Introduction to Mathematical Reasoning	MATH 300 R. Hammack	September 24, 2007 Test #1

Name: ____________________ Score: ________

Directions: Please answer each question in the space provided. Be sure to show your work when appropriate. Calculators may not be used on this test.

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1. (8 points) Write each of the following sets in set builder notation (e.g. $\{x^2 : x \in \mathbb{N}\}$, etc.).

   (a) $\{\ldots, -4, -2, 0, 2, 4, 6, 8, 10, \ldots\} = \boxed{\{2x : x \in \mathbb{Z}\}}$

   (b) $\{\ldots, -2, 3, 8, 13, 18, 23, 28, 33, \ldots\} = \boxed{\{5x-2 : x \in \mathbb{Z}\}}$

   (c) $\{-1, -4, -9, -16, -25, -36, \ldots\} = \boxed{\{-x^2 : x \in \mathbb{N}\}}$

   (d) $[2,7) = \boxed{\{x \in \mathbb{R} : 2 \leq x < 7\}}$

2. (8 points) Write each of the following sets by listing its elements between curly brackets.

   (a) $\{x^2 - 1 : x \in \mathbb{N}\} = \boxed{\{0, 3, 8, 15, 24, 35, \ldots\}}$

   (b) $\{x \in \mathbb{R} : x^2 - x = 0\} = \boxed{\{0, 1\}}$

   (c) $\{(x,y) \in \mathbb{Z} \times \mathbb{N} : x^2 = 4 \text{ and } y^2 = 9\} = \boxed{\{(2,3), (-2,3)\}}$

   (d) $\{X \in \mathcal{P}(\{a,b,c\}) : |X| = 2\} = \boxed{\{\{a,b\}, \{a,c\}, \{b,c\}\}}$

3. (8 points) Answer the following questions, where $A = \{2,3\}$, $B = \{a,b\}$, and $C = \{3,4\}$.

   (a) $(A \cap C) \times B = \boxed{\{(3,a), (3,b)\}}$

   (b) $A \cap (C \times B) = \boxed{\{\} = \emptyset}$

4. (8 points) Consider the sets $A_1 = \{0,1,2,3\}$, $A_2 = \{0,2,3,4\}$, $A_3 = \{0,3,4,5\}$, $A_4 = \{0,3,5,6\}$, and $I = \{1,2,3,4\}$.

   (a) $\bigcap_{n \in I} A_n = \boxed{\{0, 3\}}$

   (b) $\bigcup_{n \in I} A_n = \boxed{\{0, 1, 2, 3, 4, 5, 6\}}$

5. (8 points) Write truth tables for the logical connectives $\Rightarrow$ and $\Leftrightarrow$.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

6. (8 points) Suppose you know that P is false, and that the statement $(R \Rightarrow S) \Leftrightarrow (P \wedge Q)$ is true. Can the true/false values of R and S be determined? Explain. (This can be done without a truth table.)

Since P is false, it must be the case that $P \wedge Q$ is also false.
Given this and the fact that $(R \Rightarrow S) \Leftrightarrow (P \wedge Q)$ is true, the truth table for $\Leftrightarrow$ shows that $R \Rightarrow S$ is false.
But from the truth table for $\Rightarrow$, the only way that $R \Rightarrow S$ can be false is if $\boxed{\text{R is TRUE and S is FALSE.}}$

7. (8 points) Write an expression that is logically equivalent to $(\sim P) \vee (\sim Q)$ and contains only one $\sim$.

By DeMorgan's Law, $(\sim P) \vee (\sim Q)$ is logically equivalent to $\sim (P \wedge Q)$.

8. (8 points) Write out a truth table to decide if $(\sim P) \wedge (P \Rightarrow Q)$ and $\sim (Q \Rightarrow P)$ are logically equivalent.

P	Q	$\sim P$	$P \Rightarrow Q$	$Q \Rightarrow P$	$(\sim P) \wedge (P \Rightarrow Q)$	$\sim (Q \Rightarrow P)$
T	T	F	T	T	**F**	**F**
T	F	F	F	T	**F**	**F**
F	T	T	T	F	**T**	**T**
F	F	T	T	T	**T**	**F**

Since the final two columns are not the same, the two expressions $(\sim P) \wedge (P \Rightarrow Q)$ and $\sim (Q \Rightarrow P)$ are **NOT logically equivalent.**

9. (18 points) Let $x \in \mathbb{Z}$. Prove that if x is odd, then $x^2 + 1$ is even.

Proof. (Direct) Suppose that if x is odd.
By the definition of an odd number, this means $x = 2k + 1$ for some integer $k \in \mathbb{Z}$.
Then $x^2 + 1 = (2k + 1)^2 + 1 = 4k^2 + 4k + 1 + 1 = 4k^2 + 4k + 2 = 2(2k^2 + 2k + 1)$.
Letting m be the integer $m = 2k^2 + 2k + 1$, we see that the previous line gives $x^2 + 1 = 2m$.
Therefore, by definition of an even number, $x^2 + 1$ is even. ■

10. (18 points) Suppose $x, y \in \mathbb{Z}$. Prove that if xy is odd, then x and y are both odd.

Proof. (Contrapositive) Suppose that it is not the case that x and y are both odd.
This means that one or both of x are y is even.

Case 1. Suppose x is even. Then there is an integer k for which $x = 2k$.
Then $xy = (2k)y = 2(ky)$ and hence xy is even, and therefore not odd.

Case 2. Suppose y is even. Then there is an integer k for which $y = 2k$.
Then $xy = x(2k) = 2(xk)$ and hence xy is even, and therefore not odd.

Either way, we see that xy is not odd, so the proof is complete. ■