Name: ______________________ **MATH 200 MIDTERM EXAM** MARCH 17, 2022

Directions: Closed book, closed notes, no calculators. Put all phones, etc., away. You will need only a pencil or pen.

1. (10 points) Draw the graph of one function $f(x)$ meeting **all** of the following conditions.

   (a) $\lim\limits_{z \to -3} f(x) = \infty$

   (b) $\lim\limits_{z \to -\infty} f(x) = \infty$

   (c) $\lim\limits_{z \to \infty} f(x) = 2$

   (d) f is continuous on $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$.

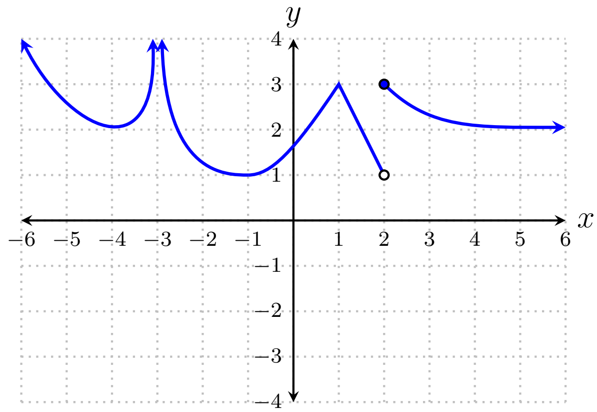
   (e) $f(-1) = 1$

   (f) $f'(-1) = 0$

   (g) $f'(1)$ does not exist

   (h) $\lim\limits_{z \to 2^-} f(x) = 1$

   (i) $\lim\limits_{z \to 2^+} f(x) = 3$



2. (24 points) Find the limits.

   (a) $$\lim_{x \to \sqrt{2}/2} \sin^{-1}(x) = \sin^{-1}\left(\sqrt{2}/2\right) = \left(\begin{array}{c}\text{angle } \theta \text{ for which} \\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \text{ and} \\ \sin(\theta) = \sqrt{2}/2\end{array}\right) = \boxed{\frac{\pi}{4}}$$

   (b) $$\lim_{x \to -\infty} \tan^{-1}(x) = \boxed{-\frac{\pi}{2}}$$

   (c) $$\lim_{z \to 3} \frac{\ln(z) - \ln(3)}{z - 3} = \boxed{\frac{1}{3}}$$

   Because if $f(x) = \ln(x)$, then $f'(x) = \lim\limits_{z \to x} \dfrac{\ln(z) - \ln(x)}{z - x} = \dfrac{1}{x}$,

   and therefore $\lim\limits_{z \to 3} \dfrac{\ln(z) - \ln(3)}{z - 3} = \dfrac{1}{3}$.

   (d) $$\lim_{x \to 3} \frac{1 - \frac{3}{x}}{x - 3} = \lim_{x \to 3} \frac{1 - \frac{3}{x}}{x - 3} \cdot \frac{x}{x} = \lim_{x \to 3} \frac{x - 3}{(x - 3)x} = \lim_{x \to 3} \frac{1}{x} = \boxed{\frac{1}{3}}$$

   (e) $$\lim_{x \to 1} \frac{1 - \frac{3}{x}}{x - 3} = \frac{1 - \frac{3}{1}}{1 - 3} = \frac{-2}{-2} = \boxed{1}$$

   (f) $$\lim_{x \to \infty} \frac{1 - \frac{3}{x}}{x - 3} = \lim_{x \to \infty} \frac{1 - \frac{3}{x}}{x - 3} \cdot \frac{x}{x} = \lim_{x \to \infty} \frac{x - 3}{(x - 3)x} = \lim_{x \to \infty} \frac{1}{x} = \boxed{0}$$

3. (6 points) Use a **limit definition** of the derivative to find the derivative of $f(x) = \dfrac{1}{3x}$.

$$\begin{aligned}
f'(x) &= \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \\
&= \lim_{z \to x} \frac{\frac{1}{3z} - \frac{1}{3x}}{z - x} \\
&= \lim_{z \to x} \frac{\frac{1}{3z} - \frac{1}{3x}}{z - x} \cdot \frac{3zx}{3zx} \\
&= \lim_{z \to x} \frac{x - z}{(z - x)3zx} \\
&= \lim_{z \to x} \frac{-1}{3zx} \\
&= \frac{-1}{3x^2}
\end{aligned}$$

$\boxed{\text{Therefore } f'(x) = \dfrac{-1}{3x^2}}$

4. (6 points) Find all x for which the tangent to the graph of $y = \dfrac{x^3}{3} - \dfrac{3x^2}{2} + 2x + 1$ has slope 20.

We need to solve the following equation.

$$\begin{aligned}
y' &= 20 \\
x^2 - 3x + 2 &= 20 \\
x^2 - 3x - 18 &= 0 \\
(x + 3)(x - 6) &= 0
\end{aligned}$$

$\boxed{\text{Thus the slope equals 20 at } x = -3 \text{ and } x = 6.}$

5. (6 points) Suppose it costs $C(x)$ dollars to build a transmitting tower that is x meters high. Suppose it happens that $C'(100) = 1000$. Explain in simple terms what this means.

$C'(x)$ is the rate of change in (dollars per meter) of the cost of building the tower x meters high.

The statement $C'(100) = 1000$ means that when the tower is 100 meters high (i,e., when x=100), the cost is changing at a rate of \$1000 per meter. At this rate it will cost an extra \$1000 to build the tower one additional meter higher.

6. (35 points) Find the derivatives of these functions. You do **not** need to simplify your answers.

(a) $f(x) = 3x^2 + e^3$ .... $\boxed{f'(x) = 6x}$

(b) $f(x) = \dfrac{4}{\sqrt{x}} = \dfrac{4}{x^{1/2}} = 4x^{-1/2}$ .... $f'(x) = 4\left(-\dfrac{1}{2}x^{-1/2-1}\right) = -2x^{-3/2} = -\dfrac{2}{x^{3/2}} = \boxed{-\dfrac{2}{\sqrt{x^3}}}$

(c) $f(x) = \tan\left(\dfrac{1}{x^2+1}\right)$ .... $f(x) = \sec^2\left(\dfrac{1}{x^2+1}\right) D_x\left[\dfrac{1}{x^2+1}\right]$

$$= \sec^2\left(\frac{1}{x^2+1}\right) \frac{0\cdot(x^2+1)^2 - 1\cdot(2x+0)}{(x^2+1)^2}$$

$$= \boxed{-\sec^2\left(\frac{1}{x^2+1}\right) \frac{2x}{(x^2+1)^2}}$$

(d) $f(x) = 3x^4\cos(x)$ .... $f'(x) = 12x^3\cos(x) + 3x^4(-\sin(x)) = \boxed{12x^3\cos(x) - 3x^4\sin(x)}$

(e) $f(x) = \left(\tan^{-1}(x)\right)^4$ .... $f'(x) = 4\left(\tan^{-1}(x)\right)^3 D_x\left[\tan^{-1}(x)\right]$

$$= 4\left(\tan^{-1}(x)\right)^3 \frac{1}{1+x^2}$$

$$= \boxed{\frac{4\left(\tan^{-1}(x)\right)^3}{1+x^2}}$$

(f) $f(x) = \dfrac{6x+1}{x^3+4x+9}$ .... $f'(x) = \dfrac{6(x^3+4x+9) - (6x+1)(3x^2+4))}{(x^3+4x+9)^2}$

$$= \frac{6x^3 + 24x + 54 - 18x^3 - 24x - 3x^2 - 4}{(x^3+4x+9)^2}$$

$$= \boxed{\frac{50 - 12x^3 - 3x^2}{(x^3+4x+9)^2}}$$

(g) $y = \sec\left(\ln\left(x^3+x\right)\right)$ .... $y' = \sec\left(\ln\left(x^3+x\right)\right)\tan\left(\ln\left(x^3+x\right)\right) D_x\left[\ln\left(x^3+x\right)\right]$

$$= \boxed{\sec\left(\ln\left(x^3+x\right)\right)\tan\left(\ln\left(x^3+x\right)\right)\cdot\frac{3x^2+1}{x^3+x}}$$

$$= \boxed{\frac{\sec\left(\ln\left(x^3+x\right)\right)\tan\left(\ln\left(x^3+x\right)\right)(3x^2+1)}{x^3+x}}$$

7. (7 points) Given the equation $x\ln(y)+x^2=5y$, find y'.

$$\begin{aligned}
x\ln(y)+x^2 &= 5y\\
D_x\Big[x\ln(y)+x^2\Big] &= D_x\Big[5y\Big]\\
\ln(y)+x\frac{y'}{y}+2x &= 5y'\\
x\frac{y'}{y}-5y' &= -2x-\ln(y)\\
y'\left(\frac{x}{y}-5\right) &= -2x-\ln(y)\\
y' &= \frac{2x+\ln(y)}{5-\dfrac{x}{y}}
\end{aligned}$$

8. (6 points) A spherical balloon is inflated at a rate of 100π cubic feet per minute. How fast is the radius increasing at the instant that the radius is 5 feet?

Let V be the balloon's volume and let r be its radius.

Know: $\dfrac{dV}{dt}=100\pi$ cubic feet per minute.

Want: $\dfrac{dr}{dt}$ at the instant $r=5$.

$$\begin{aligned}
V &= \frac{4}{3}\pi r^3\\
D_t\Big[V\Big] &= D_t\left[\frac{4}{3}\pi r^3\right]\\
\frac{dV}{dt} &= \frac{4}{3}3\pi r^2\frac{dr}{dt}\\
\frac{dV}{dt} &= 4\pi r^2\frac{dr}{dt}\\
100\pi &= 4\pi r^2\frac{dr}{dt}\\
\frac{100\pi}{4\pi r^2} &= \frac{dr}{dt}\\
\frac{dr}{dt} &= \frac{25}{r^2}
\end{aligned}$$

Answer: When $r=5$ the radius is changing at a rate of $\left.\dfrac{dr}{dt}\right|_{r=5}=\dfrac{25}{5^2}=\boxed{\text{1 foot per minute}}$

(A sphere of radius r has volume $V=\frac{4}{3}\pi r^3$ cubic units, and surface area $S=4\pi r^2$ square units.)