Intro to modular arithmetic relevant to encryption & hashing

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Basic equations in modular arithmetic

All variables represent integers. The integers a and b are congruent modulo n.

$$a \equiv b \pmod{n}$$

where “mod” in parenthesis means that the modulus operation applies to all parts of the expression. This can also be expressed as

$$a = b + kn$$

That is, a is divisible k times by n, with remainder b.

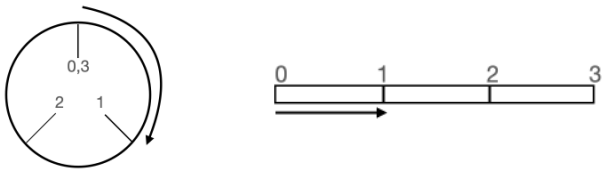
Those of us who use a 12 hour clock dial use modular arithmetic everyday. For example, the hour hand points to 6 on the dial at 6 am and 6 pm, where 6 pm is 18:00 in 24-hour time.

$$18 \equiv 6 \pmod{12}$$

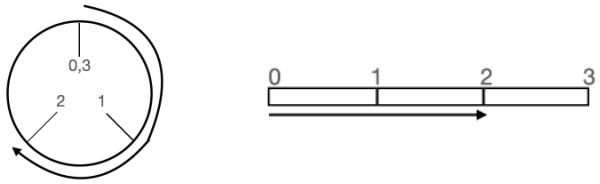
$$18 = 6 + 1(12)$$

Cryptography and hashing use very large integers. For illustration here, we use small integers. For some operations below, we will use a dial or linear scale with three divisions to simplify things. The relationships shown work for numbers of all magnitudes.

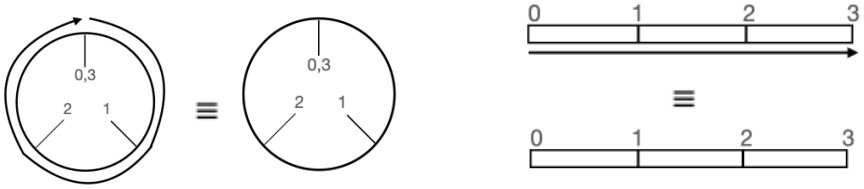
$1 \equiv 1 \pmod{3}$



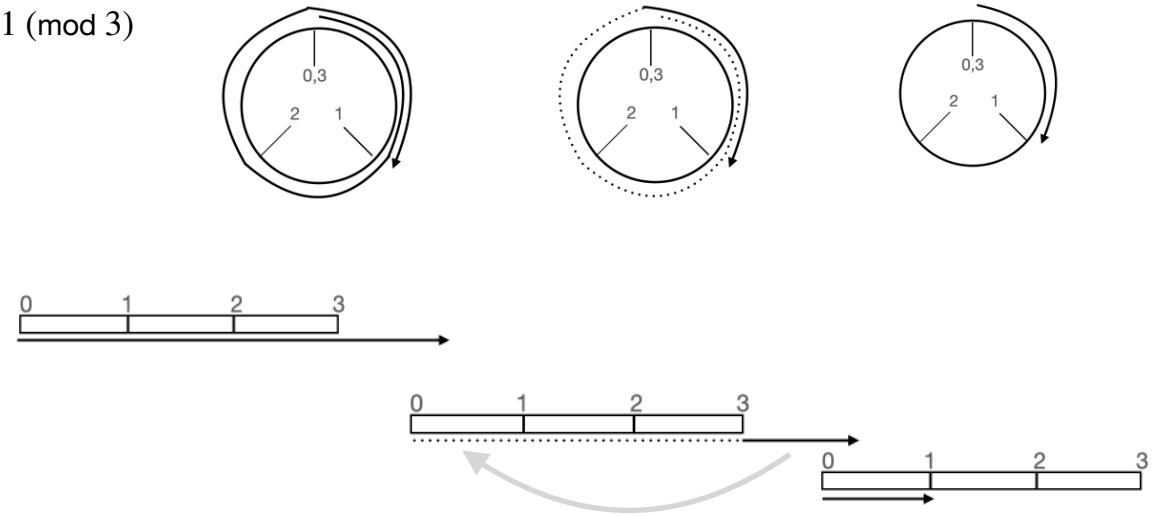
$2 \equiv 2 \pmod{3}$



$3 \equiv 0 \pmod{3}$



$4 \equiv 1 \pmod 3$



One way to think of this is that numbers greater than some multiple k of the modulus n "wraps around" so that the maximum value in the system is n. In the examples above k = 1 but k may have any integer value.

When we discuss hash algorithms, you will see that this "wrap around" feature is what keeps hash values a constant length, regardless of the length of the message being hashed.

One of the important relations in RSA cryptography is Fermat's Little Theorem.

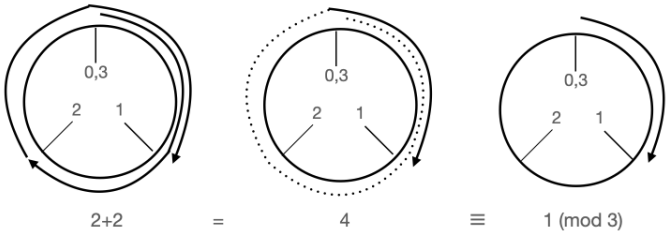
In the first relationship in the theorem, m is an integer and p is a prime number (prime integer)

$m^{(p-1)} \equiv 1 \pmod p$

As an example,

$2^{(3-1)} \equiv 1 \pmod 3$

$2^{(3-1)} = 2^2 = (2+2) \equiv 1 \pmod 3$

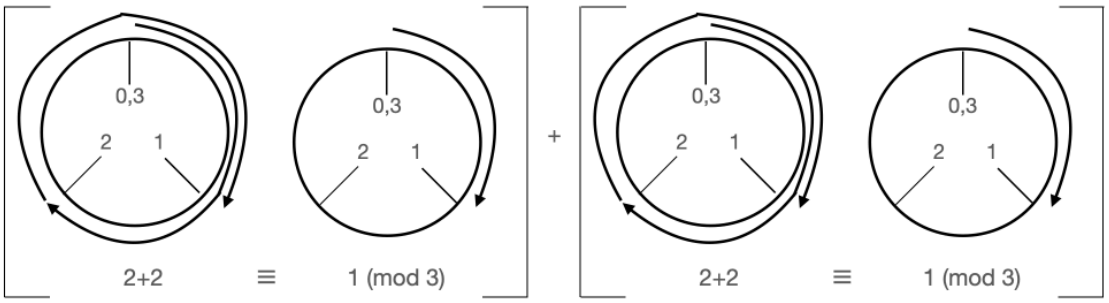


In the second relationship,

$m^p \equiv m \pmod{p}$

$2^3 \equiv 2 \pmod{3}$

$2^3 = 2(2^2) = (2+2)+(2+2) = 8 \equiv 2 \pmod{3}$



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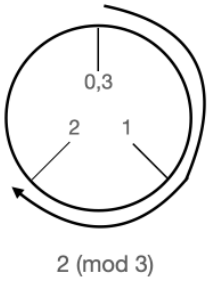

Fermat's Little Theorem is a special case of the Euler-Fermat Theorem.

In that theorem, if the following apply,

$m^{(p-1)} \equiv 1 \pmod{p}$ and $m^{(q-1)} \equiv 1 \pmod{q}$

where p and q are prime numbers,

then

$m^{(p-1)(q-1)} \equiv 1 \pmod{pq}$

This can be expressed as

$m^{\phi(n)} \equiv 1 \pmod{n}$

where $n = pq$

and ϕ is Euler's phi function

$\phi(n) = \phi(pq) = \phi(p)\phi(q) = (p-1)(q-1)$

RSA public-key encryption

The primes p and q are the basis numbers of the RSA public-key encryption method. Their choice results in the generation of the public and private keys. They are kept secret.

The public key consists of two integers, n and e. The integer e is chosen to be coprime with $n = pq$. That is, $e < n$ and e will not evenly divide n.

The private key consists of two integers, n and d. The integer d is the "modular inverse" of e, such that

$$ed \equiv 1 \pmod{\phi(n)}$$

This can also be expressed as

$$ed = 1 + k\phi(n)$$

A message can be encrypted by the public key but once encrypted, the encoded message cannot be decrypted by the public key in order to recover the original message.

The encoded message can only be decrypted by the private key.

A plain-text message can be converted to an integer by converting the message to a string of the ASCII or Unicode values of the characters in the message.

Message $m < n$ is encrypted to the encoded message c by this equation:

$$m^e \equiv c \pmod{n}$$

The holder of the private key can decrypt c in order to recover m by this equation:

$$c^d \equiv m \pmod{n}$$

This can be seen by the following development:

$$c^d = (m^e)^d \pmod{n}$$

$$= (m^e)^d = m^{(ed)} = m^{(1+k\phi(n))} = m(m^{k\phi(n)}) \pmod{n}$$

$$= m(m^{k\phi(n)}) = m(m^{\phi(n)})^k \pmod{n}$$

From the result in the section above, $m^{\phi(n)} \equiv 1 \pmod{n}$. Continuing,

$$= m(m^{\phi(n)})^k = m(1)^k \pmod{n}$$

$= m \pmod{n}$

$= m,$ since $m < n$

An example with small values: p = 3, q = 11, n = 33, $\phi(n)$ = 20, e = 3, d = 7, m = 4, c = 31.

In actual RSA encryption, the values are large such that the computation of m^e cannot be done directly on a computer before the modulo operation, as can be done with small values such as 2^3. This is because the value of m^e that would be obtained with the large values of m and e that are used would be so large that it could not be contained by a computer's method of storing numbers.

The solution of this problem is to compute the results using the modular exponentiation algorithm. In that algorithm, the largest value that must be stored during computation is m^2. This algorithm is shown below as a MATLAB function, where `key(1)` is n and `key(2)` is e for encryption and d for decryption, and $m \pmod{n}$, for example, is coded as `mod(m, n)`.

```
function r = fModExp(m,key)
    %
    % result r = m^key(2) mod key(1)
    % input m is one integer, output r is one integer
    % input key(1) and key(2) are each one integer
    % use modular exponentiation algorithm
    % since can't exponentiate directly with large numbers
    % here, only need to be able to square m and keep all significant figs

    % convert key(2) to binary char array
    % so we know which square terms in array p we need in modular exponentiation
    b = dec2bin(key(2));
    blen = length(b);

    % get results in array p of successive squares of m mod key(1)
    p(1) = m;
    for i = 2:blen
      p(i) = mod( p(i-1)^2 , key(1) );
    end

    % compute result using the p(i) required by key(2)
    % use only the powers-of-two of bit values = 1 in b = key(2)
    % increasing index in p is higher power-of-two
    % increasing index in b is lower power-of-two
    % flip order of b to match index of powers-of-two in p
    b = flip(b);

    % use only the powers-of-two of bit values in flipped b == '1'
    C = 1; % initialize product
    for i = 1:blen
      if (b(i) == '1')
         C = mod(C, key(1)) * mod(p(i), key(1));
         C = mod(C, key(1));
      end
    end
    r = C; % return r
end
```

Cryptographic Hashing

Hashing consists of processing messages of arbitrary length to a unique hash value of fixed length such that any change in the message results in a change in the hash value.

This result is valuable. You can verify that a message hasn't been tampered with by running the message through the hash algorithm and ensuring that the hash value hasn't changed from that supplied with the message. Even a tiny change in the message will result in a large and obvious change in the hash. In that case, you know that the message has been tampered with.

Hashing is used in blockchains in several ways: (1) create unique addresses for exchange of information (e.g., bitcoins), (2) create signatures of transactions for use in Merkle trees, which are incorporated in blocks in order to identify the transactions in a block of transactions, (3) to create a hash of an entire block for incorporation of that hash into the next block in the blockchain to prevent modification of previous blocks in the blockchain.

A block contains the hash of the previous block such that any change in any block changes the hashes of that and all following blocks.

Claude Shannon's information theory on cryptography states that the concepts of "confusion and diffusion" are key to hash functions.

"Confusion" is the result that the hash value has no apparent correspondence with the message. That is, by looking at the hash, you can't tell what was in the message.

"Diffusion" is that a change in any byte in the message will change most bytes in the hash at random locations. That is, a small change in the message produces large - and obvious - changes in the hash value.

Using modular arithmetic and "wrapping around" values at the end of the hash length produces a hash of constant length.

We wrote a program that executes the MD2 hash algorithm. This hash algorithm was published in 1989 by Ronald Rivest (the R in RSA encryption). The MD2 hash value length is 128 bits = 16 bytes. The most-used hash algorithm currently (2021) is the SHA-256 algorithm, which produces a 256-bit, 32-byte hash.

The MD2 hash is no longer secure because of increased computer power but, since it is relatively simple, it is useful in learning about one method in which hashes provide "confusion" and "diffusion."

In the MD2 hash, an array *S* of 256 bytes of random values is used.

“Confusion” is provided by the bitwise XOR (exclusive OR) of each message byte with a random byte in the *S*-array of random numbers to generate a byte in the hash. The apparent randomness of the hash actually contains information from the message. An XOR operation at one bit location returns 1 if either, but not both, of the input bits is 1.

“Diffusion” is provided by using the XOR result at one byte location of the message to get the next *S*-array value to use for the next byte in the message. This change propagates to the next byte, and so on. This results in a single change in the message affecting many bytes in the hash.

Padding a message (adding extra bytes to it) so it's a multiple of the number of bytes in the hash allows both short and long messages to be hashed to a fixed length result by mod (hash length).