CHAPTER 25

Derivatives of Inverse Trig Functions

Our goal is simple, and the answers will come quickly. We will derive six new derivative formulas for the six inverse trigonometric functions:

$$\frac{d}{dx}\left[\sin^{-1}(x)\right] \qquad \frac{d}{dx}\left[\tan^{-1}(x)\right] \qquad \frac{d}{dx}\left[\sec^{-1}(x)\right]$$
$$\frac{d}{dx}\left[\cos^{-1}(x)\right] \qquad \frac{d}{dx}\left[\cot^{-1}(x)\right] \qquad \frac{d}{dx}\left[\csc^{-1}(x)\right]$$

These formulas will flow from the inverse rule from Chapter 24 (page 280):

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'\big(f^{-1}(x)\big)}. \tag{25.1}$$

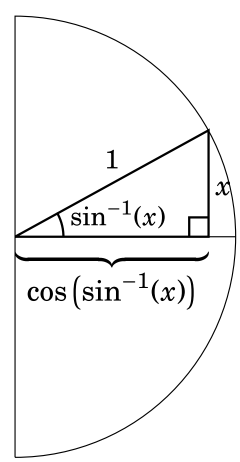
25.1 Derivatives of Inverse Sine and Cosine

Applying the inverse rule (25.1) with $f(x) = \sin(x)$ yields

$$\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\cos\big(\sin^{-1}(x)\big)}. \tag{25.2}$$

We are almost there. We just have to simplify the $\cos\big(\sin^{-1}(x)\big)$ in the denominator. To do this recall

$$\sin^{-1}(x) = \left(\begin{array}{l} \text{the angle } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ \text{for which } \sin(\theta) = x \end{array}\right).$$



Thus $\sin^{-1}(x)$ is the angle (between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$) of a the triangle on the unit circle whose opposite side is x. (Because sin of this angle equals x.) Then $\cos\big(\sin^{-1}(x)\big)$ is the length of the adjacent side. By the Pythagorean theorem this side length is $\sqrt{1-x^2}$. Putting $\cos\big(\sin^{-1}(x)\big) = \sqrt{1-x^2}$ into the above Equation (25.2), we get our latest rule:

$$\boxed{\textbf{Rule 20} \quad \frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}}$$

We reviewed $\sin^{-1}(x)$ In Section 6.1 and presented its graph on page 101. Figure 25.1 repeats the graph, along with the derivative from Rule 20.

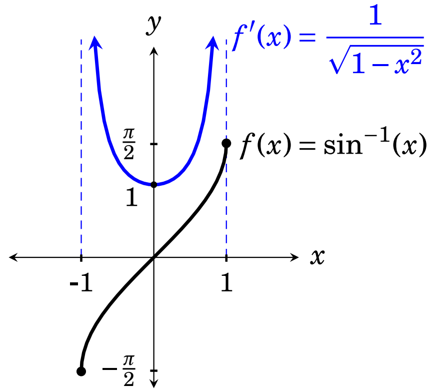


Figure 25.1. The function $\sin^{-1}(x)$ and its derivative. The derivative is always positive, reflecting the fact that the tangents to $\sin^{-1}(x)$ have positive slope. The derivative has vertical asymptotes at $x = \pm 1$, as the tangents to $\sin^{-1}(x)$ become increasingly steep as x approaches ± 1.

Now consider $\cos^{-1}(x)$. The tangents to its graph (Figure 25.2 below) have *negative* slope, and the geometry suggests that its derivative is *negative* the derivative of $\sin^{-1}(x)$. Indeed this turns out to be exactly the case. This chapter's Exercise 1 asks you to prove our next rule:

$$\boxed{\textbf{Rule 21} \quad \frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}}$$

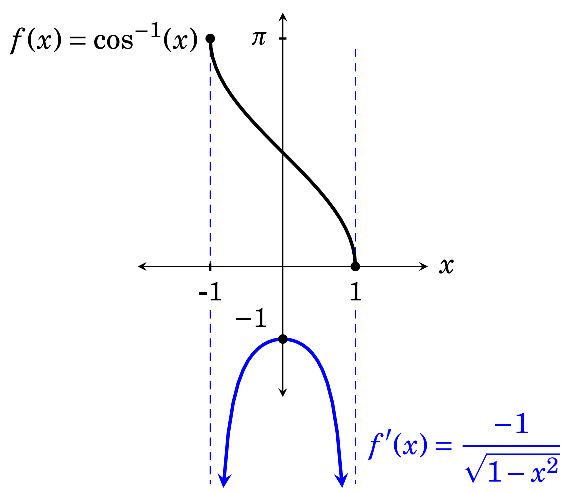


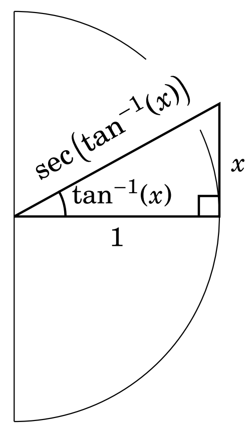
Figure 25.2. The function $\cos^{-1}(x)$ and its derivative.

25.2 Derivatives of Inverse Tangent and Cotangent

Now let's find the derivative of $\tan^{-1}(x)$. Putting $f(x) = \tan(x)$ into the inverse rule (25.1), we have $f^{-1}(x) = \tan^{-1}(x)$ and $f'(x) = \sec^2(x)$, and we get

$$\frac{d}{dx}\Big[\tan^{-1}(x)\Big] = \frac{1}{\sec^2\big(\tan^{-1}(x)\big)} = \frac{1}{\big(\sec\big(\tan^{-1}(x)\big)\big)^2}. \tag{25.3}$$

The expression $\sec\big(\tan^{-1}(x)\big)$ in the denominator is the length of the hypotenuse of the triangle to the right. (See example 6.3 in Chapter 6, page 114.) By the Pythagorean theorem, the length is $\sec\big(\tan^{-1}(x)\big) = \sqrt{1+x^2}$. Inserting this into the above Equation (25.4) yields



$$\frac{d}{dx}\Big[\tan^{-1}(x)\Big] = \frac{1}{\big(\sec\big(\tan^{-1}(x)\big)\big)^2} = \frac{1}{\big(\sqrt{1+x^2}\big)^2} = \frac{1}{1+x^2}.$$

We now have: $\boxed{\textbf{Rule 22} \quad \frac{d}{dx}\Big[\tan^{-1}(x)\Big] = \frac{1}{1+x^2}}$

We discussed $\tan^{-1}(x)$ in Chapter 6, and its graph is in Figure 6.3. Below Figure 25.3 repeats the graph, along with the derivative $\frac{1}{x^2+1}$.

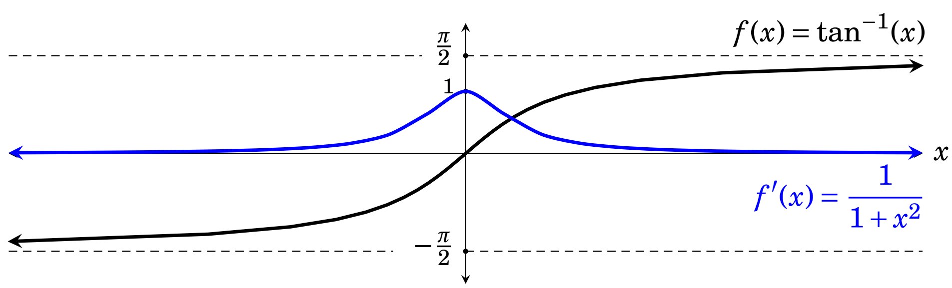


Figure 25.3. The function $\tan^{-1}(x)$ and its derivative $\frac{1}{1+x^2}$. Note $\lim_{x\to\infty} \frac{1}{1+x^2} = 0$ and $\lim_{x\to-\infty} \frac{1}{1+x^2} = 0$, reflecting the fact that the tangent lines to $y = \tan^{-1}(x)$ become closer and closer to horizontal as $x \to \pm\infty$. The derivative bumps up to 1 at $x = 0$, where the tangent to $y = \tan^{-1}(x)$ is steepest, with slope 1

Exercise 3 below asks you to mirror the above arguments to deduce:

$$\boxed{\textbf{Rule 23} \quad \frac{d}{dx}\Big[\cot^{-1}(x)\Big] = \frac{-1}{1+x^2}}$$

25.3 Derivatives of Inverse Secant and Cosecant

We reviewed $\sec^{-1}(x)$ in Section 6.3. For its derivative, put $f(x) = \sec(x)$ into the inverse rule (25.1), with $f^{-1}(x) = \sec^{-1}(x)$ and $f'(x) = \sec(x)\tan(x)$. We get

$$\begin{aligned}\frac{d}{dx}\Big[\sec^{-1}(x)\Big] &= \frac{1}{f'\big(f^{-1}(x)\big)}\\ &= \frac{1}{\sec\big(f^{-1}(x)\big)\cdot\tan\big(f^{-1}(x)\big)}\\ &= \frac{1}{\sec\big(\sec^{-1}(x)\big)\cdot\tan\big(\sec^{-1}(x)\big)}.\end{aligned}$$

Because $\sec\big(\sec^{-1}(x)\big) = x$, the above becomes

$$\frac{d}{dx}\Big[\sec^{-1}(x)\Big] = \frac{1}{x\cdot\tan\big(\sec^{-1}(x)\big)}. \tag{25.4}$$

In Example 6.5 we showed that $\tan\big(\sec^{-1}(x)\big) = \begin{cases} \sqrt{x^2-1} & \text{if } x \text{ is positive} \\ -\sqrt{x^2-1} & \text{if } x \text{ is negative} \end{cases}$
With this, Equation 25.4 above becomes

$$\frac{d}{dx}\Big[\sec^{-1}(x)\Big] = \begin{cases} \dfrac{1}{x\sqrt{x^2-1}} & \text{if } x \text{ is positive} \\ \dfrac{1}{-x\sqrt{x^2-1}} & \text{if } x \text{ is negative.} \end{cases}$$

But if x is negative, then $-x$ is positive, and the above consolidates to

$$\boxed{\textbf{Rule 24} \quad \frac{d}{dx}\Big[\sec^{-1}(x)\Big] = \frac{1}{|x|\sqrt{x^2-1}}}$$

This graph of $\sec^{-1}(x)$ and its derivative is shown in Figure 25.3.

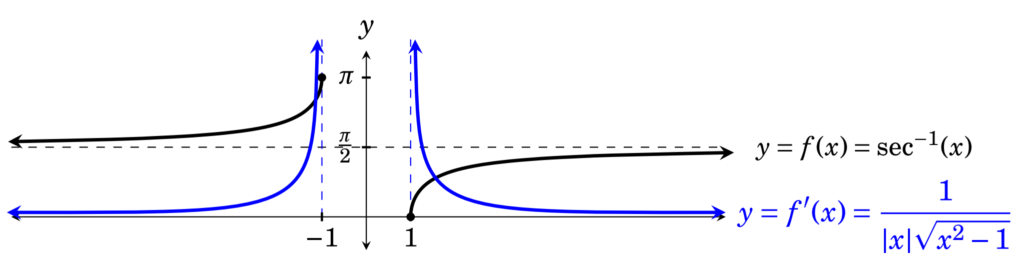


Figure 25.4. The graph of $\sec^{-1}(x)$ and its derivative. The domain of both functions is $(-\infty,-1]\cup[1,\infty)$. Note that the derivative has vertical asymptotes at $x = \pm 1$, where the tangent line to $y = \sec^{-1}(x)$ is vertical.

This chapter's Exercise 2 asks you to use reasoning similar to the above to deduce our final rule.

$$\boxed{\textbf{Rule 25} \quad \frac{d}{dx}\Big[\csc^{-1}(x)\Big] = \frac{-1}{|x|\sqrt{x^2-1}}}$$

Each of our new rules has a chain rule generalization. For example, Rule 25 generalizes as

$$\frac{d}{dx}\Big[\csc^{-1}\big(g(x)\big)\Big] = \frac{-1}{\big|g(x)\big|\sqrt{(g(x))^2-1}}g'(x) = \frac{-g'(x)}{\big|g(x)\big|\sqrt{(g(x))^2-1}}.$$

Here is a summary of this Chapter's new rules, along with their chain rule generalizations.

$\frac{d}{dx}\Big[\sin^{-1}(x)\Big] = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\Big[\sin^{-1}\big(g(x)\big)\Big] = \frac{1}{\sqrt{1-(g(x))^2}}g'(x)$
$\frac{d}{dx}\Big[\cos^{-1}(x)\Big] = \frac{-1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\Big[\cos^{-1}\big(g(x)\big)\Big] = \frac{-1}{\sqrt{1-(g(x))^2}}g'(x)$
$\frac{d}{dx}\Big[\tan^{-1}(x)\Big] = \frac{1}{1+x^2}$	$\frac{d}{dx}\Big[\tan^{-1}\big(g(x)\big)\Big] = \frac{1}{1+(g(x))^2}g'(x)$
$\frac{d}{dx}\Big[\cot^{-1}(x)\Big] = \frac{-1}{1+x^2}$	$\frac{d}{dx}\Big[\cot^{-1}\big(g(x)\big)\Big] = \frac{-1}{1+(g(x))^2}g'(x)$
$\frac{d}{dx}\Big[\sec^{-1}(x)\Big] = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}\Big[\sec^{-1}\big(g(x)\big)\Big] = \frac{g'(x)}{\big\lvert g(x)\big\rvert\sqrt{(g(x))^2-1}}$
$\frac{d}{dx}\Big[\csc^{-1}(x)\Big] = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}\Big[\csc^{-1}\big(g(x)\big)\Big] = \frac{-g'(x)}{\big\lvert g(x)\big\rvert\sqrt{(g(x))^2-1}}$

Example 25.1 $\frac{d}{dx}\Big[\sqrt{\cos^{-1}(x)}\Big] = \frac{d}{dx}\Big[\big(\cos^{-1}(x)\big)^{\frac{1}{2}}\Big] = \frac{1}{2}\big(\cos^{-1}(x)\big)^{-\frac{1}{2}}\frac{d}{dx}\Big[\cos^{-1}(x)\Big]$

$$= \frac{1}{2}\big(\cos^{-1}(x)\big)^{-\frac{1}{2}}\frac{-1}{\sqrt{1-x^2}} = \boxed{\frac{-1}{2\sqrt{\cos^{-1}(x)}\sqrt{1-x^2}}.}$$

✍

Example 25.2 $\frac{d}{dx}\Big[e^{\tan^{-1}(x)}\Big] = e^{\tan^{-1}(x)}\frac{d}{dx}\Big[\tan^{-1}(x)\Big] = e^{\tan^{-1}(x)}\frac{1}{1+x^2} = \boxed{\frac{e^{\tan^{-1}(x)}}{1+x^2}.}$

Example 25.3 $\frac{d}{dx}\Big[\tan^{-1}\big(e^x\big)\Big] = \frac{1}{1+(e^x)^2}\frac{d}{dx}\Big[e^x\Big] = \frac{1}{1+e^{2x}}e^x = \boxed{\frac{e^x}{1+e^{2x}}.}$ ✍

25.4 Summary of Derivative Rules

We have reached the end of our derivative rules. In summary, we have the following rules for specific functions. The corresponding chain rule generalizations are shown to the right.

	Rule	**Chain Rule Generalization**
Constant Rule	$\frac{d}{dx}[c] = 0$	
Identity Rule	$\frac{d}{dx}[x] = 1$	
Power Rule	$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$	$\frac{d}{dx}\left[\left(g(x)\right)^n\right] = n\left(g(x)\right)^{n-1}g'(x)$
Exp Rules	$\frac{d}{dx}\left[e^x\right] = e^x$ $\frac{d}{dx}\left[a^x\right] = \ln(a)a^x$	$\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)}g'(x)$ $\frac{d}{dx}\left[a^{g(x)}\right] = \ln(a)a^{g(x)}g'(x)$
Log Rules	$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}$ $\frac{d}{dx}\left[\log_a(x)\right] = \frac{1}{x\ln(a)}$	$\frac{d}{dx}\left[\ln\left(g(x)\right)\right] = \frac{1}{g(x)}g'(x)$ $\frac{d}{dx}\left[\log_a\left(g(x)\right)\right] = \frac{1}{g(x)\ln(a)}g'(x)$
Trig Rules	$\frac{d}{dx}\left[\sin(x)\right] = \cos(x)$ $\frac{d}{dx}\left[\cos(x)\right] = -\sin(x)$ $\frac{d}{dx}\left[\tan(x)\right] = \sec^2(x)$ $\frac{d}{dx}\left[\cot(x)\right] = -\csc^2(x)$ $\frac{d}{dx}\left[\sec(x)\right] = \sec(x)\tan(x)$ $\frac{d}{dx}\left[\csc(x)\right] = -\csc(x)\cot(x)$	$\frac{d}{dx}\left[\sin\left(g(x)\right)\right] = \cos\left(g(x)\right)g'(x)$ $\frac{d}{dx}\left[\cos\left(g(x)\right)\right] = -\sin\left(g(x)\right)g'(x)$ $\frac{d}{dx}\left[\tan\left(g(x)\right)\right] = \sec^2\left(g(x)\right)g'(x)$ $\frac{d}{dx}\left[\cot\left(g(x)\right)\right] = -\csc^2\left(g(x)\right)g'(x)$ $\frac{d}{dx}\left[\sec\left(g(x)\right)\right] = \sec\left(g(x)\right)\tan\left(g(x)\right)g'(x)$ $\frac{d}{dx}\left[\csc\left(g(x)\right)\right] = -\csc\left(g(x)\right)\cot\left(g(x)\right)g'(x)$
Inverse Trig Rules	$\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2}$ $\frac{d}{dx}\left[\cot^{-1}(x)\right] = \frac{-1}{1+x^2}$ $\frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$ $\frac{d}{dx}\left[\csc^{-1}(x)\right] = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}\left[\sin^{-1}\left(g(x)\right)\right] = \frac{1}{\sqrt{1-(g(x))^2}}g'(x)$ $\frac{d}{dx}\left[\cos^{-1}\left(g(x)\right)\right] = \frac{-1}{\sqrt{1-(g(x))^2}}g'(x)$ $\frac{d}{dx}\left[\tan^{-1}\left(g(x)\right)\right] = \frac{1}{1+(g(x))^2}g'(x)$ $\frac{d}{dx}\left[\cot^{-1}\left(g(x)\right)\right] = \frac{-1}{1+(g(x))^2}g'(x)$ $\frac{d}{dx}\left[\sec^{-1}\left(g(x)\right)\right] = \frac{1}{\lvert g(x)\rvert\sqrt{(g(x))^2-1}}g'(x)$ $\frac{d}{dx}\left[\csc^{-1}\left(g(x)\right)\right] = \frac{-1}{\lvert g(x)\rvert\sqrt{(g(x))^2-1}}g'(x)$

In addition we have the following general rules for the derivatives of combinations of functions.

Constant Multiple Rule:	$\frac{d}{dx}\left[\, cf(x) \,\right]$	$=$	$cf'(x)$
Sum/Difference Rule:	$\frac{d}{dx}\left[f(x) \pm g(x)\right]$	$=$	$f'(x) \pm g'(x)$
Product Rule:	$\frac{d}{dx}\left[f(x)g(x)\right]$	$=$	$f'(x)g(x)+f(x)g'(x)$
Quotient Rule:	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$	$=$	$\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
Chain Rule:	$\frac{d}{dx}\left[\, f\big(g(x)\big) \,\right]$	$=$	$f'(g(x))\,g'(x)$
Inverse Rule:	$\frac{d}{dx}\left[\, f^{-1}(x) \,\right]$	$=$	$\frac{1}{f'\left(f^{-1}(x)\right)}$

We used this last rule, the inverse rule, to find the derivatives of $\ln(x)$ and the inverse trig functions. After it has served these purposes it is mostly retired for the remainder of Calculus I, except for the stray exercise or quiz or test question.

This looks like a lot of rules to remember, and it is. But through practice and usage you will reach the point of using them automatically, with hardly a thought. Be sure to get enough practice!

Exercises for Chapter 25

1. Show that $\frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}$.

2. Show that $\frac{d}{dx}\left[\csc^{-1}(x)\right] = \frac{-1}{|x|\sqrt{x^2-1}}$.

3. Show that $\frac{d}{dx}\left[\cot^{-1}(x)\right] = \frac{-1}{1+x^2}$.

Find the derivatives of the given functions.

4. $\sin^{-1}\left(\sqrt{2x}\right)$ **5.** $\ln\left(\tan^{-1}(x)\right)$ **6.** $e^x \tan^{-1}(x)$

7. $\tan^{-1}\left(\pi x\right)$ **8.** $\sec^{-1}\left(\pi x\right)$ **9.** $\ln\left(\sin^{-1}(x)\right)$

10. $\cos^{-1}\left(\pi x\right)$ **11.** $\sec^{-1}\left(x^5\right)$ **12.** $e^{\tan^{-1}(\pi x)}$

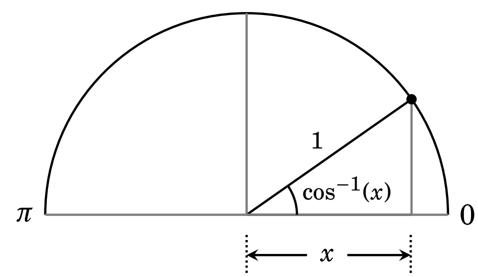
13. $\tan^{-1}\left(\ln(x)\right)+\pi$ **14.** $\tan^{-1}\left(x\sin(x)\right)$ **15.** $x\sin^{-1}\left(\ln(x)\right)$

Exercise Solutions for Chapter 25

1. Show that $\frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}$.

By the inverse rule, $\frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{1}{-\sin\left(\cos^{-1}(x)\right)}$.

Now we simplify the denominator. From the standard diagram for $\cos^{-1}(x)$ we get $\sin\left(\cos^{-1}(x)\right) = \frac{\text{OPP}}{\text{HYP}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$. With this, the above becomes $\frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}$.



3. Show that $\frac{d}{dx}\left[\cot^{-1}(x)\right] = \frac{-1}{1+x^2}$.

Suggestion: Verify the identity $\cot^{-1}(x) = \frac{\pi}{2} - \tan^{-1}(x)$. Then differentiate both sides of this.

5. $\frac{d}{dx}\left[\ln\left(\tan^{-1}(x)\right)\right] = \frac{1}{\tan^{-1}(x)}\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{\tan^{-1}(x)}\frac{1}{1+x^2} = \frac{1}{\tan^{-1}(x)\left(1+x^2\right)}$

7. $\frac{d}{dx}\left[\tan^{-1}(\pi x)\right] = \frac{\pi}{1+(\pi x)^2} = \frac{\pi}{1+\pi^2x^2}$

9. $\frac{d}{dx}\left[\ln\left(\sin^{-1}(x)\right)\right] = \frac{1}{\sin^{-1}(x)}\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sin^{-1}(x)}\frac{1}{\sqrt{1-x^2}} = \frac{1}{\sin^{-1}(x)\sqrt{1-x^2}}$

11. $\frac{d}{dx}\left[\sec^{-1}\left(x^5\right)\right] = \frac{1}{|x^5|\sqrt{\left(x^5\right)^2-1}}5x^4 = \frac{5x^4}{|x^5|\sqrt{x^{10}-1}} = \frac{5}{|x|\sqrt{x^{10}-1}}$

13. $\frac{d}{dx}\left[\tan^{-1}\left(\ln(x)\right)+\pi\right] = \frac{1}{1+\left(\ln(x)\right)^2}\frac{1}{x} = \frac{1}{x+x\left(\ln(x)\right)^2}$

15. $\frac{d}{dx}\left[x\sin^{-1}\left(\ln(x)\right)\right] = 1\cdot\sin^{-1}\left(\ln(x)\right) + x\frac{d}{dx}\left[\sin^{-1}\left(\ln(x)\right)\right]$

$$= \sin^{-1}\left(\ln(x)\right) + x\frac{1}{\sqrt{1-\left(\ln(x)\right)^2}}\frac{1}{x} = \sin^{-1}\left(\ln(x)\right) + \frac{1}{\sqrt{1-\left(\ln(x)\right)^2}}$$

CHAPTER 26

Meanings of the Derivative

In Chapter 16 we introduced the most important concept in Calculus I. The **derivative** of a function $f(x)$ is another function $f'(x)$ defined as

$$f'(x) \;=\; \lim_{h\to 0}\frac{f(x+h)-g(x)}{h} \;=\; \lim_{z\to x}\frac{f(z)-f(x)}{z-x}.$$

Given an input x for the derivative, the output $f'(x)$ equals the value of either one of these limits.

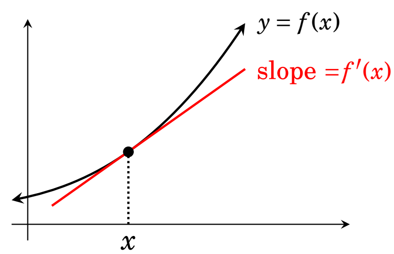
But beginning in Chapter 17 we began developing a set of derivative rules that allow us to compute $f'(x)$ without a limit *provided that $f(x)$ has a form that the rules apply to.* This program culminated in Chapter 25 with derivative rules for the six inverse trig functions. At that point we could quickly compute the derivative of nearly any function described by an algebraic expression, without using a limit.

This may well cause you to wonder why we need the limit definition. If we can find derivatives so effectively with the rules, who needs the limit?

There are two good reasons. First, if you encounter a function to which the rules do not apply, you may need to go back to the limit definition to work out its derivative. (This will not happen in Calculus I.) But the main reason we need the limit definition is that it gives the derivative a *meaning*. One meaning is *slope*. In fact we coded the limit formula for slope (Theorem 15.1) into the very definition of a derivative.

26.1 Slope

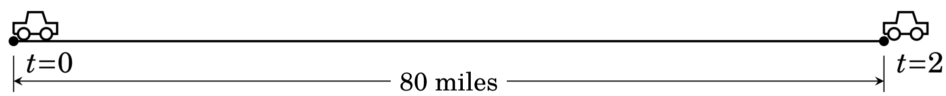
We know well at this point that given a function $f(x)$, its derivative $f'(x)$ gives the slope of the tangent to the graph of $y = f(x)$ at the point $\big(x, f(x)\big)$. This primary geometric meaning of the derivative is by now familiar and needs little review or comment.



The next derivative meaning we will explore is *velocity*.

26.2 Velocity

We will use a thought experiment to show how derivatives describe velocity. Imagine driving 80 miles on a straight highway, timing yourself with a stopwatch. At time t=0 hours you are at the starting point (below, left), and begin driving. You reach your destination at time t=2 hours (below, right).



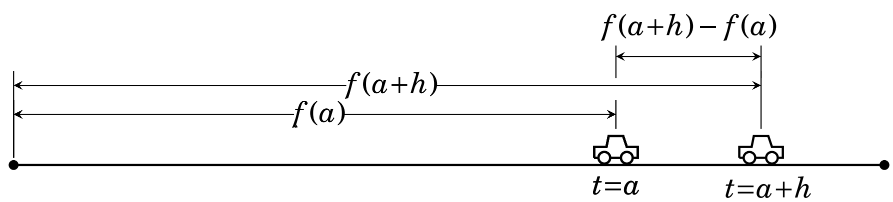
Thus your average velocity for the trip is $\frac{\text{distance traveled}}{\text{time elapsed}} = \frac{\text{80 miles}}{\text{2 hours}} = 40\,\text{mph}$. But you weren't going 40 mph the entire time. Sometimes you were going faster, sometimes slower. The question we will consider is this:

Question: *What is your exact velocity at a particular time $t = a$?*

To answer this, consider a function $f(t)$ defined as

$$f(t) = \big(\text{distance from start at time } t\big).$$

So $f(0) = 0$ and $f(2) = 80$. We don't have enough information to know other $f(a)$ values, but f is still a function whose specifics depend on how you drove.



To find the velocity at time t=a, let h be a small amount of time (say, $h = 0.1$ hours, which is 6 minutes). At time t=a you've gone $f(a)$ miles. A little later, at time $t = a+h$, you've gone a total of $f(a+h)$ miles. So in h hours between times a and $a+h$ you went $f(a+h)-f(a)$ miles. Your *average* velocity between times t=a and t=$a+h$ is thus

$$v_{\text{ave}} = \frac{\text{distance traveled}}{\text{time elapsed}} = \frac{f(a+h)-f(a)}{h}.$$

This might be pretty close to your exact velocity at t=a, but a lot can happen between times a and $a+h$. You could speed up or slow down, skewing v_{ave} away from your exact velocity. For better accuracy, make h smaller, so there's less room for velocity to change in the short time from a to $a+h$. Use $h = 0.001$ hours (3.6 seconds), or $h = 0.0001$ hours (0.36 seconds). The smaller h is, the closer v_{ave} is to the exact velocity at time a. Consequently,

Answer: Velocity at time t=a is $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$ mph.

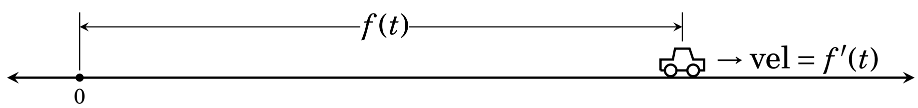
In short, your exact velocity at time t equals $f'(t)$ mph. We call $f(t)$ a **position function** because it gives the position of a moving object. The takeaway from our example is: **velocity is the derivative of position**.

Problems involving motion on a line are often modeled with the object on a number line and $f(t)$ being its location on the line at time t. Thus $f(t)$ can be positive or negative depending on whether the object is to the right or left of 0 at time t. (Also, the object doesn't necessarily have to be at the origin at time t=0.) With this convention, let's summarize our findings.

Fact 26.1 Velocity of a Moving Object:

Consider an object moving on the number line.

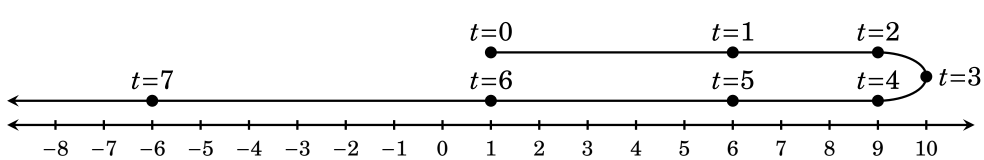
The **position function** $f(t)$ gives the object's location at time t.
The **velocity** of the object at time t is $f'(t)$.



Example 26.1 In this problem the number line is marked in meters, so any point x on the line is x meters from 0. An object moving on the number line has location $x = f(t) = 1 + 6t - t^2$ meters at time t seconds. Its velocity at time t is thus $f'(t) = 6 - 2t$ meters per second. The chart below indicates the object's position and velocity at select times.

t	(seconds)	0	1	2	3	4	5	6	7	8	$\cdots$
$f(t) = 1+6x-x^2$	(meters)	1	6	9	10	9	6	1	−6	−15	$\cdots$
$f'(t) = 6 - 2x$	(meters/sec)	6	4	2	0	−2	−4	−6	−8	−10	$\cdots$

We tally this trajectory visually on the number line (below). At time t=0 the object is at 1, moving to the right. Three seconds later it has reached 10. There it stops moving right and begins to move left. At t=7 it has reached the point −6 and continues moving left forever.



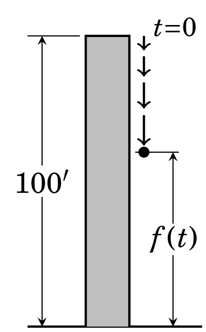
Initially, at time t=0, the object's velocity is 6 meters/sec. After that the velocity decreases steadily by 2 meters per second each second. By time t=3, it has slowed down to 0 meters per second. (This is the instant that it stops moving right and begins moving left.)

In Example 26.1 the velocity $f'(x)$ is positive when the object is moving right and negative when it is moving left. This is because when an object on a number line is moving to the right, its position is increasing; velocity is then positive because position is changing at a *positive* number of meters per second (distance units per time unit). If the object is moving left, then its position is *decreasing*; velocity is then negative because position is changing at a *negative* number of distance units per time unit.

Similarly, for vertical motion, where the number line is oriented vertically (with positive pointing up), velocity is positive when the object is moving up, and negative when the object is moving down.

Example 26.2 A ball is dropped off a 100-foot tower at time t=0 seconds. A formula from physics states that it has height of $f(t) = 100 - 16t^2$ feet at time t. Find the ball's velocity at the instant it strikes ground.

Solution The ball's velocity at time t is $f'(t) = -32t$ feet per second. To find the velocity upon impact, we will find the exact time the ball hits ground and plug that time into $f'(t)$.



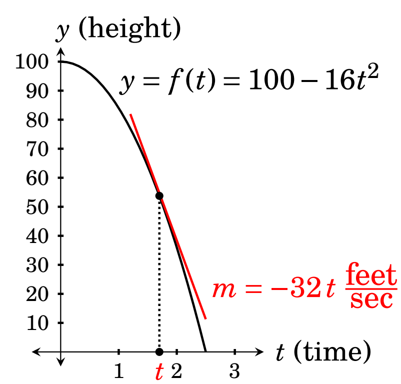
The ball hits ground at the instant its height is 0, that is, when $f(t) = 0$. So to find the time of impact we solve

$$\begin{aligned} 100 - 16t^2 &= 0 \\ 4\left(25 - 4t^2\right) &= 0 \\ 4\left(5 - 2t\right)\left(5 + 2t\right) &= 0 \end{aligned}$$

The solutions are $t = \pm 5/2$. The ball hits ground *after* t=0, so we use $t = 5/2$ seconds. So velocity on impact is

$$\boxed{f'(5/2) = -32 \cdot 5/2 = -80 \text{ feet/sec}}$$

(Negative since the ball moves down.)

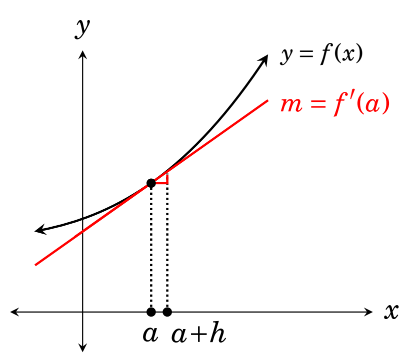


Notice that $f'(t) = -32t$ gives both the ball's velocity at time t **and** the slope of the tangent to $y = f(t)$ at t. In general, velocity at time t equals the slope of the tangent to $y = f(t)$ at t. (See the graph above.) ✍

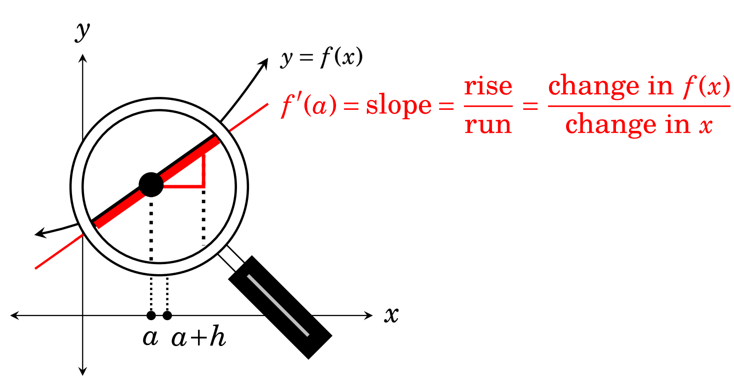
26.3 Rate of Change

We have just seen that if $f(t)$ gives the position of an object at time t, then $f'(t)$ is the object's velocity at time t. Velocity is the rate of change of distance traveled (position), so this suggests that the derivative of a function gives the rate of change of the function. We now explore this important connection.

Suppose $f(x)$ is some quantity that depends on a variable x, so changing x changes $f(x)$. Consider this at a particular value $x{=}a$, indicated below. Increasing x from a to $a{+}h$, causes $y{=}f(x)$ to go from $f(a)$ to $f(a{+}h)$. For small h, the ratio $\frac{\text{change in } f(x)}{\text{change in } x} = \frac{f(a+h)-f(a)}{h}$ is very close to the ratio $m = \frac{\text{rise}}{\text{run}} = f'(a)$ for the tangent to $y = f(x)$ at a.



Look at $\big(a, f(a)\big)$ with a magnifying glass powerful enough that the graph and the tangent are indistinguishable (below). Make the increment h tiny enough that the rise/run triangle fits into the field of vision.



Then $f'(a) = \frac{\text{change in } f(x)}{\text{change in } x}$. This ratio—*change in f per change in x*—is the rate of change of $f(x)$ with respect to x (when $x{=}a$).

Conclusion: $f'(a)$ is the rate of change of $f(x)$ with respect to x, at $x{=}a$.

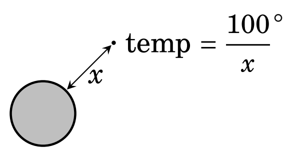
Fact 26.2 Suppose $f(x)$ is a quantity that depends on x. Then

$f'(a)$ is the rate of change of $f(x)$ with respect to x, at $x{=}a$.

We can say this more succinctly as

$f'(x)$ is the rate of change of $f(x)$ with respect to x, at x

Example 26.3 Suppose that the temperature x miles above a planet's surface is given by the function $f(x) = \frac{100}{x}$ degrees celsius. So, for instance, one mile above the surface the temperature is $f(1) = \frac{100}{1} = 100$ degrees. Two miles above the surface the temperature is $f(2) = \frac{100}{2} = 50$ degrees.



By familiar rules, the derivative is $f'(x) = -\frac{100}{x^2}$. By Fact 26.2,

$$f'(x) \;=\; -\frac{100}{x^2} \;=\; \left(\begin{array}{l}\text{rate of change in temperature } f(x)\\[1ex] \text{with respect to height } x\text{, at } x\end{array}\right).$$

To illustrate this, plug a value—say $x{=}5$ miles—into the derivative.

$$f'(5) \;=\; -\frac{100}{5^2} = -4 \text{ degrees per mile}$$

This means that at five miles above the surface, temperature is *decreasing* at a rate of 4 degrees per mile. At this rate, increasing your height by one mile will decrease the temperature by 4 degrees. Next, look at height $x = 10$:

$$f'(10) \;=\; -\frac{100}{10^2} = -1 \text{ degrees per mile}$$

This means that when you are 10 miles above the surface, temperature is *decreasing* at a rate of 1 degree per mile. At this rate, increasing your height by one mile will decrease the temperature by 1 degree. However, this does *not* mean that temperature will necessarily decrease by 1 degree if you go up one mile. The value $f'(10) = -1$ is the *instantaneous* rate of change in temperature at $x = 10$, but the rate of change $f'(x)$ changes with x, so going up will change it. It's like if you are driving 60 mph one instant. It's unlikely you will travel exactly 60 miles in the next hour, because your velocity may deviate from 60 mph during that hour. ✍

Example 26.4 In economics, the cost of producing x units of a product is modeled by a cost function $C(x)$. The so-called **marginal cost** is the derivative $C'(x)$. By Fact 26.2, marginal cost $C'(x)$ is the rate of change of cost with respect to production x.

For example, say you produce 1000 units of a product. In so doing you incur a cost of $C(1000)$ dollars. At this level of production, the marginal cost is $C'(1000)$ dollars per unit. So if you increase the level of production by one unit, expect the cost to increase by $C'(1000)$. ✍

Example 26.5 Imagine that a perfectly spherical balloon is being inflated. As this happens, its radius r (centimeters) and volume V (cubic centimeters) both increase. Recall that the volume of a sphere is

$$V = \frac{4}{3}\pi r^3.$$

Thus volume is a function of radius r, and the derivative of this function is

$$\frac{dV}{dr} = 4\pi r^2.$$

By Fact 26.2, the derivative $\frac{dV}{dr}$ is the rate of change of volume V with respect to radius r. For example, if $r = 10$, then

$$\left.\frac{dV}{dr}\right|_{r=10} = 4\pi 10^2 = 400\pi.$$

This means that when the radius is 10 centimeters, volume is increasing at a rate of 400π cubic centimeters per centimeter change in r. (If r increases from 10 centimeters to 11 centimeters, volume will increase by about 400π cubic centimeters.) ✍

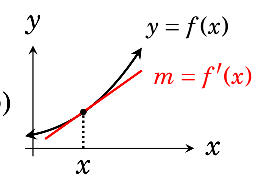
Example 26.6 Suppose $g(x)$ equals the number of gallons of fuel in your tank when you have driven your car x miles on a particular trip. Explain in practical terms what the statement $g'(100) = -0.03$ means.

Solution The derivative $g'(x)$ is the rate of change (in gallons per mile) of the number of gallons in your tank, with respect to x (distance traveled). The statement $g'(100) = -0.03$ thus means that when you have reached 100 miles traveled, the fuel your tank is changing at a rate of -0.03 gallons per mile. In other words, between the 100th and 101st mile, expect to use 0.03 gallons of fuel. ✍

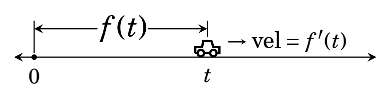
Here is a summary of the main interpretations of the derivative.

Let f be a function. Its derivative f' has the following meanings.

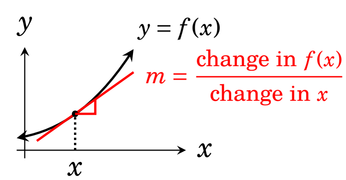
- $f'(x)$ = **slope** of tangent to $y{=}f(x)$ at $(x, f(x))$



- $f'(t)$ = (**velocity** at time t of object whose position at time t is $f(t)$)



- $f'(x)$ = (**rate of change** of quantity $f(x)$ with respect to x, at x)



26.4 More on Motion: Acceleration

We now further develop the theme of motion on a line, which was laid out in Section 26.2. There we saw that if an object's location on a line at time t is given by a function $f(t)$, then $f'(t)$ is its velocity at time t. This section explains how the *second derivative* $f''(t)$ gives the object's *acceleration*.

In Section 26.2, we used f to denote a moving object's position function. Actually, in physics it is conventional to denote a position functions with the letter s. With this convention we restate Fact 26.1 as follows:

If an object moving on the number line has position $s(t)$ at time t, then its velocity at time t is $v(t) = s'(t)$.

For example, suppose an object moving on a line is $s(t) = 1+6t-t^2$ feet from its starting point at time t seconds. Its velocity at time t is thus $s'(t) = 6-2t$ feet per second. The chart below indicates the object's position and velocity at select times.

t	(seconds)	0	1	2	3	4	5	6	7	$8 \cdots$
$s(t) = 1+6x-x^2$	(feet)	1	6	9	10	9	6	1	-6	$-15 \cdots$
$v(t) = s'(t) = 6-2x$	(feet/sec)	6	4	2	0	-2	-4	-6	-8	$-10 \cdots$

Notice how the object's velocity *decreases by* 2 *feet per second each second*. We could say that velocity changes at a rate of -2 feet per second, per second. Or, *the rate of change of velocity is* -2 *feet per second per second.*

This makes sense because from Section 26.3 we know that the derivative of a function gives its rate of change. In this sense, the rate of change of velocity $v(t) = 6-2t$ is $v'(t) = -2$, and this agrees with our observation that velocity is changing at a rate of -2 feet per second per second.

There is a name for the derivative of velocity. It is called **acceleration**. Thus the object above has an acceleration of -2 feet per second per second.

In general we denote acceleration as $a(t) = v'(t) = s''(t)$. Acceleration is the first derivative of velocity and the second derivative of position. Acceleration is the rate of change of velocity, so it is measured in distance units per time unit per time unit (feet/second/second, or meters/second/second, or miles/hour/hour, etc.). Here is a summary of motion on a line.

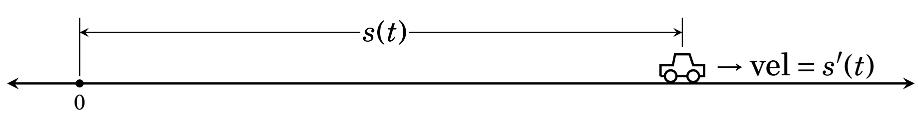
Fact 26.3 Motion on a Straight Line:

Suppose an object moving on the number line has position $s(t)$ at time t.

The object's **velocity** at time t is $v(t) = s'(t)$.

The object's **acceleration** at time t is $a(t) = v'(t)$.

The object's **speed** at time t is $|v(t)|$.



The object's *speed* is the absolute value of its velocity. Recall that *velocity* can be either positive or negative, depending on whether the object is moving right (or up) or left (or down). By contrast speed is positive. Think of it as being measured by the speedometer of a moving car. Speed is positive whether the car is moving left or right.

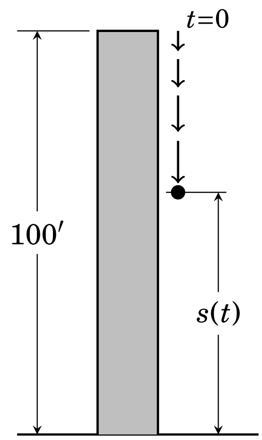
Speed and velocity are both measured in distance units per time unit (for example, miles per hour, feet per second, or meters per second).

Acceleration is the rate of change of velocity. If your acceleration is 3 feet per second per second, then your velocity is increasing at a rate of 3 feet per second, each second. However fast you are moving now, in one second you will gain another 3 feet per second in velocity.

Our next example is a continuation of Example 26.2

Example 26.7 A ball is dropped off a 100-foot tower at time t=0 seconds. A formula from physics states that it has height of $s(t) = 100 - 16t^2$ feet at time t. Find and interpret the ball's velocity and acceleration.

Solution In Example 26.2 we found that the ball strikes ground at time $t = 2.5$ seconds. We also computed its velocity at time t as $v(t) = s'(t) = -32t$ feet per second.



Its acceleration at time t is $a(t) = v'(t) = -32$ feet per second per second. Notice that acceleration is constant. This means that the ball's velocity is *decreasing* at a constant rate of 32 feet per second, each second.

This is supported in the table below, which tallies position, velocity and acceleration at various times. Notice that whatever the velocity is at any time t, one second later it has decreased by exactly 32 feet per second. So velocity is indeed decreasing at a rate of 32 feet per second per second.

t	(seconds)	0	0.5	1	1.5	2	2.5
$s(t) = 100 - 16x^2$	(feet)	100	96	84	64	36	0
$v(t) = s'(t) = -32t$	(feet/sec)	0	−16	−32	−48	−64	−80
$a(t) = v'(t) = -32$	(feet/sec/sec)	−32	−32	−32	−32	−32	−32

The constant acceleration of −32 feet per second is caused by gravity. ✍

Example 26.8 An object moves on a straight line in such a way that its distance from its starting point at time t seconds is $s(t) = 4\sqrt{t}^5$ feet. What is its velocity is when its acceleration is 30 feet per second per second?

Solution As $s(t) = 4t^{5/2}$, the object's velocity is $v(t) = 4 \cdot \frac{5}{2} t^{5/2-1} = 10t^{3/2} = 10\sqrt{t}^3$. Its acceleration is $a(t) = v'(t) = 10 \cdot \frac{3}{2} t^{3/2-1} = 15t^{1/2} = 15\sqrt{t}$. By just looking at this you can see that $t = 4$ makes the acceleration 30 feet per second per second. At this time the velocity is $v(4) = 10\sqrt{4}^3 = 80$ feet per second. ✍

Exercises for Chapter 26

1. An object moving on a straight line is $s(t) = t^2 + \sqrt{t+1} - 1$ feet from its starting point at time t seconds. Find its velocity at time $t = 8$ seconds.

2. An object moving on a straight line is $s(t) = \sqrt{t} + t^2 + 3$ feet from its starting point at time t seconds. Find its velocity at time $t = 4$ seconds.

3. An object moves on a straight line in such a way that its distance from its starting point at time t seconds is $s(t) = 3\sqrt[3]{t}^4 + 4t$ feet. How far away from the starting point is it when its velocity is 12 feet per second?

4. This problem concerns a rock that is thrown straight up in the air at time $t = 0$. At time t (in seconds) it has a height of $s(t) = 64t - 16t^2$ feet. When does the rock hit the ground? What is its velocity when it hits the ground?

5. This problem concerns a rock that is thrown off a tower at time $t = 0$. At time t (in seconds) it has a height of $s(t) = 48 + 32t - 16t^2$ feet. When does the rock hit the ground? What is its velocity when it hits the ground?

6. An object moves on a straight line in such a way that its distance from a fixed point at time t is $s(t) = t^3 - 9t^2 + 15t + 4$. Find the times t at which its velocity is 0. When is the object moving left? When is it moving right?

7. An object moves on a straight line in such a way that its distance from a fixed point at time t is $f(t) = t^3 - 15t^2 + 48t$. Find the times t at which its velocity is 0. When is the object moving left? When is it moving right?

8. An object moving on a straight line is $s(t) = \dfrac{t}{t^2+1}$ feet from its starting position at time t seconds. What is its acceleration when its velocity is zero feet per second?

9. An object moving on a straight line is $s(t) = t^3 - 3t^2$ feet from its starting point at time t seconds. What is the object's velocity at time t? What is its acceleration at time t? Find its acceleration when its velocity is -3 feet per second.

10. An object moving on the number line has a position of $s(t) = \tan^{-1}$ at time t. Find its velocity at time t. Find its acceleration at time t. For which times t is the object moving to the right? For which t is it speeding up? For which t is it slowing down?

11. An object moving on a straight line is $s(t) = 2 + t + t^3$ feet from its starting point at time t seconds. What is the object's velocity at time t? What is its acceleration at time t? Find its velocity when its acceleration is 12 feet per second per second.

12. Given that the surface area of a sphere of radius r inches is $S = 4\pi r^2$ square inches, find and interpret the rate of change of surface area S with respect to radius r.

13. Given that the area of a circle of radius r inches is $A = \pi r^2$ square inches, find and interpret the rate of change of area A with respect to radius r.

14. Suppose the temperature in a kiln at time t (in minutes) is $203+6\sqrt{t}$ degrees F. What it the rate of change of the temperature at the time $t=9$?

15. Suppose you begin a road trip at time $t=0$ and $f(t)$ gives your distance you have traveled after t hours. Your average velocity for the trip at any time t is distance traveled divided by time elapsed, which is $A(t)=f(t)/t$ mph. Suppose at time $t=2$ you have gone 100 miles, and are moving at a velocity of 20 miles per hour. What is the rate of change of your average velocity $A(t)$ at time $t=2$?

16. Is it possible for the population of a city to decrease while the rate of change of population increases?

17. Is it possible for the national debt to increase while its rate of change decreases?

18. Consider the function $g(v)$, where $g(v)$ equals your car's gas mileage when you are driving v miles per hour. Suppose $g'(60)=0.75$. If you are driving 60 mph and you want to improve your gas mileage, should you speed up or slow down?

19. Consider the function $h(x)$, where $h(x)$ equals the elevation (in feet above sea level) x miles due west of your present location. Suppose $h'(75)=5$. Explain what this means.

20. Consider the function $T(x)$, where $T(x)$ equals the the temperature of the atmosphere (in degrees C) x meters above your present location. Suppose $T'(900)=-2.5$. Explain what this means.

21. Suppose it costs $C(x)$ dollars to build a transmitting tower that is x meters high. Explain the meaning of $C'(x)$. Explain in simple terms the meaning of the statement $C'(100)=1000$.

22. Suppose $f(x)$ is the number of liters of fuel in a rocket when it is x miles above the Earth's surface. Explain in simple terms the meaning of the statement $f'(20)=-8$.

23. It takes a certain competitive eater $f(x)$ minutes to eat x hotdogs. (It's understood that x need not be an integer. For instance, $x=0.6$ is 60% of a hotdog.) Explain in simple terms (that the eater would understand) what $f'(x)$ means. It is somehow determined that $f'(30)=4$. What does that mean?

Exercise Solutions for Chapter 26

1. An object moving on a straight line is $s(t)=t^2+\sqrt{t+1}-1$ feet from its starting point at time t seconds. Find its velocity at time $t=8$ seconds.

Velocity at time t is $v(t)=s'(t)=2t+\frac{1}{2\sqrt{t+1}}$ feet per second. Therefore the velocity at time $t=8$ is $v(8)=2\cdot 8+\frac{1}{2\sqrt{8+1}}=16+\frac{1}{6}=\frac{97}{6}=16.1\bar{6}$ feet per second.

3. An object moves on a straight line in such a way that its distance from its starting point at time t seconds is $s(t)=3\sqrt[3]{t}^4+4t$ feet. How far away from the starting point is it when its velocity is 12 feet per second?

Velocity at time t is $s'(t) = 3 \cdot \frac{4}{3} t^{4/3-1} + 4 = 4t^{1/3} + 4 = 4\sqrt[3]{t} + 4$. To find the time at which the velocity is 12 feet per second, we need to solve

$$\begin{aligned} v(t) &= 12 \\ 4\sqrt[3]{t} + 4 &= 12 \\ \sqrt[3]{t} + 1 &= 3 \\ \sqrt[3]{t} &= 2 \\ t &= 2^3 = 8. \end{aligned}$$

So velocity 12 feet per second when $t = 8$ seconds. At this time the object's position is $s(8) = 3\sqrt[3]{8}^4 + 4 \cdot 8 = 3 \cdot 2^4 + 4 \cdot 8 = 80$ feet.

5. This problem concerns a rock that is thrown off a tower at time $t = 0$. At time t (in seconds) it has a height of $s(t) = 48 + 32t - 16t^2$ feet. When does the rock hit the ground? What is its velocity when it hits the ground?

The rock hits ground when $s(t) = 48 + 32t - 16t^2 = 0$ feet, so we need to solve

$$\begin{aligned} 48 + 32t - 16t^2 &= 0 \\ 3 + 2t - t^2 &= 0 \\ (3-t)(1+t) &= 0. \end{aligned}$$

The solutions are $t = 3$ and $t = -1$. But t must be positive in this problem, so the rock strikes ground at time $t = 3$. Velocity is at time t is $v(t) = s'(t) = 32 - 32t$, so velocity on impact is $v(3) = 32 - 32 \cdot 3 = -64$ feet per second.

7. An object moves on a straight line in such a way that its distance from a fixed point at time t is $f(t) = t^3 - 15t^2 + 48t$. Find the times t at which its velocity is 0. When is the object moving left? When is it moving right?

The object's velocity at time t is $f'(t) = 3t^2 - 30t + 48 = 3(t^2 - 10t + 16) = 3(t-2)(t-8)$. Thus the velocity is zero at times $t = 2$ and $t = 8$. For times t between 2 and 8 ($2 < t < 8$), the factor $(t-2)$ is positive and the factor $(t-8)$ is negative, so the velocity $f'(t) = 3(t-2)(t-8)$ is negative; hence the object is moving left when $2 < t < 8$. For other values of t, factors $(t-2)$ and $(t-8)$ are either both positive or both negative, so, $f'(t) = 3(t-2)(t-8) > 0$, meaning the object is moving right.

9. An object moving on a straight line is $s(t) = t^3 - 3t^2$ feet from its starting point at time t seconds. What is the object's velocity at time t? What is its acceleration at time t? Find its acceleration when its velocity is -3 feet per second.

Velocity at time t is $v(t) = s'(t) = 3t^2 - 6t$ feet per second.
Acceleration at time t is $a(t) = v'(t) = 6t - 6$ feet per second per second.

To find the time that the velocity is -3 feet per second, we solve $v(t) = -3$.

$$\begin{aligned} 3t^2 - 6t &= -3 \\ 3t^2 - 6t + 3 &= 0 \\ t^2 - 2t + 1 &= 0 \\ (t-1)(t-1) &= 0 \end{aligned}$$

Thus velocity is -3 feet per second when $t = 1$. At this time the acceleration is $a(1) = 6(1) - 6 = 0$ feet per second per second.

11. An object moving on a straight line is $s(t) = 2 + t + t^3$ feet from its starting point at time t seconds. What is the object's velocity at time t? What is its acceleration at time t? Find its velocity when its acceleration is 12 feet per second per second.

The velocity at time t is $v(t) = s'(t) = 1 + 3t^2$ feet per second. The acceleration at time t is $a(t) = v'(t) = 6t$ feet per second per second.

Thus at time $t = 2$ seconds, the acceleration is $a(t) = 12$ feet per second per second. At this time the velocity is $v(2) = 1 + 3 \cdot 2^2 = 13$ feet per second.

13. Given that the area of a circle of radius r inches is $A = \pi r^2$ square inches, find and interpret the rate of change of area A with respect to radius r.

The rate of change is $\frac{dA}{dt} = 2\pi r$ square inches per inch. That is, the instantaneous rate of change of area is $2\pi r$ square inches of area per inch of radius. So if (say) the radius is $r = 3$ inches, and increasing, the area is increasing at a rate of $2\pi r = 6\pi$ square inches per inch of r. If s $r = 4$ inches, and increasing, the area is increasing at a rate of $2\pi r = 8\pi$ square inches per inch of r, etc.

15. Suppose you begin a road trip at time $t = 0$ and $f(t)$ gives your distance you have traveled after t hours. Your average velocity for the trip at any time t is distance traveled divided by time elapsed, which is $A(t) = f(t)/t$ mph. Suppose at time $t = 2$ you have gone 100 miles, and are moving at a velocity of 20 miles per hour. What is the rate of change of your average velocity $A(t)$ at time $t = 2$?

The rate of change of $A(t)$ is $A'(t) = \dfrac{f'(t) \cdot t - f(t) \cdot 1}{t^2}$ (by the quotient rule). From the information given, at time $t = 2$ you've gone $f(2) = 100$ miles and are are moving at a velocity of $f'(2) = 20$ mph. Thus $A'(2) = \dfrac{f'(2) \cdot 2 - f(2) \cdot 1}{2^2} = \dfrac{20 \cdot 2 - 100}{4} = -15$ miles per hour per hour. At this rate your average velocity for the trip is *decreasing* at a rate of 15 mph per hour.

17. Is it possible for the national debt to increase while its rate of change decreases?

Yes, this is possible. For example, suppose that at time t (in years since 2020) the national debt is $y = \ln(t)$ trillion dollars, which is unrealistic, but possible (or at least conceivable). The function $y = \ln(t)$ increases as t increases. But its rate of change is $\frac{dy}{dt} = \frac{1}{t}$ trillion dollars per year, and this *decreases* as time t increases.

19. Consider the function $h(x)$, where $h(x)$ equals the elevation (in feet above sea level) x miles due west of your present location. Suppose $h'(75) = 5$. Explain what this means.

Note that $h'(x)$ is the rate of change of elevation (in feet per mile) x miles due west of your present location. Thus $h'(75) = 5$ means that 75 miles due west of your present location, elevation is increasing at a rate of 5 feet per mile.

21. Suppose it costs $C(x)$ dollars to build a transmitting tower that is x meters high. Explain the meaning of $C'(x)$. Explain in simple terms the meaning of the statement $C'(100) = 1000$.

$C'(x)$ is the rate of change, in dollars per meter (at height x) of the cost of building the tower. In other words, if the tower is currently x meters high, then the cost of increasing its height is changing at a rate of $C'(x)$ dollars per meter. If the tower is x meters high, expect it to cost $C'(x)$ dollars to increase its height by one meter. Therefore $C'(100) = 1000$ means that if the tower is 100 meters high, it should cost about \$1000 to increase the height to 101 meters.

23. It takes a certain competitive eater $f(x)$ minutes to eat x hotdogs. (It's understood that x need not be an integer. For instance, $x = 0.6$ is 60% of a hotdog.) Explain in simple terms (that the eater would understand) what $f'(x)$ means. It is somehow determined that $f'(30) = 4$. What does that mean?

The derivative $f'(x)$ is the rate of change (in minutes per hotdog) of the number of minutes it takes him to eat x hotdogs. In other words, if he's eaten x hotdogs, expect it to take him another $f'(x)$ minutes to eat the next one. Thus $f'(30) = 4$ means that once he's eaten 30 hotdogs he is eating at a rate of 4 minutes per hotdog. Expect it to take him 4 minutes to eat the 31st hotdog.

CHAPTER 31

Extrema and the First Derivative Test

This chapter builds on the previous chapter, so we begin by reviewing the main ideas from Chapter 30. Given a function f,

- $f(x)$ increases where $f'(x)$ is positive.
- $f(x)$ decreases where $f'(x)$ is negative.
- A number c in the domain of f is called a **critical point** for f if either $f'(c) = 0$ or $f'(c)$ is not defined.
- If $f(x)$ stops increasing and starts decreasing (or stops decreasing and starts increasing) at $x=c$, then c is a critical point.

Our main definition formalizes the notion a function's hills and valleys.

Definition 31.1 Suppose c is a number in the domain of a function $f(x)$.

We say f has a **local maximum** at $x=c$ if $f(x) \le f(c)$ for all x near c.
That is, the graph has a "hilltop" at $x=c$.
Moving x immediately to the right or left of c will not make $f(x)$ larger.

We say f has a **local minimum** at $x=c$ if $f(x) \ge f(c)$ for all x near c.
That is, the graph has a "valley bottom" at $x=c$.
Moving x immediately to the right or left of c will not make $f(x)$ smaller.

A local maximum or minimum is called a *local extremum*. The plurals are **local maxima**, **local minima** and **local extrema**. The function in Figure 31.1 has two local maxima, two local minima and four local extrema.

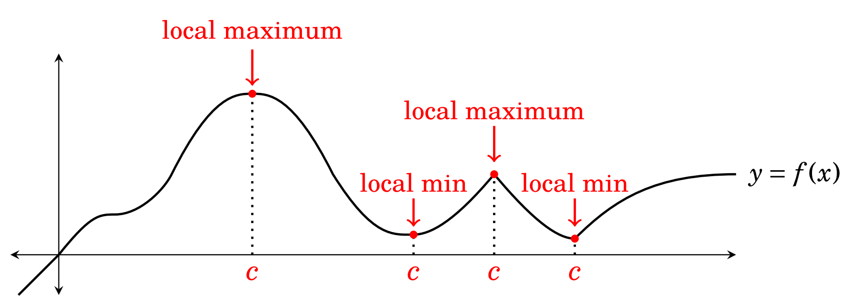


Figure 31.1. Local extrema a function.

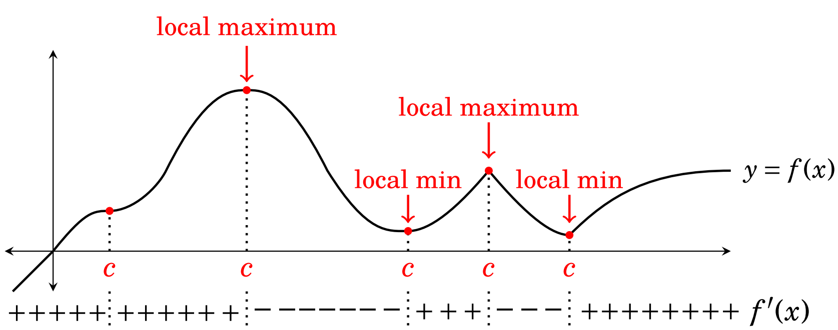
If f has a local maximum at c, then c gives the location of the hill, and $f(c)$ is the hill's height. Thus we may say that "*the local maximum is* $f(c)$" when we want to emphasize height over location. (The same remarks apply to local minima.) However, for now we will be concerned with where the extrema occur, not how high or low they happen to be.

Our primary method for finding the locations of extrema is called the **first derivative test**. It is the simple observation that maxima occur where f stops increasing and starts decreasing, so $f'(x)$ changes sign from $+$ to $-$. Minima occur where f stops decreasing and starts increasing, so $f'(x)$ goes from $-$ to $+$. Either way, the sign change happens at a critical point.

Fact 31.1 The First Derivative Test (For finding local extrema.)
Suppose c is a critical point of $f(x)$.

1. If $f'(x)$ changes from $+$ to $-$ at c, then $f(x)$ has a **local maximum** at c.
2. If $f'(x)$ changes from $-$ to $+$ at c, then $f(x)$ has a **local minimum** at c.
3. If $f'(x)$ does not change sign at c, there is no local extremum at c.

The graph below illustrates this. Five critical points c of f are shown. There is a local maximum at c when $f'(x)$ changes sign from $+$ to $-$ at c (meaning f stops rising and starts falling). There is a local minimum at c if $f'(x)$ changes from $-$ to $+$ at c (meaning f stops falling and starts rising).



Notice that at the left-most critical point c (where $f'(c) = 0$), the sign of the derivative does not change. Here the graph increases before getting to c, levels out at c, and then continues increasing. The derivative has no sign-change at c, and this corresponds to f having no extremum at c.

We now have a procedure for finding all local extrema of a function.

How to find all local extrema of a function

1. Find all critical points of the function.
2. Apply the First Derivative Test to each critical point.

Example 31.1 Find all local extrema of the function $f(x) = x^4 - 2x^2 + 2$.
Solution The first step is to find all the critical points, the numbers in the domain of f for which the derivative is zero or undefined. The derivative $f'(x) = 4x^3 - 4x$ is defined for all x, so if there are any critical points, then they will make the derivative zero.

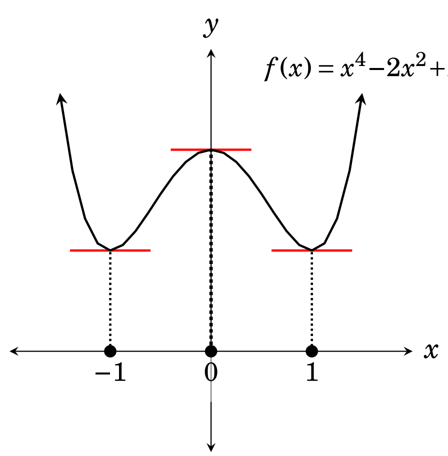
The derivative factors as $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x+1)(x-1)$, so the solutions of $f'(x) = 0$ are -1, 0 and 1. These are the critical points of f, so if f has any local extrema, they will happen at these points.

We next apply the first derivative test to each of these critical points. This requires examining the sign change of $f'(x)$ at each critical point. We use our familiar procedure of examining the factors of $f'(x) = 4x(x+1)(x-1)$.

	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$x > 1$
$4x$	$-$	$-$	$+$	$+$
$(x+1)$	$-$	$+$	$+$	$+$
$(x-1)$	$-$	$-$	$-$	$+$
$f'(x) = 4x(x+1)(x-1)$	$-$	$+$	$-$	$+$

Because the derivative changes from $-$ to $+$ at -1 and 1, the first derivative test guarantees a local minimum for f at $x = -1$ and $x = 1$. And since the derivative changes from $+$ to $-$ at 0, there is a local maximum at $x = 0$.

Answer: $f(x) = x^4 - 2x^2 + 2$ has local minima at $x = -1$ and $x = 1$, and a local maximum at $x = 0$.



The graph of $f(x) = x^4 - 2x^2 + 2$ is sketched above. ✍

Example 31.2 Find all local extrema of $f(x) = \sqrt[3]{x}^2 - \dfrac{2}{3}x$ on $(-\infty, \infty)$.

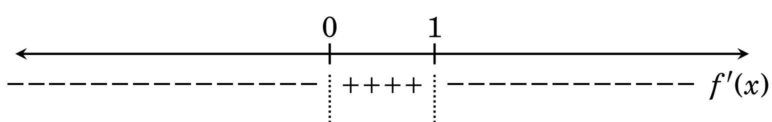
Solution The first step is to find this function's critical points, and that involves finding its derivative. Write it as $f(x) = x^{2/3} - \frac{2}{3}x$. Then

$$f'(x) \;=\; \frac{2}{3}x^{2/3-1} - \frac{2}{3} \;=\; \frac{2}{3}\left(x^{-1/3} - 1\right) \;=\; \frac{2}{3}\left(\frac{1}{\sqrt[3]{x}} - 1\right).$$

We immediately see that $x = 0$ is a critical point because $f'(0)$ is not defined (division by 0). To find any other critical points, solve the equation $f'(x) = 0$.

$$\begin{aligned}
\frac{2}{3}\left(\frac{1}{\sqrt[3]{x}} - 1\right) &= 0\\
\frac{1}{\sqrt[3]{x}} - 1 &= 0\\
\frac{1}{\sqrt[3]{x}} &= 1\\
1 &= \sqrt[3]{x}\\
1^3 &= \sqrt[3]{x}^3\\
x &= 1
\end{aligned}$$

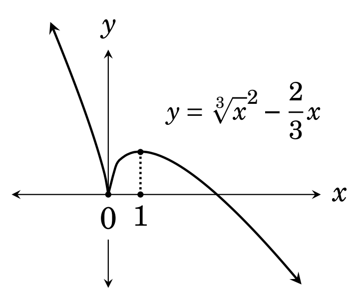
Therefore f has exactly two critical points 0 and 1, and these divide the function's domain into three intervals $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$.



To find the sign of $f'(x)$ on $(-\infty, 0)$ pick a test point, say $x = -1$, in this interval. Then $f'(-1) = \frac{2}{3}\left(\frac{1}{\sqrt[3]{-1}} - 1\right) = -\frac{4}{3}$, hence $f'(x)$ is negative on $(-\infty, 0)$.

To find the sign of $f'(x)$ on $(0, 1)$ let's use the test point $x = \frac{1}{8}$ because its a perfect cube, with cube root $\frac{1}{2}$. Then $f'\left(\frac{1}{8}\right) = \frac{2}{3}\left(\frac{1}{\sqrt[3]{1/8}} - 1\right) = \frac{2}{3}$, so $f'(x)$ is positive on $(0, 1)$. Finally, as $f'(8) = \frac{2}{3}\left(\frac{1}{\sqrt[3]{8}} - 1\right) = -\frac{1}{3}$, $f'(x)$ is negative on $(1, \infty)$.

Answer: By the first derivative test, f has a local minimum at $x = 0$ (where f' changes from $-$ to $+$) and a local maximum at $x = 1$ (where f' changes from $+$ to $-$).
The graph of $f(x) = \sqrt[3]{x}^2 - \frac{2}{3}x$ is shown on the right. ✍



Example 31.3 Find all local extrema of $y = x\sin(x)+\cos(x)-\dfrac{x^2}{4}$ on $(-\pi,\pi)$.

Solution The domain of this function is all real numbers, but we are only asked to find the extrema of it on the interval $(-\pi,\pi)$. As always, the first step is to take the derivative:

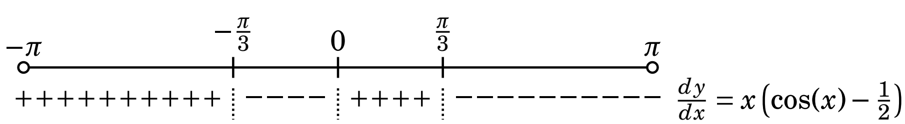
$$\frac{dy}{dx} = 1\cdot\sin(x)+x\cos(x)-\sin(x)-\frac{x}{2} = x\left(\cos(x)-\frac{1}{2}\right).$$

This derivative is defined for all values of x, so there are no critical points that make the derivative undefined. To find any other critical points, we solve the equation $\frac{dy}{dx}=0$:

$$x\left(\cos(x)-\frac{1}{2}\right) = 0.$$

Certainly $x=0$ is one solution. Familiarity with the unit circle reveals the two other solutions in $(-\pi,\pi)$. Since $\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$ and $\cos\left(-\frac{\pi}{3}\right)=\frac{1}{2}$, the two values $x=\pm\frac{\pi}{3}$ make $\frac{dy}{dx}$ zero and are hence critical points.

The three critical points $-\frac{\pi}{3}$, 0 and $\frac{\pi}{3}$ divide the interval $(-\pi,\pi)$ into four sub-intervals, as diagramed below.



Let's now pick a test point in each interval. Use $-\frac{\pi}{2}$ for the first interval, and then $-\frac{\pi}{6}$ for the second. Use $\frac{\pi}{6}$ and $\frac{\pi}{2}$ for the third and fourth intervals.

Since $\left.\frac{dy}{dx}\right|_{-\pi/2} = -\frac{\pi}{2}\left(\cos\left(-\frac{\pi}{2}\right)-\frac{1}{2}\right) = -\frac{\pi}{2}\left(0-\frac{1}{2}\right)=\frac{\pi}{4}>0$, $\frac{dy}{dx}$ is positive on $\left(-\pi,-\frac{\pi}{3}\right)$.

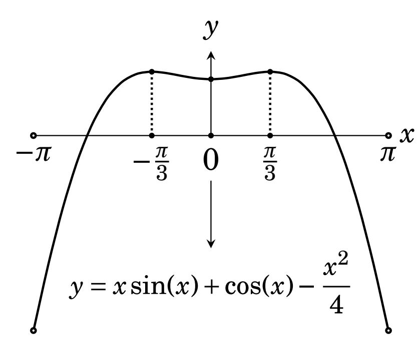
Since $\left.\frac{dy}{dx}\right|_{-\pi/6} = -\frac{\pi}{6}\left(\cos\left(-\frac{\pi}{6}\right)-\frac{1}{2}\right) = -\frac{\pi}{6}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)<0$, $\frac{dy}{dx}$ is negative on $\left(-\frac{\pi}{3},0\right)$.

Since $\left.\frac{dy}{dx}\right|_{\pi/6} = \frac{\pi}{6}\left(\cos\left(\frac{\pi}{6}\right)-\frac{1}{2}\right) = \frac{\pi}{6}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)>0$, $\frac{dy}{dx}$ is positive on $\left(0,\frac{\pi}{3}\right)$.

Since $\left.\frac{dy}{dx}\right|_{\pi/2} = \frac{\pi}{2}\left(\cos\left(\frac{\pi}{2}\right)-\frac{1}{2}\right) = \frac{\pi}{2}\left(0-\frac{1}{2}\right)=-\frac{\pi}{4}<0$, $\frac{dy}{dx}$ is negative on $\left(\frac{\pi}{3},\pi\right)$.

Answer: By the first derivative test, $y = x\sin(x)+\cos(x)-\frac{x^2}{4}$ has two local maxima, at $x=-\frac{\pi}{3}$ and $x=\frac{\pi}{3}$. There is one local maximum, at $x=0$.

The graph of $y = x\sin(x)+\cos(x)-\frac{x^2}{4}$ is shown on the right. ✍



Example 31.4 Find all local extrema of $y = \dfrac{\ln\left(x^2\right)}{x^2}$.

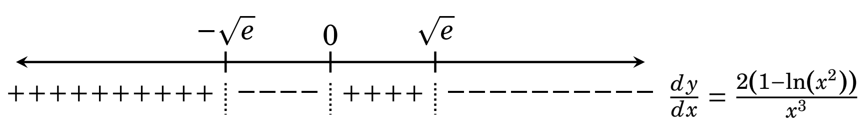
Solution: The domain of this function is the two intervals $(-\infty, 0) \cup (0, \infty)$. Any critical points will further split these intervals. The derivative is

$$\frac{dy}{dx} = \frac{\frac{2x}{x^2} \cdot x^2 - \ln\left(x^2\right) \cdot 2x}{\left(x^2\right)^2} = \frac{2x - 2x\ln\left(x^2\right)}{x^4} = \frac{2\left(1-\ln\left(x^2\right)\right)}{x^3}.$$

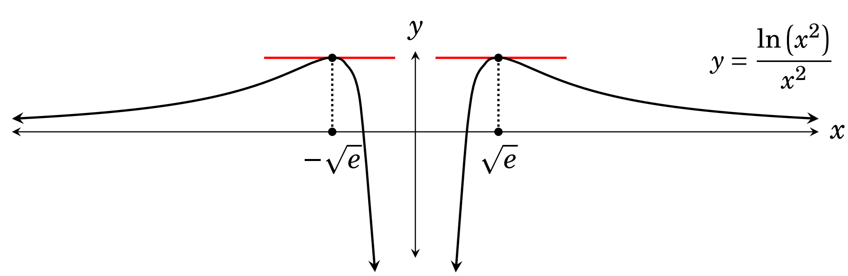
This is undefined only for $x = 0$, **but** 0 is not in the domain of $\frac{\ln(x^2)}{x^2}$, so 0 is **not a critical point**. Thus the critical points (if any) will be the x for which the derivative is zero. To find them we solve the equation $\frac{dy}{dx} = 0$.

$$\begin{aligned} \frac{2\left(1-\ln\left(x^2\right)\right)}{x^3} &= 0 \\ 1 - \ln\left(x^2\right) &= 0 \\ \ln\left(x^2\right) &= 1 \\ e^{\ln(x^2)} &= e^1 \\ x^2 &= e \qquad \text{Thus } x = \pm\sqrt{e}. \end{aligned}$$

The two critical points $x \pm \sqrt{e}$ divide the intervals $(-\infty, 0)$ and $(0, \infty)$ as shown below. Picking test points, we determine the sign of $\frac{dy}{dx}$ on each interval.



Answer: By the first derivative test, $\frac{\ln(x^2)}{x^2}$ has two local maxima, at $x = -\sqrt{e}$ and $x = \sqrt{e}$, where $\frac{dy}{dx}$ changes from + to −. There is no local minimum. (See the graph below.)



Important: Since 0 is not in the function's domain, there is no extremum at 0, even though $\frac{dy}{dx}$ changes sign there. If we mistakenly interpreted 0 as a critical point, then we would mistakenly get a local minimum at 0. ✍

Example 31.5 Find all local extrema of $f(x) = \sin(x) + x$ on $(-\infty, \infty)$.

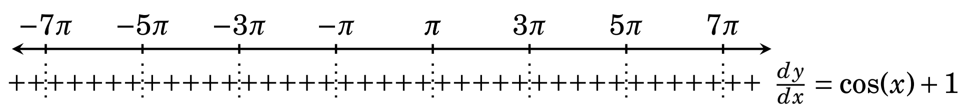
Solution The first step is to find all critical critical points. The derivative $f'(x) = \cos(x) + 1$ is defined for all real values of x, so there are no critical points c for which $f'(c)$ is not defined. Therefore all critical points of $f(x)$ will be solutions to the equation $f'(x) = 0$.

$$\begin{aligned} \cos(x) + 1 &= 0 \\ \cos(x) &= -1 \end{aligned}$$

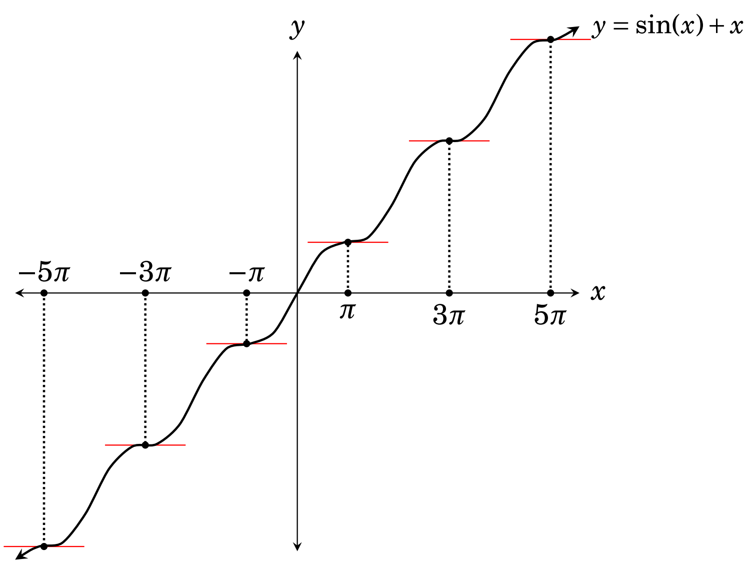
The values of x for which $\cos(x) = -1$ are the odd multiples of π (think of the unit circle), so f has infinitely many critical points, which we list as

$$\ldots -7\pi,\ -5\pi,\ -3\pi,\ -\pi,\ \pi,\ 3\pi,\ 5\pi,\ 7\pi, \ldots$$

Consider the derivative $f'(x) = \cos(x) + 1$. Because $\cos(x)$ is never less than -1, it follows that $f'(x) = \cos(x) + 1$ is never negative. (Though it equals 0 at each critical point.) Therefore $f'(x)$ never changes sign. Unless x is a critical point, $f'(x)$ is positive. This information is tallied on our usual chart, below.



Answer: The function $f(x) = \sin(x) + x$ has no local extrema at all.



The above graph of $y = \sin(x) + x$ may help make sense of our answer. Notice that the graph is continually rising, though it levels out at each critical point, where the tangent to the graph is horizontal. ✍

Exercises for Chapter 31

1. Find all local extrema of $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$.

2. Find all local extrema of $f(x) = \frac{3}{2}x^4 - x^6$.

3. Find all local extrema of $g(x) = x\sqrt{8 - x^2}$.

4. Find all local extrema of $f(x) = 5x^4 + 20x^3 + 10$.

5. Find all local extrema of $f(x) = x^2 e^{-x}$.

6. Find all local extrema of $f(x) = x^2 e^x$.

7. Find all local extrema of $f(x) = x \ln |x|$.

8. Find all local extrema of $f(x) = e^{x^3 - 12x}$.

9. Find all local extrema of $y = \tan^{-1}\left(x^2 + x - 2\right)$.

10. Find all local extrema of $y = x \tan^{-1}(x)$.

11. Find all local extrema of $y = \dfrac{1}{1 + x^2}$

12. Find all local extrema of $y = \dfrac{x}{1 + x^2}$

13. Find all local extrema of $y = \sqrt[3]{8 - x^3}$

14. Find all local extrema of $z = \ln\left|w^2 + 10w + 24\right|$

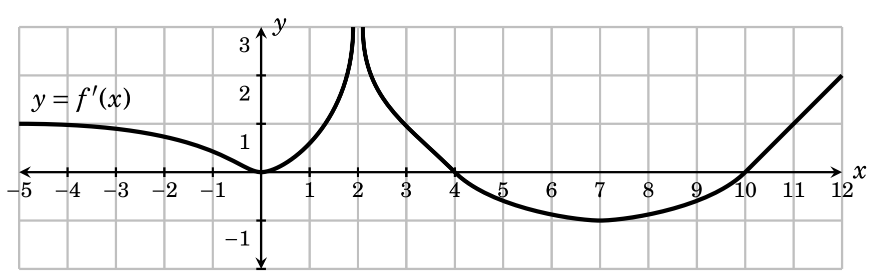
15. Find all local extrema of $f(x) = \ln\left(x^2 e^x + 1\right)$.

16. Find all local extrema of $f(x) = x^2 + \dfrac{16}{x}$ on its domain.

17. Find all local extrema of $f(x) = \ln(x) + \dfrac{3}{x} - \dfrac{1}{x^2}$ on its domain.

18. Find all local extrema of $f(x) = \sin^{-1}(x) - 2x$ on its domain.

19. The **derivative** $f'(x)$ of a function $f(x)$ with domain $(-5, 12)$ is graphed below. Answer the questions about $f(x)$.

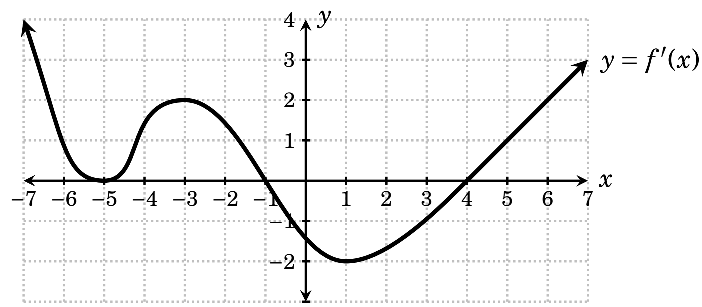


(a) State the intervals on which $f(x)$ increases.

(b) State the intervals on which $f(x)$ decreases.

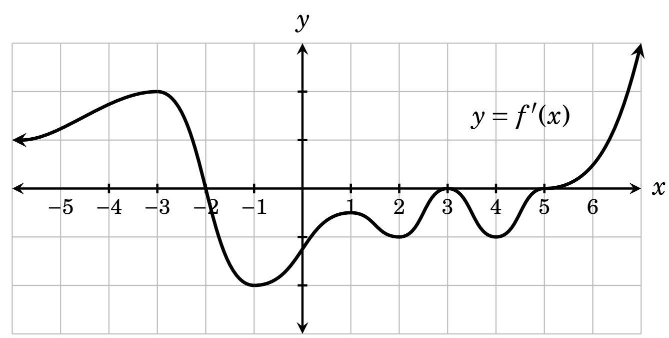
(c) List all critical points of $f(x)$.
(d) At which of its critical points does $f(x)$ have a local maximum?
(e) At which of its critical points does $f(x)$ have a local minimum?
(f) Based on this information, sketch a possible graph of $f(x)$.

20. The **derivative** $f'(x)$ of a function $f(x)$ is graphed below. Answer the questions about $f(x)$.



(a) State the intervals on which $f(x)$ increases.
(b) State the intervals on which $f(x)$ decreases.
(c) List all critical points of $f(x)$.
(d) At which of its critical points does $f(x)$ have a local maximum?
(e) At which of its critical points does $f(x)$ have a local minimum?
(f) Based on this information, sketch a possible graph of $f(x)$.

21. The **derivative** $f'(x)$ of a function $f(x)$ is graphed below. Answer the questions about $f(x)$.

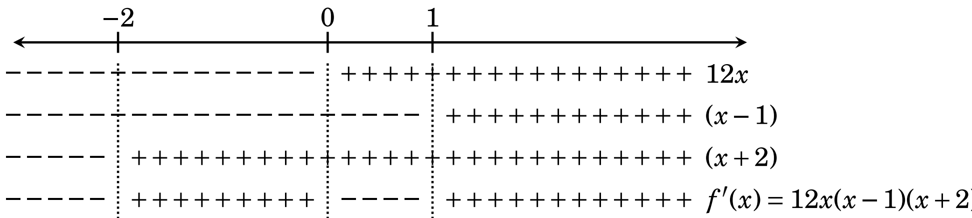


(a) State the critical points of f.
(b) State the interval(s) on which f decreases.
(c) Does f have a local maximum? Where?
(d) Does f have a local minimum? Where?

Exercises Solutions for Chapter 31

1. Find all local extrema of $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$.

The derivative is $f'(x) = 12x^3 + 12x^2 - 24x = 12x\left(x^2 + x - 2\right) = 12x(x-1)(x+2)$. The critical points are $x = 0$, $x = 1$ and $x = -2$.



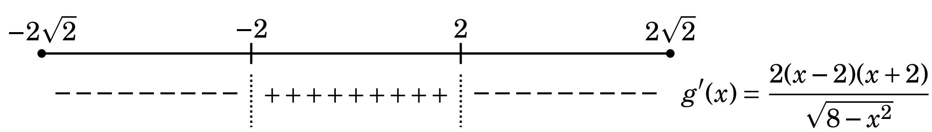
By the first derivative test, f has local minima at $x = -2$ and $x = 1$, and a local maximum at $x = 0$.

3. Find all local extrema of $g(x) = x\sqrt{8 - x^2}$.

Notice that $x - 8^2 \geq 0$ on $\left[-\sqrt{8}, \sqrt{8}\right] = \left[-2\sqrt{2}, 2\sqrt{2}\right]$, and $x - 8^2$ is negative elsewhere. Thus the domain of g is the interval $\left[-2\sqrt{2}, 2\sqrt{2}\right]$. By the product rule,

$$\begin{aligned}
g'(x) &= 1 \cdot \sqrt{8-x^2} + x \cdot \frac{-x}{\sqrt{8-x^2}} \\
&= \sqrt{8-x^2} - \frac{x^2}{\sqrt{8-x^2}} \\
&= \frac{\left(8-x^2\right) - x^2}{\sqrt{8-x^2}} \\
&= \frac{8-2x^2}{\sqrt{8-x^2}} \\
&= \frac{2(x-2)(x+2)}{\sqrt{8-x^2}}
\end{aligned}$$

This is undefined at $x = \pm\sqrt{8} = \pm 2\sqrt{2}$, but these are not critical points because the are not in the domain of g. Thus the only critical points of g are the values of x for which $g'(x) = 0$, namely $x = -2$ and $x = 2$. These divide the domain $\left[-2\sqrt{2}, 2\sqrt{2}\right]$ of g into three smaller intervals, as shown on the diagram below.



Looking at the sign of g' on these intervals, we see that the first derivative test gives a local minimum of $g(-2) = -2\sqrt{8-(-2)^2} = -4$ at $x = -2$, and a local maximum of $g(2) = 2\sqrt{8-2^2} = 4$ at $x = 2$.

5. Find all local extrema of $f(x) = x^2 e^{-x}$.

By the product rule, $f'(x) = 2xe^{-x} + x^2(-e^{-x}) = e^{-x}x(2-x)$. There are two critical points, 0 and -2. Here are the signs on the resulting intervals.

	0		2		
+ + + + + + + + + + + + + +		+ + + +		+ + + + + + + + + + + +	e^{-x}
− − − − − − − − − − − − − −		+ + + +		+ + + + + + + + + + + +	x
+ + + + + + + + + + + + + +		+ + + +		− − − − − − − − − − − −	$2-x$
− − − − − − − − − − − − − −		+ + + +		− − − − − − − − − − − −	$f'(x) = e^{-x}x(2-x)$

By the first derivative test. f has a local minimum of $f(0) = 0$ at $x = 0$, and a local maximum of $f(2) = \frac{4}{e^2}$ at $x = 2$.

7. Find all local extrema of $f(x) = x\ln|x|$.

By the product rule, the derivative is $f'(x) = 1 \cdot \ln|x| + x\frac{1}{x} = \ln|x| + 1$. This is undefined at $x = 0$, but 0 is not in the domain of f (which is all real numbers *except* 0), so 0 is not a critical point. To find the critical points, we solve $f'(x) = 0$, that is, $\ln|x| + 1 = 0$, or $\ln|x| = -1$. From this, $e^{\ln|x|} = e^{-1}$, so $|x| = \frac{1}{e}$, and the solutions are $\pm\frac{1}{e}$. Therefore there are two critical points $\pm\frac{1}{e}$. These divide the domain $(-\infty, 0) \cup (0, \infty)$ into four intervals as indicated in the chart below.

Interval	$\left(-\infty, -\frac{1}{e}\right)$	$\left(-\frac{1}{e}, 0\right)$	$\left(0, \frac{1}{e}\right)$	$\left(\frac{1}{e}, \infty\right)$
Test point a	$-e$	$-\frac{1}{e^2}$	$\frac{1}{e^2}$	e
$f'(a)$	$f'(-e) = \ln\|-e\| + 1 = 2$	$f'(-1/e^2) = -1$	$f'(1/e^2) = -1$	$f'(e) = 2$
Sign of $f'(a)$	+	−	−	+
f is	increasing	decreasing	decreasing	increasing

By the first derivative test, f has a local maximum of $f(-1/e) = \frac{1}{e}$ at $x = -\frac{1}{e}$. Also there is a local minimum of $f(1/e) = -\frac{1}{e}$ at $x = \frac{1}{e}$.

9. Find all local extrema of $y = \tan^{-1}\left(x^2 + x - 2\right)$.

The derivative is $f'(x) = \dfrac{2x+1}{1 + \left(x^2 + x - 2\right)^2}$. This is defined for all x and equals 0 only for $x = -\frac{1}{2}$. Thus $x = -\frac{1}{2}$ is the only critical point. Note that the denominator of $f'(x)$ is always positive, to the sign of $f'(x)$ is determined by the numerator $2x + 1$. Thus $f'(x) < 0$ on $(\infty, -1/2)$, and Thus $f'(x) > 0$ on $(-1/2, \infty)$. Therefore f has a local maximum of $f(-1/2)$ at $x = -\frac{1}{2}$. There is no local minimum.

11. Find all local extrema of $y = \dfrac{1}{1+x^2}$. The derivative is $y' = \dfrac{-2x}{\left(1+x^2\right)^2}$.

This is positive when x is negative, and negative when x is positive. Thus the derivative changes from positive to negative at $x = 0$, so the function has a local maximum at $x = 0$. There is no local minimum.

13. Find all local extrema of $y = \sqrt[3]{8 - x^3}$.

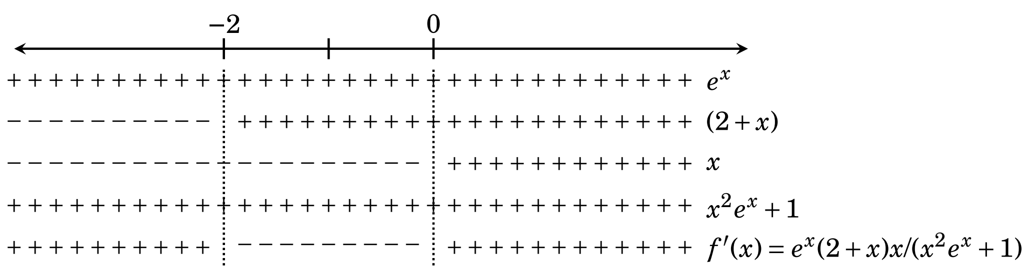
The derivative is $y' = \frac{1}{3}(8-x^3)^{-2/3}(-3x^2) = \frac{-x^2}{\sqrt[3]{8-x^3}^2}$. This is zero when $x=0$ and undefined when $x=2$. For all other x the derivative is defined and negative. Thus there are exactly two critical points, 0 and 2. But since the derivative is negative for all x, there are no local extrema.

15. Find all local extrema of $f(x) = \ln\left(x^2e^x+1\right)$.

To find the critical points, we compute $f'(x)$ and examine what x values would make it undefined or zero.

$$f'(x) = \frac{2xe^x + x^2e^x \ +0}{x^2e^x+1} = \frac{e^x(2x+x^2)}{x^2e^x+1} = \frac{e^x(2+x)x}{x^2e^x+1}$$

The domain of the derivative is all real numbers, so no x values make it undefined. But it will be zero for $x=0$ and $x=-2$, so these are the critical points. Next we examine the signs of the various factors of $f'(x)$.



By the first derivative test, f has a local maximum at $x=-2$ and a local minimum at $x=0$.

17. Find all local extrema of $f(x) = \ln(x) + \frac{3}{x} - \frac{1}{x^2}$ on its domain.

The domain of this function is the interval $(0,\infty)$. To find the critical points, look at $f'(x) = \frac{1}{x} - \frac{3}{x^2} + \frac{2}{x^3}$. This is defined for all x in the domain of f, so the only critical points we will find are those we will find by solving $f'(x)=0$.

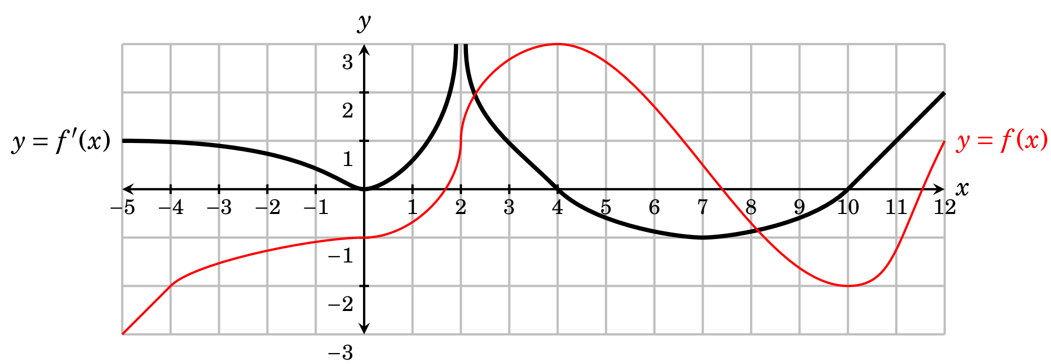
$$\begin{aligned} \frac{1}{x} - \frac{3}{x^2} + \frac{2}{x^3} &= 0 \\ x^3\left(\frac{1}{x} - \frac{3}{x^2} + \frac{2}{x^3}\right) &= 0\cdot x^3 \\ x^2-3x+2 &= 0 \\ (x-1)(x-2) &= 0 \end{aligned}$$

Therefore the critical points are $x=1$ and $x=2$. This breaks the domain $(0,\infty)$ of f into three intervals, namely $(0,1),(1,2)$ and $(2,\infty)$. We will decide the sign of $f'(x)$ on these three intervals with test points.

Interval	$(0,1)$	$(1,2)$	$(2,\infty)$
Test point a	$1/2$	$3/2$	3
$f'(a)$	$f'(1/2)=2$	$f'(3/2)=-2/27$	$f'(3)=2/27$
Sign of $f'(a)$	$+$	$-$	$+$

By the first derivative test, f has a local maximum at $x=1$ and a local minimum at $x=2$.

19. The **derivative** $f'(x)$ of a function $f(x)$ with domain $(-5,12)$ is graphed below. Answer the questions about $f(x)$.



(a) $f(x)$ increases on $(-5,0)$, $(0,2)$, $(2,4)$ and $(10,12)$ where $f'(x)>0$.

(b) $f(x)$ decreases on $(4,10)$ because $f'(x)$ is negative there.

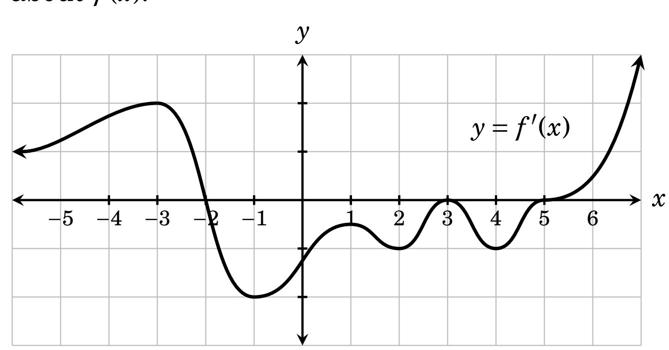
(c) The critical points of $f(x)$ are the values for which $f'(x)$ equals 0 or is undefined. These values are $x=0$, $x=2$, $x=4$ and $x=5$.

(d) $f(x)$ has a local max at $x=4$ because $f'(x)$ changes sign from $+$ to $-$ there.

(e) $f(x)$ has a local min at $x=10$ because $f'(x)$ changes sign from $-$ to $+$ there.

(f) A possible graph of $f(x)$ is sketched in red.

21. The **derivative** $f'(x)$ of a function $f(x)$ is graphed below. Answer the questions about $f(x)$.



(a) The critical points of f are $\boxed{-2, 3 \text{ and } 5}$ because $f'(x)=0$ for these x values.

(b) f increases on the intervals $\boxed{(-\infty,-2)\ \&\ (5,\infty)}$ since $f'(x)>0$ there.

(c) f decreases on the intervals $\boxed{(-2,3)\ \&\ (3,5)}$ since $f'(x)<0$ there

(d) Does f have a local maximum? $\boxed{\text{Yes, at } x=-2}$ as $f'(x)$ goes + to − there.

(e) Does f have a local minimum? $\boxed{\text{Yes, at } x=5}$ as $f'(x)$ goes − to + there.

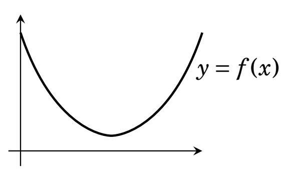
CHAPTER 32

Concavity and the Second Derivative Test

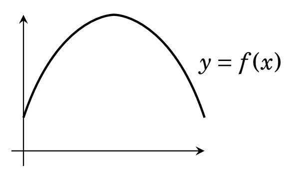
Given a function f, we've learned that its derivative f' tells us something about the shape of the graph of f, namely where it is increasing and decreasing. The function f increases where f' is positive and decreases where f' is negative.

This chapter investigates what the *second derivative* f'' tells us about the shape of graph of $f(x)$. As we will see, f'' gives information about the *concavity* of the graph of f.

But first, a few informal definitions and ideas. A curve is **concave up** if it has the shape of a bowl that would hold water. It is **concave down** if it has the shape of an upside down bowl. This is illustrated below.

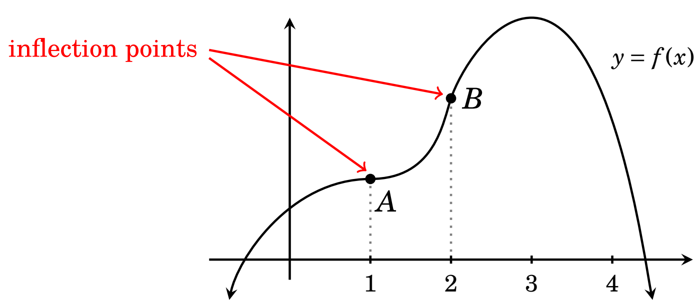


concave up



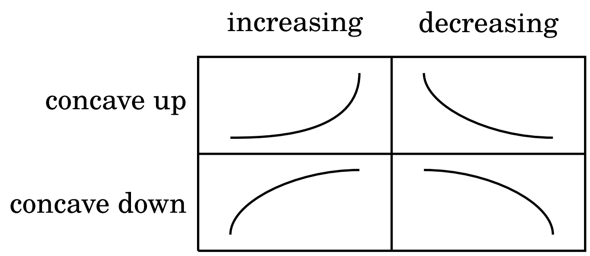
concave down

The graph of a function can be concave up on some intervals and concave down on others. The graph shown below is concave down on the intervals $(-\infty, 1)$ and $(2, \infty)$. It is concave up on $(1, 2)$.



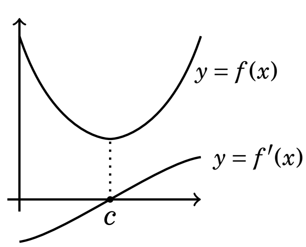
A point on the graph at which the concavity changes is called an **inflection point**. The graph above has two inflection points A (where the concavity changes from down to up), and B (where the concavity changes from up to down).

Note that a graph (or a portion of a graph) that is concave up (or down) can be increasing or decreasing. The four possibilities are shown in the chart below.



Let's now investigate how concavity is determined by the sign of the second derivative. We'll consider the concave up and down situations side-by-side and record our conclusion at the bottom of the page.

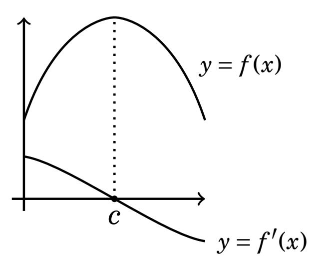
First consider the concave up graph of a function $y = f(x)$ shown below. The tangents to $y = f(x)$ have negative slope to the left of c, but positive slopes to the right of c. As x moves past c the tangent slope at $(x, f(x))$ increases, changing from negative to positive.



Therefore the derivative f' *increases.* Since the derivative of an increasing function is positive, $f''(x) > 0$.

Conclusion: A function is concave up wherever its second derivative is positive.

Now consider the concave down graph of a function $y = f(x)$ shown below. The tangents to $y = f(x)$ have positive slope to the left of c, and negative slopes the right of c. As x moves past c the tangent slope at $(x, f(x))$ decreases, changing from positive to negative.



Thus the derivative f' *decreases.* As the derivative of a decreasing function is negative, $f''(x) < 0$.

Conclusion: A function is concave down where its second derivative is negative.

Fact 32.1 Concavity

A function $f(x)$ is concave up on an interval if $f''(x) > 0$ on the interval.
A function $f(x)$ is concave down on an interval if $f''(x) < 0$ on the interval.

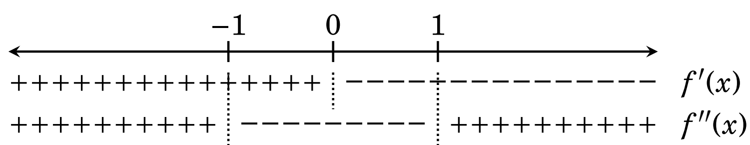
Example 32.1 Consider the function $f(x) = \dfrac{6}{x^2+3}$. Find the intervals on which this function is increasing, decreasing, concave up and concave down. Find all extrema, inflection points, and sketch the graph.

Solution: By the quotient rule, the derivative is $f'(x) = \dfrac{-12x}{\left(x^2+3\right)^2}$. The denominator is positive for any x (because it is squared), so the sign of $f'(x)$ is the same as the sign of its numerator $-12x$. Thus $f'(x)$ is positive when x is negative and negative when x is positive. We tally this in the chart below. By the first derivative test, the point $(0, f(0)) = (0,2)$ is a local maximum.

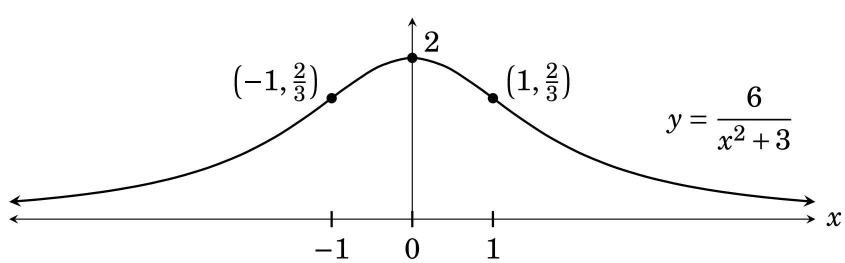
To find the concavity we investigate the second derivative, which is

$$\begin{aligned} f''(x) &= \frac{-12\left(x^2+3\right)^2 - (-12x)2\left(x^2+3\right)2x}{\left(x^2+3\right)^4} = \frac{-12\left(x^2+3\right)\left(\left(x^2+3\right)-4x^2\right)}{\left(x^2+3\right)^4} \\ &= \frac{-12\left(3-3x^2\right)}{\left(x^2+3\right)^3} = \frac{-36\left(1-x^2\right)}{\left(x^2+3\right)^3} = \frac{-36(1+x)(1-x)}{\left(x^2+3\right)^3}. \end{aligned}$$

The denominator is positive for all x, so the sign of $f''(x)$ is the sign of its numerator. The factored numerator tells us $f''(x)$ is zero if $x = \pm 1$, and that $f''(x)$ is positive on $(-\infty,-1)$ and $(1,\infty)$, and negative on $(-1,1)$. We tally this information in the chart below.

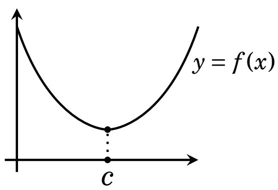


The concavity changes at 1 and 1, so the inflection points are $\left(-1, f(-1)\right) = \left(-1, \frac{2}{3}\right)$ and $\left(1, f(1)\right) = \left(1, \frac{2}{3}\right)$. From the chart, f is increasing and concave up on $(-\infty,-1)$. It is increasing and concave down on $(-1,0)$, and decreasing and concave down on $(0,1)$. On $(1,\infty)$, f is decreasing and concave up.



Note also that $f(x)$ is positive for all x, and $\lim_{x\to\pm\infty} f(x) = 0$, so the line $y = 0$ is a horizontal asymptote. All of this information yields the graph above. ✍

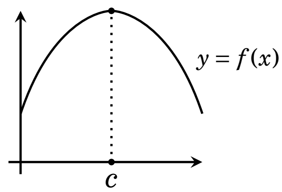
There is an interesting link between concavity and local extrema. Suppose a function f has a critical point c for which $f'(c) = 0$. Observe (as illustrated below) that f has a local minimum at c if its graph is concave up there. And f has a local maximum at c if it is concave down at c.



Local minimum at c

$f(x)$ is concave up

$f''(x) > 0$



Local maximum at c

$f(x)$ concave down

$f''(x) < 0$

Therefore there will be a local minimum at c if $f''(c)$ is positive, and a local minimum at c if $f'(c)$ is negative. This simple observation is called the **second derivative test** for identifying local extrema. It does the same thing as the first derivative test, but it uses the second derivative instead of the first derivative.

The Second Derivative Test (for finding local extrema of a function)

Suppose c is a critical point of $f(x)$ for which $f'(c) = 0$

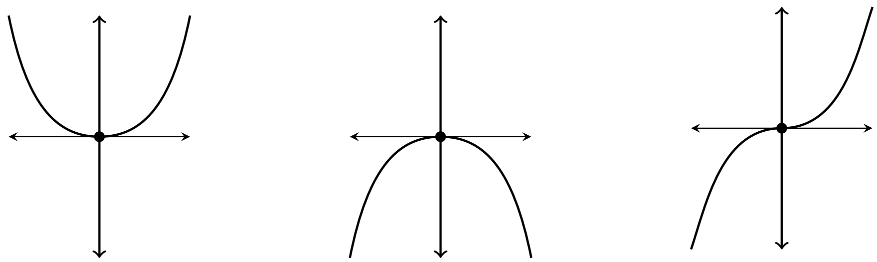
- If $f''(c) > 0$, then $f(x)$ has a local minimum at c.
- If $f''(c) < 0$, then $f(x)$ has a local maximum at c.
- If $f''(c) = 0$, then there is no conclusion. (Use first derivative test.)

The second derivative test can be (as we will see) easier to use than the first derivative test, but it does have some drawbacks. First, it only applies to critical points for which $f'(c) = 0$. If f has any critical points c for which $f'(c)$ is not defined, then the second derivative test says nothing about them.

Another drawback of the second derivative test is that it is inconclusive if $f''(c) = 0$. In this case there could be a minimum, a maximum, or no extremum at all at c, and the second derivative test can't distinguish between these possibilities. Figure 32.1 explains why this is so.

If ever the second derivative test is inconclusive (or if you have a critical point c for which $f'(c)$ is not defined), then you have to resort to the first derivative test.

Local min. at 0 | **Local max. at 0** | **No extremum at 0**



$f(x) = x^4$	$f(x) = -x^4$	$f(x) = x^3$
$f'(x) = 4x^3$	$f'(x) = -4x^3$	$f'(x) = 3x^2$
$f''(x) = 12x^2$	$f''(x) = -12x^2$	$f''(x) = 6x$
$f''(0) = 0$	$f''(0) = 0$	$f''(0) = 0$

Figure 32.1. If $f''(c) = 0$, the function f could have a local maximum, a local minimum, or neither at c. Thus $f''(c) = 0$ tells us nothing about extrema at c. Therefore the second derivative test is inconclusive when $f''(c) = 0$.

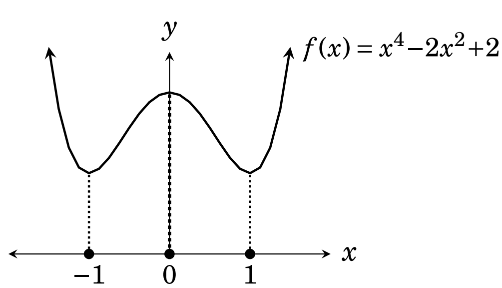
Let's work some examples using the second derivative test to find local extrema. We'll do the same examples we did in Chapter 31, but this time we'll use the second derivative test instead of the first derivative test.

Example 32.2 Find all local extrema of the function $f(x) = x^4 - 2x^2 + 2$.

Solution The derivative is $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1)$, from which we read off the critical points as -1, 0 and 1. Moreover, these critical points make f' zero, so the second derivative test will apply to them.

The second derivative is $f''(x) = 12x^2 - 4$. We now test each critical point.

Because $f''(-1) = 12(-1)^2 - 4 = 8 > 0$, there is a local minimum at -1.
Because $f''(0) = 12 \cdot 0^2 - 4 = -4 < 0$, there is a local maximum at 0.
Because $f''(1) = 12 \cdot 1^2 - 4 = 8 > 0$, there is a local minimum at 1.



This is exactly the same answer we got in Example 31.1. Notice how much easier our work was with the second derivative test. ✍

Example 32.3 Find all local extrema of $f(x) = \sqrt[3]{x^2} - \frac{2}{3}x$ on $(-\infty, \infty)$.

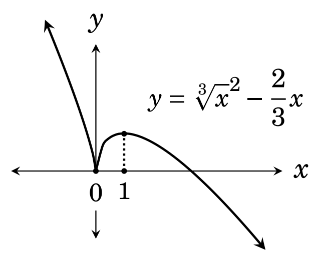
Solution We solved this using the first derivative test in Example 31.2, but now we will try it with the second derivative test. The derivative is

$$f'(x) \;=\; \frac{2}{3}x^{2/3-1} - \frac{2}{3} \;=\; \frac{2}{3}\left(x^{-1/3} - 1\right) \;=\; \frac{2}{3}\left(\frac{1}{\sqrt[3]{x}} - 1\right).$$

We can read off the critical points as 0 (because $f'(0)$ is undefined) and 1 (because $f'(1) = 0$). However, the second derivative test will not apply to the critical point 0. Nonetheless, let's apply it to the critical point 1.

The second derivative is $f''(x) = -\frac{2}{9}x^{-1/3-1} = -\frac{2}{9\sqrt[3]{x^4}}$. Because $f''(1) = -\frac{2}{9} < 0$, the second derivative test guarantees a local maximum at 1.

So the second derivative test has given us a partial answer. There is a local maximum at 1, but it didn't pick up the local minimum at 0. For that we need to use the first derivative test, as in Example 31.2. ✍



Example 32.4 Find all local extrema of $y = x\sin(x) + \cos(x) - \frac{x^2}{4}$ on $(-\pi, \pi)$.

Solution The domain of this function is all real numbers, but we are only asked about its extrema of it on $(-\pi, \pi)$. The derivative is

$$\frac{dy}{dx} \;=\; 1 \cdot \sin(x) + x\cos(x) - \sin(x) - \frac{x}{2} \;=\; x\left(\cos(x) - \frac{1}{2}\right).$$

As was noted in Example 31.3, the critical points are the values in $(-\pi, \pi)$ that make this zero, namely $x = 0$ and $x = \pm\frac{\pi}{3}$. The second derivative is

$$\frac{d^2y}{dx^2} \;=\; 1 \cdot \left(\cos(x) - \frac{1}{2}\right) - x\sin(x) \;=\; \cos(x) - \frac{1}{2} - x\sin(x).$$

Next, plug the critical points into the second derivative and note the signs.

$$\left.\frac{d^2y}{dx^2}\right|_{-\pi/3} \;=\; \cos\left(-\frac{\pi}{3}\right) - \frac{1}{2} - \left(-\frac{\pi}{3}\sin\left(-\frac{\pi}{3}\right)\right) \;=\; \frac{1}{2} - \frac{1}{2} - \frac{\pi}{3}\frac{\sqrt{3}}{2} < 0 \qquad \text{(negative)}$$

$$\left.\frac{d^2y}{dx^2}\right|_{0} \;=\; \cos(0) - \frac{1}{2} - 0\sin(0) \;=\; \frac{1}{2} > 0 \qquad \text{(positive)}$$

$$\left.\frac{d^2y}{dx^2}\right|_{\pi/3} \;=\; \cos\left(\frac{\pi}{3}\right) - \frac{1}{2} - \frac{\pi}{3}\sin\left(\frac{\pi}{3}\right) \;=\; \frac{1}{2} - \frac{1}{2} - \frac{\pi}{3}\frac{\sqrt{3}}{2} < 0 \qquad \text{(negative)}$$

The function has local maxima at $-\frac{\pi}{3}$ and $\frac{\pi}{3}$, and a local minimum at 0. ✍

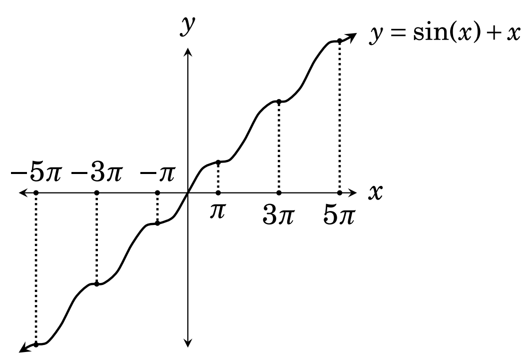
Example 32.5 Find all local extrema of $f(x) = \sin(x) + x$ on $(-\infty, \infty)$.

Solution We already solved this problem in Example 31.5, where we found that the function has no local extrema at all. Here we will attempt to do it again with the second derivative test.

The derivative $f'(x) = \cos(x) + 1$ is defined for all real values of x, so there are no critical points c for which $f'(c)$ is not defined. Therefore the critical points of f will be numbers for which the derivative $\cos(x) + 1$ equals 0. These are the odd multiples of π (the numbers c for which $\cos(c) = -1$),

$$\ldots -7\pi,\ -5\pi,\ -3\pi,\ -\pi,\ \pi,\ 3\pi,\ 5\pi,\ 7\pi, \ldots$$

The second derivative is $f''(x) = -\sin(x)$, and because $f''(k\pi) = -\sin(k\pi) = 0$ for integer multiples of π, we conclude $f''(c) = 0$ for all critical points of f. Thus the second derivative test is entirely inconclusive. To solve this problem we must revert to the first derivative test, which was done in Example 31.5.



There we found that f continues rising forever, and has no extrema. ✍

Exercises for Chapter 32

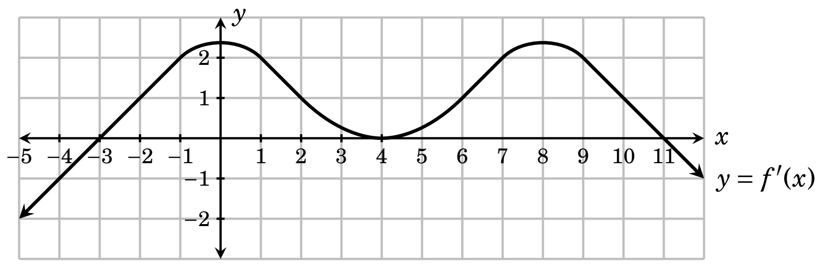
1. This problem concerns the function $f(x) = (x-2)e^x$. Find the intervals on which $f(x)$ increases/decreases. Find the intervals on which $f(x)$ is concave up/down.
2. This problem concerns the function $f(x) = 3x^{2/3} - 2x$. Find the intervals on which $f(x)$ increases/decreases. Find the intervals on which $f(x)$ is concave up/down.
3. Use the second derivative test to find the local extrema of $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$.
4. Use the second derivative test to find the local extrema of $f(x) = \frac{3}{2}x^4 - x^6$.
5. Use the second derivative test to find the local extrema of $f(x) = 5x^4 + 20x^3 + 10$.
6. Use the second derivative test to find thelocal extrema of $f(x) = x^2 e^{-x}$.
7. Use the second derivative test to find the local extrema of $f(x) = x^2 e^x$.

8. Use the second derivative test to find thelocal extrema of $f(x) = x\ln|x|$.

9. Use the second derivative test to find the local extrema of $f(x) = e^{x^2-2x}$.

10. Use the second derivative test to find the local extrema of $y = \tan^{-1}\left(x^2+x-2\right)$.

11. The graph $y = f'(x)$ of <u>the derivative</u> of a function $f(x)$ is shown. Answer the questions <u>about $f(x)$.</u>



(a) State the intervals on which f increases and decreases.

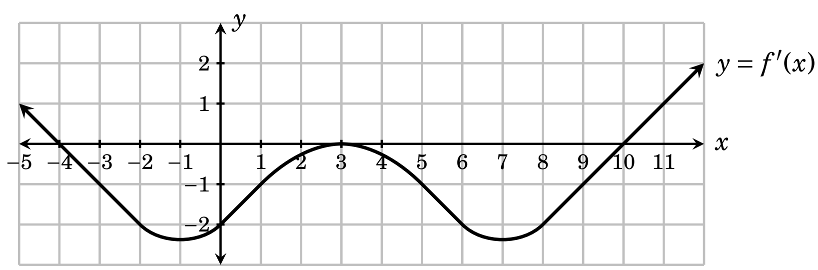
(b) List all critical points of f.

(c) At each critical point, say if f has a local maximum, minimum, or neither.

(d) State the intervals on which f is concave up and concave down.

(e) Based on this information, sketch a possible graph of f.

12. The graph $y = f'(x)$ of <u>the derivative</u> of a function $f(x)$ is shown. Answer the questions <u>about $f(x)$.</u>



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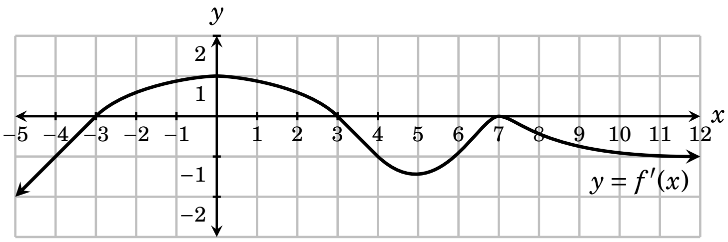
13. The graph $y = f'(x)$ of <u>the derivative</u> of a function $f(x)$ is shown. Answer the questions <u>about $f(x)$.</u>



(a) State the intervals on which f increases and decreases.

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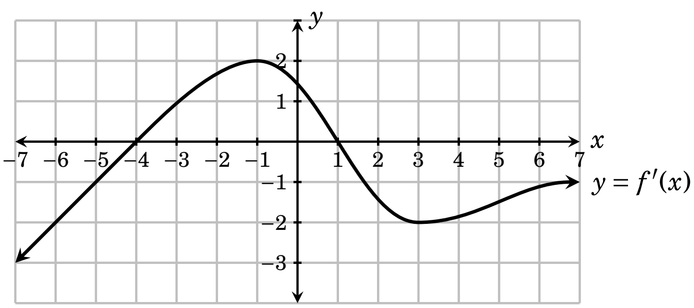
(c) At each critical point, say if f has a local maximum, minimum, or neither.

(d) State the intervals on which f is concave up and concave down.

(e) Based on this information, sketch a possible graph of f.

14. The graph $y = f'(x)$ of the derivative of a function $f(x)$ is shown. Answer the questions about $f(x)$.



(a) State the intervals on which f increases and decreases.

(b) List all critical points of f.

(c) At each critical point, say if f has a local maximum, minimum, or neither.

(d) State the intervals on which f is concave up and concave down.

(e) Based on this information, sketch a possible graph of f.

Exercises Solutions or Chapter 32

1. This problem concerns the function $f(x) = (x-2)e^x$. Find the intervals on which $f(x)$ increases/decreases. Find the intervals on which $f(x)$ is concave up/down.

By the product rule, $f'(x) = e^x + (x-2)e^x = e^x(x-1)$. This is negative on $(-\infty, 1)$ and positive on $(1, \infty)$. Therefore f decreases on $(-\infty, 1)$ and increases on $(1, \infty)$.

The second derivative is $f''(x) = e^x(x-1) + e^x = xe^x$. This is negative on $(-\infty, 0)$ and positive on $(0, \infty)$. Thus f is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

3. Use the second derivative test to find the local extrema of $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$.

Since $f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2) = 12x(x-1)(x+2)$, the critical points as $x = 1$ $x = 0$ and $x = -2$. The second derivative is $f''(x) = 36x^2 + 24x - 24$. Since $f''(1) = 36 \cdot 1^2 + 24 \cdot 1 - 24 = 36 > 0$, the function f has a a local minimum of $f(1) = -3$ at $x = 1$. Since $f''(0) = 36 \cdot 0^2 + 24 \cdot 0 - 24 = -24 < 0$, the function f has a a local maximum of $f(0) = 2$ at $x = 0$. Since $f''(-2) = 36 \cdot (-2)^2 + 24 \cdot (-2) - 24 = 72 > 0$, the function f has a a local minimum of $f(-2) = 34$ at $x = -2$.

5. Use the second derivative test to find the local extrema of $f(x) = 5x^4 + 20x^3 + 10$.

Since $f'(x) = 20x^3 + 60x^2 = 20x^2(x+3)$, the critical points as $x = 0$ and $x = -3$. The second derivative is $f''(x) = 60x^2 + 120x$. Since $f''(-3) = 60 \cdot (-3)^2 + 120 \cdot (-3) = 180 > 0$, the second derivative test says $f(x)$ has a local minimum of $f(-3) = -125$ at $x = -3$. However, $f''(0) = 60 \cdot 0^2 + 120 \cdot 0 = 0$, so the second derivative test gives no conclusion about a local extremum at 0. (Using the first derivative test, you will find that $f'(x) > 0$ on either side of 0, so there is no extremum at $x = 0$.)

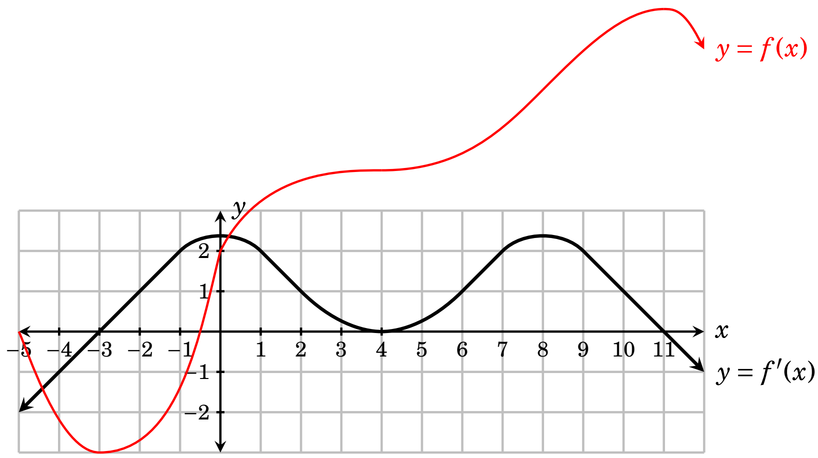
7. Use the second derivative test to find the local extrema of $f(x) = x^2 e^x$.

Since $f'(x) = 2xe^x + x^2 e^x = e^x(2x + x^2) = e^x x(2+x)$, we can read off the critical points as $x = 0$ and $x = -2$. The second derivative is $f''(x) = e^x(2x + x^2) + e^x(1 + 2x) = e^x\left(x^2 + 4x + 1\right)$. Since $f''(0) = e^0 = 1 > 0$, the function $f(x)$ has a local minimum of $f(0) = 0^2 e^0 = 0$ at $x = 0$. Since $f''(-2) = e^{-2}\left((-2)^2 + 4(-2) + 1\right) = \frac{-3}{e^2} < 0$, the function $f(x)$ has a local maximum of $f(-2) = (-2)^2 e^{-2} = \frac{4}{e^2}$ at $x = -2$.

9. Use the second derivative test to find the local extrema of $f(x) = e^{x^2 - 2x}$.

Since $f'(x) = e^{x^2-2x}(2x-2)$, the only critical point is $x = 1$. The second derivative is $f''(x) = e^{x^2-2x}(2x-2)(2x-2) + e^{x^2-2x}2 = e^{x^2-2x}\left(4x^2 - 8x + 6\right)$. Now observe that $f''(1) = e^{1^2 - 2\cdot 1}(4 \cdot 1^2 - 8 \cdot 1 + 6) = \frac{2}{e} > 0$. Thus the function f has a local minimum of $f(1) = e^{1^2-2} = \frac{1}{e}$ at $x = 1$.

11. The graph $y = f'(x)$ of the derivative of a function $f(x)$ is shown. Answer the questions about $f(x)$.



(a) $f(x)$ increases on $(-3,4)$ and $(4,11)$ because $f'(x) > 0$ there. Also, $f(x)$ decreases on $(-\infty,-3)$ and $(11,\infty)$ because $f'(x) < 0$ there.

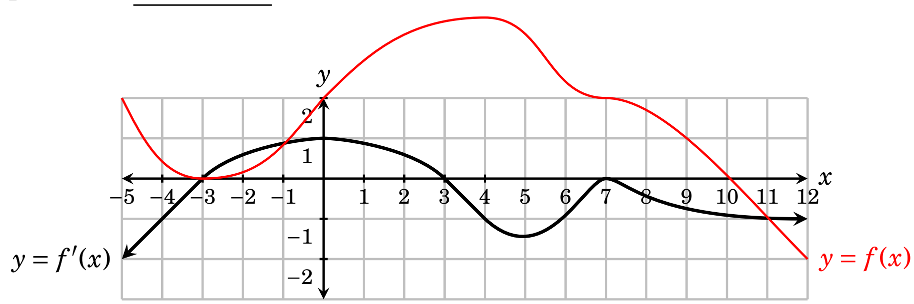
(b) The critical points of $f(x)$ are $-3,4,11$

(c) f has a local minimum at -3 because $f'(x)$ changes sign from $-$ to $+$ there. f has no extrema at $x = 4$ because $f'(x)$ does not change sign there.

(d) f is concave up on $(-\infty,0)$ and $(4,8)$ because f' is increasing on these intervals and consequently $f''(x) > 0$ there. Also, f is concave down on $(0,4)$ and $(8,\infty)$ because f' is decreasing on these intervals and consequently $f''(x) < 0$ there.

(e) A possible sketch of f is shown above.

13. The graph $y = f'(x)$ of the derivative of a function $f(x)$ is shown. Answer the questions about $f(x)$.



(a) The function f increases on $(-3,3)$ because $f'(x) > 0$ there. Also, f decreases on $(-\infty,-3)$, $(3,7)$ and $(7,\infty)$ because $f'(x)$ is negative on those intervals.

(b) The critical points of f are -3, 3 and 7.

(c) The function f has a local minimum at -3 because $f'(x)$ changes from $-$ to $+$ there. Also f has a local maximum at 3 because $f'(x)$ changes from $+$ to 1 there. There is no extremum at 7 because $f'(x)$ does not change sign there.

(d) The function f is concave up on $(-\infty,0)$ and $(5,7)$ because f' is increasing there (and hence $f''(x) > 0$). Also, f is concave down on $(0,5)$ and $(7,\infty)$ because f' is increasing there (and hence $f''(x) > 0$).

(e) A possible graph of f is sketched above.

CHAPTER 38

Antiderivatives

We are at a major turning point of the course. Up until now our primary focus has been on the process of *differentiation*. Given a function $f(x)$, find its derivative $f'(x)$.

$$f(x) \xrightarrow{\text{differentiation}} f'(x)$$

Starting now, we shift our focus to the reverse process. If you know the derivative $f'(x)$, can you find the function $f(x)$? This reverse process is called **antidifferentiation** or **integration.**

$$f(x) \xleftarrow[\text{antidifferentiation}]{\text{integration}} f'(x)$$

With this goal in mind, we state the chapter's main definition.

Definition 38.1
A function $F(x)$ is an **antiderivative** of $f(x)$ if $D_x\Big[F(x)\Big] = f(x)$.

$$F(x) \xrightarrow{\text{differentiate}} f(x)$$

antiderivative of $f(x)$ $\qquad$ derivative of $F(x)$

For example, let's find an antiderivative of the function $f(x) = 2x$. We ask: what function could we differentiate that would produce a derivative of $2x$?

$$? \xrightarrow{\text{differentiate}} 2x$$

In essence we are asking $D_x[?] = 2x$. Because $D_x[x^2] = 2x$, the function x^2 is an antiderivative of $2x$.

Actually, there are *lots* of functions whose derivatives are $2x$:

$$\begin{aligned} D_x[x^2] &= 2x \\ D_x[x^2+1] &= 2x \\ D_x[x^2-2] &= 2x \\ D_x[x^2+\pi] &= 2x \end{aligned}$$

In general, if C is any constant whatsoever,

$$D_x[x^2+C] = 2x.$$

Thus the function $f(x)=2x$ has infinitely many antiderivatives $F(x)=x^2+C$. Their graphs are the graph of $y=x^2$ raised (or lowered) by C units.

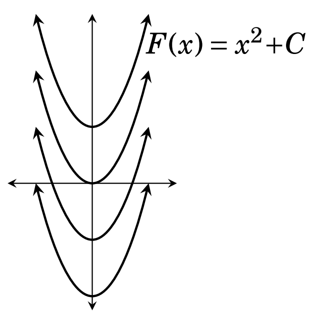


Figure 38.1. The antiderivatives of the function $f(x)=2x$.

Likewise, a little reverse engineering tells us that the antiderivatives of the function $f(x)=3x^2$ are the functions $F(x)=x^3+C$ (where C is a constant) because $D_x[x^2+C]=3x^2$. Their graphs are indicated below.

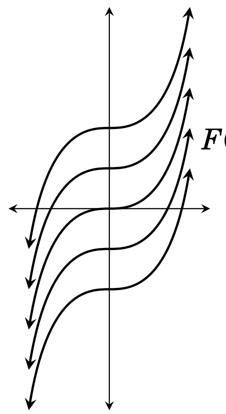


Figure 38.2. The antiderivatives of the function $f(x) = 3x^2$.

So if $f(x)$ is a function that has an antiderivative $F(x)$, then $f(x)$ has not one but infinitely many antiderivatives $F(x)+C$, where C is any constant. There is a special name and notation for these antiderivatives.

Definition 38.2 Suppose f is a continuous function on an interval. The set of all antiderivatives of f is called the **indefinite integral** of f. This set of functions is denoted by

$$\int f(x)\,dx.$$

Thus $\int f(x)dx$ stands for the set of all functions whose derivative is $f(x)$. We typically write $\int f(x)dx = F(x)+C$ where C denotes a constant and $D_x\big[F(x)+C\big] = f(x)$. We read $\int f(x)dx$ as "*the indefinite integral of $f(x)$ dx*" or just "*the integral of $f(x)$ dx*".

Example: $\displaystyle\int 2x\,dx = x^2 + C$, where C is a constant.

Example: $\displaystyle\int 3x^2\,dx = x^3 + C$, where C is a constant.

Given $\int f(x)dx$, the process of finding $F(x)+C$ is called **integration**. In the examples above we found $F(x)+C$ simply from our experience with differentiation, but we will shortly develop a set of integration formulas.

In the expression $\int f(x)dx$, the symbol $\int$ is called the **integral sign** and the function $f(x)$ (the function being integrated) is called the **integrand**. The dx is called a *differential*. We will have more to say about differentials in Chapter 39, but for now it's best to think of the dx as punctuation, like a closing parenthesis.

Remember that $\int f(x)\,dx = F(x) + C$ means that $D_x\Big[F(x)+C\Big] = f(x)$. This is so important that we will display it in a box and revisit it many times.

$$\boxed{\int f(x)\,dx = F(x)+C \quad \Longleftrightarrow \quad D_x\Big[F(x)+C\Big] = f(x)}$$

Example: $\displaystyle\int x^3\,dx = \frac{1}{4}x^4 + C$ because $\displaystyle D_x\left[\frac{1}{4}x^4 + C\right] = x^3$.

Example: $\displaystyle\int \frac{2x}{1+x^2}\,dx = \ln\big(1+x^2\big) + C$ because $\displaystyle D_x\Big[\ln\big(1+x^2\big)+C\Big] = \frac{2x}{1+x^2}$.

Example: $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$ because $\displaystyle D_x\Big[\tan^{-1}(x)+C\Big] = \frac{1}{1+x^2}$.

These examples were somewhat ad hoc. But they underscore the fact that we need a systematic set of rules for finding indefinite integrals. We will begin that task now. Fortunately the task is relatively easy, because for every derivative rule we get a corresponding integral rule by "running the derivative rule in reverse." We will start with the power rule.

For powers $n \neq -1$ we get an immediate rule for $\int x^n\,dx$ as follows.

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \text{ because } D_x\left[\frac{1}{n+1}x^{n+1}+C\right] = \frac{1}{n+1}(n+1)x^n = x^n.$$

This is called the *power rule for integration*.

$$\boxed{\textbf{Power Rule for Integration:} \quad \int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad (\text{provided } n \neq -1)}$$

Example: $\displaystyle\int x^3\,dx = \frac{1}{3+1}x^{3+1} + C = \frac{1}{4}x^4 + C$ ✍

Example: $\displaystyle\int x^8\,dx = \frac{1}{8+1}x^{8+1} + C = \frac{1}{9}x^9 + C$ ✍

Example: $\displaystyle\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{1}{1/2+1}x^{1/2+1} + C = \frac{2}{3}x^{3/2} + C = \frac{3}{2}\sqrt{x}^3 + C$ ✍

Example: $\displaystyle\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{1}{-3+1}x^{-3+1} + C = -\frac{1}{2}x^{-2} + C = \frac{1}{2x^2} + C$ ✍

The power rule for integration breaks down for $n = -1$ because it would read $\int x^{-1} dx = \frac{1}{-1+1}x^{-1+1} + C$, and this involves division by zero.

So is there a formula for $\int x^{-1}dx$? That is, is there a formula for $\int \frac{1}{x}dx$? The answer would have to be a function $F(x)+C$ whose derivative is $\frac{1}{x}$. We don't have to look far: Because $D_x\big[\ln|x|+C\big] = \frac{1}{x}$, we have our next rule.

Power Rule for n = −1: $\displaystyle\int \frac{1}{x}\,dx \;=\; \ln|x|+C$

What other easy integration rules are within reach? If c is a constant (possibly different from C), then $D_x\big[cx+C\big] = c$. This yields another rule.

Constant Rule for Integration: $\displaystyle\int c\,dx \;=\; cx+C$

Example: $\displaystyle\int 5\,dx \;=\; 5x+C$

Example: $\displaystyle\int \sqrt{2}\,dx \;=\; \sqrt{2}x+C$

Any derivative rule $D_x\big[F(x)\big] = f(x)$ yields an integration rule $\int f(x)\,dx = F(x)+C$. For example, from $D_x\big[\sin(x)\big] = \cos(x)$ we get $\int \cos(x)\,dx = \sin(x)+C$. From $D_x\big[\cos(x)\big] = -\sin(x)$ we get $\int \sin(x)\,dx = -\cos(x)+C$.

Here is a list, beginning with the three formulas from the previous page. Each rule below has the form $\int f(x)\,dx = F(x)+C$. Check that each rule is correct by verifying $D_x\big[F(x)+C\big] = f(x)$.

Integration Rules

$$\int c\,dx = cx + C$$

$$\int x^n\,dx = \; = \frac{1}{n+1}x^{n+1} + C \quad (\text{if } n \neq -1)$$

$$\int x^{-1}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int b^x\,dx = \frac{1}{\ln(b)}b^x + C$$

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int \cos(x)\,dx = \sin(x) + C$$

$$\int \sec^2(x)\,dx = \tan(x) + C$$

$$\int \csc^2(x)\,dx = -\cot(x) + C$$

$$\int \sec(x)\tan(x)\,dx = \sec(x) + C$$

$$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}|x| + C$$

These integration rules, or “backwards derivative rules” are easy to internalize and memorize because of our experience with derivative rules.

But notice that there are certain glaring omissions. For instance, there is no formula for $\int \tan(x)\,dx$ because we don't have a derivative rule of form $D_x\Big[F(x)\Big] = \tan(x)$. Formulas for $\int \tan(x)\,dx$, $\int \cot(x)\,dx$, $\int \sec(x)\,dx$, $\int \ln(x)\,dx$, etc., will have to wait until Calculus II.

Never forget that $\int f(x)\,dx = F(x) + C$ means $D_x\big[F(x) + C\big] = f(x)$. So if you integrate $f(x)$, then differentiate the result, you get $f(x)$ back. In symbols,

$$D_x\left[\int f(x)\,dx\right] \;=\; f(x).$$

One consequence of this is another integration rule.

Constant Multiple Rule for Integration: If c is a constant, then

$$\int c f(x)\,dx \;=\; c\int f(x)\,dx.$$

To check that this is correct, we can differentiate the right-hand side and see if we get the integrand $c\,f(x)$ from in the integral on the left. Doing so,

$$D_x\left[c\int f(x)\,dx\right] \;=\; c\,D_x\left[\int f(x)\,dx\right] \;=\; c\,f(x).$$

Since we got $c\,f(x)$, the formula is correct.

The constant multiple rule combines with the other integration formulas:

Example: $\displaystyle\int 7x^3\,dx \;=\; 7\int x^3\,dx \;=\; 7\frac{1}{3+1}x^{3+1} + C \;=\; \frac{7}{4}x^4 + C$

Example: $\displaystyle\int \frac{\pi}{x}\,dx \;=\; \pi\int \frac{1}{x}\,dx \;=\; \pi\ln|x| + C$

Another integration rule comes from reversing the sum-difference rule for derivatives. Check it by differentiating the right side to get $f(x)\pm g(x)$:

Sum-Difference Rule for Integration:

$$\int \big(f(x) \pm g(x)\big)\,dx \;=\; \int f(x)\,dx \;\pm\; \int g(x)\,dx.$$

A great many indefinite integrals can be found by combining the rules on this page with those on previous page. For example:

$$\begin{aligned}
\int (5x^2 + 2x)\,dx &= \int 5x^2\,dx + \int 2x\,dx && \text{(sum-difference rule)} \\
&= 5\int x^2\,dx + 2\int x^1\,dx && \text{(const. mult. rule, twice)} \\
&= 5\frac{1}{3}x^3 + 2\frac{1}{2}x^2 + C && \text{(power rule, twice)} \\
&= \frac{5}{3}x^3 + x^2 + C
\end{aligned}$$

In the third step we just added a C to the end, rather than getting two constants (one from each integral) and adding them.

We did this example in four steps, but after some practice you'll work problems like this in one step.

Example: $\int \left(\pi\cos(x) - 3\sec^2(x) - 3x^2 + 4\right) dx \;=\; \pi\sin(x) - 3\tan(x) - x^3 + 4x + C.$
Always remember that you can check your answer to an integration problem by differentiating your answer and seeing if that produces the integrand. In this example, $D_x\left[\pi\sin(x) - 3\tan(x) - x^3 + 4x + C\right] \;=\; \pi\cos(x) - 3\sec^2(x) - 3x^2 + 4$. That is the integrand, so we know we integrated correctly.

Work enough exercises that you can do such problems readily.

Our examples here and the exercises below use the variable x exclusively. But any variable can be used.

Example: $\int \left(u + 3e^u\right) du \;=\; \frac{1}{2}u^2 + 3e^u + C$. Don't forget that the differential du must match the variable u (use du here, and not dx).

The next example illustrates that occasionally some algebraic manipulation is needed to bring a problem to a form that matches a rule.

Example: Find $\int \left(w^2 - 3w\right)(w+1)\,dw$.
This does not match any integration rules, but we can put it into a manageable form by multiplying the binomials before integrating.

$$\int \left(w^2 - 3w\right)(w+1)\,dw \;=\; \int \left(w^3 - 2w^2 - 3w\right) dw \;=\; \frac{w^4}{4} - \frac{2w^3}{3} - \frac{3w^2}{2} + C$$

There is no product rule for integration, that is, no rule for $\int f(x)g(x)\,dx$. Reversing the product rule for derivatives would yield

$$\int \left(f'(x)g(x) + f(x)g'(x)\right)\,dx \;=\; f(x)g(x) + C.$$

This is not very useful, as few functions match the form $f'(x)g(x) + f(x)g'(x)$. But in Chapter 44 we will see that reversing the *chain rule* is very useful.

In conclusion, we have begun exploring integration, the opposite process of differentiation. This theme will occupy us for the remainder of the course.

$$\begin{tikzcd} F(x)+C \arrow[r, bend left, "D_x\ \ \text{differentiation}"] & f(x) \arrow[l, bend left, "\text{integration}\ \ \int dx"] \end{tikzcd}$$

Exercises for Chapter 38

Find the indicated indefinite integrals.

1. $\int (5x+3+x^4)\,dx$

2. $\int (x^5+x+1)\,dx$

3. $\int (4x^5+x+2)\,dx$

4. $\int (x^3+3x+5)\,dx$

5. $\int (5x^2+2+\sin(x))\,dx$

6. $\int (2e^x+x^4+\sec(x)\tan(x))\,dx$

7. $\int (7+x^6+\sec^2(x))\,dx$

8. $\int (3x^2+\sin(x)+3)\,dx$

9. $\int (e^x+e+\csc^2(x))\,dx$

10. $\int 5x^{-1}\,dx$

11. $\int 3\sec(x)\tan(x)\,dx$

12. $\int (4x^3+\cos(x)+1)\,dx$

13. $\int (\sqrt[3]{x}+\cos(x))\,dx$

14. $\int \sqrt[5]{x^3}\,dx$

15. $\int (e^x+x^4+3)\,dx$

16. $\int (\sec^2(x)+3\sin(x))\,dx$

17. $\int (x^3+2x+e^x)\,dx$

18. $\int 6\sqrt{x}\,dx$

19. $\int \left(4x+\frac{1}{x}+\sin(x)\right)dx$

20. $\int \left(\frac{1}{x^3}+\sqrt{x}\right)dx$

21. $\int \frac{5}{1+x^2}\,dx$

22. $\int \left(\frac{1}{x}+\cos(x)\right)dx$

23. $\int \frac{2}{x\sqrt{x^2-1}}\,dx$

24. $\int \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$

25. $\int \frac{\pi}{\sqrt{1-x^2}}\,dx$

26. $\int \left(\sqrt[3]{x}+\frac{1}{x^4}\right)dx$

27. $\int \frac{\pi}{3+3x^2}\,dx$

28. $\int x+\frac{1}{\sqrt{1-x^2}}\,dx$

29. $\int \frac{1}{\sqrt{x^5}}\,dx$

30. $\int \left(x^2+\frac{1}{x^2}+e\right)dx$

31. $\int \frac{e^{2x}+e^x}{e^x}\,dx$

32. $\int \left(x^4+\frac{1}{x}+\sqrt{2}\right)dx$

33. $\int \frac{1}{x^2}\,dx$

34. $\int \frac{x^2+1}{2x}\,dx$

35. $\int \frac{x^3-3x^2+1}{x^2}\,dx$

36. $\int (x^2+1)(2x+1)\,dx$

37. Is the equation $\int x\cos(x)\,dx = x\sin(x)+\cos(x)+C$ true or false?

38. Is the equation $\int \left(\cos(x)\frac{1}{x}-\sin(x)\ln(x)\right)dx = \cos(x)\ln(x)+C$ true or false?

39. Is the equation $\int \frac{\sin(1/x)}{x^2}\,dx = \cos\left(\frac{1}{x}\right)+C$ true or false?

40. Is the equation $\int x\cos(x)\,dx = \frac{x^2}{2}\sin(x)+C$ true or false?

41. If $f(x)$ and $g(x)$ are differentiable functions, find $\int (f'(x)g(x)+f(x)g'(x))\,dx$.

42. If $f(x)$ and $g(x)$ are differentiable functions, find $\int f'(g(x))g'(x)\,dx$.

43. If $f(x)$ and $g(x)$ are differentiable functions, find $\int \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}\,dx$.

Exercise Solutions for Chapter 38

1. $\int \left(5x+3+x^4\right)\,dx = \frac{5x^2}{2}+3x+\frac{x^5}{5}+C$

3. $\int \left(4x^5+x+2\right)\,dx = \frac{2x^6}{3}+\frac{x^2}{2}+2x+C$

5. $\int \left(5x^2+2+\sin(x)\right)\,dx = \frac{5x^3}{3}+2x-\cos(x)+C$

7. $\int \left(7+x^6+\sec^2(x)\right)\,dx = 7x+\frac{x^7}{7}+\tan(x)+C$

9. $\int \left(e^x+e+\csc^2(x)\right)\,dx = e^x+ex-\cot(x)+C$

11. $\int 3\sec(x)\tan(x)\,dx = 3\sec(x)+C$

13. $\int \left(\sqrt[3]{x}+\cos(x)\right)\,dx = \int \left(x^{1/3}+\cos(x)\right)\,dx = \frac{1}{1/3+1}x^{1/3+1}+\sin(x)+C = \frac{1}{4/3}x^{4/3}+\sin(x)+C$
$= \frac{3}{4}\sqrt[3]{x}^4+\sin(x)+C$

15. $\int \left(e^x+x^4+3\right)\,dx = e^x+\frac{x^5}{5}+3x+C$

17. $\int \left(x^3+2x+e^x\right)\,dx = \frac{x^4}{4}+x^2+e^x+C$

19. $\int \left(4x+\frac{1}{x}+\sin(x)\right)\,dx = 2x^2+\ln|x|-\cos(x)+C$

21. $\int \frac{5}{1+x^2}\,dx = 5\int \frac{1}{1+x^2} = 5\tan^{-1}(x)+C$

23. $\int \frac{2}{x\sqrt{x^2-1}}\,dx = 2\int \frac{1}{x\sqrt{x^2-1}}\,dx = 2\sec^{-1}|x|+C$

25. $\int \frac{\pi}{\sqrt{1-x^2}}\,dx = \pi\int \frac{1}{\sqrt{1-x^2}}\,dx = \pi\sin^{-1}(x)+C$

27. $\int \frac{\pi}{3+3x^2}\,dx = \frac{\pi}{3}\int \frac{1}{1+x^2}\,dx = \frac{\pi}{3}\tan^{-1}(x)+C$

29. $\int \frac{1}{\sqrt{x}^5}\,dx = \int x^{-5/2}dx = \frac{1}{-5/2+1}x^{-5/2+1}+C = \frac{1}{-3/2}x^{-3/2}+C = -\frac{2}{3\sqrt{x}^3}+C$

31. $\int \frac{e^{2x}+e^x}{e^x}\,dx = \int \frac{e^{2x}}{e^x}+\frac{e^x}{e^x}\,dx \int \left(e^x+1\right)\,dx = e^x+x+C$

33. $\displaystyle\int \frac{1}{x^2}\,dx = \int x^{-2}dx = \frac{1}{-2+1}x^{-2+1} + C = -x^{-1} + C = -\frac{1}{x} + C$

35. $\displaystyle\int \frac{x^3 - 3x^2 + 1}{x^2}\,dx = \int \left(x - 3 + x^{-2}\right)dx = \frac{x^2}{2} - 3x - \frac{1}{x} + C$

37. Is the equation $\displaystyle\int x\cos(x)\,dx = x\sin(x) + \cos(x) + C$ true or false?

Since $D_x\Big[x\sin(x) + \cos(x) + C\Big] = \sin(x) + x\cos(x) - \sin(x) = x\cos(x)$, this is true.

39. Is the equation $\displaystyle\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}dx = \cos\left(\frac{1}{x}\right) + C$ true or false?

Since $D_x\left[\cos\left(\frac{1}{x}\right) + C\right] = -\sin\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = \dfrac{\sin\left(\frac{1}{x}\right)}{x^2}$ equals the integrand, this is **true**.

41. $\displaystyle\int \left(f'(x)g(x) + f(x)g'(x)\right)dx = f(x)g(x) + C$ because $D_x\Big[f(x)g(x) + C\Big] = f'(x)g(x) + f(x)g'(x)$.

43. $\displaystyle\int \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}\,dx = \frac{f(x)}{g(x)} + C$ because $D_x\left[\dfrac{f(x)}{g(x)} + C\right] = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.