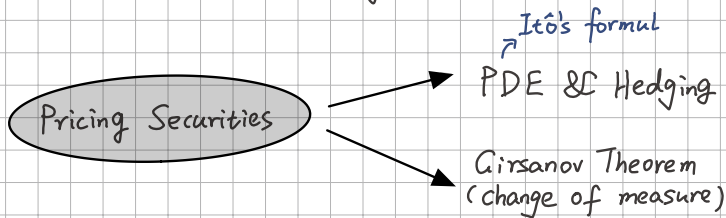


§ Risk Neutral Pricing



Change of measure & Girsanov's Theorem.

Def 7.1.1 (The Girsanov Theorem)

Two probability measures \mathbb{P} and $\tilde{\mathbb{P}}$ are said to be equivalent if for $\forall A \in \mathcal{F}$, we have $\mathbb{P}(A) = 0 \Leftrightarrow \tilde{\mathbb{P}}(A) = 0$

for example,

Let Z be a random variable, $Z > 0$ and $E[Z] = 1$. Define

$$\tilde{\mathbb{P}}(A) = E[Z 1_A] = \int_A Z d\mathbb{P} \quad \forall A \in \mathcal{F} \quad (*)$$

Then $\tilde{\mathbb{P}}$ is a probability measure, equivalent to \mathbb{P} .

Remark:

The assumption $E[Z] = 1$ is required to guarantee $\tilde{\mathbb{P}}(\Omega) = 1$

Def 7.1.2. If $\tilde{\mathbb{P}}$ is defined as $(*)$, then we write

$$d\tilde{\mathbb{P}} = Z d\mathbb{P} \quad , \quad Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \quad \text{for } (*)$$

and Z is called the density of $\tilde{\mathbb{P}}$ w.r.t. \mathbb{P} .

Example: $\Omega = \{\omega_1, \omega_2\}$ $\mathbb{P}(\{\omega_1\}) = \frac{1}{2} = \mathbb{P}(\{\omega_2\})$

$$Z(\omega_1) = \frac{2}{3} \quad Z(\omega_2) = \frac{4}{3} \Rightarrow \tilde{\mathbb{P}} = ?$$

$$\tilde{\mathbb{P}}(\{\omega_1\}) = E[1_{\{\omega_1\}} Z]$$

$$= Z(w_1) \cdot \underbrace{1_{\{w_1\}}}_{=1} P(\{w_1\}) + Z(w_2) \cdot \underbrace{1_{\{w_2\}}}_{=0} P(\{w_2\})$$

$$= Z(w_1) \cdot P(\{w_1\}) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$$

$$\tilde{P}(\{w_2\}) = E[1_{\{w_2\}} Z]$$

$$= Z(w_2) \cdot P(\{w_2\}) = \frac{4}{3} \times \frac{1}{2} = \frac{2}{3}$$

Proof of lemma (\tilde{P} is a prob measure, and is equivalent to P)

① \tilde{P} is a probability measure $\tilde{P}(\Omega) = 1$

$$\tilde{P}(\Omega) = E[Z 1_{\Omega}] = E[Z] = 1$$

Let $A_1, \dots, A_N \in \mathcal{F}$ disjoint. Then

$$\tilde{P}(A_1 \cup \dots \cup A_N) = \tilde{P}(A_1) + \dots + \tilde{P}(A_N)$$

$$\begin{aligned} &\parallel \\ E[Z 1_{A_1 \cup \dots \cup A_N}] &= E[Z 1_{A_1} + \dots + Z 1_{A_N}] \\ &= 1_{A_1} + 1_{A_2} + \dots + 1_{A_N} \text{ (disjoint)} \\ &= E[Z 1_{A_1}] + \dots + E[Z 1_{A_N}] \\ &= \tilde{P}(A_1) + \dots + \tilde{P}(A_N) \end{aligned}$$

$$\tilde{P}(A) \in [0, 1]$$

$$\parallel$$

$$E[Z 1_A] \geq 0 \quad E[Z 1_A] \leq E[Z] = 1$$

$\geq 0 \quad \geq 0 \quad \leq 1$

② Fix $A \in \mathcal{F}$

assume $P(A) = 0$ since $P(A) = E[1_A] = 0$, thus $1_A = 0$

$\Rightarrow E[Z \cdot 1_A] = 0$, that is $\tilde{P}(A) = 0$

assume $\tilde{P}(A) = 0 \Rightarrow Z \cdot 1_A = 0 \Rightarrow 1_A = 0 \Rightarrow E[1_A] = P(A) = 0$

\parallel

$$E[Z \cdot 1_A] = 0$$

≥ 0

Special choice of Z

Suppose $T > 0$ is fixed, and Z is a MG such that $Z_T > 0$ and $E[Z_T] = 1$

Define a new measure $\tilde{\mathbb{P}}$ via $d\tilde{\mathbb{P}} = Z_T d\mathbb{P}$. We will denote expectation and conditional expectations w.r.t. $\tilde{\mathbb{P}}$ by $\tilde{\mathbb{E}}$.

i.e. $\tilde{\mathbb{E}}[X]$, $\tilde{\mathbb{E}}[X|G]$

In particular, \forall r.v. X . $\tilde{\mathbb{E}}[X] = E[Z_T X] = \int Z_T X d\mathbb{P}$

Theorem 7.1.3. (Gameron, Martin, Girsanov)

Let $(b_t)_{t \geq 0}$ be an adapted process, W_t be a SBM. and define

$$\tilde{W}_t := W_t + \int_0^t b_s ds$$

Let Z be the process

for this chapter, Z is of this form, and is a MG.

$$Z_t := \exp\left(-\int_0^t b_s ds - \frac{1}{2} \int_0^t b_s^2 ds\right)$$

and define a new measure $d\tilde{\mathbb{P}} = Z_T d\mathbb{P}$.

(under certain conditions)

In our setting, Z is a MG, and \tilde{W} is a BM under $\tilde{\mathbb{P}}$.

Note that $Z_0 = 1$ and $E[Z_T] = 1$

proof: denote $M_t := \int_0^t b_s dW_s$ i.e. $dM_t = b_t dW_t$

$$f(t, x) = \exp\left(-x - \frac{1}{2} \int_0^t b_s^2 ds\right)$$

So, $Z_t = f(t, M_t)$

$$\Rightarrow \partial_t f(t, x) = f(t, x) \left(-\frac{1}{2} b_t^2\right)$$

$$\partial_x f(t, x) = f(t, x) (-1)$$

$$\partial_x^2 f(t, x) = f(t, x), \quad [M, M]_t = \int_0^t b_s^2 ds$$

Itô's formula leads to

$$\begin{aligned} dz_t &= df(t, M_t) = \partial_t f dt + \partial_x f dM_t + \frac{1}{2} \partial_x^2 f d[M, M]_t \\ &= Z_t \left(-\frac{1}{2} b_t^2 dt - b_t dW_t + \frac{1}{2} b_t^2 dt \right) \\ &= -Z_t b_t dW_t \end{aligned}$$

$\Rightarrow Z_t$ is a MG.

Remark:

$$\tilde{\mathbb{E}}[X] = \mathbb{E}[XZ], \text{ however } \tilde{\mathbb{E}}[X|G] \neq \mathbb{E}[XZ|G]$$

Lemma 7.1.7: (Bayes Theorem)

Let X be a r.v. and $d\tilde{P} = Z_T dP$. Let G be a σ -algebra, $G \subseteq \mathcal{F}$

Then,

$$\tilde{\mathbb{E}}[X|G] = \frac{\mathbb{E}[Z_T X | G]}{\mathbb{E}[Z_T | G]}$$

In particular, if $G = \mathcal{F}_s$ and Z is defined as above. Let $0 \leq s \leq t \leq T$. If X is a \mathcal{F}_t -measurable r.v. then

$$\tilde{\mathbb{E}}[X | \mathcal{F}_s] = \frac{1}{Z_s} \mathbb{E}[Z_t X | \mathcal{F}_s]$$

Lemma 7.1.2. An adapted process M is a martingale under \tilde{P} if and only if MZ is a martingale under P .

Risk Neutral Pricing

Recall $dS_t = \alpha_t S_t dt + \beta_t S_t dW_t$ (stock price modeled by a generalized GBM).

$$dB_t = B_t R_t dt, \text{ i.e. } B_t = B_0 \exp\left(\int_0^t R_s ds\right)$$

$$B_0 = 1$$

$\rightarrow B_t D_t = B_0$ is always a M.G.

Define $D_t = \exp(-\int_0^t R_s ds)$ the discount process.

Def 7.2.1. (Risk-neutral measure). A risk-neutral measure \tilde{P}

is a probability measure satisfying

$$\triangleright \forall A \in \mathcal{F}, P(A) = 0 \Leftrightarrow \tilde{P}(A) = 0$$

\triangleright under \tilde{P} , $D_t S_t$ is a MG.

Here, we try to find \tilde{P} using Girsanov's theorem.

$$d(S_t D_t) = S_t dD_t + D_t dS_t + \underbrace{d[S, D]_t}_{\rightarrow 0, \text{ because } D \text{ has finite variation}}$$

$$= -R_t S_t D_t dt + D_t d(\alpha S_t dt + \sigma S_t dW_t)$$

$$= (\alpha_t - R_t) D_t S_t dt + D_t S_t \sigma_t dW_t$$

$$= \underbrace{\frac{\alpha_t - R_t}{\sigma_t}}_{\theta_t} \cdot \sigma_t D_t S_t dt + D_t S_t \sigma_t dW_t$$

$$= \sigma_t D_t S_t (\theta_t dt + dW_t)$$

where $\theta_t = \frac{\alpha_t - R_t}{\sigma_t}$ is the market price of risk.

Define a new process, $d\tilde{W}_t = dW_t + \theta_t dt$, and observe

$$d(S_t D_t) = \sigma_t D_t S_t d\tilde{W}_t \quad (*)$$

and \tilde{W}_t is a BM under \tilde{P} by Girsanov's theorem.

Prop. 7.2.1. Set

$$Z_t = \exp\left(-\int_0^t \theta_s dW_s - \frac{1}{2} \int_0^t \theta_s^2 ds\right)$$

Then $d\tilde{P} = Z_t dP$ is a risk-neutral measure.

Proof: By Girsanov theorem, \tilde{W} is a BM under \tilde{P} , and thus by

(*) , $D_t S_t$ is a MG. under \tilde{P} .

Theorem 7.2.2. (Risk-neutral pricing formula)

Let V_t be a \mathcal{F}_t -measurable r.v. Let \tilde{P} be our risk-neutral

measure from prop 7.2.1. Then the arbitrage free price of the option with payoff V_T at $T > 0$ is given by

$$V_t = \tilde{\mathbb{E}} \left[\exp \left(- \int_t^T R_s ds \right) \cdot V_T \mid \mathcal{F}_t \right]$$

Proof:

Note that under $\tilde{\mathbb{P}}$, $D_t V_t$ is a M.G. (prop 7.2.1)

$$D_t V_t = \tilde{\mathbb{E}} [D_T V_T \mid \mathcal{F}_t]$$

$$\begin{aligned} \Rightarrow V_t &= \frac{1}{D_t} \tilde{\mathbb{E}} [D_T V_T \mid \mathcal{F}_t] \\ &= \tilde{\mathbb{E}} \left[\frac{D_T}{D_t} V_T \mid \mathcal{F}_t \right] \\ &= \tilde{\mathbb{E}} \left[\exp \left(- \int_0^T R_s ds + \int_0^t R_s ds \right) \cdot V_T \mid \mathcal{F}_t \right] \\ &= \tilde{\mathbb{E}} \left[\exp \left(- \int_t^T R_s ds \right) V_T \mid \mathcal{F}_t \right] \end{aligned}$$

Dynamics of S_t under $\tilde{\mathbb{P}}$:

$$\begin{aligned} dS_t &= \alpha_t S_t dt + \sigma_t S_t dW_t \\ &= (\alpha_t S_t - \underbrace{\sigma_t S_t \theta_t}_{S_t(\alpha_t - R_t)}) dt + \sigma_t S_t \underbrace{d\tilde{W}_t}_{= dW_t + \theta_t dt} \\ &= (\alpha_t S_t - \alpha_t S_t + R_t S_t) dt + \sigma_t S_t d\tilde{W}_t \\ &= R_t S_t dt + \sigma_t S_t d\tilde{W}_t \quad (***) \end{aligned}$$

Since under $\tilde{\mathbb{P}}$, \tilde{W} is a BM, so S is a GBM with drift R_t

\Rightarrow this is the reason why BS-call price does not include σ .

Lemma 7.2.1. Let Δ be an adapted process and let X_t be the wealth process of self-financing portfolio that holds Δ_t shares of stock. Then $D_t X_t$ is a M.G. under $\tilde{\mathbb{P}}$.

proof: from the self-financing condition: (6.1.7)

$$dX_t = \Delta_t dS_t + R_t(X_t - \Delta_t S_t)dt$$

using $(\ast \ast)$,

→ continuous version.

for the discrete version, it's naive

$$dX_t = \Delta_t G_t S_t d\tilde{W}_t + R_t X_t dt$$

that $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$

Thus, by product rule,

$$\begin{aligned} d(D_t X_t) &= X_t dD_t + D_t dX_t + d[D, X]_t \\ &= -R_t X_t D_t dt + D_t \Delta_t G_t S_t d\tilde{W}_t \\ &\quad + D_t R_t X_t dt + 0 \\ &= D_t \Delta_t G_t S_t d\tilde{W}_t \end{aligned}$$

$\Rightarrow D_t X_t$ is a MG under $\tilde{\mathbb{P}}$.

proof of theorem 7.2.2.

Suppose X_t is the wealth process of the replicating portfolio at time t . Then by definition $V_t = X_t$ and by lemma 7.2.1.

$D_t X_t$ is a MG under $\tilde{\mathbb{P}}$, thus

$$\begin{aligned} V_t = X_t &= \frac{1}{D_t} D_t X_t = \frac{1}{D_t} \tilde{\mathbb{E}}[D_T X_T | \mathcal{F}_t] \\ &= \tilde{\mathbb{E}}\left[\frac{D_T}{D_t} X_T | \mathcal{F}_t\right] \\ &= \tilde{\mathbb{E}}\left[\frac{D_T}{D_t} V_T | \mathcal{F}_t\right] \end{aligned}$$

7.3. Black-Scholes Formula

Dynamics under $\tilde{\mathbb{P}}$

$$dS_t = S_t(r dt + \sigma d\tilde{W}_t) \quad \sigma, r \text{ fixed and } > 0$$

$$dB_t = rB_t dt \quad \tilde{W}_t \text{ is a BM under } \tilde{\mathbb{P}}$$

$$\begin{aligned} \Rightarrow S_T &= S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma \tilde{W}_T} \\ &= S_0 e^{(r - \frac{\sigma^2}{2})t + \sigma \tilde{W}_t} \cdot e^{(r - \frac{\sigma^2}{2})(T-t) + \sigma(\tilde{W}_T - \tilde{W}_t)} \\ &= \underbrace{S_t}_{\mathcal{F}_t\text{-measurable}} \cdot e^{\underbrace{(r - \frac{\sigma^2}{2})(T-t) + \sigma(\tilde{W}_T - \tilde{W}_t)}_{\text{independent of } \mathcal{F}_t}} \end{aligned}$$

Arbitrage-free price of European call using the Risk-neutral pricing formula.

$$\begin{aligned} V_t &= \tilde{\mathbb{E}}[(S_T - K)^+ e^{-r(T-t)} | \mathcal{F}_t] \\ &= e^{-r(T-t)} \tilde{\mathbb{E}}[(S_T - K)^+ | \mathcal{F}_t] \\ &= e^{-r(T-t)} \tilde{\mathbb{E}}[S_t \cdot e^{(r - \frac{\sigma^2}{2})(T-t) + \sigma(\tilde{W}_T - \tilde{W}_t)} - K)^+ | \mathcal{F}_t] \end{aligned}$$

independence lemma \rightarrow

$$= e^{-r(T-t)} \int_{-\infty}^{+\infty} \underbrace{S_t e^{(r - \frac{\sigma^2}{2})(T-t) + \sigma \sqrt{T-t} \cdot y}}_{\substack{\text{pdf of standard} \\ \text{normal}}} - K)^+ \cdot \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}}_{\text{pdf of standard normal}} dy$$

Set $S_t = x$ and define

$$d_{\pm}(t, x) = \frac{1}{\sigma \sqrt{t}} \left(\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)t \right)$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy = \frac{1}{\sqrt{2\pi}} \int_{-x}^{\infty} e^{-\frac{y^2}{2}} dy$$

therefore

$$C(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-d_-}^{\infty} x \cdot \exp\left(-\frac{\sigma^2}{2}t + \sigma \sqrt{t} \cdot y - \frac{y^2}{2}\right) dy - e^{-rt} K \Phi(d_-)$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \int_{-d_-}^{\infty} x \cdot \exp\left(-\frac{(y - \frac{\sigma\sqrt{\tau}}{2})^2}{2}\right) dy - e^{-r\tau} K \Phi(d_-) \\
&= x \Phi(d_+) - e^{-r\tau} K \Phi(d_-)
\end{aligned}$$

⇒ the arbitrage-free price of European call is

$$V_t = S_t \Phi(d_+) - e^{-r(\tau-t)} K \Phi(d_-)$$

Similarly,

The risk neutral pricing formula says that the price of a European put is

$$P(t, S_t) = \tilde{\mathbb{E}}[e^{-r(\tau-t)} (K - S_\tau)^+ | \mathcal{F}_t]$$

Set $\tau = T - t$

$$\Rightarrow P(t, S_t) = \tilde{\mathbb{E}}[e^{-r\tau} (K - S_\tau)^+ | \mathcal{F}_t]$$

In the B-S market: $S_t = S_0 \exp\left((r - \frac{\sigma^2}{2})t + \sigma \tilde{W}_t\right)$

where \tilde{W} is a BM under $\tilde{\mathbb{P}}$, then

$$\begin{aligned}
P(t, S_t) &= e^{-r\tau} \tilde{\mathbb{E}}[(K - S_0 \exp((r - \frac{\sigma^2}{2})T + \sigma \tilde{W}_T))^+ | \mathcal{F}_t] \\
&= e^{-r\tau} \tilde{\mathbb{E}}[(K - S_t \exp((r - \frac{\sigma^2}{2})\tau + \sigma(\tilde{W}_T - \tilde{W}_t))^+ | \mathcal{F}_t] \\
&= \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} (K - S_t \exp((r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau} \cdot y))^+ \cdot e^{-\frac{y^2}{2}} dy
\end{aligned}$$

Set $S_t = x$, define

$$d_{\pm}(\tau, x) := \frac{1}{\sigma\sqrt{\tau}} \left(\ln\left(\frac{x}{K}\right) + (r \pm \frac{\sigma^2}{2})\tau \right), \text{ and}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy = \frac{1}{\sqrt{2\pi}} \int_{-x}^{\infty} e^{-\frac{y^2}{2}} dy$$

then

$$\begin{aligned}
p(t, x) &= \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} (K - x \cdot \exp((r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau} \cdot y))^+ e^{-\frac{y^2}{2}} dy \\
&= \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{-\infty}^{-d_-} (K - x \cdot \exp((r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau} \cdot y)) \cdot e^{-\frac{y^2}{2}} dy
\end{aligned}$$

$$\begin{aligned}
&= K \cdot e^{-rT} \Phi(-d_-) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_-} x \cdot \exp\left(-\frac{6^2}{2}T + 6\sqrt{T} \cdot y - \frac{y^2}{2}\right) dy \\
&= K \cdot e^{-rT} \Phi(-d_-) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_-} x \cdot \exp\left(-\frac{(y - 6\sqrt{T})^2}{2}\right) dy \\
&= K \cdot e^{-rT} \Phi(-d_-) - \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{-d_+} \exp\left(-\frac{y'^2}{2}\right) dy' \\
&= K \cdot e^{-rT} \Phi(-d_-) - x \cdot \Phi(-d_+)
\end{aligned}$$

Remark:

BS option prices depend on

- time to maturity: $\tau = T - t$
- strike price: K
- (constant) interest rate: r
- price of the underlying asset: S_t
- (constant) volatility: σ

Hedging a short call

Suppose we sell a call with value $c(t, x)$. We want to build a replicating portfolio

\Rightarrow invest $\partial_x c$ into the asset S_t and put the rest into money market account B .

$$\begin{aligned}
&c(t, x) - \partial_x c \\
&= x \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-) - x \Phi(d_+) \\
&= -K e^{-r(T-t)} \Phi(d_-) \leq 0 \rightarrow \text{to hedge the call, we will have to borrow money.}
\end{aligned}$$

For $t \rightarrow T$

$$\begin{aligned}d_+ &= \frac{1}{\sigma\sqrt{T-t}} \left(\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t) \right) \\&= \frac{1}{\sigma} \left(\underbrace{\frac{1}{\sqrt{T-t}} \ln\left(\frac{S_t}{K}\right)}_{\begin{cases} > 0 \text{ if } S_t > K \\ \leq 0 \text{ if } S_t \leq K \end{cases}} + \underbrace{\left(r + \frac{\sigma^2}{2}\right) \cdot \sqrt{T-t}}_{\rightarrow 0 \text{ as } t \rightarrow T} \right)\end{aligned}$$

$$= \begin{cases} +\infty & \text{if } S_t > K \\ -\infty & \text{if } S_t \leq K \end{cases}$$

$$\text{thus } \Phi(d_+) = \begin{cases} 1 & \text{if } S_t > K \\ 0 & \text{if } S_t \leq K \end{cases}$$

that is, if $S_t > K$, we will invest S_t when $t \rightarrow T$. if $S_t \leq K$, we will not invest in S_t when $t \rightarrow T$.

An example of arbitrage (?)

Assume the stock price is X_0 & we short $\partial_x C(t, X_0)$ shares of stock & buy a call valued $C(t, X_0)$, we invest

$$M = X_0 \partial_x C(t, X_0) - C(t, X_0)$$

What happens if Stock price changes from X_0 to x ?

The portfolio value is

$$C(t, x) - \partial_x C(t, X_0) \cdot x + M$$

$$= C(t, x) - \partial_x C(t, X_0) \cdot x + X_0 \partial_x C(t, X_0) - C(t, X_0)$$

$$= \underbrace{C(t, x)} - \underbrace{C(t, X_0)} - \underbrace{\partial_x C(t, X_0)} (x - X_0) \geq 0$$

this inequality is wrong because we pretend t to be constant