

MATH 501, Section 10 Solutions

2. Find the cosets of the subgroup $4\mathbb{Z}$ of $2\mathbb{Z}$.

There are two cosets:

$$4\mathbb{Z} = \{\dots, -8, -4, 0, 4, 8, 12, 16, \dots\} = \{4k | k \in \mathbb{Z}\}$$

$$2 + 4\mathbb{Z} = \{\dots, -6, -2, 2, 6, 10, 14, 18, \dots\} = \{2 + 4k | k \in \mathbb{Z}\}$$

4. Find all cosets of the subgroup $\langle 4 \rangle = \{0, 4, 8\}$ of \mathbb{Z}_{12} .

$$\langle 4 \rangle = \{0, 4, 8\}$$

$$1 + \langle 4 \rangle = \{1, 5, 9\}$$

$$2 + \langle 4 \rangle = \{2, 6, 10\}$$

$$3 + \langle 4 \rangle = \{3, 7, 11\}$$

12. Find the index of the subgroup $\langle 3 \rangle = \{0, 3, 6, 9, 12, 15, 18, 21\}$ in \mathbb{Z}_{24} .

$$\text{The index is } \frac{|\mathbb{Z}_{24}|}{|\langle 3 \rangle|} = \frac{24}{8} = 3.$$

16. Consider $\mu = (1, 2, 4, 5)(3, 6) \in S_6$. Find the index of the subgroup $\langle \mu \rangle$.

$$\text{Note } \langle \mu \rangle = \{(1, 2, 4, 5)(3, 6), (1, 4)(2, 5), (1, 5, 2, 1)(3, 6), ()\}$$

$$\text{The index is } \frac{|S_6|}{|\langle \mu \rangle|} = \frac{6!}{4} = 180.$$

29. Suppose $H \leq G$ has the property that $g^{-1}hg \in H$ for all $h \in H$ and $g \in G$. Show that $gH = Hg$ for every element $g \in G$.

Proof. Assume g is an arbitrary but fixed element of G .

First we show $gH \subseteq Hg$. Suppose $x \in gH$. By definition of gH , this means $x = gh$ for some $h \in H$. Then $x = gh = (g^{-1})^{-1}h = (g^{-1})^{-1}hg^{-1}g = [(g^{-1})^{-1}hg^{-1}]g$. The property that H is assumed to have gives $(g^{-1})^{-1}hg^{-1} \in H$, so $x = h'g$ for $h' = (g^{-1})^{-1}hg^{-1} \in H$. Consequently $x \in Hg$, so $gH \subseteq Hg$.

Next we show $Hg \subseteq gH$. Suppose $x \in Hg$. By definition of Hg , this means $x = hg$ for some $h \in H$. Then $x = hg = g(g^{-1}hg)$. By assumption, $g^{-1}hg \in H$, so $x = gh'$ for $h' = g^{-1}hg \in H$. This means $x \in gH$, so $Hg \subseteq gH$.

Since $gH \subseteq Hg$ and $Hg \subseteq gH$, it follows that $gH = Hg$.