MATH 501, Section 2 Solutions

(2)
$(a * b) * c = b * c = a$
$a * (b * c) = a * a = a$
Even though we've shown that $(a*b)*c = a*(b*c)$, that's no guarantee that the opeation $*$ is associative. We would have to show $(x * y) * z = x * (y * z)$ for *all* possible values of x, y and z. In fact, note that $(d * a) * b = b * b = c$ is unequal to $d * (a * b) = d * b = e$, so $*$ is **NOT ASSOCIATIVE.**

(4) The operation $*$ is **NOT COMMUTATIVE** because, for instance, $e * b = b$ but $b * e = c$.

(6) Suppose the following partial table is for an associative binary operation on $S = \{a, b, c, d\}$.

$*$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	d	c	d
d				

The missing line should give the values of $d * x$ for the various x. To fill in this line, use the fact that the table gives $c * b = d$, together with the fact that $*$ is associative:
$d * a = (c * b) * a = c * (b * a) = c * b = d$
$d * b = (c * b) * b = c * (b * b) = c * a = c$
$d * c = (c * b) * c = c * (b * c) = c * c = c$
$d * d = (c * b) * d = c * (b * d) = c * d = d$

Thus the completed table is as follows

$*$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	d	c	d
d	d	c	c	d

(10) Consider the binary operation on $\mathbb{Z}$ defined as $a * b = 2^{ab}$.
This is **COMMUTATIVE** because $a * b = 2^{ab} = 2^{ba} = b * a$ for all $a, b \in \mathbb{Z}$.
This is **NOT ASSOCIATIVE** because, in particular
$0 * (1 * 2) = 0 * (2^{1 \cdot 2}) = 0 * 4 = 2^{0 \cdot 4} = 2^0 = 1$ but
$(0 * 1) * 2 = (2^{0 \cdot 1}) * 2 = 1 * 2 = 2^{1 \cdot 2} = 2^2 = 4$.

(36) Suppose $*$ is an associative binary operation on a set S, and $H = \{a \in S | a * x = x * a \text{ for all } x \in S\}$. Show H is closed under $*$.

Proof. Suppose that a and b are two arbitrary elements of H. To show H is closed, we must show that $a * b \in H$. And to show $a * b$ is in H we must show $a * b$ satisfies the requirement for being in H, that is we must show $(a * b) * x = x * (a * b)$ for every element x in S.

Let x be an arbitrary element of S. The fact that a and b are in H means

$$a * x = x * a \qquad (1)$$
$$b * x = x * b \qquad (2)$$

Using (1) and (2) together with associativity of $*$, we deduce

$$(a * b) * x = a * (b * x) = a * (x * b) = (a * x) * b = (x * a) * b = x * (a * b).$$

Thus $(a * b) * x = x * (a * b)$, which means $a * b \in H$, so H is closed.