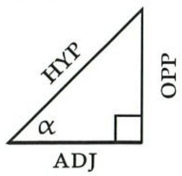
Section 8.3 Trigonometric Integrals

Our goal in this section is to learn how to evaluate integrals involving trig functions.

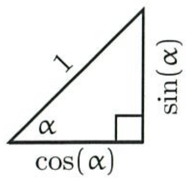
Basic Definitions



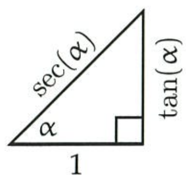
$\sin(\alpha) = \dfrac{\text{OPP}}{\text{HYP}}$ $\quad\tan(\alpha) = \dfrac{\text{OPP}}{\text{ADJ}}$ $\quad\sec(\alpha) = \dfrac{\text{HYP}}{\text{ADJ}}$

$\cos(\alpha) = \dfrac{\text{ADJ}}{\text{HYP}}$ $\quad\cot(\alpha) = \dfrac{\text{ADJ}}{\text{OPP}}$ $\quad\csc(\alpha) = \dfrac{\text{HYP}}{\text{OPP}}$

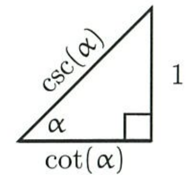
Basic Trig Triangles and Identities



$\sin^2(\alpha) + \cos^2(\alpha) = 1$



$\tan^2(\alpha) + 1 = \sec^2(\alpha)$



$\cot^2(\alpha) + 1 = \csc^2(\alpha)$

Addition and Double-Angle Identities

$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$ $\quad \sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$ $\quad \sin^2(\alpha) = \frac{1}{2}(1 - \cos(2\alpha))$

$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$ $\quad \cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$ $\quad \cos^2(\alpha) = \frac{1}{2}(1 + \cos(2\alpha))$

Basic Trig Integrals

$\int \sin(u)\,du = -\cos(u) + C$ $\qquad \int \sin^2(u)\,du = \frac{u}{2} - \frac{\sin(2u)}{4} + C$

$\int \cos(u)\,du = \sin(u) + C$ $\qquad \int \cos^2(u)\,du = \frac{u}{2} + \frac{\sin(2u)}{4} + C$

$\int \tan(u)\,du = \ln|\sec(u)| + C$ $\qquad \int \tan^2(u)\,du = \tan(u) - u + C$

Derived Below

$\int \cot(u)\,du = \ln|\sin(u)| + C$ $\qquad \int \cot^2(u)\,du = -\cot(u) - u + C$

$\int \sec(u)\,du = \ln|\sec(u) + \tan(u)| + C$ $\qquad \int \sec^2(u)\,du = \tan(u) + C$

$\int \csc(u)\,du = -\ln|\csc(u) - \cot(u)| + C$ $\qquad \int \csc^2(u)\,du = -\cot(u) + C$

$\int \sec(u)\tan(u)\,du = \sec(u) + C$

$\int \csc(u)\cot(u)\,du = -\csc(u) + C$

- $\int \tan^2(u)\,du = \int \sec^2(u) - 1\,du = \tan(u) - u + C$
- $\int \cot^2(u)\,du = \int \csc^2(u) - 1\,du = -\cot(u) - u + C$
- $\int \sin^2(u)\,du = \int \frac{1}{2}(1 - \cos(2u))\,du = \frac{u}{2} - \frac{\sin(2u)}{4} + C$
- $\int \cos^2(u)\,du = \int \frac{1}{2}(1 + \cos(2u))\,du = \frac{u}{2} + \frac{\sin(2u)}{4}$

Ex $\int \frac{\cot^2(\ln(x))}{x}\,dx =$

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

$= \int \cot^2(u)\,du$

$= -\cot(u) - u + C$

$= \boxed{-\cot(\ln(u)) - \ln(u) + C}$

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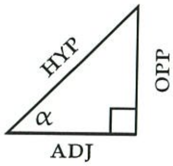
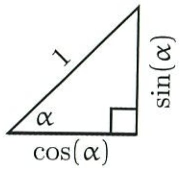
Basic Definitions



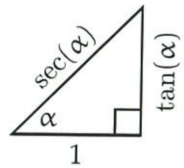
$$\sin(\alpha) = \frac{\text{OPP}}{\text{HYP}} \qquad \tan(\alpha) = \frac{\text{OPP}}{\text{ADJ}} \qquad \sec(\alpha) = \frac{\text{HYP}}{\text{ADJ}}$$

$$\cos(\alpha) = \frac{\text{ADJ}}{\text{HYP}} \qquad \cot(\alpha) = \frac{\text{ADJ}}{\text{OPP}} \qquad \csc(\alpha) = \frac{\text{HYP}}{\text{OPP}}$$

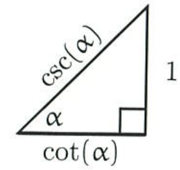
Basic Trig Triangles and Identities



$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$



$$\tan^2(\alpha) + 1 = \sec^2(\alpha)$$



$$\cot^2(\alpha) + 1 = \csc^2(\alpha)$$

Addition and Double-Angle Identities

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \qquad \sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \qquad \sin^2(\alpha) = \frac{1}{2}(1-\cos(2\alpha))$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \qquad \cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \qquad \cos^2(\alpha) = \frac{1}{2}(1+\cos(2\alpha))$$

Basic Trig Integrals

$$\int \sin(u)\,du = -\cos(u) + C \qquad \int \sin^2(u)\,du = \frac{u}{2} - \frac{\sin(u)\cos(u)}{2} + C$$

$$\int \cos(u)\,du = \sin(u) + C \qquad \int \cos^2(u)\,du = \frac{u}{2} + \frac{\sin(u)\cos(u)}{2} + C$$

$$\int \tan(u)\,du = \ln|\sec(u)| + C \qquad \int \tan^2(u)\,du = \tan(u) - u + C$$

$$\int \cot(u)\,du = \ln|\sin(u)| + C \qquad \int \cot^2(u)\,du = -\cot(u) - u + C$$

Derived below

$$\int \sec(u)\,du = \ln|\sec(u)+\tan(u)| + C \qquad \int \sec^2(u)\,du = \tan(u) + C$$

$$\int \csc(u)\,du = -\ln|\csc(u)-\cot(u)| + C \qquad \int \csc^2(u)\,du = -\cot(u) + C$$

$$\int \sec(u)\tan(u)\,du = \sec(u) + C$$

$$\int \csc(u)\cot(u)\,du = -\csc(u) + C$$

Ex. $\int \frac{\cot^2(\ln(x))}{x}\,dx$

$u = \ln(x)$, $du = \frac{1}{x}\,dx$

$$= \int \cot^2(u)\,du = -\cot(u) - u + C = \boxed{-\cot(\ln(x)) - \ln(x) + C}$$

$$\int \tan^2(u)\,du = \int \sec^2(u) - 1\,du = \tan(u) - u + C$$

$$\int \cot^2(u)\,du = \int \csc^2(u) - 1\,du = -\cot(u) - u + C$$

$$\int \sin^2(u)\,du = \int \frac{1}{2}(1-\cos(2u))\,du = \frac{1}{2}\left(u - \frac{\sin(2u)}{2}\right) + C = \frac{1}{2}\left(u - \sin(u)\cos(u)\right) + C$$

$$\int \cos^2(u)\,du = \int \frac{1}{2}(1+\cos(2u))\,du = \frac{1}{2}\left(u + \frac{\sin(2u)}{2}\right) + C = \frac{1}{2}\left(u + \sin(u)\cos(u)\right) + C$$

But what should we make of $\int \sin^5 x \, dx$? Familiar substitutions don't work. We have to be clever and use some identities.

$$\int \sin^5 x \, dx = \int \sin^4 x \sin x \, dx = \int (\sin^2 x)^2 \sin x \, dx = \int (1-\cos^2 x)^2 \sin x \, dx$$

$u = \cos x$
$du = -\sin x \, dx$

$$= -\int (1-u^2)^2 \, du = -\int (1 - 2u^2 + u^4) \, du = -\left(u - \tfrac{2}{3}u^3 + \tfrac{u^4}{4}\right) + C$$

$$= \boxed{-\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{4}\cos^4 x + C}$$

That wasn't so bad, but note we would be in trouble if the power 5 had been even. Also if the power was at all large this method wouldn't have been adequate. For general integrals of form $\int \cos^n x \, dx$ and $\int \sin^n x \, dx$ we use

REDUCTION FORMULAS

$$\int \sin^n(x) \, dx = -\frac{1}{n} \sin^{n-1}(x) \cos(x) + \frac{n-1}{n} \int \sin^{n-2}(x) \, dx$$

$$\int \cos^n(x) \, dx = \frac{1}{n} \cos^{n-1}(x) \sin(x) + \frac{n-1}{n} \int \cos^{n-2}(x) \, dx$$

(These are derived using integration by parts – see that section)

Ex $\int \sin^2(x) \, dx = -\frac{1}{2} \sin(x) \cos(x) + \frac{1}{2} \int dx = \frac{1}{2} x - \frac{1}{2} \sin(x) \cos(x)$

Ex

$$\int \cos^4 x \, dx = \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \int \cos^2 x \, dx$$

$$= \frac{1}{4} \cos^3 x \sin x + \frac{3}{4}\left(\frac{1}{2}x + \frac{1}{4} \sin 2x\right) + C$$

$$= \frac{1}{4} \cos^3 x \sin x + \frac{3}{8} x + \frac{3}{16} \sin 2x + C$$

Check: $\frac{d}{dx}\left[\text{"}\right] = -\frac{1}{4} 3 \cos^2 x \sin^2 x + \frac{1}{4} \cos^3 x \cos x + \frac{3}{8} + \frac{3}{16} \cos(2x) 2$

$$= -\frac{3}{4} \cos^2 x \sin^2 x + \frac{1}{4} \cos^4 x + \frac{3}{8} + \frac{3}{8} \cos(2x)$$

$$= -\frac{3}{4} \cos^2 x \sin^2 x + \frac{1}{4} \cos^4 x + \frac{3}{8} + \frac{3}{8} \cos^2 x - \frac{3}{8} \sin^2 x$$

$$= -\frac{3}{4} \cos^2 x (1 - \cos^2 x) + \frac{1}{4} \cos^4 x + \frac{3}{8} + \frac{3}{8} \cos^2 x - \frac{3}{8}(1 - \cos^2 x) = \cos^4 x$$

Ex $\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx$

$u = \cos x$
$du = -\sin x \, dx$

$$= \int (1-\cos^2 x)\cos^2 x \sin x \, dx$$

$$= \int (1-u^2)\, u^2 (-1)\, du$$

$$= \int (u^4 - u^2)\, du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \boxed{\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.}$$

Ex $\int \sin^2 x \cos^5 x \, dx = \int \sin^2 x (\cos^2 x)^2 \cos x \, dx$

$u = \sin x$
$du = \cos x \, dx$

$$= \int \sin^2 x (1-\sin^2 x)^2 \cos x \, dx$$

$$= \int u^2 (1-u^2)^2 \, du = \int u^2(1-2u^2+u^4)\, du$$

$$= \int (u^2 - 2u^4 + u^6)\, du = \frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7} + C$$

$$= \boxed{\frac{\sin^3 x}{3} - 2\frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + C}$$

Ex $\int \sin^2 x \cos^2 x \, dx = \int \frac{1}{2}(1-\cos 2x)\frac{1}{2}(1+\cos 2x)\, dx$

$u = 2x$
$du = 2\, dx$

$$= \frac{1}{4}\int (1-\cos^2 2x)\, dx = \frac{1}{4}\int \sin^2 2x \, dx$$

$$= \frac{1}{8}\int \sin^2(2x)\, 2\, dx = \frac{1}{8}\int \sin^2(u)\, du$$

$$= \frac{1}{8}\left(\frac{1}{2}u - \frac{1}{2}\sin(u)\cos(u)\right) + C$$

$$= \frac{1}{8}\left(x - \frac{1}{2}\sin(2x)\cos(2x)\right) + C$$

$$= \frac{x}{8} - \frac{1}{16}\sin(2x)\cos(2x) + C$$

$$\int \tan^3 x \sec^3 x \, dx = \int \tan^2 x \sec^2 x \quad \sec x \tan x \, dx$$

$$= \int (\sec^2(x) - 1)\sec^2 x \, (\sec x \tan x)\, dx$$

$u = \sec x$
$du = \sec x \tan x \, dx$

$$= \int \left(\sec^4 x - \sec^2 x\right)(\sec x \tan x)\, dx$$

$$= \int (u^4 - u^2)\, du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C.$$

$$\int \tan^3 x \sec^4 x \, dx = \int \tan^3 x \sec^2 x \sec^2 x \, dx$$

$u = \tan x$
$du = \sec^2 x \, dx$

$$= \int \tan^3(x)\left(\tan^2(x) + 1\right)\sec^2(x)\, dx$$

$$= \int \left(\tan^5(x) + \tan^3(x)\right)\sec^2(x)\, dx$$

$$= \int (u^5 + u^3)\, du = \frac{u^6}{6} + \frac{u^4}{4} + C$$

What about $\int \tan^2(x) \sec^3(x)\, dx$?
See next page.

$$= \frac{\tan^6 x}{6} + \frac{\tan^4 x}{4} + C$$

$$\int \sin(\pi x)\cos(3\pi x)\, dx = \int \frac{1}{2}\left[\sin(\pi x - 3\pi x) + \sin(\pi x + 3\pi x)\right] dx$$

$$= \frac{1}{2}\int \left(\sin(-2\pi x) + \sin(4\pi x)\right) dx$$

$$= \frac{1}{2}\left(\frac{1}{2\pi}\cos(-2\pi x) - \frac{1}{4\pi}\cos(4\pi x)\right) + C$$

$$= \cancel{\frac{-1}{4\pi}\sin(-2\pi x) + \frac{1}{8\pi}\sin(4\pi x) + C}$$

REDUCTION FORMULAS

$$\boxed{\begin{aligned} \int \tan^n x\,dx &= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx \\ \int \sec^n x\,dx &= \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x\,dx \end{aligned}}$$

Ex $\displaystyle\int \tan^2 x \sec x\,dx$

$$= \int (\sec^2 x - 1) \sec x\,dx$$

$$= \int \sec^3 x\,dx - \int \sec x\,dx$$

$$= \frac{\sec x \tan x}{2} + \frac{1}{2}\int \sec x\,dx - \int \sec x\,dx$$

$$= \frac{\sec x \tan x}{2} + \cancel{\tfrac{1}{2} \sec x \tan x} - \frac{1}{2}\int \sec x\,dx$$

$$= \frac{\sec x \tan x}{2} - \frac{1}{2}\ln|\sec x + \tan x| + C$$

Ex $\displaystyle\int \tan^2(x)\sec^3(x)\,dx = \int (\sec^2(x) - 1)\sec^3(x)\,dx$

$= \displaystyle\int \sec^5(x)\,dx - \int \sec^3(x)\,dx$ = Now use reduction formulas!

= next page

$$= \int \sec^5(x)\,dx - \int \sec^3(x)\,dx$$

$$= \frac{\sec^3(x)\tan(x)}{4} + \frac{5-2}{5-1}\int \sec^3(x)\,dx - \int \sec^3(x)\,dx$$

$$= \frac{\sec^3(x)\tan(x)}{4} - \frac{1}{4}\int \sec^3(x)\,dx$$

$$= \frac{\sec^3(x)\tan(x)}{4} - \frac{1}{4}\left(\frac{\sec(x)\tan(x)}{2} + \frac{3-2}{3-1}\int \sec(x)\,dx\right)$$

$$= \boxed{\frac{\sec^3(x)\tan(x)}{4} - \frac{\sec(x)\tan(x)}{8} - \frac{1}{8}\ln\left|\sec(x)+\tan(x)\right| + C}$$