Fall 2018 Research

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1 Theorems

Lemma 1.1. *Suppose G or H is even palindromic, then $G \times H$ is even palindromic.*

Proof. Suppose, without loss of generality, that G is even palindromic. Let $\alpha : G \to G$ defined by $x \mapsto \alpha(x)$ be even palindromic. Then $\varphi : G \times H \to G \times H$ defined by $(x, y) \mapsto (\alpha(x), y)$ is an even palindromic involution. Since α has no fixed points, $G \times H$ is even palindromic. □

Lemma 1.2. *Suppose G and H are odd palindromic, then $G \times H$ is odd palindromic.*

Proof. Suppose G and H are odd palindromic. Then G has an involution α with exactly one fixed vertex x_0, so $\alpha(x_0) = x_0$. Likewise, H has an involution β with exactly one fixed vertex y_0, so $\beta(y_0) = y_0$. Then $\varphi : G \times H \to G \times H$ defined by $(x, y) \mapsto (\alpha(x), \beta(y))$ is an involution of $G \times H$ with exactly one fixed vertex (x_0, y_0), so $G \times H$ is odd palindromic. □

Theorem 1.3. *(Handbook of Product Graphs Theorem 8.18)*

Suppose φ is an automorphism of a connected nonbipartite R-thin graph G that has a prime factorization $G = G_1 \times G_2 \times ... \times G_k$. Then there exists a permutation of π of $\{1, 2, ..., k\}$, together with isomorphisms $\varphi_i : G_{\pi(i)} \to G_i$ such that $\varphi(x_1, x_2, ..., x_k) = (\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), ..., \varphi_k(x_{\pi(k)}))$. Thus $Aut(G)$ is generated by the automorphisms of the prime factors and transpositions of isomorphic factors. Consequently, $Aut(G)$ is isomorphic to the automorphism group of the disjoint union of the prime factors of G.

Lemma 1.4. *Suppose G and H are both connected, nonbipartite, R-thin, and prime. Then:*

1. *$G \times H$ is even palindromic if and only if G or H is even palindromic*

2. *$G \times H$ is odd palindromic if and only if G and H are odd palindromic*

Proof. Suppose $G \times H$ is palindromic. Then there exists an involution $\varphi : G \times H \to G \times H$ with at most one fixed point. Suppose G and H are both connected, nonbipartite, R-thin, and prime.

Case 1: φ operates such that $\varphi(x, y) = (\alpha(x), \beta(y))$ where $\alpha : G \to G$ and $\beta : H \to H$. Observe that $\varphi(x, y) = (\alpha(x), \beta(y))$ and $\varphi^2(x, y) = (\alpha^2(x), \beta^2(y)) = (x, y)$, since $G \times H$ is palindromic. Thus, $\alpha^2(x) = x$ and $\beta^2(y) = y$ which means α and β are involutions.

Now, suppose φ has no fixed points. Then $\varphi(x, y) \neq (x, y)$ for all $(x, y) \in V(G \times H)$. Thus, $\alpha(x) \neq x$ for all $x \in G$ or $\beta(y) \neq y$ for all $y \in H$. So α or β has no fixed points. Therefore, one of G or H is even palindromic.

Now, suppose φ has one fixed point, so $\varphi(x, y) = (x, y)$. Then $\alpha(x) = x$ and $\beta(y) = y$. Since φ may have at most one fixed point, both α and β have exactly one fixed point. If one of α or β has more than one fixed point, say α, then $\alpha(x) = x$ and $\alpha(x') = x'$ for some $x \neq x'$. Thus, $\varphi(x, y) = (x, y)$ and $\varphi(x', y) = (x', y)$. Therefore, φ would have more than one fixed point. Since that cannot be the case, both G and H are odd palindromic.

Case 2: φ operates such that $\varphi(x, y) = (\alpha(y), \beta(x))$ where $\alpha : H \to G$ and $\beta : G \to H$ such that $G \cong H$. We will show Case 2 cannot happen.

Since φ is an involution, $\varphi^2(x, y) = (x, y)$. Thus, if $\varphi(x, y) = (\alpha(y), \beta(x))$, then $\varphi^2(x, y) = \varphi(\alpha(y), \beta(x)) = (\alpha(\beta(x)), \beta(\alpha(y)) = (x, y)$. So $\alpha(\beta(x)) = x$ and $\beta(\alpha(y)) = y$. Therefore, $\beta(x) = \alpha^{-1}(x)$ and $\alpha(y) = \beta^{-1}(y)$. We can rewrite φ as $\varphi(x, y) = (\alpha(y), \alpha^{-1}(x))$.

Suppose φ is an involution, and φ has no fixed points. Then $\varphi(x, y) \neq (x, y)$, so fix an (x, y) such that $\varphi(x, y) = (x', y')$ for some $(x', y') \in G \times H$ where $(x', y') \neq (x, y)$. Note that we will never have a situation such as $\varphi(x, y) = (\alpha(y), \alpha^{-1}(x)) = (x', y)$ where one coordinate element is fixed, since this implies that $\alpha(y) = x'$ and $\alpha^{-1}(x) = y$ which means α^{-1} cannot be the inverse of α. Now, when $\varphi(x, y) = (x', y')$, this means $\alpha(y) = x'$ and $\alpha^{-1}(x) = y'$. Since φ is an involution, we have $\varphi(x', y') = (x, y)$. So $\alpha(y') = x$ and $\alpha^{-1}(x') = y$. Now, note that $\varphi(x, y') = (\alpha(y'), \alpha^{-1}(x)) = (x, y')$. So (x, y') is a fixed point of φ. This contradicts our original assumption that φ has no fixed points.

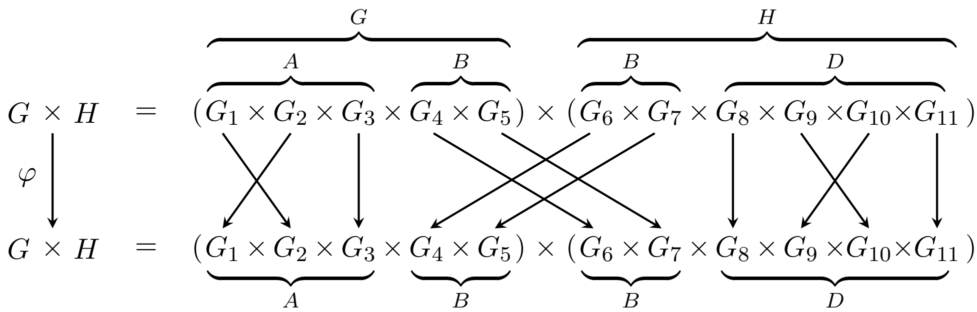
Thus, φ must have a fixed point. So, suppose φ is an involution and has exactly one fixed point. Observe from our previous construction that our fixed point was at (x, y'). However, observe that $\varphi(x', y) = (\alpha(y), \alpha^{-1}(x')) = (x', y)$. Thus, (x', y) is also a fixed point of φ. Therefore, φ has more than one fixed point.

Thus, case 2 cannot happen. Therefore, G or H is even palindromic, or G and H are both odd palindromic. □

Lemma 1.5. *Suppose $G \times H$ is palindromic, and G and H are both connected, nonbipartite, and R-thin. Then G or H is even palindromic, or G and H are both odd palindromic.*

Proof. Suppose $G \times H$ is palindromic with palindromic involution φ. Consider the connected, nonbipartite, R-thin, prime factorings $G = G_1 \times \cdots \times G_j$ and $H = G_{j+1} \times \cdots \times G_k$, so we have an involution φ of $G \times H = (G_1 \times \cdots \times G_j) \times (G_{j+1} \times \cdots \times G_k)$.

Utilizing Theorem 1.3, the involution φ permutes the prime factors of this product such that the permutation π satisfies $\pi^2 = \text{id}$. Using commutativity of $\times$, we can group together the prime factors G_i of G for which $1 < \pi(i) \leq j$, and call their product A. Note that $A = K_1^*$, where K_1^* is a single vertex with a loop, if no such factors G_i exist. The same applies for the graphs B and D defined below. Let B be the product of the remaining factors G_i of G. Also group together the prime factors G_i of H for which $j + 1 < \pi(i) \leq k$, and call their product D. The direct product of the remaining factors of H is then a graph isomorphic to B. The structure of φ under this scheme is as indicated below, where the arrows represent isomorphisms $\varphi_i : G_{\pi(i)} \to G_i$ between factors.



We have now coordinatized G and H as $G = A\times B$ and $H = B\times D$, and φ is an involution of $G\times H = (A\times B)\times(B\times D)$ for which $\varphi\big((a,b),(b',d)\big) = \big((\alpha(a),\beta(b')),(\gamma(b),\delta(d))\big)$, for automorphisms $\alpha : A \to A$, $\beta,\gamma : B \to B$ and $\delta : D \to D$. But because φ^2 is the identity, it must be that $\alpha^2 = \mathrm{id}$, $\gamma = \beta^{-1}$ and $\delta^2 = \mathrm{id}$. Thus we have involutions α and δ of A and D, respectively, and

$$\varphi\big((a,b),(b',d)\big) = \big((\alpha(a),\beta(b')),(\beta^{-1}(b),\delta(d))\big), \tag{1}$$

From (1) it is evident that the fixed points of φ (if any) are precisely

$$\big((a_0,\beta(b)),(b,d_0)\big) \quad \text{with } \alpha(a_0) = a_0,\ \delta(d_0) = d_0, \text{ and } b \in V(B). \tag{2}$$

Thus φ has a fixed point if and only if both α and δ have fixed points. Further, if φ has a fixed point, then it has exactly $|V(B)|$ of them.

Now suppose $G \times H$ is even palindromic. Let φ be an even palindromic involution of $G \times H$ (having no fixed point). From (2), at least one of α or δ has no fixed point, so suppose it is α. Then α is an even palindromic involution of A, so A is even palindromic. By the first part of the theorem, $G = A \times B$ is even palindromic. Similarly H is even palindromic if δ has no fixed points.

Suppose $G\times H$ is odd palindromic. Let φ be an odd palindromic involution whose sole fixed point is $\big((a_0,\beta(b_0)),(b_0,d_0)\big)$. The remark following (2) implies φ has at least $|V(B)|$ fixed points, so $B = K_1$. Thus we can drop B from our discussion, so $G = A$, $H = D$ and $\varphi(a,d) = \big(\alpha(a),\delta(d)\big)$. We now have involutions $\alpha : G \to G$ and $\delta : H \to H$ with fixed points a_0 and d_0, respectively. Also (a_0,d_0) is a fixed point of φ. If the involution

α of G had a second fixed point a_1, then (a_0, d_0) and (a_1, d_0) would be two distinct fixed points of φ. Thus a_0 is the only fixed point of α, so α (hence also G) is odd palindromic. By the same reasoning H is odd palindromic. □

We say that vertices x and y of a graph are in relation R, written xRy, provided that each has the same open neighborhood, that is, $N_G(x) = N_G(y)$. It is straightforward to check that R is an equivalence relation on G.

Proposition 1.6. *R is an equivalence relation on G.*

Proof. To see that R is reflexive, let $x \in V(G)$. Then $N(x) = N(x)$, so $xRx\ \forall x \in V(G)$.

To see that R is symmetric, let $x, y \in V(G)$. Then if $N(x) = N(y)$, then $N(y) = N(x)$, so xRy implies that $yRx\ \forall x, y \in V(G)$.

To see that R is transitive, suppose xRy and yRz. Then $N(x) = N(y)$ and $N(y) = N(z)$. Thus, $N(x) = N(z)$, so xRz. □

We will refer to an R-equivalence class of $V(G)$ as an **R-class**.

Proposition 1.7. *The subgraph induced on an R-class is either completely disconnected or is the complete graph with loops at each vertex.*

Proof. Let $x_1, ..., x_n$ be vertices in the R-class $[x]$. Thus, $N(x_1) = N(x_2) = ... = N(x_n)$.

Case 1: $x_1 \notin N(x_1)$

If x_1 is not in its own neighborhood (i.e. there is no loop at x_1), then $x_1 \notin N(x_2) = ... = N(x_n)$. Likewise, if $x_2 \notin N(x_2)$, then $x_2 \notin N(x_1) = N(x_3) = ... = N(x_n)$. Now, assume towards a contradiction that $x_1x_2 \in E(G)$. Then $x_1 \in N(x_2)$, so $x_1 \in N(x_1)$ in order for $N(x_1) = N(x_2)$, but we assumed $x_1 \notin N(x_1)$, so no such edge can exist, and such is true for all pairs of vertices in $[x]$. So, the subgraph induced on $[x]$ has no edges in it, thus it is completely disconnected.

Case 2: $x_1 \in N(x_1)$

If $x_1 \in N(x_1)$, then $x_1 \in N(x_2) = N(x_3) = ... = N(x_n)$. So, x_1 has a loop, and if $x_1 \in N(x_1)$, then $x_1 \in N(x_2)$ and $x_2 \in N(x_1)$, thus $x_2 \in N(x_2)$. So, x_2 also has a loop, and it must be that $x_1x_2 \in E(G)$. Thus, if $x_1 \in N(x_1)$, it must be in the neighborhood of every other vertex of $[x]$, so every other vertex of $[x]$ must be in their own neighborhoods, so each of them will have a loop, and they all must be connected to one another in order for their neighborhoods to remain equal. Thus, the subgraph induced on $[x]$ is the complete graph with loops on each vertex. $\square$

Let G/R be the graph whose vertices are the R-classes of a graph G in Γ_0 (where Γ_0 is the set of finite graphs in which loops are admitted). A graph is called **R-thin** if all of its R-classes contain just one vertex. Let $[x] = \{y \in V(G) | N_G(x) = N_G(y)\}$. Check that G/R is always R-thin.

Proposition 1.8. *G/R is always R-thin.*

Proof. Suppose not. Then for two distinct $[x], [y] \in V(G/R)$, we must have $N([x]) = N([y])$. The neighborhoods of $[x]$ and $[y]$ correspond to neighborhoods of the vertices in G, and vertices in G/R are adjacent only if their corresponding vertices were adjacent in G. So, in G, we must have that $\forall x \in [x]$ and $\forall y \in [y]$, $N(x) = N(y)$. Thus, all such x and y are in the same R-class, which means that $[x] = [y]$. Thus, G/R is always R-thin. $\square$

Because R is defined in terms of the adjacency structure of a graph, given an isomorphism $\varphi : G \to H$, we have xR_Gy if and only if $\varphi(x)R_H\varphi(y)$. So φ maps equivalence classes of R_G to equivalence classes of R_H, and, in particular, $\varphi([x]) = [\varphi(x)]$.

It is the case that for any isomorphism $\varphi : G \to H$, there is a corresponding isomorphism $\tilde{\varphi} : G/R \to H/R$ such that $\tilde{\varphi}([x]) = [\varphi(x)]$. (cite HoPG Prop 8.3)

However, the existence of the isomorphism $\tilde{\varphi} : G/R \to H/R$ does not necessarily mean there is an isomorphism $\varphi : G \to H$. But, if we let $|X| = |\tilde{\varphi}(X)|$ for each $X \in V(G/R)$, then we can lift $\tilde{\varphi}$ to an isomorphism $\varphi : G \to H$ by declaring φ to restrict to a bijection $X \to \tilde{\varphi}(X)$ for each X.

We will make use of the following propositions:

Proposition 1.9. $N_{G\times H}((x, y)) = N_G(x) \times N_H(y)$

Proof. For $x \in V(G)$, consider $N_G(x) = \{u \mid ux \in E(G)\}$, and for $y \in V(H)$, consider

$N_H(y) = \{v \mid vy \in E(H)\}$. Then:

$$\begin{aligned}
(u,v) \in N_G(x) \times N_H(y) &\iff u \in N_G(x), v \in N_H(y) \\
&\iff xu \in E(G), yv \in E(H) \\
&\iff (x,y)(u,v) \in E(G \times H) \\
&\iff (u,v) \in N_{G\times H}((x,y))
\end{aligned}$$

Thus, $N_{G\times H}((x,y)) = N_G(x) \times N_N(y)$. □

Proposition 1.10. *If G and H in Γ_0 has no isolated vertices, then $V((G \times H)/R) = \{X \times Y \mid X \in V(G/R),\ Y \in V(H/R)\}$. In particular, $[(x,y)] = [x] \times [y]$.*

Proof. For some arbitrary $[(x,y)] \in (G \times H)/R$, observe that $(x',y') \in [(x,y)] \iff N_{G\times H}((x',y') = N_{G\times H}((x,y))$. Then, by Proposition 1.9, $(x',y') \in [(x,y)] \iff N_G(x') \times N_H(y') = N_G(x) \times N_H(y)$. So, $(x',y') \in [(x,y)] \iff N_G(x') = N_G(x)$ and $N_H(y') = N_H(y)$, since there are no isolated vertices. Then, $(x',y') \in [(x,y)] \iff x' \in [x]$ and $y' \in [y]$, so $(x',y') \in [x] \times [y]$. Therefore, $[(x,y)] = [x] \times [y]$. □

Proposition 1.11. *For $G \in \Gamma_0$, $xy \in E(G) \iff [x][y] \in E(G/R)$.*

Proof. By definition, if $xy \in E(G)$ then $[x][y] \in E(G/R)$.

Now, let $[x][y] \in E(G/R)$. Then there exists an $x'y' \in E(G)$ such that $x' \in [x]$, $x' \in N(y')$, $y' \in [y]$, and $y' \in N(x')$. Now, $N(x') = N(x)$ and $N(y') = N(y)$. Thus, $x' \in N(y') = N(y)$ and $y' \in N(x') = N(x)$, so $xy \in E(G)$. □

Proposition 1.12. $(G \times H)/R \cong G/R \times H/R$ *with isomorphism* $[(x,y)] \mapsto ([x],[y])$.

Proof. Consider $[(x,y)][(x',y')] \in E(G\times H)/R)$. By Proposition 5, $[(x,y)][x',y'] \in E(G\times H)/R) \iff (x,y)(x',y') \in E(G \times H) \iff xx' \in E(G), yy' \in E(H) \iff [x][x'] \in E(G/R), [y][y'] \in E(H/R) \iff ([x],[y])([x'],[y']) \in E(G/R \times H/R)$. Thus $[(x,y)] \mapsto ([x],[y])$. So $(G \times H)/R \cong G/R \times H/R$. □

Theorem 1.13. *Suppose G and H are connected and non-bipartite. Then:*

1. *G or H is even palindromic if and only if $G \times H$ is even palindromic*

2. G and H are odd palindromic if and only if $G \times H$ is odd palindromic

Proof. If G or H (say G) is even palindromic, then there exists an even palindromic involution α of G, so $(x, y) \mapsto (\alpha(x), y)$ is an even palindromic involution of $G \times H$. Now, suppose G and H are odd palindromic. Then G has an odd palindromic involution α with fixed point x_0, and H has an odd palindromic involution β with fixed point y_0. Then $(x, y) \mapsto (\alpha(x), \beta(y))$ is an odd palindromic involution of $G \times H$ who sole fixed point is (x_0, y_0).

Part I (Involution Structure) Let $\varphi : G \times H \to G \times H$ be an involution. By the remarks preceding this theorem, φ induces an automorphism $\tilde{\varphi}$ of the R-thin graph $(G \times H)/R \cong G/R \times H/R$. Because φ is an involution, we have $\tilde{\varphi}^2 = id$.

Take prime factorings $G/R = G_1 \times ... \times G_j$ and $H/R = G_{j+1} \times ... \times G_k$. Then $\tilde{\varphi}$ is an automorphism (of order 2, or possibly of order 1, if φ fixes each R-class) of the graph $G/R \times H/R = (G_1 \times ... \times G_j) \times (G_{j+1} \times G_k)$. Now, $\tilde{\varphi}$ permutes the prime factors of this product in the sense of Theorem 1.3, where the permutation π satisfies $\pi^2 = id$. As in the proof of Lemma 1.5 $\square$