Lecture 8

Understanding EM using Resistor-inductor (RL) circuit

GEOL 4397: Electromagnetic Methods for Exploration
GEOL 6398: Special Problems

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Sept. 18th/20th, 2018

UNIVERSITY of HOUSTON
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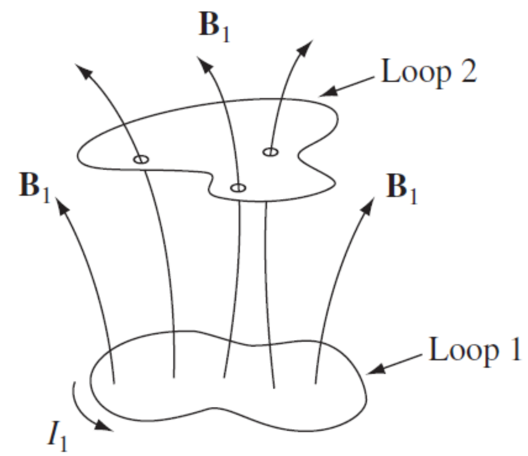
EARTH AND ATMOSPHERIC SCIENCES

Outline

- Inductance
- RL circuit under DC
- RL circuit under AC
- Understanding frequency domain EM using RL circuit
- Understanding time domain EM using RL circuit

Inductance

- Imagine two loops of wire at rest
- If we run a steady current I_1 around loop 1, it produces a magnetic field $\boldsymbol{B}_1$



Griffiths, 4th edition, pp 322

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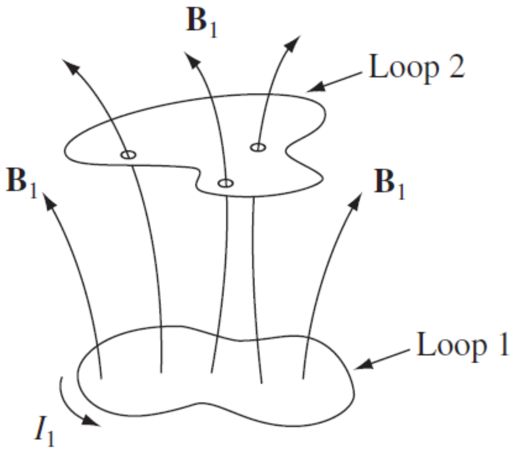

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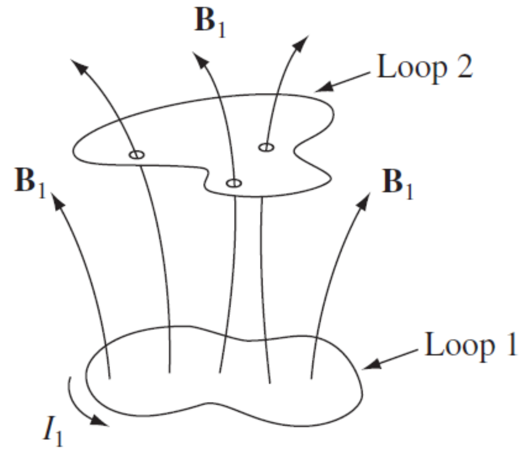

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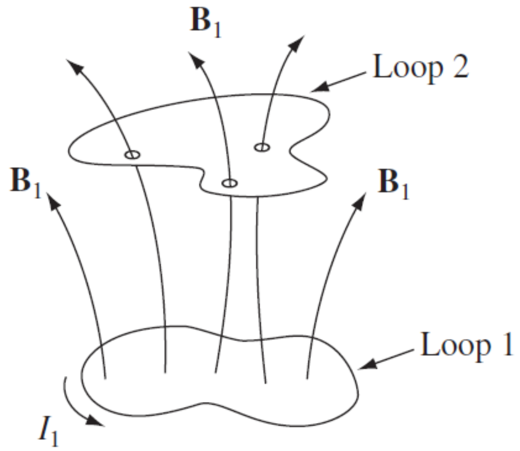

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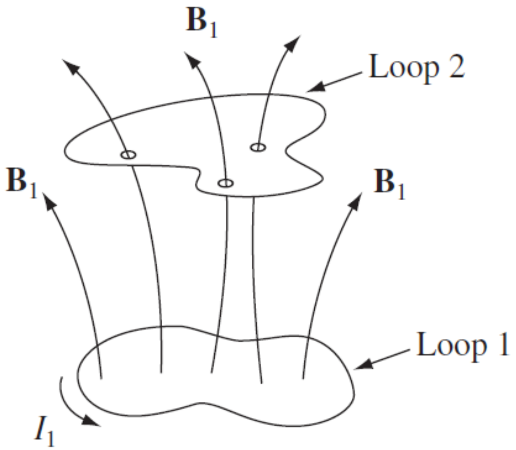

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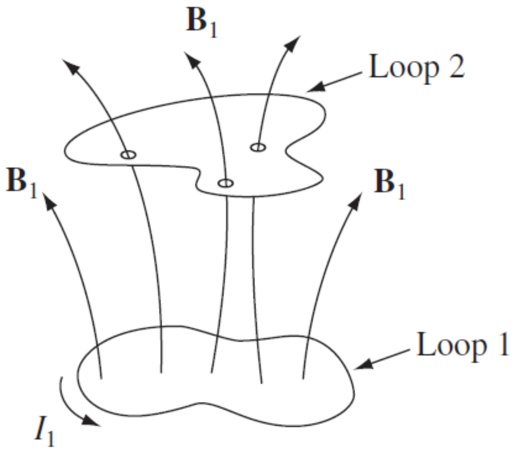

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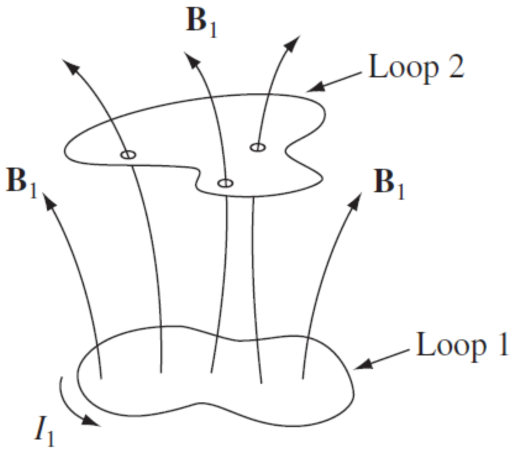

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- Therefore,

$$\Phi_2 = M_{21} I_1$$

where M_{21} is the constant of proportionality known as the mutual inductance of the two loops

Griffiths, 4th edition, pp 322

Mutual inductance (optional)

- We derive a formula for M_{21} using vector potential and Stokes' theorem

$$\Phi_2 = \int \boldsymbol{B}_1 \cdot d\boldsymbol{a}_2 = \int (\nabla \times \boldsymbol{A}_1) \cdot d\boldsymbol{a}_2 = \oint \boldsymbol{A}_1 \cdot d\boldsymbol{l}_2$$

- According to eq 5.66 in Griffiths' book (fourth edition)

$$\boldsymbol{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\boldsymbol{l}_1}{r}$$

- Therefore,

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \left(\oint \frac{d\boldsymbol{l}_1}{r} \right) \cdot d\boldsymbol{l}_2$$

- Therefore,

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\boldsymbol{l}_1 \cdot d\boldsymbol{l}_2}{r}$$

Griffiths, 4th edition, pp 322

Mutual inductance

Observations:

- M_{21} is a purely geometrical quantity, having to do with the sizes, shapes, and relative positions of the two loops

- $M_{21} = M_{12}$

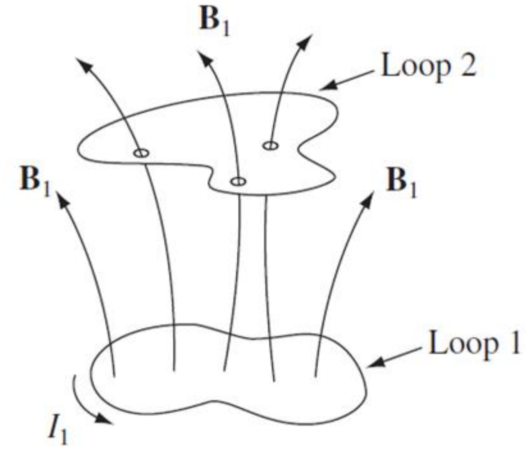
  Whatever the shapes and positions of the loops, the flux through loop 2 due to current I in loop 1 is identical to the flux through loop 1 due to current I in loop 2.

  We can then drop the subscripts, and call them both M.

Griffiths, 4th edition, pp 323

EMF

- If we vary the current in loop 1, the flux through loop 2 will vary accordingly.



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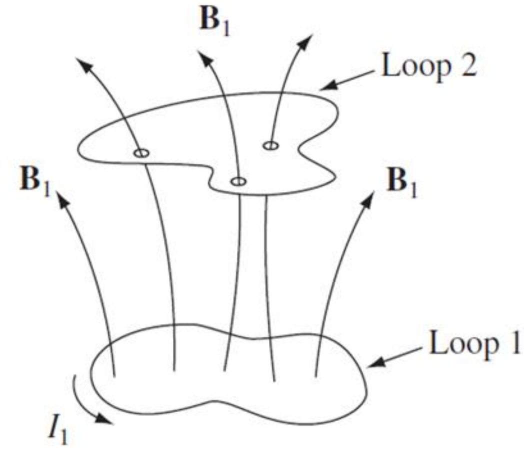

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- According to Faraday's law, the changing flux will induce an emf in loop 2

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Griffiths, 4th edition, pp 324

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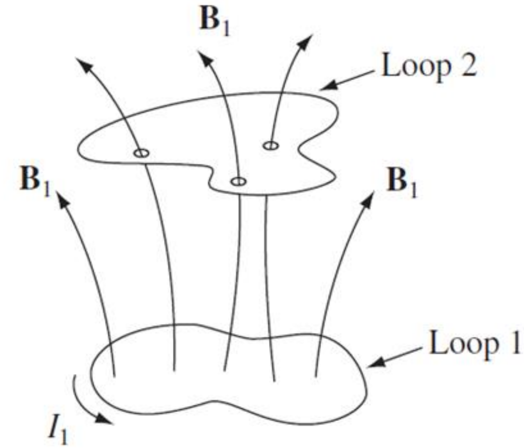

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- Here we assume that the currents change slowly enough that the system can be considered quasi-static, and Biot-Savart law still holds.
- This is the EMF induced in one circuit by a current flowing in another circuit (amazing!).

Griffiths, 4th edition, pp 324

Self inductance

- Changing current in loop 1 not only induces an emf in loop 2, but also induces an emf in itself.

Griffiths, 4^{th} edition, pp 324

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- Changing current in loop 1 not only induces an emf in loop 2, but also induces an emf in itself.
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- If the current changes, the emf induced in the loop is

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Back EMF

- The minus sign in the previous equation reflects Lenz's law, which says that the emf is in such a direction as to oppose any change in current.

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- Therefore, it is also called back emf.
- Whenever you try to alter the current in a wire, you must fight against this back emf.
- Inductance plays somewhat the same role in electric circuits that mass plays in mechanical system: The greater L is, the harder it is to change the current, just as the larger the mass, the harder it is to change an object's velocity.

Griffiths, 4th edition, pp 325

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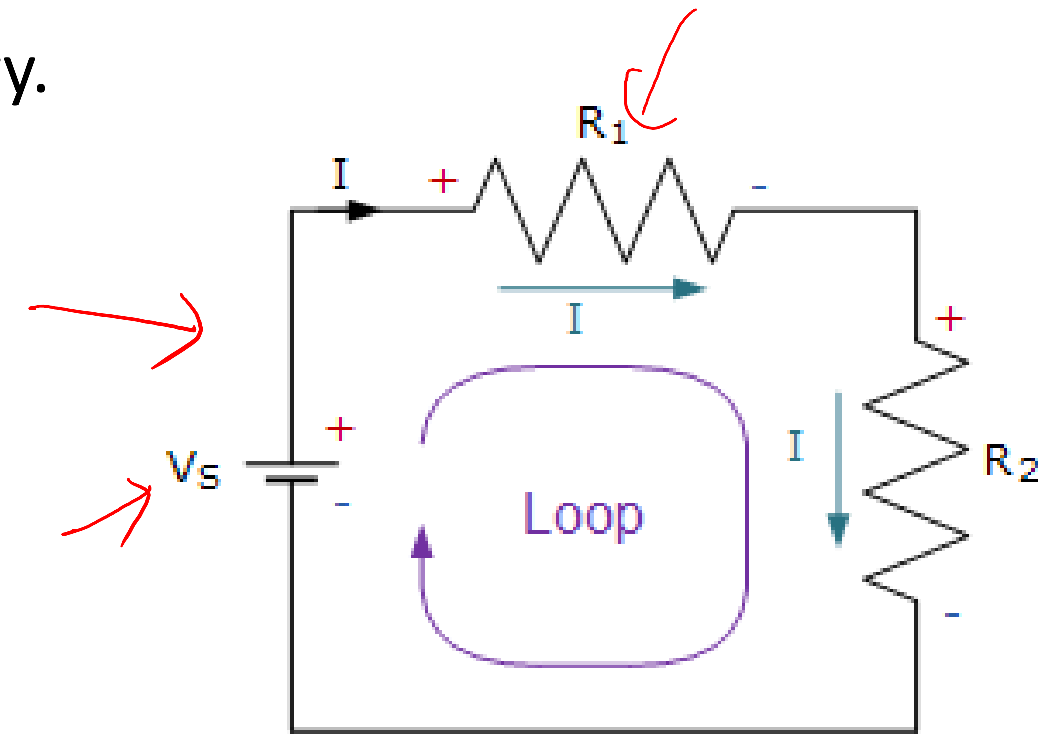
Kirchhoff's voltage law (KVL)

- The algebraic sum of all the voltages (or, potential differences) around any close loop in a circuit is 0.
- Conservation of energy.



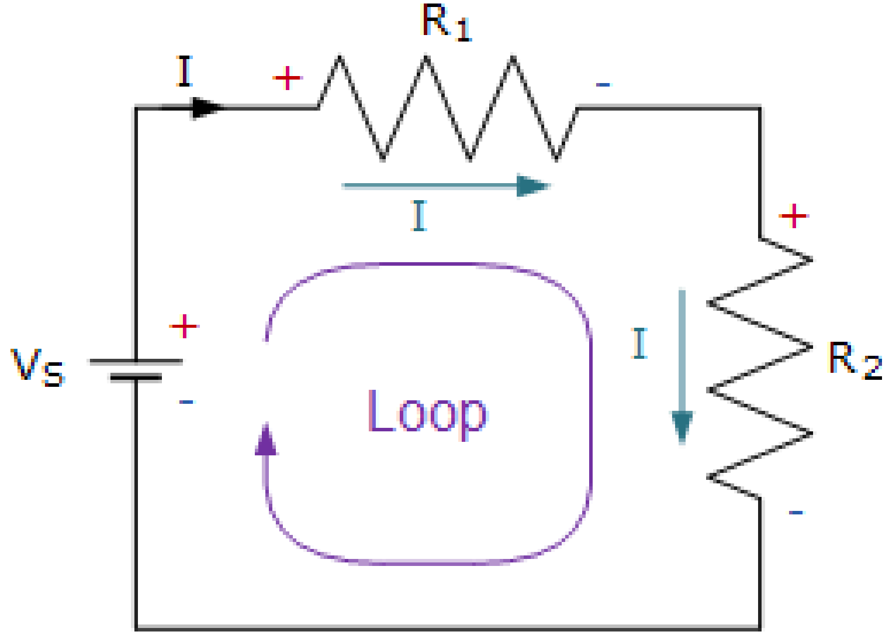
https://www.electronics-tutorials.ws/dccircuits/kirchhoffs-voltage-law.html

Kirchhoff's voltage law (KVL)

- The algebraic sum of all the voltages (or, potential differences) around any close loop in a circuit is 0.
- Conservation of energy.

- Current must be the same because of series connection
- The voltage drop across R_1 : IR_1
- The voltage drop across R_2 : IR_2
- Suppose the voltage from the battery is V_s
- Then,

$$V_s - IR_1 - IR_2 = 0$$



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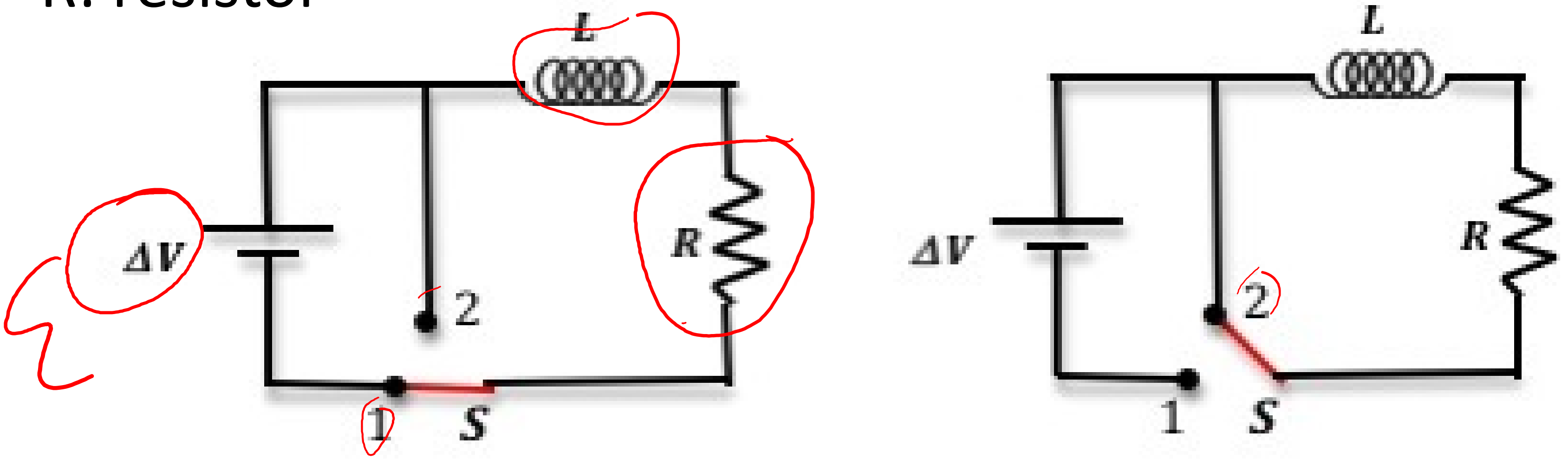
DC RL circuit

L: inductor

R: resistor

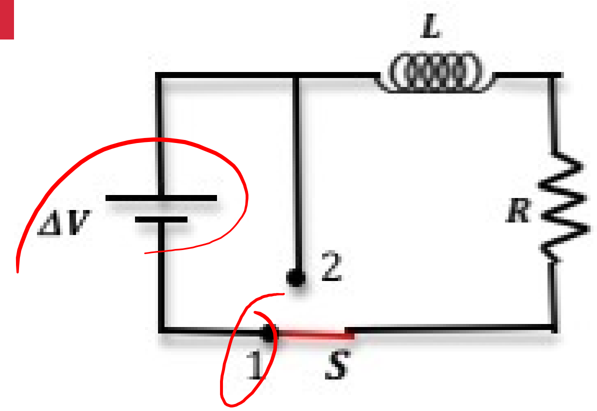


(a) (b)

Inductance: the tendency of a circuit to oppose any changes in the current (or magnetic flux).

http://www.webassign.net/labsgraceperiod/ncsulcpem2/lab_7/manual.html

DC RL circuit: energizing



- From a physical point of view, what physical phenomena would happen if the switch is closed?

http://www.physics.louisville.edu/cldavis/phys299/notes/mag_LR.html

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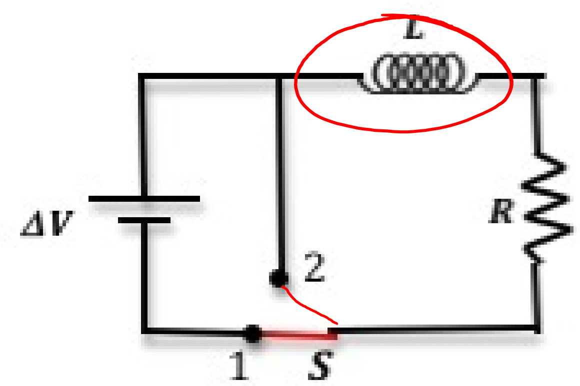

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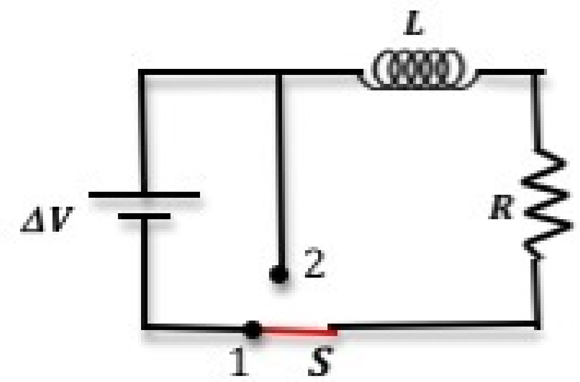

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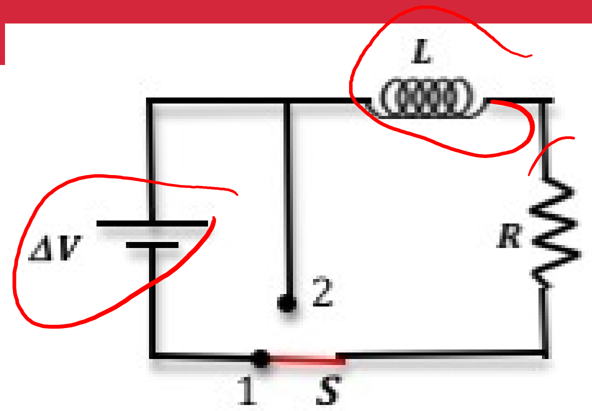

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- Now, let us write out the Kirchhoff's loop equation.

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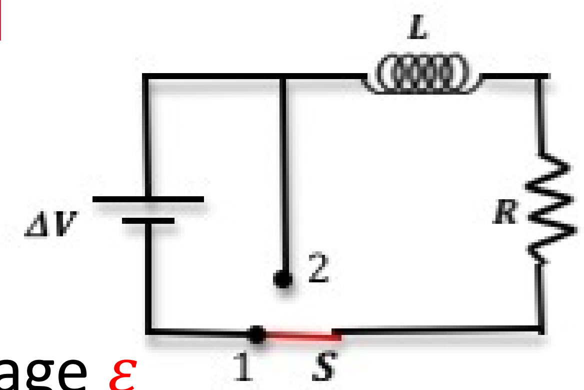

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Griffiths, 4th edition, pp 326

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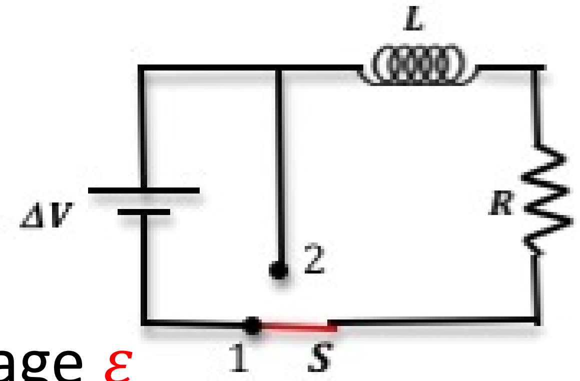

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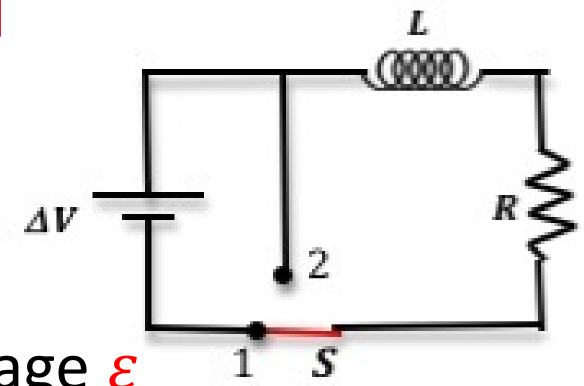

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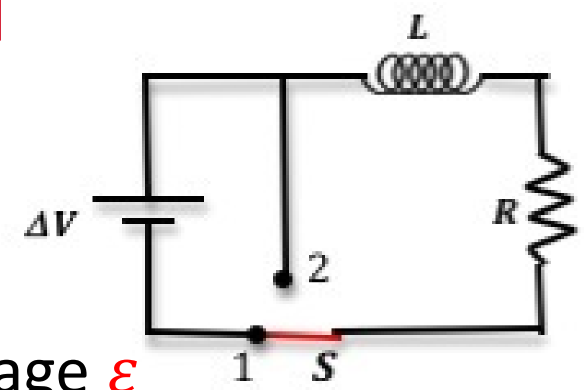

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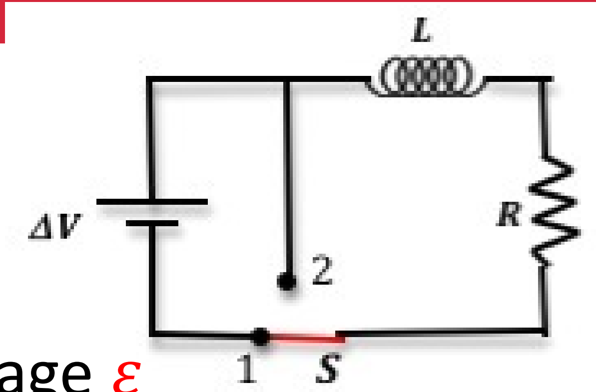

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- Solving the above differential equation, we obtain

$$I(t) = \frac{\varepsilon}{R}\left[1 - e^{-t/\tau}\right]$$

where $\tau = \frac{L}{R}$ known as time constant

Griffiths, 4th edition, pp 326

http://www.physics.louisville.edu/cldavis/phys299/notes/mag_LR.html

In-class exercise

- Verify the solution

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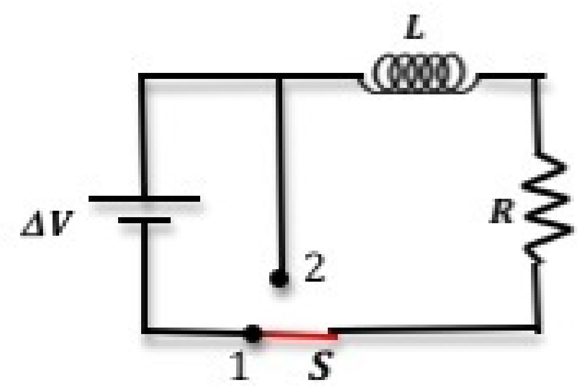

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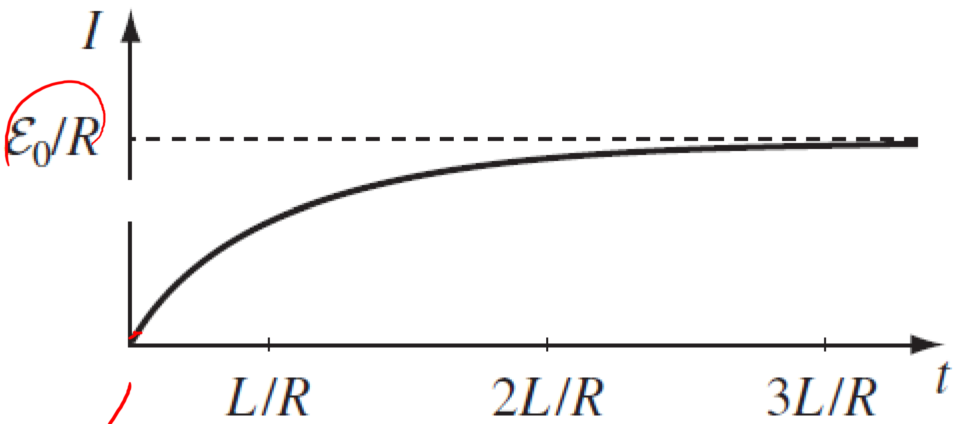

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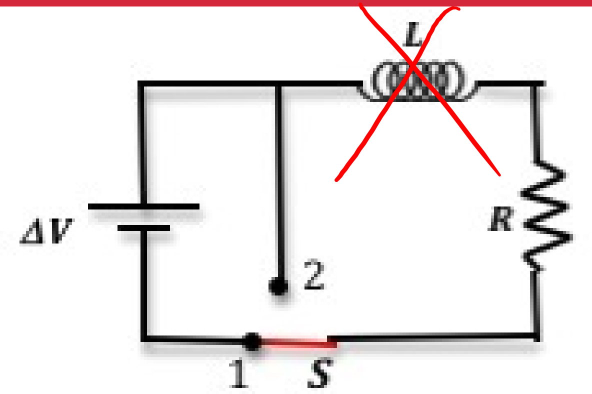

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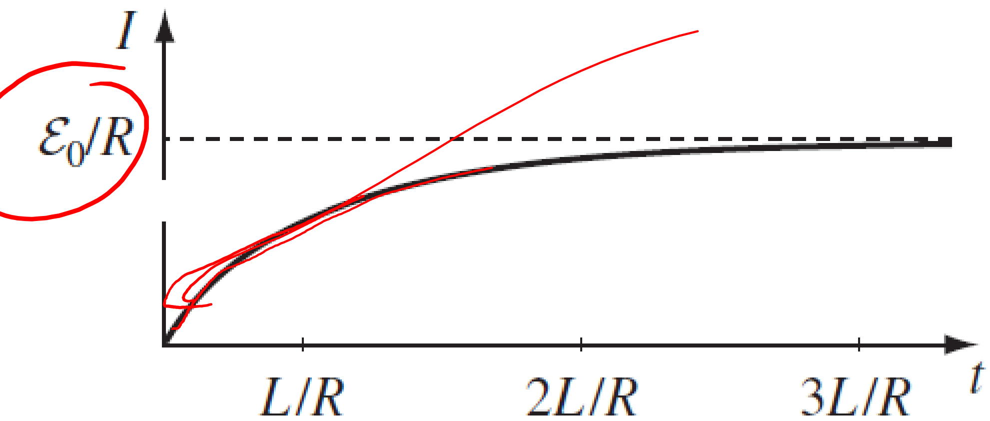

- Had there been no inductance in the circuit, the current would have jumped immediately to ε/R
- In practice, every circuit has some self-inductance
- Time constant tells you how long it takes for the current to reach a substantial fraction (roughly two thirds) of its final value.

Griffiths, 4th edition, pp 326

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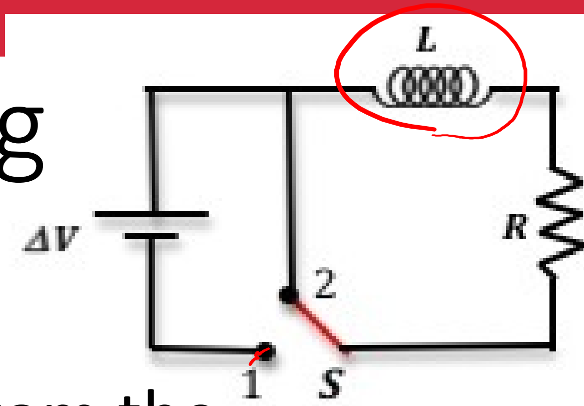
DC RL circuit: de-energizing



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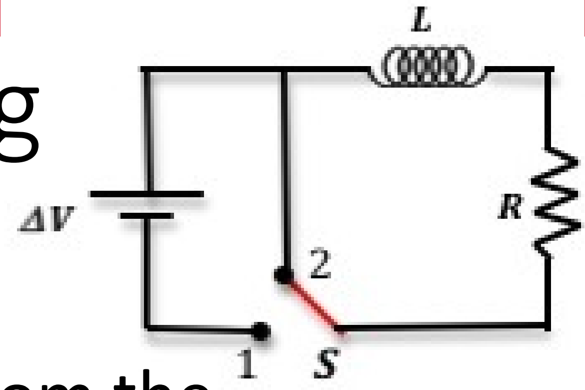

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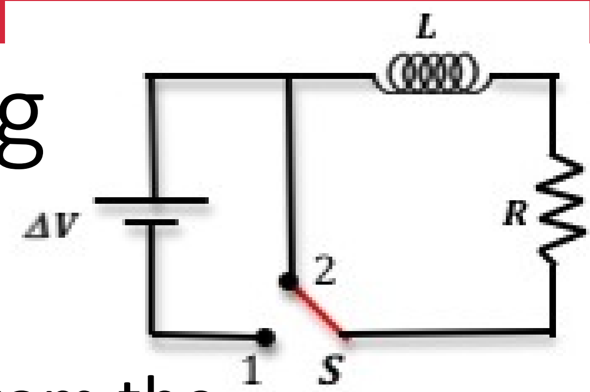

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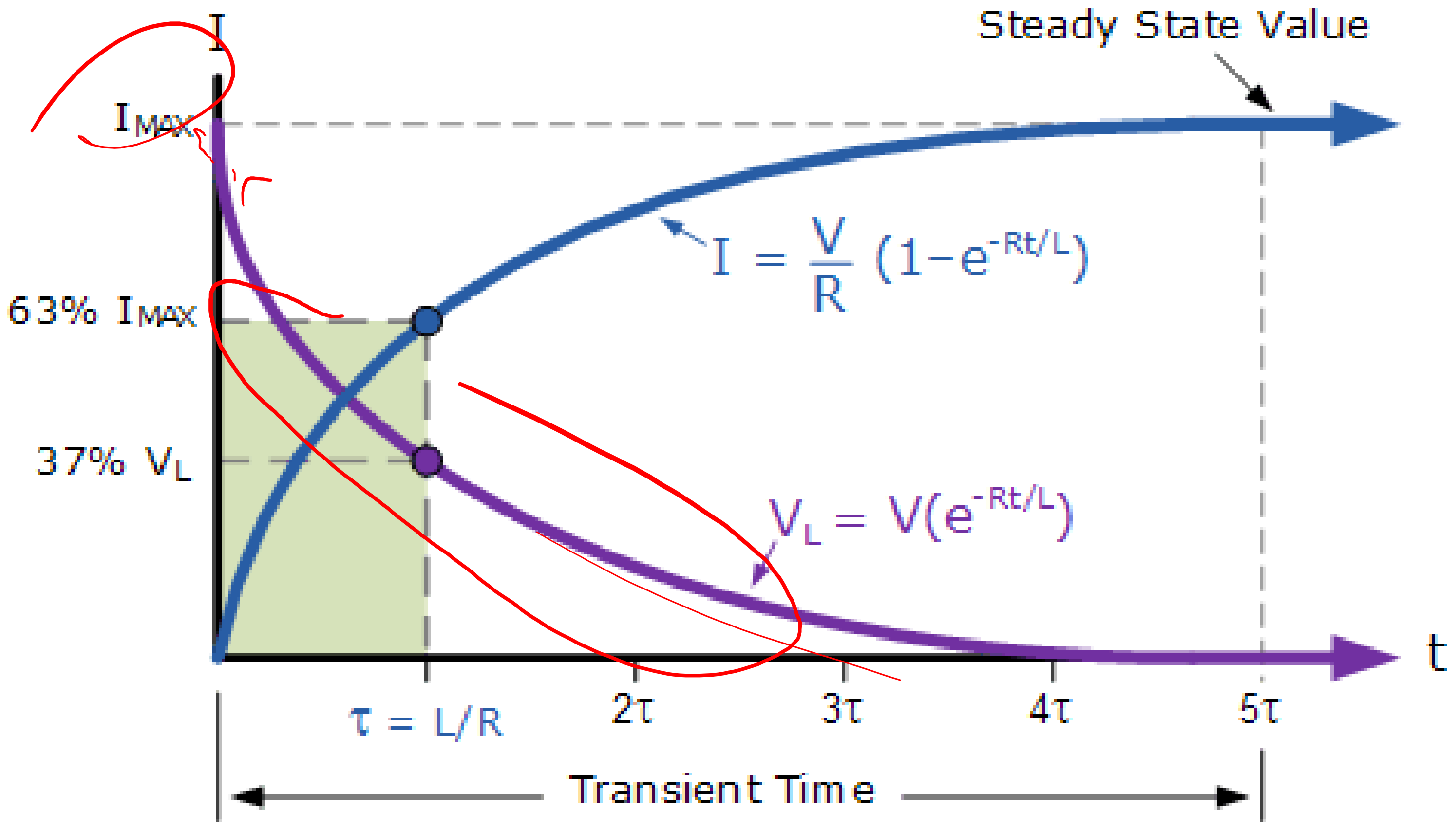

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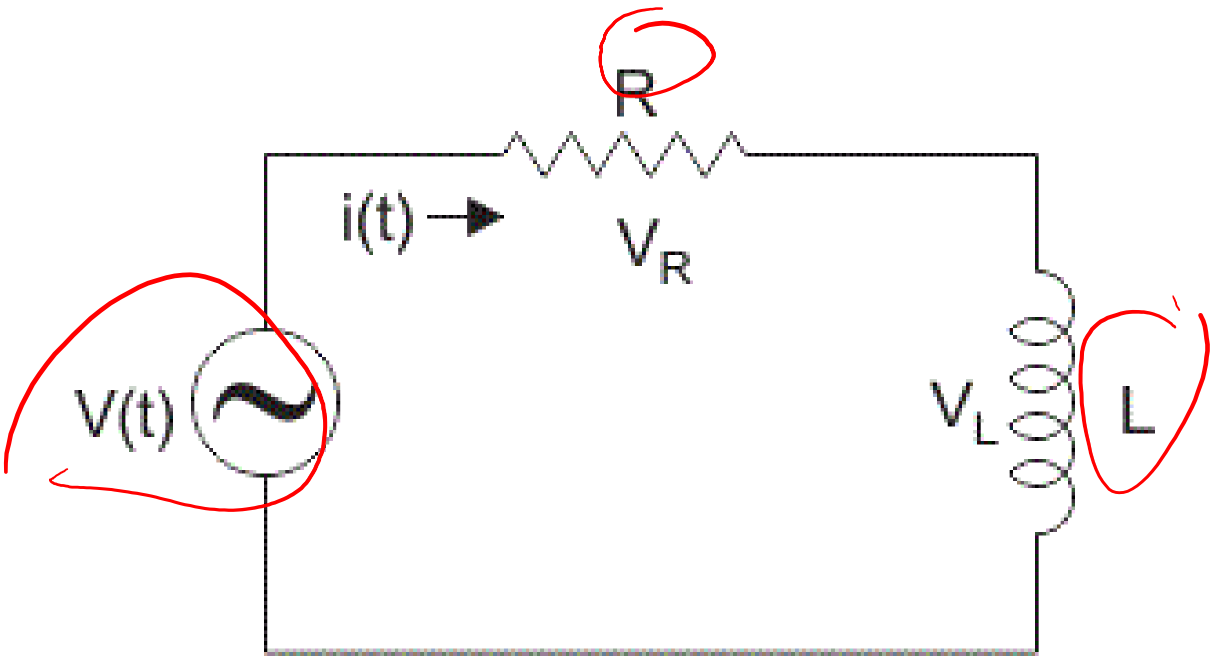
Resources

- https://physics.info/circuits-rl/
- http://www.physics.louisville.edu/cldavis/phys299/notes/mag_LR.html
- http://www.webassign.net/labsgraceperiod/ncsulcpem2/lab_7/manual.html
- https://www.electrical4u.com/rl-series-circuit/
- http://web.mit.edu/viz/EM/visualizations/coursenotes/modules/guide12.pdf
- https://www.york.cuny.edu/academics/departments/earth-and-physical-sciences/physics-lab-manuals/physics-ii/alternating-current-rl-circuits

AC RL circuit



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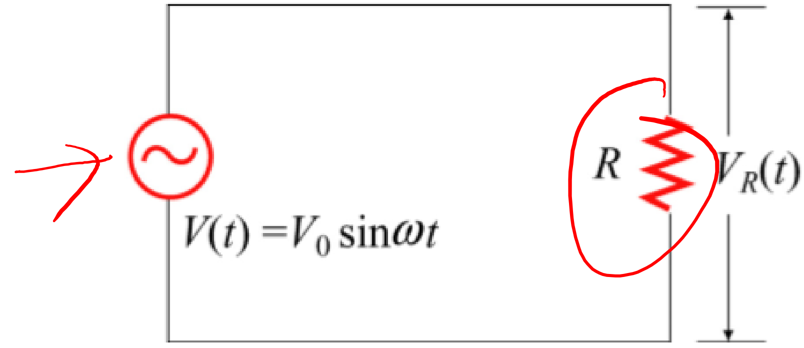
Simple AC circuits

Purely resistive load

A purely resistive circuit corresponds to infinite capacitance C = ∞ and zero inductance L = 0

Applying Kirchhoff's loop rule yields



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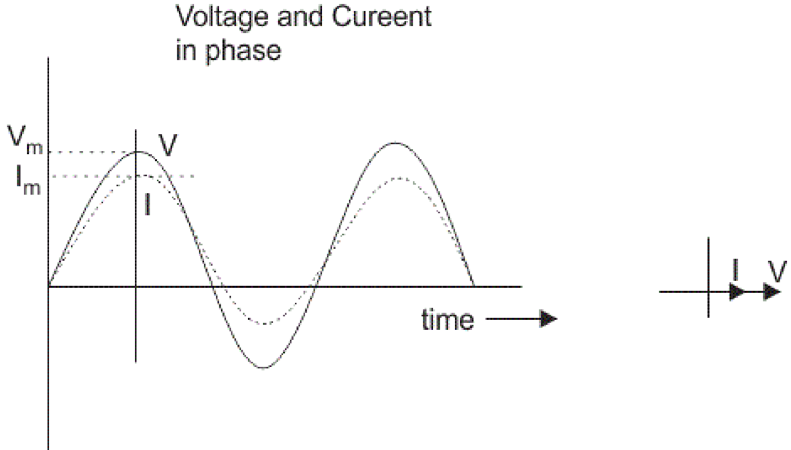


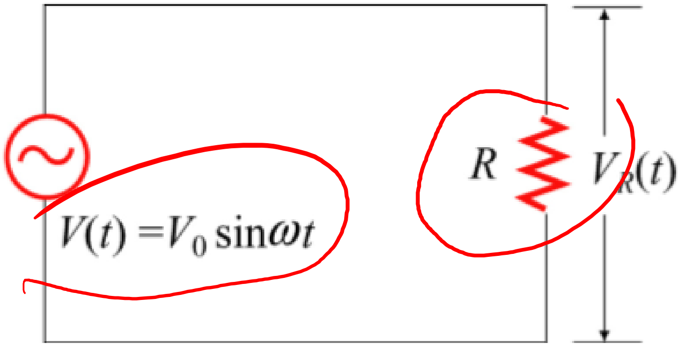
https://www.electrical4u.com/rl-series-circuit/

Purely resistive load

A purely resistive circuit corresponds to infinite capacitance C = ∞ and zero inductance L = 0

Applying Kirchhoff's loop rule yields

$$V(t) = V_R(t) = I_R(t)R$$



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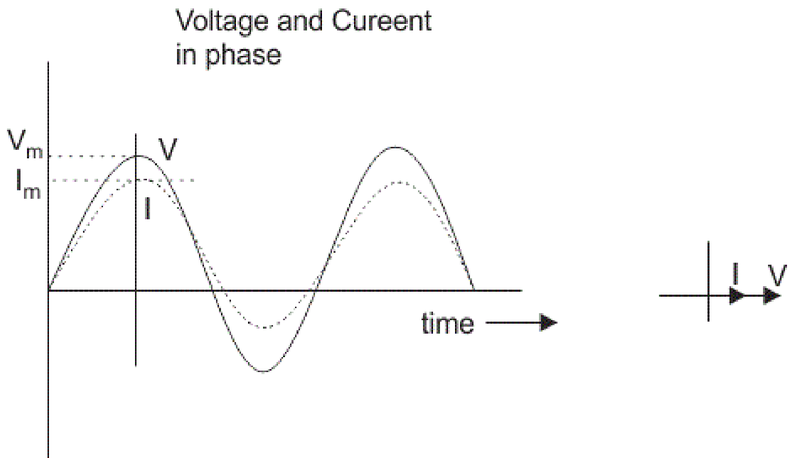


https://www.electrical4u.com/rl-series-circuit/

Purely resistive load

A purely resistive circuit corresponds to infinite capacitance C = ∞ and zero inductance L = 0

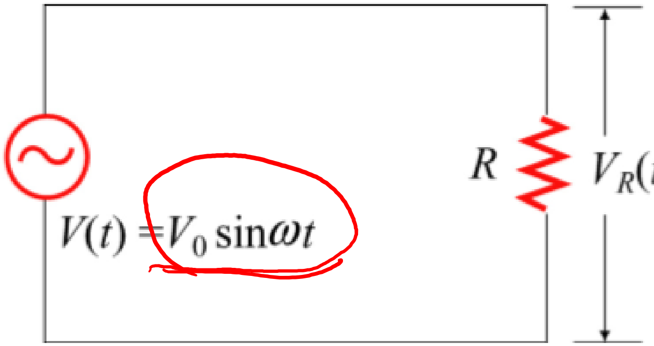
Applying Kirchhoff's loop rule yields

$$V(t) = V_R(t) = I_R(t)R$$

The current is then

$$I_R(t) = \frac{V_R(t)}{R} = \frac{V_0 \sin(wt)}{R} = I_o \sin(wt)$$

In this case, $I_R(t)$ and $V_R(t)$ are in phase with each other, meaning that they reach their maximum or minimum values at the same time.



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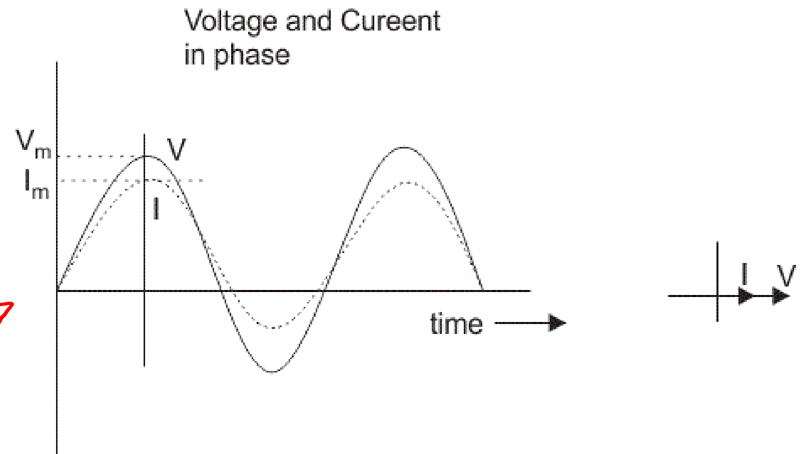
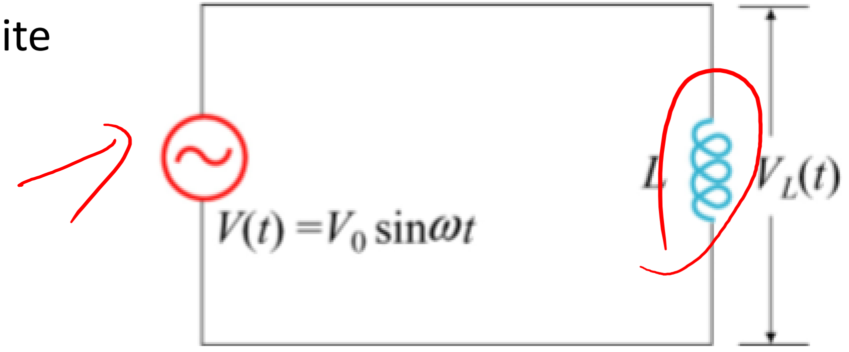


https://www.electrical4u.com/rl-series-circuit/

Purely inductive load

A purely resistive circuit corresponds to infinite capacitance C = ∞ and zero resistance R = 0

Applying Kirchhoff's loop rule yields



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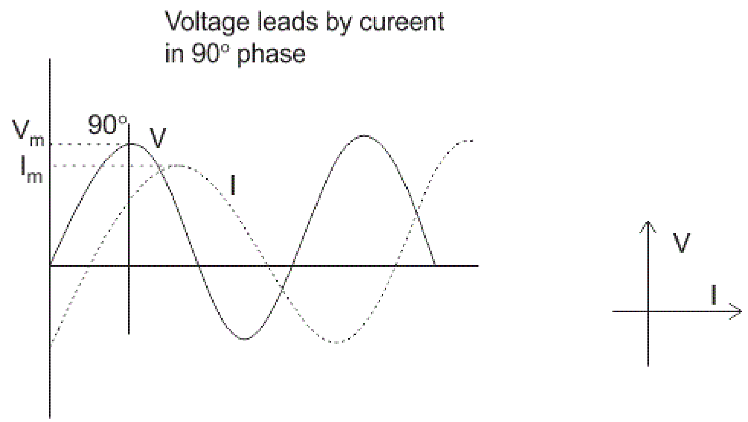


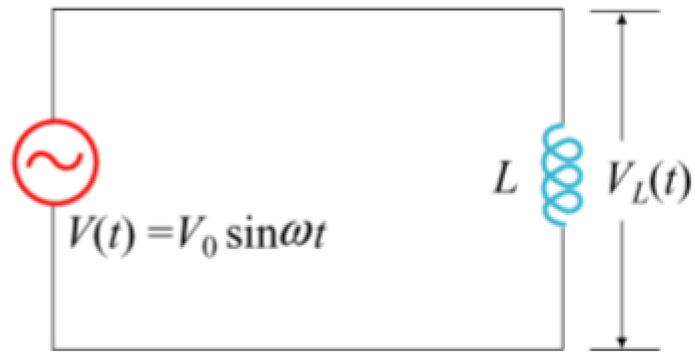
https://www.electrical4u.com/rl-series-circuit/

Purely inductive load

A purely resistive circuit corresponds to infinite capacitance C = ∞ and zero resistance R = 0

Applying Kirchhoff's loop rule yields

$$V(t) - L\frac{dI_L}{dt} = 0$$



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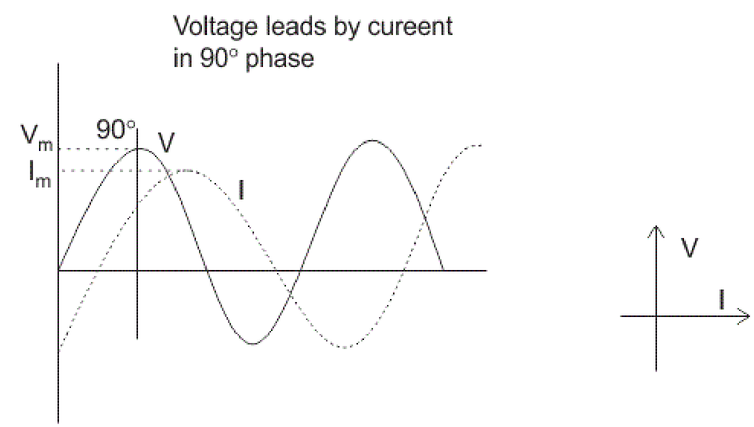
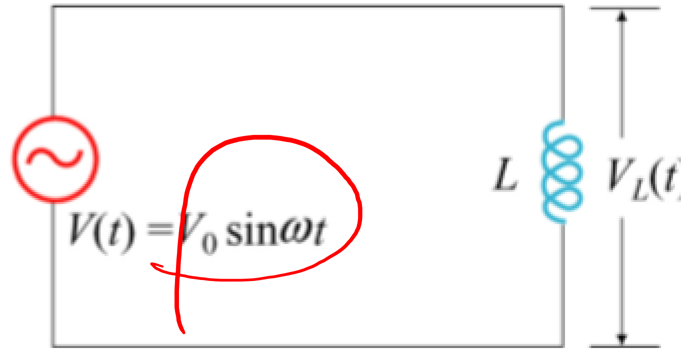


https://www.electrical4u.com/rl-series-circuit/

Purely inductive load

A purely resistive circuit corresponds to infinite capacitance C = ∞ and zero resistance R = 0



http://web.mit.edu/viz/EM/visualizations/coursenotes/modules/guide12.pdf

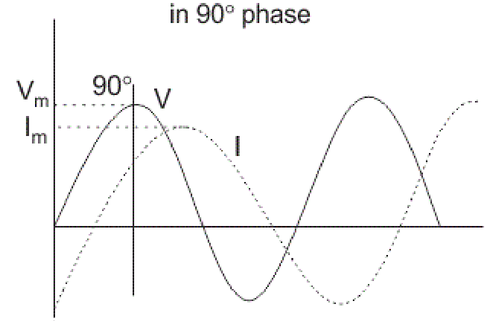
Applying Kirchhoff's loop rule yields

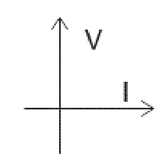
$$V(t) - L\frac{dI_L}{dt} = 0$$

Then, $\frac{dI_L}{dt} = \frac{V(t)}{L} = \frac{V_0 \sin(wt)}{L}$

$$I_L(t) = \int dI_L = \frac{V_0}{L}\int \sin(wt)\,dt = -\left(\frac{V_0}{wL}\right)\cos(wt) = \left(\frac{V_0}{wL}\right)\sin(wt - \frac{\pi}{2})$$

Voltage leads by cureent in 90° phase





https://www.electrical4u.com/rl-series-circuit/

Purely inductive load

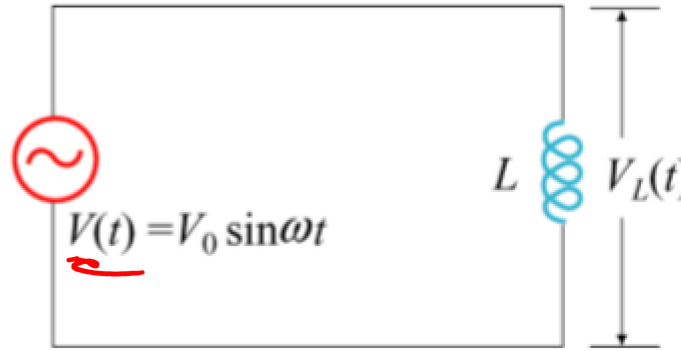
A purely resistive circuit corresponds to infinite capacitance C = ∞ and zero resistance R = 0

Applying Kirchhoff's loop rule yields

$$V(t) - L\frac{dI_L}{dt} = 0$$

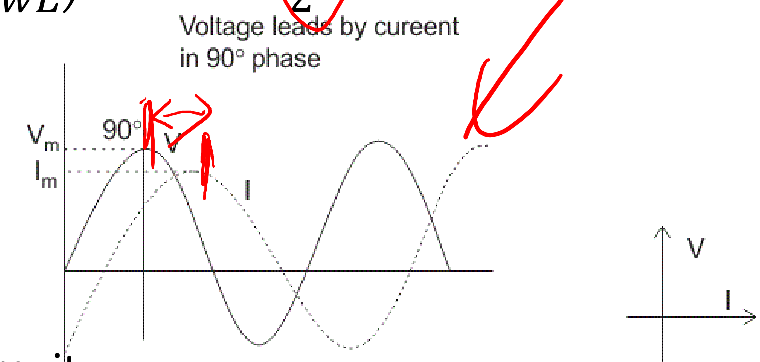
Then, $\frac{dI_L}{dt} = \frac{V(t)}{L} = \frac{V_0 \sin(wt)}{L}$

$$I_L(t) = \int dI_L = \frac{V_0}{L}\int \sin(wt)\,dt = -\left(\frac{V_0}{wL}\right)\cos(wt) = \left(\frac{V_0}{wL}\right)\sin\left(wt - \frac{\pi}{2}\right)$$



http://web.mit.edu/viz/EM/visualizations/coursenotes/modules/guide12.pdf

In this case, the current $I_L(t)$ is out of phase with $V_L(t)$ by $\frac{\pi}{2}$; it reaches its maximum after $V_L(t)$ by one quarter of a cycle.



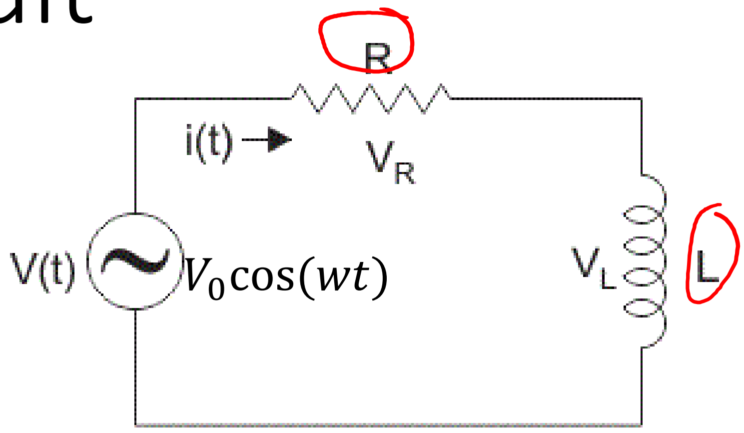
The current lags voltage by $\frac{\pi}{2}$ in a purely inductive circuit.

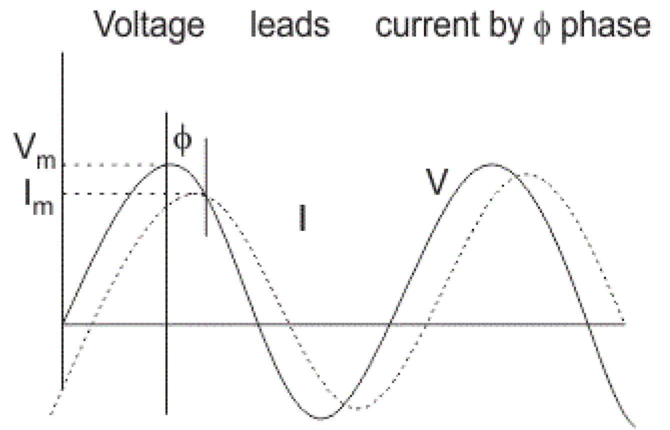
https://www.electrical4u.com/rl-series-circuit/

RL circuits under AC

AC RL circuit

Apply Ohm's law



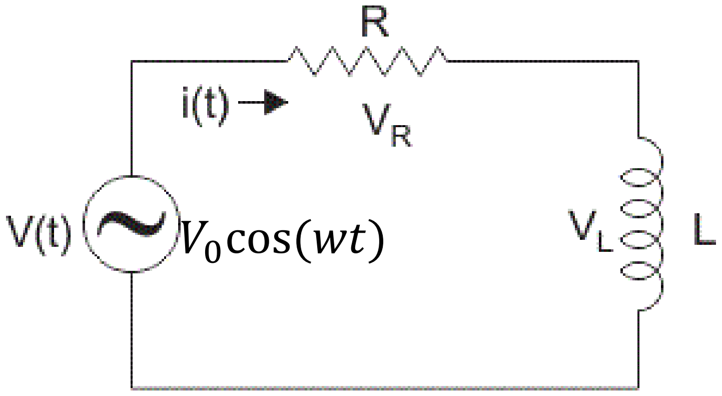


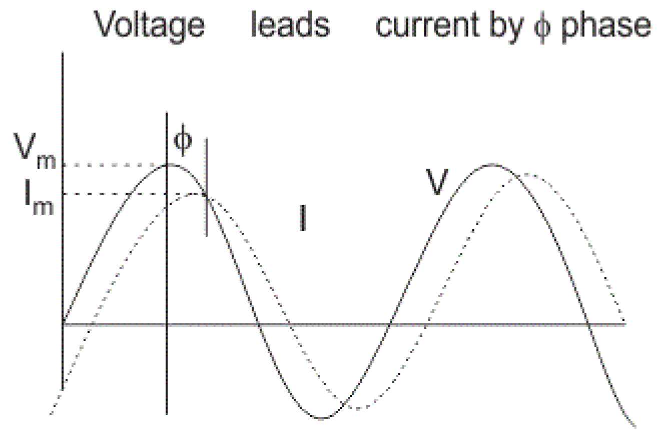
https://www.electrical4u.com/rl-series-circuit/

AC RL circuit

Apply Ohm's law

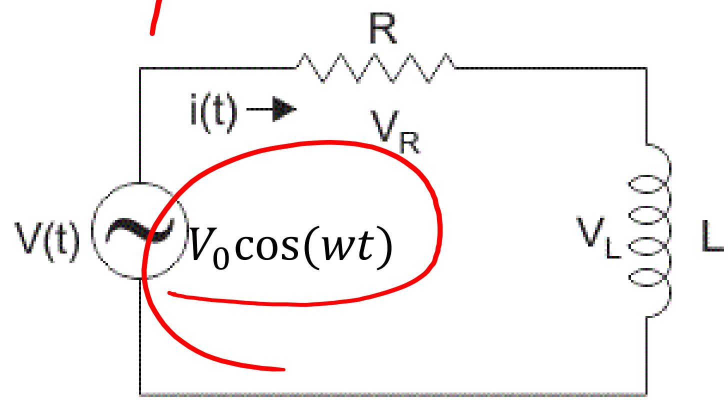
$$V_0\cos(wt) - L\frac{dI(t)}{dt} = \mathrm{I(t)R}$$





https://www.electrical4u.com/rl-series-circuit/

AC RL circuit



Apply Ohm's law

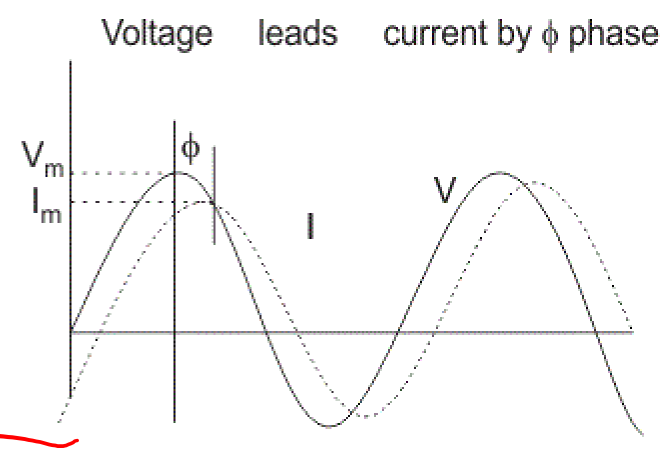
$$V_0\cos(wt) - L\frac{dI(t)}{dt} = \mathrm{I(t)R}$$

Now we need to solve this differential equation
The current $I(t)$ should have the following form:

$$I(t) = I_0\cos(wt - \phi)$$

Substitute into the above differential equation

$$V_0\cos(wt) = -wLI_0\sin(wt - \phi) + RI_0\cos(wt - \phi)$$



The right hand side is equal to (https://goo.gl/hCYkUv)

$$I_0\sqrt{R^2 + w^2L^2}\cos(wt - \phi - \theta)$$

where $\theta = tan^{-1}(-\frac{wL}{R})$

Comparing it with the left hand side term, we obtain

$I_0 = \frac{V_0}{\sqrt{R^2+w^2L^2}}$ and $\phi = -\theta = -\,tan^{-1}\left(-\frac{wL}{R}\right) = tan^{-1}\left(\frac{wL}{R}\right)$

https://www.electrical4u.com/rl-series-circuit/

A different derivation (optional)

Compare with Ohm's law $I = \frac{V}{R}$, The denominator looks like some measure of electrical impedance

- Previously we obtained that $I_0 = \frac{V_0}{\sqrt{R^2+w^2L^2}}$
- Use complex number, the electrical impedance of a RL circuit can be written as

$$Z(w) = R + iwL$$

Another way of understanding this is to consider the LR circuit in time domain $V(t) = I(t)R + L\frac{\partial I(t)}{\partial t}$. In frequency domain, assuming current $I = I_0 e^{iwt}$, we have $V = IR + iwLI = I(R + iwL)$

- Assume the AC voltage is $V_0 e^{iwt}$
- Then the current is simply

$$I = \frac{V_0 e^{iwt}}{R+iwL}$$

- How does it compare with what we obtained previously?

$$I = \frac{V_0 e^{iwt}}{R + iwL} = \frac{V_0 (R - iwL) e^{iwt}}{R^2 + w^2 L^2}$$

$$= \left(\frac{V_0 R}{R^2 + w^2 L^2} - i\frac{V_0 wL}{R^2 + w^2 L^2}\right) e^{iwt} = \frac{V_0}{\sqrt{R^2 + w^2 L^2}} e^{i\phi} e^{iwt}$$

where $\phi = tan^{-1}(-\frac{wL}{R})$

AC RL circuit: conclusions

- In case of pure resistive circuit, the phase angle between voltage and current is zero
- In case of pure inductive circuit, phase angle is 90^o

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AC RL circuit: conclusions

- In case of pure resistive circuit, the phase angle between voltage and current is zero
- In case of pure inductive circuit, phase angle is 90^o
- When a circuit has both a resistor and an inductor, the phase angle is between 0^o to 90^o.

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AC RL circuit: conclusions

- In case of pure resistive circuit, the phase angle between voltage and current is zero
- In case of pure inductive circuit, phase angle is 90°
- When a circuit has both a resistor and an inductor, the phase angle is between 0° to 90°.
- If R >> L, e.g., a strong resistor, the phase lag is 0
- If R << L, e.g., a perfect conductor, the phase lag is 90°

https://www.electrical4u.com/rl-series-circuit/

Lecture 9

Understanding EM using Resistor-inductor (RL) circuit

GEOL 4397: Electromagnetic Methods for Exploration
GEOL 6398: Special Problems

Jiajia Sun, Ph.D.

Sept. 20th, 2018

UNIVERSITY of HOUSTON
YOU ARE THE PRIDE
EARTH AND ATMOSPHERIC SCIENCES

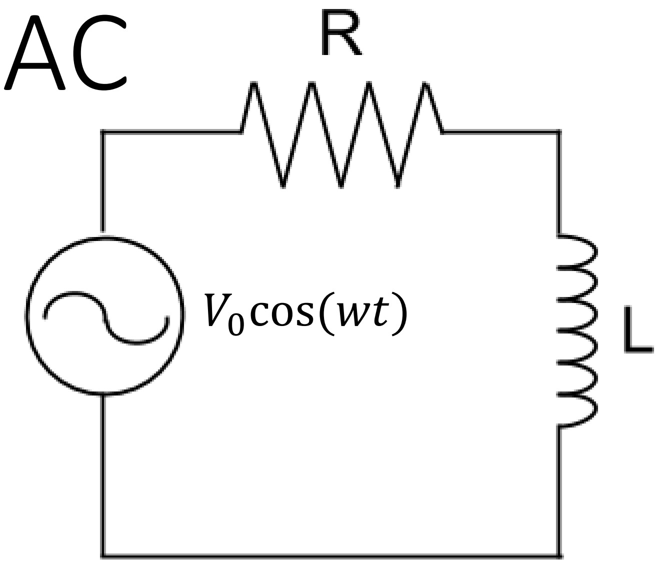
Outline

- Inductance
- RL circuit under DC
- RL circuit under AC
- Understanding frequency domain EM using RL circuit
- Understanding time domain EM using RL circuit

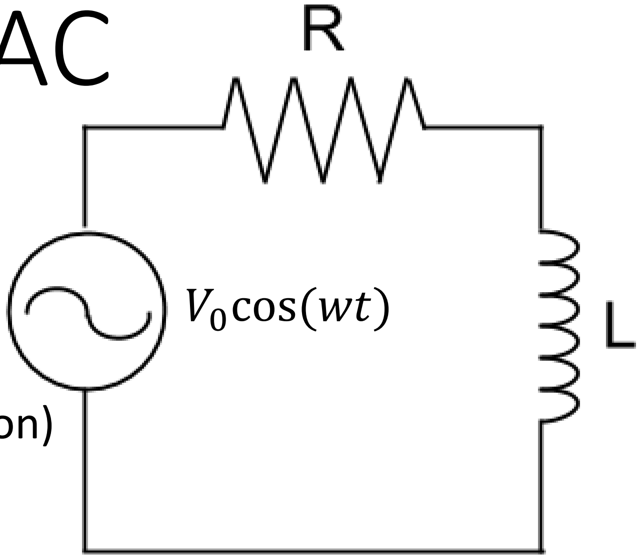
Recap: RL circuit under AC

Kirchhoff equation:

$$V(t) = I(t)R + L\frac{\partial I(t)}{\partial t}$$



Recap: RL circuit under AC



Kirchhoff equation:

$$V(t) = I(t)R + L\frac{\partial I(t)}{\partial t}$$

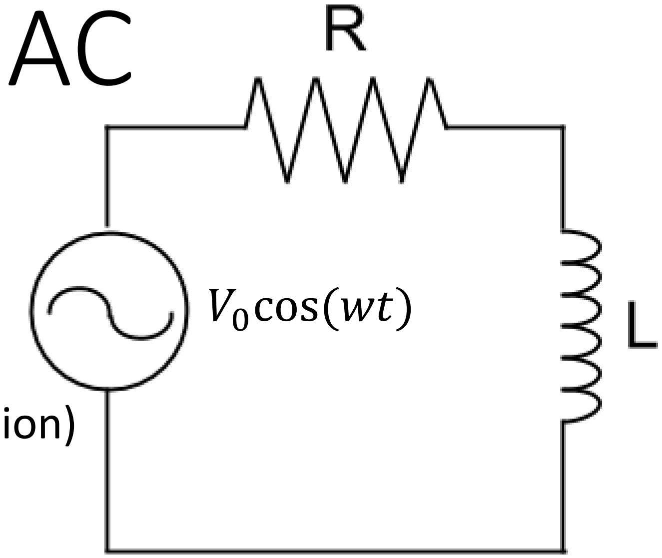
We can solve the above equation (with some initial condition) to get the current:

$$I(t) = I_0 \cos(wt - \phi)$$

where

$$I_0 = \frac{V_0}{\sqrt{R^2+w^2L^2}} \text{ and } \phi = tan^{-1}\left(\frac{wL}{R}\right)$$

Recap: RL circuit under AC



Kirchhoff equation:

$$V(t) = I(t)R + L\frac{\partial I(t)}{\partial t}$$

We can solve the above equation (with some initial condition) to get the current:

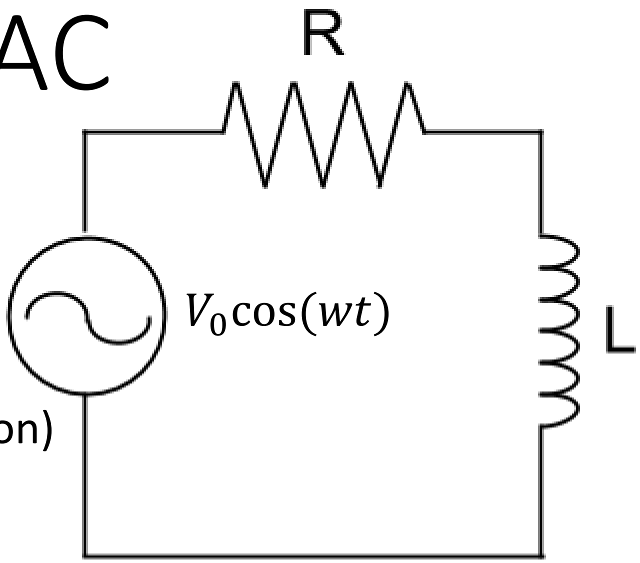
Magnitude Phase

$$I(t) = I_0\cos(wt - \phi)$$

where

$$I_0 = \frac{V_0}{\sqrt{R^2+w^2L^2}} \text{ and } \phi = tan^{-1}\left(\frac{wL}{R}\right)$$

Recap: RL circuit under AC



Kirchhoff equation:

$$V(t) = I(t)R + L\frac{\partial I(t)}{\partial t}$$

We can solve the above equation (with some initial condition) to get the current:

Magnitude Phase

$$I(t) = I_0 \cos(wt - \phi)$$

where

$$I_0 = \frac{V_0}{\sqrt{R^2+w^2L^2}} \text{ and } \phi = tan^{-1}\left(\frac{wL}{R}\right)$$

Another convenient way of expressing current is:

$$I(t) = I_0 e^{i(wt-\phi)}$$

The current lags behind voltage by ϕ in a RL circuit under AC.

Time-domain vs. Frequency domain EM

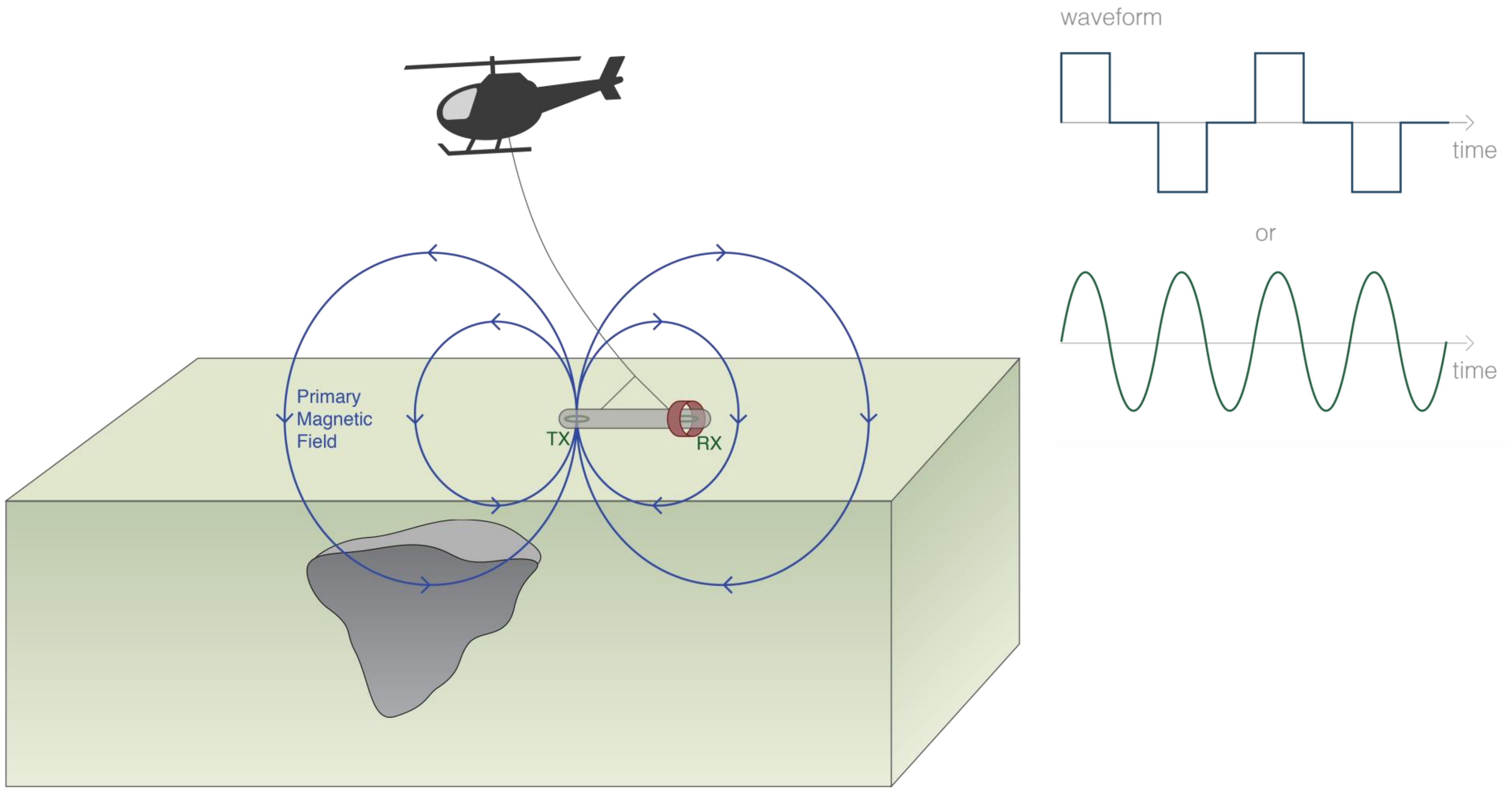
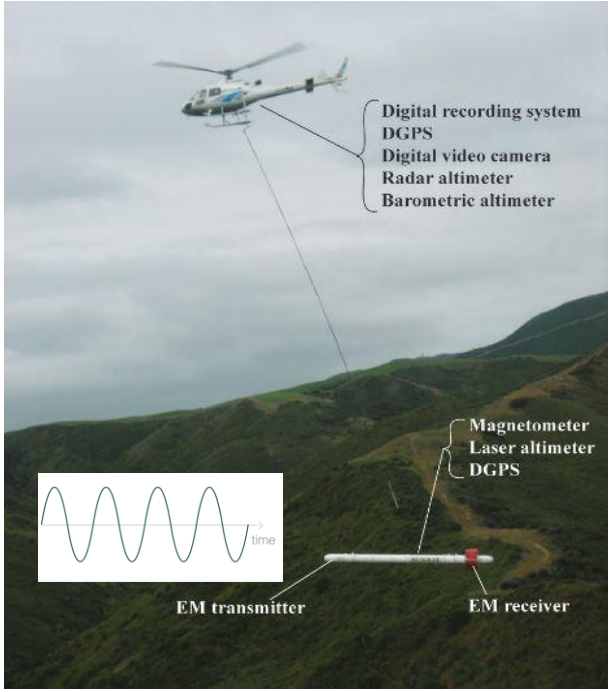
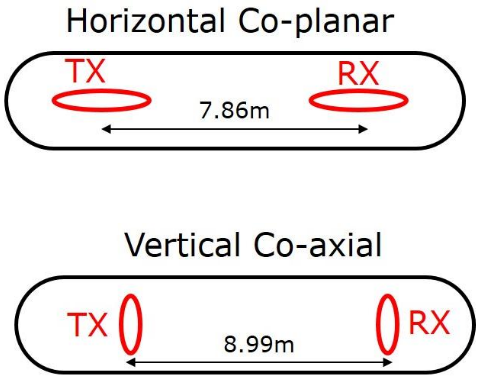


Image credit: Doug Oldenburg, Seogi Kang and Linsey Heagy from UBC-GIF

Airborne EM systems

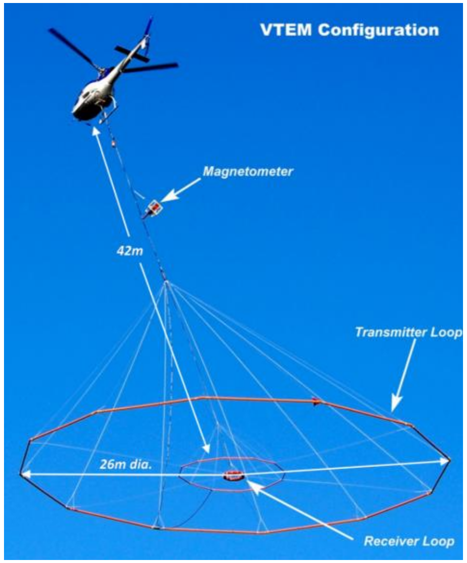
Resolve system (2008)

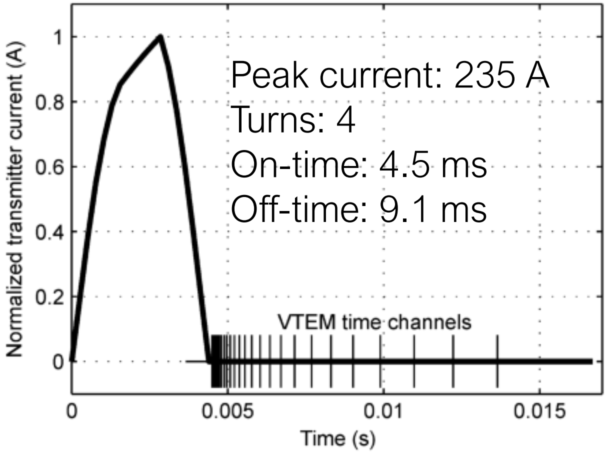




Area = 535 m^2

VTEM (2007)





Horizontal Co-planar (HCP) frequencies:
- 382, 1822, 7970, 35920 and 130100 Hz

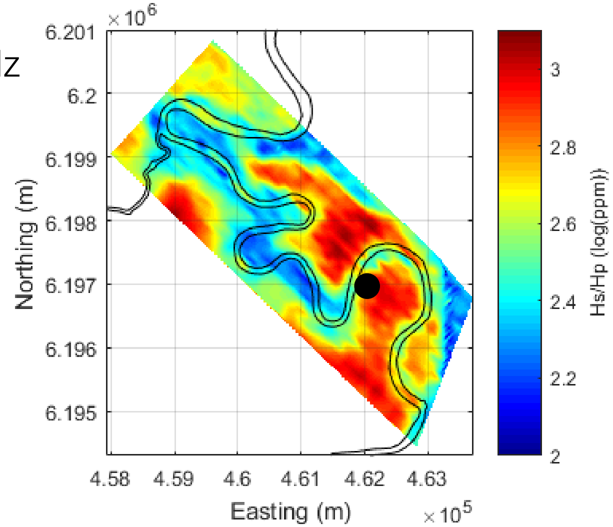
Vertical Co-axial (VCA) frequencies:
- 3258 Hz

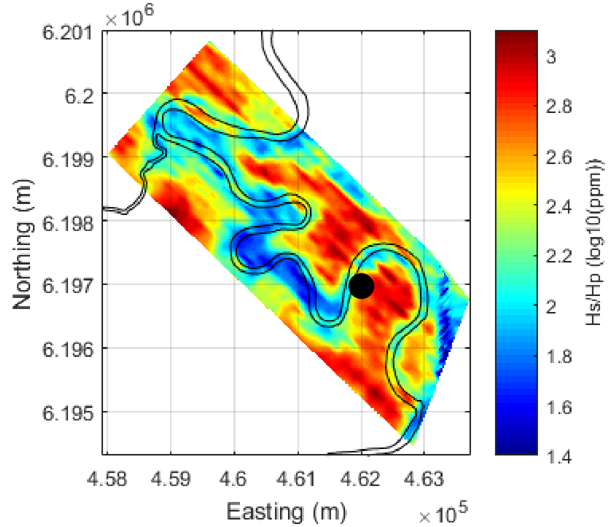
Horizontal Co-planar (HCP) data

In-Phase (Real)

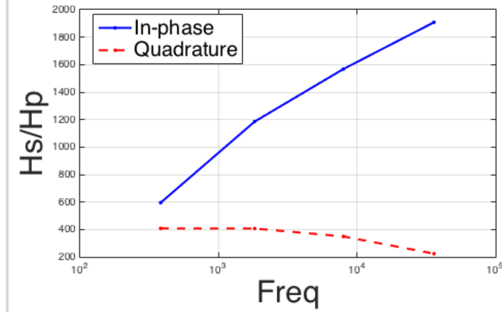
Quadrature (Imaginary)

382 Hz

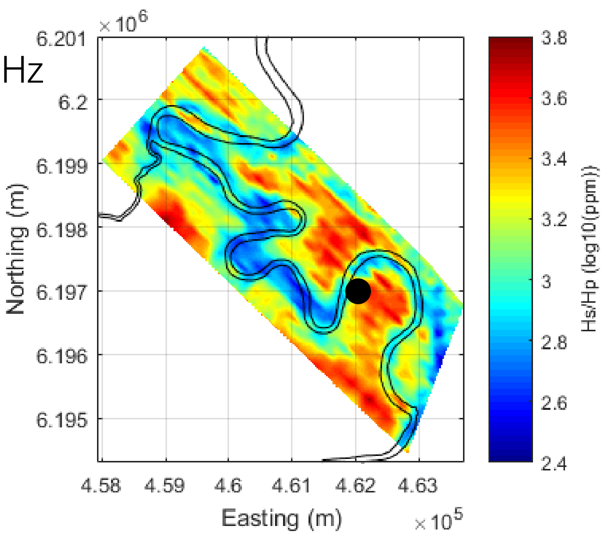


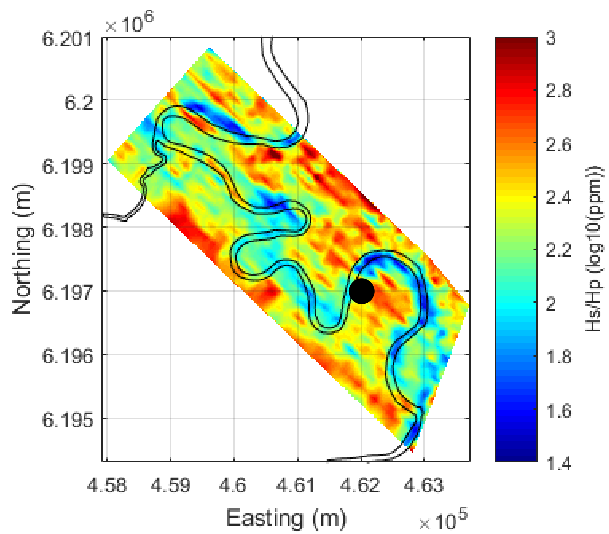


Sounding curve

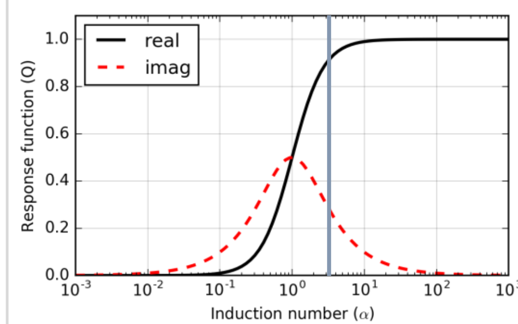


35920 Hz

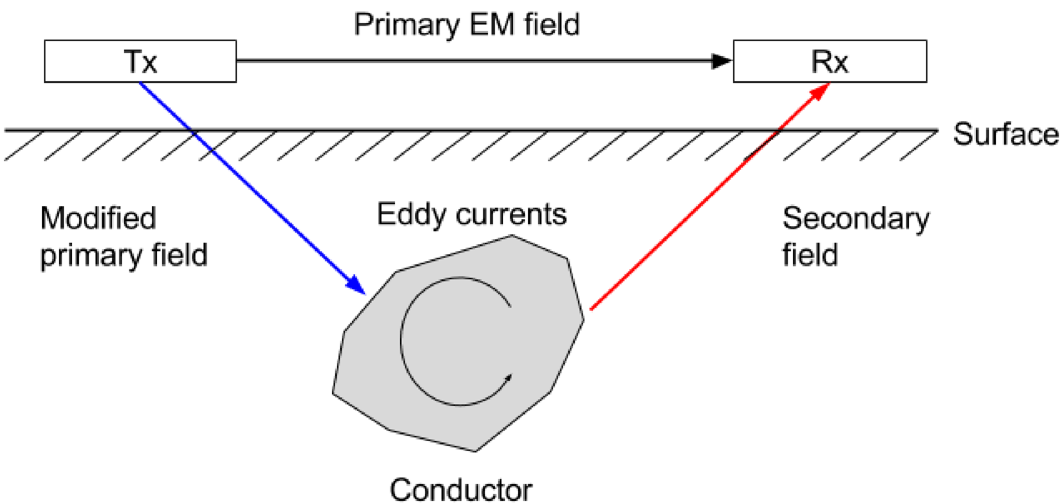


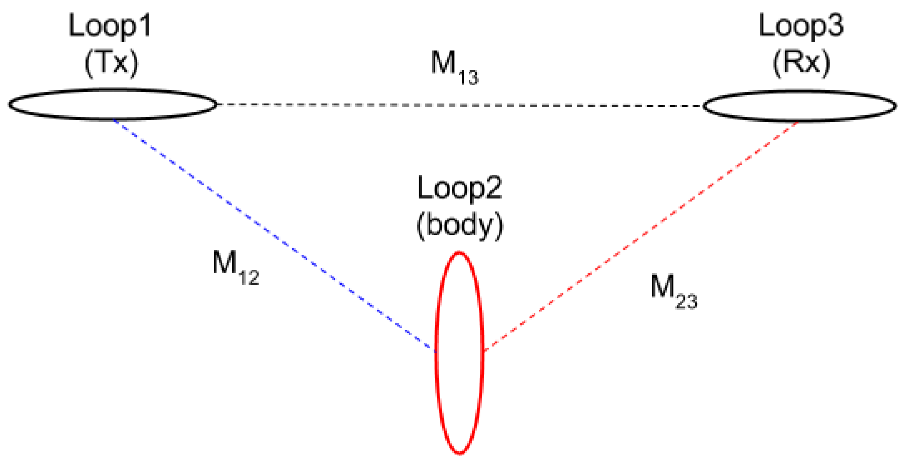


Response curve



Understand frequency domain EM induction using RL circuits



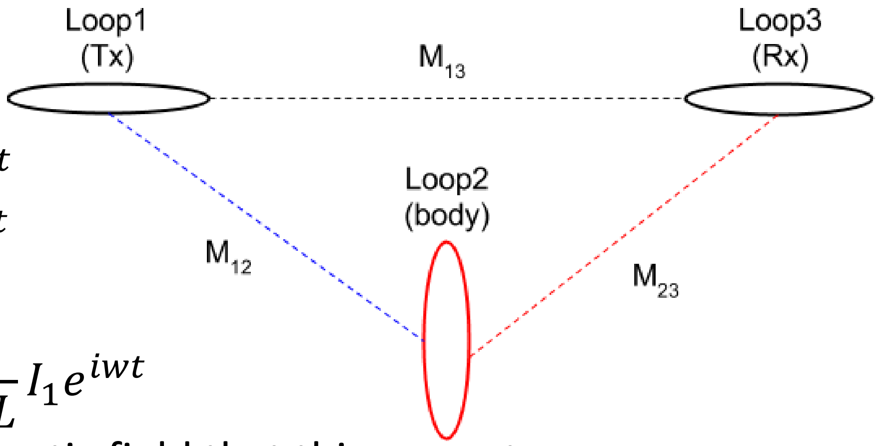


https://em.geosci.xyz/content/maxwell3_fdem/circuitmodel_for_eminduction/index.html

3-Loop system

- Suppose alternating current, $I_1 e^{iwt}$, is flowing in the Tx (Loop 1)
- This current generates an alternating magnetic field in the surrounding environment (e.g., the Earth's subsurface)
- which in turn induces an EMF both in the body (Loop 2) and the Rx (Loop 3)
- Recall that $\varepsilon_2 = -\frac{d\Phi_2}{dt} = -M_{12}\frac{dI_1}{dt}$
- Therefore,

$$\varepsilon_3^p = -iwM_{13}I_1e^{iwt}$$
$$\varepsilon_2 = -iwM_{12}I_1e^{iwt}$$



- What is the current in the body (Loop 2)?

$$I_2 = \frac{\varepsilon_2}{R + iwL} = -\frac{iwM_{12}}{R + iwL} I_1 e^{iwt}$$

- We are interested only in the secondary magnetic field that this current produces, and particularly the EMF induced in the Rx

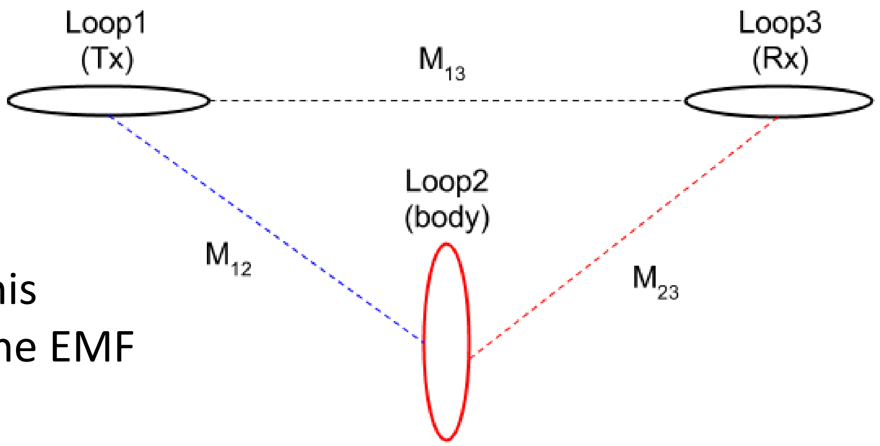
$$\varepsilon_3^s = -M_{23}\frac{dI_2}{dt} = -iwM_{23}I_2$$

https://em.geosci.xyz/content/maxwell3_fdem/circuitmodel_for_eminduction/derive_response_function.html

3-Loop system

$$\varepsilon_3^p = -iwM_{13}I_1e^{iwt}$$

$$\varepsilon_3^s = -M_{23}\frac{dI_2}{dt} = -iwM_{23}I_2$$



- In most cases, EM instrument measures this anomalous voltage by comparing it with the EMF induced by the primary field
- That is, it measures $\frac{\varepsilon_3^s}{\varepsilon_3^p}$

$$\frac{\varepsilon_3^s}{\varepsilon_3^p}=\frac{iwM_{23}I_2}{iwM_{13}I_1e^{iwt}} = \frac{M_{23}I_2}{M_{13}I_1e^{iwt}} = \frac{M_{23}(-\frac{iwM_{12}}{R+iwL}I_1e^{iwt})}{M_{13}I_1e^{iwt}}$$

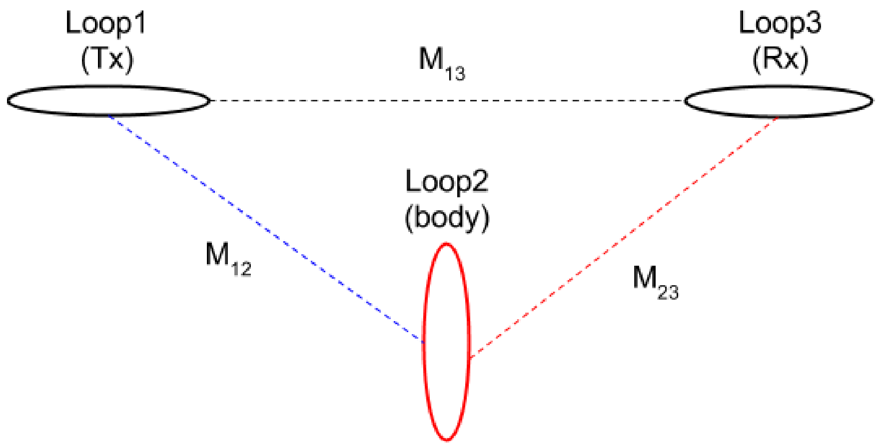
$$= -\frac{M_{12}M_{23}}{M_{13}}\frac{iw}{R+iwL} = -\frac{M_{12}M_{23}}{M_{13}L}\frac{iwL}{R+iwL} = -\frac{M_{12}M_{23}}{M_{13}L}\frac{\frac{iwL}{R}}{1+\frac{iwL}{R}} =$$

$$-\frac{M_{12}M_{23}}{M_{13}L}\frac{i\alpha}{1+i\alpha} = -\frac{M_{12}M_{23}}{M_{13}L}\frac{\alpha^2+i\alpha}{1+\alpha^2}$$

where $\alpha = \frac{wL}{R}$ is dimensionless and termed induction number.

https://em.geosci.xyz/content/maxwell3_fdem/circuitmodel_for_eminduction/derive_response_function.html

3-Loop system



$$\frac{\varepsilon_3^s}{\varepsilon_3^p} = -\frac{M_{12}M_{23}}{M_{13}L}\frac{\alpha^2+i\alpha}{1+\alpha^2} \quad \text{where } \alpha = \frac{wL}{R}$$

$$\frac{\varepsilon_3^s}{\varepsilon_3^p} = CQ(\alpha)$$

$$C = -\frac{M_{12}M_{23}}{M_{13}L}$$

Coupling coefficient determined by geometry

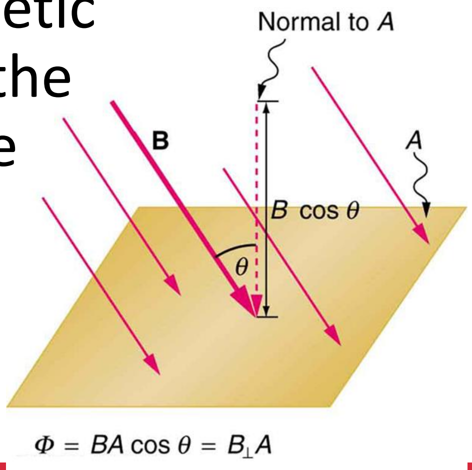
$$Q = \frac{\alpha^2 + i\alpha}{1 + \alpha^2}$$

Response function relates to the target body

https://em.geosci.xyz/content/maxwell3_fdem/circuitmodel_for_eminduction/derive_response_function.html

How about magnetic field?

- We know that $\Phi_2 = M_{12} I_1$
- Suppose the loop is small enough that the magnetic field within it is uniform
- Then $\Phi_2 = BA$ where A is the area of the loop and B is one component of the induced magnetic field that depends on the coil orientation. For example, if coil is horizontal (parallel with the ground surface), then it measures the vertical component of the magnetic field. If it happens to align with the direction of the magnetic field, it measures the magnitude of the magnetic field.
- Thus, $B = \frac{M_{12}}{A} I_1$

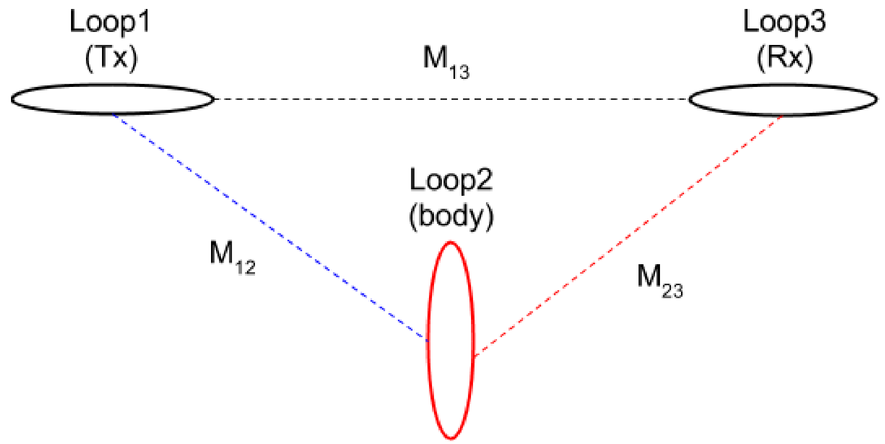


How about magnetic field?

- $B = \frac{M_{12}}{A} I_1$
- Therefore,

$$B_p^3 = \frac{M_{13}}{A} I_1$$

$$B_s^3 = \frac{M_{23}}{A} I_2$$



- Therefore,

$$\frac{B_S^3}{B_P^3} = \frac{M_{23} I_2}{M_{13} I_1} = \frac{M_{23}\left(-\frac{iwM_{12}}{R+iwL} I_1 e^{iwt}\right)}{M_{13} I_1 e^{iwt}} = -\frac{M_{12} M_{23}}{M_{13}} \frac{iw}{R+iwL} = -\frac{M_{12} M_{23}}{M_{13} L} \frac{\alpha^2 + i\alpha}{1+\alpha^2} = \frac{\varepsilon_3^S}{\varepsilon_3^P}$$

Similarly, $\frac{H_S^3}{H_p^3} = \frac{\varepsilon_3^S}{\varepsilon_3^p} = CQ(\alpha)$

- Therefore, the fields and the voltages can be used interchangeably when measured with a coil.

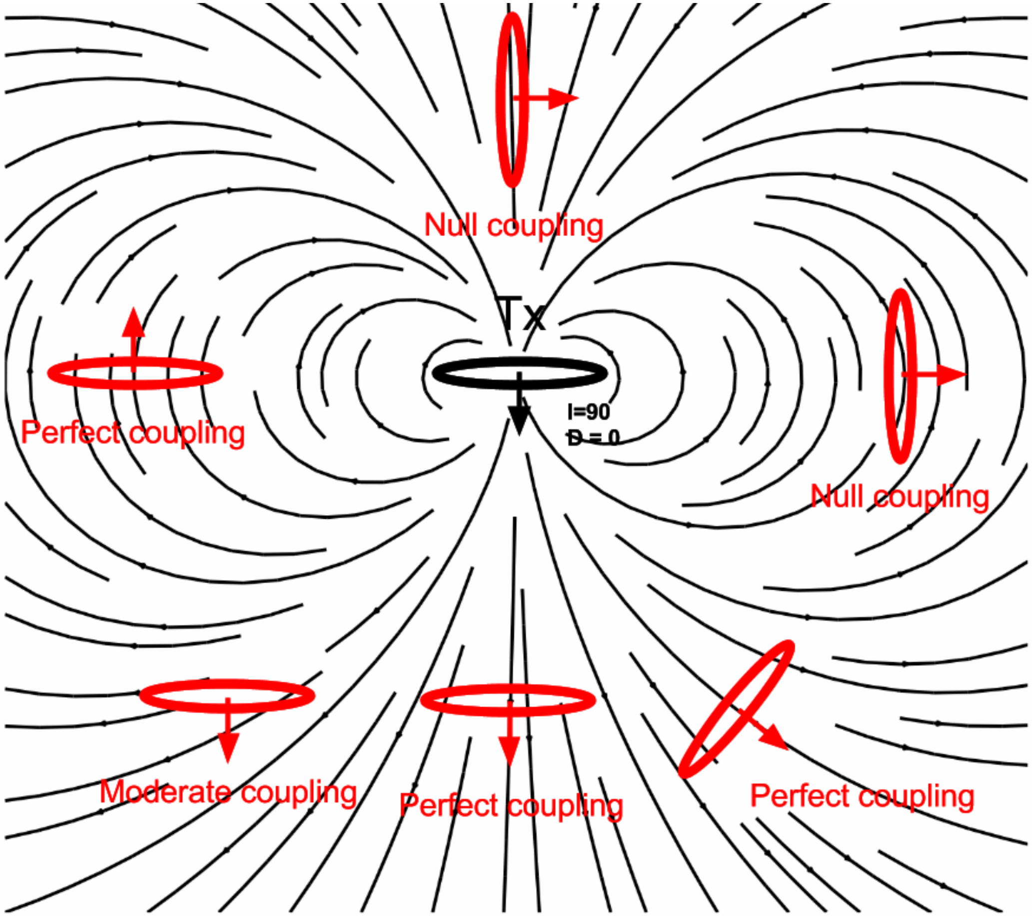
This is what we have learned so far

$$\frac{H_s^3}{H_p^3} = \frac{\varepsilon_3^s}{\varepsilon_3^p} = CQ(\alpha)$$

$$C = -\frac{M_{12}M_{23}}{M_{13}L}$$ Coupling coefficient

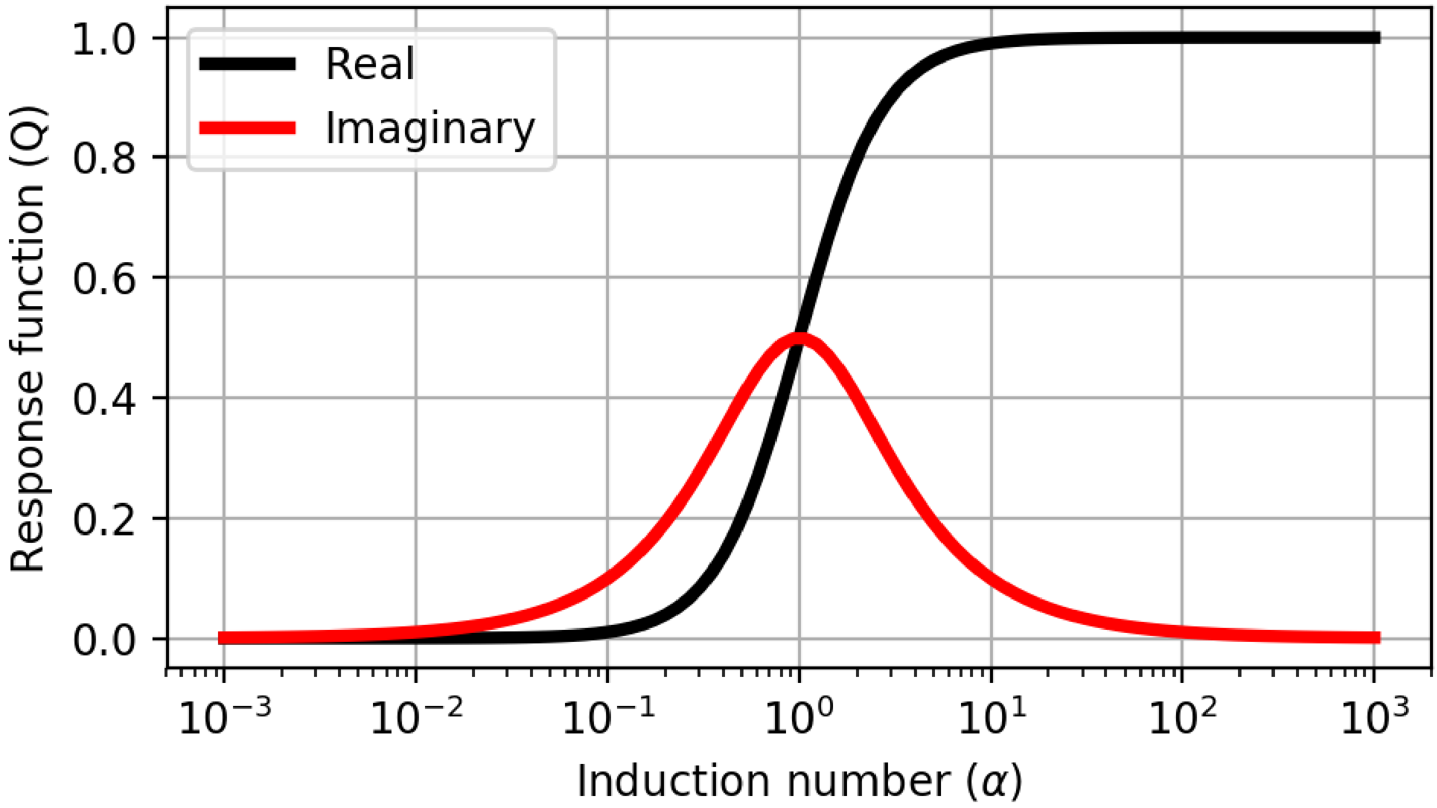
$$Q = \frac{\alpha^2 + i\alpha}{1 + \alpha^2}$$ Response function

Coupling between loops



https://em.geosci.xyz/content/maxwell3_fdem/circuitmodel_for_eminduction/understanding_harmonicEMresponse.html

Response function



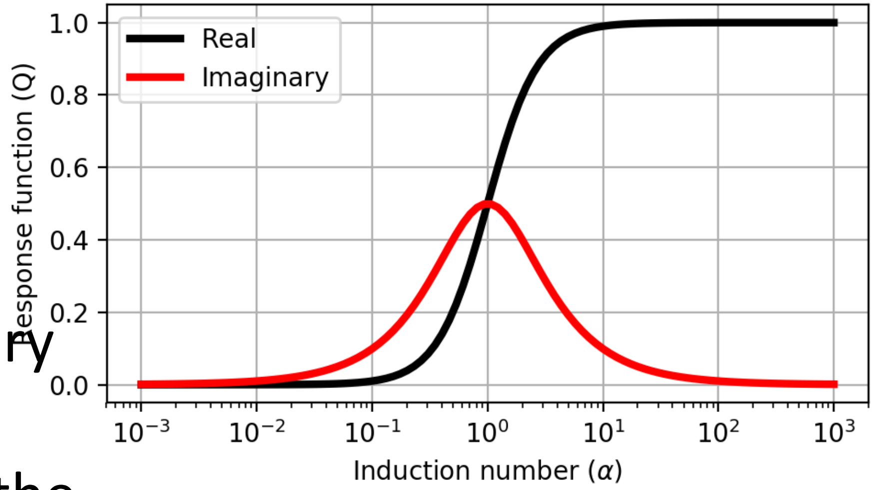
https://em.geosci.xyz/content/maxwell3_fdem/circuitmodel_for_eminduction/derive_response_function.html

Understanding the response function

$$Q = \frac{\alpha^2 + i\alpha}{1+\alpha^2} \quad \text{with } \alpha = \frac{wL}{R}$$

when $\alpha \ll 1$

$$Q \approx i\alpha$$



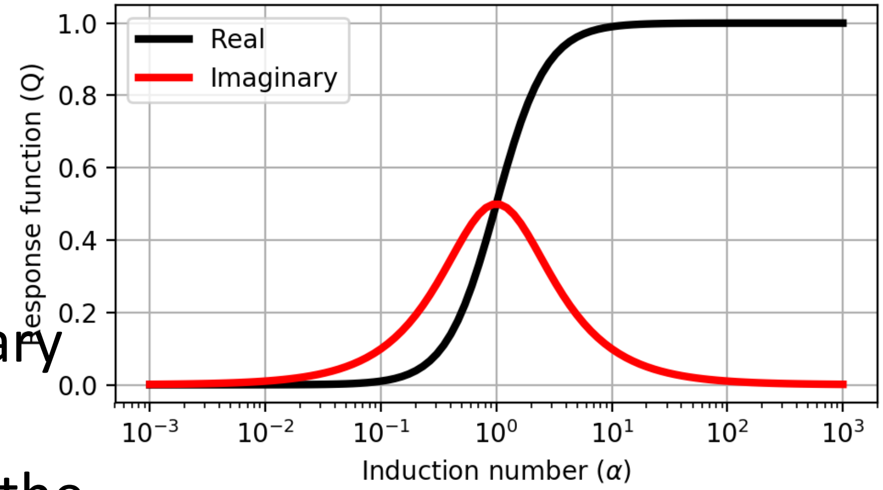
- The EM response is purely imaginary and small.
- The amount of current induced in the body will also be small. Remember that $|I| = \frac{\varepsilon_2}{\sqrt{R^2 + w^2 L^2}}$

Understanding the response function

$$Q = \frac{\alpha^2 + i\alpha}{1+\alpha^2} \quad \text{with } \alpha = \frac{wL}{R}$$

Resistive limit: when $\alpha \ll 1$

$$Q \approx i\alpha$$



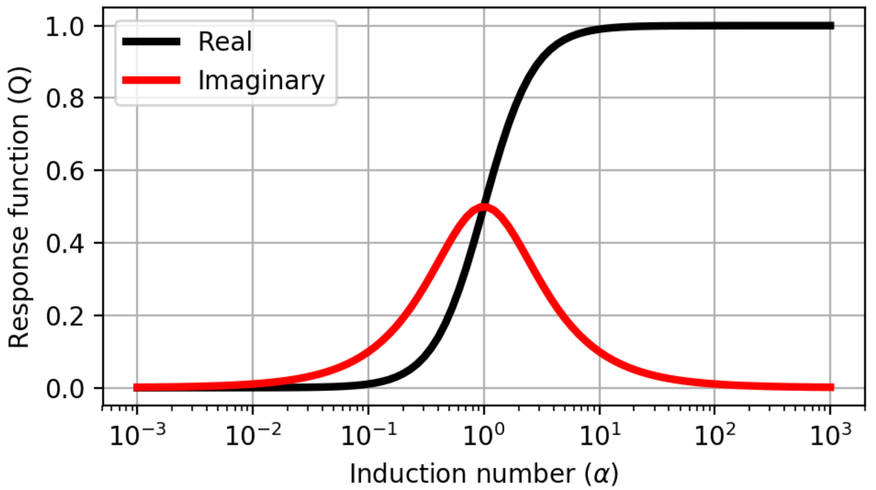
- The EM response is purely imaginary and small.
- The amount of current induced in the body will also be small. Remember that $|I| = \frac{\varepsilon_2}{\sqrt{R^2 + w^2 L^2}}$

Understanding the response function

$$Q = \frac{\alpha^2 + i\alpha}{1+\alpha^2} \quad \text{with } \alpha = \frac{wL}{R}$$

when $\alpha \gg 1$

$$Q \approx 1$$



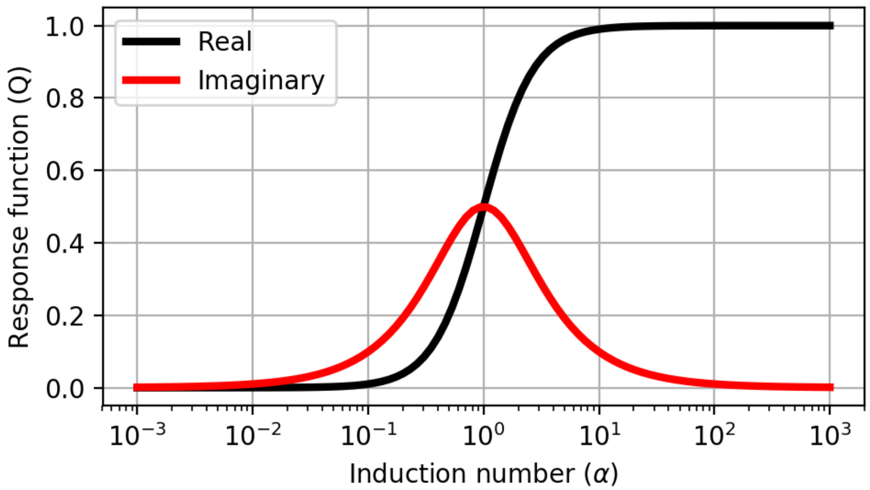
- The EM response is largely real-valued.
- The imaginary part is very small.

Understanding the response function

$$Q = \frac{\alpha^2 + i\alpha}{1+\alpha^2} \quad \text{with } \alpha = \frac{wL}{R}$$

Inductive limit: when $\alpha \gg 1$

$$Q \approx 1$$

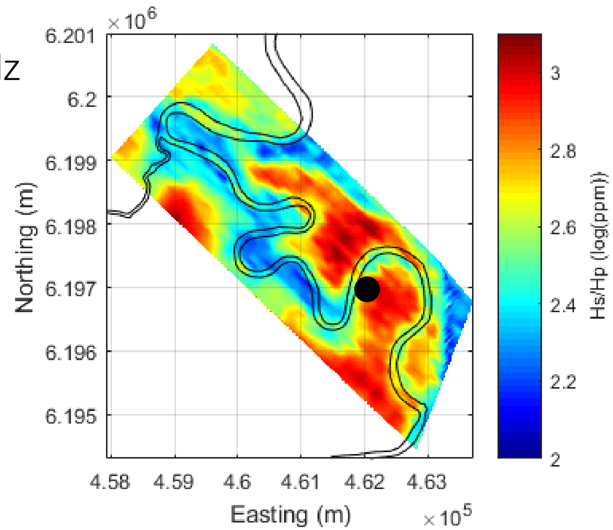


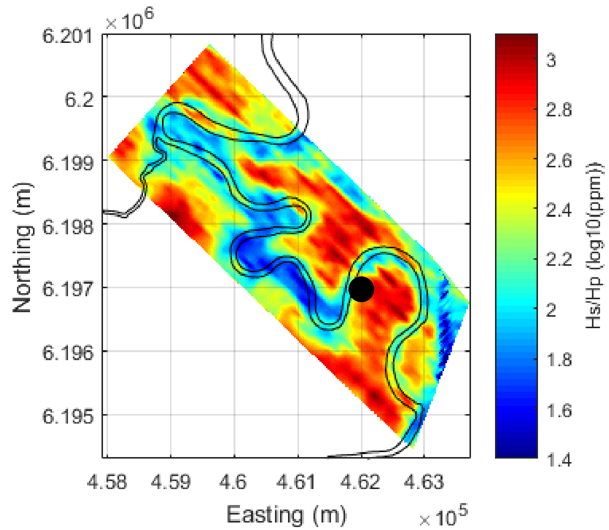
- The EM response is largely real-valued.
- The imaginary part is very small.

Horizontal Co-planar (HCP) data

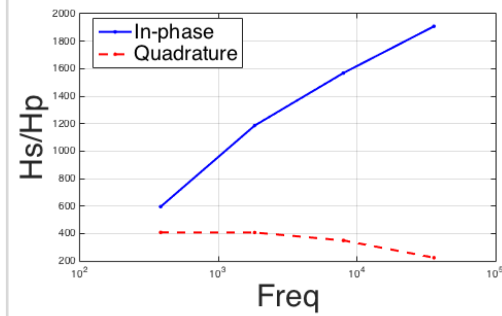
In-Phase (Real) | Quadrature (Imaginary)

382 Hz

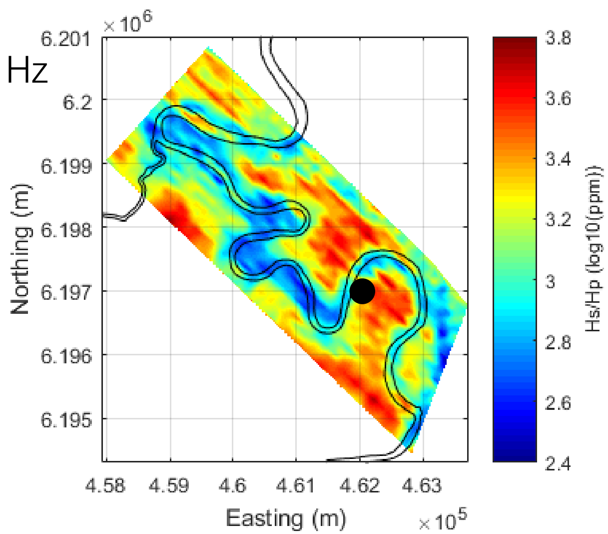


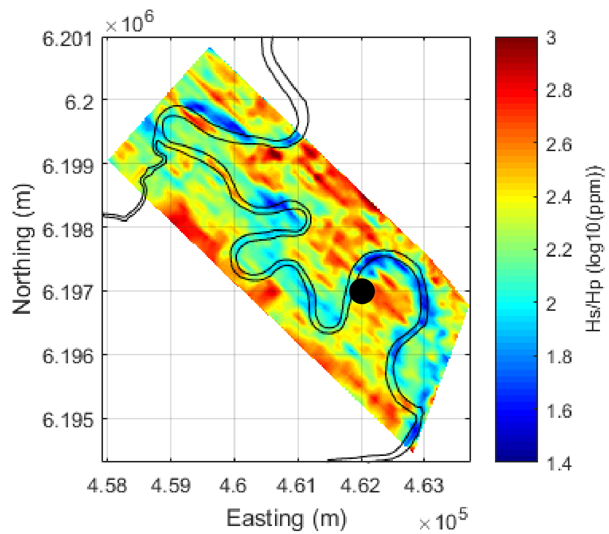


Sounding curve

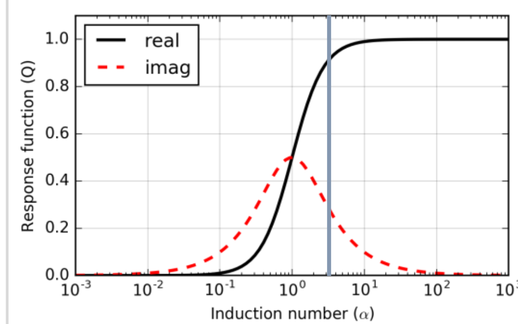


35920 Hz



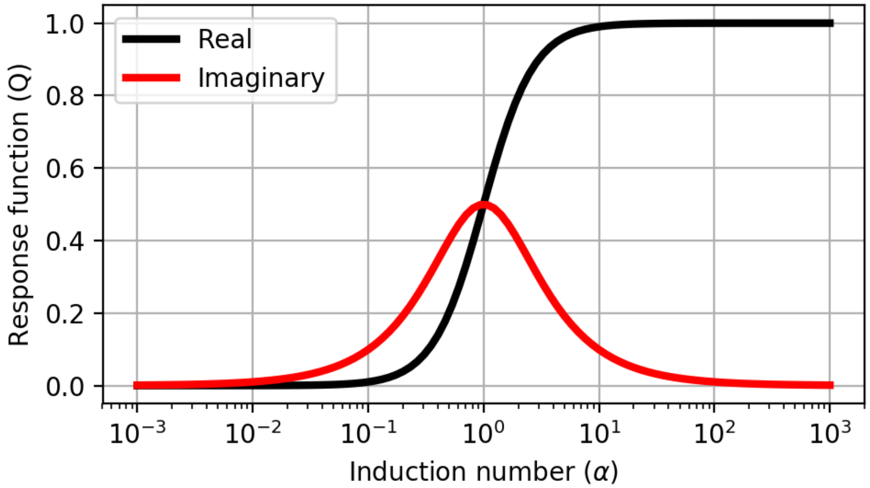


Response curve



Understanding the harmonic EM response (optional)

$$\frac{H_s^3}{H_p^3} = -\frac{M_{12}M_{23}}{M_{13}L}\frac{\alpha^2+i\alpha}{1+\alpha^2} \text{ with } \alpha = \frac{wL}{R}$$



The phase:

$$\tan(\theta) = \frac{1}{\alpha}$$

How to obtain ϕ?

From $\tan(\theta) = \frac{1}{\alpha}$, we obtain $\cot(\theta) = \alpha$

which is equal to $\tan(\frac{\pi}{2} - \theta) = \alpha$

Therefore, $\frac{\pi}{2} - \theta \pm k\pi = tan^{-1}(\frac{wL}{R})$. Therefore, $\theta = \frac{\pi}{2} - tan^{-1}(\frac{wL}{R}) \pm k\pi$

We know that θ must be negative, because we know that the secondary magnetic field lags behind the primary field. But in the meantime, $\tan(\theta)$ must be positive. That leaves us with $-\pi < \theta < -\frac{\pi}{2}$.

Therefore, $\theta = \frac{\pi}{2} - tan^{-1}\left(\frac{wL}{R}\right) - \pi = -\frac{\pi}{2} - tan^{-1}\left(\frac{wL}{R}\right)$

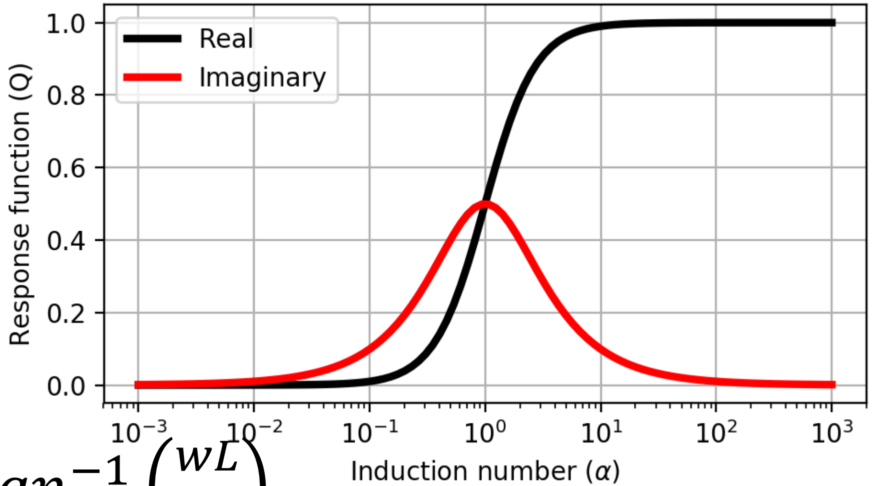
Understanding the harmonic EM response (optional)

$$\frac{H_s^3}{H_p^3} = -\frac{M_{12}M_{23}}{M_{13}L}\frac{\alpha^2+i\alpha}{1+\alpha^2} \text{ with } \alpha = \frac{wL}{R}$$

The phase:

$$\tan(\theta) = \frac{1}{\alpha}$$

$$\theta = -\frac{\pi}{2} - tan^{-1}\left(\frac{wL}{R}\right)$$



The phase lag is therefore, $\psi = \frac{\pi}{2} + tan^{-1}\left(\frac{wL}{R}\right)$

The lag of $\frac{\pi}{2}$ is due to the inductive coupling between Loop 1 and Loop 2 (i.e., time-varying magnetic field induces EMF in Loop 2), whereas the additional phase lag $tan^{-1}\left(\frac{wL}{R}\right)$ comes from the fact that Loop 2 is acting as a RL circuit when applied to an EMF.

For a very good conductor, $\frac{wL}{R} \to \infty$, $\psi \to \pi$. The phase of secondary field is 180° behind the primary field. For a very good resistor, the phase lag is 90°.

In-phase vs Out-of-phase

- Phase lag $\psi = \frac{\pi}{2} + tan^{-1}\left(\frac{wL}{R}\right) = \frac{\pi}{2} + \phi$

$$H_s^3 = |H_s^3| \cos\left(wt - \left(\frac{\pi}{2} + \phi\right)\right)$$

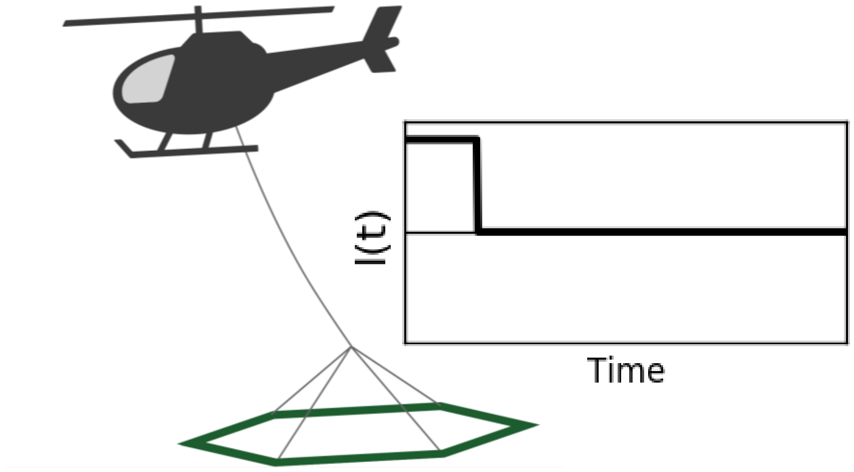
$$= |H_s^3| \left(\cos\left(wt - \frac{\pi}{2}\right)\cos(\phi) + \cos(wt - \pi)\sin(\phi)\right)$$

The component of H_s^3 that is 180º out of phase with H^p is $|H_s^3|\sin(\phi)$. This is called Real or In-phase component.
The component 90º out of phase is $|H_s^3|\cos(\phi)$. This is called Imaginary, Out-of-phase, or quadrature component.

https://em.geosci.xyz/content/maxwell3_fdem/circuitmodel_for_eminduction/understanding_harmonicEMresponse.html

Understand time domain EM induction using RL circuits

Synthetic airborne TEM data



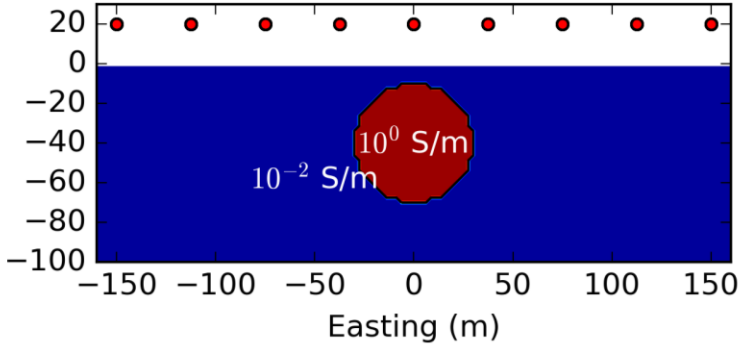
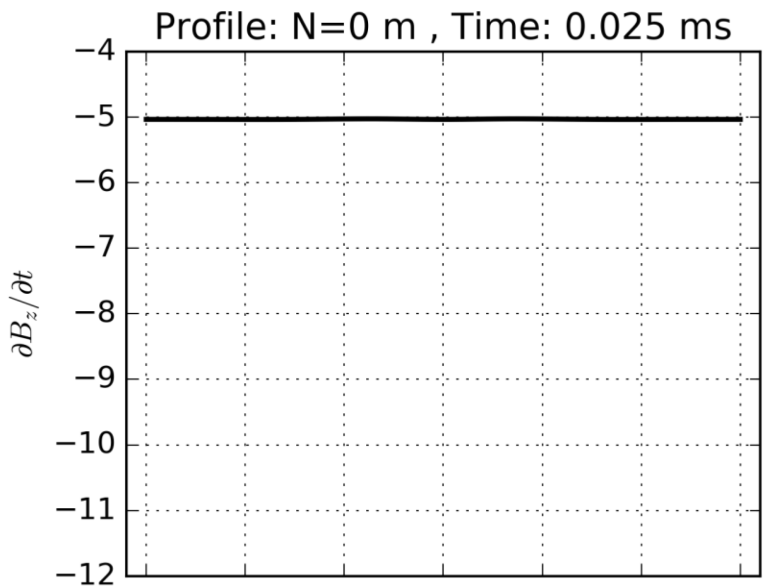
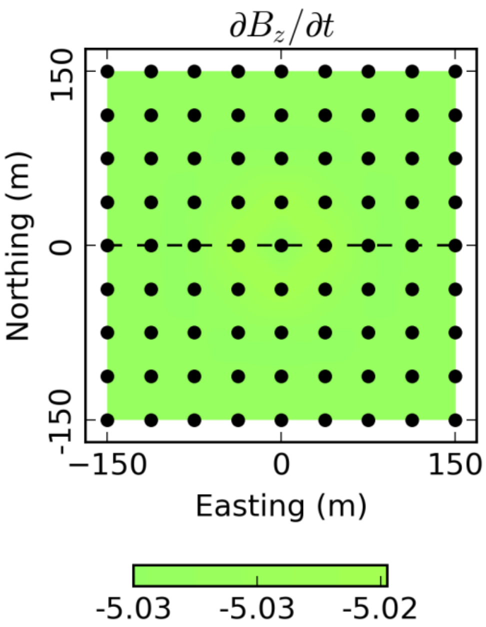


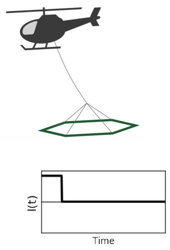
Image credit: Doug Oldenburg, Seogi Kang and Linsey Heagy from UBC-GIF

Data profile



Data map





Conductivity

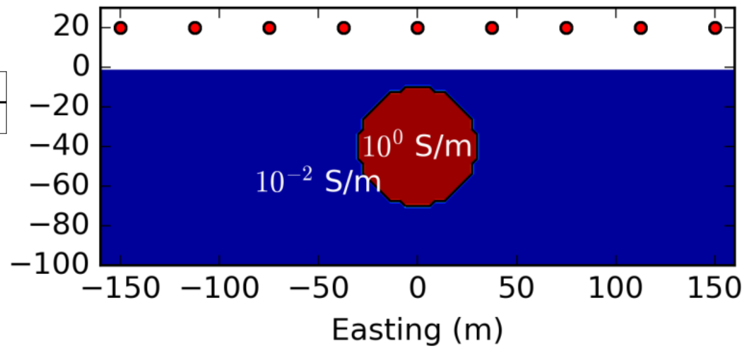


Image credit: Doug Oldenburg, Seogi Kang and Linsey Heagy from UBC-GIF

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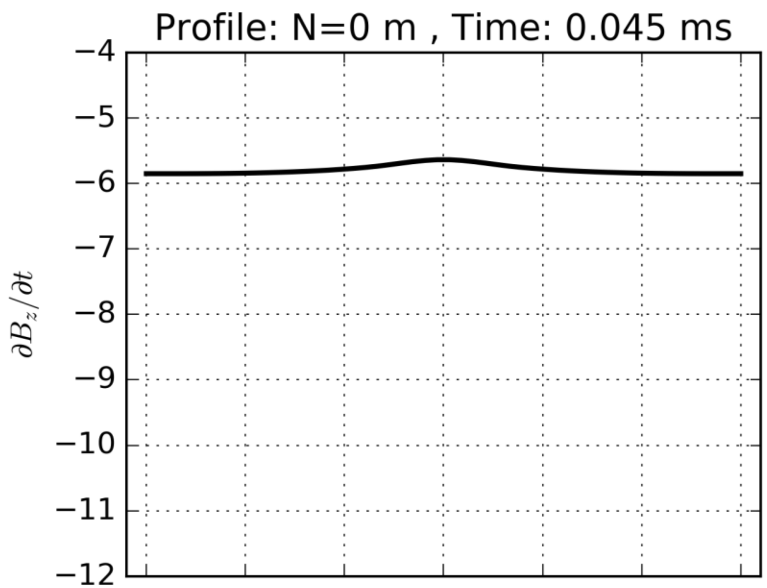

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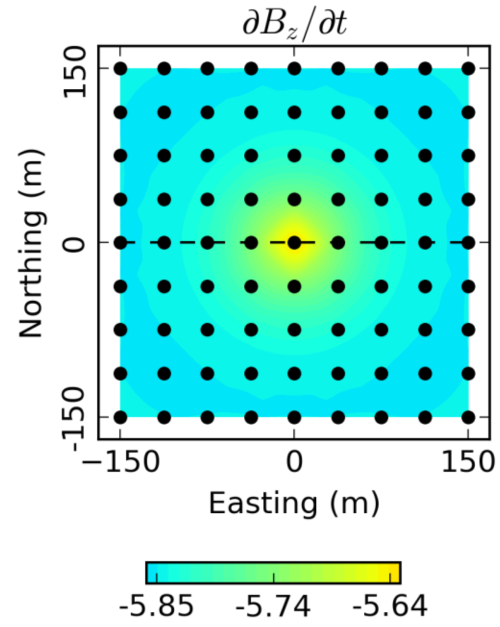
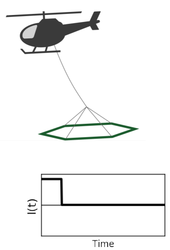
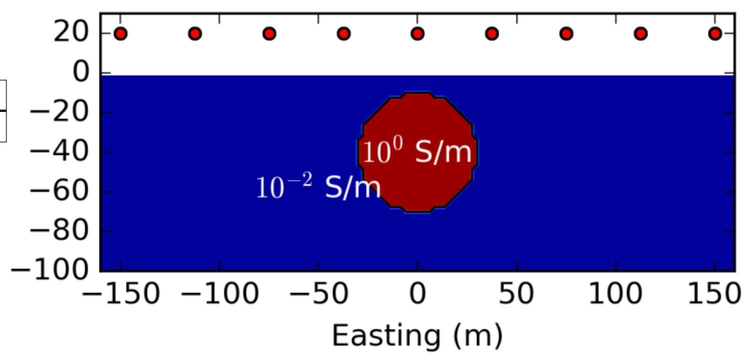




Conductivity



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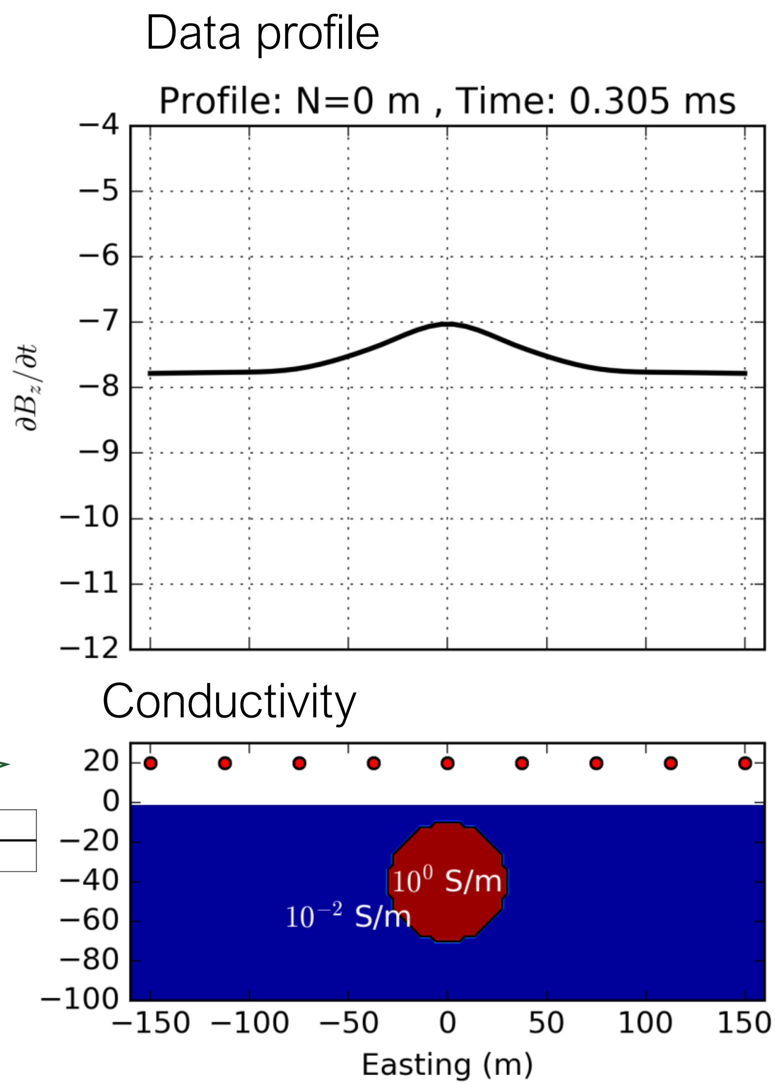

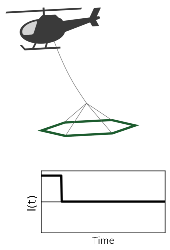

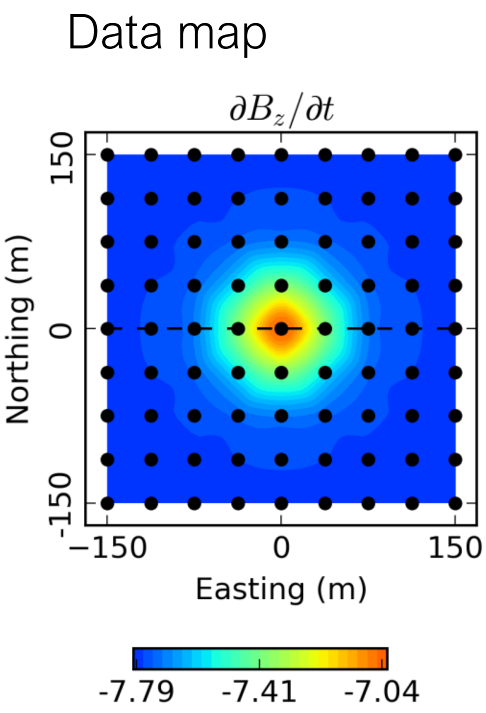

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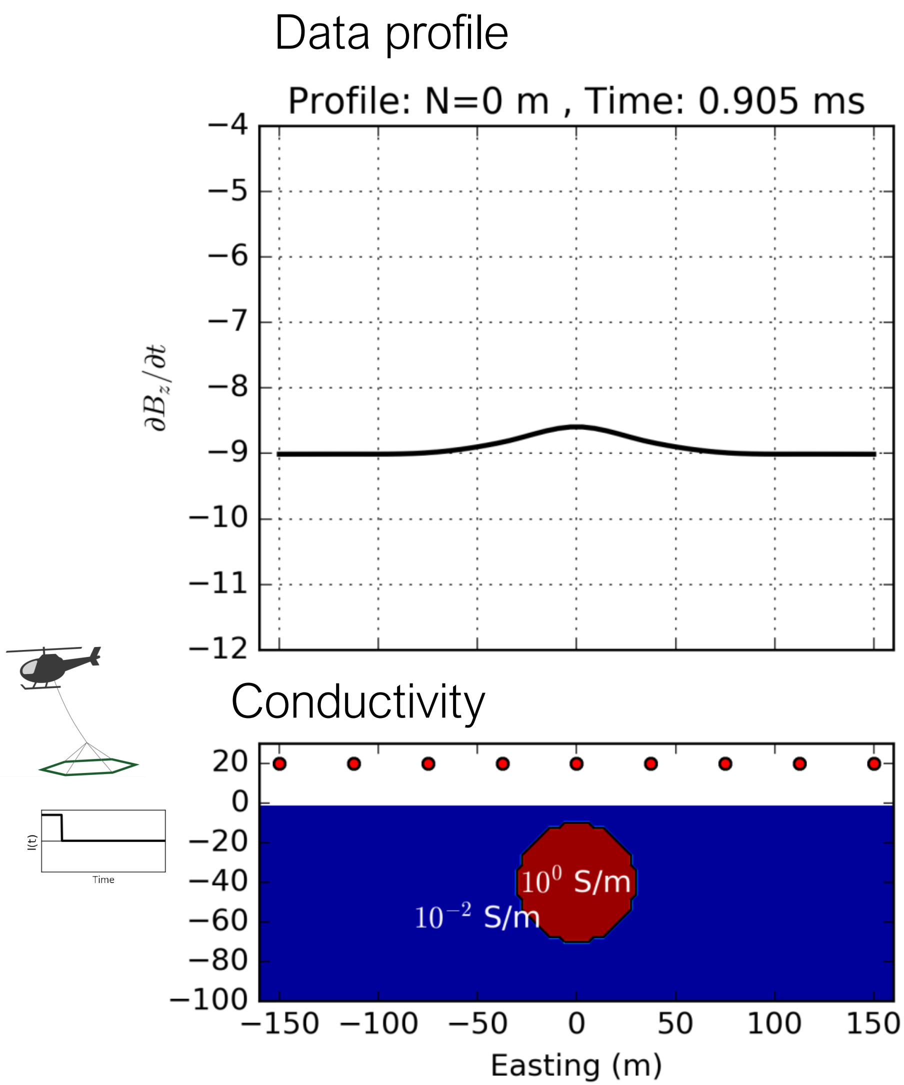

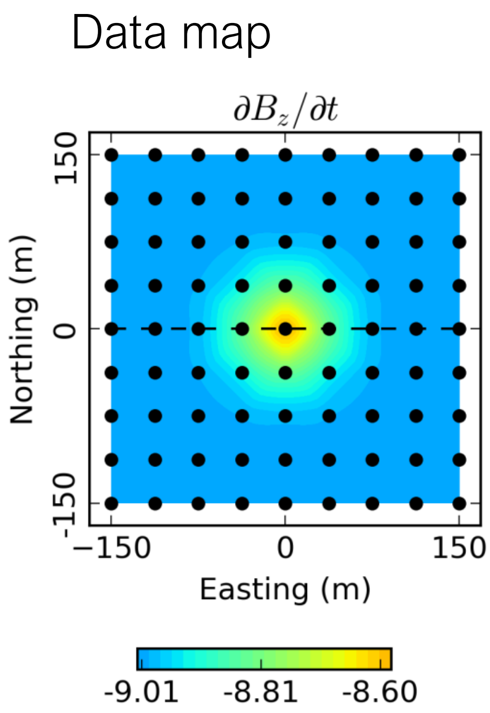

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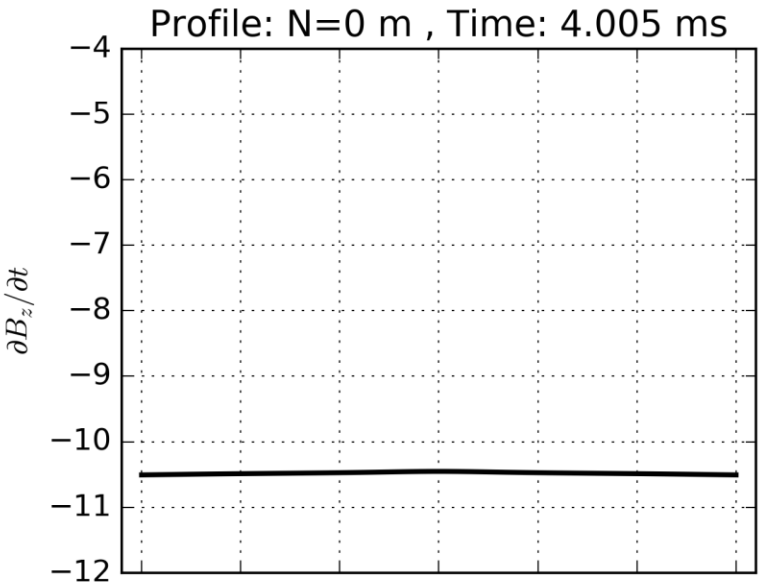
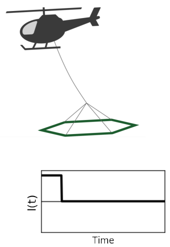
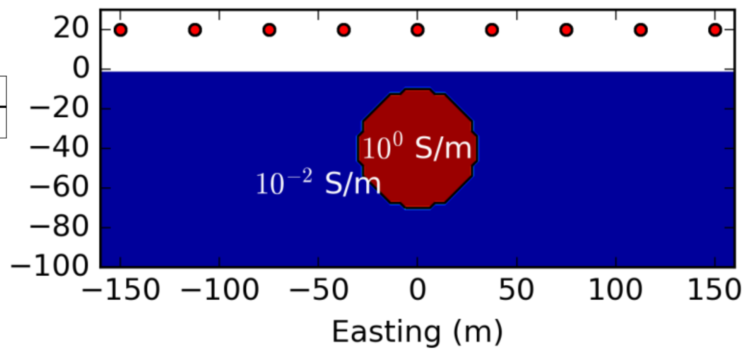
Data profile

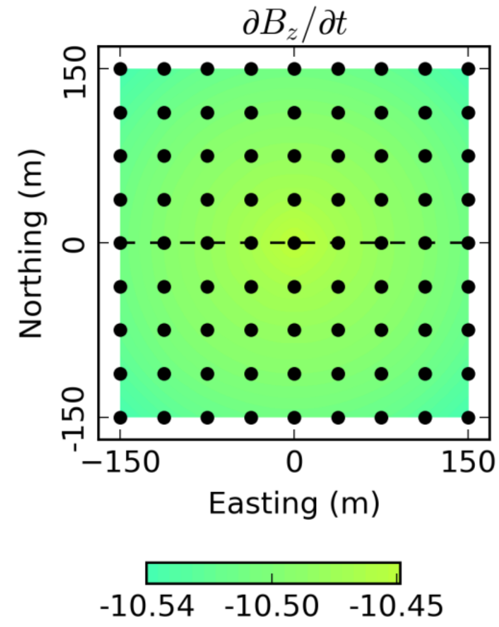




Conductivity



Data map



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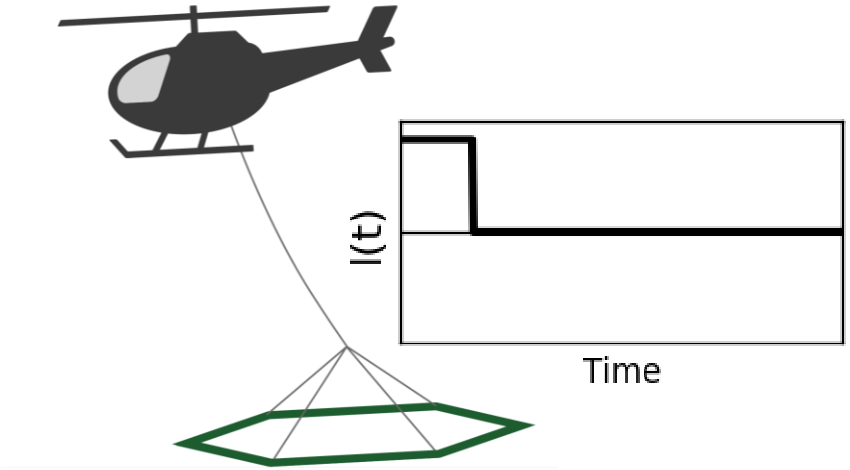
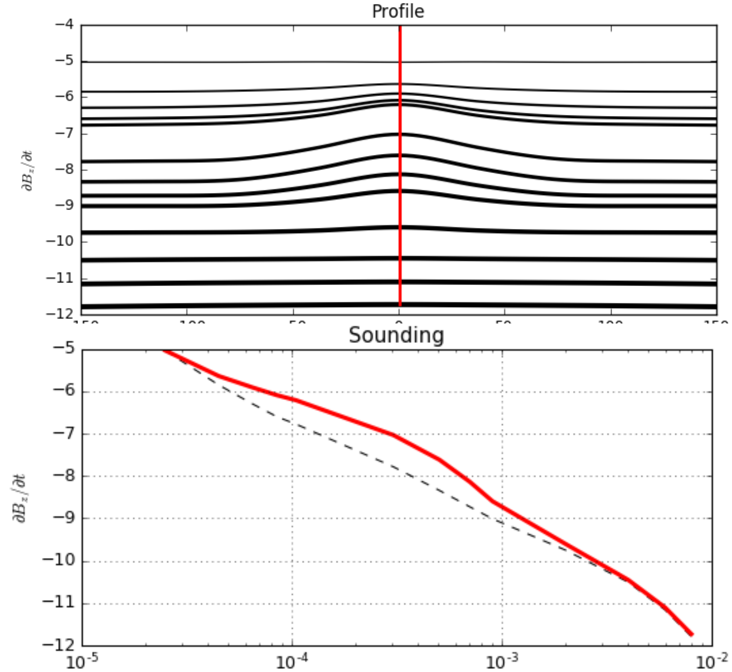
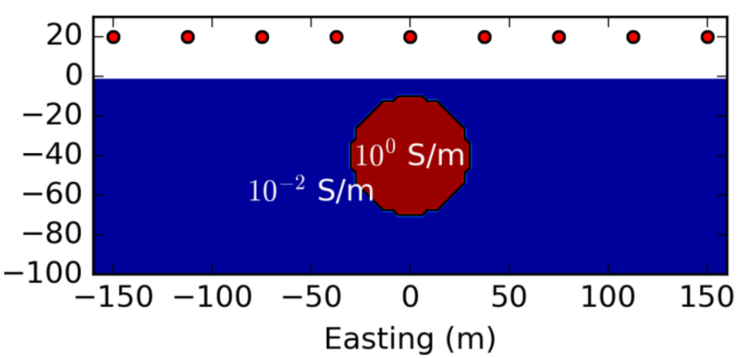
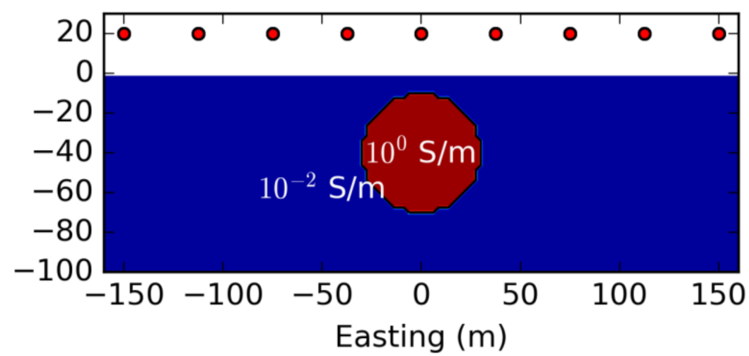
Synthetic airborne TEM data



Data profile







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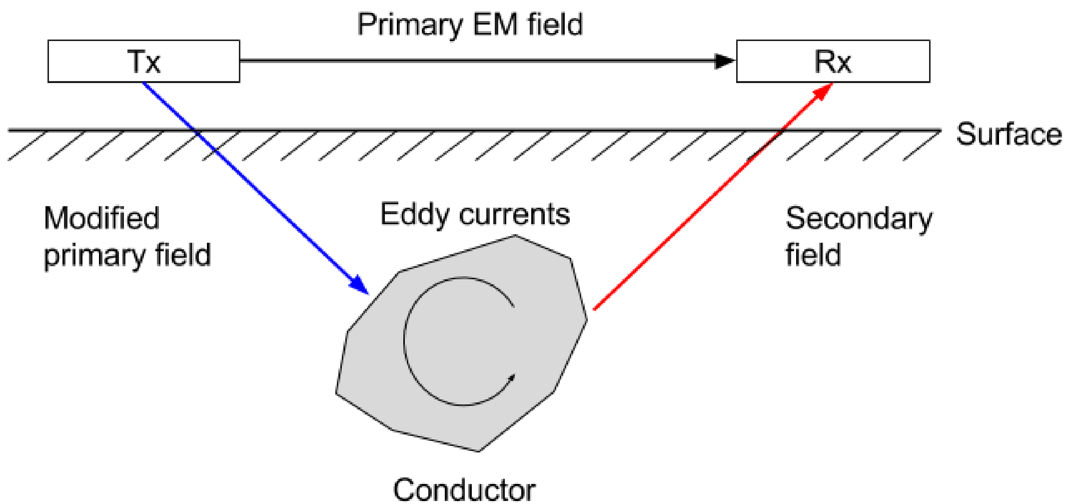
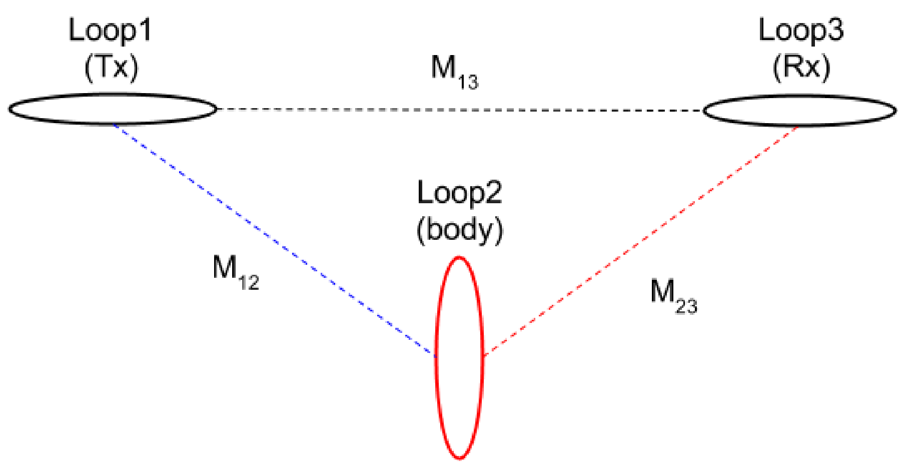
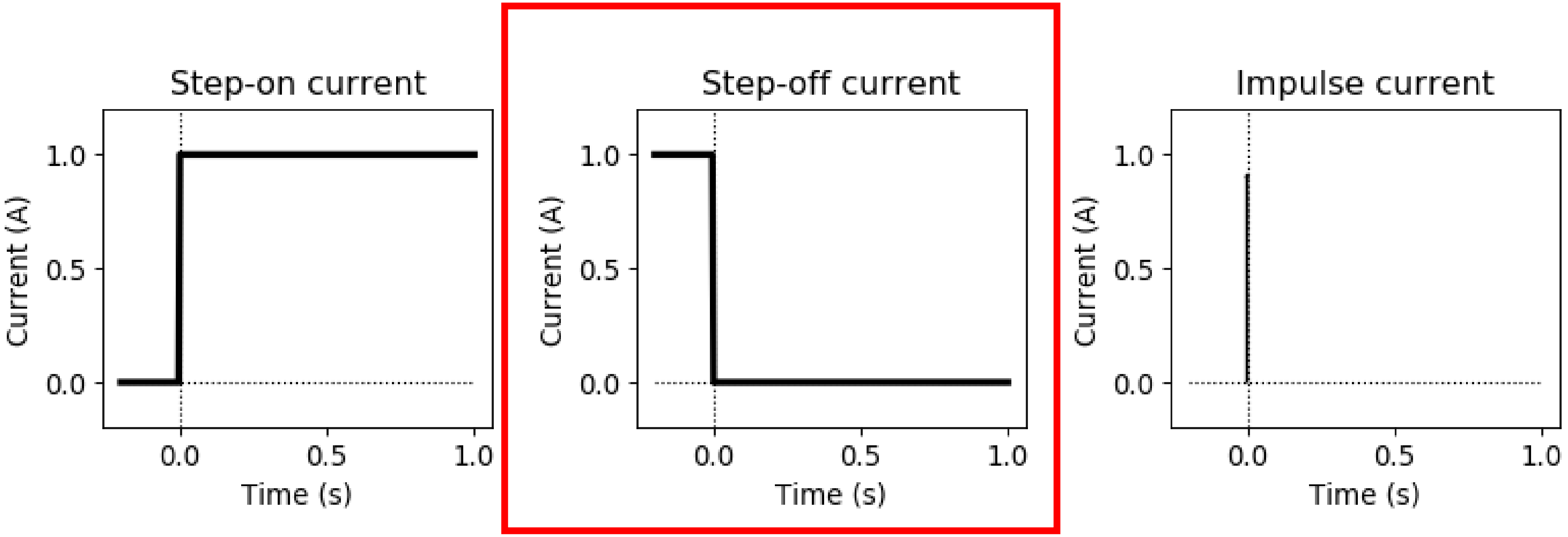




https://em.geosci.xyz/content/maxwell3_fdem/circuitmodel_for_eminduction/index.html

Step-off current

- Let us consider a step-off current in the transmitter



- Mathematically, $I(t) = I_1(1 - u(t))$, where $u(t)$ is Heaviside step function.

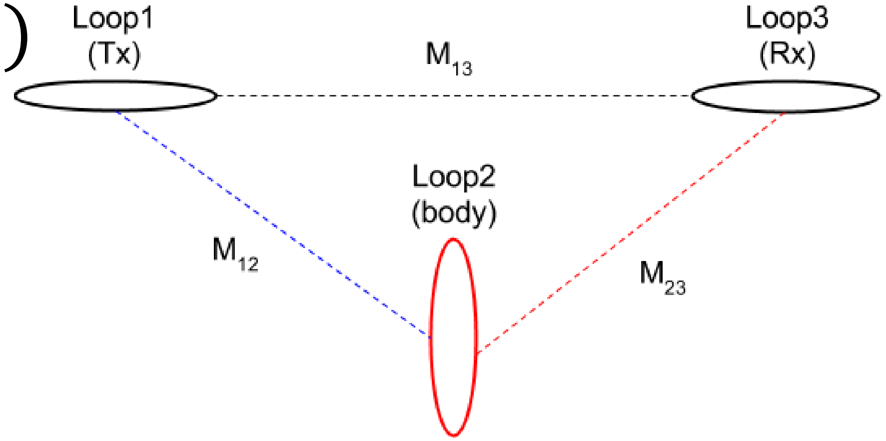
Primary voltage in Loop 3

- The magnetic flux through Loop 3 is

$$\Phi_{13} = M_{13}I(t) = M_{13}I_1(1 - u(t))$$

- Thus, the primary voltage is

$$\varepsilon_3^P = -\frac{\partial \Phi_{13}}{\partial t} = M_{13}I_1\delta(t)$$

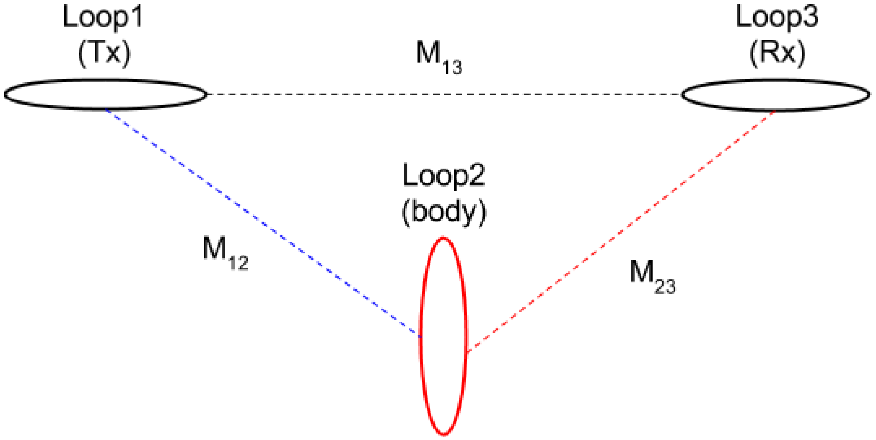


EMF in Loop 2

- What is the EMF in Loop 2 due to the step-off current in Loop 1?

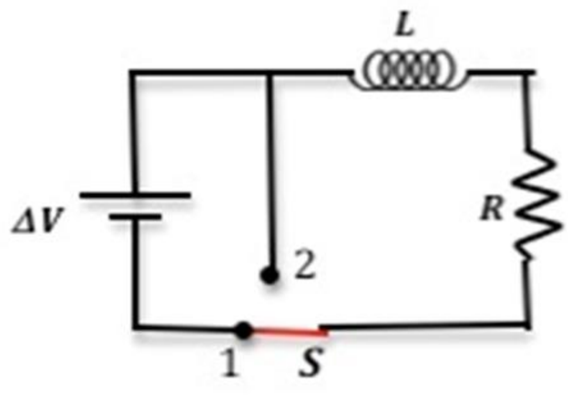
$$\Phi_{12} = M_{12}I(t) = M_{12}I_1(1 - u(t))$$

$$\varepsilon_2 = -\frac{\partial \Phi_{12}}{\partial t} = M_{12}I_1\delta(t)$$



How about current in Loop 2?

- Consider Loop 2 as a RL circuit
- Applying KVL, we have

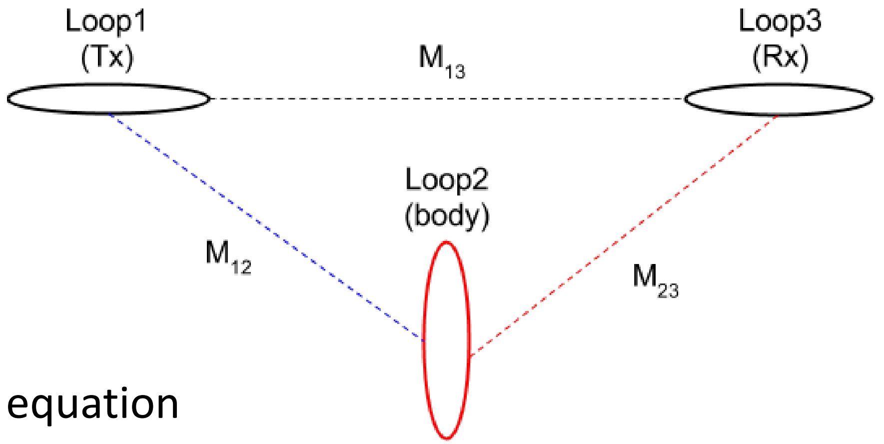


$$\varepsilon_2 - L\frac{dI_2(t)}{dt} - I_2(t)R = 0$$

- Solving the above equation, we obtain

$$I_2(t) = \frac{M_{12}I_1}{L}e^{-t/\tau}, (t > 0)$$

time constant $\tau = \frac{L}{R}$



See next slide for how to solve the differential equation

Calculate current in Loop 2 (Optional)

Applying KVL, we have

$$\varepsilon_2 - L\frac{dI_2(t)}{dt} - I_2(t)R = 0$$

Remember

$$\varepsilon_2 \ = -\frac{\partial \Phi_{12}}{\partial t} = M_{12}I_1\delta(t)$$

Therefore,

$$M_{12}I_1\delta(t) = L\frac{dI_2(t)}{dt} + I_2(t)R$$

Apply Fourier transform to both sides

$$\mathcal{F}[\delta(t)] = 1$$

$$\mathcal{F}[I_2(t)] = \widetilde{I_2}(\omega)$$

Therefore,

$$M_{12}I_1 = L \cdot i\omega \cdot \widetilde{I_2}(\omega) + \widetilde{I_2}(\omega)R$$

It then follows that,

$$\widetilde{I_2}(\omega) = \frac{M_{12}I_1}{R + i\omega L}$$

In order to get $I_2(t)$, apply the inverse Fourier transform

$$I_2(t) = \mathcal{F}^{-1}\left[\widetilde{I_2}(\omega)\right] = \mathcal{F}^{-1}\left[\frac{M_{12}I_1}{R + i\omega L}\right]$$

Note that

$\mathcal{F}[e^{-at}u(t)] = \frac{1}{a+i\omega}$, where $u(t)$ is Heaviside step function

Thus, $\mathcal{F}^{-1}\left[\frac{1}{a+i\omega}\right] = e^{-at}u(t)$

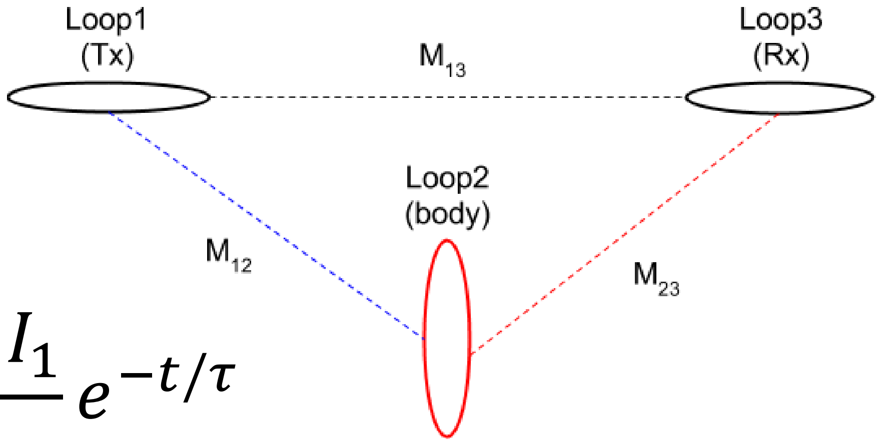
Therefore,

$$I_2(t) = \mathcal{F}^{-1}\left[\frac{M_{12}I_1}{R+i\omega L}\right] = M_{12}I_1 \cdot \mathcal{F}^{-1}\left[\frac{1}{L} \cdot \frac{1}{\frac{1}{\tau}+i\omega}\right] =$$

$\frac{M_{12}I_1 e^{-\frac{t}{\tau}}}{L}u(t) = \frac{M_{12}I_1}{L}e^{-t/\tau}$, $(t > 0)$, where $\tau = \frac{L}{R}$

Secondary voltage in Loop 3

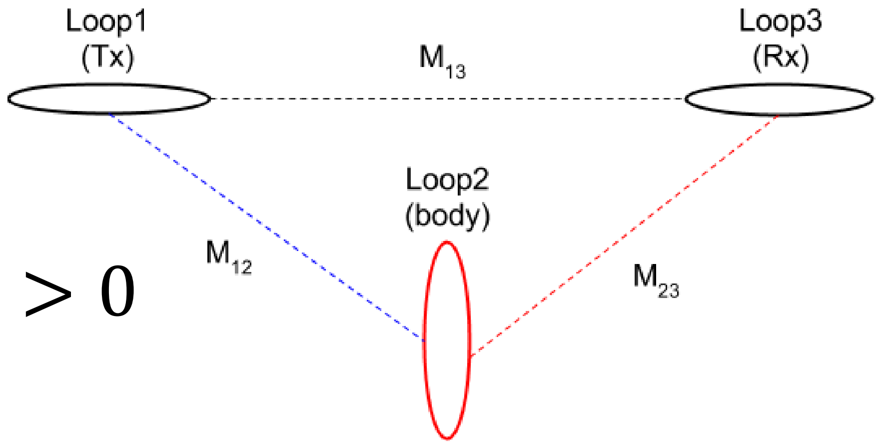
- Now that we have obtained the current in Loop 3, how about the secondary EMF in Loop 3?



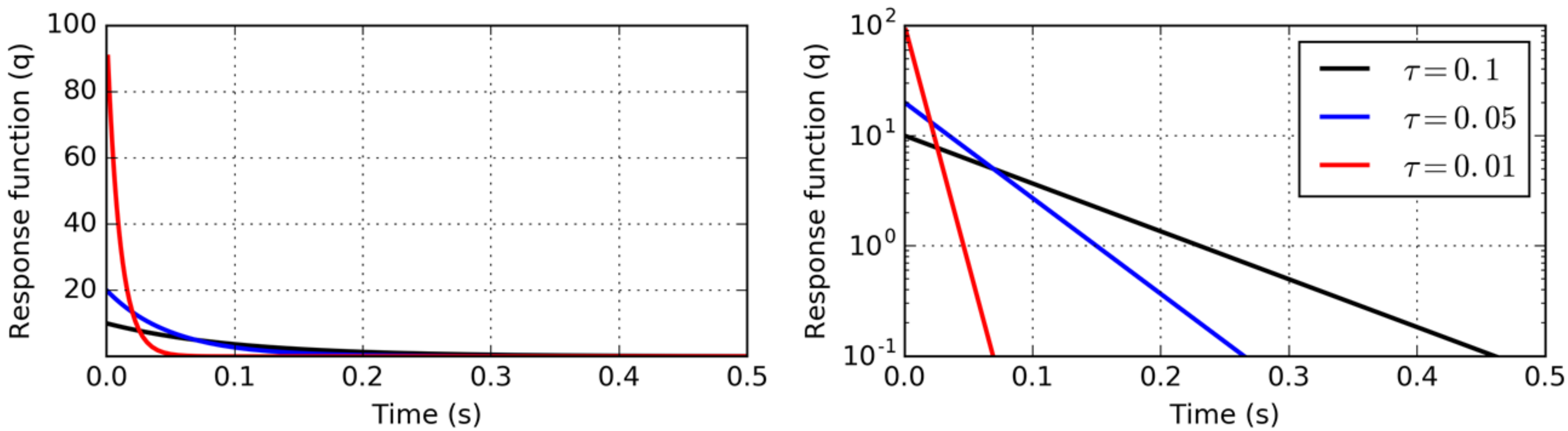
$$\Phi_{23} = M_{23} I_2(t) = M_{23} \frac{M_{12} I_1}{L} e^{-t/\tau}$$

$$\varepsilon_3^S = -\frac{\partial \Phi_{23}}{\partial t} = \frac{M_{12} M_{23}}{L} \frac{I_1}{\tau} e^{-t/\tau}, t > 0$$

Secondary voltage in Loop 3



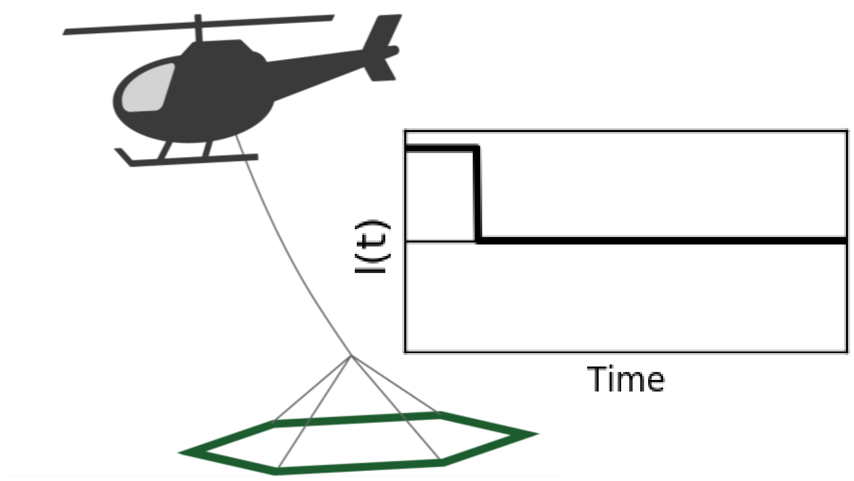
$$\varepsilon_3^S = -\frac{\partial \Phi_{23}}{\partial t} = \frac{M_{12}M_{23}}{L}\frac{I_1}{\tau}e^{-t/\tau}, t > 0$$



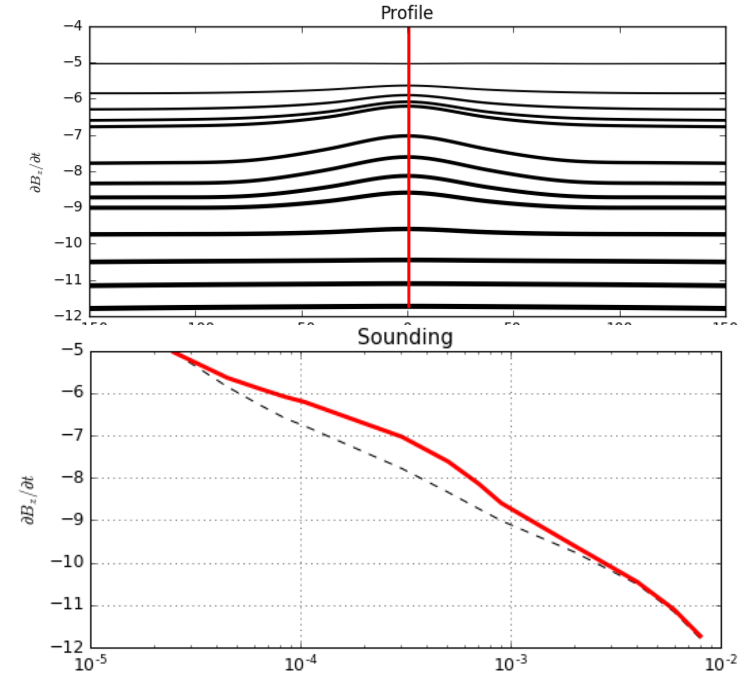
The larger the τ, the slower it decays. Remember $\tau = \frac{L}{R}$.

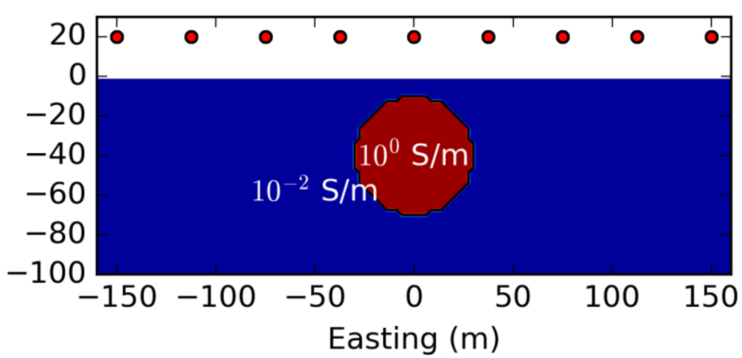
Therefore, the more conductive the subsurface, the slower it decays.

Synthetic airborne TEM data



Data profile





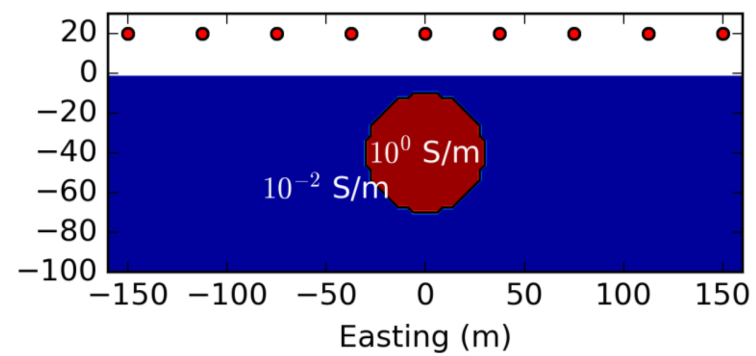


Image credit: Doug Oldenburg, Seogi Kang and Linsey Heagy from UBC-GIF