Homework assignments for the course "Financial Computing with C++"

Master of Science in Computational Finance
Carnegie Mellon University

Spring 2025

Contents

Homework 1: explicit data curves and root finding

While implementing the functions below, you need to account for the singularities of the type 0/0.

Explicit data curves

Volatility curve computed from variance curve

Input:

$V = (V(t))_{t \geq t_0}$: the variance curve.

t_0 : the initial time given as year fraction.

Output: the continuously compounded volatility curve $\Sigma = (\Sigma(t))_{t \geq t_0}$.

We recall that

$$V(t) = \Sigma^2(t)(t - t_0), \quad t \geq t_0.$$

The Svensson discount curve

Input:

$(c_i)_{i=0,1,2,3}$: the constant coefficients of the Svensson model of interest rates.

$(\lambda_i)_{i=1,2}$: the strictly positive mean-reversion rates, $\lambda_1 \neq \lambda_2$.

t_0 : the initial time given as year fraction.

Output: the discount curve

$$D(t) = e^{-\gamma(t)(t-t_0)}, \quad t \geq t_0,$$

where the yield curve has the Svensson form:

$$\begin{aligned} \gamma(t) = c_0 + c_1 \frac{1 - e^{-\lambda_1(t-t_0)}}{\lambda_1(t-t_0)} + c_2 \left(\frac{1 - e^{-\lambda_1(t-t_0)}}{\lambda_1(t-t_0)} - e^{-\lambda_1(t-t_0)} \right) \\ + c_3 \left(\frac{1 - e^{-\lambda_2(t-t_0)}}{\lambda_2(t-t_0)} - e^{-\lambda_2(t-t_0)} \right), \quad t \geq t_0. \end{aligned}$$

Forward prices for a stock that pays dividends

Input:

$S(t_0):$ the spot price.

$(t_m)_{m=1,\dots,M}:$ the dividend times, $t_1 > t_0$.

$(Q_m)_{m=1,\dots,M}:$ the cash dividend payments.

$D = (D(t))_{t \geq t_0}:$ the discount curve.

$t_0:$ the initial time = the issue time for the forward.

Output:

$F = (F(t))_{t \in [t_0, t_M]}:$ the forward prices for the stock.

The buyer pays forward price $F(t)$ at delivery time t and then receives the stock at t. It costs nothing to enter a forward contract at its issue time t_0.

Root finding

Price of the clique option in the Black model

Input:

$K:$ the strike.

$(t_m)_{m=1,\dots,M}:$ the averaging times, $t_M < T$.

$T:$ the maturity.

$(B(t_0,t))_{t \geq t_0}:$ the initial discount curve.

$(F(t_0,t))_{t \geq t_0}:$ the initial forward curve.

$\lambda \geq 0:$ the mean-reversion rate.

$\sigma > 0:$ the volatility of the spot.

$t_0:$ the initial time.

Output: the price of the clique option in the Black model at t_0.

The payoff of the clique option at the maturity T is given by

$$V(T) = \frac{1}{M} \sum_{m=1}^{M} \max(S(t_m) - K, 0),$$

where $S(t)$ is the spot price at t.

We use the stationary form of the general Black model. The interest rates are deterministic. The forward price $F(t, u)$ computed at time t for the contract with delivery time u evolves as

$$dF(t, u) = F(t, u) e^{-\lambda(u-t)} \sigma dW_t, \quad t \in [t_0, u],$$

where W is a Brownian motion. Solving this linear SDE, we obtain that

$$S(t) = F(t, t) = F(t_0, t) \exp\left(\sigma \int_{t_0}^{t} e^{-\lambda(t-s)} dW_s - \frac{1}{2} \Sigma^2(t_0, t)(t - t_0) \right),$$

where the average variance of the forward price $F(\cdot, t)$ on $[t_0, t]$ is given by

$$\Sigma^2(t_0, t) = \sigma^2 \frac{1}{t - t_0} \int_{t_0}^{t} e^{-2\lambda(t-s)} ds = \sigma^2 \frac{1 - e^{-2\lambda(t-t_0)}}{2\lambda(t - t_0)}, \quad t \geq t_0.$$

Vega for the clique option in the Black model

Input:

K : the strike.

$(t_m)_{m=1,\dots,M}$: the averaging times, $t_M < T$.

T : the maturity.

$(B(t_0, t))_{t \geq t_0}$: the initial discount curve.

$(F(t_0, t))_{t \geq t_0}$: the initial forward curve.

$\lambda \geq 0$: the mean-reversion rate.

$\sigma > 0$: the volatility of the spot.

t_0 : the initial time.

Output: the derivative of the price of the clique option in the Black model at t_0 with respect to σ.

Implied volatility for the clique option in the Black model

Input:

K : the strike.

$(t_m)_{m=1,\dots,M}$: the averaging times, $t_M < T$.

T : the maturity.

P : the price of the clique.

$(B(t_0,t))_{t\geq t_0}$: the initial discount curve.

$(F(t_0,t))_{t\geq t_0}$: the initial forward curve.

$\lambda \geq 0$: the mean-reversion rate.

t_0 : the initial time.

$\sigma_0 > 0$: the initial guess for the volatility.

`cfl::RootD:` the root-finding routing of polishing type. It requires the computation of the function and its derivative.

Output:

σ : the implied volatility of the clique option with the given parameters computed in the Black model:

$$\text{theoretical price}\,(\sigma) = \text{market price}.$$

Homework 2: interpolation and least-squares fitting

While implementing the functions below, you need to account for the singularities of the type 0/0.

Interplation of data curves

Discount curve obtained by interpolation of yields

Input:

$(t_m)_{m=1,\dots,M}$: the maturities, $t_m < t_{m+1}$,

$(D_m)_{m=1,\dots,M}$: the discount factors,

r_0 : the initial short-term interest rate.

t_0 : the initial time, $t_0 < t_1$.

$\mathcal{I}$: an interpolation method, a class `cfl::Interpolation`.

Output: the discount curve

$$D(t) = \exp(-\gamma(t)(t-t_0)), \quad t \in [t_0, t_M],$$

where the yield curve $\gamma = \gamma(t)$ is the $\mathcal{I}$-interpolation of the market yields to maturity

$$\gamma(t) = \mathcal{I}((t_m)_{m=0,1,\dots,M}, (\gamma_m)_{m=0,1,\dots,M}), \quad t \in [t_0, t_M],$$
$$\gamma_0 = r_0, \quad \gamma_m = -\frac{\log(D_m)}{t_m - t_0}, \quad m = 1, \dots, M,$$

Forward curve obtained by log interpolation

Input:

S_0 : the market spot price,

$(t_m)_{m=1,\dots,M}$: the maturities, $t_m < t_{i+1}$,

$(F_m)_{m=1,\dots,M}$: the market forward prices,

t_0 : the initial time, $t_0 < t_1$,

$\mathcal{I}$: an interpolation method, a class `cfl::Interpolation`.

Output: the forward curve

$$F(t) = \exp(L(t)), \quad t \in [t_0, t_M],$$

where the function $L = L(t)$ is the $\mathcal{I}$-interpolation of the logs of the market spot and forward prices:

$$L(t) = \mathcal{I}((t_m)_{m=0,1,\dots,M}, (\log F_m)_{m=0,\dots,M}), \quad F_0 = S_0.$$

Volatility curve obtained by interpolation of variances

Input:

$(t_m)_{m=1,\dots,M}$: the maturities, $t_m < t_{m+1}$,

$(\sigma_m)_{m=1,\dots,M}$: the implied volatilities,

t_0 : the initial time, $t_0 < t_1$,

$\mathcal{I}$: an interpolation method.

Output: the volatility curve

$$\sigma(t) = \sqrt{\frac{v(t)}{t - t_0}}, \quad t \in [t_0, t_M],$$

where the variance function $v = v(t)$ is the $\mathcal{I}$-interpolation of the market variances:

$$\begin{aligned} v(t) &= \mathcal{I}((t_m)_{m=0,1,\dots,M}, (v_m)_{m=0,\dots,M}), \quad t \in [t_0, t_M], \\ v_m &= \sigma_m^2 (t_m - t_0), \quad m = 1, \dots, M, \quad v_0 = 0. \end{aligned}$$

Least-squares fitting of data curves

Forward curve by fitting of cost-of-carry rates

Input:

S_0 : the market spot price.

$(t_m)_{m=1,\dots,M}$: the maturities of the forward contracts, $t_m < t_{m+1}$.

$(F_m)_{m=1,\dots,M}$: the market forward prices.

t_0 : the initial time, $t_0 < t_1$.

$\mathcal{L}$: a fitting method for cost-of-carry rates, a class `cfl::Fit`.

Output:

$F = F(t)$: the fitted forward curve.

$\epsilon = \epsilon(t)$: the error function of the fit for the forward curve.

$\mathbf{P} = (\widehat{\mathbf{c}}, \mathbf{\Gamma}, \chi^2)$: the parameters of the fit for the cost-of-carry rate curve. They contain the fitted constants $\widehat{\mathbf{c}}$, the covariance matrix $\mathbf{\Gamma}$, and the total fitting error χ^2.

The forward curve has the form:

$$F(t) = S_0 \exp(q(t)(t - t_0)), \quad t \geq t_0,$$

where the cost-of-carry function $q = q(t)$ is the result of the $\mathcal{L}$-fit of the market cost-of-carry rates:

$$q(t) = \mathcal{L}((t_m)_{m=1,\dots,M}, (q_m)_{m=1,\dots,M}),$$
$$q_m = \frac{\log(F_m/S_0)}{t_m - t_0}, \quad m = 1, \dots, M.$$

Forward curve obtained by fitting of cost-of-carry rates in the Black model

Input:

$S_0 = S(t_0)$: the market spot price.

$(t_m)_{m=1,\dots,M}$: the maturities of forward contracts, $t_m < t_{m+1}$.

$(F_m)_{m=1,\dots,M}$: the market forward prices.

$\lambda \geq 0$: the mean-reversion rate.

$\sigma > 0$: the short-term volatility.

t_0 : the initial time, $t_0 < t_1$.

Output:

$F = F(t)$: the fitted forward curve.

$\epsilon = \epsilon(t)$: the error function of the fit for the forward curve.

$\mathbf{P} = (\theta, \Gamma, \chi^2)$: the parameters of the fit for the cost-of-carry rate curve. They contain the fitted constant θ, its variance Γ, and the total fitting error χ^2.

Returns the forward curve in the form:

$$F(t) = S_0 \exp(q(t)(t - t_0)), \quad t \geq t_0,$$

where the cost-of-carry rate $q = q(t)$ fits the market cost-of-carry rates in the Black model. Under the Black model, the log of spot price has the form:

$$\log S(t) = \log S_0 + X(t), \quad t \geq t_0,$$

where $X = (X(t))$ is an OU process:

$$dX(t) = (\theta - \lambda X(t))dt + \sigma dB(t), \quad t \geq t_0, \quad X(t_0) = 0.$$

Here $B = (B(t))$ is a Brownian motion and the drift θ is the fitting parameter. Computations show that

$$q(t) = \theta \frac{1 - e^{-\lambda(t-t_0)}}{\lambda(t - t_0)} + \frac{\sigma^2}{2} \frac{1 - e^{-2\lambda(t-t_0)}}{2\lambda(t - t_0)}, \quad t \geq t_0.$$

Forward curve by fitting of cost-of-carry rates in the 2-factor Black model

Input:

$S_0 = S(t_0)$: the market spot price.

$(t_m)_{m=1,\dots,M}$: the maturities of forward contracts, $t_m < t_{m+1}$.

$(F_m)_{m=1,\dots,M}$: the market forward prices.

$\lambda_1 > 0$: the second mean-reversion rate.

$\sigma_0 > 0$: the first short-term volatility.

$\sigma_1 > 0$: the second short-term volatility; $\sigma_1 \neq \sigma_0$.

t_0 : the initial time, $t_0 < t_1$.

Output:

$F = F(t)$: the fitted forward curve.

$\epsilon = \epsilon(t)$: the error function of the fit for the forward curve.

$\mathbf{P} = ((\theta_0, \theta_1), \Gamma, \chi^2)$: the fitted constants, their covariance matrix, and the total fitting error.

Returns the forward curve in the form:

$$F(t) = S(t_0) \exp(q(t)(t - t_0)), \quad t \geq t_0,$$

where the cost-of-carry rate curve $q = q(t)$ fits the market cost-of-carry rates in the 2-factor Black model. Under this model, the log of spot price has the form:

$$\log S(t) = \log S_0 + X_0(t) + X_1(t), \quad t \geq t_0,$$

where $X_0 = (X_0(t))$ and $X_1 = (X_1(t))$ are independent OU processes:

$$\begin{aligned} dX_0(t) &= \theta_0 dt + \sigma_0 dB_0(t), \quad X_0(t_0) = 0, \\ dX_1(t) &= (\theta_1 - \lambda_1 X_1(t))dt + \sigma_1 dB_1(t), \quad X_1(t_0) = 0. \end{aligned}$$

Here B_0 and B_1 are independent Brownian motions and drifts θ_0 and θ_1 are the fitting parameters. Computations show that

$$q(t) = \theta_0 + \theta_1 \frac{1 - e^{-\lambda_1(t-t_0)}}{\lambda_1(t - t_0)} + \frac{\sigma_0^2}{2} + \frac{\sigma_1^2}{2} \frac{1 - e^{-2\lambda_1(t-t_0)}}{2\lambda_1(t - t_0)}, \quad t \geq t_0.$$

Homework 3: options on a stock

The issue time for all options coincides with the initial time. The maturities and coupon, barrier, and exercise times are strictly greater than the initial time.

Straddle option

K : the strike.

T : the maturity.

The straddle option is the sum of the standard European put and call options with the same strike and maturity. The payoff at T is given by

$$V(T) = |S(T) - K|,$$

where $S(T)$ is the price of the stock at T.

American butterfly

K : the middle strike.

W : the size of wing ($W < K$).

$(t_m)_{m=1,\dots,M}$: the exercise times.

A holder of the option can exercise it at any time t_m. In this case, he gets

1. a long position in the call with strike $K - W$,
2. a long position in the call with strike $K + W$,
3. two short positions in the call with strike K.

If not exercised, the option expires worthless.

Equity linked note with buy-back provision

N : the notional.

K : the strike.

q : the participation rate.

p : the buy-back percentage of the notional; usually $p < 1$.

$(t_m)_{m=1,\ldots,M}$: the buy-back times.

T : the maturity, $t_M < T$.

At maturity T, the holder gets the payment

$$V_T = N\left(1 + q\max\left(\frac{S_T}{K} - 1, 0\right)\right),$$

if the buy-back provision has not been used. The buy-back feature allows the holder to sell the note back at any of the buy-back times $(t_m)_{m=1,\ldots,M}$ for the amount pN.

Down-and-rebate option

N : the notional.

L : the lower barrier.

$(t_m)_{m=1,\ldots,M}$: the barrier times.

The option pays the notional N at the first barrier time t_m when the spot price $S(t_m)$ stays below the lower barrier L. Otherwise, the option expires worthless.

Corridor option

N : the notional.

L : the lower barrier.

U : the upper barrier.

$(t_m)_{m=1,\ldots,M}$: the barrier times.

The last barrier time t_M is the maturity of the option. At the maturity, the option pays the product of the notional on the percentage of the barrier times when the price of the stock is less than the upper barrier and is greater than the lower barrier:

$$X(t_M) = N\frac{1}{M}\sum_{m=1}^{M} 1_{\{L<S(t_m)<U\}}.$$

Here, $S(t)$ is the price of the stock at time t.

Up-and-in American put

U : the upper barrier.

$(u_i)_{i=1,\ldots,N_1}$: the barrier times.

K : the strike.

$(v_j)_{j=1,\ldots,N_2}$: the exercise times ($v_1 > u_1$, $v_{N_2} \geq u_{N_1}$).

The option converts to the American put option (with the given set of exercise times) at the first barrier time u_i when the stock price $S(u_i)$ is above the upper barrier U. If $u_i = v_j$, that is, the barrier time is also an exercise time, then the option can still be exercised at v_j. If the price of the stock stays below U for all barrier times, then the option expires worthless.

Homework 4: options on interest rates

The issue time for all options coincides with the initial time. The maturities and coupon, barrier, and exercise times are strictly greater than the initial time.

Forward Swap Lock

T : the maturity.

Parameters of underlying swap:

> N : the notional.
>
> R : the fixed rate.
>
> δt : the interval of time between the payments given as year fraction.
>
> M : the total number of payments.
>
> **side**: the side of the swap contract, that is, whether one pays "fixed" and receives "float" or otherwise.

At maturity T, a holder of the contract enters into the interest rate swap with the parameters above and the issue time T.

Hint. Write the algorithm with only one event time t_0.

Cap set in arrears

N : the notional.

C : the cap rate.

δt : the interval of time between the payments given as year fraction.

M : the total number of payments.

Brief description: the current caplet is determined by the *current* float rate (not the float rate from the previous period as in the case of the standard cap).

We assume that today is the issue time and denote this time by t_0. The payment times are given by

$$t_m = t_0 + m\delta t, \quad m = 1, \dots, M.$$

At payment time t_m, a holder receives the *caplet set in arrears*

$$N\delta t \max(r(t_m, t_m + \delta t) - C, 0),$$

where $r(s,t)$ is the float rate computed at time s for maturity t.

American put on forward rate agreement

$N:$ the notional.

$R:$ the fixed rate.

$\delta t:$ the time interval for the loan as year fraction.

$(t_m)_{m=1,\dots,M}:$ the exercise times.

The owner of the option has the right to sell the forward rate agreement at any exercise time t_m. In this case,

1. At time t_m, he receives notional N.
2. At time $t_m + \delta t$, he pays notional plus fixed interest, that is, the amount $N(1 + R\delta t)$.

Hint. Write the algorithm with the vector of event times

$$\{t_0, \underbrace{(t_m)_{m=1,\dots,M}}_{\text{exercise times}}\},$$

where t_0 is the initial time.

Callable capped floater

$N:$ the notional.

$C:$ the cap rate.

$\delta t:$ the interval of time between the payments given as year fraction.

$M:$ the total number of payments.

$\delta r:$ the spread over float rate.

We assume that today is the issue time of the capped floater and denote this time by t_0. Let $r(s,t)$ be the float rate computed at s for maturity t. The payment times are given by

$$t_m = t_0 + m\delta t, \quad m = 1, \ldots, M.$$

At payment time t_{m+1},

1. The holder *receives* the coupon

$$N\delta t \min(r(t_m, t_{m+1}) + \delta r, C).$$

2. The seller *has the right to cancel* the contract. In this case, in addition to the above coupon he pays the notional. No payments will be made in the future. Note that the option can not be terminated at issue time.

If the contract has not been terminated before, then at maturity t_M the holder receives the coupon above plus the notional.

Putable and callable bond

Coupon bond:

$N:$ the notional.

$R:$ the coupon rate.

$\delta t:$ the interval of time between the payments given as year fraction.

$M:$ the total number of coupon payments.

$U:$ the repurchase price of the bond as percentage of the notional. After the coupon payment, the issuer of the bond can repurchase the bond from the holder by paying NU. Typically, this payment is greater than the notional ($U > 1$).

$L:$ the redemption price of the bond as percentage of the notional. After the coupon payment, the holder of the bond can sell it back to the issuer for amount LN. Typically, this amount is less than the notional ($L < 1$).

Denote by t_0 the current time and by $(t_m)_{m=1,\dots,M}$ the coupon times:

$$t_m = t_0 + m\delta t, \quad m = 1, \dots, M.$$

1. At maturity $T = t_M$, if the bond has not been terminated before, the owner of the bond receives coupon $NR\delta t$ and notional N.

2. At a coupon time t_m other than the maturity, if the bond has not been terminated before,

   (a) The owner of the bond receives coupon $NR\delta t$.

   (b) The owner of the bond has the right to redeem the bond. In this case he receives amount LN from the issuer of the bond and the bond is terminated.

   (c) The issuer of the bond has the right to repurchase the bond. In this case the holder of the bond receives amount UN and the bond is terminated.

   Note that the events take place in their respective order:

$$\text{coupon} \longrightarrow \text{redemption} \longrightarrow \text{repurchase}.$$

Futures on cheapest bond to deliver

This problem is motivated by the existing futures contract on US treasury bonds.

Input: the parameters of the futures contract.

$T:$ the maturity of the futures contract.

$M:$ the number of settlement times between today and the maturity.

Bonds to deliver with indexes $j = 1, \dots, J$. We assume that all the bonds are issued at T (the maturity of the futures contract). The parameters of the bond with index j have the form:

N_j : the notional.
R_j : the coupon rate.
$(\delta t)_j$: the interval of time between the payments given as year fraction.
M_j : the number of coupon payments.

Output: futures price $F(t_0)$ computed at the initial time.

We assume that the settlement times are given by

$$t_m = t_0 + m\delta t, \quad m = 1, \dots, M,$$

where t_0 is the initial time and

$$\delta t = \frac{T - t_0}{M}.$$

Notice that the settlement times include T, but do not contain t_0.

The futures contract involves the following transactions:

1. It costs nothing to enter into either a long or a short position.
2. At time t_m before the maturity, $m = 1, \dots, M - 1$,
   (a) the buyer (long position) pays futures price $F(t_{m-1})$ established at the previous trading day,
   (b) the seller (short position) pays futures price $F(t_m)$ established at the current trading day.
3. At maturity $T = t_M$,
   (a) the buyer (long position) pays futures price $F(t_{M-1})$ established at previous trading day,
   (b) the seller (short position) delivers one of the available coupon bonds. Note that the seller has the right to choose which bond to deliver.

Homework 5: explicit data curves and root finding

While implementing the functions below, you need to account for the singularities of the type 0/0.

Explicit data curves

Forward price curve for an annuity

Input:

q : the coupon rate.

δt : the time interval between coupon payments.

T : the maturity.

$D = (D(t))_{t \geq t_0}$: the discount curve.

t_0 : the initial time.

is_clean : the boolean parameter specifying the type of the prices: "clean" or "dirty". The dirty price is the actual amount paid in a transaction. The clean price is the difference between the dirty price and the accrued interest. If t_i is the previous coupon time (or the initial time if no coupons have been paid so far) and t is the settlement time, then the accrued interest is given by

$$A(t) = q(t - t_i).$$

Output:

$F = (F(t))_{t \in [t_0, T]}$: the forward prices for the annuity. Here t is the maturity of the contract and t_0 is the issue time.

The annuity pays coupons $q\delta t$ at times $(t_i)_{i=1,\dots,M}$ such that

$$t_0 < t_1 \leq t_0 + \delta t, \quad t_{i+1} - t_i = \delta t, \quad t_M = T.$$

The buyer pays forward price $F(t)$ at delivery time t and then receives coupons $q\delta t$ at payments times $t_i > t$.

Forward swap rates

Input:

$\delta t :$ the time interval between payments.

$M :$ the number of payments.

$D = (D(t))_{t\geq t_0} :$ the discount curve; t_0 is the initial time.

Output:

$R^f(t) = R^f(t; t_0, \delta t, M)_{t\geq t_0} :$ the forward swap rates computed at t_0 in the contract with period δt and number of payments M.

It costs nothing to enter the forward swap contract with maturity t. At time t, the swap is issued with a notional amount N, the number of payments M, the period between payments δt, and the fixed rate $R^f(t)$.

Root finding

Yield to maturity for a cash flow with `cfl::Root`

Input:

$(P_m)_{m=1,\dots,M} :$ the cash flow. Some payments can be negative, which may lead to counter-intuitive results.

$(t_m)_{m=1,\dots,M} :$ the payment times, $t_1 > t_0$.

$S(t_0) :$ the initial value of the cash flow.

$t_0 :$ the initial time.

$\gamma_0 :$ the initial lower bound for the YTM.

$\gamma_1 :$ the initial upper bound for the YTM.

`cfl::Root:` the root-finding routing of bracketing type.

Output:

$\gamma :$ the yield to maturity for the cash flow.

The yield to maturity γ solves the equation:

$$S(t_0) = \sum_{m=1}^{M} P_m e^{-\gamma(t_m - t_0)}.$$

Forward YTM curve for an annuity with `cfl::RootD`

Input:

$q:$ the coupon rate.

$\delta t:$ the time interval between coupon payments.

$T:$ the maturity, $T > t_0$.

$D = (D(t))_{t\geq t_0}:$ the discount curve.

$t_0:$ the initial time.

`cfl::RootD:` the root-finding routing of polishing type. It requires the computation of the function and its derivative. We use the coupon rate q as the initial guess for the YTM.

Output:

$\gamma = (\gamma(t))_{t\in[t_0,T)}:$ the forward yields to maturity for the annuity.

The annuity pays coupons $q\delta t$ at times $(t_m)_{m=1,\dots,M}$ such that

$$t_0 < t_1 \leq t_0 + \delta t, \quad t_{m+1} - t_m = \delta t, \quad t_M = T.$$

For $t \in [t_0, T)$, the forward yield to maturity $\gamma(t)$ solves the equation:

$$F(t) = q\delta t \sum_{t_m > t} e^{-\gamma(t)(t_m - t)},$$

where $F(t)$ is the "dirty" forward price computed at t_0 for delivery at t. In the forward contract, the buyer pays forward price $F(t)$ at delivery time t and then receives coupons $q\delta t$ at payments times $t_m > t$.

Homework 6: interpolation and least-squares fitting

While implementing the functions below, you need to account for the singularities of the type 0/0.

Interplation of data curves

Discount curve obtained from swap rates by log interpolation

Input:

δt : the time interval between payments given as year fraction.

$(R_m)_{m=1,\dots,M}$: the vector of swap rates, where R_m is the current swap rate in the contract with time interval δt and the number of payments m.

t_0 : the initial time given as year fraction.

$\mathcal{I}$: an interpolation method, a class `cfl::Interpolation`.

Output: the discount curve $D = D(t)$ on $[t_0, t_0 + M\delta t]$. It is obtained by the following procedure:

1. We compute the discount factors D_m for maturities $t_m = t_0 + m\delta t$, $m = 1, \dots, M$.
2. We apply the $\mathcal{I}$-interpolation to $(t_m, \log D_m)$, $m = 0, 1, \dots, M$, where $D_0 = 1$.

Forward exchange curve obtained by the interpolation of cost-of-carry rates

Input:

S_0 : the current spot exchange rate.

$(D^d_m)_{m=1,\dots,M}$: the discount factors in domestic currency.

$(D^f_m)_{m=1,\dots,M}$: the discount factors in foreign currency.

$(t_m)_{m=1,\dots,M}$: the maturities of the discount factors, $t_0 < t_1$.

t_0 : the initial time given as year fraction.

$\mathcal{I}$: an interpolation method, a class `cfl::Interpolation`.

Output: the forward exchange curve

$$F(t) = S_0 \exp(q(t)(t - t_0)), \quad t \in [t_0, t_M],$$

where the cost-of-carry rate curve $q = (q(t))_{t \in [t_0, t_M]}$ is obtained by the $\mathcal{I}$-interpolation of the market cost-of-carry rates $(q(t_m))_{m=0,1,\dots,M}$. We assume that

$$q(t_0) = q(t_1).$$

Least-squares fitting of data curves

Discount curve by an approximate fit of discount factors

Input:

$(t_m)_{m=1,\dots,M}$: the maturities of discount factors, $t_1 > t_0$.

$(D_m)_{m=1,\dots,M}$: the discount factors.

t_0 : the initial time given as year fraction.

$\mathcal{L}$: a fitting method for the yields, a class `cfl::Fit`.

Output:

$D = D(t)$: the fitted discount curve.

$\epsilon = \epsilon(t)$: the error function of the fit for the discount curve.

$\mathbf{P} = (\widehat{\mathbf{c}}, \mathbf{\Gamma}, \chi^2)$: the parameters of the fit for the yield curve. They consist of the fitted constants $\widehat{\mathbf{c}}$, the covariance matrix $\mathbf{\Gamma}$, and the total fitting error χ^2.

The fitted discount curve has the form:

$$D(t) = \exp(-\gamma(t)(t - t_0)), \quad t \geq t_0,$$

where the yield curve $\gamma = \gamma(t)$ is the result of the minimization:

$$\chi^2 = \min_{\mathcal{L}} \sum_{m=1}^{M} w_m (\gamma(t_m) - \gamma_m)^2, \tag{1}$$

with the fitting method $\mathcal{L}$ and the weights

$$w_m = D_m^2 (t_m - t_0)^2, \quad m = 1, \dots, M.$$

Here $(\gamma_m)_{m=1,\dots,M}$ are the market yields:

$$\gamma_m = -\frac{\log D_m}{t_m - t_0}, \quad m = 1, \dots, M.$$

Expression (1) is the first order approximation to the (non-linear) least-squares fitting of the discount curve $D = D(t)$ to the market discount factors (D_m). The underlying statistical model has the form:

$$\gamma_m = f(t_m, \mathbf{c}^0) + \frac{\kappa}{\sqrt{w_m}} \epsilon_m, \quad m = 1, \dots, M,$$

where (ϵ_m) are independent standard gaussian random variables with mean 0 and variance 1, $f = f(t, \mathbf{c})$ is the family of fitting functions from method $\mathcal{L}$, $\mathbf{c}^0$ is the true value of the parameter, and $\kappa > 0$ is the *unknown* normalizing volatility: `bChi2 = true`.

Discount curve by the Svensson fit of yields

Input:

$(t_m)_{m=1,\dots,M}$: the maturities, $t_m < t_{m+1}$.

$(D_m)_{m=1,\dots,M}$: the discount factors.

$\lambda_1 > 0$: the first mean-reversion rate.

$\lambda_2 > 0$: the second mean-reversion rate, $\lambda_2 \neq \lambda_1$.

t_0 : the initial time, $t_0 < t_1$.

Output:

$D = D(t)$: the fitted discount curve.

$\epsilon = \epsilon(t)$: the error function of the fit for the discount curve.

$\mathbf{P} = ((c_j), \Gamma, \chi^2)$: the fitted constants, their covariance matrix, and the total fitting error.

The discount curve is given by

$$D(t) = \exp(-\gamma(t)(t - t_0)), \quad t \geq t_0,$$

where the yield curve $\gamma = \gamma(t)$ has the Svensson form:

$$\begin{aligned}\gamma(t; c_0, c_1, c_2, c_3) = c_0 + c_1 \frac{1 - e^{-\lambda_1(t-t_0)}}{\lambda_1(t - t_0)} + c_2 \left(\frac{1 - e^{-\lambda_1(t-t_0)}}{\lambda_1(t - t_0)} - e^{-\lambda_1(t-t_0)} \right) \\ + c_3 \left(\frac{1 - e^{-\lambda_2(t-t_0)}}{\lambda_2(t - t_0)} - e^{-\lambda_2(t-t_0)} \right), \quad t \geq t_0,\end{aligned}$$

and constants c_0, c_1, c_2, c_3 are the result of the least-squares fit of the market yields:

$$\chi^2 = \min_{c_0,c_1,c_2,c_3} \sum_{m=1}^{M} (\gamma(t_m; c_0, c_1, c_2, c_3) - \gamma_m)^2,$$

$$\gamma_m = -\frac{\log D_m}{t_m - t_0}, \quad m = 1, \ldots, M.$$

Homework 7: options on a stock

The issue time for all options coincides with the initial time. The maturities and coupon, barrier, and exercise times are strictly greater than the initial time.

BOOST (Banking On Overall Stability) option

N : the notional.

L : the lower barrier.

U : the upper barrier.

$(t_m)_{m=1,\dots,M}$: the barrier times.

The option terminates at the first barrier time, when the price of the stock hits either of the barriers, that is, at the barrier time t_k such that

$$k = \min\{m = 1, \dots, M : \ S(t_m) > U \text{ or } S(t_m) < L\}.$$

At the exit time t_k, the holder receives the payoff

$$V(t_k) = N\frac{k-1}{M},$$

the product of the notional on the percentage of the barrier times the price of the stock spends inside the barriers.

If the price of the stock never exits the barriers, then at the last barrier time t_M, the holder receives the notional amount N.

Asset swaption

T : the maturity.

Parameters of underlying asset swap:

N : the notional.

δr : the spread over float rate.

δt : the interval of time between the payments given as year fraction.

M : the total number of payments.

side: the side of the swap contract, that is, whether one pays returns on the performance of the stock and receives "float rate + spread" or otherwise.

At maturity T, the owner of the option has the right to enter into the asset swap contract with the parameters defined above and the issue time T.

The settlement times of the swap are given by

$$t_m = T + m\delta t, \quad m = 1, \dots, M.$$

At a settlement time $t_{m+1} = t_m + \delta t$,

1. One side pays "float rate + spread" given by

$$N(r(t_m, t_m + \delta t) + \delta r)\delta t,$$

where $r(s,t)$ is the float rate computed at s for maturity t.

2. The other side pays the return on the notional N produced by the stock between t_m and t_{m+1}, that is, the amount

$$\frac{N}{S(t_m)}(S(t_{m+1}) - S(t_m)) = N\left(\frac{S(t_{m+1})}{S(t_m)} - 1\right),$$

where $S(t)$ is the price of the stock at time t.

Cancellable currency swap

Parameters of underlying swap:

N^{dom} : the notional in domestic currency (USD).

N^{for} : the notional in foreign currency (EUR).

R^{dom} : the fixed rate in the swap (USD).

δr^{for} : the spread for foreign float (EUR rate).

δt : the interval of time between payments given as year fraction.

M : the total number of payments.

Brief description: At initial time t_0 we pay notional N^{for} in EUR and receive notional N^{dom} in USD. Later, at payment times we pay fixed rate R^{dom} in USD and receive float rate r^{for} plus spread δr^{for} in EUR. At maturity we also pay notional N^{dom} in USD and receive notional N^{for} in EUR. At payment times we have the right to terminate the swap. In this case, the last transaction takes place at the next payment time and consists of interest payments and notionals (the same payments as at the maturity).

We denote by $(t_m)_{m=1,\dots,M}$ the payment times of the swap:

$$t_m = t_0 + m\delta t, \quad m = 1, \dots, M.$$

1. At initial time t_0, we pay N^{for} EUR and receive N^{dom} USD.
2. At payment time t_{m+1},
   (a) We receive float interest plus spread in EUR:

   $$N^{for}(r^{for}(t_m, t_m + \delta t) + \delta r^{for})\delta t \quad \text{(EUR)}$$

   where $r^{for}(t_m, t_m + \delta t)$ is the float rate for EUR at t_m for the period of δt years.

   (b) We pay fixed interest in USD:

   $$N^{dom}R^{dom}\delta t.$$

   Then,

   (a) If the swap has been terminated at t_m or, if the current time t_{m+1} is the maturity, we pay notional N^{dom} USD and receive notional N^{for} EUR. There are no payments after that.

   (b) Otherwise, we have the right to terminate the swap. In this case, the payments at t_{m+2} will still take place.

In the single asset model in `cfl`, the forward curve defines the forward exchange rate. The exchange rate is the number of units of domestic currency needed to buy one unit of foreign currency.

Strike of variance swap

$T:$ the maturity,

$M:$ the number of times used in the computation of the variance.

We assume that initial time t_0 is the issue time for the swap and denote

$$t_m = t_0 + m\delta t, \quad m = 1, \ldots, M,$$

the sample times, where

$$\delta t = \frac{T - t_0}{M}.$$

The payoff of the variance swap at maturity $T = t_M$ is given by

$$V(T) = \frac{1}{T - t_0} \sum_{m=1}^{M} \left(\ln \frac{S(t_m)}{S(t_{m-1})} \right)^2 - K^2,$$

where $S(t_m)$ is the spot price at t_m and K is the strike. Compute K using the fact that it costs nothing to enter the variance swap. In other words, K^2 is the forward price on the annualized variance of the stock price.

Homework 8: options on interest rates

The issue time for all options coincides with the initial time. The maturities and coupon, barrier, and exercise times are strictly greater than the initial time.

Futures on average rate

The futures of these type are traded, for example, on EUREX, where they are called *EONIA* (Effective Over Night Interest Average) futures.

$T:$ the maturity of the futures.

$M:$ the number of payment (settlement) times.

The set of payment times in the contract is given by

$$t_m = t_0 + m\delta t, \quad m = 1, \dots, M,$$

where t_0 is the initial (issue) time and

$$\delta t = \frac{T - t_0}{M}.$$

At maturity $T = t_M$, the futures price is given by

$$F(T) = 1 - \frac{1}{M} \sum_{m=1}^{M} r(t_m, t_m + \delta t),$$

where $r(s, t)$ is the float rate computed at s for maturity t.

Callable Range Accrued Note (CRAN)

Note parameters:

$N:$ the notional.

$\delta t:$ the interval of time between the coupon payments given as year fraction.

$R:$ the coupon rate.

M : the total number of coupon payments.

Lookup range:

L : the total number of lookup times in the coupon period.

U : the upper barrier for the float rate.

D : the lower barrier for the float rate.

Δ : the period for the float rate given as year fraction.

Brief description: Each coupon period has several look up dates. For example, coupon period may be monthly with weekly lookups. At the end of coupon period, the paid coupon is proportional to the number of lookup times when the float rate is inside of the range specified by the barriers. After the coupon, the issuer has the right to redeem the note for the notional N.

Denote by $(u_m)_{m=1,\dots,M}$ the payment times of the note:

$$u_m = t_0 + m\delta t, \quad m = 1, \dots, M.$$

Fix a payment time u_{m+1} and denote by $(v_l)_{l=1,\dots,L}$ the lookup dates in the interval (u_m, u_{m+1}):

$$v_l = u_m + \delta t \frac{l}{L+1}, \quad l = 1, \dots, L.$$

Note that the lookup dates are strictly inside of the coupon period (u_m, u_{m+1}):

$$u_m < v_1 < \cdots < v_L < u_{m+1}.$$

At time u_{m+1}, the holder of the note receives the product of the fixed coupon $NR\delta t$ on the fraction of lookup times $(v_l)_{l=1,\dots,L}$, when Δ-period float rate was inside of the range (D, U). In other words, the payment at u_{m+1} is given by

$$NR\delta t \frac{1}{L} \sum_{l=1}^{L} 1_{\{D < r(v_l, v_l + \Delta) < U\}},$$

where $r(s, s+\Delta)$ is the market float rate computed at s for maturity $s+\Delta$.

After the coupon payment, the issuer of the note can terminate it by paying back to the holder the notional N.

If the note has not been terminated before, then at maturity u_M, in addition to the coupon payment, the holder of the note also receives the notional N.

Constant maturity swaption

T : the maturity of the option.

Parameters of underlying swap:

N : the notional.

R : the fixed rate.

δs : the interval of time between payments given as year fraction.

M : the total number of payments.

side: the side of the swap contract, that is, whether one pays "fixed" and receives "float" or otherwise.

L : the number of periods in the standard swap contract (with the same period δs) that determines the floating rate in CMS (constant maturity swap).

At maturity T, a holder of the option can enter into the underlying CMS issued at T. The difference between CMS and standard "plain vanilla" swap is in the way the floating rate is computed:

1. In CMS, the float interest is paid according to the current swap rate for L periods δs. That is, the floating rate transacted at time $T+(m+1)\delta s$ is the swap rate determined at $T+m\delta s$ for L periods δs.

2. In standard swap, the float interest is paid according to the current float rate.

American constant maturity swaption

$(t_m)_{m=1,\ldots,M}$: the exercise times.

Parameters of underlying swap:

N : the notional.

R : the fixed rate.

δs : the interval of time between the payments given as year fraction.

M : the total number of payments.

side: the side of the swap contract, i.e., whether one pays "fixed" and receives "float" or otherwise.

L : the number of periods δs that determines the floating rate in CMS.

A holder can enter into the underlying CMS at any exercise time t_m. This time then becomes the issue time of the CMS. The difference between CMS and the standard "plain vanilla" swap is in the way the floating rate is computed.

1. In CMS, the float interest at $t_{m+1} = t_m + \delta s$ is paid according to the market swap rate $r(t_m, \delta s, L)$ computed at t_m for the swap expiring after L periods of length δs.

2. In standard swap, the float interest at $t_{m+1} = t_m + \delta s$ is paid according to the float rate $r(t_m, t_m + \delta s)$ computed at t_m for period δs.