

## MATH 501, Section 19 Solutions

4. Find all the solutions of  $x^2 + 2x + 4 = 0$  in  $\mathbb{Z}_6$ .

Since  $\mathbb{Z}_6$  has only 6 elements, we can try them all:

$$0^2 + 2 \cdot 0 + 4 = 4$$

$$1^2 + 2 \cdot 1 + 4 = 1$$

$$2^2 + 2 \cdot 2 + 4 = 0$$

$$3^2 + 2 \cdot 3 + 4 = 1$$

$$4^2 + 2 \cdot 4 + 4 = 4$$

$$5^2 + 2 \cdot 5 + 4 = 3$$

Thus there is only one solution,  $x = 2$ .

10. What is the characteristic of  $\mathbb{Z}_6 \times \mathbb{Z}_{15}$ ?

This is the smallest integer  $n$  for which  $n(1, 1) = (n, n) = (0, 0)$ .

So you can see that integer  $n$  must be divisible by both 6 and 15.

The smallest such integer is 30. Thus the characteristic is 30.

14. Notice that  $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ , so  $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$  is a zero divisor in  $M_2(\mathbb{Z})$ .

20. Show that the characteristic of an integral domain  $D$  is either 0 or a prime number.

First, let's rewrite the statement in the form If A then B.

Here is the statement we must prove:

If  $D$  is an integral domain, then its characteristic is either 0 or prime.

Proof (By contradiction):

Suppose that it is not true that the characteristic is either 0 or prime.

Then the characteristic is a positive non-prime number.

Thus the characteristic can be written as a product  $mn$  of two positive integers.

Then  $mn(1) = 0$ , by definition of characteristic. This equation is

$$0 = 1 + 1 + 1 + \dots + 1 \quad (mn \text{ times})$$

$$0 = (1 + 1 + \dots + 1) + (1 + 1 + \dots + 1) + \dots + (1 + 1 + \dots + 1) \quad (m \text{ groups of } n \text{ 1's})$$

$$0 = n1 + n1 + \dots + n1 \quad (m \text{ times})$$

$$0 = n1 \cdot 1 + n1 \cdot 1 + \dots + n1 \cdot 1 \quad (a1 = a)$$

$$0 = (n1)(1 + 1 + \dots + 1) \quad (\text{distributive property})$$

$$0 = (n1)(m1)$$

But neither  $n1$  nor  $m1$  is 0 for each of  $m$  and  $n$  is smaller than the characteristic  $mn$ .

This means  $D$  has zero divisors  $m1$  and  $n1$ , contradicting the fact that  $D$  is an integral domain. This contradiction proves the theorem.