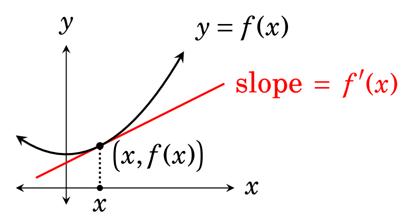
CHAPTER 18

Notation, Examples, Differentiability

Let's pause for a moment to reflect on what we have done in Part 3. We defined the important idea of the **derivative** of a function $f(x)$, which is another function $f'(x)$, for which

$$\begin{aligned} f'(x) &= \lim_{z \to x} \frac{f(z)-f(x)}{z-x} \\ &= \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \\ &= \left(\begin{array}{l} \text{Slope of tangent to} \\ \text{graph of } y = f(x) \\ \text{at point } (x, f(x)) \end{array} \right). \end{aligned}$$



We remarked that another notation for the derivative of $f(x)$ is $D_x\big[f(x)\big]$, and we developed five rules for computing derivatives of certain functions without resorting to limits. (Here c is a constant.)

Constant Rule:	$D_x\Big[c\Big] = 0$
Identity Rule:	$D_x\Big[x\Big] = 1$
Power Rule:	$D_x\Big[x^n\Big] = nx^{n-1}$
Sum-Difference Rule:	$D_x\Big[f(x) \pm g(x)\Big] = f'(x) \pm g'(x)$
Constant Multiple Rule:	$D_x\Big[cf(x)\Big] = cf'(x)$

More rules will come. But in this chapter we pause our quest for rules to discuss some important issues regarding derivatives. We will begin by describing some of the many different notations for derivatives. Then we will look at some instructive examples of procedures that will be especially important later in the course. Finally we will examine what is called *differentiability*, a property that a function may possess that is related to whether (or where) its derivative exists.

18.1 Notation

There are a great many different notations for a function's derivative. For example, take a function $y = f(x)$. In addition to the notation $f'(x) = D_x\big[f(x)\big]$ we may write $f'(x) = y'$. Often the symbol $\frac{d}{dx}$ is used in place of D_x, so $D_x\big[f(x)\big] = \frac{d}{dx}\big[f(x)\big]$. Variants of this include $\frac{d}{dx}\big[f(x)\big] = \frac{df}{dx} = \frac{dy}{dx}$.

What we are saying is that given a function $y = f(x)$, its derivative can be denoted in the following ways:

$$f'(x) \;=\; y' \;=\; D_x\big[f(x)\big] \;=\; \frac{d}{dx}\Big[f(x)\Big] \;=\; \frac{df}{dx} \;=\; \frac{dy}{dx}.$$

Be attentive to variables. If the function is $w = g(z)$, then its derivative is

$$g'(z) \;=\; w' \;=\; D_z\big[g(z)\big] \;=\; \frac{d}{dz}\Big[g(z)\Big] \;=\; \frac{dg}{dz} \;=\; \frac{dw}{dz}.$$

If the function is $y = h(\theta)$, then its derivative is denoted

$$h'(\theta) \;=\; y' \;=\; D_\theta\big[h(\theta)\big] \;=\; \frac{d}{d\theta}\Big[h(\theta)\Big] \;=\; \frac{dh}{d\theta} \;=\; \frac{dy}{d\theta}.$$

For example, the derivative of the function $y = x^3$ can be denoted in any of the following ways:

$$y' = 3x^2, \qquad D_x\big[x^3\big] = 3x^2, \qquad \frac{d}{dz}\Big[x^3\Big] = 3x^2, \qquad \frac{dy}{dx} = 3x^2.$$

The notation $f'(x) = \frac{dy}{dx}$ is very common and (as we'll see often in this course) convenient. Figure 18.1 suggests its origin. We will return to the idea expressed in this picture several times in this book, as the significance of the notation emerges.

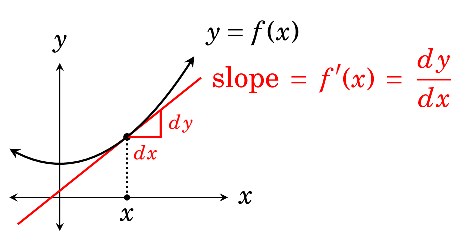


Figure 18.1. The rational for the notation $f'(x) = \frac{dy}{dx}$. The tangent line to $y = f(x)$ at $\big(x, f(x)\big)$ has slope $f'(x)$. Draw a right triangle whose hypotenuse is tangent to $y = f(x)$ at $\big(x, f(x)\big)$. Call its side lengths dx and dy, as shown. The quotient $\frac{dy}{dx}$ is rise over run for the tangent, so its slope is $f'(x) = \frac{dy}{dx}$.

The $\frac{dy}{dx}$ or y' notation comes in handy when we are working with a function but haven't designated it with a letter like f or g. For example, say we are working with the function $y = x^3 + x$, and need to compute its derivative. It would be wrong (or at least inconsistent) to say that the derivative is $f'(x) = 3x^2 + 1$, because the function wasn't called f. But it makes perfect sense to refer to its derivative as $\frac{dy}{dx} = 3x^2 + 1$ or $y' = 3x^2 + 1$.

But the notations $\frac{dy}{dx}$ and y' do have one unsettling feature. In $f'(x)$ there is an x for the input value, but this is lacking in $\frac{dy}{dx}$ and y'. Say $y = f(x) = x^3 + x$, so $f'(x) = \frac{dy}{dx} = y' = 3x^2 + 1$. Then we can, for instance, plug in 2 to $f'(x)$ to get

$$f'(2) = 3 \cdot 2^2 + 1 = 13.$$

But where would we plug in the 2 to $\frac{dy}{dx}$ or y'? For this the convention is to write $\left.\frac{dy}{dx}\right|_{x=2}$ or $y'\big|_{x=2}$, each of which means the same thing as $f'(2)$. Thus

$$\left.\frac{dy}{dx}\right|_{x=2} = 3 \cdot 2^2 + 1 = 13.$$

Now that we've discussed notation, let's look at some examples. They will illustrate not just our new notation, but also some common procedures that you'll find yourself carrying out often later in the course.

18.2 Some Instructive Examples

One particularly useful procedure in applications is to find the values of x for which the tangent line to the graph $y = f(x)$ of a function at $\big(x, f(x)\big)$ has slope 0. Our first example examines this.

Example 18.1 Find all x for which the tangent to $y = 2x^3 - 3x^2 - 36x$ at $\big(x, f(x)\big)$ is horizontal.

Solution: The tangent line's slope at $\big(x, f(x)\big)$ equals $\frac{dy}{dx} = 6x^2 - 6x - 36$. We want to find the x for which the tangent line at $\big(x, f(x)\big)$ is horizontal, that is, has slope 0. This happens provided that $\frac{dy}{dx} = 0$, or $6x^2 - 6x - 36 = 0$.

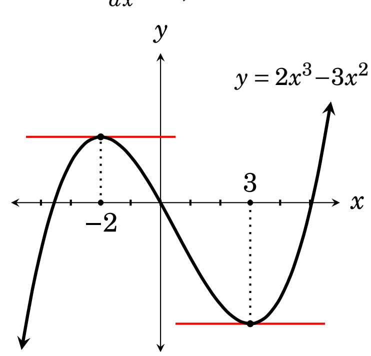
So we can find x by solving

$$6x^2 - 6x - 36 = 0.$$

Dividing by 6 and factoring gives:

$$\begin{aligned} x^2 - x - 6 &= 0 \\ (x+3)(x-2) &= 0 \end{aligned}$$

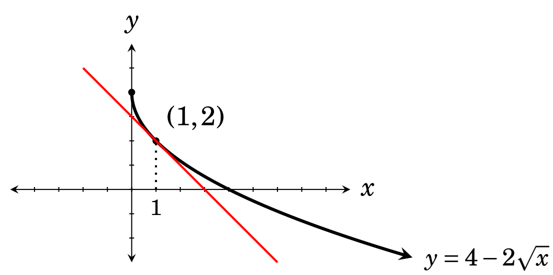
Thus the tangent line has slope 0 (is horizontal) at $x = -3$ and $x = 2$, This is supported by the graph. ✍



There will be occasions in this course where you will need to find the *equation* of a tangent line, not just its slope. Let's look at such an example.

Example 18.2 Find the equation of the tangent line to $y = 4 - 2\sqrt{x}$ at the point $(1,2)$.

Solution: Though it's not necessary for the solution of this problem, let's start by drawing a graph. We can get a quick sketch of the graph of $y = 4-2\sqrt{x}$ by graph shifting. It is the familiar graph $y = \sqrt{x}$ reflected across the x-axis, scaled vertically by a factor of 2, and moved up 4 units, as shown below.



Notice that the point $(1,2)$ really is on the graph of $y = 4-2\sqrt{x}$, because when $x = 1$, we have $y = 4-2\sqrt{1} = 2$. The tangent line at $(1,2)$ is sketched in. We need to find its equation $y = mx+b$.

This requires finding its slope m. Since the derivative gives slope, we need to compute the derivative of $y = 4-2\sqrt{x}$. Write this as $y = 4-2x^{1/2}$ so that we are ready to use the power rule. The derivative of $y = 4-2x^{1/2}$ is

$$\frac{dy}{dx} = 0-2\cdot\frac{1}{2}x^{1/2-1} = -x^{-1/2} = -\frac{1}{\sqrt{x}}.$$

Thus the slope of the tangent to $y = 4-2\sqrt{x}$ at any x is $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$. We are interested in the tangent at x=1, so it's slope is

$$m = \left.\frac{dy}{dx}\right|_{x=1} = -\frac{1}{\sqrt{1}} = -1.$$

So the tangent line in question has slope $m = -1$ and it passes through the point $(1,2)$. Using the point-slope formula for the equation of a line, we get

$$\begin{aligned} y - y_0 &= m(x-x_0) \\ y-2 &= -1(x-1) \\ y &= -x+3 \end{aligned}$$

Answer: The tangent line to $y = 4-2\sqrt{x}$ at $(1,2)$ has equation $y = -x+3$. ✍

Example 18.3 Let $f(x) = 3x^2 + 200$ and $g(x) = 2x^3 - 72x$. Find all x for which the slope of the tangent to $y = f(x)$ at $(x, f(x))$ equals the slope of the tangent to $y = g(x)$ at $(x, g(x))$.

Solution: The slope of the tangent to $y = f(x)$ at $(x, f(x))$ is $f'(x)$. The slope of the tangent to $y = g(x)$ at $(x, g(x))$ is $g'(x)$. Thus we seek all x for which

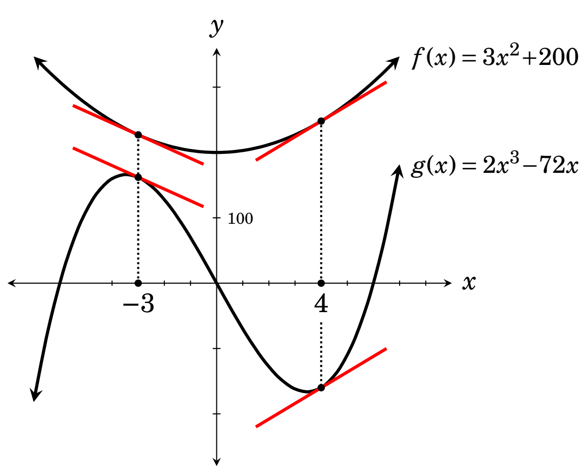
$$\begin{aligned} f'(x) &= g'(x) \\ 6x + 0 &= 6x^2 - 72. \end{aligned}$$

So we need to solve this equation for x. This is a quadratic, so let's try getting 0 to one side and factoring.

$$\begin{aligned} 0 &= 6x^2 - 6x - 72 \\ 0 &= x^2 - x - 12 && \text{(divide both sides by 6)} \\ 0 &= (x+3)(x-4) \end{aligned}$$

The solutions are $x = -3$ and $x=4$. These are the values for which $f'(x) = g'(x)$.

Answer: The tangents to functions $f(x) = 3x^2 + 200$ and $g(x) = 2x^3 - 72x$ have equal slopes at $x = -3$ and $x = 4$.

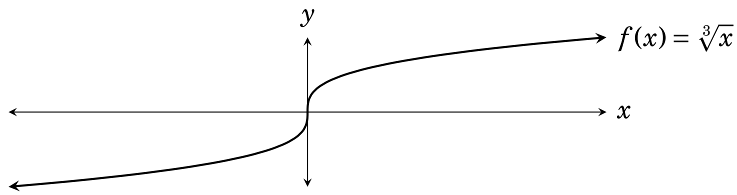


The above graphs offer visual evidence that our answer is correct. ✍

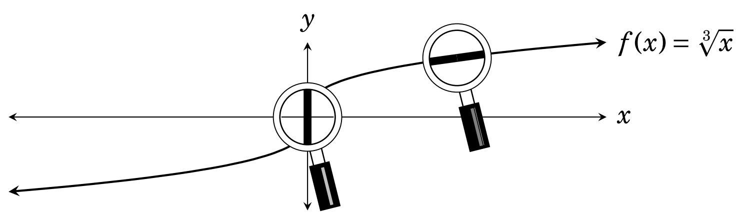
Problems like the ones above will be a continual theme in this course. The functions will become more complex and varied, and deeper meanings and applications will emerge. For now it is good to get some practice by working the exercises.

18.3 Differentiability

Consider the function $f(x) = \sqrt[3]{x}$, graphed below. Apparently, every tangent line to its graph has positive slope, with one possible exception. The tangent at the origin $(0,0)$ appears to be vertical, with *undefined* slope.



Looking at the graph close up with a powerful magnifying glass would reveal that it looks like a line with positive slope everywhere except at $(0,0)$. There it looks like a vertical line, as shown below.



Based on this evidence, we would expect that the derivative $f'(x)$ (which gives slope) is positive for all values of x except for $x = 0$. We would expect $f'(0)$ to be undefined, because a vertical line has undefined slope. Indeed, by the power rule, the derivative of $f(x) = \sqrt[3]{x} = x^{1/3}$ is

$$f'(x) = \frac{1}{3}x^{1/3-1} \;=\; \frac{1}{3}x^{-2/3} \;=\; \frac{1}{3x^{2/3}} \;=\; \frac{1}{3\sqrt[3]{x}^2}.$$

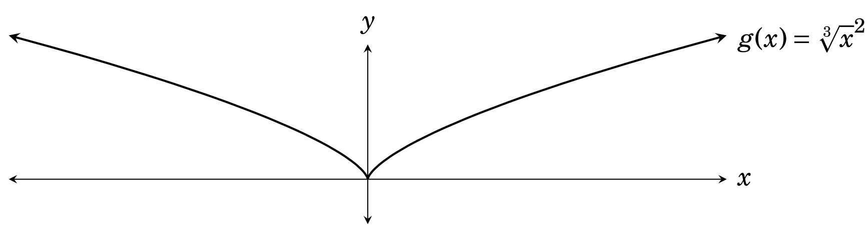
As expected, $f'(x)$ is positive for any nonzero x because the squared $\sqrt[3]{x}^2$ in the denominator is positive. And as expected, $f'(0) = \frac{1}{3\sqrt[3]{0}^2} = \frac{1}{0}$ is undefined.

So here is a function f whose domain is all real numbers $(-\infty,\infty)$, but the domain of its derivative f' is $(-\infty,0)\cup(0,\infty)$. That is, $f(0)$ is defined, but $f'(0)$ is not defined. In such a situation we say f is **differentiable** at any $x \neq 0$, but that is is **not differentiable** at 0. Here is the exact definition.

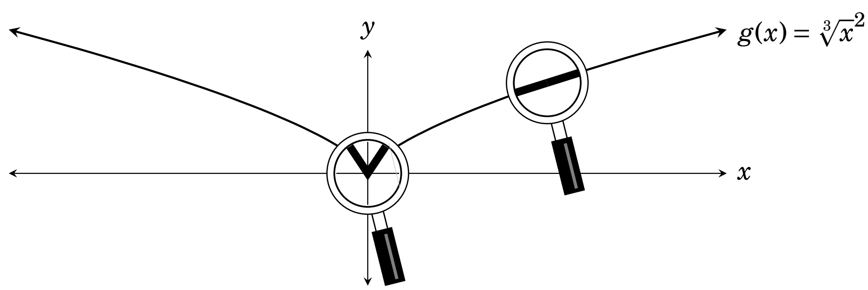
> **Definition 18.1** We say function f is **differentiable** at a number a if $f'(a)$ is defined. Otherwise f is **not differentiable** at a.
>
> Equivalently, f is differentiable at a number a if $f'(a) = \lim\limits_{z\to a} \frac{f(z)-f(a)}{z-a}$ exists, and it is not differentiable if this limit DNE.

If a function is differentiable at all numbers in an interval, then we say that it is **differentiable on the interval**. Thus the function $f(x) = \sqrt[3]{x}$ from the previous page is differentiable on the intervals $(-\infty, 0)$ and $(0, \infty)$. That is, it is differentiable on $(-\infty, 0) \cup (0, \infty)$.

So we have seen that if the graph of f has a vertical tangent line at $x = a$, then f is not differentiable at a. There is another way that a function might not be differentiable. To understand it consider the function $g(x) = \sqrt[3]{x}^2$, which is the square of $f(x) = \sqrt[3]{x}$. Its graph is somewhat similar to the graph of $y = \sqrt[3]{x}$ except that all y values are squared and thus become positive. Note that the graph of g has a sharp corner or "cusp" at $(0,0)$.



Taking a powerful magnifying glass and looking at the graph close-up at $(0,0)$ would reveal that the graph does not look like a straight line there at all. It looks like a bent line. Here the problem is not that the tangent line is vertical, but that there is no tangent line at all!

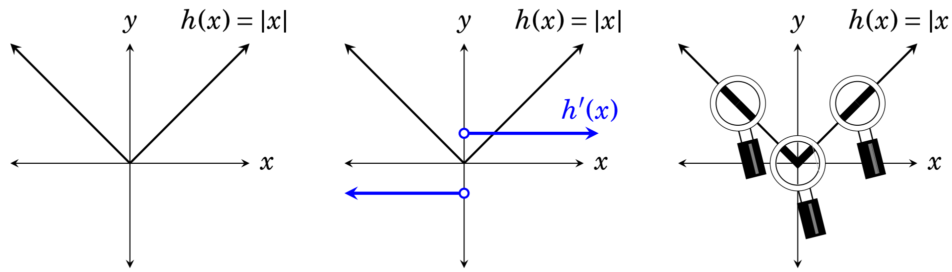


So something strange is going to happen with $g'(0)$. Let's find out. By the power rule, the derivative of $g(x) = \sqrt[3]{x}^2 = x^{2/3}$ is

$$g'(x) = \frac{2}{3}x^{2/3-1} = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}} = \frac{2}{3\sqrt[3]{x}}.$$

Indeed, $g'(0) = \frac{2}{3\sqrt[3]{0}} = \frac{1}{0}$ is not defined. But $g'(x)$ *is* defined for $x \neq 0$. Therefore this function g is not differentiable at $x = 0$, but it *is* differentiable at any other x. Thus g is differentiable on $(-\infty, 0) \cup (0, \infty)$.

Our next example is another familiar function that has a cusp, the absolute value function $h(x) = |x|$, graphed below, left. (If you are unsure of why the graph looks this way, see the discussion on page 19.) The graph is a line with slope -1 meeting a line with slope 1 at a right angle at the origin.



Therefore (because the derivative gives slope) we have $h'(x) = -1$ for $x < 0$, and $h'(x) = 1$ for $x > 0$. Thus $|x|$ is differentiable on $(-\infty, 0) \cup (0, \infty)$. The derivative h' is graphed above (middle).

But is $|x|$ differentiable at 0? That is, is $h'(0)$ defined? By definition, $h'(0) = \lim\limits_{z \to 0} \frac{|z|-|0|}{z-0}$, so we are asking whether this limit exists. Exercise 23 asks you to do the left- and right-hand limits. You will find that $\lim\limits_{z \to 0^-} \frac{|z|-|0|}{z-0} = -1$ and $\lim\limits_{z \to 0^+} \frac{|z|-|0|}{z-0} = 1$. These two limits don't agree, so $h'(0) = \lim\limits_{z \to 0} \frac{|z|-|0|}{z-0}$ DNE. Thus the function $h(x) = |x|$ is **not differentiable at 0**.

One important fact is that differentiability implies continuity, as the theorem below asserts. This makes sense intuitively: If f is differentiable at a, then its graph looks line a line there, so it's not going to "jump" at a.

Theorem 18.1 If f is differentiable at $x = a$, then f is continuous at $x = a$.

Proof. Suppose f is differentiable at a. Then $f'(a) = \lim\limits_{z \to a} \frac{f(z)-f(a)}{z-a}$, and this limit exists. We must show that f is continuous at $x = a$. So by definition of continuity (Definition 11.1), we must show $\lim\limits_{z \to a} f(z) = f(a)$. Notice that

$$f(z) \;=\; \frac{f(z)-f(a)}{z-a}(z-a)+f(a).$$

Taking limits of both sides and using limit laws,

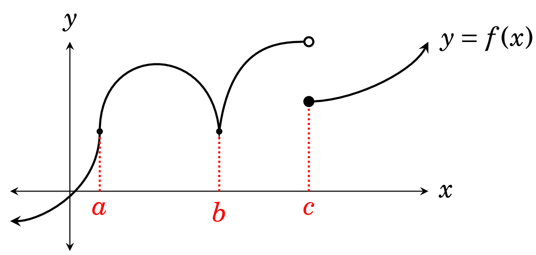
$$\begin{aligned}\lim_{z \to a} f(z) \;&=\; \left(\lim_{z \to a} \frac{f(z)-f(a)}{z-a}\right) \cdot \left(\lim_{z \to a}(z-a)\right) + \lim_{z \to a} f(a)\\ &=\; f'(a) \cdot 0 + f(a) = f(a).\end{aligned}$$

Thus $\lim\limits_{z \to a} f(z) = f(a)$, which means f is continuous at a. ■

Theorem 18.1 says that if f is differentiable at $x=a$, then f is continuous at $x=a$. In particular, this means that if f is **not** continuous at $x=a$, then it is **not** differentiable at $x=a$. This makes sense because if f is discontinuous at $x=a$, then its graph—close up at a—certainly does not look like a line.

In summary, we have seen three ways that a function can fail to be differentiable: because the tangent is vertical, because there is no tangent and because the function is discontinuous.

For example, the function graphed below is not differentiable at $x=a$ because its tangent is vertical there. It is not differentiable at $x=b$ because there is no tangent there. It is not differentiable at $x=c$ because it is not continuous there. We would say that this function is differentiable on the intervals $(-\infty,a)$, (a,b), (b,c) and (c,∞).



Naturally, in a calculus course we want our functions to be differentiable. But many functions that have derivatives will not be differentiable at certain points, as we have seen. Later—particularly in Part 4 of this text—we will see that the points at which a function is *not* differentiable can reveal valuable information in certain circumstances.

Exercises for Chapter 18

These exercises are *cumulative*, covering all material from chapters 16 to 18.

In Exercises 1–6, find all x for which the tangent to the given function at $(x, f(x))$ is horizontal.

1. $f(x) = 2x^3 - 3x^2 - 12x + 4$

2. $f(x) = 2\sqrt{x} - x$

3. $y = x^3 - 4x^2 + 5x$

4. $y = x^3 + 3x^2 + x + 1$

5. $y = \dfrac{1}{x} + x$

6. $y = \dfrac{1}{x^2} + x^2$

7. Find all x for which the tangent to the graph of $y = \dfrac{x^3}{3} + \dfrac{3x^2}{2} - 2x + 1$ has slope 8.

8. Find all x for which the tangent to the graph of $y = \dfrac{x^3}{3} + \dfrac{3x^2}{2} - 2x + 1$ has slope 7.

In Exercises 9–12, two functions f and g are given. Find all x for which the tangent to $y = f(x)$ at $(x, f(x))$ is parallel to the tangent to $y = g(x)$ at $(x, g(x))$.

9. $y = x^2 + 2x^3$ and $y = x^2 - 2x^3 + 48x$

10. $f(x) = x^3 - 3x$ and $g(x) = 3x^2 + 6x$

11. $y = x^2$ and $y = x^3$

12. $y = x^2$ and $y = \sqrt{x}$

In Exercises 13–16, find the equation of the line tangent to the graph of the function at the given point.

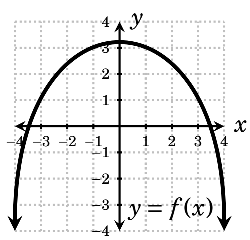
13. $f(x) = x^2 - 3x + 4$ at $(3, 4)$

14. $f(x) = \sqrt{x}$ at $(9, 3)$

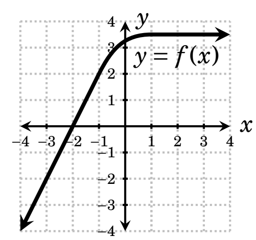
15. $f(x) = \dfrac{1}{x} + x$ at $\big(2, f(2)\big)$

16. $f(x) = \dfrac{1}{\sqrt{x}}$ at $\big(9, f(9)\big)$

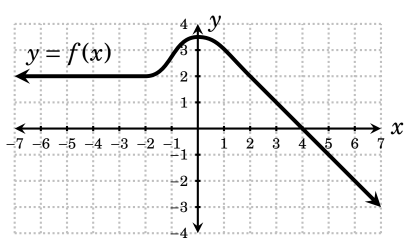
17. For the function graphed below, find $\lim\limits_{h\to 0} \frac{f(3+h)-f(3)}{h}$ and $\lim\limits_{h\to 0} \frac{f(h)-f(0)}{h}$.



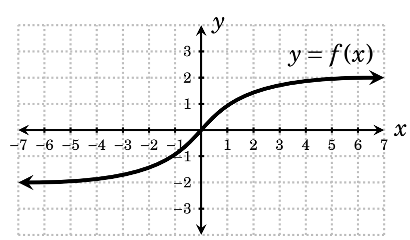
18. For the function graphed below, find $\lim\limits_{h\to 0} \frac{f(-3+h)-f(-3)}{h}$ and $\lim\limits_{h\to 0} \frac{f(2+h)-f(2)}{h}$.



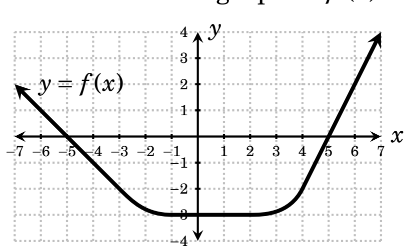
19. A function $f(x)$ is graphed. below. Sketch the graph of $f'(x)$.



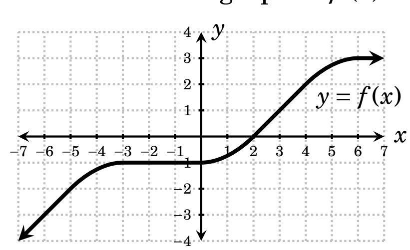
20. A function $f(x)$ is graphed below. Sketch of the graph of $f'(x)$.



21. A function $f(x)$ is graphed below. Sketch the graph of $f'(x)$.



22. A function $f(x)$ is graphed below. Sketch the graph of $f'(x)$.



23. Consider the absolute value function $h(x) = |x|$. We have not developed a rule for its derivative, but we can find one with the limit definition of the derivative. For $x < 0$, use the limit definition to show that $h'(x) = -1$. For $x > 0$, use the limit definition to show that $h'(x) = 1$. Conclude that

$$h'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Then investigate differentiability at $x = 0$. Conclude that $f'(0)$ is not defined because in its limit definition $h'(0) = \lim_{z \to 0} \frac{|z|-|0|}{z-0}$, the left- and right-hand limits are unequal, so the limit DNE.

Exercise Solutions for Chapter 18

1. Suppose $f(x) = 2x^3 - 3x^2 - 12x + 4$. Find all x for which the tangent to $y = f(x)$ at the point $(x, f(x))$ is horizontal.

The tangent line at $(x, f(x))$ has slope $f'(x)$, so we need to solve $f'(x) = 0$.

$$\begin{aligned} 6x^2 - 6x - 12 &= 0 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \end{aligned}$$

Therefore the tangent line is horizontal (has slope 0) for $x = 2$ and $x = -1$.

3. Find all x for which the tangent to $y = x^3 - 4x^2 + 5x$ at $(x, f(x))$ is horizontal.

The answer will be the solutions of $y' = 0$, which is $3x^2 - 8x + 5 = 0$. This factors as $(x-1)(3x-5) = 0$. Therefore the tangent is horizontal at the two values $x = 1$ and $x = 5/3$.

5. Find all x for which the tangent to $y = \dfrac{1}{x} + x$ at the point $(x, f(x))$ is horizontal.

The answer will be the solutions of $y' = 0$, that is, $-\dfrac{1}{x^2} + 1 = 0$, or $1 = \dfrac{1}{x^2}$. Cross-multiplying yields $x^2 = 1$, so there are two solutions $x = \pm 1$.

7. Find all x for which the tangent to the graph of $y = \dfrac{x^3}{3} + \dfrac{3x^2}{2} + -2x + 1$ has slope 8.

The answer will be the solutions of the equation $y' = 8$, that is,

$$\begin{aligned} x^2 + 3x - 2 &= 8 \\ x^2 + 3x - 10 &= 0 \\ (x+5)(x-2) &= 0 \end{aligned}$$

So the tangent has slope 8 at the points $x = -5$ and $x = 2$.

9. Suppose $f(x) = x^2 + 2x^3$ and $g(x) = x^2 - 2x^3 + 48x$. Find all x for which the tangent to $y = f(x)$ at $(x, f(x))$ is parallel to the tangent to $y = g(x)$ at $(x, g(x))$.

The tangent lines being parallel means they have the same slope, so for this we need to solve $f'(x) = g'(x)$.

$$\begin{aligned} 2x + 6x^2 &= 2x - 6x^2 + 48 \\ 12x^2 &= 48 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

Thus the tangent lines have the same slope for $x = 2$ and $x = -2$.

11. Find the values of x at which $y = x^2$ and $y = x^3$ have the same slope.

Call these two functions $f(x) = x^2$ and $g(x) = x^3$. The answer to the question will be the solutions of the equation $f'(x) = g'(x)$, which is $2x = 3x^2$, or $2x - 3x^2 = 0$. This factors as $x(2 - 3x) = 0$, so the solutions are $x = 0$ and $x = 2/3$.

Thus the functions have the same slope at $x = 0$ and $x = 2/3$.

13. Find the equation of the line tangent to $f(x) = x^2 - 3x + 4$ at the point $(3, 4)$.

The slope of the tangent line to $f(x)$ at $(x, f(x))$ is $f'(x) = 2x - 3$. Thus the tangent line at $(3, 4)$ has slope $m = f'(3) = 2 \cdot 3 - 3 = 3$. By the point-slope formula for a line, the equation of the tangent line is

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 4 &= 3(x - 3) \\ y &= 3x - 5. \qquad \longleftarrow \text{(answer)} \end{aligned}$$

15. Find the equation of the line tangent to $f(x) = \dfrac{1}{x} + x$ at $\big(2, f(2)\big)$.

The line goes through $\big(2, f(2)\big) = (2, 1/2 + 2) = (2, 5/2)$. The slope at x is $f'(x) = \dfrac{-1}{x^2} + 1$, so the slope at $x = 2$ is $f'(2) = \dfrac{-1}{2^2} + 1 = \dfrac{3}{4}$. By the point-slope formula, tangent is

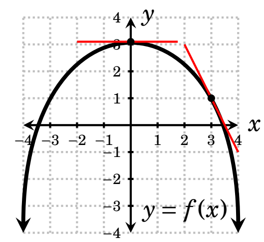
$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - \frac{5}{2} &= \frac{3}{4}(x - 2) \\ y &= \frac{3}{4}x + 1. \qquad \longleftarrow \text{(answer)} \end{aligned}$$

17. For the function graphed below, find $\lim\limits_{h\to 0}\frac{f(3+h)-f(3)}{h}$ and $\lim\limits_{h\to 0}\frac{f(h)-f(0)}{h}$.

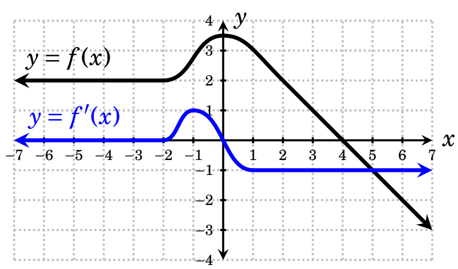
Recall that $\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$.

Thus $\lim\limits_{h\to 0}\frac{f(3+h)-f(3)}{h} = f'(3) = -2$ because the tangent line to $f(x)$ at $x=3$ (sketched in) has slope -2.

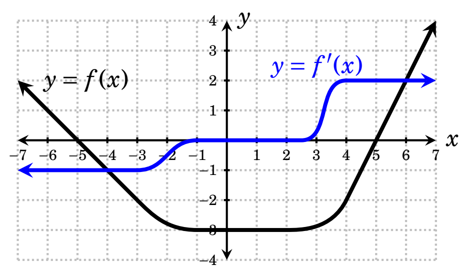
Also $\lim\limits_{h\to 0}\frac{f(h)-f(0)}{h} = \lim\limits_{h\to 0}\frac{f(0+h)-f(0)}{h} = f'(0) = 0$ because the tangent line to $f(x)$ at $x=0$ (sketched in) has slope 0.



19. A function $f(x)$ is graphed below. Sketch the graph of $f'(x)$.



21. A function $f(x)$ is graphed below. Sketch the graph of $f'(x)$.



23. Find the derivative of the absolute value function $h(x) = |x|$ and investigate its differentiability at $x=0$.

We use the limit definition $h'(x) = \lim\limits_{z\to x}\frac{h(z)-h(x)}{z-x} = \lim\limits_{z\to x}\frac{|z|-|x|}{z-x}$.

First suppose $x<0$. In the limit, z approaches the negative number x, so we can assume that z is negative too. Then $|x| = -x$ and $|z| = -z$. In this case $h'(x) = \lim\limits_{z\to x}\frac{|z|-|x|}{z-x} = \lim\limits_{z\to x}\frac{-z-(-x)}{z-x} = \lim\limits_{z\to x}\frac{-(z-x)}{z-x} = \lim\limits_{z\to x} -1 = -1$.

Next suppose $x>0$. In the limit, z approaches the positive number x, so we can assume that z is positive too. Then $|x| = x$ and $|z| = z$. In this case $h'(x) = \lim\limits_{z\to x}\frac{|z|-|x|}{z-x} = \lim\limits_{z\to x}\frac{z-x}{z-x} = \lim\limits_{z\to x} 1 = 1$.

We have now shown that $h'(x) = \begin{cases} -1 & \text{if } x<0 \\ 1 & \text{if } x>0. \end{cases}$

Next we investigate differentiability at $x=0$. Note that $h'(0) = \lim\limits_{z\to 0}\frac{|z|-|0|}{z-0} = \lim\limits_{z\to 0}\frac{|z|}{z}$, provided the limit exists. Let's examine this limit from the left and from the right. From the left, $\lim\limits_{z\to 0^-}\frac{|z|}{z} = \lim\limits_{z\to 0}\frac{-z}{z} = -1$. From the right, $\lim\limits_{z\to 0^+}\frac{|z|}{z} = \lim\limits_{z\to 0}\frac{z}{z} = 1$. Since the limits do not agree, it follows that $h'(0) = \lim\limits_{z\to 0}\frac{|z|}{z}$ DNE. Thus $h'(0)$ is not defined, and hence $|x|$ is not differentiable at $x=0$.