§. Stochastic Integration

4.2 Quadratic Variation

Def. 4.2.1. Let M be a stochastic process. Then

$$[M,M]_T(\omega) := \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} (\Delta M_i(\omega))^2$$

where $\Delta M_i := M_{t_{i+1}} - M_{t_i}$ and $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$

for Brownian motion W, $[W,W]_t = t$

lemma. 4.2.1 $M_t = W_t^2 - t$ is a martingale

proof: • Adapted: M_t is $\mathcal{F}_t$-measurable because W_t is $\mathcal{F}_t$-measurable and M_t is a function of W_t and t.

• Martingale property: check $E[M_t \mid \mathcal{F}_s] = M_s \ \forall t \geq s$

$$\begin{aligned}
E[M_t \mid \mathcal{F}_s] &= E[W_t^2 - t \mid \mathcal{F}_s] \\
&= E[(W_t - W_s + W_s)^2 - t \mid \mathcal{F}_s] \\
&= E[(W_t - W_s)^2 + 2(W_t - W_s)W_s + W_s^2 - t \mid \mathcal{F}_s] \\
&= E[\underbrace{(W_t - W_s)^2}_{\text{ind of } \mathcal{F}_s} \mid \mathcal{F}_s] + E[2\underbrace{(W_t - W_s)}_{\text{ind of } \mathcal{F}_s} \cdot \underbrace{W_s}_{\mathcal{F}_s\text{-measurable}} \mid \mathcal{F}_s] \\
&\quad + E[\underbrace{W_s^2}_{\mathcal{F}_s\text{-measurable}} \mid \mathcal{F}_s] - E[\underbrace{t}_{\mathcal{F}_s\text{-measurable}} \mid \mathcal{F}_s] \\
&= E[(W_t - W_s)^2] + 2W_s E[W_t - W_s] \\
&\quad + W_s^2 - t \\
&= t - s + W_s^2 - t \\
&= W_s^2 - s \\
&= M_s
\end{aligned}$$

Theorem 4.2.1 Let M be a martingale w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$

Then for all $t \geq 0$, $E[M_t^2] < \infty \Leftrightarrow E[[M,M]_t] < \infty$

In this case, $(M_t^2 - [M,M]_t)_{t\geq 0}$ is a martingale w.r.t. the same filtration, and. $E[M_t^2] - E[M_0^2] = E[[M,M]_t]$

Example: $M = W$ is a martingale $\Rightarrow (W_t^2 - t)_{t\geq 0}$ is a martingale

Theorem 4.2.2. If M is a martingale and A_t is a continuous adapted increasing stochastic process s.t. $A_0 = 0$ and $(M_t^2 - A_t)_{t\geq 0}$ is a martingale, then

$$A_t = [M,M]_t \quad \text{(proof is omitted)}$$

Example: $M = W$, $A_t = t$

A_t is continuous, adapted, $\mathcal{F}_0$-measurable, increasing, and $A_0 = 0$

$$M_t^2 - A_t = W_t^2 - t \text{ is a martingale} \Rightarrow A_t = [W,W]_t$$

Remark:

- intuition for first variation and quadratic variation:
  divide the interval $[0,T]$ into $T/(\delta t)$ intervals of size δt.
  if X has finite first variation, then on each subinterval $(k\delta t, (k+1)\delta t)$ the increment of X should be of order δt.
  Similarly. X has finite quadratic variation $\Rightarrow$ increment $\propto \sqrt{\delta t}$
- If a continuous process has finite first variation, its quadratic variation will necessarily be zero.
  If a continuous process has finite and non-zero quadratic variation, its first variation will necessarily be infinite

4.3. Construction of Itô integral.

Let W be a standard BM, $(\mathcal{F}_t)_{t\geq 0}$ be the Brownian filtration and D be an adapted process.

Let $(\Delta_t)_{t\geq 0}$ be our position in S at time t, that is invest $\Delta_t S_t$ at time t and the value of portfolio time $t+1$ is $\Delta_t \cdot S_{t+1}$

however, almost any continuous martingale S will not have finite first variation, thus we need the Itô integral

$$\Rightarrow \text{PnL} = \sum_{i=0}^{n-1} \Delta_i (S_{i+1} - S_i) \xrightarrow{n\to\infty} \int \Delta_t \cdot dS_t$$

lemma 4.3.1 let $\Pi = \{0 = t_0 < t_1 < \cdots < t_n\}$ be an increasing sequence of times and assume that D is constant on $[t_i, t_{i+1})$ $\forall i$ (i.e. the asset is only traded at time $t_0, \cdots t_n$)

let
$$I_T^\Pi = \sum_{i=0}^{n-1} D_{t_i} \Delta W_i + D_{t_n}(W_T - W_{t_n}) \quad \text{if } T \in [t_n, t_{n+1})$$

where $\Delta W_i = W_{t_{i+1}} - W_{t_i}$

denote the cumulative earnings up to time T, then

$$E[(I_T^\Pi)^2] = E\Big[\sum_{i=0}^{n-1} D_{t_i}^2 (t_{i+1} - t_i) + D_{t_n}^2 (T - t_n)\Big] \text{ if } T \in [t_n, t_{n+1})$$

Moreover, I^Π is a martingale and.

$$[I^\Pi, I^\Pi]_T = \sum_{i=0}^{n-1} D_{t_i}^2 (t_{i+1} - t_i) + D_{t_n}^2 (T - t_n) \text{ if } T \in [t_n, t_{n+1})$$

Theorem 4.3.1 If $\int_0^T D_t^2 dt < \infty$, then as $\|\Pi\| \to 0$, the process I^Π converge to a cost process I given by

$$I_T := \lim_{\|\Pi\|\to 0} I_T^\Pi = \int_0^T D_t \, dW_t$$

Note that D should be adapted, and is sampled at the left endpoint of the time interval, i.e. terms in the sum are $D_{t_i}(W_{t_{i+1}} - W_{t_i})$

This is called the Itô integral of D w.r.t. W.

If further, $E[\int_0^T D_t^2 dt] < \infty$, then the process I is a martingale and the quadratic variation $[I, I]$ satisfies

$$[I, I]_T = \int_0^T D_t^2 dt \text{ almost surely.}$$

Properity. 4.3.1 (linearity)

If D^1, D^2 are two adapted process, $\lambda \in \mathbb{R}$, then

$$\int_0^T (D_t^1 + \lambda D_t^2) dW_t = \int_0^T D_t^1 dW_t + \lambda \int_0^T D_t^2 dW_t$$

(Itô Isometry)

If $E[\int_0^T D_t^2 dt] < \infty$, then

$$E[(\int_0^T D_t dW_t)^2] = E[\int_0^T D_t^2 dt]$$

Example: $D_t = 1$, then

$$E[(\underbrace{\int_0^T D_t dW_t}_{W_T - W_0})^2] = E[\underbrace{\int_0^T 1^2 dt}_{T}]$$

$$\Rightarrow E[W_T^2] = T$$

Remark:

positivity is not preserved by Itô integrals. Namely, if $D^1 \le D^2$, there is no reason to expect $\int_0^T D_t^1 dW_t \le \int_0^T D_t^2 dW_t$.

Def (GBM). we define Geometric Brownian Motion S as

$$dS_t = \sigma S_t dW_t + \lambda S_t dt \quad , \sigma, \lambda \in \mathbb{R}$$

$\Rightarrow$ stochastic differential equation (SDE)

$$\int_0^T 1 \, dS_t = \int_0^T \sigma S_t dW_t + \int_0^T \lambda S_t dt$$

$$\Rightarrow S_T - S_0 = \int_0^T \sigma S_t dW_t + \int_0^T \lambda S_t dt$$

4.4. Itô formula

Goal: Compute $\int_0^T W_t \, dW_t = ?$

Def 4.4.1 Let b, σ be adapted process. Then a process X defined as

Riemann integral   Itô integral

$$X_T = X_0 + \int_0^T b_t \, dt + \int_0^T \sigma_t \, dW_t \qquad X_0 \in \mathbb{R}$$

is called an Itô process if X_0 is deterministic (not random) and for all $T \geq 0$ $E\left[\int_0^T \sigma_t^2 \, dt\right] < \infty$ and $\int_0^T |b_t| \, dt < \infty$

Remark:

the equation above is equivalent to

$$dX_t = b_t \, dt + \sigma_t \, dW_t$$

properity 4.4.1 The quadratic variation of X is

$$[X, X]_T = \int_0^T \sigma_t^2 \, dt$$

Def 4.4.2. Process X which can be decomposed as $X = A + M$
(semi-MG)
where M is a martingale and A has finite (first) variation are called semi-martingale.

- M is called the martingale part of X
- A is called the finite variation part of X

Prop 4.4.2 The semi-MG decomposition is unique, that is

if $X = A_1 + M_1 = A_2 + M_2^{(*)}$, then

$$A_1 = A_2, \quad M_1 = M_2$$

(where A_1, A_2 are finite variation processes, M_1, M_2 are martingales) proof: by (*) we have

$$\underbrace{A_1 - A_2}_{\text{finite variation}} = \underbrace{M_1 - M_2}_{MG} := M$$

thus M is a MG with finite variation $\Rightarrow [M,M]_T = 0$

$\overset{4.2.1}{\Longrightarrow}$ $E[M_t^2] = E[[M,M]_T] = 0$

($M^2 - [M,M]_t$ is a M.G.) $\Rightarrow$ $M_t^2 \geq 0$ thus $M_t = 0$, that is $A_1 = A_2$

$M_1 = M_2$

Prop 4.4.3 Let X be an Itô process, then

X is a MG $\Leftrightarrow$ $b_t = 0$ $\forall t \geq 0$ (i.e. $X_T = X_0 + \int_0^T \sigma_t \, dW_t$)

Proof: If $b_t = 0$ $\forall t \geq 0$ $\Rightarrow$ X is a M.G.

Suppose X is a M.G. Define

$$A_T := \int_0^T b_t \, dt = \underbrace{X_T - X_0}_{M.G.} - \underbrace{\int_0^T \sigma_t \, dW_t}_{M.G.}$$

$\Rightarrow$ A is a martingale. (also a semi-MG.)

$$A = A + 0 = 0 + A \quad \text{thus } A = 0 \Leftrightarrow b_t = 0$$

Def 4.4.3. We define the integral of D w.r.t. X by

$$\int_0^T D_t \, dX_t := \int_0^T D_t \cdot b_t \, dt + \int_0^T D_t \sigma_t \, dW_t$$

where $dX_t = b_t \, dt + \sigma_t \, dW_t$

[Given an adapted process D, we interpret X as the price of an asset, and D as our position in it, which can be positive or negative).

Theorem 4.4 (Itô formula) → Recall that if $f: \mathbb{R} \to \mathbb{R}$ is continuously differentiable $f(y) - f(x) = \int_x^y \frac{\partial f}{\partial x}(z) \, dz$

If $f: [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is such that

$t \mapsto f(t,x)$ is cont. differentiable $\forall x \in \mathbb{R}$ $\partial_t f(t,x)$ exists

$x \mapsto f(t,x)$ is twice cont. differentiable.

$\forall t \in \mathbb{R}$. $\partial_x f(t,x)$, $\partial_x^2 f(t,x)$ exist and if X is an Itô process, then

$$f(T, X_T) - f(0, X_0) = \int_0^T \partial_t f(t, X_t)\,dt + \int_0^T \partial_x f(t, X_t)\,dX_t + \underbrace{\frac{1}{2}\int_0^T \partial_x^2 f(t, X_t)\,d[X, X]_t}_{\text{Itô correction term}}$$

Remark:

- $\partial_t f(t, X_t)$ stands for taking derivative of $f(t, X_t)$ w.r.t. t and then substitute X_t, Similar for $\partial_x f(t, X_t)$, $\partial_x^2 f(t, X_t)$
- The Itô formula is simply a version of the chain rule for stochastic processes.

Stochastic form:

$$df(t, X_t) = \partial_t f(t, X_t)\,dt + \partial_x f(t, X_t)\,dX_t + \frac{1}{2}\partial_x^2 f(t, X_t)\,d[X, X]_t$$

Substitute (the Itô process) $\begin{cases} dX_t = b_t\,dt + \sigma_t\,dW_t \\ d[X, X]_t = \sigma_t^2\,dt \end{cases}$ we have

$$\begin{aligned} \Rightarrow df(t, X_t) &= \partial_t f(t, X_t)\,dt + \partial_x f(t, X_t)\,[b_t\,dt + \sigma_t\,dW_t] + \frac{1}{2}\partial_x^2 f(t, X_t)\cdot\sigma_t^2\,dt \\ &= \left[\partial_t f(t, X_t) + \partial_x f(t, X_t)\cdot b_t + \frac{1}{2}\partial_x^2 f(t, X_t)\,\sigma_t^2\right]dt + \partial_x f(t, X_t)\cdot\sigma_t\,dW_t \end{aligned}$$

if the term before dt is zero, $f(t, X_t)$ is a martingale, otherwise not

4.5 Examples

ex 1. write W^2 as a sum of dt and dW_t - integral

use the Itô formula and substitue $f(t,X) = X^2$, $X = W_t$

$\Rightarrow \partial_t f(t,x) = 0$, $\partial_x f(t,x) = 2x$. $\partial_x^2 f(t,x) = 2$

$\Rightarrow [W,W]_t = t$

By Itô formula

$$d(W_t^2) = d(f(t,W_t)) = \underset{\nearrow 0}{\partial_t f(t,W_t)}dt + \underset{\nearrow 2W_t}{\partial_x f(t,W_t)}dW_t$$

$$+ \frac{1}{2}\underbrace{\partial_x^2 f(t,W_t)}_{2} d\underbrace{[W,W]_t}_{t}$$

$$= 2W_t dW_t + dt$$

calculating the integral from 0 to T

$$W_T^2 - 0 = \int_0^T 2W_t dW_t + \int_0^T dt = \int_0^T 2W_t dW_t + T$$

$$\Rightarrow \int_0^T W_t dW_t = \frac{1}{2}(W_T^2 - T)$$

Q: calculate $[W^2, W^2]_T$

$$[W^2,W^2]_T = \int_0^T (2W_t)^2 dt = 4\int_0^T W_t^2 dt$$

not constant. a r.v.

ex.2. Let $M_t = W_t$, $N_t = W_t^2 - t$, is $M \cdot N$ a M.G ?

first method is to verify $E[M_t \cdot N_t \mid \mathcal{F}_s] = M_s \cdot N_s \ \forall t \geq s$

another method is to use the Itô formula.

Note $M_t \cdot N_t = W_t(W_t^2 - t) = W_t^3 - W_t \cdot t$

$f(t,x) = x^3 - tx$ $\quad X_t = W_t \quad [X,X]_t = t$

$\Rightarrow \partial_x f(t,x) = 3x^2 - t \quad \partial_t f(t,x) = -x \quad \partial_x^2 f(t,x) = 6x$

$$\Rightarrow df(t,x) = \partial_t f(t,X_t)\,dt + \partial_x f(t,X_t)\,dX_t + \frac{1}{2}\partial_x^2 f(t,X_t)\,d[X,X]_t$$

$$= -X\,dt + (3X^2 - t)\,dX_t + 3X\,dt$$

$$= 2X\,dt + (3X^2 - t)\,dX_t$$

$$= 2W_t\,dt + (3W_t^2 - t)\,dW_t$$

$$M_T N_T = \int_0^T 2W_t\,dt + \int_0^T (3W_t^2 - t)\,dW_t$$

Since the finite (first) Variation part is not zero, $M_T N_T$ is not a martingale.

ex3. Let $X_t = t\sin(W_t)$. Is $X^2 - [X,X]$ a martingale?

let $f(t,x) = t\sin(x) \quad \partial_t f = \sin x \quad \partial_x f = t\cos x$

$\partial_x^2 f = -t\sin x$

$$\Rightarrow dX_t = \sin W_t\,dt + t\cos W_t\,dW_t - \frac{1}{2}t\sin W_t\,dt$$

$$= \left(\sin W_t - \frac{1}{2}t\sin W_t\right)dt + t\cos W_t\,dW_t$$

$$\Rightarrow d[X,X] = t^2\cos^2 W_t\,dt$$

$$dX_t^2 = 2X_t\,dX_t + d[X,X]_t$$

$$\Rightarrow d[X^2 - [X,X]] = 2X_t\,dX_t$$

$$= 2t\sin(W_t)\cdot\left[\sin(W_t) - \frac{1}{2}t\sin(W_t)\right]dt + 2t^2\sin(W_t)\cdot\cos(W_t)\cdot dW_t$$

Since the dt term above is not 0, thus $X^2 - [X,X]$ is not a martingale