MATH 211 | Test #2 ♠ | November 16, 2017

Name: Richard | R. Hammack | Score: 100

Directions No calculators. Please put all phones, etc., away.

1. (4 points) Complete the following truth tables.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

2. (12 points) Complete the truth table to decide if $P \vee (Q \wedge R)$ and $(\sim Q \vee \sim R) \Rightarrow P$ are logically equivalent.

P	Q	R	$(Q \wedge R)$	$P \vee (Q \wedge R)$	$\sim Q$	$\sim R$	$(\sim Q \vee \sim R)$	$(\sim Q \vee \sim R) \Rightarrow P$
T	T	T	T	T	F	F	F	T
T	T	F	F	T	F	T	T	T
T	F	T	F	T	T	F	T	T
T	F	F	F	T	T	T	T	T
F	T	T	T	T	F	F	F	T
F	T	F	F	F	F	T	T	F
F	F	T	F	F	T	F	T	F
F	F	F	F	F	T	T	T	F

Are they logically equivalent? Why or why not? Their columns agree, so YES they are logically equivalent.

3. (6 points) Suppose the statement $\sim\big(S \Rightarrow (P \vee Q \vee \sim R)\big)$ is true.
Find the truth values of P, Q, R and S. (This can be done without a truth table.)

If this is true, Then $S \Rightarrow (P \vee Q \vee \sim R)$ is False which means S is true and $P \vee Q \vee \sim R$ is false. Now The only way $P \vee Q \vee \sim R$ can be false is if all of P, Q and $\sim R$ are false. Thus:

$$\boxed{S = T, \quad P = F, \quad Q = F, \quad R = T}$$

4. (12 points) This problem concerns the following statement.
P: There is a number $n \in \mathbb{Z}$ for which $m|n$ for every $m \in \mathbb{Z}$.

(a) Is the statement P true or false? **Explain.**

It is true because there is a number $n=0$ in $\mathbb{Z}$ for which $m|n$ for every m.

($m|0$ for any # m)

(b) Write the statement P in symbolic form.

$\exists\, n \in \mathbb{Z}, \forall\, m \in \mathbb{Z}, m|n.$

(c) Form the negation $\sim P$ of your answer from (b), and simplify.

$\sim(\exists\, n \in \mathbb{Z}, \forall\, m \in \mathbb{Z}, m|n) = \forall\, n \in \mathbb{Z} \sim(\forall\, m \in \mathbb{Z}, m|n)$

$= \forall\, n \in \mathbb{Z}, \exists\, m \in \mathbb{Z} \sim(m|n)$

$= \boxed{\forall\, n \in \mathbb{Z}, \exists\, m \in \mathbb{Z}, m \nmid n}$

(d) Write the negation $\sim P$ as an English sentence.
(The sentence may use mathematical symbols.)

For any $n \in \mathbb{Z}$, there exists some number $m \in \mathbb{Z}$ for which $m \nmid n$.

5. (6 points) Complete the first and last lines of each of the following proof outlines.

Proposition: If P, then Q. **Proof:** (Direct) Suppose P ⋮ Therefore Q. ■	**Proposition:** If P, then Q. **Proof:** (Contradiction) Suppose $P \wedge \sim Q$ ⋮ Therefore $C \wedge \sim C$. ■	**Proposition:** If P, then Q. **Proof:** (Contrapositive) Suppose $\sim Q$ ⋮ Therefore $\sim P$. ■

6. (15 points) Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$.
Prove: If $a \equiv b \pmod{n}$, then $ab \equiv b^2 \pmod{n}$. [Use direct proof.]

Proof (Direct) Suppose $a \equiv b \pmod{n}$,
This means $4 \mid (a-b)$, so $a-b = 4c$ for some $c \in \mathbb{Z}$,
Now multiply both sides by b, as follows:

$$a-b = 4c$$
$$(a-b)b = 4cb$$
$$ab - b^2 = 4(cb).$$

This shows $ab - b^2 = 4k$ for $k = cb \in \mathbb{Z}$, so
consequently $4 \mid (ab - b^2)$.
Therefore $ab \equiv b^2 \pmod{4}$. ∎

7. (15 points) Suppose $a, b, c \in \mathbb{Z}$. **Prove:** If $a \nmid bc$, then $a \nmid b$ and $a \nmid c$. [Use contrapositive.]

$\sim((a \nmid b) \wedge (a \nmid c))$
$= \sim(a \nmid b) \vee \sim(a \nmid c)$
$= a \mid b \vee a \mid c$

Proof (Contrapositive)
Suppose it is not true that $a \nmid b$ and $a \nmid c$
Then $a \mid b$ or $a \mid c$.

CASE I Suppose $a \mid b$. Then $b = ak$ for some $k \in \mathbb{Z}$.
Multiply both sides by c to get $bc = akc$. Then
$bc = a \cdot (kc)$ for $kc \in \mathbb{Z}$, and that means $a \mid bc$.

CASE II Suppose $a \mid c$. Then $c = al$ for some $l \in \mathbb{Z}$,
Multiply both sides by b to get $bc = alb$. Then
$bc = a \cdot (lb)$ and that means $a \mid bc$.

In either case we got $a \mid bc$.
Therefore $a \mid bc$.

$\sim((a \text{ odd}) \wedge (b \text{ odd}))$

8. (15 points) **Prove:** If $4 \mid (a^2 + b^2)$, then a and b are not both odd. [Use contradiction.]

Proof Suppose for the sake of contradiction that $4 \mid (a^2+b^2)$ but a and b are both odd.

Then $a = 2c+1$ and $b = 2d+1$ for $c, d \in \mathbb{Z}$.

Also $a^2+b^2 = 4k$ for some $k \in \mathbb{Z}$, by definition of divides. Thus:

$$a^2 + b^2 = 4k$$
$$(2c+1)^2 + (2d+1)^2 = 4k$$
$$4c^2+4c+1 + 4d^2+4d+1 = 4k$$
$$4c^2+4c+4d^2+4d-4k = 2$$
$$2c^2+2c+2d^2+2d-2k = 1 \quad \text{(divide by 2)}$$
$$2(c^2+c+d^2+d-k) = 1$$

As $c^2+c+d^2+d-k \in \mathbb{Z}$, this shows that 1 is even, which is a contradiction. ∎

9. (15 points) Suppose $a, b, c \in \mathbb{Z}$. **Prove:** If $a \mid b$ and $a \mid (b+c)$, then $a \mid c$.

Proof (Direct) Suppose $a \mid b$ and $a \mid (b+c)$.

This means $b = ak$ and $b+c = al$ for some $k, l \in \mathbb{Z}$.

From $b = ak$ and $b+c = al$, we get $ak + c = al$, so $c = al - ak$, i.e. $c = a(l-k)$. Consequently we have $c = am$ for $m = l-k \in \mathbb{Z}$.

Consequently $a \mid c$ ∎