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Minimum cycle bases of direct products of complete graphs

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Abstract

This paper presents a construction of a minimum cycle basis for the direct product of two complete graphs on three or more vertices. With the exception of two special cases, such bases consist entirely of triangles.

Keywords: Combinatorial problems; Minimum cycle basis; Graph direct product

1. Introduction

Cycle bases and minimum cycle bases for graphs have a long history in applied discrete mathematics, going back at least as far as G. Kirchhoff's 1847 treatise on electrical networks [10]. More recently, minimal cycle bases have been employed in structural flexibility analysis [9], and in search-and-retrieval of molecular structures in chemical information systems [1,2]. Spurred by these and other applications, much attention has focused on the problem of constructing minimum cycle bases of graphs. Although Horton [5] presents a polynomial algorithm that finds a minimum cycle basis in any graph, the algorithmic approach can lead us to miss deeper connections between the structures of graphs and their cycle bases. Consequently, some authors have recently examined the problem of direct (non-algorithmic) constructions of minimum cycle bases for certain classes of graphs [8,11,12].

In particular, several papers have addressed the problem of finding minimum cycle bases for the various graph-theoretical products. In the article *Minimum Cycle Bases of Product Graphs* [7], W. Imrich and P. Stadler construct minimum cycle bases for Cartesian and strong products of graphs. F. Berger solves the same problem for the lexicographical product [1]. The corresponding construction for the direct product appears to be intriguingly subtle, for there seems to be no clear connection between the minimum cycle bases of the product and of its factors. Currently only partial results have been established. (See, for instance, [4], which constructs an MCB for the direct product of two bipartite graphs.) The present contribution addresses the case of the direct product $K_p \times K_q$ of two complete graphs with $p, q \geqslant 3$. The situation is fairly complex even in this simple setting, but this work could possibly provide insights into the general case.

Recall that if $G = (V(G), E(G))$ is a simple graph, its *cycle space*, denoted $\mathcal{C}(G)$, is the vector space (over the two-element field $\mathbb{F}_2 = \{0, 1\}$) whose elements are the subsets $X \subseteq E(G)$ for which each vertex of G is incident with an even number of the elements of X. Ad-

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dition in $\mathcal{C}(G)$ is symmetric difference of sets, and zero is the empty set. Elements of $\mathcal{C}(G)$ are called *generalized cycles*, or just *cycles*. The dimension of $\mathcal{C}(G)$ is $\nu(G) = |E(G)| - |V(G)| + c$, where c is the number of components of G. (See [3], Theorem 1.9.6.) A basis $\mathcal{B}$ of $\mathcal{C}(G)$ is called a *cycle basis of* G, and its *length* is $l(\mathcal{B}) = \sum_{C\in\mathcal{B}} |C|$. Among all cycle bases of G, one with smallest possible length is called a *minimum cycle basis*, or *MCB*.

The *direct product* of graphs G and H is the graph $G \times H$ whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose edges are $(w, x)(y, z)$ where $wy \in E(G)$ and $xz \in E(H)$. Observe $G \times H$ has $2|E(G)||E(H)|$ edges and $|V(G)||V(H)|$ vertices, so $\nu(G \times H) = 2|E(G)||E(H)| - |V(G)||V(H)| + 1$ if $G \times H$ is connected. The vertices of the complete graph K_p are denoted as $\{1, 2, 3, \ldots, p\}$. Thus, the edges of $K_p \times K_q$ are exactly the pairs $(i, j)(k, \ell)$ with $1 \leqslant i, k \leqslant p$, and $1 \leqslant j, \ell \leqslant q$, and $i \neq k$ and $j \neq \ell$. Note that for $p, q \geqslant 3$, $K_p \times K_q$ is connected, so

$$\nu(K_p \times K_q) = 2\binom{p}{2}\binom{q}{2} - pq + 1.$$

Any triangle $(a, d)(b, e)(c, f)(a, d)$ in $K_p \times K_q$ projects uniquely to two triangles $abca$ and $defd$ in K_p and K_q, respectively. Conversely, given such triangles $abca$ and $defd$, the graph $K_p \times K_q$ has six triangles that project onto them, namely $(\pi(a), d)(\pi(b), e)(\pi(c), f)(\pi(a), d)$, where π is any permutation of $\{a, b, c\}$. Consequently, $K_p \times K_q$ has exactly $6\binom{p}{3}\binom{q}{3}$ triangles. The following pages show that whenever this number is at least as large as $\nu(K_p \times K_q)$, the triangles will span $\mathcal{C}(K_p \times K_q)$, and an MCB can be extracted from them.

2. Two special cases

Here we describe MCBs for $K_3 \times K_3$ and $K_3 \times K_4$. These graphs are special in that their MCBs contain both triangles and squares. By contrast, the subsequent section will show that if $p, q \geqslant 4$, any MCB for $K_p \times K_q$ consists solely of triangles. The arguments in this section use ideas from homology theory, but they do not employ the machinery of this theory, for the benefit of readers who are not conversant with algebraic topology. Topologists will see how the arguments can be shortened.

Consider $\Gamma = K_3 \times K_3$. Fig. 1 shows an embedding of Γ on the torus, with vertices (i, j) abbreviated as ij. (The torus is presented as an identification space, with opposite sides of the large outer-most square identified. Note that these four sides are not edges of Γ.) This embedding has six triangular regions, two square regions and one 10-gon region. Denote by $\mathcal{T}$ the set of six triangular subgraphs of Γ that bound of the triangular regions of the embedding. Elements of $\mathcal{T}$ are pairwise edge-disjoint, so $\mathcal{T}$ is a linearly independent set in $\mathcal{C}(\Gamma)$. Note that $\mathcal{T}$ contains every triangle in Γ, so if it is enlarged to a cycle basis of Γ then longer cycles must be appended to it; if all such larger cycles are squares, we will have an MCB. Let $\mathcal{S}$ denote the (independent) set of the two square subgraphs that bound the square regions of the embedding. Set $\mathcal{R} = \{(3, 2)(2, 3)(1, 2)(2, 1)(3, 2), (2, 2)(3, 3)(1, 2)(3, 1)(2, 2)\}$. Observe that $\mathcal{T} \cup \mathcal{S} \cup \mathcal{R}$ is linearly independent: Suppose $T + S + R = 0$, where T, S and R are in the spans of $\mathcal{T}$, $\mathcal{S}$ and $\mathcal{R}$, respectively. Note $T = 0$ because each nonzero element of $\mathrm{Span}(\mathcal{T})$ has edges that do not belong to elements of $\mathcal{S} \cup \mathcal{R}$, and therefore cannot be canceled by S and R. Next, $S = 0$ because each nonzero element of $\mathrm{Span}(\mathcal{S})$ has edges belonging to no element of $\mathcal{R}$. Thus $T = S = R = 0$, proving independence. Since $|\mathcal{T} \cup \mathcal{S} \cup \mathcal{R}| = 10 = \nu(K_3 \times K_3)$, it follows $\mathcal{T} \cup \mathcal{S} \cup \mathcal{R}$ is an MCB of $K_3 \times K_3$ consisting of six triangles and four squares.

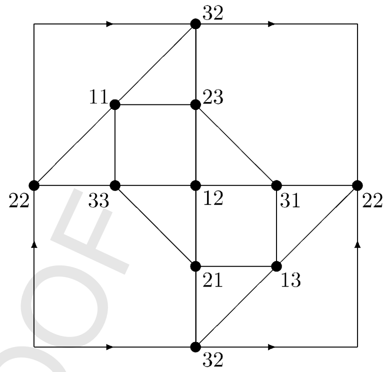

Fig. 1. $K_3 \times K_3$ on the torus.

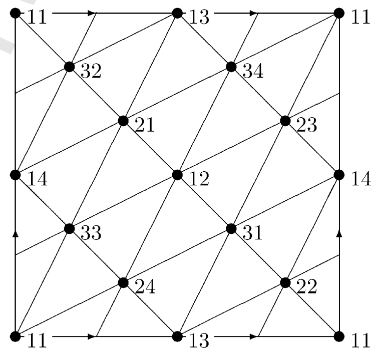

Fig. 2. $K_3 \times K_4$ on the torus.

Next, consider $\Lambda = K_3 \times K_4$. Fig. 2 shows an embedding of Λ on the torus, again with vertices (i, j)

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abbreviated as ij. The boundaries of the 24 regions of this embedding are the 24 triangles in Λ. Now, if all these triangles are added together, their edges will cancel pair-by-pair and give 0, so they are not an independent set. However, we claim that a set of any 23 of them is independent. Let $\mathcal{T}$ be a subset of $\mathcal{C}(\Lambda)$ consisting of 23 of the 24 triangles in Λ. Then $\mathcal{T}$ is linearly independent because any sum of its elements that equals 0 cannot include the three triangles in $\mathcal{T}$ with edges belonging to the missing triangle (for such edges could not then be canceled); nor can the sum include triangles with edges belonging to these three triangles, etc., and it follows from connectivity of the torus that only the trivial linear combination of elements of $\mathcal{T}$ can equal 0. Since $\nu(\Lambda) = 25$, just two more cycles are needed to enlarge $\mathcal{T}$ to an MCB, and since all the triangles have been used up, the extra cycles must have length greater than 3. Set $\mathcal{R} = \{(1,1)(3,3)(1,4)(3,2)(1,1), (1,1)(2,4)(1,3)(2,2)(1,1)\}$. (These squares are representatives of the two standard generators of the first homology group of the torus.) It is straightforward—if tedious—to check that $\mathcal{T} \cup \mathcal{R}$ is linearly independent and is therefore an MCB of $K_3 \times K_4$.

These two examples have proved

Proposition 1. *The product $K_3 \times K_3$ has an MCB consisting of six triangles and four squares. The product $K_3 \times K_4$ has an MCB consisting of 23 triangles and two squares.*

3. A basis of triangles

Suppose $p, q \geqslant 4$, or $p = 3$ and $q \geqslant 5$. Here is a construction of an MCB of $K_p \times K_q$ that consists entirely of triangles.

For each integer $m \in \{1, 2, 3, \ldots, q\}$, let G_m be the subgraph of $K_p \times K_q$ induced on the vertices $\{(i, j) \mid 2 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant q,\ j \neq m\}$. Thus $G_m \cong K_{p-1} \times K_{q-1}$, and is connected because its factors are connected, and at least one has an odd cycle (Theorem 5.29 of [6]). For each $2 \leqslant m \leqslant q$, let T_m be a spanning tree of G_m. Further, T_2 and T_3 are required to have a special form. Each vertex of form $(i, 1)$ is required to have degree 1 in T_2. Also, T_3 is chosen to have no edges of form $(i, 1)(j, 2)$. (One easily checks that such trees exist for the stated values of p and q.)

Define the following edge sets in $K_p \times K_q$.

$$
\begin{aligned}
S_1 &= E(G_1),\\
S_2 &= E(T_2) \cup \big[E(G_2) - E(G_1)\big],\\
S_3 &= E(T_3) \cup \big[E(G_3) - \big(E(G_1) \cup E(G_2)\big)\big],\\
S_4 &= E(T_4),\\
S_5 &= E(T_5),\\
&\ \vdots\\
S_q &= E(T_q).
\end{aligned}
$$

Observe that these sets have been constructed so that the subgraph of $K_p \times K_q$ induced on edges $(S_1 \cup S_2 \cup \cdots \cup S_{m-1}) \cap S_m$ is acyclic for $2 \leqslant m \leqslant q$. This is obvious for $4 \leqslant m \leqslant q$. For $m = 2$, note that

$$
\begin{aligned}
S_1 \cap S_2 &= E(G_1) \cap \big[E(T_2) \cup \big[E(G_2) - E(G_1)\big]\big]\\
&= \big[E(G_1) \cap E(T_2)\big]\\
&\quad \cup \big[E(G_1) \cap \big[E(G_2) - E(G_1)\big]\big]\\
&= \big[E(G_1) \cap E(T_2)\big] \cup \emptyset
\end{aligned}
$$

is acyclic. Also

$$
\begin{aligned}
(S_1 \cup S_2) \cap S_3 &= (S_1 \cup S_2) \cap \big[E(T_3)\\
&\qquad \cup \big[E(G_3) - \big(E(G_1) \cup E(G_2)\big)\big]\big]\\
&= \big[(S_1 \cup S_2) \cap E(T_3)\big] \cup \big[(S_1 \cup S_2)\\
&\qquad \cap \big[E(G_3) - \big(E(G_1) \cup E(G_2)\big)\big]\big]\\
&= \big[(S_1 \cup S_2) \cap E(T_3)\big] \cup \emptyset
\end{aligned}
$$

is acyclic.

In what follows, the cardinalities of the S_m will be needed, so these are now computed. First, since $G_1 \cong K_{p-1} \times K_{q-1}$ we have:

$$|S_1| = 2\binom{p-1}{2}\binom{q-1}{2}. \tag{1}$$

For S_2, note $E(G_2) - E(G_1) = \{(i, 1)(k, \ell)\colon\ 2 \leqslant i, k \leqslant p, i \neq k, 3 \leqslant \ell \leqslant q\}$ contains exactly $(p-1)(p-2)(q-2)$ edges. By choice of T_2, each of its $p-1$ vertices of form $(i, 1)$ has degree 1 in T_2, so $|E(T_2) \cap [E(G_2) - E(G_1)]| = p - 1$. Hence $|S_2| = |T_2| + |E(G_2) - E(G_1)| - (p-1)$, so

$$
\begin{aligned}
|S_2| &= (p-1)(q-1) - 1 + (p-1)(p-2)(q-2)\\
&\quad - (p-1).
\end{aligned}
\tag{2}
$$

Now consider S_3. Note $E(G_3) - (E(G_1) \cup E(G_2)) = \{(i, 1)(k, 2)\colon\ 2 \leqslant i, k \leqslant p, i \neq k\}$ has cardinality $(p-1)(p-2)$. Recall that T_3 was chosen to contain none of these edges, so

$$|S_3| = (p-1)(q-1) - 1 + (p-1)(p-2). \tag{3}$$

Lastly, note that

$$|S_4| = |S_5| = \cdots = |S_q| = (p-1)(q-1) - 1. \tag{4}$$

Now form the following sets of triangles in $K_p \times K_q$.

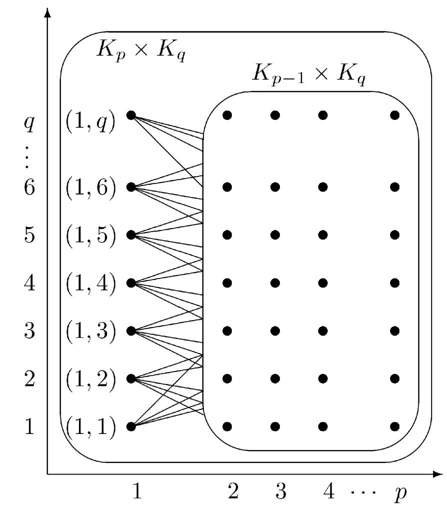

Fig. 3.

$$\begin{aligned}
\mathcal{B}_1 &= \big\{(1,1)(i,j)(k,\ell)(1,1) \mid (i,j)(k,\ell) \in E(S_1)\big\},\\
\mathcal{B}_2 &= \big\{(1,2)(i,j)(k,\ell)(1,2) \mid (i,j)(k,\ell) \in E(S_2)\big\},\\
\mathcal{B}_3 &= \big\{(1,3)(i,j)(k,\ell)(1,3) \mid (i,j)(k,\ell) \in E(S_3)\big\},\\
&\ \vdots\\
\mathcal{B}_q &= \big\{(1,q)(i,j)(k,\ell)(1,q) \mid (i,j)(k,\ell) \in E(S_q)\big\}.
\end{aligned}$$

Fig. 3 illustrates these sets. Any triangle in $\mathcal{B}_m$ is incident with vertex $(1,m)$ while its opposite edge is in S_m. Notice that the sets $\mathcal{B}_m$ are pairwise disjoint and $|\mathcal{B}_m| = |S_m|$ for $1 \leqslant m \leqslant q$. Moreover, observe that each $\mathcal{B}_m$ is linearly independent in $\mathcal{C}(K_p \times K_q)$: Each edge in S_m is on exactly one triangle in $\mathcal{B}_m$, and each triangle in $\mathcal{B}_m$ contains exactly one edge of S_m. Consequently no nonempty subset of $\mathcal{B}_m$ can sum to 0, since such a sum will always contain edges in S_m.

Next, we argue that $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_q$ is a linearly independent set by showing for each $2 \leqslant m \leqslant q$ that $\mathrm{Span}(\mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_{m-1}) \cap \mathrm{Span}(\mathcal{B}_m) = 0$. Suppose C is in this intersection. Since $C \in \mathrm{Span}(\mathcal{B}_m)$, any vertex of form $(1,j)$ incident with an edge of C must be of form $(1,m)$. But as no cycle in $\mathrm{Span}(\mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_{m-1})$ has an edge incident with such a vertex, no edge of C is incident with a vertex of form $(1,j)$. Then, from the definition of the sets $\mathcal{B}_j$, the fact that $C \in \mathrm{Span}(\mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_{m-1})$ means $C \subseteq S_1 \cup S_2 \cup \cdots \cup S_{m-1}$. The fact that $C \in \mathrm{Span}(\mathcal{B}_m)$ means $C \subseteq S_m$. Thus C is in the acyclic subgraph $(S_1 \cup S_2 \cup \cdots \cup S_{m-1}) \cap S_m$, so $C = 0$.

Finally, to show $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_q$ is a basis, we confirm $|\mathcal{B}| = \nu(K_p \times K_q)$. Using Eqs. (1)–(4), as well as the substitutions $r = p-1$ and $s = q-1$ (to streamline the calculation), note

$$\begin{aligned}
|\mathcal{B}| &= \sum_{m=1}^{q} |\mathcal{B}_m|\\
&= \sum_{m=1}^{q} |S_m| = |S_1| + |S_2| + |S_3| + \sum_{m=4}^{q} |S_m|\\
&= 2\binom{r}{2}\binom{s}{2} + \big[rs - 1 + r(r-1)(s-1) - r\big]\\
&\quad + \big[rs - 1 + r(r-1)\big] + (s-2)(rs-1)\\
&= 2\binom{r}{2}\binom{s}{2} + r(r-1)s + rs(s-1)\\
&\quad + 2rs - rs - s - r\\
&= 2\binom{r}{2}\binom{s}{2} + 2\binom{r}{2}s + 2r\binom{s}{2}\\
&\quad + 2rs - (r+1)(s+1) + 1\\
&= 2\left[\binom{r}{2} + r\right]\left[\binom{s}{2} + s\right]\\
&\quad - (r+1)(s+1) + 1\\
&= 2\left[\binom{r}{2} + \binom{r}{1}\right]\left[\binom{s}{2} + \binom{s}{1}\right]\\
&\quad - (r+1)(s+1) + 1\\
&= 2\binom{r+1}{2}\binom{s+1}{2} - (r+1)(s+1) + 1\\
&= 2\binom{p}{2}\binom{q}{2} - pq + 1\\
&= \nu(K_p \times K_q).
\end{aligned}$$

It follows that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_q$ is a basis of triangles for $\mathcal{C}(K_p \times K_q)$, and is therefore an MCB. This completes the construction, and proves the following proposition.

Proposition 2. *If $p, q \geqslant 4$, or if $p = 3$ and $q \geqslant 5$, then any MCB of $K_p \times K_q$ consists of $2\binom{p}{2}\binom{q}{2} - pq + 1$ triangles.*

We do not address the problem of finding an MCB for $K_2 \times K_q$. Indeed, to the author's knowledge, this problem still open, though it probably would not be difficult to resolve. However, since such products are bipartite, the approach would likely differ from the one used here. Moreover, it may be more fruitful to examine the more general problem of graphs of form $G \times H$ where G is bipartite, for, as [4] suggests, bipartiteness of a factor tends to simplify the cycle structure of the product.

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