MATH 201
Integration by Substitution

You learned the **Substitution Rule** in Calculus I, but it is so important in Calculus II that we are going to spend some time reviewing it.

The **substitution rule** is an integration rule that is the chain rule in reverse. Think of it as the *chain rule for integration*. To apply it, you must first have command of the basic integration formulas.

Basic Integration Formulas

$\int c\,du = cu + C$	$\int \sec^2(u)\,du = \tan(u) + C$
$\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C$	$\int \csc^2(u)\,du = -\cot(u) + C$
$\int \dfrac{1}{u}\,du = \ln\lvert u\rvert + C$	$\int \sec(u)\tan(u)\,du = \sec(u) + C$
$\int e^u\,du = e^u + C$	$\int \csc(u)\cot(u)\,du = -\csc(u) + C$
$\int b^u\,du = \dfrac{1}{\ln(b)}b^u + C$	$\int \dfrac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) + C$
$\int \sin(u)\,du = -\cos(u) + C$	$\int \dfrac{1}{1+u^2}\,du = \tan^{-1}(u) + C$
$\int \cos(u)\,du = \sin(u) + C$	$\int \dfrac{1}{u\sqrt{u^2-1}}\,du = \sec^{-1}\lvert u\rvert + C$

Substitution Rule

$$\text{If } u = g(x), \text{ then } \int f(g(x))\,g'(x)\,dx = \int f(u)\,du.$$

Substitution Rule for Definite Integrals

$$\text{If } u = g(x), \text{ then } \int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

EXAMPLES

1. $\int \sin(x^2)\, 2x\, dx = \int \sin(u)\, du = -\cos(u) + C = \boxed{-\cos(x^2) + C}$

$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\, dx$

2. $\int x \sin(x^2)\, dx = \int \sin(x^2)\, x\, dx = \int \sin(u)\, \frac{1}{2}\, du$

$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\, dx \rightarrow \frac{1}{2}\, du = x\, dx$

$= \frac{1}{2} \int \sin(u)\, du$

$= \frac{1}{2}(-\cos(u)) + C$

$= \boxed{-\frac{1}{2}\cos(x^2) + C}$

3. $\int \frac{\sec(\ln(x)) \tan(\ln(x))}{x}\, dx =$

$u = \ln(x)$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}\, dx$

$= \int \sec(\ln(x)) \tan(\ln(x))\, \frac{1}{x}\, dx$

$= \int \sec(u) \tan(u)\, du$

$= \sec(u) + C = \boxed{\sec(\ln(x)) + C}$

4. $\int \sqrt{e^x}\, e^x\, dx = \int \sqrt{u}\, du$

$u = e^x$

$\frac{du}{dx} = e^x$

$du = e^x\, dx$

$= \int u^{1/2}\, du = \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{u^{3/2}}{3/2} + C$

$= \frac{2}{3}\sqrt{u}^{\,3} + C = \boxed{\frac{2}{3}\sqrt{e^x}^{\,3} + C}$

5. $\int \frac{3x^2 + 8x - 1}{x^3 + 4x^2 - x + 1}\, dx = \int \frac{1}{x^3 + 4x^2 - x + 1}(3x^2 + 8x - 1)\, dx$

$u = x^3 + 4x^2 - x + 1$

$\frac{du}{dx} = 3x^2 + 8x - 1$

$du = (3x^2 - 8x - 1)\, dx$

$= \int \frac{1}{u}\, du = \ln|u| + C$

$= \boxed{\ln|x^3 + 4x^2 - x + 1|}$

6. $\int \frac{5x+2}{25x^2+20x+4}dx = \int \frac{1}{25x^2+20x+4}(5x+2)\,dx = \int \frac{1}{u}\frac{1}{10}\,du$

$u = 25x^2+20x+4$

$\frac{du}{dx} = 50x+20$

$du = (50x+20)\,dx \Rightarrow \frac{1}{10}du = (5x+2)\,dx$

$= \frac{1}{10}\ln|u| + C = \boxed{\frac{1}{10}\ln|25x^2+20x+4|+C}$

7. $\int \frac{1}{25x^2+20x+4}dx = \int \frac{1}{(5x+2)(5x+2)}dx = \int \frac{1}{(5x+2)^2}dx$

$u = 5x+2$

$\frac{du}{dx} = 5$

$du = 5\,dx \rightarrow dx = \frac{1}{5}du$

$= \int \frac{1}{u^2}\frac{1}{5}du = \frac{1}{5}\int u^{-2}du = \frac{1}{5}\frac{u^{-2+1}}{-2+1} + C$

$= -\frac{1}{5}u^{-1} + C = \frac{-1}{5u} + C = \frac{-1}{5(5x+2)} + C$

$= \boxed{\frac{-1}{25x+10} + C}$

8. $\int \frac{1}{25x^2+20x+5}dx = \int \frac{1}{25x^2+20x+4+1}dx$

$u = 5x+2$

$du = 5dx$

$\frac{1}{5}du = dx$

$= \int \frac{1}{(5x+2)^2+1}dx = \int \frac{1}{u^2+1}\frac{1}{5}du$

$= \frac{1}{5}\tan^{-1}(u) + C = \boxed{\frac{1}{5}\tan^{-1}(5x+2)+C}$

9. $\int_1^2 \frac{5}{(5x-1)^2}dx = \int_1^2 (5x-1)^{-2}\,5\,dx$

$u = 5x-1$

$\frac{du}{dx} = 5$

$du = 5dx$

$= \int_{5\cdot1-1}^{5\cdot2-1} u^{-2}du = \left[\frac{u^{-2+1}}{-2+1}\right]_4^9 = \left[\frac{-1}{u}\right]_4^9 = -\frac{1}{9} - \left(-\frac{1}{4}\right)$

$= -\frac{1}{9} + \frac{1}{4} = -\frac{4}{36} + \frac{9}{36} = \boxed{\frac{5}{36}}$

10. $\int_0^1 \frac{x+1}{x^2+2x+2}dx =$

$\int_0^1 \frac{1}{x^2+2x+2}(x+1)\,dx = \int_{0^2+2\cdot0+2}^{1^2+2\cdot1+2} \frac{1}{2u}\,du$

$u = x^2+2x+2$

$\frac{du}{dx} = 2x+2$

$du = (2x+2)\,dx$

$\frac{1}{2}du = (x+1)\,dx$

$= \frac{1}{2}\int_2^5 \frac{1}{u}du = \frac{1}{2}\left[\ln|u|\right] = (\ln|5| - \ln|2|)/2$

$= \boxed{\frac{\ln\left(\frac{5}{2}\right)}{2}}$

Some Useful Integration Rules

Next we are going to derive some integration formulas that will be useful in Calculus II. In these exercises, $a \neq 0$ is a constant. Use the substitution $u = ax$ to get the answers on the right.

In each case your substitution will work like this:

$$\boxed{u = ax \qquad \frac{du}{dx} = a \qquad du = a\,dx \qquad dx = \frac{1}{a}du}$$

1. $\int e^{ax}\,dx = \int e^{u} \frac{1}{a} du = \frac{1}{a}\int e^{u} du = \frac{1}{a} e^{u} + C = \frac{1}{a}e^{ax} + C$

2. $\int \sin(ax)\,dx = \int \sin(u) \frac{1}{a} du = \frac{1}{a}\int \sin(u)\,du = \frac{1}{a}(-\cos(u) + C) = -\frac{1}{a}\cos(ax) + C$

3. $\int \cos(ax)\,dx = \int \cos(u)\frac{1}{a} du = \frac{1}{a}\int \cos(u)\,du = \frac{1}{a}\sin(u) + C = \frac{1}{a}\sin(ax) + C$

4. $\int \sec^2(ax)\,dx = \int \sec^2(u)\frac{1}{a} du = \frac{1}{a}\int \sec^2(u)\,du = \frac{1}{a}\tan(u) + C = \frac{1}{a}\tan(ax) + C$

5. $\int \csc^2(ax)\,dx = \int \csc^2(u)\frac{1}{a} du = \frac{1}{a}\int \csc^2(u)\,du = \frac{1}{a}\left(-\cot(u) + C\right) = -\frac{1}{a}\cot(ax) + C$

6. $\int \sec(ax)\tan(ax)\,dx = \int \sec(u)\tan(u)\frac{1}{a} du = \frac{1}{a}\sec(u) + C = \frac{1}{a}\sec(ax) + C$

7. $\int \csc(ax)\cot(ax)\,dx = \int \csc(u)\cot(u)\frac{1}{a} du = \frac{1}{a}\left(-\csc(u) + C\right) = -\frac{1}{a}\csc(ax) + C$

For the examples on this page, use the substitution $u = \frac{1}{a}x$, as follows.

$u = \frac{1}{a}x$ (so $x = au$)	$\frac{du}{dx} = \frac{1}{a}$	$dx = a\,du$

8. $$\int \frac{1}{\sqrt{a^2 - x^2}}dx = \int \frac{dx}{\sqrt{a^2-(au)^2}} = \int \frac{dx}{\sqrt{a^2-a^2u^2}} = \int \frac{1}{\sqrt{a^2(1-u^2)}}dx$$

$$= \int \frac{1}{|a|\sqrt{1-u^2}}a\,du = \frac{a}{|a|}\int \frac{du}{\sqrt{1-u^2}} = \frac{a}{|a|}\sin^{-1}(u)+C = \boxed{\sin^{-1}\left(\frac{x}{a}\right)+C}$$

If $a>0$

$\frac{a}{|a|}=1$ if $a>0$

9. $$\int \frac{1}{a^2 + x^2}dx = \int \frac{1}{a^2+(au)^2}a\,du = \int \frac{1}{a^2+a^2u^2}a\,du$$

$$= \int \frac{1}{(1+u^2)a^2}a\,du = \frac{1}{a}\int \frac{1}{1+u^2}du = \frac{1}{a}\tan^{-1}(u)+C = \boxed{\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C}$$

10. $$\int \frac{1}{x\sqrt{x^2 - a^2}}dx = \int \frac{1}{\cancel{a}u\sqrt{(au)^2-a^2}}\cancel{a}\,du = \int \frac{1}{u\sqrt{a^2(u^2-1)}}du$$

$$= \int \frac{1}{|a|u\sqrt{u^2-1}}du = \frac{1}{|a|}\int \frac{1}{u\sqrt{u^2-1}}du = \frac{1}{|a|}\sec^{-1}\left|\frac{x}{a}\right|+C = \boxed{\frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|+C}$$

If $a>0$

You should regard the examples on this page as giving **new formulas** for integrals that you will see often. For instance, we will apply # 9 above in the next example.

Example: $\int \frac{1}{7+x^2}dx = \int \frac{1}{\sqrt{7}^2+x^2}dx = \boxed{\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{x}{\sqrt{7}}\right)+C}$ (using #9 above)

Example: $\int \frac{1}{\sqrt{16-x^2}}dx = \int \frac{1}{\sqrt{4^2-x^2}}dx = \boxed{\sin^{-1}\left(\frac{x}{4}\right)+C}$ (using #8 above)

TAKEAWAY

Remember these general integration formulas. Use them in applying the Substitution Rule.

General Integration Formulas

$\int c\ du = cu + C$	$\int \sec^2(u)\ du = \tan(u) + C$
$\int u^n\ du = \dfrac{u^{n+1}}{n+1} + C$	$\int \csc^2(u)\ du = -\cot(u) + C$
$\int \dfrac{1}{u}\ du = \ln\lvert u\rvert + C$	$\int \sec(u)\tan(u)\ du = \sec(u) + C$
$\int e^u\ du = e^u + C$	$\int \csc(u)\cot(u)\ du = -\csc(u) + C$
$\int b^u\ du = \dfrac{1}{\ln(b)} b^u + C$	$\int \dfrac{1}{\sqrt{a^2-u^2}}\ du = \sin^{-1}\left(\dfrac{u}{a}\right) + C \quad (a > 0)$
$\int \sin(u)\ du = -\cos(au) + C$	$\int \dfrac{1}{a^2+u^2}\ du = \dfrac{1}{a}\tan^{-1}\left(\dfrac{u}{a}\right) + C$
$\int \cos(u)\ du = \sin(au) + C$	$\int \dfrac{1}{u\sqrt{u^2-a^2}}\ du = \dfrac{1}{a}\sec^{-1}\left\lvert\dfrac{u}{a}\right\rvert + C \quad (a > 0)$

Substitution Rule

$$\text{If } u = g(x), \text{ then } \int f\big(g(x)\big)\, g'(x)\, dx = \int f(u)\, du.$$

Substitution Rule for Definite Integrals

$$\text{If } u = g(x), \text{ then } \int_a^b f\big(g(x)\big)\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$