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**PART I.** Prove the following statements.

1. Prove that an integer  $a$  is even if and only if  $a^2 + 2a + 9$  is odd.

**Proof.** First we will show that if  $a$  is even, then  $a^2 + 2a + 9$  is odd. We use direct proof.  
Suppose  $a$  is even. Then  $a = 2k$  for some integer  $k$ , and

$$a^2 + 2a + 9 = (2k)^2 + 2(2k) + 9 = 4k^2 + 4k + 8 + 1 = 2(2k^2 + 2k + 4) + 1.$$

This shows that  $a^2 + 2a + 9$  is twice an integer plus 1, so it is odd.

Conversely, we will show that if  $a^2 + 2a + 9$  is odd, then  $a$  is even.

We use contrapositive proof; that is we will assume  $a$  is not even and show  $a^2 + 2a + 9$  is not odd.  
Suppose  $a$  is not even, so it is odd, and thus  $a = 2k + 1$  for some integer  $k$ . Then

$$\begin{aligned} a^2 + 2a + 9 &= (2k + 1)^2 + 2(2k + 1) + 9 \\ &= 4k^2 + 4k + 1 + 4k + 2 + 9 \\ &= 4k^2 + 8k + 12 \\ &= 2(2k^2 + 4k + 6). \end{aligned}$$

This shows that  $a^2 + 2a + 9$  is twice an integer, so it is even.

The proof is now complete.

2. Suppose  $A, B$  and  $C$  are nonempty sets. Prove that if  $A \times B \subseteq B \times C$ , then  $A \subseteq C$ .

**Proof.** We will use direct proof. Suppose  $A \times B \subseteq B \times C$ .

In what follows we show  $A \subseteq C$ .

Suppose  $a \in A$ .

Since  $B$  is not empty, there is an element  $b \in B$ , so  $(a, b) \in A \times B$ . (By definition of  $\times$ .)

But since  $A \times B \subseteq B \times C$ , it follows that  $(a, b) \in B \times C$ . (By definition of  $\subseteq$ .)

In particular, this gives us  $a \in B$ , so it now follows that  $(a, a) \in A \times B$ . (By definition of  $\times$ .)

But again, since  $A \times B \subseteq B \times C$ , it we get  $(a, a) \in A \times C$ . (By definition of  $\subseteq$ .)

In particular, this means  $a \in C$ . (By definition of  $\times$ .)

We've now shown  $a \in A$  implies  $a \in C$ , so  $A \subseteq C$ .

3. Use induction to prove that  $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ .

We will prove this with mathematical induction.

- (1) When  $n = 1$  the statement is  $1^3 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$ , which is true.  
 (2) Now assume the statement is true for some integer  $n = k \geq 1$ , that is assume

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}.$$

Observe that this implies the statement is true for  $n = k + 1$ , as follows:

$$\begin{aligned} 1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 + (k+1)^3 &= \\ (1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3) + (k+1)^3 &= \\ \frac{k^2(k+1)^2}{4} + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2(k^2 + 4(k+1)^1)}{4} \\ &= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4} \\ &= \frac{(k+1)^2((k+1)+1)^2}{4} \end{aligned}$$

Therefore  $1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$ ,  
 which means the statement is true for  $n = k + 1$ .

This completes the proof by mathematical induction.

4. There exists a set  $X$  for which  $\mathbb{Z} \in X$ ,  $\mathbb{N} \in \mathcal{P}(X)$  and  $\mathbb{R} \in \mathcal{P}(X)$ .

**Proof.** Consider the set  $X = \{\mathbb{Z}\} \cup \mathbb{R}$ .

(That is,  $X$  contains every real number, *and* it also contains the *set* of all integers.)

We have  $\mathbb{N} \subseteq X$  and  $\mathbb{R} \subseteq X$ , and this means  $\mathbb{N} \in \mathcal{P}(X)$  and  $\mathbb{R} \in \mathcal{P}(X)$ .

Also, we have  $\mathbb{Z} \in \{\mathbb{Z}\}$ , so  $\mathbb{Z} \in \{\mathbb{Z}\} \cup \mathbb{R} = X$ .

5. Use induction to prove that  $24|(5^{2n} - 1)$  for every integer  $n \geq 0$ .

**Proof.** The proof is by mathematical induction.

- (1) For  $n = 0$ , the statement is  $24|(5^{2 \cdot 0} - 1)$ . This simplifies to  $24|0$ , which is true.
- (2) Now assume the statement is true for some integer  $n = k \geq 1$ , that is assume  $24|(5^{2k} - 1)$ .  
This means  $5^{2k} - 1 = 24a$  for some integer  $a$ , and from this we get  $5^{2k} = 24a + 1$ .  
Now observe that

$$\begin{aligned} 5^{2(k+1)} - 1 &= \\ 5^{2k+2} - 1 &= \\ 5^2 5^{2k} - 1 &= \\ 5^2(24a + 1) - 1 &= \\ 25(24a + 1) - 1 &= \\ 25 \cdot 24a + 25 - 1 &= 24(25a + 1) \end{aligned}$$

This shows  $5^{2(k+1)} - 1 = 24(25a + 1)$ , which means  $24|5^{2(k+1)} - 1$ .

This completes the proof by mathematical induction.

**PART II.** Decide if the following statements are true or false. Prove the true statements; disprove the false ones.

7. If  $A, B$  and  $C$  are sets, then  $A \cup (B - C) = (A \cup B) - (A \cup C)$ .

This is FALSE. Here is a counterexample:

Let  $A = B = C = \{1\}$ .

Then  $A \cup (B - C) = \{1\}$ .

Also  $(A \cup B) - (A \cup C) = \emptyset$ .

This example shows that it is not always true that  $A \cup (B - C) = (A \cup B) - (A \cup C)$ .

8. Suppose  $a$  and  $b$  are integers. If  $a|b$  and  $b|a$ , then  $a = b$ .

This is FALSE. Here is a counterexample:

Let  $a = 2$  and  $b = -2$ .

Then  $a|b$  and  $b|a$ , but  $a \neq b$ .

9. If  $A, B, C$  are sets and  $A \cap B \cap C = \emptyset$ , then  $|A \cup B \cup C| = |A| + |B| + |C|$ .

This is FALSE. Here is a counterexample:

Let  $A = \{1, 2\}$ ,  $B = \{2, 3\}$  and  $C = \{3, 1\}$ .

Then  $|A \cup B \cup C| = |\{1, 2, 3\}| = 3 \neq 6 = |A| + |B| + |C|$ .