1.

(a) Since $n = 40 > 30$, we can construct the 95% confidence interval as

$$\bar{x} \pm Z_{0.025} \frac{s}{\sqrt{n}}$$

$$\Rightarrow \left[14.3 - 1.96 \times \frac{4.2}{\sqrt{40}},\ 14.3 + 1.96 \times \frac{4.2}{\sqrt{40}}\right]$$

$$\Leftrightarrow [12.9984,\ 15.6016]$$

(b) Since $n = 10 < 30$, and we do not have other information about the distribution, we cannot find the confidence interval

(c) Since $n = 10 < 30$, and X_i are normal we can construct the 95% confidence interval as

$$\bar{x} \pm t_{0.025, n-1} \frac{s}{\sqrt{n}}$$

$$\Rightarrow \left[14.3 - 2.262 \times \frac{4.2}{\sqrt{10}},\ 14.3 + 2.262 \times \frac{4.2}{\sqrt{10}}\right]$$

$$\Leftrightarrow [11.2957,\ 17.3043]$$

2. (a) $M_1 = \sum_{i=1}^{n} \frac{x_i}{n} = E[x] = \frac{3}{4}\theta$

thus $\hat{\theta}_1 = \frac{4}{3}\bar{X}$

(b) $E[\hat{\theta}_1] = \frac{4}{3}E[\bar{x}]$

$= \frac{4}{3}E\left[\sum_{i=1}^{n}\frac{x_i}{n}\right]$

$= \frac{4}{3n}\cdot\sum_{i=1}^{n}E[x_i] = \frac{4}{3n}\cdot n\cdot\frac{3}{4}\theta = \theta$

(c) $E[\hat{\theta}_2] = E[X_{(n)}]$

$= \int_0^\theta x\cdot\frac{3nx^{3n-1}}{\theta^{3n}}\,dx = \int_0^\theta \frac{3nx^{3n}}{\theta^{3n}}\,dx$

$= \frac{3n}{(3n+1)\theta^{3n}}\cdot X^{3n+1}\Big|_0^\theta$

$= \frac{3n\theta}{3n+1} \neq \theta$

$\text{bias}(\hat{\theta}_2) = E[\hat{\theta}_2] - \theta = -\frac{\theta}{3n+1}$

(d) for $\hat{\theta}_1$, $MSE(\hat{\theta}_1) = Var(\hat{\theta}_1) = Var\left(\frac{4}{3n}\sum_{i=1}^{n}\frac{x_i}{n}\right)$

$= \frac{16}{9n^2}Var\left(\sum_{i=1}^{n}x_i\right)$

$= \frac{16}{9n^2}\sum_{i=1}^{n}Var(x_i)$

$= \frac{16}{9n^2}\times\frac{3n}{80}\theta^2 = \frac{\theta^2}{15n}$

for $\hat{\theta}_2$, $MSE(\hat{\theta}_2) = Var(X_{(n)}) + bias(\hat{\theta}_2)^2$

$$= \frac{\theta^2}{(3n+1)^2} + \frac{3n\theta^2}{(3n+2)(3n+1)^2}$$

$$= \frac{(6n+2)\theta^2}{(3n+1)^2(3n+2)} = \frac{2\theta^2}{(3n+1)(3n+2)}$$

then $\frac{MSE(\hat{\theta}_2)}{MSE(\hat{\theta}_1)} = \frac{30n}{(3n+1)(3n+2)} = \frac{30}{(3+\frac{1}{n})(3n+2)}$

$$= \frac{30}{(9n+\frac{2}{n}+9)}$$

where $\frac{\partial(9n+\frac{2}{n}+9)}{\partial n} > 0$ for $\forall$ $n \geq 3$

thus $\frac{30}{9n+\frac{2}{n}+9} \leq \frac{30}{9\times 3+\frac{2}{3}+9} < 1$

therefore, for $n \geq 3$, $MSE(\hat{\theta}_2) < MSE(\hat{\theta}_1)$

3. for $X \sim Ga(\alpha, \beta)$ $E[X] = \frac{\alpha}{\beta}$ $E[X^2] = Var[X] + E^2[X]$

$$= \frac{\alpha}{\beta^2} + \frac{\alpha^2}{\beta^2}$$

$$= \frac{\alpha^2+\alpha}{\beta^2}$$

thus $\frac{\hat{\alpha}}{\hat{\beta}} = \hat{\mu}_1 = \bar{X}$

$$\frac{\hat{\alpha}^2+\hat{\alpha}}{\hat{\beta}^2} = \hat{\mu}_2 = \sum_{i=1}^{n} \frac{X_i^2}{n}$$

$$\Rightarrow \quad \hat{\alpha} = \frac{1}{\frac{\hat{\mu}_2}{\hat{\mu}_1^2} - 1} = \frac{\hat{\mu}_1^2}{\hat{\mu}_2 - \hat{\mu}_1^2} = \frac{\bar{X}^2}{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

$$\text{and } \hat{\beta} = \frac{\hat{\alpha}}{\bar{X}} = \frac{\bar{X}}{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

4.