Section 8.4 Trigonometric Substitutions

Motivational problem: $\int \sqrt{1-x^2}\,dx = \ ?$

Familiar substitutions and integration by parts don't work.

Idea: Try trig substitution that eliminates the radical.

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta = \int \cos\theta\cos\theta\,d\theta$$

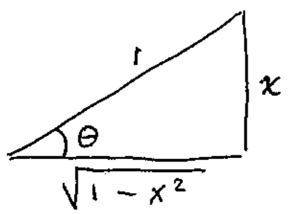
$$= \int \cos^2\theta\,d\theta = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$$

$$= \boxed{\frac{1}{2}\sin^{-1}(x) + \frac{1}{2}x\sqrt{1-x^2} + C}$$

$x = \sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = \cos\theta\,d\theta$

$\theta = \sin^{-1}(x)$



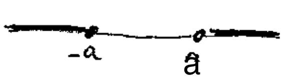
Check: $\frac{d}{dx}\left[\frac{1}{2}\sin^{-1}(x) + \frac{1}{2}x\sqrt{1-x^2} + C\right]$

$$= \frac{1}{2}\frac{1}{\sqrt{1-x^2}} + \frac{1}{2}\sqrt{1-x^2} + \frac{1}{2}x\frac{-2x}{2\sqrt{1-x^2}} + 0$$

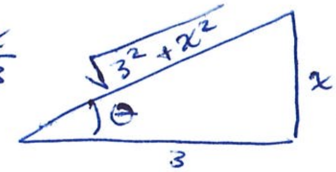
$$= \frac{1}{2\sqrt{1-x^2}} + \frac{\sqrt{1-x^2}\sqrt{1-x^2}}{2\sqrt{1-x^2}} - \frac{2x^2}{2\sqrt{1-x^2}} = \frac{2-2x^2}{2\sqrt{1-x^2}} = \frac{1-x^2}{\sqrt{1-x^2}} = \sqrt{1-x^2}$$

Simplifications:

Expression	Substitution	Result
$\sqrt{a^2-x^2}$	$x = a\sin\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$\sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$\sqrt{a^2 + a^2\tan^2\theta} = a\sec\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta \quad \begin{cases} 0 \le \theta < \frac{\pi}{2} & \text{if } x \ge a \\ \frac{\pi}{2} < \theta \le \pi & \text{if } x \le -a \end{cases}$	$\sqrt{a^2\sec^2\theta - a^2} = a\sqrt{\tan^2\theta} = a\lvert\tan\theta\rvert$



Ex $\displaystyle\int \frac{1}{x\sqrt{9+x^2}}\,dx$ $\qquad x = 3\tan\theta$, $\quad dx = 3\sec^2\theta\,d\theta$ $\qquad \tan\theta = \frac{x}{3}$

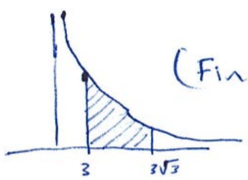


$$= \int \frac{3\sec^2\theta\,d\theta}{3\tan\theta\sqrt{9+(3\tan\theta)^2}} = \int \frac{\sec^2\theta\,d\theta}{\tan\theta\;3\sec\theta}\,d\theta = \frac{1}{3}\int\frac{\sec\theta}{\tan\theta}\,d\theta$$

$$= \frac{1}{3}\int \csc\theta\,d\theta = -\frac{1}{3}\ln\left|\csc\theta+\cot\theta\right| + C = \boxed{\frac{-1}{3}\ln\left|\frac{\sqrt{9+x^2}}{x}+\frac{3}{x}\right|}$$

Ex $\displaystyle\int_3^{3\sqrt{3}} \frac{1}{x\sqrt{9+x^2}}\,dx = -\frac{1}{3}\ln\left|\frac{\sqrt{9+x^2}}{x}+\frac{3}{x}\right|_3^{3\sqrt{3}} =$

$$-\frac{1}{3}\ln\left|\frac{\sqrt{9+27}}{3\sqrt{3}}+\frac{3}{3\sqrt{3}}\right| + \frac{1}{3}\ln\left|\frac{\sqrt{9+9}}{3}+\frac{3}{3}\right| = \boxed{\frac{-1}{3}\ln\left|\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right| + \frac{1}{3}\ln\left|\sqrt{2}+1\right|}$$



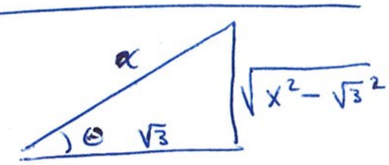
(Find the area)

Ex $\displaystyle\int_3^{3\sqrt{3}} \frac{1}{x\sqrt{9+x^2}}\,dx = \frac{1}{3}\int_{\pi/4}^{\pi/3} \csc\theta\,d\theta = -\frac{1}{3}\ln\left|\csc\theta+\cot\theta\right|\Big]_{\pi/4}^{\pi/3}$

$$= -\frac{1}{3}\ln\left|\csc\frac{\pi}{3}+\cot\frac{\pi}{3}\right| + \frac{1}{3}\ln\left|\csc\frac{\pi}{4}+\cot\frac{\pi}{4}\right|$$

$$= \boxed{\frac{1}{3}\ln\left|\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right| + \frac{1}{3}\ln\left|\sqrt{2}+1\right|}$$

Ex $\displaystyle\int \frac{dx}{x^4\sqrt{x^2-3}}$ $\qquad x = \sqrt{3}\sec\theta$, $\quad dx = \sqrt{3}\sec\theta\tan\theta\,d\theta$ $\qquad \sec\theta = \frac{x}{\sqrt{3}}$



$$= \int \frac{\sqrt{3}\sec\theta\tan\theta\,d\theta}{(\sqrt{3}\sec\theta)^4\sqrt{(\sqrt{3}\sec\theta)^2-3}} = \int \frac{\sqrt{3}\sec\theta\tan\theta\,d\theta}{9\sec^4\theta\,\sqrt{3}\tan\theta} = \frac{1}{9}\int\frac{d\theta}{\sec^3\theta}\,d\theta = \frac{1}{9}\int\cos^3\theta\,d\theta$$

$$\frac{1}{9}\int\cos^2\theta\cos\theta\,d\theta = \frac{1}{9}\int(1-\sin^2\theta)\cos\theta\,d\theta = \frac{1}{9}\int(1-u^2)\,du$$

$$= \frac{1}{9}\left(u-\frac{u^3}{3}\right)+C = \frac{1}{9}\sin\theta - \frac{1}{27}\sin^3\theta + C = \boxed{\frac{1}{9}\frac{\sqrt{x^2-3}}{x} - \frac{1}{27}\left(\frac{\sqrt{x^2-3}}{x}\right)^3 + C}$$

Notice that some familiar formulas fall out of this.

Formulas:

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

Reason: $u = a\sin\theta$
$du = a\cos\theta\, d\theta$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-(a\sin\theta)^2}} = \int \frac{a\cos\theta}{a\cos\theta}\, d\theta$$

$$= \int d\theta = \theta + C = \sin^{-1}\left(\frac{u}{a}\right) + C$$

So if you understand how to use this technique, you dont need to remember these formulas.

Integrals involving $ax^2 + bx + c$

$$\int \frac{dx}{\sqrt{5+6x-x^2}} = \int \frac{dx}{\sqrt{\sqrt{14}^2-(x-3)^2}} = \int \frac{dx}{\sqrt{\sqrt{14}^2-u^2}} = \sin^{-1}\left(\frac{u}{\sqrt{14}}\right) + C$$

$u = x-3$
$du = dx$

$$= \boxed{\sin^{-1}\left(\frac{x-3}{\sqrt{14}}\right) + C}$$

$5 + 6x - x^2$

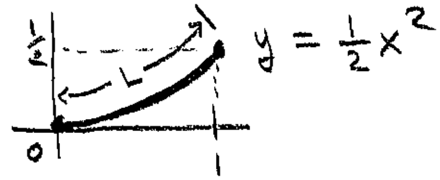
$= -(x^2-6x) + 5$

$= -(x^2-6x+9) + 14$

$= -(x-3)^2 + 14$

$= \sqrt{14}^2 - (x-3)^2$

$\underline{\text{Ex}}$ Find the length:



$$L = \int_0^1 \sqrt{1 + (f'(x))^2}\, dx = \int_0^1 \sqrt{1 + x^2}\, dx$$

$x = \tan\theta$
$dx = \sec^2\theta\, d\theta$

$$= \int_{\tan^{-1}(0)}^{\tan^{-1}(1)} \sqrt{1 + \tan^2\theta}\ \sec^2\theta\, d\theta = \int_0^{\pi/4} \sqrt{\sec^2\theta}\ \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \sec^3\theta\, d\theta = \left[\frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left(\sqrt{2}\cdot 1 + \ln|\sqrt{2}+1|\right) - \frac{1}{2}\left(1\cdot 0 + \ln|1+0|\right)$$

$$= \boxed{\frac{\sqrt{2}}{2} + \ln|\sqrt{2}+1| \text{ units}}$$

$$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta$$

$u = \sec\theta \quad dv = \sec^2\theta\, d\theta$
$du = \sec\theta\tan\theta\, d\theta \quad v = \tan\theta$

$$= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\, d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta - \sec\theta\, d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta$$

$$2\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \int \sec\theta\, d\theta$$

$$\boxed{\int \sec^3\theta\, d\theta = \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)}$$