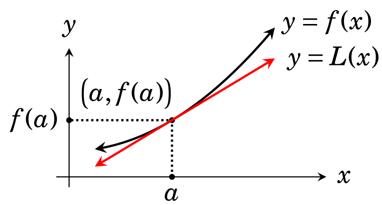
CHAPTER 36

Linear Approximation and Newton's Method

We return to the main idea of calculus: Up close, nonlinear functions look linear. At this point we know to refine this mantra slightly: Up close, *differentiable* nonlinear functions look linear. Zooming in on a point $(a, f(a))$ on such a function's graph, we see the tangent line at $x = a$. Because it is a line, this tangent is the graph of a linear function $L(x) = mx + b$.



In this chapter we first compute the linear function $L(x)$, then use it to develop *Newton's method*, a technique for solving equations with calculus.

36.1 Linear Approximation

Let's find $L(x)$. The tangent line has slope $f'(a)$ and it passes through the point $(x_0, y_0) = (a, f(a))$. By the point-slope formula, its equation is

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - f(a) &= f'(a)(x-a) \\ y &= f(a) + f'(a)(x-a). \qquad (*) \end{aligned}$$

Here x is the independent variable and a, $f(a)$ and $f'(a)$ are constants. We could go one step further and write it as $y = mx + b$:

$$y = \underbrace{f'(a)}_{m} \cdot x + \underbrace{\big(f(a) - af'(a)\big)}_{b}$$

But for simplicity we will use Equation $(*)$. That is, the linear function

$$L(x) = f(a) + f'(a)(x-a),$$

when graphed, is the tangent line to $y = f(x)$ at a.

This function $L(x)$ is called the **linear approximation of f at a.** It is the best approximation to the (possibly complex) function $f(x)$ at a by a (simple) linear function.

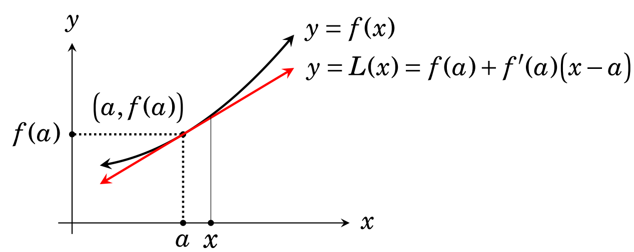
So if x is close to a, the graph of $L(x)$ is almost indistinguishable from the graph of $f(x)$. Hence $f(x) \approx L(x)$ for such x. (The symbol "$\approx$" means "*approximately equal to*.") We summarize this as follows.

Fact 36.1 (Linear Approximation Formula)

Given a function $f(x)$ that is differentiable at a point $x = a$, its **linear approximation** at a is the function

$$L(x) = f(a) + f'(a)\big(x - a\big).$$

The graph of $L(x)$ is the tangent line to f at a, and $f(x) \approx L(x)$ for x linear a.

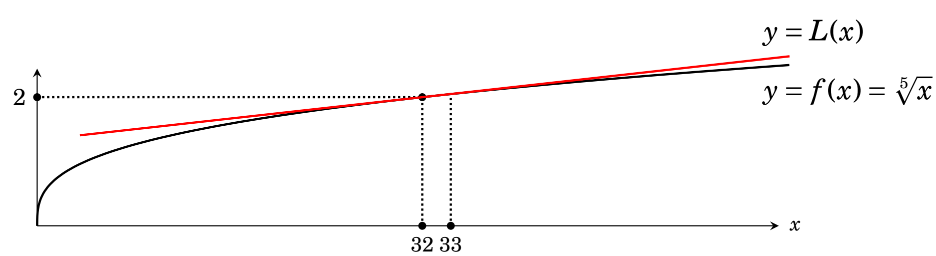


Example 36.1 Find the linear approximation of $f(x) = \sqrt[5]{x}$ at $a = 32$. Then use it to find an approximate value of $\sqrt[5]{33}$.

Here $f(x) = \sqrt[5]{x}$, so $f'(x) = 1/\sqrt[5]{x}^4$ while $a = 32$. The linear approximation is

$$\begin{aligned} L(x) &= f(32) + f'(32)\big(x - 32\big) \\ &= \sqrt[5]{32} + \frac{1}{5\sqrt[5]{32}^4}\big(x - 32\big) \\ &= 2 + \frac{1}{80}\big(x - 32\big). \end{aligned}$$

For comparison, the graphs of f and L are sketched below.



Notice how $f(x) \approx L(x)$ when x is near 32. In particular consider $x = 33$. We have $\sqrt[5]{33} = f(33) \approx L(33) = 2 + \frac{1}{80}(33-32) = \frac{161}{80} = 2.0124$. A calculator gives $\sqrt[5]{33} \approx 2.01234661$, so our approximation is good to three decimal places. ✍

The linear approximation formula is rarely used to approximate function values (as in the previous example) because this can usually be done easily by other means. But it is a significant theoretical model. In the next section it is an essential ingredient in *Newton's method*, a technique for solving equations.

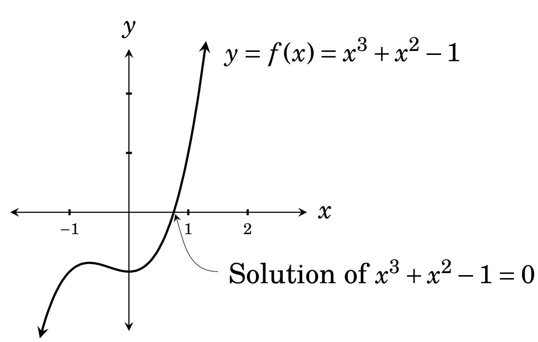
36.2 Newton's Method

Newton's method is a means of applying calculus to the algebraic problem of solving equations. It was invented by Isaac Newton (1642–1727), who used it to solve equations for which standard techniques (like factoring) fail.

To motivate Newton's method, consider the problem of solving

$$x^3 + x^2 - 1 = 0.$$

Try for a moment to factor this or isolate x; you will see that it is difficult. Since factoring doesn't work, lets take a visual approach. The solutions of $x^3+x^2-1=0$ are the x-intercepts of the function $f(x) = x^3+x^2-1$, so let's analyze the problem by drawing a quick graph of $f(x)$. Using our standard graphing techniques,[1] we get the following graph.



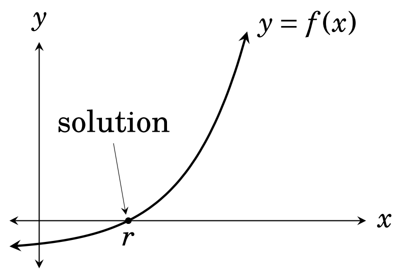
Notice that $f(0) = -1$ is below the x-axis, but $f(1) = 1$ is above the x-axis. Thus the x-intercept of $y = f(x)$ lies somewhere between $x = 0$ and $x = 1$. From the graph, it looks like the x-intercept is about $x = 0.8$.

So we see that $x^3 + x^2 - 1 = 0$ has one solution, approximately $x = 0.8$.

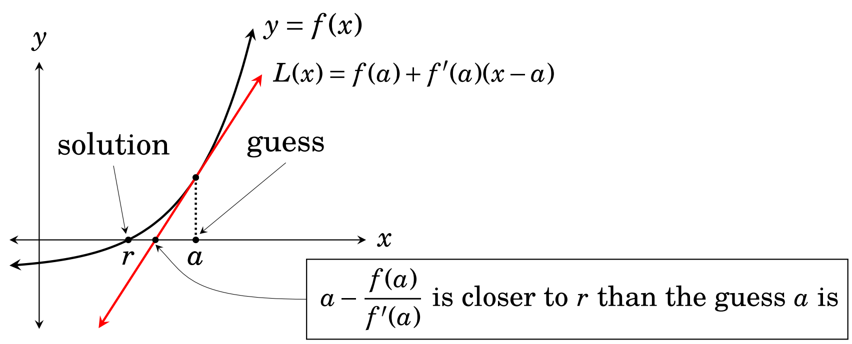
[1]Details: As $f'(x) = 3x^2 + 2x = x(3x+2)$, the critical points are 0 and $-2/3$. Also f increases on $(-\infty, -2/3) \cup (0, \infty)$ and decreases on $(-2/3, 0)$. Thus $f(-2/3) = -23/27$ is a local maximum and $f(0) = -1$ is a local minimum. From this information, we get sketch the graph as above.

Newton's method is a means of solving equations like this one, of the form $f(x) = 0$. We will next describe Newton's method in general, then illustrate it by finding the exact solution of $x^3 + x^2 - 1 = 0$.

In Newton's method, the objective is to solve an equation of form $f(x) = 0$. A solution is a number r for which $f(r) = 0$, so the solutions are precisely the x-intercepts of $y = f(x)$, as shown below.



The first step in Newton's method is to make a guess at a solution r. Say our guess is some number a. The guess is probably wrong, that is, $r \neq a$. But then an iterative process (described below) improves the guess, moving it closer and closer to r. The central idea is that the linear approximation $L(x)$ to $f(x)$ at a is the best linear fit to $y = f(x)$ at a, so its x-intercept will be closer to the actual solution r than is a. (See the diagram below.)



By Fact 36.1, the linear approximation to f at a is $L(x) = f(a) + f'(a)(x-a)$. To find its x-intercept we just solve $f(a) + f'(a)(x-a) = 0$ for x. Doing so,

$$\begin{aligned} f(a) + f'(a)(x-a) &= 0 \\ f(a) + f'(a)x - f'(a)a &= 0 \\ f'(a)x &= f'(a)a - f(a) \\ x &= \frac{f'(a)a - f(a)}{f'(a)} = a - \frac{f(a)}{f'(a)} \end{aligned}$$

The takeaway is that (as illustrated in the above diagram), if a number a is close to r, then the number $a - \frac{f(a)}{f'(a)}$ will be even closer.

Newton's method applies this in a feedback loop, giving a sequence of numbers $a_1, a_2, a_3, \ldots$ that get closer and closer to r, as follows.

If a_1 is our first guess at r, then the number $a_2 = a_1 - \dfrac{f(a_1)}{f'(a_1)}$ is closer to r.

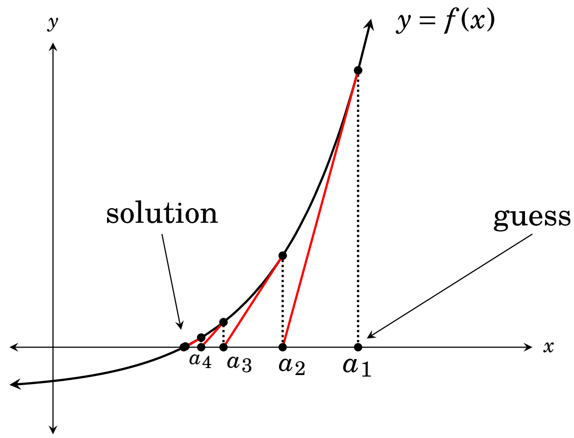
Now that we have a_2, the number $a_3 = a_2 - \dfrac{f(a_2)}{f'(a_2)}$ is even closer to r.

Putting all this together, we get Newton's method.

Fact 36.2 (Newton's Method)

To find a solution of the equation $f(x) = 0$, first make a reasonable guess a_1 at the desired solution. Then compute the following numbers:

$$\begin{aligned} a_2 &= a_1 - \frac{f(a_1)}{f'(a_1)} \\ a_3 &= a_2 - \frac{f(a_2)}{f'(a_2)} \\ a_4 &= a_3 - \frac{f(a_3)}{f'(a_3)} \\ &\vdots \\ a_{n+1} &= a_{n-1} - \frac{f(a_n)}{f'(a_n)} \\ &\vdots \end{aligned}$$



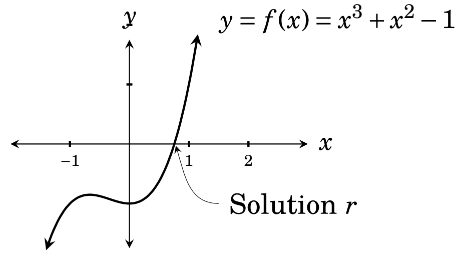
Then the solution is $r = \lim\limits_{n \to \infty} a_n$. In particular, this means that a_n is a very good approximation to the solution r when n is large.

Note: Newton's method applies only to equations of the exact form $f(x) = 0$, where f is differentiable.

Obviously, this works better the closer the guess a_1 is to the actual solution. For a well-chosen a_1, it is not unrealistic to expect a_6 or a_7 to already approximate the solution with 10 decimal places of accuracy.

Example 36.2 Solve $x^3 + x^2 - 1 = 0$.

This is the equation we began the chapter with. It has form $f(x) = 0$, where $f(x) = x^3 + x^2 - 1 = 0$, so Newton's method applies. As noted earlier, there is only one solution, approximately $x = 0.8$, so our initial guess is $a_1 = 0.8$.



In Newton's method, if we know a_n then

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} = a_n - \frac{a_n^3 + a_n^2 - 1}{3a_n^2 + 2a_n}.$$

Because $a_1 = 0.8$ we get

$$a_2 = a_1 - \frac{a_1^3 + a_1^2 - 1}{3a_1^2 + 2a_1} = 0.8 - \frac{(0.8)^3 + (0.8)^2 - 1}{3 \cdot (0.8)^2 + 2 \cdot (0.8)} = 0.756\overline{81}.$$

Now that we know $a_2 = 0.756\overline{81}$ we get (rounding off to 8 decimal places):

$$a_3 = a_2 - \frac{a_2^3 + a_2^2 - 1}{3a_2^2 + 2a_2} = 0.756\overline{81} - \frac{(0.756\overline{81})^3 + (0.756\overline{81})^2 - 1}{3 \cdot (0.756\overline{81})^2 + 2 \cdot (0.756\overline{81})} \approx 0.75488147.$$

Plugging this value for x_3 into the expression for x_4 yields

$$a_4 = a_3 - \frac{a_3^3 + a_3^2 - 1}{3a_3^2 + 2a_3} = 0.75488147 - \frac{(0.75488147)^3 + (0.75488147)^2 - 1}{3 \cdot (0.75488147)^2 + 2 \cdot (0.75488147)} \approx 0.75487766.$$

Next, using the value of a_4 to compute a_5,

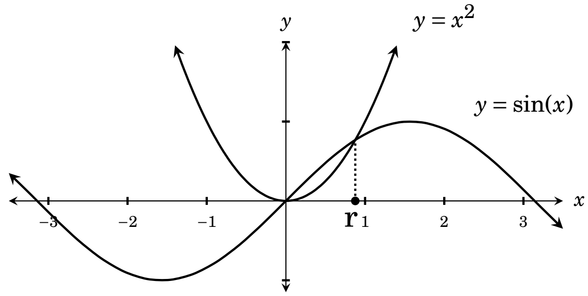
$$a_5 = a_4 - \frac{a_4^3 + a_4^2 - 1}{3a_4^2 + 2a_4} = 0.75487766 - \frac{(0.75487766)^3 + (0.75487766)^2 - 1}{3 \cdot (0.75487766)^2 + 2 \cdot (0.0.75487766)} \approx 0.75487766.$$

Notice that a_4 and a_5 (rounded off to eight decimal places) are exactly the same. Theoretically they are different (and a_5 is closer to the actual solution) but the difference doesn't appear until after the eighth decimal place. Thus 0.75487766 is within 0.00000001 of the solution of $x^3 + x^2 - 1 = 0$.

Answer The solution of the equation $x^3 + x^2 - 1 = 0$, to eight decimal places, is $r = 0.75487766$. ✍

Example 36.3 Solve $\sin(x) = x^2$.

Traditional algebraic techniques are useless for this problem. There is simply no way to isolate the x. However, one solution is obvious. The equation is true if $x = 0$. This is evident if we plot the graphs of $\sin(x)$ and x^2. The graphs cross at $x = 0$.



But there is another crossing point $x = r$ that is just slightly to the left of 1. Thus the equation has a second solution r that is about 0.9 or so. Let's use Newton's method to find this r.

The first step is to write the equation $\sin(x) = x^2$ as $f(x) = 0$. This is easy. The equation becomes $\sin(x) - x^2 = 0$, with $f(x) = \sin(x) - x^2$.

An initial guess of $a_1 = 1$ seems reasonably close to the solution.

Setting up Newton's method, $a_{n+1} = a_n - \dfrac{f(a_n)}{f'(a_n)} = a_n - \dfrac{\sin(a_n) - a_n^2}{\cos(a_n) - 2a_n}$

Starting with $a_1 = 1$ and applying this iterative yields the following sequence. (Be sure your calculator is in radian mode.)

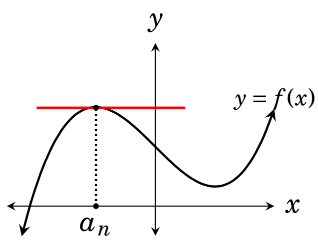
a_n
$a_1 = 1$
$a_2 = 0.8913959$
$a_3 = 0.8769848$
$a_4 = 0.8767262$
$a_5 = 0.8767262$

At a_5 we appear to be accurate within eight decimal places.

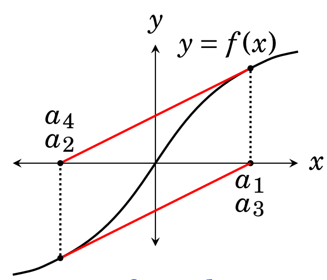
Answer The solutions of $\sin(x) = x^2$ are $x = 0$ and $x \approx 0.8767262$. ✍

Applying Newton's method is somewhat of an art, as it can involve some guesswork and careful analysis of graphs. There are a few pitfalls to watch out for.

For instance, if you are unlucky enough to reach an a_n for which $f'(a_n) = 0$ (below, left) then the next step in Newton's method will involve division by zero, and the program crashes. And watch out for infinite loops. In the example shown on the right below, the numbers $a_1, a_2, a_3, \ldots$ just bounce back and forth forever, never approaching the actual solution of $x = 0$.

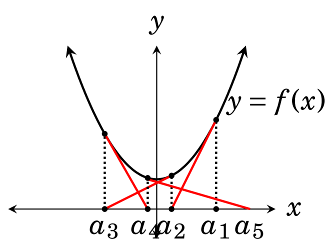


program crashes

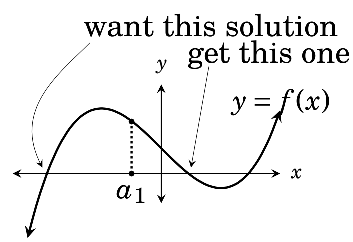


infinite loop

Uncritical use can lead to nonsense. If there are actually no solutions (below left) then obviously Newton's method will produce garbage. And a badly placed a_1 can lead to an unexpected solution (below, right).



garbage in garbage out



uncritical use

Exercises for Chapter 36

1. Find the linear approximation to $\sqrt{x}$ at 81. Then use it to find a pencil-and-paper estimate of $\sqrt{80}$.

2. Find the linear approximation to $\sqrt[4]{x}$ at 10000. Then use it to find a pencil-and-paper estimate of $\sqrt[4]{10001}$.

3. Find the linear approximation to e^x at 0. Then use it to find a pencil-and-paper estimate of $e^{0.1}$.

4. Find the linear approximation to $\sin(x)$ at 0. Then use it to find a pencil-and-paper estimate of $\sin 0.1$.

5. Suppose $f(x)$ is a function, and $f(31) = 106$ and $f'(31) = -5$. Based on this information, find an approximate value of $f(32)$.

Solutions for Chapter 36 Exercises

1. Find the linear approximation to $\sqrt{x}$ at 81. Then use it to find a pencil-and-paper estimate of $\sqrt{80}$.

Solution Let $f(x) = \sqrt{x}$, so $f'(x) = \frac{1}{2\sqrt{x}}$. The linear approximation for $f(x)$ at 81 is $L(x) = f(81) + f'(81)\big(x - 81\big) = \sqrt{81} + \frac{1}{2\sqrt{81}}(x - 81) = 9 + \frac{1}{2\cdot 9}(x - 81) = 9 + \frac{1}{18}(x - 81)$. Then $\sqrt{80} \approx L(80) = 9 + \frac{1}{18}(80 - 81) = 9 - \frac{1}{18} = \boxed{\frac{161}{18}}$.

3. Find the linear approximation to e^x at 0. Then use it to find a pencil-and-paper estimate of $e^{0.1}$.

Solution Let $f(x) = e^x$, so $f'(x) = e^x$ also. The linear approximation formula at 0 is $L(x) = f(0) + f'(0)\big(x - 0\big) = e^0 + e^0(x - 0) = 1 + x$. Then $e^{0.1} \approx L(0.1) = 1 + 0.1 = \boxed{1.1}$.

5. Suppose $f(x)$ is a function, and $f(31) = 106$ and $f'(31) = -5$. Based on this information, find an approximate value of $f(32)$.

Solution The linear approximation formula $L(x) = f(a) + f'(a)\big(x - a\big)$ at $a = 31$ is $L(x) = f(31) + f'(31)\big(x - 31\big) = 106 - 5(x - 31)$. So $f(32) \approx L(32) = 106 - 5(32 - 31) = \boxed{101}$.