CHAPTER 30

Increase-Decrease

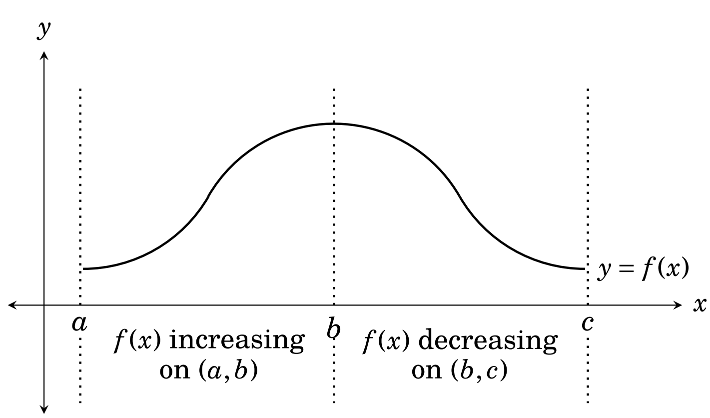
Part 3 of this course dealt with derivatives of functions: what they are, how to compute them, what they mean and how to work with them. In Part 4 the focus now shifts to how derivatives are useful. In Chapters 30 through 34 the theme is what the derivative f' tells us about the function f. Here the primary interest will be the behavior of some function f, and the derivative is a *tool* that gives information about f.

In this chapter we examine one of the most immediate things f' tells us about f: where f increases and where f decreases.

Definition 30.1 Suppose $f(x)$ is a function defined on some interval I.

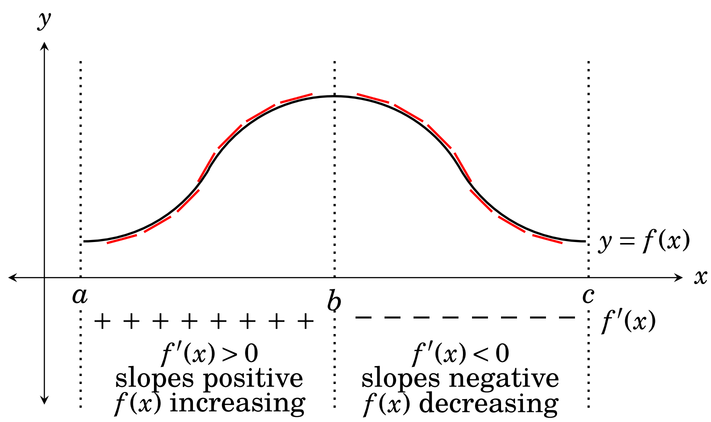
- $f(x)$ **increases** on I if x moving to the right on I causes $f(x)$ to increase. (That is, if x and x' are on I and $x' > x$, then $f(x') > f(x)$.)
- $f(x)$ **decreases** on I if x moving to the right on I causes $f(x)$ to decrease. (That is, if x and x' are on I and $x' > x$, then $f(x') < f(x)$.)

For example, the function f below increases on the interval (a,b) and it decreases on the interval (b,c).



For another example, your familiarity with the parabola $f(x) = x^2$ tells you that this function decreases on $(-\infty,0)$ and increases on $(0,\infty)$.

A function's derivative tells where the function increases and where it decreases. Consider the function $f(x)$ graphed on the previous page, shown again below. Notice that, as $f(x)$ *increases* on (a,b), the slopes of its tangent lines are *positive*. And as $f(x)$ *decreases* on (b,c), the slopes of its tangent lines are *negative*. (We have called attention to this by putting a row of $++++$ or $----$ to show where $f'(x)$ is positive or negative.)



So positive derivative means the function increases; negative derivative means the function decreases. Let's record this very useful, far-reaching (and obvious!) fact.

Fact 30.1 Suppose $f(x)$ is a function defined on some interval I.

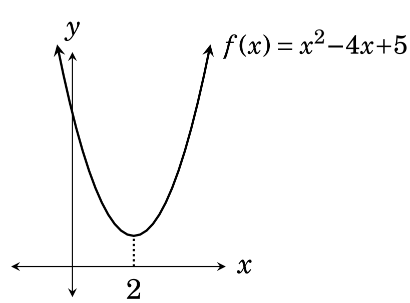
- $f(x)$ increases on I if $f'(x) > 0$ for all x in I.
- $f(x)$ decreases on I if $f'(x) < 0$ for all x in I.

Example 30.1 Find the intervals on which the function $f(x) = x^2 - 4x + 5$ increases/decreases.

Solution Fact 30.1 says that we can find an answer by looking at the derivative, which is $f'(x) = 2x - 4 = 2(x-2)$. By inspection, $f'(x) = 2(x-2)$ is negative when $x < 2$, and it is positive when $x > 2$. This means $f'(x)$ is positive on $(2,\infty)$, and negative on $(-\infty,2)$.

Answer: The function $f(x)=x^2-4x+5$ decreases on the interval $(-\infty,2)$ and increases on the interval $(2,\infty)$.

We got this answer from looking at the derivative alone, not a graph. To underscore that our answer is correct the graph shown on the right. ✍



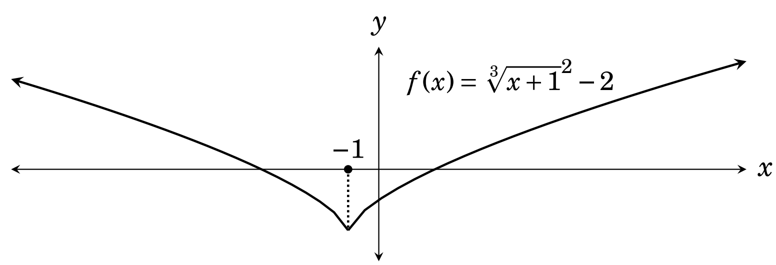
Example 30.2 Find the intervals on which the function $f(x) = \sqrt[3]{x+1}^2 - 2$ increases/decreases.

Solution Fact 30.1 says that we can get an answer by looking at the sign of the derivative. Since, $f(x) = (x+1)^{2/3} - 2$, the generalized power rule gives

$$f'(x) = \frac{2}{3}(x+1)^{-1/3}\frac{d}{dx}\Big[x+1\Big] = \frac{2}{3\sqrt[3]{x+1}}.$$

The sign of $f'(x)$ is controlled by the cube root $\sqrt[3]{x+1}$ in the denominator. Notice that $\sqrt[3]{x+1}$ is negative when $x+1<0$, and it is positive when $x+1>0$. In other words, $\sqrt[3]{x+1}$ is negative when $x<-1$, and it is positive when $x>-1$. Therefore $f'(x) = \frac{2}{3\sqrt[3]{x+1}}$ is negative when $x<-1$, and positive when $x>-1$.

Answer: The function $f(x) = \sqrt[3]{x+1}^2 - 2$ decreases on the interval $(-\infty,-1)$ (where $f'(x)$ is negative) and it increases on $(-1,\infty)$ (where $f'(x)$ is positive).



To check this answer let's draw a quick sketch of the graph of $f(x) = \sqrt[3]{x+1}^2 - 2$. It is the graph of $y = \sqrt[3]{x}^2$ moved 1 unit left and 2 units down. (See above.) Indeed this graph decreases on $(-\infty,-1)$ and increases on $(-1,\infty)$.

Notice that the graph of $f(x)$ has a cusp at -1. This makes sense because $f'(-1) = -\frac{2}{3\sqrt[3]{-1+1}} = -\frac{2}{0}$ does not exist, so $f(x)$ has no tangent at $x=-1$. ✍

Examples 30.1 and 30.2 illuminate a very significant fact about what happens at the point that a function switches from decreasing to increasing (or increasing to decreasing).

In Example 30.1, $f(x)$ stopped decreasing and started increasing at $x=2$, and $f'(2)=0$. The function "bottomed out" at 2 with a horizontal tangent.

In Example 30.2, $f(x)$ stopped decreasing and started increasing at $x=-1$, and $f'(-1)$ was not defined. The function "hits bottom with a kink" at -1.

These two examples illustrate the two possibilities that signal a switch in increase/decrease. Draw the graph of any continuous $f(x)$, like the one in Figure 30. It will be the case that whenever $f(x)$ switches increase/decrease at some number c, then either $f'(c)=0$ or $f'(c)$ does not exist.

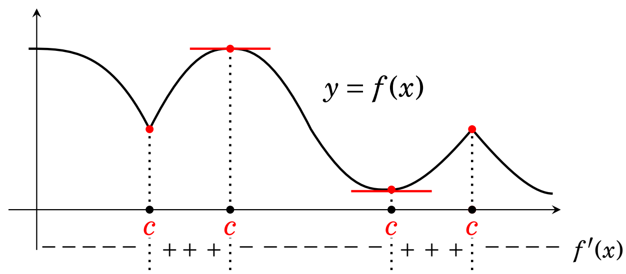


Figure 30.1. If a function $f(x)$ changes from decreasing to increasing (or increasing to decreasing) at a number $x{=}c$, then either $f'(c) = 0$ or $f'(c)$ is not defined. Such a number c is called a **critical point** for $f(x)$.

The reason for this should be intuitively clear: Suppose that $f(x)$ switches increase/decrease at $x = c$. If it happened that $f'(c)$ were positive, then $f(x)$ would continue rising through c. If $f'(c)$ were negative, then $f(x)$ would continue falling through c. Because neither of these two alternatives holds, we conclude that $f'(c)$ is neither positive nor negative. There are only two ways this can happen: either $f'(c) = 0$ or $f'(c)$ simply doesn't exist.

So the values $x = c$ that make a function's derivative zero or undefined are going to play an important role. They are called *critical points*.

Definition 30.2 A number c *in the domain of a function* f is called a **critical point** for f if either $f'(c) = 0$ or $f'(c)$ is not defined.

We summarize our observations as the following fact.

Fact 30.2 If a function $f(x)$ switches from increasing to decreasing (or decreasing to increasing) at a number $x = c$, then c is a critical point for $f(x)$.

With this we have a simple procedure to find the intervals on which a function increase or decreases. (We will assume that any function f under discussion here is differentiable on its domain, except possibly at a discrete set of points at which its derivative is not defined.)

To find the intervals on which a function f increases or decreases

1. Find all critical points of the function.
2. The critical points divide the function's domain into a set of intervals.
3. For each interval, check if $f'(x) > 0$. If so, f increases on this interval. Otherwise, if $f'(x) < 0$, then f decreases on this interval.

Example 30.3 Find the intervals on which $f(x) = 3x^4 - 4x^3 - 12x^2 + 24$ is increasing/decreasing.

Solution The first step is to find the critical points, the values of x that make the derivative zero or undefined. To find them we must examine the derivative, $f'(x) = 12x^3 - 12x^2 - 24x$. This polynomial is defined for all real x, so there are no critical points that make $f'(x)$ undefined. To find the critical points that make $f'(x)$ zero, we solve the equation $f'(x) = 0$:

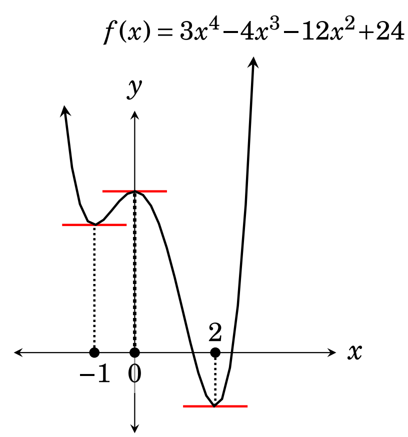
$$\begin{aligned} 12x^3 - 12x^2 - 24x &= 0 \\ 12x\left(x^2 - x - 2\right) &= 0 \\ 12x(x-2)(x+1) &= 0. \end{aligned}$$

So the derivative factors as $f'(x) = 12x(x-2)(x+1)$, and we can see that the critical points are $x = 0$, $x = 2$ and $x = -1$. They divide the number line into four intervals, as shown in the diagram below.

	-1	0	2	
− − − − − − − − − −	− − − −	+ + + + + + + + + +	+ + + + + + +	$12x$
− − − − − − − − − −	− − − −	− − − − − − − − − −	+ + + + + + +	$(x-2)$
− − − − − − − − − −	+ + + +	+ + + + + + + + + +	+ + + + + + +	$(x+1)$
− − − − − − − − − −	+ + + +	− − − − − − − − − −	+ + + + + + +	$f'(x) = 12x(x-2)(x+1)$

For each factor of the derivative, we indicate the intervals on which it is negative (−) or positive (+). For example, the factor $12x$ is negative on the interval $(-\infty, 0)$ and positive on $(0, \infty)$. Once this is done for all factors, we can read off the sign of $f'(x)$ for each of the four intervals. For example, on $(-\infty, -1)$, $f'(x)$ is a product of three negatives, so it is negative (−). From this chart we can read off our answer.

Answer: $f(x) = 3x^4 - 4x^3 - 12x^2 + 24$ increases on the intervals $(-1, 0)$ and $(2, \infty)$. It decreases on the intervals $(-\infty, -1)$ and $(0, 2)$.



Note: our final answer does not involve $f'(x)$ at all. The derivative was just a tool used to get the answer

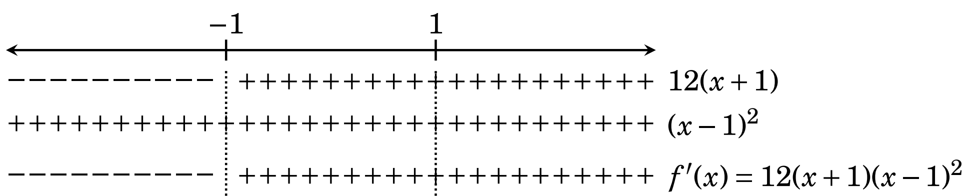
The function is sketched on the right. Notice the zero slope at the critical points. ✍

Example 30.4 Find the intervals on which $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 18$ is increasing/decreasing.

Solution The first step is to find the critical points, and to find them we must examine the derivative, $f'(x) = 12x^3 - 12x^2 - 12x + 12$. This polynomial is defined for all real x, so there are no critical points that make $f'(x)$ undefined. To find the critical points that make $f'(x)$ zero, we solve the equation $f'(x) = 0$:

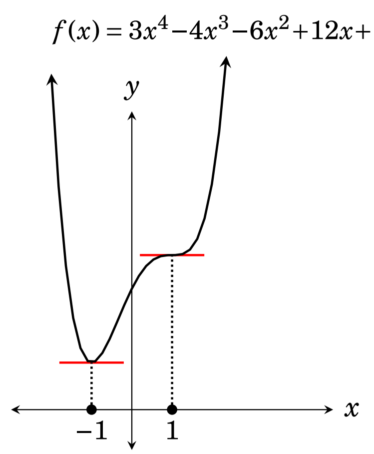
$$\begin{aligned}
12x^3 - 12x^2 - 12x + 12 &= 0 \\
12\left(x^3 - x^2 - x + 1\right) &= 0 \\
12\Big(x^2(x-1) - (x-1)\Big) &= 0 \\
12\left(x^2 - 1\right)(x-1) &= 0 \\
12(x+1)(x-1)(x-1) &= 0 \\
12(x+1)(x-1)^2 &= 0.
\end{aligned}$$

So the derivative factors as $f'(x) = 12(x+1)(x-1)^2$, and the critical points are $x = -1$, and $x = 1$. They divide the number line into three intervals, as shown in the diagram below.



As indicated, the factor $12(x+1)$ is negative for $x < -1$ and positive for $x > -1$. But the factor $(x-1)^2$ is *never negative*, because it is squared. Therefore, the derivative $f'(x) = 12(x+1)(x-1)^2$ is negative when $x < -1$, and it is positive when $x > -1$.

Answer: $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 18$ decreases the interval $(-\infty, -1)$ and increases on $(-1, 1)$ and $(1, \infty)$.



Notice how the derivative does not change signs at $x = 1$, even though $f'(1) = 0$. The function $f(x)$ (graphed on the right) rises before getting to $x = 1$, then levels out at $x = 1$, then continues rising. Given this, it is allowable to say that $f(x)$ increases on the interval $(-1, \infty)$. ✍

Example 30.5 Find the intervals on which the function $f(x) = e^{(3\sqrt[3]{x}^2 - 4x)}$ is increasing/decreasing.

Solution The first step is to find all critical points, and this involves examining $f'(x)$. By the chain rule (or generalized exponential rule),

$$f'(x) = \frac{d}{dx}\left[e^{(3\sqrt[3]{x}^2-4x)}\right] \;=\; e^{(3\sqrt[3]{x}^2-4x)}\frac{d}{dx}\left[3\sqrt[3]{x}^2 - 4x\right] \;=\; e^{(3\sqrt[3]{x}^2-4x)}\left(\frac{2}{\sqrt[3]{x}} - 4\right).$$

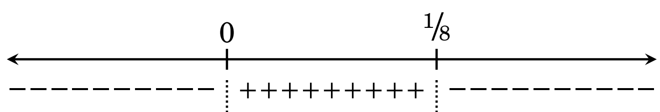
From this we can see that $x = 0$ is a critical point, for $f'(0)$ involves division by zero, so $f'(0)$ is not defined. But $f'(x)$ is defined for all other x, so $x = 0$ is the only critical point of $f(x)$ that makes $f'(x)$ undefined. Any other critical point will make $f'(x)$ zero, so to find them we solve the equation $f'(x) = 0$:

$$e^{(3\sqrt[3]{x}^2-4x)}\left(\frac{2}{\sqrt[3]{x}} - 4\right) \;=\; 0$$

Since e to any power is positive, we can divide both sides of this equation by the nonzero expression $2e^{(3\sqrt[3]{x}^2-4x)}$, getting

$$\begin{aligned} \frac{1}{\sqrt[3]{x}} - 2 &= 0 \\ \frac{1}{\sqrt[3]{x}} &= 2 \\ \frac{1}{2} &= \sqrt[3]{x} \\ x &= \frac{1}{8} \end{aligned}$$

Thus we have just two critical points $x = 0$ and $x = \frac{1}{8}$. These divide the domain of f into three intervals, $(-\infty, 0)$, $(0, \frac{1}{8})$ and $(\frac{1}{8}, \infty)$.

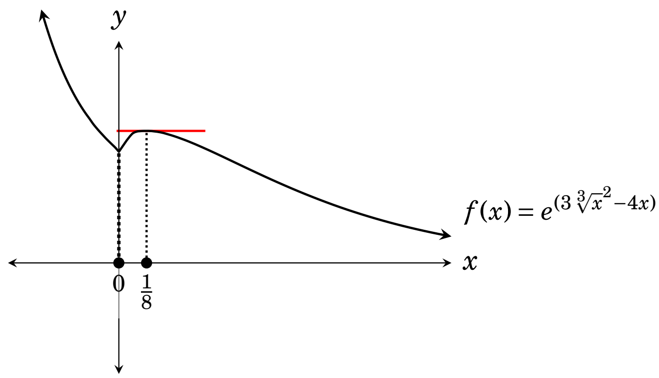


$f'(x) = e^{(3\sqrt[3]{x}^2-4x)}\left(\frac{2}{\sqrt[3]{x}} - 4\right)$

An alternative approach to finding the sign of $f'(x)$ on these intervals is to select a “test point" in each interval and plug it into $f'(x)$. For example:

- 1 is in $(\frac{1}{8}, \infty)$, and $f'(1) = e^{(3-4)}\left(\frac{2}{\sqrt[3]{1}} - 4\right) < 0$, so $f'(x)$ is negative on $(\frac{1}{8}, \infty)$.
- -1 is in $(-\infty, 0)$, and $f'(-1) = e^{(-3+4)}\left(\frac{2}{\sqrt[3]{-1}} - 4\right) < 0$, so $f'(x)$ is negative on $(\frac{1}{8}, \infty)$.
- $\frac{1}{27}$ is in $(0, \frac{1}{8})$, and $f'\left(\frac{1}{27}\right) > 0$, so $f'(x)$ is positive on $(0, \frac{1}{8})$.

Answer: The function $f(x) = e^{(3\sqrt[3]{x}^2-4x)}$ increases on $(0, \frac{1}{8})$, and decreases on $(-\infty, 0)$ and $(\frac{1}{8}, \infty)$.



The function $f(x) = e^{(3\sqrt[3]{x}^2 - 4x)}$ has been sketched with a graphing utility above. Notice that there is a cusp at the critical point 0, where $f'(0)$ is not defined. And the slope is zero at the critical point $1/8$, where $f'(1/8) = 0$. ✍

In all of this chapter's examples the domain of the function has been all real numbers, $(-\infty, \infty)$, and the critical points split $(-\infty, \infty)$ into smaller intervals. By contrast, the function $f(x) = \frac{1}{x} + x$ from Example 5 below has domain $(-\infty, 0) \cup (0, \infty)$, and its critical points will further split these two intervals into smaller intervals. Test your understanding by working this exercise.

Exercises for Chapter 30

1. Find the intervals on which $y = x^4 - 8x^2 + 16$ increases/decreases.

2. Find the intervals on which $y = x^3 - 27x + 36$ increases/decreases.

3. Find the intervals on which $f(x) = (x-2)e^x$ increases/decreases.

4. Find the intervals on which $y = \sqrt{x} - x$ increases/decreases.

5. Find the intervals on which $y = \frac{1}{x} + x$ increases/decreases.

6. Find the intervals on which $y = e^x - x$ increases/decreases.

7. Find the intervals on which $y = \ln\left(x^2 + 10x + 26\right)$ increases/decreases.

8. Find the intervals on which $y = \tan^{-1}\left(x^2 + 10x + 24\right)$ increases/decreases.

9. Find the intervals on which $y = \tan^{-1}\left(\sqrt[3]{x}^2 + 3\right)$ increases/decreases.

10. Find the intervals on which $y = x \ln|x|$ increases/decreases.

Exercises Solutions for Chapter 30

1. Find the intervals on which $y = x^4 - 8x^2 + 16$ increases/decreases.

The derivative is $f'(x) = 4x^3 - 16x = 4x(x^2-4) - 4x(x-2)(x+2)$. From this we can see that there are three critical points, 0, -2 and 2. These divide the domain $(-\infty,\infty)$ of f into four intervals, $(-\infty,-2)$, $(-2,0)$, $(0,2)$ and $(2,\infty)$.

Let's pick a test point a in each interval to determine the sign of f' on that interval. This is tabulated in the table below.

Interval	$(-\infty,-2)$	$(-2,0)$	$(0,2)$	$(2,\infty)$
Test point a	-3	-1	1	3
$f'(a)$	$f'(-3) = -60$	$f'(-1) = 12$	$f'(1) = -12$	$f'(3) = 60$
Sign of $f'(a)$	$-$	$+$	$-$	$+$
f is	decreasing	increasing	decreasing	increasing

Answer: f increases on $(-2,0)$ and $(2,\infty)$, and decreases on $(\infty,-2)$ and $(0,2)$.

3. Find the intervals on which $f(x) = (x-2)e^x$ increases/decreases.

By the product rule, the derivative is $f'(x) = 1\cdot e^x + (x-2)e^x = e^x(1+x-2) = e^x(x-1)$. Since e^x is positive for any x, we can just look at this and see that there is only one critical point, $x = 1$. This critical point divides the domain $(-\infty,\infty)$ of f into two intervls $(-\infty,1)$ and $(1,\infty)$. By inspection, $f'(x)$ is negative on $(-\infty,1)$, and positive on $(1,\infty)$.

Answer: f decreases on $(-\infty,1)$ and increases on $(1,\infty)$.

5. Find the intervals on which $y = \frac{1}{x} + x$ increases/decreases.

Observe that the domain of this function is $(-\infty,0)\cup(0,\infty)$. Its derivative is $\frac{dy}{dx} = -\frac{1}{x^2} + 1$, and this is zero if $x = \pm 1$. The critical points $x = \pm 1$ divide the domain into intervals $(-\infty,-1)$, $(-1,0)$, $(0,1)$ and $(1,\infty)$. Let's pick a test point a in each interval to determine the sign of f' on that interval. This is tabulated in the table below.

Interval	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
Test point a	-2	$-1/2$	$1/2$	2
$f'(a)$	$f'(-2) = 3/4$	$f'(-1/2) = -3$	$f'(1/2) = -3$	$f'(2) = 3/4$
Sign of $f'(a)$	$+$	$-$	$-$	$+$
f is	increasing	decreasing	decreasing	increasing

Answer: f decreases on $(-1,0)$ and $(1,0)$, and increases on $(-\infty,-1)$ and $(1,\infty)$.

7. Find the intervals on which $y = \ln\left(x^2+10x+26\right)$ increases/decreases.

Notice that $x^2+10x+26 = (x^2+10x+25)+1 = (x+5)^2+1 > 0$, so $\ln\left(x^2+10x+26\right)$ is defined for all x. Hence the domain of this function is $(-\infty,\infty)$. The derivative is $\frac{dy}{dx} = \frac{2x+10}{x^2+10x+26} = \frac{2(x+5)}{x^2+10x+26}$, and the only critical point is $x = -5$. This splits the domain into two intervals $(-\infty,-5)$ and $(-5,\infty)$.

Interval	$(-\infty,-5)$	$(-5,\infty)$
Test point a	-6	0
$f'(a)$	$f'(-6)=\frac{-2}{2}<0$	$f'(0)=\frac{10}{26}$
Sign of $f'(a)$	$-$	$+$
f is	decreasing	increasing

Thus the function decreases on $(-\infty,-5)$ and increases on $(-5,\infty)$.

9. Find the intervals on which $y=\tan^{-1}\left(\sqrt[3]{x}^2+3\right)$ increases/decreases.

The derivative is $\frac{dy}{dx}=\frac{1}{1+\left(\sqrt[3]{x}^2+3\right)^2}\cdot\frac{2}{3\sqrt[3]{x}}$. This is never zero, but it is undefined for $x=0$. Thus $x=0$ is the only critical point, splitting the domain into two intervals $(-\infty,0)$ and $(0,\infty)$. The derivative is negative on the first interval and positive on the second. Therefore the function decreases on $(-\infty,0)$ and increases on $(0,\infty)$.