Section 10.3 Infinite Series

An *infinite sequence* is an ∞ list of #'s $\frac{1}{2}\ \frac{1}{4}\ \frac{1}{8}\ \frac{1}{16} \cdots$

An *infinite series* is an ∞ sum of #'s

$$\sum_{K=1}^{\infty} \frac{1}{2^K} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = 1$$

$$\sum_{K=1}^{\infty} u_K = u_1 + u_2 + u_3 + u_4 + u_5 + \cdots = S = ?$$

When does it make sense to say an ∞ series equals a #?
To answer this question we will make a definition.

Given an ∞ series $\sum_{K=1}^{\infty} u_K$ we define:

$$S_1 = u_1$$
$$S_2 = u_1 + u_2$$
$$S_3 = u_1 + u_2 + u_3$$
$$S_4 = u_1 + u_2 + u_3 + u_4$$
$$\vdots$$
$$S_n = u_1 + u_2 + u_3 + \cdots + u_n = \sum_{K=1}^{n} u_K \leftarrow n^{th} \text{ partial sum}$$

$$S_1\ S_2\ S_3\ S_4 \cdots S_n \cdots = \{S_K\}_{K=1}^{\infty} \leftarrow \text{sequence of partial sums}$$

If $\{S_n\}_{n=1}^{\infty}$ has limit S then $\sum_{K=1}^{\infty} u_K = S$,
and we say the series *converges* to S.
Otherwise we say the series *diverges*.

Ex $\sum_{K=1}^{\infty} \frac{6}{10^K} = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \cdots$

$$S_1 = .6$$
$$S_2 = .6 + .06 = .66$$
$$S_3 = .6 + .06 + .006 = .666$$
$$\vdots$$
$$S_n = .6 + .06 + \cdots .000\cdot06 = .666\cdots6$$

$\{S_n\} = .6\ \ .66\ \ .666\ \ .6666\ \ .66666 \ldots$

$\lim_{n\to\infty} S_n = .666666\overline{6} = \frac{2}{3}$

Therefore $\sum \frac{6}{10^K} = \frac{2}{3}$

Important Example

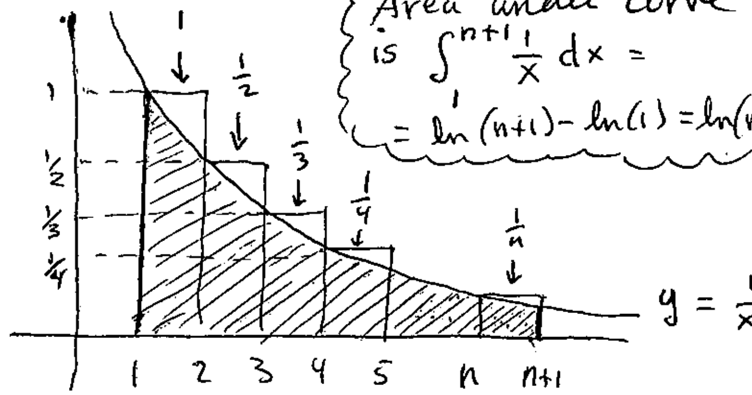
Harmonic Series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

Note:

$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln(1+n)$

So $\lim_{n\to\infty} S_n \geq \lim_{n\to\infty} \ln(1+n) = \infty$

Area under curve is $\int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1) - \ln(1) = \ln(n+1)$



$\therefore$ $\boxed{\text{Harmonic Series \underline{diverges}}}$

Ex $\sum_{k=1}^{\infty} (-1)^k = -1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 + 1 \cdots = ?$

$$\left.\begin{aligned} S_1 &= -1 \\ S_2 &= 0 \\ S_3 &= -1 \\ S_4 &= 0 \\ S_5 &= -1 \\ &\vdots \end{aligned}\right\} \lim_{n\to\infty} S_n \text{ DNE, so } \sum_{k=1}^{\infty} (-1)^k \text{ \underline{diverges}}$$

Theorem 10.8

1. $\sum a_k = A \implies \sum c a_k = c \sum a_k = cA$
2. $\sum a_k = A$ and $\sum b_k = B \implies \sum (a_k + b_k) = \sum a_k + \sum b_k = A + B$
3. If $\sum a_k$ diverges, and $c \neq 0$ then $\sum c a_k$ diverges
4. If one of $\sum a_k$ and $\sum b_k$ diverges, then $\sum (a_k \pm b_k)$ diverges
5. $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=M}^{\infty} a_k$ either both converge or both diverge.

A *geometric* series is one of form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots$$

a = "initial term"

r = "ratio"

Ex $\quad 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots \qquad \begin{cases} a = 3 \\ r = \frac{1}{2} \end{cases}$

Ex $\quad 1 - 2 + 4 - 8 + \cdots \qquad \begin{cases} a = 1 \\ r = (-2) \end{cases}$

Let's find a formula for geometric series.

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots$$

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n$$

$$rS_n = ar + ar^2 + ar^3 + ar^4 + ar^5 + \cdots + ar^n + ar^{n+1}$$

$$S_n - rS_n = a - ar^{n+1}$$

$$S_n(1-r) = a(1 - r^{n+1})$$

$$S_n = \frac{a(1-r^{n+1})}{1-r}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^{n+1})}{1-r} = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{DNE} & \text{if } |r| \geq 1 \end{cases}$$

Theorem The geometric series $\sum_{k=1}^{\infty} ar^k$ diverges if $|r| \geq 1$.

If $|r| < 1$, $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

Ex $\quad 6 + \frac{6}{5} + \frac{6}{25} - \frac{6}{125} + \cdots = \sum_{k=0}^{\infty} 6\left(\frac{-1}{5}\right)^k = \frac{6}{1 - \frac{-1}{5}} = \frac{6}{1 + \frac{1}{5}} = \frac{6}{\frac{6}{5}} = 5$

$= \sum^{\infty} \cdots^{k}\ 6$

Telescoping Series

Ex $\displaystyle\sum_{K=1}^{\infty}\left(\frac{1}{K+3}-\frac{1}{K+4}\right) = \lim_{n\to\infty} S_n$

$$= \lim_{n\to\infty}\left(\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{8}\right)+\cdots+\left(\frac{1}{n+3}-\frac{1}{n+4}\right)\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{4}-\frac{1}{n+4}\right) = \boxed{\frac{1}{4}}$$

Ex $\displaystyle\sum_{K=2}^{\infty}\frac{2}{K^2-1} = \frac{2}{3}+\frac{2}{8}+\frac{2}{15}+\frac{2}{24}+\cdots = ?$

$$\left.\begin{aligned}\frac{2}{K^2-1} &= \frac{2}{(K+1)(K-1)} = \frac{A}{K+1}+\frac{B}{K-1}\\ 2 &= A(K-1)+B(K+1)\end{aligned}\right\}\Rightarrow \boxed{\begin{aligned}B&=1\\A&=-1\end{aligned}}$$

$$\therefore \boxed{\frac{2}{K^2+1} = \frac{1}{K-1}-\frac{1}{K+1}}$$

$$\sum_{k=2}^{\infty}\frac{2}{K^2-1} = \sum_{K=2}^{\infty}\left(\frac{1}{K-1}-\frac{1}{K+1}\right) = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty}\left(\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\cdots+\left(\frac{1}{n-2}-\frac{1}{n}\right)+\left(\frac{1}{n-1}+\frac{1}{n+1}\right)\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{n}+\frac{1}{n+1}\right) = 1+\frac{1}{2}+0+0 =$$

$$= \boxed{\frac{3}{2}}$$