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Zero Divisors among Digraphs

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Abstract A digraph C is called a zero divisor if there exist non-isomorphic digraphs A and B for which $A \times C \cong B \times C$, where the operation is the direct product. In other words, C being a zero divisor means that cancellation property $A \times C \cong B \times C \Rightarrow A \cong B$ fails. Lovász proved that C is a zero divisor if and only if it admits a homomorphism into a disjoint union of directed cycles of prime lengths.

Thus any digraph C that is homomorphically equivalent to a directed cycle (or path) is a zero divisor. Given such a zero divisor C and an arbitrary digraph A, we present a method of computing all solutions X to the digraph equation $A \times C \cong X \times C$.

This work extends and generalizes some earlier results by R. Hammack and K. Toman [Cancellation of direct products of digraphs, *Discusiones Mathematicae Graph Theory*, **31**, 2011, in press].

Keywords digraphs · direct product of digraphs · cancellation

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1 Introduction

The article [1] solves the following variation of the cancellation problem for the direct product of graphs: Given graphs A and C, find all graphs B for which $A \times C \cong B \times C$.

The analogous problem where A, B and C are digraphs presents some special challenges, and a complete solution is not yet realized. The article [2] solves the problem for those digraphs C which are homomorphically equivalent to a single arc $\overrightarrow{P_2}$. (Such C are of special interest because they are the most "pathological" of all zero divisors, in a sense that will be explained in Section 3 below.)

The current article solves the problem for a more general class of digraphs C, namely those that are homomorphically equivalent to directed cycles or paths of arbitrary lengths. Specifically, given a digraph A and a digraph C which is homomorphically equivalent to a directed path or cycle, we classify those digraphs B for which $A \times C \cong B \times C$.

We first fix the notation by recalling some relevant concepts. A *digraph* A is a binary relation $E(A)$ on a finite vertex set $V(A)$, that is, a subset $E(A) \subseteq V(A) \times V(A)$. For brevity, an ordered pair $(a, a') \in E(A)$ is denoted aa', and is visualized as an arrow pointing from a to a'. Elements of $E(A)$ are called *arcs*. A reflexive arc aa is called a *loop*. A *graph* is a digraph that is symmetric (as a relation). We use the usual notation for graphs; in particular K_n is the complete graph on n vertices.

Given a positive integer n, the *directed cycle* $\overrightarrow{C_n}$ is the digraph with vertices $\{0, 1, 2, \ldots, n-1\}$ and arcs $\{01, 12, 23, \ldots, (n-1)0\}$. Thus $\overrightarrow{C_1}$ consists of a single vertex with a loop, and $\overrightarrow{C_2} = K_2$. The *directed path* $\overrightarrow{P_n}$ is $\overrightarrow{C_n}$ with one arc removed. Figure 1 shows some of these digraphs.

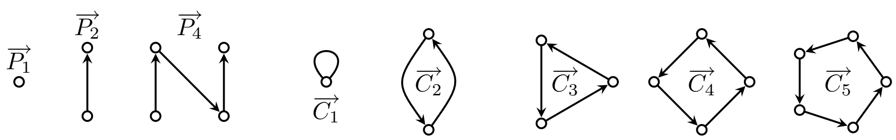

Fig. 1 Some digraphs

We denote the condition of X being a sub-digraph of A as $X \subseteq A$. A digraph A is *strongly connected* if for every pair a, a' of its vertices there is a sub-digraph $\overrightarrow{P_n} \subseteq A$ beginning at a and ending at a'. A digraph is *connected* if any a and a' are joined by a path, each arc of which has arbitrary orientation. The *connected components* (respectively *strongly connected components*) of A are the maximal sub-digraphs of A that are connected (respectively strongly connected).

If A and B are digraphs, then $A+B$ denotes the disjoint union of A and B. The disjoint union of n copies of A is denoted nA. A *homomorphism* $\varphi : A \to B$ is a map $\varphi : V(A) \to V(B)$ for which $aa' \in E(A)$ implies $\varphi(a)\varphi(a') \in E(B)$.

Digraphs A and B are *homomorphically equivalent* if there are homomorphisms $A \to B$ and $B \to A$.

The *direct product* of two digraphs A and B is the digraph $A \times B$ whose vertex set is the Cartesian product $V(A) \times V(B)$ and whose arcs are the pairs $(a, b)(a', b')$ with $aa' \in E(A)$ and $bb' \in E(B)$. We assume the reader to be familiar with direct products and homomorphisms. For standard references see [4] and [3].

2 Cancellation Laws

Lovász [5] defines a digraph C to be a *zero divisor* if there exist non-isomorphic digraphs A and B for which $A \times C \cong B \times C$. For example, Figure 2 shows that $\overrightarrow{C_3}$ is a zero divisor: If $A = \overrightarrow{C_3}$ and $B = 3\overrightarrow{C_1}$, then clearly $A \ncong B$, yet $A \times \overrightarrow{C_3} \cong B \times \overrightarrow{C_3}$. (Both products are isomorphic to three copies of $\overrightarrow{C_3}$.) Here is the main result concerning zero divisors.

Theorem 1 (Lovász [5], Theorem 8) *A digraph C is a zero divisor if and only if there is a homomorphism $\varphi : C \to \overrightarrow{C_{p_1}} + \overrightarrow{C_{p_2}} + \overrightarrow{C_{p_3}} + \cdots + \overrightarrow{C_{p_k}}$ for prime numbers $p_1, p_2, \ldots, p_k$.*

Thus, in particular, $\overrightarrow{C_n}$ with $n > 1$ is a zero divisor. (Even if n is not prime, there is an $\frac{n}{p}$-fold homomorphic cover $\varphi : \overrightarrow{C_n} \to \overrightarrow{C_p}$ for any prime divisor p of n.) Also each $\overrightarrow{P_n}$ is a zero divisor, for clearly there is a homomorphism $\overrightarrow{P_n} \to \overrightarrow{C_p}$ for any n and p.

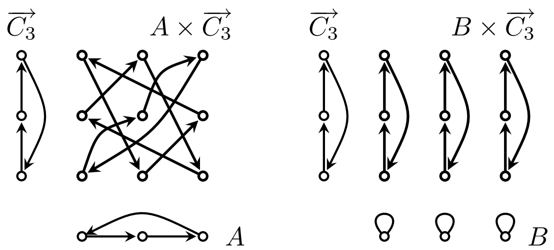

Fig. 2 Example of a zero divisor

Theorem 1 can be regarded as cancellation law for the direct product, as it gives exact conditions on C under which $A \times C \cong B \times C$ necessarily implies $A \cong B$. By contrast, the present article focuses on ways that cancellation can fail. Given a digraph A and a natural number n, we will describe a method of finding all digraphs B for which $A \times \overrightarrow{P_n} \cong B \times \overrightarrow{P_n}$, as well as all digraphs B for which $A \times \overrightarrow{C_n} \cong B \times \overrightarrow{C_n}$. Further, given a digraph C that is homomorphically equivalent to $\overrightarrow{P_n}$ or $\overrightarrow{C_n}$, we describe how to find all B for which $A \times C \cong B \times C$.

Theorem 1 characterizes zero divisors as those digraphs C which admit a homomorphism $C \to \overrightarrow{C_{p_1}} + \overrightarrow{C_{p_2}} + \cdots + \overrightarrow{C_{p_k}}$. If C is connected, such a homomorphism has an image in just one directed cycle, so it can be regarded as a homomorphism $C \to \overrightarrow{C_p}$. Often there are only finitely many p for which homomorphisms $C \to \overrightarrow{C_p}$ exist. But for some C it may happen that there is a homomorphism $C \to \overrightarrow{C_p}$ for each prime number p. Then, by taking $p > |V(C)|$, we see that C admits a homomorphism $C \to \overrightarrow{P_n}$ for some n. Conversely, since there are homomorphisms $\overrightarrow{P_n} \to \overrightarrow{C_p}$ for any n and p, the existence of a homomorphism $C \to \overrightarrow{P_n}$ guarantees a homomorphism $C \to \overrightarrow{C_p}$ for every p. Therefore connected zero divisors C can be divided into two distinct and mutually exclusive types: On one hand there are those that admit a homomorphism $C \to \overrightarrow{P_n}$ for some n (and thus a homomorphism $C \to \overrightarrow{C_p}$ for all p); on the other hand there are those that admit homomorphisms $C \to \overrightarrow{C_p}$ for only finitely many prime numbers p.

This suggests that the expressions $A \times \overrightarrow{P_n} \cong B \times \overrightarrow{P_n}$ and $A \times \overrightarrow{C_n} \cong B \times \overrightarrow{C_n}$ are of fundamental importance in the study of zero divisors, and motivates the results of the present article.

Our methods will require the following theorems due to Lovász.

Theorem 2 (Lovász [5], Theorem 6) *Let A, B, C and D be digraphs. If $A \times C \cong B \times C$ and there is a homomorphism from D to C, then $A \times D \cong B \times D$.*

Theorem 3 (Lovász [5], Theorem 7) *Let A, B, C be digraphs. If $A \times C \cong B \times C$, then there is an isomorphism from $A \times C$ to $B \times C$ of the form $(a, c) \mapsto (\beta(a, c), c)$.*

3 Permuted Digraphs

Given a digraph A, we denote the set of permutations of $V(A)$ as $\mathrm{Perm}(V(A))$. The next definition is central to the remainder of this paper. For a permutation $\alpha \in \mathrm{Perm}(V(A))$, we define the *permuted digraph* A^α as follows.

Definition 1 Given a digraph A and $\alpha \in \mathrm{Perm}(V(A))$, the *permuted digraph* A^α has vertices $V(A^\alpha) = V(A)$. Its arcs are $E(A^\alpha) = \{a\alpha(a') : aa' \in E(A)\}$. Thus $aa' \in E(A)$ if and only if $a\alpha(a') \in E(A^\alpha)$, and $aa' \in E(A^\alpha)$ if and only if $a\alpha^{-1}(a') \in E(A)$.

Figure 3 shows several examples. In the upper part of the figure, the cyclic permutation (0124) of the vertices of $\overrightarrow{C_6}$ yields a permuted graph $\overrightarrow{C_6}^{(0124)} = 2\overrightarrow{C_3}$. The permuted digraph $\overrightarrow{C_6}^{(23)}$ is also shown. The lower part of the figure shows a digraph A and two of its permuted digraphs. For another example, note that $A^{\mathrm{id}} = A$ for any digraph A. We remark that it may be possible that $A^\alpha \cong A$ for some non-identity permutation α. For instance, $\overrightarrow{C_6}^{(024)} \cong \overrightarrow{C_6}$.

The following fundamental result about permuted digraphs was proved in [2]. We omit its proof here because it will be a consequence of our more general Theorem 4 below.

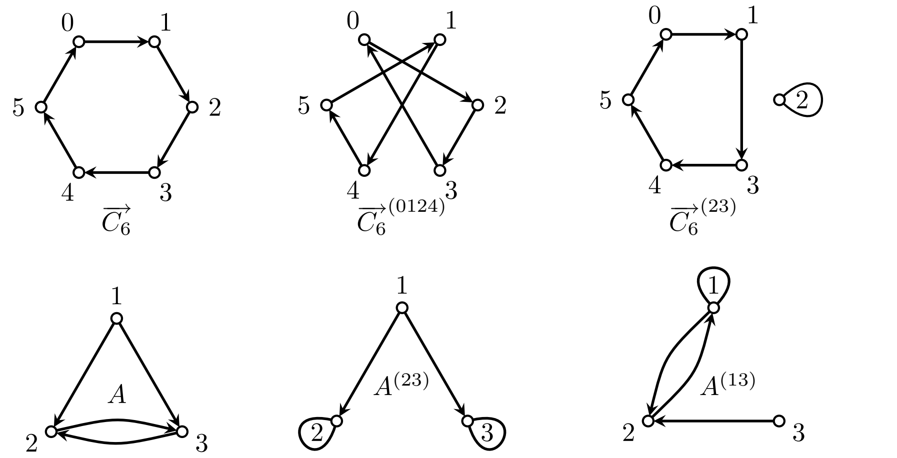

Fig. 3 Examples of permuted digraphs

Proposition 1 *If A and B are digraphs, then $A \times \overrightarrow{P_2} \cong B \times \overrightarrow{P_2}$ if and only if $B \cong A^\alpha$ for some $\alpha \in \mathrm{Perm}(V(A))$.*

This yields a corollary that describes a relationship that must hold between A and B whenever $A \times C \cong B \times C$.

Corollary 1 *Suppose A, B and C are digraphs and C has at least one arc. If $A \times C \cong B \times C$, then $B \cong A^\alpha$ for some $\alpha \in \mathrm{Perm}(V(A))$.*

Proof Suppose $A \times C \cong B \times C$. Since C has at least one arc, there is a homomorphism $\overrightarrow{P_2} \to C$. Theorem 2 implies $A \times \overrightarrow{P_2} \cong B \times \overrightarrow{P_2}$. Proposition 1 now guarantees a permutation $\alpha \in \mathrm{Perm}(V(A))$ for which $B \cong A^\alpha$. □

If there happens to be a homomorphism $C \to \overrightarrow{P_2}$ (that is if C is homomorphically equivalent to $\overrightarrow{P_2}$) then the converse of the above corollary becomes true. Indeed, if $B \cong A^\alpha$, then Proposition 1 guarantees $A \times \overrightarrow{P_2} \cong B \times \overrightarrow{P_2}$, whence Theorem 2 gives $A \times C \cong B \times C$. We thus get a second corollary.

Corollary 2 *If C is homomorphically equivalent to $\overrightarrow{P_2}$, then $A \times C \cong B \times C$ if and only if $B \cong A^\alpha$ for some $\alpha \in \mathrm{Perm}(V(A))$.*

Corollaries 1 and 2 show that $A \times C \cong B \times C$ implies $B \cong A^\alpha$ for some permutation α, but the converse holds only if C is homomorphically equivalent to an arc $\overrightarrow{P_2}$. Thus digraphs C that are homomorphically equivalent to an arc are the most "pathological" of all zero divisors in the sense that for a given A there are potentially $|V(A)|!$ digraphs $B = A^\alpha \ncong A$ for which $A \times C \cong B \times C$. For other digraphs C we expect fewer such B. In other words, cancellation of $A \times C \cong B \times C$ is "most likely" to fail if C is homomorphically equivalent to an arc.

In general if A, C and α are arbitrary, we do not expect that $A \times C \cong A^\alpha \times C$ unless there is some special relationship between A, C and α. To describe this relationship we will need a construction called the *factorial* of a digraph.

4 The Digraph Factorial

The following definition was introduced in [2].

Definition 2 Given a digraph A, its *factorial* is another digraph, denoted as $A!$, and is defined as follows. The vertex set is $V(A!) = \mathrm{Perm}(V(A))$. Given two permutations $\alpha, \beta \in V(A!)$, there is an arc from α to β provided that $aa' \in E(A) \iff \alpha(a)\beta(a') \in E(A)$ for all pairs $a, a' \in V(A)$. We denote an arc from α to β as $(\alpha)(\beta)$ to avoid confusion with composition.

We remark in passing that $A!$ is a subgraph of the digraph exponential A^A. (See Section 2.4 of [3].) Observe that the definition implies there is a loop at $\alpha \in V(A!)$ if and only if α is an automorphism of A. In particular any $A!$ has a loop at the identity id.

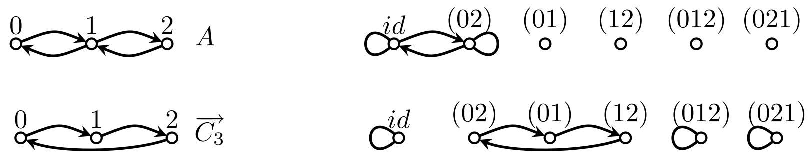

Fig. 4 Examples of digraphs and their factorials

Figure 4 shows some examples of digraph factorials. For another example, which explains the origins of the term "factorial," let K_n^* be the complete (symmetric) graph with a loop at each vertex and note that

$$K_n^*! \cong K_n^* \times K_{n-1}^* \times K_{n-2}^* \times \cdots \times K_3^* \times K_2^* \times K_1^*.$$

The components of the factorial hold a special significance, as the next proposition indicates.

Proposition 2 *If λ and μ are in the same component of $A!$, then $A^\mu \cong A^\lambda$.*

Proof Suppose $(\alpha)(\beta) \in E(A!)$. It suffices to show that $A^\alpha \cong A^\beta$. Observe that

$$\begin{aligned} aa' \in E(A^\beta) &\iff a\beta^{-1}(a') \in E(A) \iff \alpha(a)\beta\beta^{-1}(a') \in E(A) \\ &\iff \quad \alpha(a)a' \in E(A) \quad \iff \quad \alpha(a)\alpha(a') \in E(A^\alpha). \end{aligned}$$

Thus $\alpha : A^\beta \to A^\alpha$ is an isomorphism. $\square$

The converse of Proposition 2 is generally false, so Proposition 2 does not completely characterize the conditions under which $A^\lambda \cong A^\mu$. Instead the characterization involves the following relation $\simeq$ on $V(A!)$.

Definition 3 Suppose A is a digraph and $\lambda, \mu \in V(A!)$. Then $\lambda \simeq \mu$ if and only if there is an arc $(\alpha)(\beta) \in E(A!)$ for which $\mu = \alpha^{-1}\lambda\beta$.

It is proved in [2] that this is an equivalence relation which obeys the following:

Proposition 3 *If A is a digraph and $\lambda, \mu \in \mathrm{Perm}(V(A))$, then $A^\lambda \cong A^\mu$ if and only if $\lambda \simeq \mu$.*

5 Results

We are now ready to prove our main results. We begin with a result that—given a digraph A and a natural number n—characterizes those digraphs B for which $A \times \overrightarrow{P_n} \cong B \times \overrightarrow{P_n}$. In what follows, $\overrightarrow{P_n}$ has vertices $0, 1, 2, \ldots, n-1$, and edges $01, 12, 23, \ldots, (n-2)(n-1)$.

Theorem 4 *Suppose A and B are digraphs, and $n > 1$. Then $A \times \overrightarrow{P_n} \cong B \times \overrightarrow{P_n}$ if and only if $B \cong A^{\alpha}$, where α is a vertex of a directed walk of length $n-2$ in the factorial $A!$.*

Proof Suppose that $B \cong A^{\alpha}$, where α is a vertex of a directed walk of length $n-2$ in $A!$. Call this walk $(\alpha_1)(\alpha_2)\cdots(\alpha_{n-1})$ where $\alpha = \alpha_i$ for some i. By Proposition 2, $B \cong A^{\alpha_1}$, so we just need to show $A \times \overrightarrow{P_n} \cong A^{\alpha_1} \times \overrightarrow{P_n}$. Define a map $\varphi : V(A \times \overrightarrow{P_n}) \to V(A^{\alpha_1} \times \overrightarrow{P_n})$ as

$$\varphi(a,i) = \begin{cases} (\alpha_1\alpha_2\cdots\alpha_i(a), i) & \text{if } i \neq 0 \\ (a,i) & \text{if } i = 0. \end{cases}$$

Clearly this is a bijection because each α_i is a permutation on the vertices of A. We need to show that it is an isomorphism. First consider edges of $A \times \overrightarrow{P_n}$ that have form $(a,0)(a',1)$. Note that $(a,0)(a',1) \in E(A \times \overrightarrow{P_n})$ if and only if $(a,0)(\alpha_1(a'),1) \in E(A^{\alpha_1} \times \overrightarrow{P_n})$ if and only if $\varphi(a,0)\varphi(a',1) \in E(A^{\alpha_1} \times \overrightarrow{P_n})$.

The remaining edges of $A \times \overrightarrow{P_n}$ have form $(a,i)(a',i+1)$, for $1 \leq i < n-1$. For these,

$$\begin{aligned}
& (a,i)(a',i+1) \in E(A \times \overrightarrow{P_n}) \\
\iff\ & aa' \in E(A) \\
\iff\ & \alpha_i(a)\alpha_{i+1}(a') \in E(A) \qquad\qquad \text{(since } (\alpha_i)(\alpha_{i+1}) \in E(A!)\text{)} \\
\iff\ & \alpha_{i-1}\alpha_i(a)\,\alpha_i\alpha_{i+1}(a') \in E(A) \\
& \vdots \\
\iff\ & \alpha_1\cdots\alpha_i(a)\,\alpha_2\alpha_3\cdots\alpha_{i+1}(a') \in E(A) \\
\iff\ & \alpha_1\alpha_2\cdots\alpha_i(a)\,\alpha_1\alpha_2\cdots\alpha_{i+1}(a') \in E(A^{\alpha_1}) \\
\iff\ & \big(\alpha_1\alpha_2\cdots\alpha_i(a),\, i\big)\big(\alpha_1\alpha_2\cdots\alpha_{i+1}(a'),\, i+1\big) \in E(A^{\alpha_1} \times \overrightarrow{P_n}) \\
\iff\ & \varphi(a,i)\varphi(a',i+1) \in E(A^{\alpha_1} \times \overrightarrow{P_n}).
\end{aligned}$$

Hence φ is a isomorphism.

Conversely, assume that $A \times \overrightarrow{P_n} \cong B \times \overrightarrow{P_n}$. By Theorem 3, there is an isomorphism $\varphi : A \times \overrightarrow{P_n} \to B \times \overrightarrow{P_n}$ of the form $\varphi(a,i) = (\beta(a,i), i)$. For each index $0 \leq i < n-1$, define $\beta_i : V(A) \to V(B)$ as $\beta_i(a) = \beta(a,i)$. Since φ is an isomorphism, it follows readily that each β_i is a bijection. For any $aa' \in E(A)$ and $i \in \{0, \ldots, n-2\}$ we have

$$\begin{aligned}
aa' \in E(A) &\iff (a,\, i)(a',\, i+1) \in E(A \times \overrightarrow{P_n}) \\
&\iff \varphi(a,\, i)\varphi(a',\, i+1) \in E(B \times \overrightarrow{P_n}) \\
&\iff (\beta_i(a),\, i)(\beta_{i+1}(a'),\, i+1) \in E(B \times \overrightarrow{P_n}) \\
&\iff \beta_i(a)\beta_{i+1}(a') \in E(B).
\end{aligned} \tag{1}$$

Let $0 < i < n-1$. Using the above Equivalence (1), we find that $aa' \in E(A)$ if and only if $\beta_i(a)\beta_{i+1}(a') \in E(B)$ if and only if $\beta_{i-1}^{-1}\beta_i(a)\beta_i^{-1}\beta_{i+1}(a') \in E(A)$. By Definition 2 we now have an arc $(\beta_{i-1}^{-1}\beta_i)(\beta_i^{-1}\beta_{i+1})$ in $A!$. Consequently $A!$ has a directed walk

$$(\beta_0^{-1}\beta_1)(\beta_1^{-1}\beta_2)(\beta_2^{-1}\beta_3)\cdots(\beta_{n-2}^{-1}\beta_{n-1})$$

of length $n-2$ whose first vertex is $\beta_0^{-1}\beta_1$.

To complete the proof, we need to show that $B \cong A^{\alpha}$ for some permutation α on this walk. In fact, we will show that $\beta_0 : A^{\beta_0^{-1}\beta_1} \to B$ is an isomorphism. Indeed

$$\begin{aligned} aa' \in E(A^{\beta_0^{-1}\beta_1}) &\iff a\,(\beta_0^{-1}\beta_1)^{-1}(a') \in E(A) \quad \text{(by definition of } A^{\beta_0^{-1}\beta_1}\text{)} \\ &\iff a\,\beta_1^{-1}\beta_0(a') \in E(A) \\ &\iff \beta_0(a)\beta_1\beta_1^{-1}\beta_0(a') \in E(B) \qquad \text{(by Equivalence (1))} \\ &\iff \beta_0(a)\beta_0(a') \in E(B). \end{aligned}$$

This completes the proof. □

Notice that Proposition 1 is the special case $n = 2$ of Theorem 4. Indeed, if $n = 2$, then a walk of length $n-2$ in $A!$ is a single vertex of $A!$, that is, a permutation α of $V(A)$, and Theorem 4 reduces to Proposition 1.

Corollary 3 *Suppose a digraph C is homomorphically equivalent to $\overrightarrow{P_n}$. Then $A \times C \cong B \times C$ if and only if $B \cong A^{\alpha}$, where α is on a directed walk of length $n-2$ in the factorial of A.*

Proof Let C be homomorphically equivalent to $\overrightarrow{P_n}$. By Theorem 2, $A \times C \cong B \times C$ if and only if $A \times \overrightarrow{P_n} \cong B \times \overrightarrow{P_n}$. The corollary then follows from Theorem 4. □

Corollary 3 and Proposition 3 combine to give the following.

Theorem 5 *Suppose A and C are digraphs, and C is homomorphically equivalent to $\overrightarrow{P_n}$. Let*

$$\Upsilon_n = \{\alpha \in V(A!) : \alpha \textit{ is on a directed walk of length } n-2 \textit{ in } A!\}.$$

Form a partition $\Upsilon = [\alpha_1] \cup [\alpha_2] \cup \ldots \cup [\alpha_k]$ of Υ_n, where each $[\alpha_i]$ is the $\simeq$-equivalence class (Definition 3) containing a representative α_i. Then the isomorphism classes of digraphs B for which $A \times C \cong B \times C$ are precisely $B = A^{\alpha_i}$ for $1 \le i \le k$.

Next we develop analogues of these results where the path $\overrightarrow{P_n}$ is replaced by a directed cycle $\overrightarrow{C_n}$. A definition is necessary.

A *null-walk* in $A!$ is a closed walk $(\alpha_0)(\alpha_1)(\alpha_2)(\alpha_3)\ldots(\alpha_{n-1})(\alpha_0)$, where $(\alpha_i)(\alpha_{i+1}) \in E(A!)$ for each i (arithmetic modulo n) and $\alpha_0\alpha_1\alpha_2\alpha_3\cdots\alpha_{n-1} = \text{id}$. (Null-walks are not particularly rare; any closed directed walk $W = (\alpha_0)(\alpha_1)(\alpha_2)\ldots(\alpha_{n-1})(\alpha_0)$ in $A!$ can be extended to a null-walk by traversing W k times, where k is the order of the permutation $\alpha_0\alpha_1\alpha_2\ldots\alpha_{n-1}$.)

Theorem 6 *If A and B are digraphs, and $n \geq 1$, then $A \times \overrightarrow{C_n} \cong B \times \overrightarrow{C_n}$ if and only if $B \cong A^\alpha$, where α is on a null-walk of length n in the factorial $A!$.*

Proof Suppose $B \cong A^\alpha$, where α is on a null-walk $(\alpha_0)(\alpha_1)(\alpha_2)\dots(\alpha_{n-1})(\alpha_0)$ in the factorial. By Proposition 2, $B \cong A^{\alpha_0}$, so it suffices to show $A \times \overrightarrow{C_n} \cong A^{\alpha_0} \times \overrightarrow{C_n}$. We construct this isomorphism as follows. Define a map $\varphi : A \times \overrightarrow{C_n} \to A^{\alpha_0} \times \overrightarrow{C_n}$ such that

$$\varphi(a, i) = (\alpha_0\alpha_1 \cdots \alpha_i(a),\, i).$$

Because each α_i is a permutation on the vertices of A, it follows that φ is a bijection.

Knowing that the arcs of the null-walk are arcs in $A!$, we can conclude

$$\begin{aligned}
aa' \in E(A) &\iff \alpha_i(a)\,\alpha_{i+1}(a') \in E(A) \\
&\iff \alpha_{i-1}\alpha_i(a)\,\alpha_i\alpha_{i+1}(a') \in E(A) \\
&\quad \vdots \\
&\iff \alpha_0\alpha_1 \cdots \alpha_{i-1}\alpha_i(a)\,\alpha_1\alpha_2 \cdots \alpha_i\alpha_{i+1}(a') \in E(A) \\
&\iff \alpha_0\alpha_1 \cdots \alpha_{i-1}\alpha_i(a)\,\alpha_0\alpha_1\alpha_2 \cdots \alpha_i\alpha_{i+1}(a') \in E(A^{\alpha_0})
\end{aligned}$$

for any non-negative i, where the index arithmetic is done modulo n. When $i = n-1$, this reduces to $aa' \in E(A) \iff a\alpha_0(a') \in E(A^{\alpha_0})$, as the vertices of the null-walk multiply to the identity.

The above observations imply

$$\begin{aligned}
&(a, i)(a', i+1) \in E(A \times \overrightarrow{C_n}) \\
\iff &\big(\alpha_0\alpha_1 \cdots \alpha_i(a),\, i\big)\big(\alpha_0\alpha_1 \cdots \alpha_{i+1}(a'),\; i+1\big) \in E(A^{\alpha_0} \times \overrightarrow{C_n}) \\
\iff &\varphi(a, i)\varphi(a', i+1) \in E(A^{\alpha_0} \times \overrightarrow{C_n}),
\end{aligned}$$

so we have an isomorphism $\varphi : A \times \overrightarrow{C_n} \to A^{\alpha_0} \times \overrightarrow{C_n}$.

Conversely, suppose $A \times \overrightarrow{C_n} \cong B \times \overrightarrow{C_n}$. By Theorem 3, we are guaranteed an isomorphism $\varphi : A \times \overrightarrow{C_n} \to B \times \overrightarrow{C_n}$ of the form $\varphi(a, i) = (\beta_i(a), i)$. Since φ is an isomorphism, it follows that each $\beta_i : V(A) \to V(B)$ is bijective. We now argue as before. For any $aa' \in E(A)$,

$$\begin{aligned}
aa' \in E(A) &\iff (a, i)(a', i+1) \in E(A \times \overrightarrow{C_n}) \\
&\iff \varphi(a, i)\varphi(a', i+1) \in E(B \times \overrightarrow{C_n}) \\
&\iff (\beta_i(a), i)(\beta_{i+1}(a'), i+1) \in E(B \times \overrightarrow{C_n}) \\
&\iff \beta_i(a)\beta_{i+1}(a') \in E(B),
\end{aligned} \tag{2}$$

where the index arithmetic is done modulo n. By Equivalence (2), $aa' \in E(A)$ if and only if $\beta_i(a)\beta_{i+1}(a') \in E(B)$ if and only if $\beta_{i-1}^{-1}\beta_i(a)\beta_i^{-1}\beta_{i+1}(a') \in E(A)$. Consequently $(\beta_{i-1}^{-1}\beta_i)(\beta_i^{-1}\beta_{i+1})$ is an arc of $A!$ for any $i \in \{0, 1, \dots, n-1\}$ which produces the closed walk $(\beta_0^{-1}\beta_1)(\beta_1^{-1}\beta_2)(\beta_2^{-1}\beta_3)\cdots(\beta_{n-1}^{-1}\beta_0)(\beta_0^{-1}\beta_1)$ in $A!$. The permutations in this walk multiply up to the identity, so in fact this is a null-walk.

To complete the proof, we need to show that $B \cong A^\alpha$ for some permutation α on this walk. In fact, we can show that $\beta_0 : A^{\beta_0^{-1}\beta_1} \to B$ is an isomorphism exactly as was done at the end of the proof of Theorem 4, but using Equivalence (2) instead of Equivalence (1). □

To illustrate this theorem, consider $A = \overrightarrow{C_3}$ whose factorial is given in Figure 4. The factorial contains a null-walk (02)(01)(12)(02)(01)(12)(02) of length six. Theorem 6 guarantees $\overrightarrow{C_3} \times \overrightarrow{C_6} \cong \overrightarrow{C_3}^{(02)} \times \overrightarrow{C_6}$ and this is borne out in Figure 5.

Note also that the closed directed walk (02)(01)(12)(02) of length three in $A!$ is not a null-walk, as $(02)(01)(12) = (01) \neq \mathrm{id}$. Indeed $A!$ had no null-walk of length three. The theorem predicts $\overrightarrow{C_3} \times \overrightarrow{C_3} \not\cong \overrightarrow{C_3}^{(02)} \times \overrightarrow{C_3}$, and this is in fact the case, as the reader may verify.

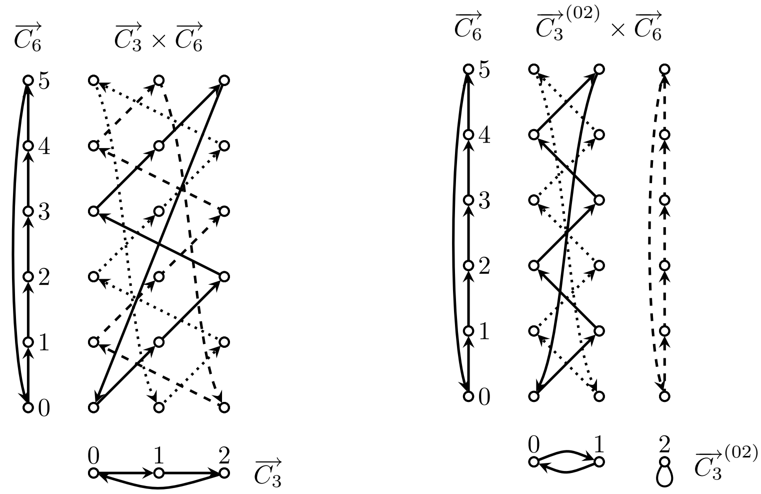

Fig. 5 Isomorphic products guaranteed by Theorem 6

Corollary 4 *Suppose a digraph C is homomorphically equivalent to $\overrightarrow{C_n}$. Then $A \times C \cong B \times C$ if and only if $B \cong A^\alpha$, where the factorial $A!$ contains a null-walk of length n through α.*

The proof repeats the argument used in Corollary 2. As in that case, our findings are summarized in a theorem.

Theorem 7 *Suppose A and C are digraphs, and C is homomorphically equivalent to $\overrightarrow{C_n}$. Let*

$$\Upsilon_n = \{\alpha \in A! : \alpha \text{ lies on a null-walk of length } n \text{ in } A!\}.$$

Consider the partition $\Upsilon = [\alpha_1] \cup [\alpha_2] \cup \ldots \cup [\alpha_k]$ of Υ_n, where each $[\alpha_i]$ is the $\simeq$-equivalence class containing the representative α_i. Then the digraphs B for which $A \times C \cong B \times C$ are precisely $B = A^{\alpha_i}$ for $1 \leq i \leq k$.

Final Remarks Our methods give a complete set of solutions X to the digraph equation $A \times C \cong X \times C$, where C is a zero divisor that is homomorphically equivalent to a directed path or cycle.

For more general types of zero divisors C, our methods give only partial solutions. As noted earlier, any zero divisor either has a homomorphism into some directed path $\overrightarrow{P_n}$, or it has homomorphisms into finitely many directed cycles $\overrightarrow{C_p}$ of prime lengths. For such C, Theorem 2 implies that any solution of $A \times \overrightarrow{P_n} \cong X \times \overrightarrow{P_n}$ (respectively $A \times \overrightarrow{C_p} \cong X \times \overrightarrow{C_p}$) is a solution to $A \times C \cong X \times C$. The results of this paper show how to find these solutions, but they do not guarantee that there may not be *more* solutions to $A \times C \cong X \times C$. Thus it remains to unravel the mysteries of zero divisors that are not homomorphically equivalent to directed paths or cycles.

References

1. R. Hammack, On direct product cancellation of graphs, *Discrete Mathematics*, **309**: 2538–2543, 2009.
2. R. Hammack and K. Toman, Cancellation of direct products of digraphs, *Discusiones Mathematicae Graph Theory*, **31**, 2011, in press.
3. P. Hell and J. Nešetřil, *Graphs and Homomorphisms,* Oxford Lecture Series in Mathematics, Oxford U. Press, 2004.
4. W. Imrich, S. Klavžar, *Product Graphs: Structure and recognition*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley and Sons, New York, 2000.
5. L. Lovász, *On the cancellation law among finite relational structures*, Period. Math. Hungar. **1**(2) (1971) 145–156.