k+1)+1)^2}{4}$, which means the statement is true for $n = k + 1$. Thus the result follows by mathematical induction. ■