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## MATH 200 MIDTERM EXAM



Oct. 17, 2024

Directions: Closed book, closed notes, no calculators. Put all phones, etc., away. You will need only a pencil or pen.

1. (10 points) Answer the questions about the function  $f$  graphed below.

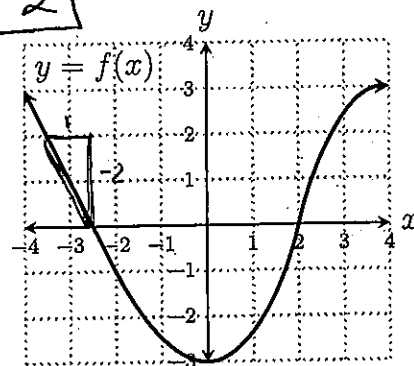
$$(a) \lim_{x \rightarrow \infty} f\left(\frac{1}{x}\right) = f\left(\lim_{x \rightarrow \infty} \frac{1}{x}\right) = f(0) = \boxed{-3}$$

$$(b) \lim_{h \rightarrow 0} \frac{f(-3+h) - f(-3)}{h} = f'(-3) = \frac{\text{rise}}{\text{run}} = \frac{-2}{1} = \boxed{-2}$$

$$(c) \lim_{x \rightarrow -2} \frac{f(x)}{(1+f(x))^2} = \frac{\text{approaching } -1}{\text{approaching } 0, \text{ positive}} = \boxed{-\infty}$$

$$(d) \lim_{x \rightarrow 2} \frac{\sin(f(x)) + 1}{f(x) + 1} = \frac{\sin(0) + 1}{0 + 1} = \frac{1}{1} = \boxed{1}$$

$$(e) \lim_{x \rightarrow 2} \frac{\sin(f(x))}{f(x)} = \boxed{1}$$



2. (20 points) Find the limits

$$(a) \lim_{x \rightarrow 0^+} \sin^{-1}(x-1) = \sin^{-1}(0-1) = \sin^{-1}(-1) = \boxed{-\frac{\pi}{2}}$$

$$(b) \lim_{x \rightarrow e} 5 \ln(x^3) = 5 \lim_{x \rightarrow e} \ln(x^3) = 5 \ln\left(\lim_{x \rightarrow e} x^3\right) = 5 \ln(e^3) = 5 \cdot 3 = \boxed{15}$$

$$(c) \lim_{x \rightarrow 3} \frac{x-3}{x^2-7x+12} = \lim_{x \rightarrow 3} \frac{x-3}{(x-3)(x-4)} = \lim_{x \rightarrow 3} \frac{1}{x-4} = \frac{1}{3-4} = \boxed{-1}$$

$$(d) \lim_{x \rightarrow 1} \frac{\frac{1}{x} - 1}{x-1} = \lim_{x \rightarrow 1} \frac{\frac{1-x}{x}}{x-1} \cdot \frac{x}{x} = \lim_{x \rightarrow 1} \frac{1-x}{(x-1)x} = \lim_{x \rightarrow 1} \frac{-(x-1)}{(x-1)x} = \lim_{x \rightarrow 1} \frac{-1}{x} = \frac{-1}{1} = \boxed{-1}$$

3. (7 points) Use a limit definition of the derivative to find the derivative of  $f(x) = \sqrt{1-x}$ .

$$\begin{aligned}
 f'(x) &= \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} = \lim_{z \rightarrow x} \frac{\sqrt{1-z} - \sqrt{1-x}}{z - x} \\
 &= \lim_{z \rightarrow x} \frac{\sqrt{1-z} - \sqrt{1-x}}{z - x} \cdot \frac{\sqrt{1-z} + \sqrt{1-x}}{\sqrt{1-z} + \sqrt{1-x}} \\
 &= \lim_{z \rightarrow x} \frac{1 - z - (1 - x)}{(z - x)(\sqrt{1-z} + \sqrt{1-x})} = \lim_{z \rightarrow x} \frac{-z + x}{(z - x)(\sqrt{1-z} + \sqrt{1-x})} \\
 &= \lim_{z \rightarrow x} \frac{-(z - x)}{(z - x)(\sqrt{1-z} + \sqrt{1-x})} = \lim_{z \rightarrow x} \frac{-1}{\sqrt{1-z} + \sqrt{1-x}} = \frac{-1}{\sqrt{1-x} + \sqrt{1-x}} \\
 &= \boxed{\frac{-1}{2\sqrt{1-x}}} \quad \text{Therefore } \boxed{f'(x) = \frac{-1}{2\sqrt{1-x}}}
 \end{aligned}$$

4. (7 points) An object moving on a straight line is  $s(t) = t^3 - 3t^2$  feet from its starting point at time  $t$  seconds. Find its velocity when its acceleration is 12 feet per second per second.

Velocity:  $v(t) = s'(t) = 3t^2 - 6t$

Acceleration:  $a(t) = v'(t) = 6t - 6$

To find when acceleration is 12, solve  $a(t) = 12 \Rightarrow$

$$6t - 6 = 12 \Rightarrow 6t = 18 \Rightarrow \boxed{t = 3 \text{ seconds}}$$

Thus acceleration is 12 at  $t = 3$  seconds.

At this time the velocity is  $v(3) = 3 \cdot 3^2 - 6 \cdot 3 = \boxed{9 \text{ ft/sec}}$

5. (7 points) Suppose  $f(x) = x^2 + 2x^3$  and  $g(x) = x^2 - 2x^3 + 48x$ . Find all  $x$  for which the tangent to  $y = f(x)$  at  $(x, f(x))$  is parallel to the tangent to  $y = g(x)$  at  $(x, g(x))$ .

Solve  $f'(x) = g'(x)$

$$2x + 6x^2 = 2x - 6x^2 + 48$$

$$12x^2 - 48 = 0$$

$$12(x^2 - 4) = 0$$

$$12(x - 2)(x + 2) = 0$$

$$\boxed{x = 2 \quad x = -2}$$

Answer:

Tangents are parallel when

$$x = -2 \text{ and } x = 2$$

6. (35 points) Find the derivatives of these functions. You do **not** need to simplify your answers.

$$(a) f(x) = \frac{\sqrt{2}}{x} + \pi x = \sqrt{2} x^{-1} + \pi x, \quad f'(x) = -\sqrt{2} x^{-2} + \pi = \pi - \frac{\sqrt{2}}{x^2}$$

$$(b) f(x) = \cos(x) \sin(x) \quad f'(x) = -\sin(x) \sin(x) + \cos(x) \cos(x) = \cos^2(x) - \sin^2(x)$$

$$(c) f(x) = \cos(\sin(x)) \quad f'(x) = -\sin(\sin(x)) \cdot \cos(x)$$

$$(d) f(x) = \tan^{-1}(-x) \quad f'(x) = \frac{1}{1+(-x)^2} (-1) = \frac{-1}{1+x^2}$$

$$(e) f(x) = \ln(e^{x^2-3x} + x) \quad f'(x) = \frac{e^{x^2-3x} (2x-3) + 1}{e^{x^2-3x} + x}$$

$$(f) f(x) = \frac{1}{x^2+5x-7} = (x^2+5x-7)^{-1} \quad f'(x) = -(x^2+5x-7)^{-2} (2x+5) = \frac{-2x-5}{(x^2+5x-7)^2}$$

$$(g) f(x) = \sqrt{\frac{x+1}{x-1}}^3 = \left(\frac{x+1}{x-1}\right)^{\frac{3}{2}}$$

$$f'(x) = \frac{3}{2} \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}} \frac{1(x-1) - (x+1)(1)}{(x-1)^2} = \frac{3}{2} \sqrt{\frac{x+1}{x-1}} \frac{-2}{(x-1)^2} = -3 \sqrt{\frac{x+1}{x-1}} \cdot \frac{1}{(x-1)^2}$$

7. (7 points) Given the equation  $\frac{x}{y} = y^5 + x$ , find  $y'$ .

$$D_x \left[ \frac{x}{y} \right] = D_x [y^5 + x]$$

$$\frac{y - xy'}{y^2} = 5y^4 y' + 1$$

$$y - xy' = y^2 (5y^4 y' + 1)$$

$$y - xy' = 5y^6 y' + y^2$$

$$y - y^2 = 5y^6 y' + xy'$$

$$y - y^2 = (5y^6 + x) y'$$

$$y' = \frac{y - y^2}{5x^6 + x}$$

8. (7 points) Find the derivative of  $f(x) = x^{\ln(x)}$ .

← Logarithmic Differentiation.

$$y = x^{\ln(x)}$$

$$\ln(y) = \ln(x^{\ln(x)})$$

$$\ln(y) = \ln(x) \cdot \ln(x)$$

$$D_x [\ln(y)] = D_x [\ln(x) \cdot \ln(x)]$$

$$\frac{y'}{y} = \frac{1}{x} \ln(x) + \ln(x) \frac{1}{x}$$

$$\frac{y'}{y} = 2 \frac{\ln(x)}{x}$$

$$y' = 2y' \frac{\ln(x)}{x}$$

$$y' = \frac{2x^{\ln(x)} \ln(x)}{x}$$