MATH 501, Section 27 Solutions

2. Find all prime and maximal ideals in $\mathbb{Z}_{12}$.

   Maximal Ideals: $N = \{0, 2, 4, 6, 8, 10\}$ and $N = \{0, 3, 6, 9\}$.
   Prime Ideals: $N = \{0, 2, 4, 6, 8, 10\}$ and $N = \{0, 3, 6, 9\}$.

   Reason: In each case $\mathbb{Z}_{12}/N$ is isomorphic to the field (and integral domain) $\mathbb{Z}_2$ or $\mathbb{Z}_3$. For any other ideal N, $\mathbb{Z}_{12}/N$ is not an integral domain.

4. Find all prime and maximal ideals in $\mathbb{Z}_2 \times \mathbb{Z}_4$.

   Maximal ideals: $N = \{(0, 0), (0, 1), (0, 2), (0, 3)\}$ and $N = \{(0, 0), (0, 2), (1, 0), (1, 2)\}$.
   The prime ideals are the same.

   Reason: In each case $(\mathbb{Z}_2 \times \mathbb{Z}_4)/N \cong \mathbb{Z}_2$, a field and integral domain.

8. Find all $c \in \mathbb{Z}_5$ for which $\mathbb{Z}_5[x]/\langle x^2 + x + c\rangle$ is a field.

   By Theorem 27.2, this will be a field provided $x^2 + x + c$ is irreducible.

   Checking:
   $x^2 + x + 0 = x(x + 1)$ is reducible.
   $x^2 + x + 1$ is irreducible because it has no zeros in $\mathbb{Z}_5$.
   $x^2 + x + 2$ is irreducible because it has no zeros in $\mathbb{Z}_5$.
   $x^2 + x + 3 = (x + 4)(x + 2)$ is reducible.
   $x^2 + x + 4 = (x + 3)(x + 3)$ is reducible.

   Thus $\mathbb{Z}_5[x]/\langle x^2 + x + c\rangle$ is a field provided $c = 1$ or $c = 2$.

24. Let R be a finite commutative ring with unity. Show that every prime ideal in R is a maximal ideal.

    Proof. Suppose N is a prime ideal in the finite commutative ring R.
    Thus R/N must be a finite integral domain by Theorem 27.15.
    Then Theorem 19.11 says any finite integral domain is a field, so R/N is a field.
    But then, by Theorem 27.9, N must be maximal.