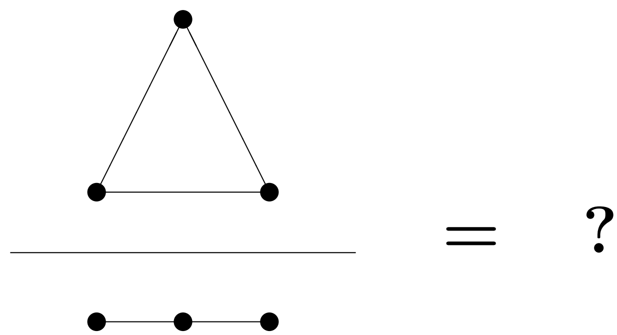
What does a graph fraction look like?



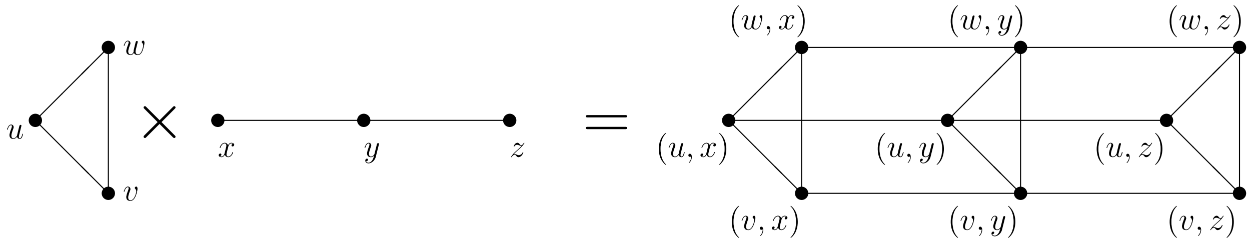
Richard Hammack
Virginia Commonwealth University

Based on the paper “Fractional Graphs”

to appear in

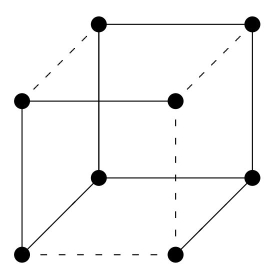
Australasian Journal of Combinatorics

Graphs are *multiplied* with the Cartesian Product

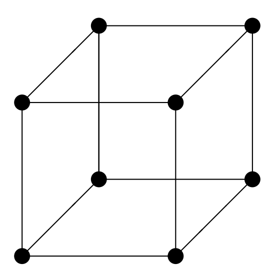


To encode information about numerator and denominator, graph edges are 2-colored:

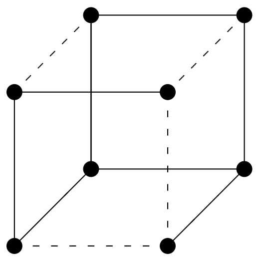
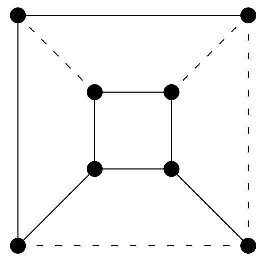
Colored graph:

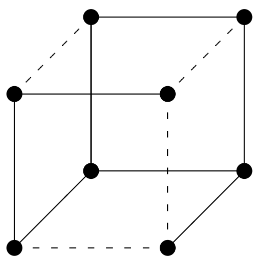
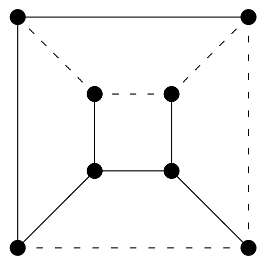


Trivially colored graph:

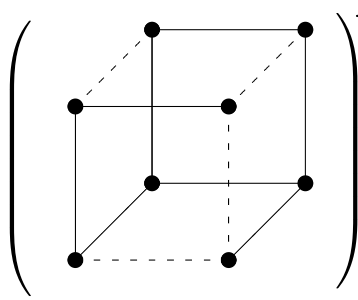
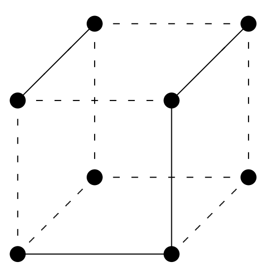


Colored graphs are *equal* if there's a color-preserving isomorphism between them

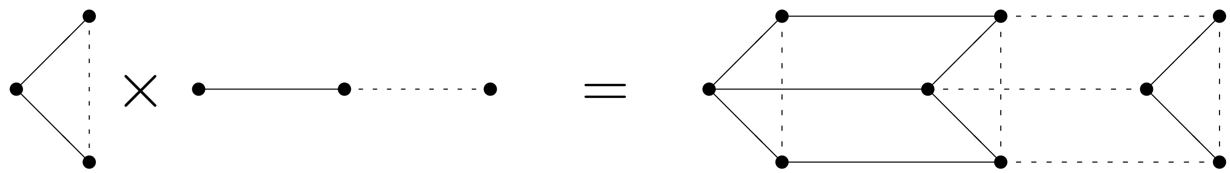
 $=$ 

 $\neq$ 

Definition: G^{-1} is G with colors interchanged.

 $^{-1} =$ 

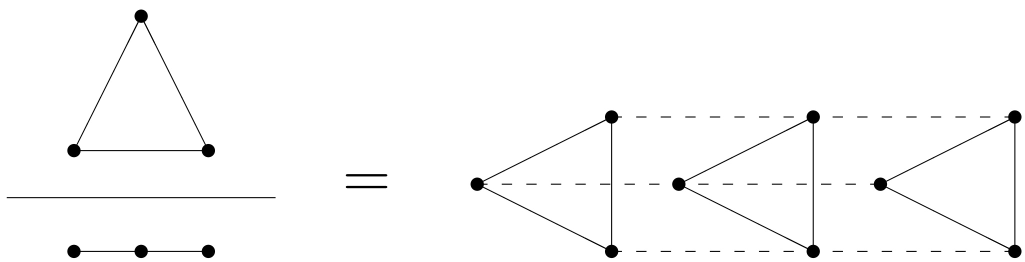
How colored graphs are multiplied:

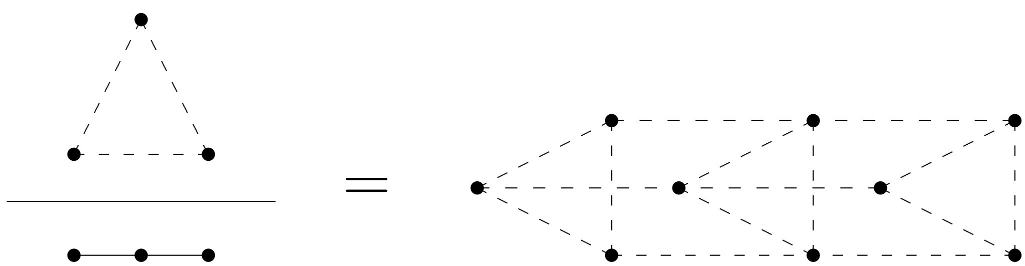


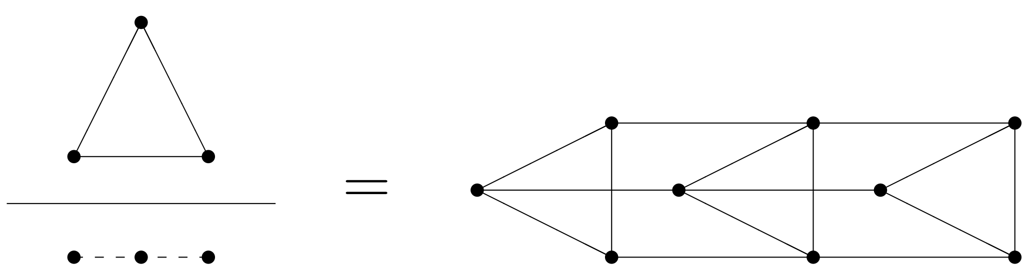
Lemma: If G, H and K are colored graphs, then

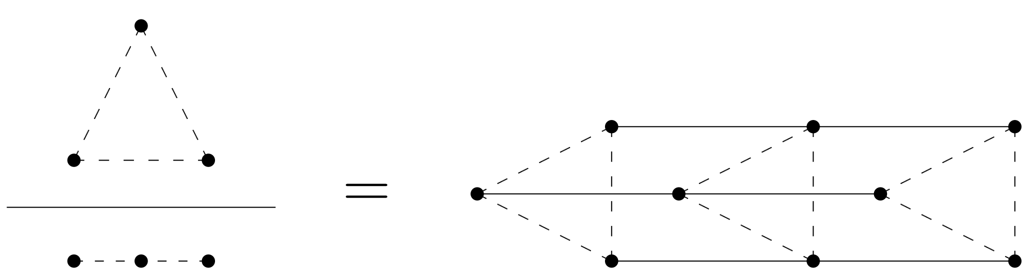
1. $G \times (H \times K) = (G \times H) \times K$
2. $G \times H = H \times G$
3. $(G \times H)^{-1} = G^{-1} \times H^{-1}$
4. $(G^{-1})^{-1} = G$

Definition: $\dfrac{G}{H} = G \times H^{-1}$









Properties:

1. $$\frac{F}{G} \times \frac{H}{K} = \frac{F \times H}{G \times K}$$

2. $$\frac{F/G}{H/K} = \frac{F \times K}{G \times H}$$

3. $$\left(\frac{G}{H}\right)^{-1} = \frac{H}{G}$$

4. $$\frac{I}{G} = G^{-1} \qquad (I = \bullet)$$

5. $$\frac{G}{I} = G$$

Proof of 1:

$$
\begin{aligned}
\frac{F}{G} \times \frac{H}{K} &= (F \times G^{-1}) \times (H \times K^{-1}) \\
&= (F \times H) \times (G^{-1} \times K^{-1}) \\
&= (F \times H) \times (G \times K)^{-1} \\
&= \frac{F \times H}{G \times K}
\end{aligned}
$$

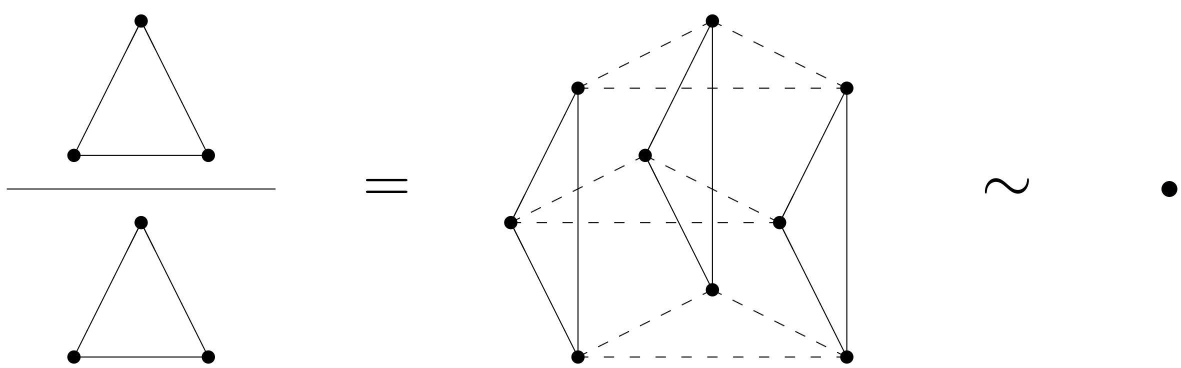
(Proofs of 2–5 are similar.)

A group of graphs

$$\mathbb{G} = \left\{ \; \frac{G}{H} \;\; \middle| \;\; G, H \text{ are connected and trivially colored} \right\}$$

Equivalence relation on $\mathbb{G}$:

$$\frac{F}{G} \sim \frac{H}{K} \iff F \times K = G \times H$$



$$\mathbb{G}^* = \left\{ \; \left[\frac{G}{H} \right] \;\; \middle| \;\; \frac{G}{H} \in \mathbb{G} \right\} \qquad \text{(set of equivalence classes)}$$

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$\mathbb{G}^*$ is a group:

$$\left[\frac{F}{G}\right] \times \left[\frac{H}{K}\right] = \left[\frac{F}{G} \times \frac{H}{K}\right] \qquad \text{(well-defined and associative)}$$

$$[\bullet] \times \left[\frac{G}{H}\right] = \left[\frac{G}{H}\right] \qquad \text{($\bullet$ is the identity)}$$

$$\left[\frac{G}{H}\right] \times \left[\frac{H}{G}\right] = [\bullet] \qquad \text{(everything has an inverse)}$$

Theorem: $\mathbb{G}^* \cong \mathbb{Q}^*$

Idea behind proof:

$\mathbb{Q}^*$ = free abelian group generated on the prime numbers.

$\mathbb{G}^*$ = free abelian group generated on the prime graphs.