§ Stochastic Integration

4.2 Quadratic Variation

Def. 4.2.1. Let M be a stochastic process. Then

$$[M,M]_T(\omega) := \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} (\Delta M_i(\omega))^2$$

where $\Delta M_i := M_{t_{i+1}} - M_{t_i}$ and $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$

for Brownian motion W, $[W,W]_t = t$

Lemma. 4.2.1 $M_t = W_t^2 - t$ is a martingale

proof:
- Adapted: M_t is $\mathcal{F}_t$-measurable because W_t is $\mathcal{F}_t$-measurable and M_t is a function of W_t and t.
- Martingale property: check $E[M_t \mid \mathcal{F}_s] = M_s \ \forall t \geq s$

$$\begin{aligned}
E[M_t \mid \mathcal{F}_s] &= E[W_t^2 - t \mid \mathcal{F}_s] \\
&= E[(W_t - W_s + W_s)^2 - t \mid \mathcal{F}_s] \\
&= E[(W_t - W_s)^2 + 2(W_t - W_s)W_s + W_s^2 - t \mid \mathcal{F}_s] \\
&= E[\underbrace{(W_t - W_s)^2}_{\text{ind of } \mathcal{F}_s} \mid \mathcal{F}_s] + E[2\underbrace{(W_t - W_s)}_{\text{ind of } \mathcal{F}_s} \cdot \underbrace{W_s}_{\mathcal{F}_s\text{-measurable}} \mid \mathcal{F}_s] \\
&\quad + E[\underbrace{W_s^2}_{\mathcal{F}_s\text{-measurable}} \mid \mathcal{F}_s] - E[\underbrace{t}_{\mathcal{F}_s\text{-measurable}} \mid \mathcal{F}_s] \\
&= E[(W_t - W_s)^2] + 2W_s E[W_t - W_s] \\
&\quad + W_s^2 - t \\
&= t - s + W_s^2 - t \\
&= W_s^2 - s \\
&= M_s
\end{aligned}$$

Theorem 4.2.1 Let M be a martingale w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$

Then for all $t\geq 0$, $E[M_t^2]<\infty \Leftrightarrow E[[M,M]_t]<\infty$

In this case, $(M_t^2-[M,M]_t)_{t\geq 0}$ is a martingale w.r.t. the same filtration, and $E[M_t^2]-E[M_0^2]=E[[M,M]_t]$

Example: $M=W$ is a martingale $\Rightarrow (W_t^2-t)_{t\geq 0}$ is a martingale

Theorem 4.2.2. If M is a martingale and A_t is a continuous adapted increasing stochastic process s.t. $A_0=0$ and $(M_t^2-A_t)_{t\geq 0}$ is a martingale, then

$$A_t=[M,M]_t$$ (proof is omitted)

Example: $M=W$, $A_t=t$

A_t is continuous, adapted, $\mathcal{F}_0$-measurable, increasing, and $A_0=0$

$M_t^2-A_t=W_t^2-t$ is a martingale $\Rightarrow A_t=[W,W]_t$

Remark:

- intuition for first variation and quadratic variation:
  divide the interval $[0,T]$ into $T/(\delta t)$ intervals of size δt.
  if X has finite first variation, then on each subinterval $(k\delta t,(k+1)\delta t)$ the increment of X should be of order δt.
  Similarly, X has finite quadratic variation $\Rightarrow$ increment $\propto \sqrt{\delta t}$
- If a continuous process has finite first variation, its quadratic variation will necessarily be zero.
  If a continuous process has finite and non-zero quadratic variation, its first variation will necessarily be infinite

4.3. Construction of Itô integral.

Let W be a standard BM, $(\mathcal{F}_t)_{t\geq 0}$ be the Brownian filtration and D be an adapted process.

Let $(\Delta_t)_{t\geq 0}$ be our position in S at time t, that is invest $\Delta_t S_t$ at time t and the value of portfolio time $t+1$ is $\Delta_t \cdot S_{t+1}$

however, almost any continuous martingale S will not have finite first variation, thus we need the Itô integral

$$\Rightarrow \quad PnL = \sum_{i=0}^{n-1} \Delta_i (S_{i+1} - S_i) \xrightarrow{n\to\infty} \int \Delta_t \cdot dS_t$$

lemma 4.3.1 let $\Pi = \{0 = t_0 < t_1 < \cdots < t_n\}$ be an increasing sequence of times and assume that D is constant on $[t_i, t_{i+1})$ $\forall i$ (i.e. the asset is only traded at time $t_0, \cdots t_n$)

let
$$I_T^\Pi = \sum_{i=0}^{n-1} D_{t_i} \Delta W_i + D_{t_n}(W_T - W_{t_n}) \quad \text{if } T \in [t_n, t_{n+1})$$

where $\Delta W_i = W_{t_{i+1}} - W_{t_i}$

denote the cumulative earnings up to time T, then

$$E[(I_T^\Pi)^2] = E\left[\sum_{i=0}^{n-1} D_{t_i}^2 (t_{i+1} - t_i) + D_{t_n}^2 (T - t_n)\right] \text{ if } T \in [t_n, t_{n+1})$$

Moreover, I^Π is a martingale and

$$[I^\Pi, I^\Pi]_T = \sum_{i=0}^{n-1} D_{t_i}^2 (t_{i+1} - t_i) + D_{t_n}^2 (T - t_n) \text{ if } T \in [t_n, t_{n+1})$$

Theorem 4.3.1 If $\int_0^T D_t^2 dt < \infty$, then as $\|\Pi\| \to 0$, the process I^Π converge to a cost process I given by

$$I_T := \lim_{\|\Pi\|\to 0} I_T^\Pi = \int_0^T D_t \, dW_t$$

Note that D should be adapted, and is sampled at the left endpoint of the time interval, i.e. terms in the sum are $D_{t_i}(W_{t_{i+1}} - W_{t_i})$

This is called the Itô integral of D w.r.t. W.

If further, $E[\int_0^T D_t^2\,dt]<\infty$, then the process I is a martingale and the quadratic variation $[I,I]$ satisfies $[I,I]_T=\int_0^T D_t^2\,dt$ almost surely. Besides, $E[\int_0^T D_t\,dW_t]=0$

properity. 4.3.1 (linearity)

If D^1, D^2 are two adapted process, $\lambda\in\mathbb{R}$, then

$$\int_0^T (D_t^1+\lambda D_t^2)\,dW_t=\int_0^T D_t^1\,dW_t+\lambda\int_0^T D_t^2\,dW_t$$

(Itô Isometry)

If $E[\int_0^T D_t^2\,dt]<\infty$, then

$$E[(\int_0^T D_t\,dW_t)^2]=E[\int_0^T D_t^2\,dt]$$

Example: $D_t=1$, then

$$E[(\underbrace{\int_0^T D_t\,dW_t}_{W_T-W_0})^2]=E[\underbrace{\int_0^T 1^2\,dt}_{T}]$$

$$\Rightarrow\quad E[W_T^2]=T$$

Remark:

positivity is not preserved by Itô integrals. Namely, if $D^1\le D^2$, there is no reason to expect $\int_0^T D_t^1\,dW_t\le\int_0^T D_t^2\,dW_t$.

Def (GBM). We define Geometric Brownian Motion S as

$$dS_t=\sigma S_t\,dW_t+\lambda S_t\,dt\quad,\ \sigma,\lambda\in\mathbb{R}$$

$\Rightarrow$ stochastic differential equation (SDE)

$$\int_0^T 1\,dS_t=\int_0^T \sigma S_t\,dW_t+\int_0^T\lambda S_t\,dt$$

$$\Rightarrow\quad S_T-S_0=\int_0^T\sigma S_t\,dW_t+\int_0^T\lambda S_t\,dt$$

4.4. Itô formula

Goal: Compute $\int_0^T W_t \, dW_t = ?$

Def 4.4.1 Let b, σ be adapted process. Then a process X defined as

Riemann integral Itô integral

$$X_T = X_0 + \int_0^T b_t \, dt + \int_0^T \sigma_t \, dW_t \qquad X_0 \in \mathbb{R}$$

is called an Itô process if X_0 is deterministic (not random) and for all $T \geq 0$ $E\left[\int_0^T \sigma_t^2 \, dt\right] < \infty$ and $\int_0^T |b_t| \, dt < \infty$

Remark:

the equation above is equivalent to

$$dX_t = b_t \, dt + \sigma_t \, dW_t$$

properity 4.4.1 The quadratic variation of X is

$$[X, X]_T = \int_0^T \sigma_t^2 \, dt$$

Def 4.4.2. (semi-MG) Process X which can be decomposed as $X = A + M$ where M is a martingale and A has finite (first) variation are called semi-martingale.

- ▷ M is called the martingale part of X
- ▷ A is called the finite variation part of X

Prop 4.4.2 The semi-MG decomposition is unique, that is

if $X = A_1 + M_1 = A_2 + M_2$ (*), then

$$A_1 = A_2, \quad M_1 = M_2$$

(where A_1, A_2 are finite variation processes, M_1, M_2 are martingales) proof: by (*) we have

$$\underbrace{A_1 - A_2}_{\text{finite variation}} = \underbrace{M_1 - M_2}_{MG} := M$$

thus M is a MG with finite variation $\Rightarrow [M,M]_T = 0$

$$\overset{4.2.1}{\Longrightarrow} E[M_t^2] = E[[M,M]_T] = 0$$

($M^2 - [M,M]_t$ is a M.G.) $\Rightarrow$ $M_t^2 \geq 0$ thus $M_t = 0$, that is $A_1 = A_2$, $M_1 = M_2$

Prop 4.4.3 Let X be an Itô process, then

X is a MG $\Leftrightarrow b_t = 0 \ \forall t \geq 0$ (i.e. $X_T = X_0 + \int_0^T \sigma_t \, dW_t$)

Proof: If $b_t = 0 \ \forall t \geq 0 \Rightarrow X$ is a M.G.

Suppose X is a M.G. Define

$$A_T := \int_0^T b_t \, dt = \underbrace{X_T - X_0}_{M.G.} - \underbrace{\int_0^T \sigma_t \, dW_t}_{M.G.}$$

$\Rightarrow A$ is a martingale. (also a semi-MG.)

$$A = A + 0 = 0 + A \quad \text{thus } A = 0 \Leftrightarrow b_t = 0$$

Def 4.4.3. We define the integral of D w.r.t. X by

$$\int_0^T D_t \, dX_t := \int_0^T D_t \cdot b_t \, dt + \int_0^T D_t \sigma_t \, dW_t$$

where $dX_t = b_t \, dt + \sigma_t \, dW_t$

[Given an adapted process D, we interpret X as the price of an asset, and D as our position in it, which can be positive or negative).

Theorem 4.4 (Itô formula) → Recall that if $f: \mathbb{R} \to \mathbb{R}$ is continuously differentiable $f(y) - f(x) = \int_x^y \frac{\partial f}{\partial x}(z) \, dz$

If $f: [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is such that

$t \mapsto f(t,x)$ is cont. differentiable $\forall x \in \mathbb{R}$ $\partial_t f(t,x)$ exists

$x \mapsto f(t,x)$ is twice cont. differentiable. $\forall t \in \mathbb{R}$. $\partial_x f(t,x)$, $\partial_x^2 f(t,x)$ exist and if X is an Itô process, then

$$f(T, X_T) - f(0, X_0) = \int_0^T \partial_t f(t, X_t)\,dt + \int_0^T \partial_x f(t, X_t)\,dX_t + \underbrace{\frac{1}{2}\int_0^T \partial_x^2 f(t, X_t)\,d[X, X]_t}_{\text{Itô correction term}}$$

Remark:

- $\partial_t f(t, X_t)$ stands for taking derivative of $f(t, X_t)$ w.r.t. t and then substitute X_t. Similar for $\partial_x f(t, X_t)$, $\partial_x^2 f(t, X_t)$
- The Itô formula is simply a version of the chain rule for stochastic processes.

Stochastic form:

$$df(t, X_t) = \partial_t f(t, X_t)\,dt + \partial_x f(t, X_t)\,dX_t + \frac{1}{2}\partial_x^2 f(t, X_t)\,d[X, X]_t$$

Substitute (the Itô process) $\begin{cases} dX_t = b_t\,dt + \sigma_t\,dW_t \\ d[X, X]_t = \sigma_t^2\,dt \end{cases}$ we have

$$\Rightarrow df(t, X_t) = \partial_t f(t, X_t)\,dt + \partial_x f(t, X_t)\,[b_t\,dt + \sigma_t\,dW_t] + \frac{1}{2}\partial_x^2 f(t, X_t)\cdot\sigma_t^2\,dt$$

$$= [\partial_t f(t, X_t) + \partial_x f(t, X_t)\cdot b_t + \frac{1}{2}\partial_x^2 f(t, X_t)\sigma_t^2]\,dt + \partial_x f(t, X_t)\cdot\sigma_t\,dW_t$$

if the term before dt is zero, $f(t, X_t)$ is a martingale, otherwise not

4.5 Examples

ex 1. write W^2 as a sum of dt and dW_t - integral

use the Itô formula and substitue $f(t,X) = X^2$, $X = W_t$

$\Rightarrow \partial_t f(t,x) = 0$, $\partial_x f(t,x) = 2x$. $\partial_x^2 f(t,x) = 2$

$\Rightarrow [W,W]_t = t$

By Itô formula

$$d(W_t^2) = d(f(t,W_t)) = \underbrace{\partial_t f(t,W_t)}_{\nearrow 0} dt + \underbrace{\partial_x f(t,W_t)}_{\nearrow 2W_t} dW_t$$

$$+ \frac{1}{2} \underbrace{\partial_x^2 f(t,W_t)}_{\hookrightarrow 2} d\underbrace{[W,W]_t}_{t}$$

$$= 2W_t dW_t + dt$$

calculating the integral from 0 to T

$$W_T^2 - 0 = \int_0^T 2W_t dW_t + \int_0^T dt = \int_0^T 2W_t dW_t + T$$

$$\Rightarrow \int_0^T W_t dW_t = \frac{1}{2}(W_T^2 - T)$$

Q: calculate $[W^2, W^2]_T$

$$[W^2, W^2]_T = \int_0^T (2W_t)^2 dt = 4\int_0^T W_t^2 dt$$

↳ not constant.
a r.v.

ex.2. Let $M_t = W_t$, $N_t = W_t^2 - t$, is $M \cdot N$ a M.G ?

first method is to verify $E[M_t \cdot N_t \mid \mathcal{F}_s] = M_s \cdot N_s \ \forall t \geq s$

another method is to use the Itô formula.

Note $M_t \cdot N_t = W_t(W_t^2 - t) = W_t^3 - W_t \cdot t$

$f(t,x) = x^3 - tx$ $X_t = W_t$ $[x,x]_t = t$

$\Rightarrow \partial_x f(t,x) = 3x^2 - t \quad \partial_t f(t,x) = -x \quad \partial_x^2 f(t,x) = 6x$

$$\Rightarrow df(t,x) = \partial_t f(t,X_t)dt + \partial_x f(t,X_t)dX_t + \frac{1}{2}\partial_x^2 f(t,X_t)d[X,X]_t$$

$$= -X dt + (3X^2 - t)dX_t + 3X dt$$

$$= 2X dt + (3X^2 - t)dX_t$$

$$= 2W_t dt + (3W_t^2 - t)dW_t$$

$$M_T N_T = \int_0^T 2W_t dt + \int_0^T (3W_t^2 - t)dW_t$$

Since the finite (first) Variation part is not zero,

$M_T N_T$ is not a martingale.

ex3. Let $X_t = t\sin(W_t)$. Is $X^2 - [X,X]$ a martingale?

let $f(t,x) = t\sin(x) \quad \partial_t f = \sin x \quad \partial_x f = t\cos x$

$\partial_x^2 f = -t\sin x$

$$\Rightarrow dX_t = \sin W_t dt + t\cos W_t dW_t - \frac{1}{2}t\sin W_t dt$$

$$= \left(\sin W_t - \frac{1}{2}t\sin W_t\right)dt + t\cos W_t dW_t$$

(then, $d[X,X] = t^2\cos^2 W_t dt$)

$$dX_t^2 = 2X_t dX_t + d[X,X]_t$$

$$\Rightarrow d[X^2 - [X,X]] = 2X_t dX_t$$

$$= 2t\sin(W_t)\cdot\left[\sin(W_t) - \frac{1}{2}t\sin(W_t)\right]dt + 2t^2\sin(W_t)\cdot\cos(W_t)\cdot dW_t$$

Since the dt term above is not 0, thus $X^2 - [X,X]$ is not a martingale

4.6.1 Multidimensional Itô

Def 4.6.1 (quadratic covariation)

Let X, Y be two Itô process. Define

$$[X,Y]_T = \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} (X_{i+1} - X_i)(Y_{i+1} - Y_i)$$

$$\Pi = \{0 = t_0 < \cdots < t_n = T\}$$

the quadratic covariation

lemma: $\forall a, b \in \mathbb{R}$ $\quad ab = \frac{1}{4}[(a+b)^2 - (a-b)^2]$

$\Rightarrow$ let $a = (X_{i+1} - X_i)$ $\quad b = (Y_{i+1} - Y_i)$

thus $[X,Y]_T = \frac{1}{4}[[X+Y, X+Y]_T - [X-Y, X-Y]_T]$

Prop 4.6.1 (Product Rule) For Itô processes X, Y, we have

$$d(XY) = XdY + YdX + d[X,Y]$$

Prop 4.6.2 If X is an Itô process and A is adapted process of finite variation, then $[X, A] = 0$

(Note that $[X \pm A, X \pm A] = [X, X]$)

Prop 4.6.4. If X, Y, Z are Itô processes and $\alpha \in \mathbb{R}$, then

(Bi-linearity) $[X, Y + \alpha Z] = [X, Y] + \alpha [X, Z]$

prop 4.6.6 Let X, Y be two continuous martingales (e.g. Itô processes) w.r.t. a common filtration $(\mathcal{F}_t)_{t \geq 0}$ such that $E[X_t^2] < \infty$ and $E[Y_t^2] < \infty$, if X, Y are independent, then $[X, Y] = 0$

the converse is not true, for example $X_t = \int_0^t 1_{\{W_s \geq 0\}} dW_s$

$Y_t = \int_0^t 1_{\{W_s < 0\}} dW_s$, $[X, Y] = 0$ but $X \not\perp Y$

Theorem 4.6.1 Multidimensional Itô process

Let $X^1, X^2, \cdots, X^n$ be Itô processes and $X = (X^1, X^2, \cdots X^n)$

Let $f: [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}$ $\quad (t, x) \to f(t, x)$

be C^1 in t ($\partial_t f$ exists)

be C^2 in X^i ($\partial_{x_i} f = \partial_i f$, $\partial_{x_i}^2 f = \partial_i^2 f$ exist)

Then

$$f(T, X_T) = f(0, X_0) + \int_0^T \partial_t f(t, X_t) dt + \sum_{i=1}^n \int_0^T \partial_i f(t, X_t) dX_t^i$$
$$+ \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \int_0^T \partial_i \partial_j f(t, X_t) d[X^i, X^j]_t$$

or

$$df(t, X_t) = \partial_t f(t, X_t) dt + \sum_{i=0}^n \partial_i f(t, X_t) dX_t^i$$
$$+ \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \partial_i \partial_j f(t, X_t) d[X^i, X^j]_t$$

Remark: we most often use the two dimensions case

$$df(t, X_t, Y_t) = \partial_t f dt + \partial_x f dX_t + \partial_Y f dY_t$$
$$+ \frac{1}{2} [\partial_x^2 f d[X, X]_t + 2 \partial_x \partial_y f d[X, Y]_t + \partial_y^2 f d[Y, Y]_t$$

Prop 4.6.3. Let M, N be Martingales. Then

(1) $M \cdot N - [M, N]$ is a MG

(when $M = N$, $M^2 - [M, M]$ is a MG, which is mentioned before)

(2) If A is adapted process of finite variation such that $A_0 = 0$ and $MN - A$ is a MG, then $A = [M, N]$

Example: $M = W$, $N = -W$ $\quad [W, -W]_t = -[W, W]_t = -t$

(Typically, M, N are MG $\not\Rightarrow$ $M \cdot N$ is MG, eg $M = N = W$)

Prop 4.6.5. let X^1, X^2 be Itô processes, σ^1, σ^2 be adapted processes and $I_t^j = \int_0^t \sigma_s^j \, dX_s^j \quad j=1,2.$

Then $[I^1, I^2]_t = \int_0^t \sigma_s^1 \sigma_s^2 \, d[X^1, X^2]_s$

Def 4.6.2. We say that $W = (W^1, W^2, \cdots W^n)$ is a n-dimensional standard BM if

① each W^j is a standard BM $\forall j = 1, \cdots n$

② $\forall i \neq j$ W^i and W^j are independent.

Example.

- for a 2-dimensional sBM $W = (W^1, W^2)$

$[W^1, W^2]_t = 0 \qquad [W^1, W^1]_t = [W^2, W^2]_t = t$

$$df(W^1, W^2) = \partial_1 f(W^1, W^2)\, dW^1 + \partial_2 f(W^1, W^2)\, dW^2 + \frac{1}{2}\left[\partial_1^2 f(W^1, W^2)\, dt + \partial_2^2 f(W^1, W^2)\, dt\right]$$

Theorem (Lévy)

Assume that M is a MG, $M_0 = 0$ and

$$d[M^i, M^j]_t = \begin{cases} dt & i = j \\ 0 & i \neq j \end{cases}$$

Then M is n-dimensional Brownian motion

Remark:

$$\underbrace{\mathrm{Cov}(X_t, Y_t)}_{\text{a scalar}} \neq \underbrace{[X, Y]_t}_{\text{a random variable}}$$