Q: Find the MLE of $\theta = (\alpha, \sigma^2)$. Approximate the distribution of $\hat{\theta}$

The $Y_1, Y_2, \cdots Y_n$ are i.i.d. $N(\Delta\mu, \Delta\sigma^2)$,

Let $\tau_1 = \Delta\mu$ and $\tau_2 = \Delta\sigma^2$, $\tau = (\tau_1, \tau_2)$

We already know that $\hat{\tau}_{MLE}$ is approximately normal with covariance

$$I^{-1}(\tau_0)/n = \begin{bmatrix} \tau_2/n & 0 \\ 0 & \frac{2\tau_2^2}{n} \end{bmatrix} \quad \text{where } \tau_2 = \sigma^2$$

To get the covariance for MLE of (α, σ^2), we can use the Δ-method

$$\alpha = g_1(\tau) = \mu + \frac{\sigma^2}{2} = \frac{\tau_1}{\Delta} + \frac{\tau_2}{2\Delta}$$

$$\sigma^2 = g_2(\tau) = \frac{\tau_2}{\Delta}$$

By invariance prop.

$$\hat{\alpha}_{MLE} = g_1(\hat{\tau}_{MLE}) = \frac{\hat{\tau}_1}{\Delta} + \frac{\hat{\tau}_2}{2\Delta}$$

$$\hat{\sigma}^2_{MLE} = \frac{\hat{\tau}_2}{\Delta}$$

The matrix G in this case: $m=2$ $p=2$, so $G_{2\times 2}$

$$G = \begin{bmatrix} \frac{\partial g_1(\tau)}{\partial \tau_1} & \frac{\partial g_1(\tau)}{\partial \tau_2} \\ \frac{\partial g_2(\tau)}{\partial \tau_1} & \frac{\partial g_2(\tau)}{\partial \tau_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\Delta} & \frac{1}{2\Delta} \\ 0 & \frac{1}{\Delta} \end{bmatrix}$$

thus $G I^{-1}(\tau) G^T / n$

$$= \begin{bmatrix} \frac{\tau_2}{n\Delta^2} + \frac{\tau_2^2}{2n\Delta^2} & \frac{\tau_2^2}{n\Delta^2} \\ \frac{\tau_2^2}{n\Delta^2} & \frac{2\tau_2^2}{n\Delta^2} \end{bmatrix}$$

note $T = n\Delta$

$\sigma^2 = \frac{\tau_2}{\Delta}$

$$= \begin{bmatrix} \frac{\sigma^2}{T} + \frac{\sigma^4}{2n} & \frac{\sigma^4}{n} \\ \frac{\sigma^4}{n} & \frac{2\sigma^4}{n} \end{bmatrix}$$

The MLE for (α, σ^2) is approximately bivariate normal distribution based on this covariance matrix

Q: What does this result suggest regarding how large n should be chosen for the best estimation of σ^2? What about for α?

$V(\hat{\sigma}^2) = \frac{2\sigma^4}{n}$ ⇒ in order to estimate σ^2 more precisely, we can sample $[0, T]$ more finitely.

But for α, its estimation is limited by T:

$V(\hat{\alpha}) = \frac{\sigma^2}{T} + \frac{\sigma^4}{2n} \approx \frac{\sigma^2}{T}$ for large n, holding T constant

so we need a larger time interval to estimate α better.

Typical value of σ is around 0.2, so if we want $SE(\hat{\alpha}) \leq 0.01$ we need $T \geq 400$ years