MATH 501, Section 26 Solutions

4. Give addition and multiplication tables for $2\mathbb{Z}/8\mathbb{Z}$. Is this group isomorphic to $\mathbb{Z}_4$?

$+$	$0+2\mathbb{Z}$	$2+2\mathbb{Z}$	$4+2\mathbb{Z}$	$6+2\mathbb{Z}$
$0+2\mathbb{Z}$	$0+2\mathbb{Z}$	$2+2\mathbb{Z}$	$4+2\mathbb{Z}$	$6+2\mathbb{Z}$
$2+2\mathbb{Z}$	$2+2\mathbb{Z}$	$4+2\mathbb{Z}$	$6+2\mathbb{Z}$	$0+2\mathbb{Z}$
$4+2\mathbb{Z}$	$4+2\mathbb{Z}$	$6+2\mathbb{Z}$	$0+2\mathbb{Z}$	$2+2\mathbb{Z}$
$6+2\mathbb{Z}$	$6+2\mathbb{Z}$	$0+2\mathbb{Z}$	$2+2\mathbb{Z}$	$4+2\mathbb{Z}$

$\cdot$	$0+2\mathbb{Z}$	$2+2\mathbb{Z}$	$4+2\mathbb{Z}$	$6+2\mathbb{Z}$
$0+2\mathbb{Z}$	$0+2\mathbb{Z}$	$0+2\mathbb{Z}$	$0+2\mathbb{Z}$	$0+2\mathbb{Z}$
$2+2\mathbb{Z}$	$0+2\mathbb{Z}$	$4+2\mathbb{Z}$	$0+2\mathbb{Z}$	$4+2\mathbb{Z}$
$4+2\mathbb{Z}$	$0+2\mathbb{Z}$	$0+2\mathbb{Z}$	$0+2\mathbb{Z}$	$0+2\mathbb{Z}$
$6+2\mathbb{Z}$	$0+2\mathbb{Z}$	$4+2\mathbb{Z}$	$0+2\mathbb{Z}$	$4+2\mathbb{Z}$

This ring is not isomorphic to $\mathbb{Z}_4$. One way to see this is to observe that $\mathbb{Z}_4$ contains a multiplicative identity 1 satisfying $1 \cdot a = a$ for all $a \in \mathbb{Z}_4$. But the multiplication table above shows $2\mathbb{Z}/8\mathbb{Z}$ has no such identity. $\boxed{\text{Thus } 2\mathbb{Z}/8\mathbb{Z} \ncong \mathbb{Z}_4.}$

14. Give an example to show that a factor ring of a ring with zero divisors may be an integral domain.

Consider $\mathbb{Z}_4 = \{0, 1, 2, 3\}$, which has a zero divisor 2. Notice that $I = \{0, 2\}$ is an ideal in $\mathbb{Z}_4$, for you can immediately check $ax \in I$ for all $a \in \mathbb{Z}_4$ and $x \in I$. Then $\mathbb{Z}_4/I = \{0 + I, 1 + I\}$ has only one nonzero entry $1 + I$, and $(1 + I)(1 + I) = 1 + I \neq 0 + I$. Thus $\mathbb{Z}_4/I$ has no zero divisors and is hence an integral domain.

18. Show that a homomorphism from a field to a ring is either one-to-one or maps everything to 0.

Proof. Suppose $\varphi : F \to R$ is a homomorphism from a field F to a ring R.
Now, either $\varphi(1) = 0$ or $\varphi(1) \neq 0$. Let's look at these cases separately.

If $\varphi(1) = 0$, then for any $x \in F$ we have $\varphi(x) = \varphi(x \cdot 1) = \varphi(x) \cdot \varphi(1) = \varphi(x) \cdot 0 = 0$, so φ sends everything to 0.

If $\varphi(1) \neq 0$, then for any nonzero $x \in F$ we have $\varphi(x)\varphi(x^{-1}) = \varphi(xx^{-1}) = \varphi(1) \neq 0$, hence $\varphi(x)\varphi(x^{-1}) \neq 0$, whence $\varphi(x) \neq 0$. Thus φ sends no nonzero elements to 0, which means $\ker(\varphi) = \{0\}$, so φ is one-to-one. (Corollary 26.6)

26. Suppose R is a commutative ring and $a \in R$. Show $I_a = \{x \in R \mid ax = 0\}$ is an ideal in R.

Proof. We must show that I_a is an additive subgroup of R and $bI_a \subseteq I_a$ and $I_a b \subseteq I_a$ for all $b \in R$.

To see that I_a is an additive subgroup of R, observe the following.

(a) I_a is closed under addition: If $x, y \in I_a$ then $ax = 0$ and $yx = 0$. Then $0 = ax + ay = a(x + y)$. But $a(x + y) = 0$ means $x + y \in I_n$.

(b) The additive identity 0 is in I_a because $a0 = 0$.

(c) If $x \in I_n$, then $ax = 0$, hence $a(-x) = -(ax) = -0 = 0$, meaning $-x \in I_a$.

To see that $bI_a \subseteq I_a$, let $y \in bI_a$, so $y = bx$ where $x \in I_a$ (which means $ax = 0$).
Then $ay = a(bx) = (ab)x = (ba)x = b(ax) = b0 = 0$, and therefore $y \in I_a$, meaning $bI_a \subseteq I_a$.

To see that $I_a b \subseteq I_a$, let $y \in I_a b$, so $y = xb$ where $x \in I_a$ (which means $ax = 0$).
Then $ay = a(xb) = (ax)b = 0b = 0$, and therefore $y \in I_a$, meaning $I_a b \subseteq I_a$.

Therefore I_a is an ideal of R.