Introduction to Mathematical Reason	Test #2 MATH 300	March 11, 2008
Name: ____________	R. Hammack	Score: ______

Directions: Please answer the questions in the space provided.

1. Suppose $a, b, c, d \in \mathbb{Z}$ and $n \in \mathbb{N}$. Prove that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$. (Suggestion: Try direct proof.)

   Proof. (Direct) Suppose $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$.
   By definition of congruence modulo n, this means $n|(a-b)$ and $n|(c-d)$.
   By definition of divisibility, $a - b = nk$ and $c - d = n\ell$ for some $k, \ell \in \mathbb{Z}$.
   Therefore we have $a = b + nk$ and $c = d + n\ell$. Consequently,

$$\begin{aligned} ac &= (b+nk)(d+n\ell) \\ ac &= bd + bn\ell + nkd + n^2k\ell \\ ac - bd &= bn\ell + nkd + n^2k\ell \\ ac - bd &= n(b\ell + kd + nk\ell). \end{aligned}$$

   Since $b\ell + kd + nk\ell \in \mathbb{Z}$, it follows from the above equation that $n|(ac-bd)$.
   This means that $ac \equiv bd \pmod{n}$. ■

2. Suppose $a, b \in \mathbb{Z}$. If $a^2(b^2 - 2b)$ is odd, then both a and b are odd. (Suggestion: Try contrapositive proof.)

   Proof. (Contrapositive) Suppose it is not the case that a and b are odd.
   Then, by DeMorgan's Law, a is even or b is even. Let us look at these cases separately.

   Case 1. Suppose a is even. Then $a = 2c$ for some integer c.
   Thus $a^2(b^2 - 2b) = (2c)^2(b^2 - 2b) = 2(2c^2(b^2 - 2b))$, which is even.

   Case 2. Suppose b is even. Then $b = 2c$ for some integer c.
   Thus $a^2(b^2 - 2b) = a^2((2c)^2 - 2(2c)) = 2(a^2(2c^2 - 2c))$, which is even.

   Thus in either case $a^2(b^2 - 2b)$ is even, so it is not odd. ■

   (NOTE: A third case where both a and b are even is not necessary.
   In that case a is even, a scenario addressed in Case 1.)

3. Prove: If $a, b \in \mathbb{Z}$, then $a^2 - 4b - 2 \neq 0$.
(Suggestion: Contradiction may be easiest.)

Proof. Suppose for the sake of contradiction that $a, b \in \mathbb{Z}$ but $a^2 - 4b - 2 = 0$.
Then we have $a^2 = 4b + 2 = 2(2b + 1)$, which means a^2 is even.
Therefore a is even also, so $a = 2c$ for some integer c. Plugging this back into $a^2 - 4b - 3 = 0$ gives us

$$\begin{aligned}
(2c)^2 - 4b - 2 &= 0 \\
4c^2 - 4b - 2 &= 0 \\
4c^2 - 4b &= 2 \\
2c^2 - 2b &= 1 \\
2(c^2 - b) &= 1
\end{aligned}$$

From this last equation, we conclude that 1 is an even number, a contradiction. ■

4. Suppose $a, b, c \in \mathbb{Z}$, and $a \neq 0$. Prove the following statement: If $a \nmid bc$, then $a \nmid b$ and $a \nmid c$.

Proof. (Contrapositive) Assume that it is not true that $a \nmid b$ and $a \nmid c$.
Then $a \mid b$ or $a \mid c$. Thus $b = ak$ or $c = ak$ for some $k \in \mathbb{Z}$. Consider these cases separately.
Case 1. If $b = ak$, then multiply both sides by c to get $bc = a(kc)$, which means $a \mid bc$.
Case 2. If $c = ak$, then multiply both sides by b to get $bc = a(kb)$, which means $a \mid bc$.
Thus, in either case $a \mid bc$, so it is not true that $a \nmid b$. ■