Complementary Composition of Graphs

Ghidewon Abay-Asmerom
Richard H. Hammack
Dewey T. Taylor

Department of Mathematics and Applied Mathematics
Virginia Commonwealth University
Richmond, VA 23284-2014, USA

ghidewon@vcu.edu
rhammack@vcu.edu
dttaylor2@vcu.edu

Abstract. This paper introduces a construction that generalizes the composition (or lexicographic product) of two graphs. We investigate some graph invariants for these graphs, including total distance, independence number, chromatic number and domination number.

Key words: composition of graphs, lexicographic product of graphs, complementary product of graphs

1 Introduction

The *Cartesian product* of two simple graphs G and H is the graph $G \times H$ whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose edges are all pairs $(g,h)(g',h')$ for which either $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$.

Recently Haynes, Hennings, Slater and van der Merwe [2] introduced a generalization of the Cartesian product, which they call the *complementary product.* Given subsets $R \subseteq V(G)$ and $S \subseteq V(H)$, they define the complementary product $G(R) \times H(S)$ to be the graph whose vertex set is $V(G) \times V(H)$ and for which $(g,h)(g',h')$ is an edge precisely if one of the following conditions holds. (Here $\overline{G}$ denotes the complement of a graph G.)

1. $g = g' \in R$ and $hh' \in E(H)$

2. $g = g' \notin R$ and $hh' \in E(\overline{H})$
3. $h = h' \in S$ and $gg' \in E(G)$
4. $h = h' \notin S$ and $gg' \in E(\overline{G})$

This extends the definition of the Cartesian product in the sense that if $R = V(G)$ and $S = V(H)$, then $G(R) \times H(S) = G \times H$.

Haynes et al. investigate a number of properties of the complementary product. However, to simplify the discussion they restrict their attention to complementary products $G(R) \times H(S)$ for which $R = V(G)$, $H = K_2$ and $|S| = 1$. In this case they denote the complementary product as $G\overline{G}$, and call it the *complementary prism*. They compute a number of graph invariants for complementary prisms, including diameter, total distance, independence number, clique number, chromatic number, packing number and domination number.

Given their work, it is natural to ask whether there exist constructions that extend other graph products in a manner analogous to how the complementary product extends the Cartesian product. Besides the Cartesian product, there are three other primary graph products; the strong product, the direct product, and composition. We have found no consistent way of generalizing the strong and direct products, however, the ideas of [2] apply quite naturally to composition. We will call the resulting generalization the *complementary composition* of graphs.

The remainder of this paper is organized as follows: we review the standard definition of graph composition, and then introduce the definition of complementary composition and a construction analogous to the complementary prism. (However, our definition will be somewhat more general in that we will replace H with the complete bipartite graph $K_{a,b}$ rather than K_2.) We then compute graph invariants parallel to the results in [2].

The composition of simple graphs H and G is the graph $H \circ G$ whose vertex set is the Cartesian product $V(H) \times V(G)$ and whose edges are $(h, g)(h', g')$ where either $h = h'$ and $gg' \in E(G)$, or $hh' \in E(H)$. Composition also goes by various names such as substitution and lexicographic product [5]. Figure 1 shows an example.

One can view the composition of H and G as follows: replace each vertex h of H by a copy G_h of G, and then join every vertex in G_h to every vertex in $G_{h'}$ as long as hh' is an edge of H. In this sense, $H \circ G$ can be viewed as formed by "substituting" G into H, and is therefore analogous to composition of two functions. Indeed, like composition of functions, graph composition is associative but not necessarily commutative.

Composition was first introduced in 1959 by Frank Harary [1] who showed

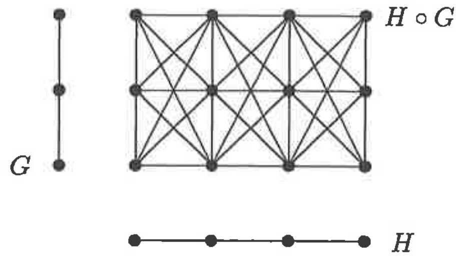

Figure 1: Composition of two graphs

that the automorphism group of $H \circ G$ is the composition of the automorphism groups of H and G. Additional early work was done by Sabidussi [4], and since then composition has been investigated extensively. See [3] for a survey of this product.

We now introduce our primary construction. It is entirely analogous to the complementary product of [2], but based on composition rather than the Cartesian product.

Let G and H be simple graphs and let $R \subseteq V(G)$ and $S \subseteq V(H)$. The *complementary composition* of H and G is the graph $H(S) \circ G(R)$ whose vertex set is the Cartesian product $V(H) \times V(G)$ and whose edges are pairs $(h, g)(h', g')$ for which one of the following conditions holds:

1. $h = h' \in S$ and $gg' \in E(G)$
2. $h = h' \notin S$ and $gg' \in E(\overline{G})$
3. $hh' \in E(H)$, and $g \in R$ or $g' \in R$

Figure 2 shows an example, where vertices in R and S are darkened. As in the composition of graphs, the construction of the complementary composition can also be viewed as follows: first every vertex $h \in S$ is replaced by a copy G_h of G and every vertex $h \in V(H) - S$ is replaced by a copy $\overline{G_h}$ of $\overline{G}$. Next every vertex (h, g) of G_h or $\overline{G}_h$ is joined to all vertices (h', g') so long as $g \in S$ and $hh' \in E(H)$.

2 The Complementary Prism

We now turn our focus to the complementary composition, $H(S) \circ G(R)$, where $H(S)$ is the complete bipartite graph $K_{a,b}(S)$, S is the partite set with a vertices and $R = V(G)$. We call this graph the *complementary prism*. To simplify notation a little, we denote $K_{a,b}(S) \circ G(R)$ by $K_{a,b}(S) \circ G$.

Thus,

$$
\begin{aligned}
\operatorname{dist}(K_{a,b}(S)\circ G) &= \frac{1}{2}\sum_{w}\operatorname{dist}(w) \\
&= \frac{1}{2}\left[\sum_{h\in S, g\in V(G)}\operatorname{dist}((h,g)) + \sum_{h\notin S, g\in V(G)}\operatorname{dist}((h,g))\right] \\
&= \frac{a}{2}\left[\sum_{g\in V(G)}(n(2a+b)-deg_G(g)-2)\right] \\
&\quad +\frac{b}{2}\left[\sum_{g\in V(G)}(n(a+2b)-deg_{\overline{G}}(g)-2)\right] \\
&= \frac{a}{2}\left[n^2(2a+b)-2q-2n\right] \\
&\quad +\frac{b}{2}\left[n^2(a+2b)-2(\binom{n}{2}-q)-2n\right] \\
&= n^2(a^2+ab+b^2)-(a+b)n+(b-a)q-b\binom{n}{2}
\end{aligned}
$$

■

It turns out that the complementary prism $K_{a,b}(S)\circ G$ is always connected. This easily follows from the next result on the connectivity of complementary products in general.

Proposition 4 *Let H and G be two nontrivial graphs with $S \subseteq V(H)$, and $R \subseteq V(G)$. Then $H(S)\circ G(R)$ is connected if and only if $R \neq \emptyset$ and H is connected.*

Proof. Suppose first that $R = \emptyset$. By the of complementary composition, for any pair of vertices h and h' of H, there will be no edges running between vertices of G_h and $G_{h'}$. Thus $H(S)\circ G(R)$ has at least as many components as there are vertices in H. Since H is nontrivial, $H(S)\circ G(R)$ has at least two components and thus $H(S)\circ G(R)$ is disconnected. Similarly, if H is disconnected and h and h' are vertices of H in different components of H, then G_h and $G_{h'}$ are in two different components of $H(S)\circ G(R)$.

Conversely, suppose that $R \neq \emptyset$ and H is connected. Let (h,g) and (h',g') be vertices of $H(S)\circ G(R)$. Since there is an $h-h'$ path in H, it is not hard to find a $(h,g)-(h',g')$ path in $H(S)\circ G(R)$. Hence $H(S)\circ G(R)$ is connected. ■

Corollary 2 *Let G be any graph. Then $K_{a,b}(S)\circ G$ is always connected.*

Recall that the *independence number* of a graph G is the integer $\alpha(G)$ which equals the cardinality of the largest set of pairwise nonadjacent vertices in G.

Proposition 5 $\alpha(K_{a,b}(S) \circ G) = \max\{a\alpha(G), b\alpha(\overline{G})\}$.

Proof. Let S and S' be the partite sets of $K_{a,b}$ with a and b elements respectively. Let I_G and $I_{\overline{G}}$ denote maximally independent sets in G and $\overline{G}$ respectively. In addition, let $I_S = \{(h,g)|h \in S \text{ and } g \in I_G\}$ and $I_{S'} = \{(h,g)|h \in S' \text{ and } g \in I_{\overline{G}}\}$. First we show that I_S and $I_{S'}$ are independent sets in $K_{a,b}(S) \circ G$. Let (h,g) and (h',g') be two distinct vertices in I_S. Suppose that $h \neq h'$. Since the vertices of S are independent, (h,g) and (h',g') must be independent. Now suppose that $h = h'$. Since g and g' are in I_G, an independent set, (h,g) and (h',g') are not adjacent. Thus I_S is an independent set and similarly $I_{S'}$ is also an independent set. Since $|I_S| = a\alpha(G)$ and $|I_{S'}| = b\alpha(\overline{G})$, we have $\alpha(K_{a,b}(S) \circ G) \geq \max\{a\alpha(G), b\alpha(\overline{G})\}$.

In the other direction, suppose I is an independent set in $K_{a,b}(S) \circ G$. Then either $I \subseteq \{(h,g)|h \in S\}$ or $I \subseteq \{(h,g)|h \in S'\}$, since (h,g) is adjacent to (h',g') whenever $hh' \in E(K_{a,b})$ and this is always the case when $h \in S$ and $h' \in S'$. Moreover it follows from the definitions that if $I \subseteq \{(h,g)|h \in S\}$, then $\{g|(h,g) \in I\}$ is independent in G, so $|I| \leq a\alpha(G)$. Likewise, if $I \subseteq \{(h,g)|h \in S'\}$, then $\{g|(h,g) \in I\}$ is independent in $\overline{G}$, so $|I| \leq b\alpha(\overline{G})$. Then $\alpha(K_{a,b}(S) \circ G) \leq |I| \leq \max\{a\alpha(G), b\alpha(\overline{G})\}$, which completes the proof. ■

We now turn to the chromatic number of a complementary prism. Let $\chi(G)$ stand for the chromatic number of a graph G.

Proposition 6 $\chi(K_{a,b}(S) \circ G) = \chi(G) + \chi(\overline{G})$.

Proof. Observe that each copy G_h can be minimally colored using $\chi(G)$ colors and each copy $\overline{G}_h$ can be minimally colored using $\chi(\overline{G})$ colors. Since the subgraphs G_h and $G_{h'}$ are pairwise non-adjacent whenever h and h' are distinct vertices in S, it follows that we can color the set $\{(h,g)|h \in S, g \in V(G)\}$ with $\chi(G)$ colors. Similarly we can color $\{(h,g)|h \in S', g \in V(G)\}$ with $\chi(\overline{G})$ colors, and it follows $K_{a,b}(S) \circ G$ can be colored with $\chi(G)+\chi(\overline{G})$ colors. Thus $\chi(K_{a,b}(S) \circ G) \leq \chi(G) + \chi(\overline{G})$.

On the other hand, consider an arbitrary coloring of $K_{a,b}(S) \circ G$. If $h \in S$ and $h' \in S'$, then each vertex in G_h is adjacent to every vertex in $\overline{G}_{h'}$. Thus, no vertex color in G_h can be used to color vertices of $\overline{G}_{h'}$. Also, the vertices $\{(h,g)|h \in S\}$ must be colored with no fewer than $\chi(G)$ colors, for otherwise some fiber G_h gets colored with fewer than $\chi(G)$ colors. Similarly $\{(h,g)|h \in S'\}$ must be colored with no fewer than $\chi(\overline{G})$ colors. Thus our coloring uses no fewer than $\chi(G) + \chi(\overline{G})$ colors, so $\chi(K_{a,b}(S) \circ G) \geq \chi(G) + \chi(\overline{G})$. ■

Recall that the *domination number* of G is the integer $\gamma(G)$ that equals the cardinality of the smallest set $X \subseteq G$ for which every vertex of G is either in X or adjacent to an element of X.

Proposition 7 $\gamma(K_{a,b}(S) \circ G) = \begin{cases} 1 & \textit{if } \gamma(G) = 1 \textit{ and } a = 1 \\ 1 & \textit{if } \gamma(\overline{G}) = 1 \textit{ and } b = 1 \\ 2 & \textit{otherwise.} \end{cases}$

Proof. Suppose that $\gamma(G) = 1$ and that $a = 1$. Let g be the vertex of G that is adjacent to every other vertex of G. Let h be the one vertex in S. Then the vertex (h, g) is adjacent to every vertex in $K_{a,b}(S) \circ G$ and $\gamma(K_{a,b}(S) \circ G) = 1$. Similar arguments hold for the case $\gamma(\overline{G}) = 1$ and $b = 1$.

Now, suppose $\gamma(G) > 1$ or $a > 1$, and $\gamma(\overline{G}) > 1$ or $b > 1$. In this case no single vertex dominates all other vertices. Choose vertices (h, g) and (h', g'), where h and h' are in different partite sets. Then it is easy to see that these two vertices dominate all vertices of $K_{a,b}(S) \circ G$. Thus $\gamma(K_{a,b}(S) \circ G) = 2$. ■

References

[1] F. Harary, *On the Group of the Composition of Two Graphs*, Duke Math J. 26 (1959), 29-34.

[2] T. W. Haynes, M. A. Henning, P. J. Slater and L. C. van der Merwe *The Complementary Product of Two Graphs*, Bulletin of the ICA 51 (2007), 21-30.

[3] W. Imrich and S. Klavžar, *Product Graphs: Structure and recognition*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley and Sons, (2000).

[4] G. Sabidussi, *The Composition of Graphs*, Duke Math J. 26 (1959), 693-696.

[5] G. Sabidussi, *The Lexicographic Product of Graphs*, Duke Math J. 28 (1961), 573-578.