Lecture 6
Complex Variables & Fourier Transform

GEOL 4397: Electromagnetic Methods for Exploration
GEOL 6398: Special Problems

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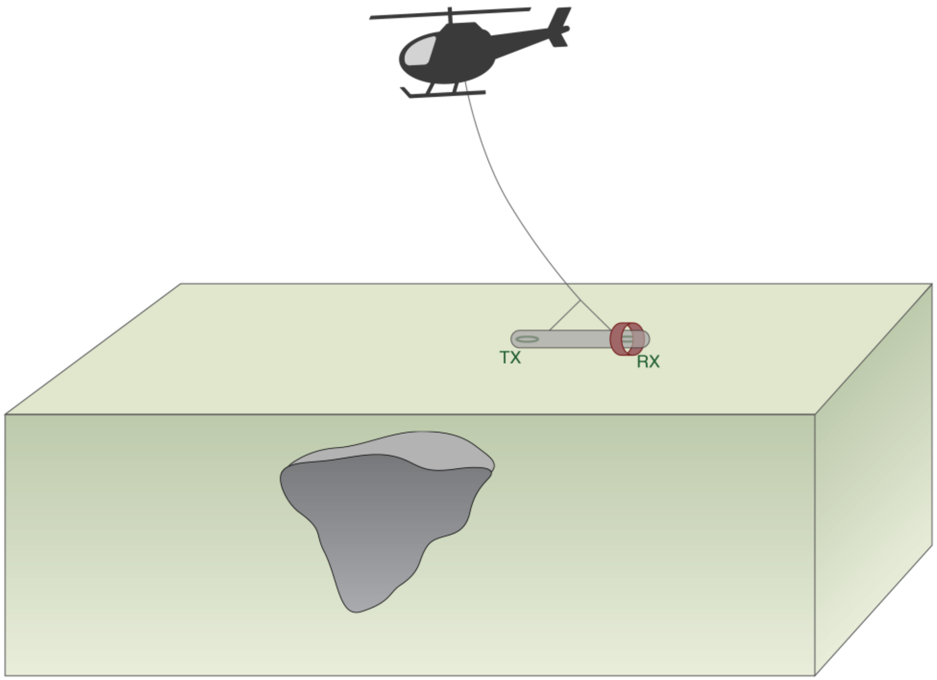
UNIVERSITY of HOUSTON
YOU ARE THE PRIDE
EARTH AND ATMOSPHERIC SCIENCES

Agenda

- Motivation
- Frequency and phase
- In-phase vs. out-of-phase
- Complex variables
- Fourier Transform

Basic Experiment

- **Setup:**
  - transmitter and receiver are in a towed bird



Basic Experiment

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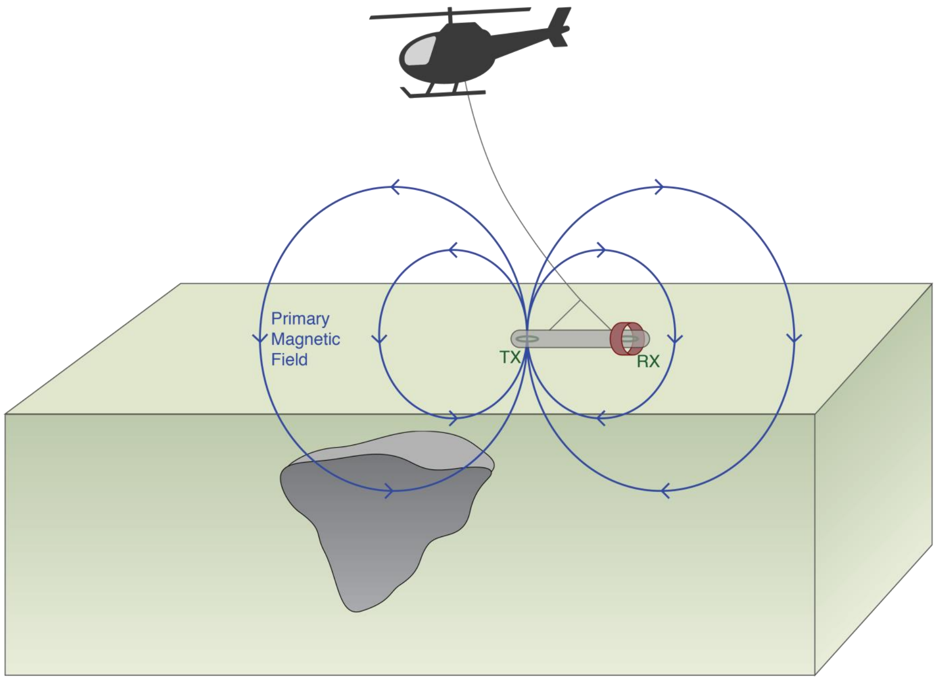
  - transmitter and receiver are in a towed bird
- **Primary:**
  - Transmitter produces a primary magnetic field



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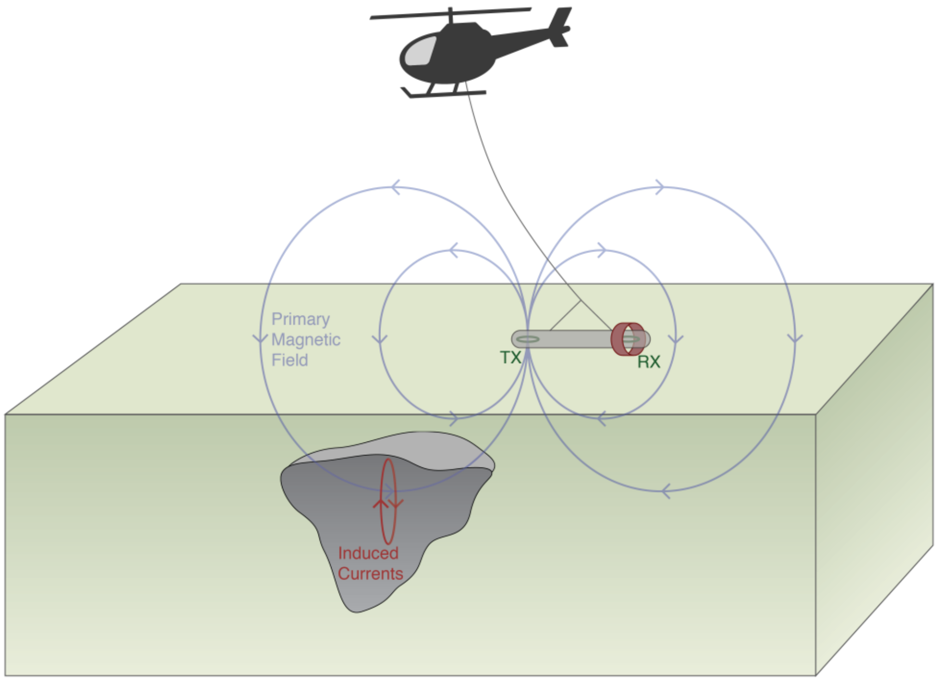
- **Induced Currents:**
  - Time varying magnetic fields generate electric fields everywhere and currents in conductors



Basic Experiment

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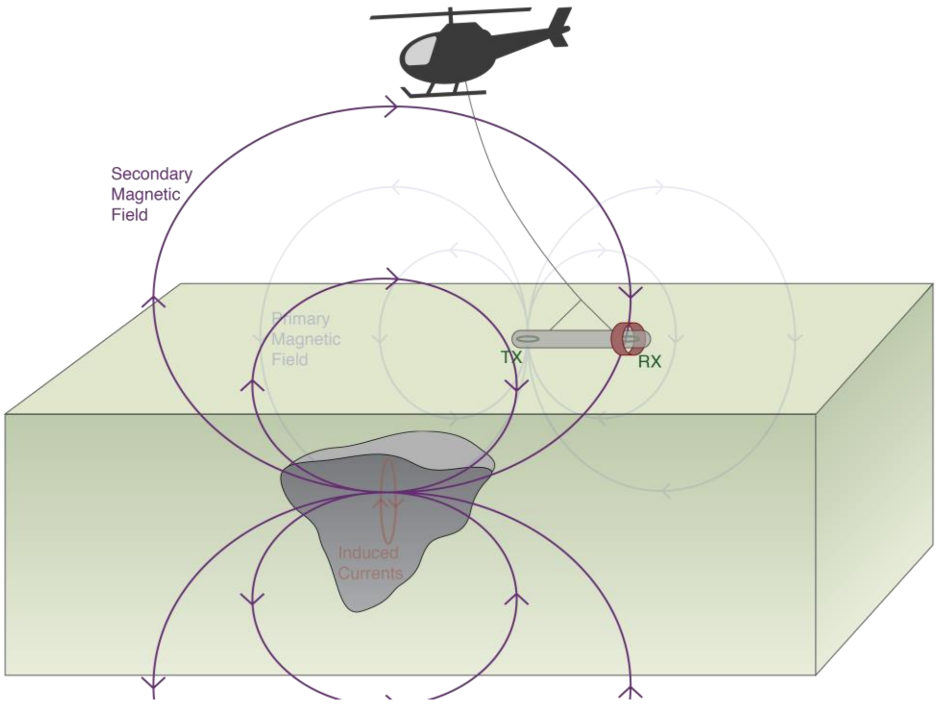
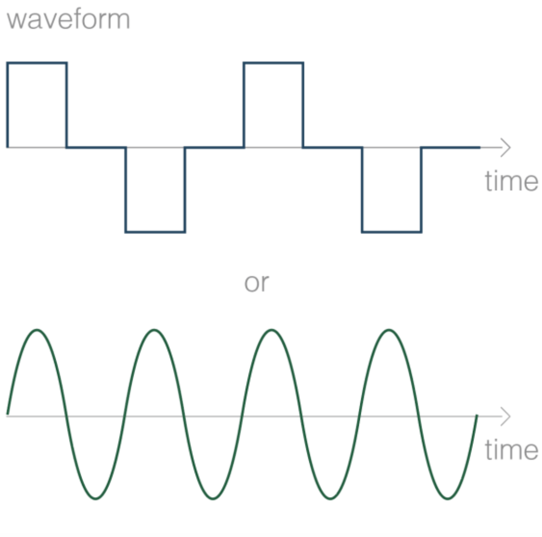
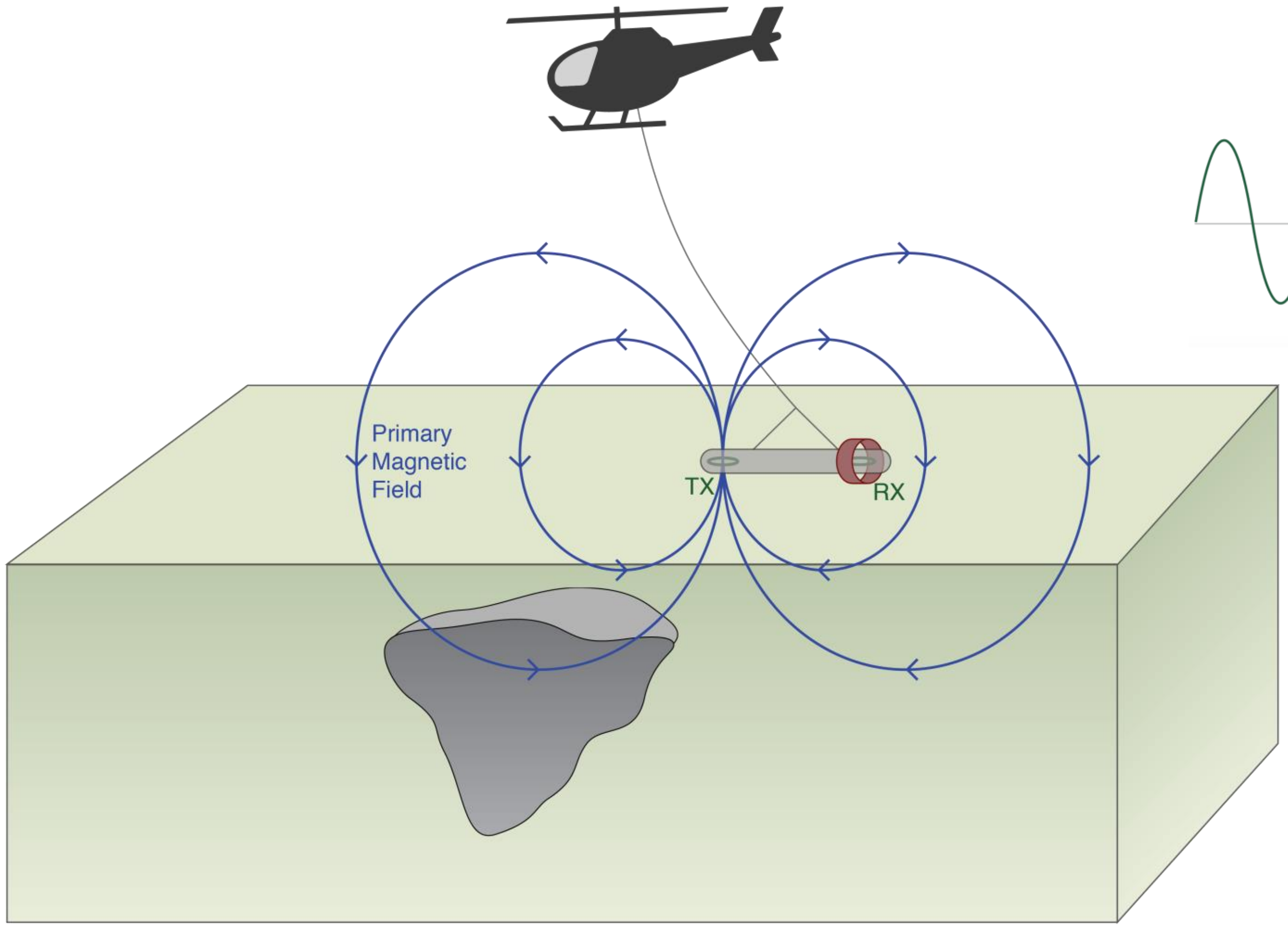
  - transmitter and receiver are in a towed bird
- **Primary:**
  - Transmitter produces a primary magnetic field
- **Induced Currents:**
  - Time varying magnetic fields generate electric fields everywhere and currents in conductors
- **Secondary Fields:**
  - The induced currents produce a secondary magnetic field.

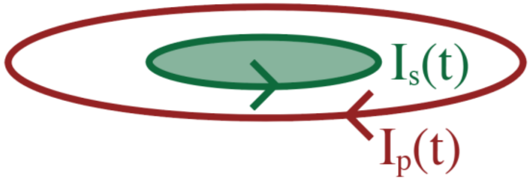


Transmitter





Two Coil Example: Harmonic

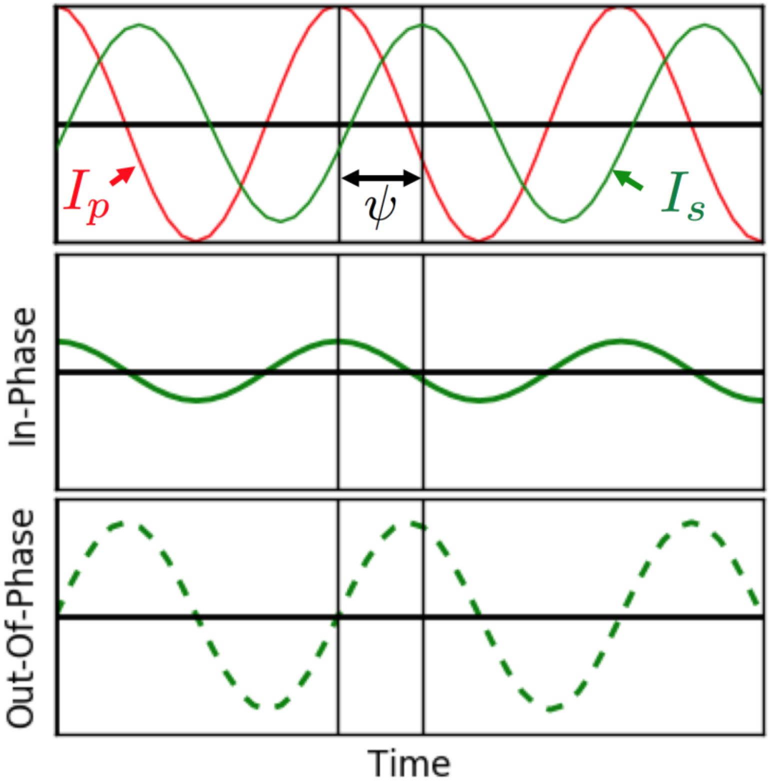


Induced Currents

$$I_p(t) = I_p \cos \omega t$$

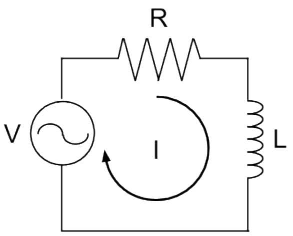
$$I_s(t) = I_s \cos(\omega t - \psi)$$

$$= \underbrace{I_s \cos \psi}_{\substack{\text{In-Phase} \\ \text{Real}}} \cos \omega t + \underbrace{I_s \sin \psi}_{\substack{\text{Out-of-Phase} \\ \text{Quadrature} \\ \text{Imaginary}}} \sin \omega t$$



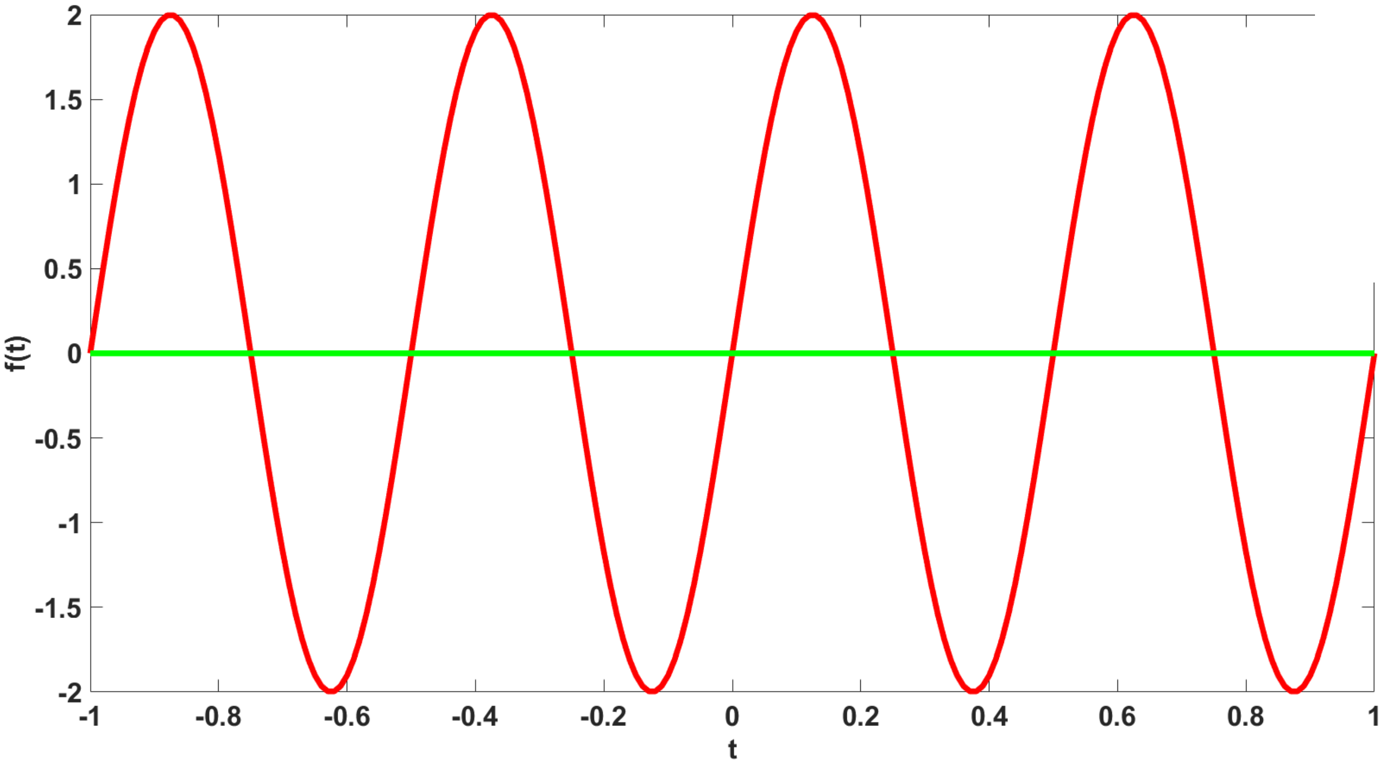
Phase Lag

$$\psi = \frac{\pi}{2} + \tan^{-1}\left(\frac{\omega L}{R}\right)$$



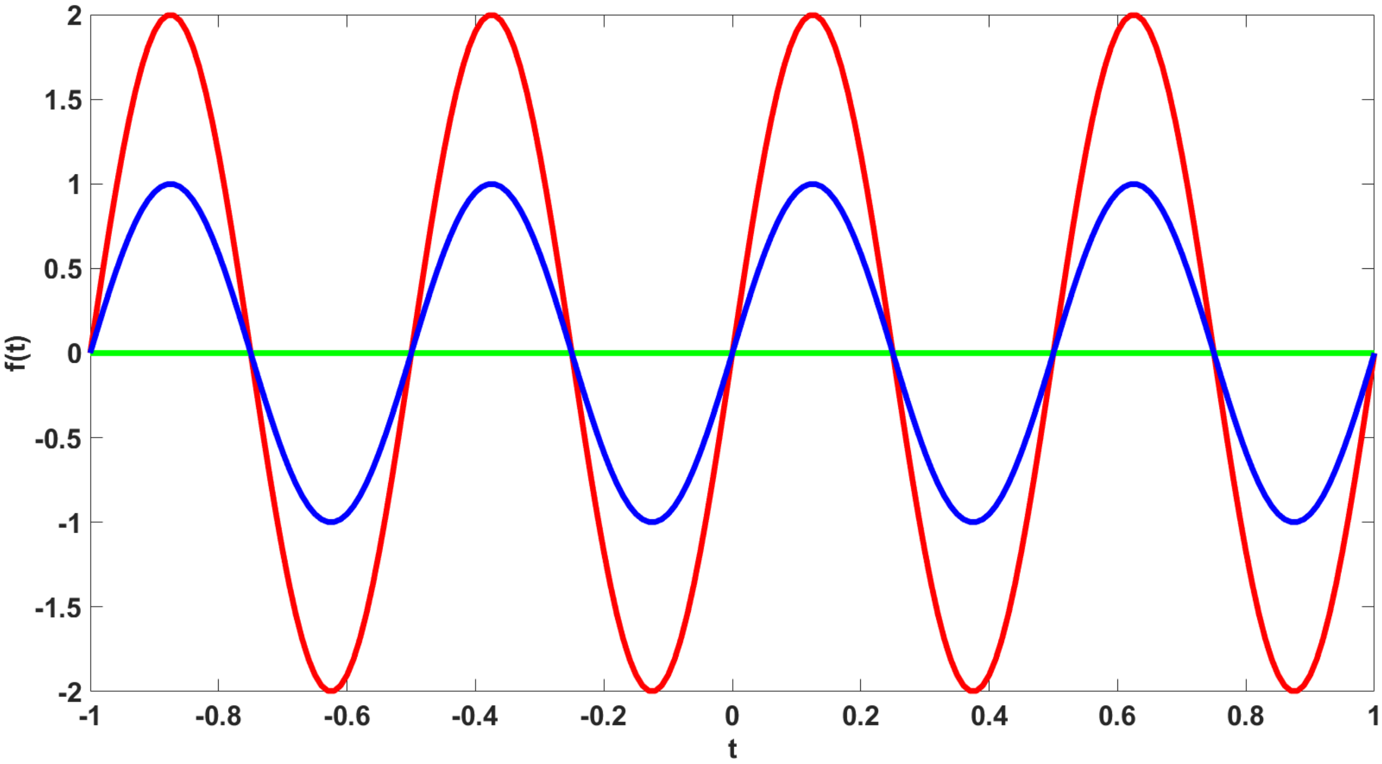
A simple waveform

$$f(t) = 2.0 * \sin(2 * \pi * 2 * t)$$



Another waveform

$$f(t) = 1.0 * \sin(2 * \pi * 2 * t)$$



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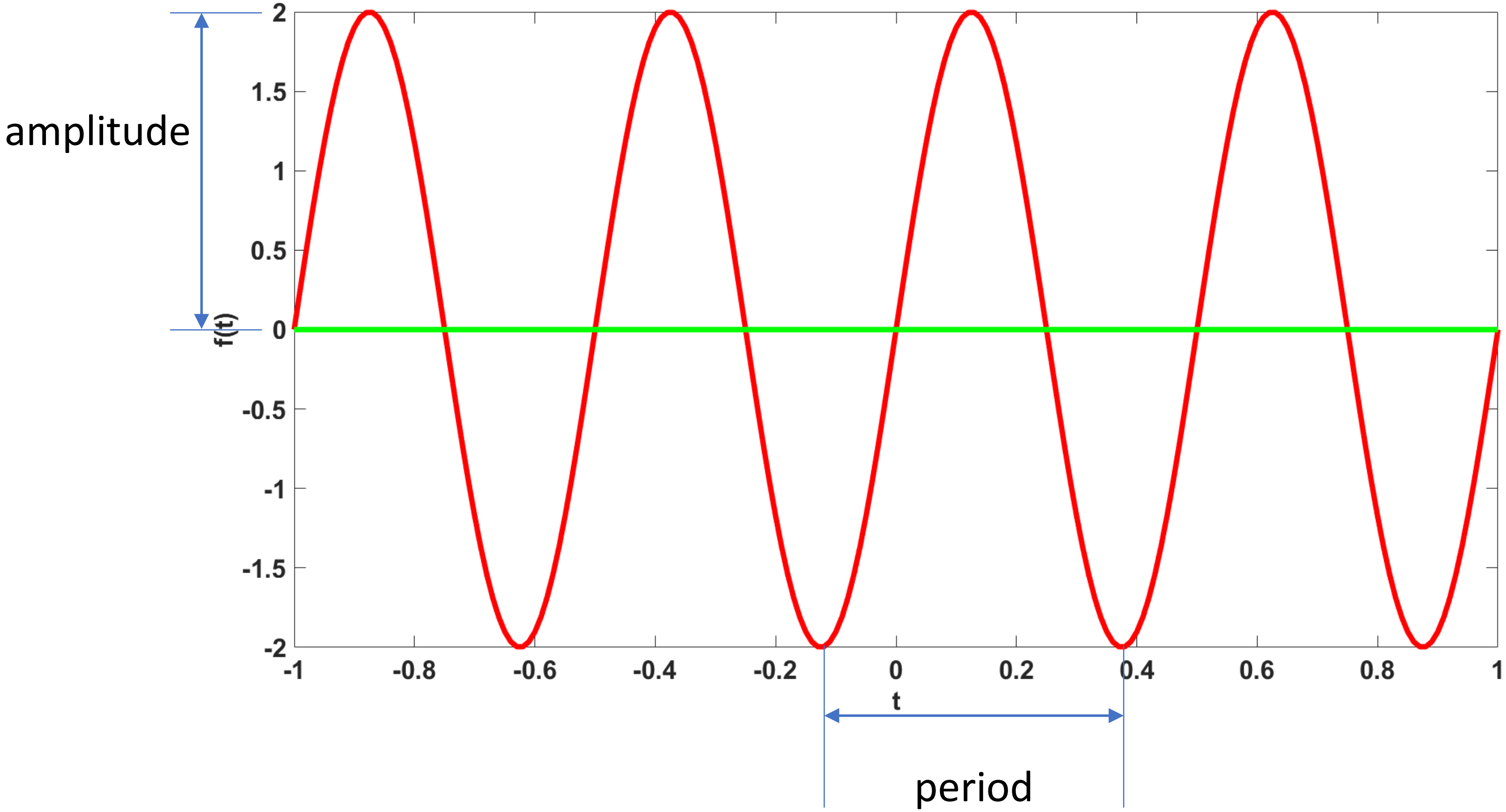
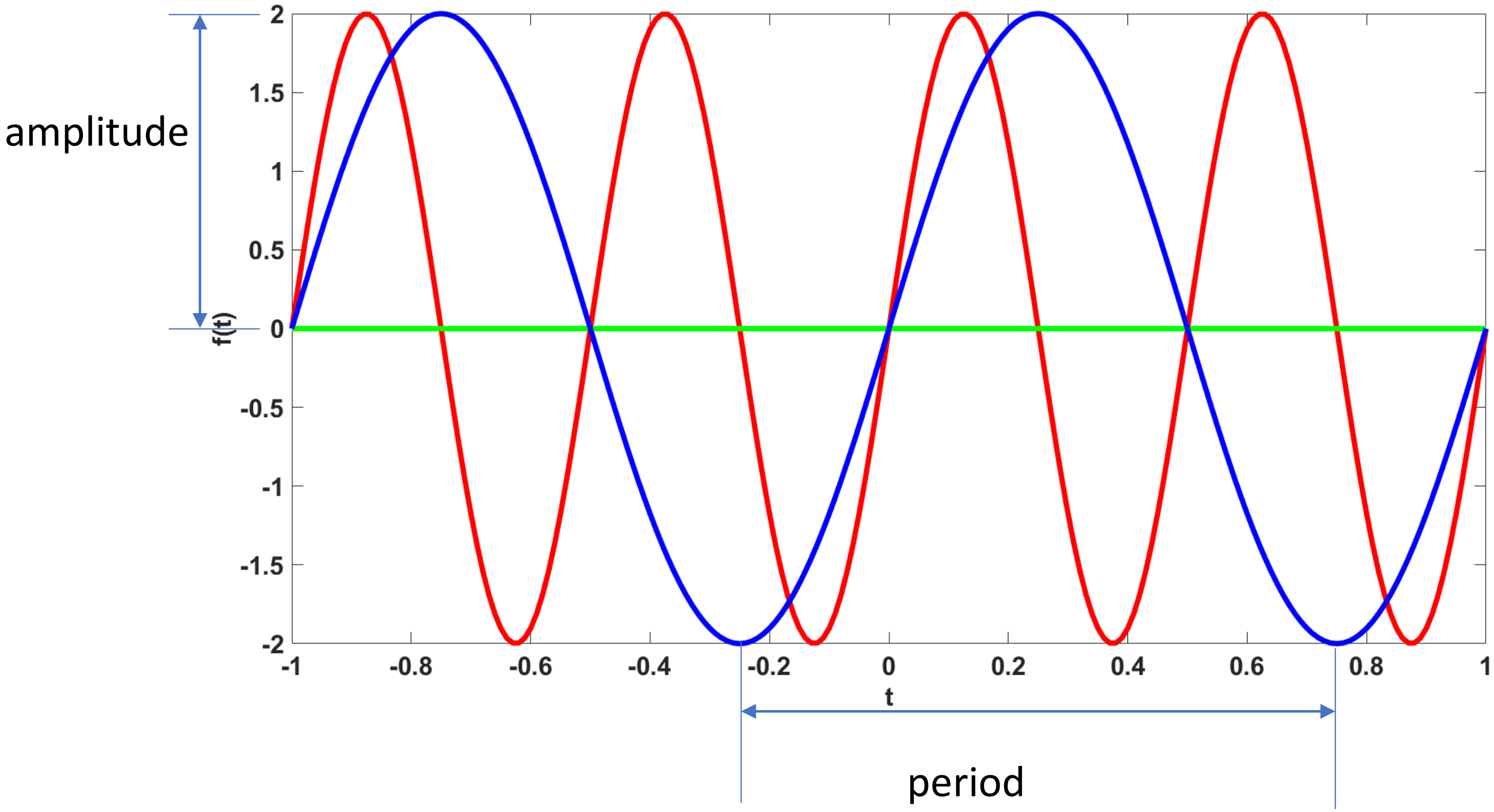
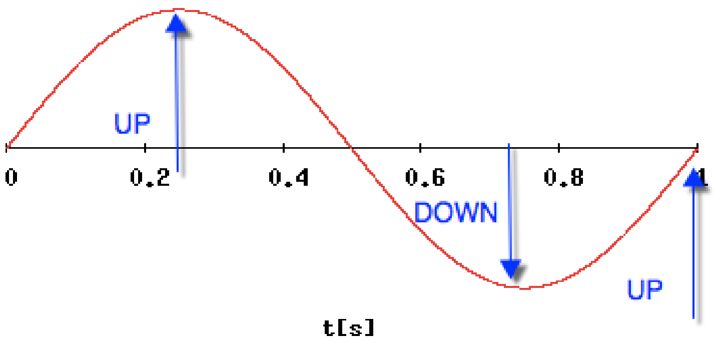


A different waveform

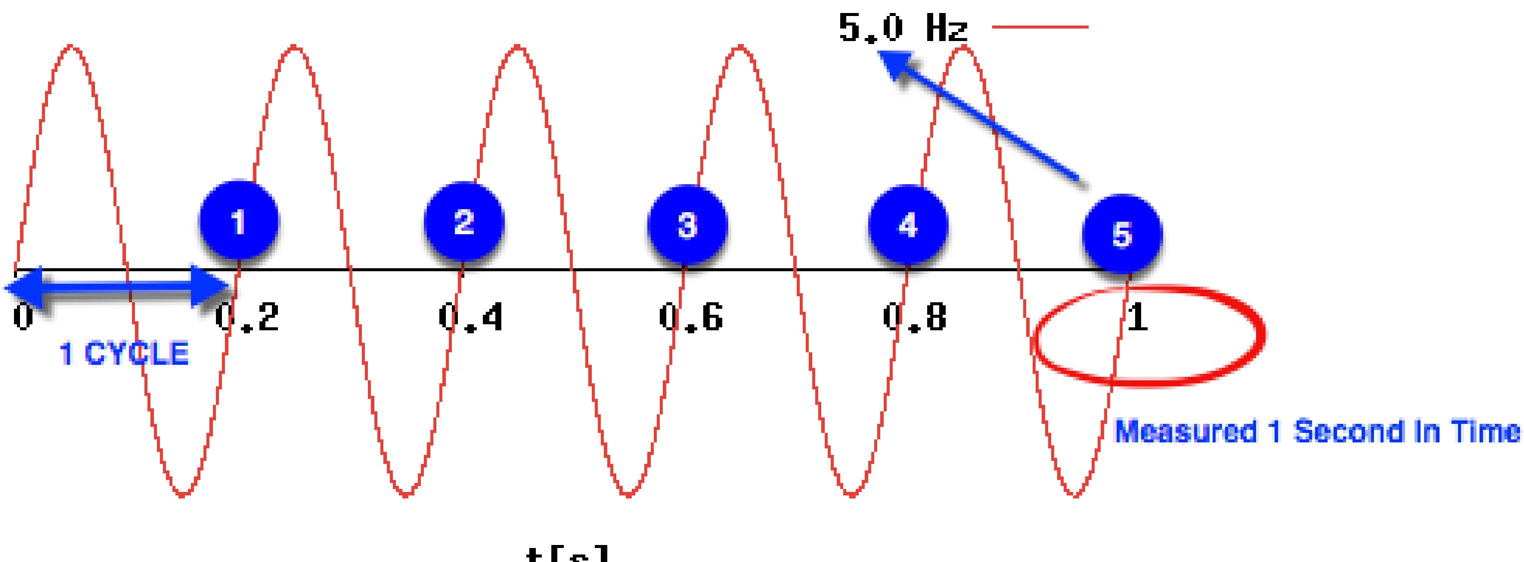
$$f(t) = 2.0 * \sin(2 * \pi * 1 * t)$$



Frequency



- 1 period = 1 cycle
- Frequency: # of cycles in 1 second
- SI unit: Hz



https://community.arubanetworks.com/t5/Technology-Blog/Frequency-Cycle-Wavelength-Amplitude-and-Phase/ba-p/222900

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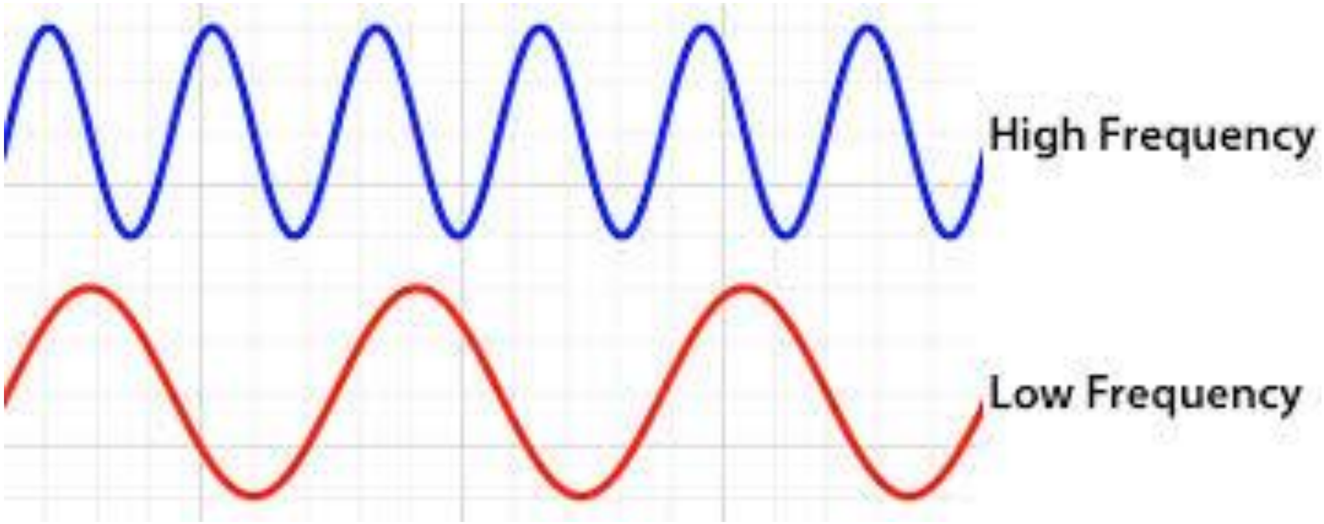

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A simple waveform

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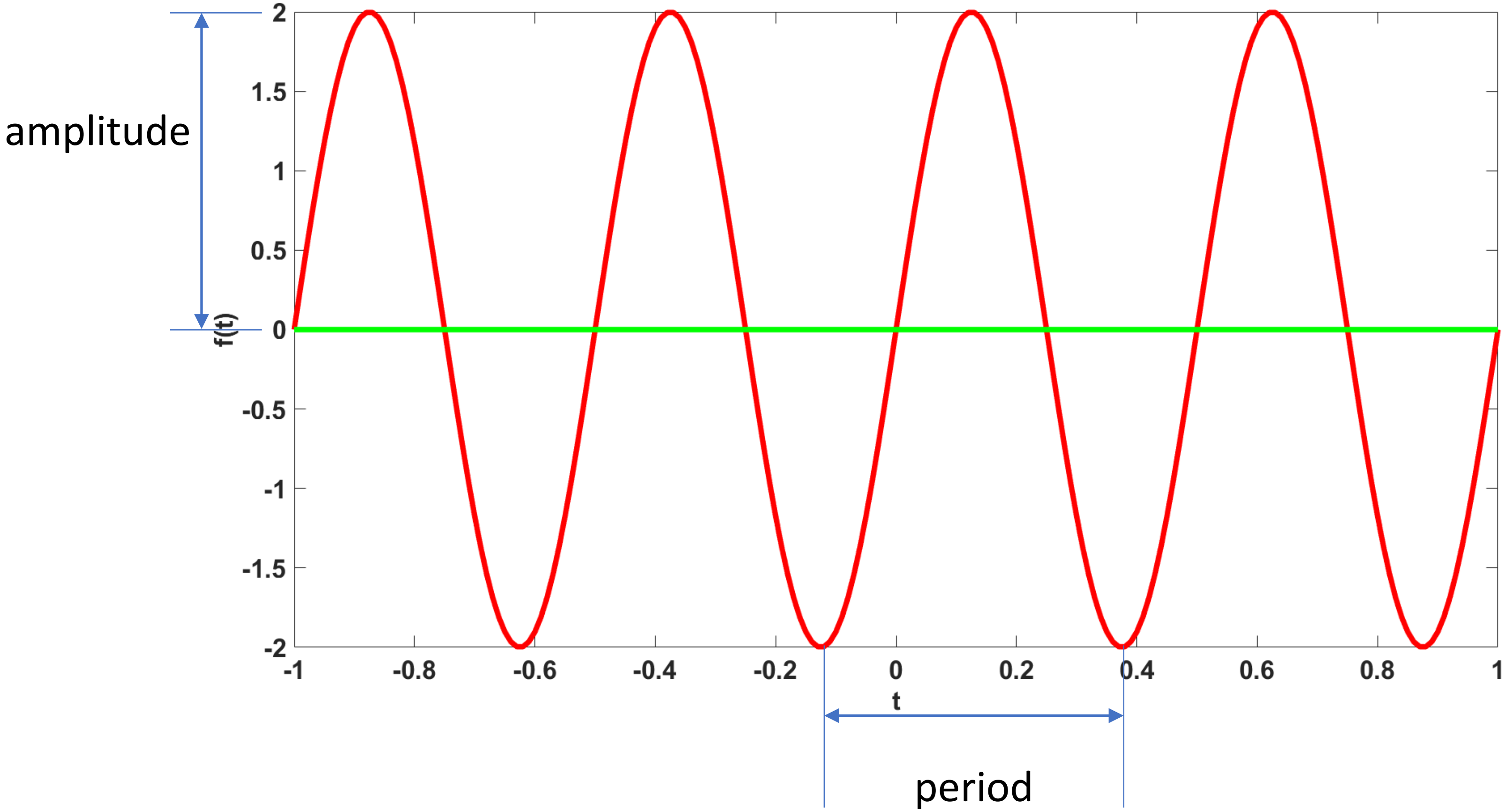
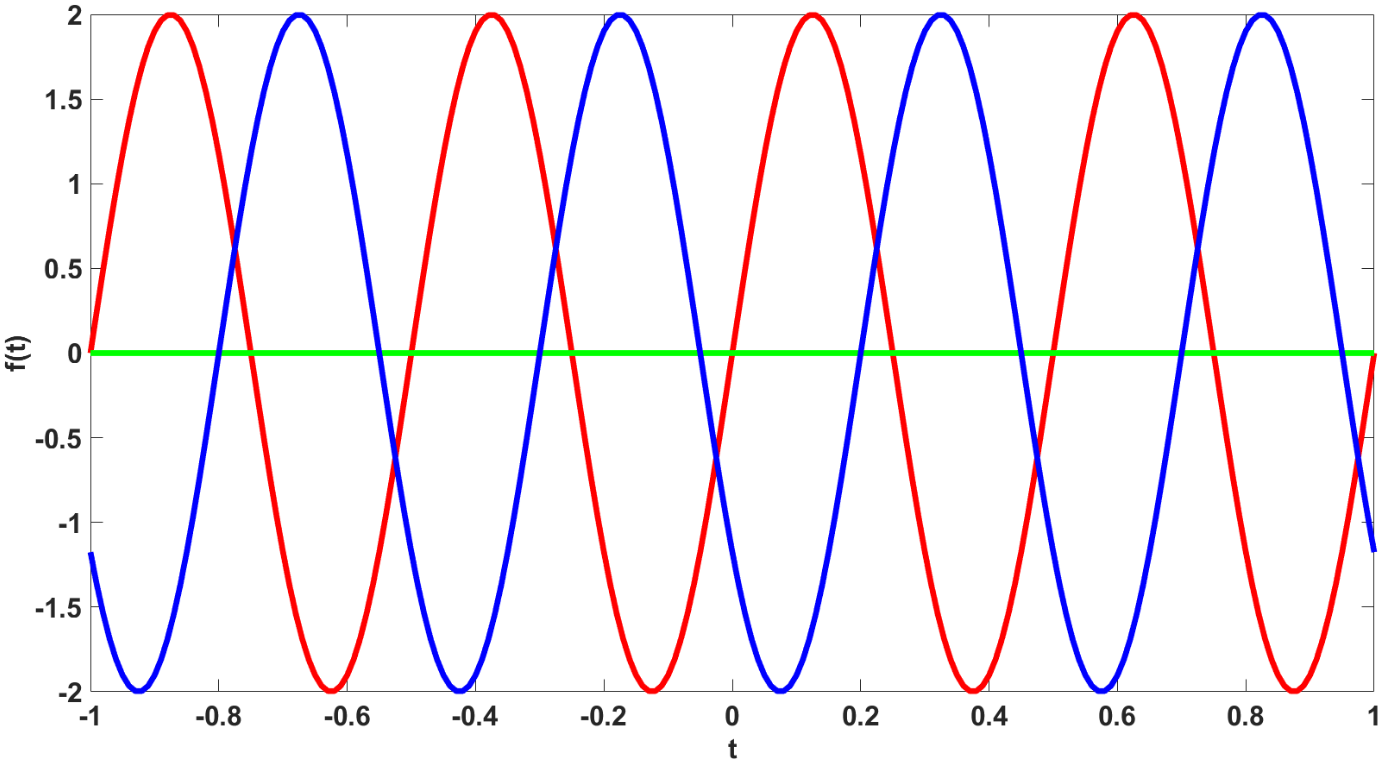


A different waveform

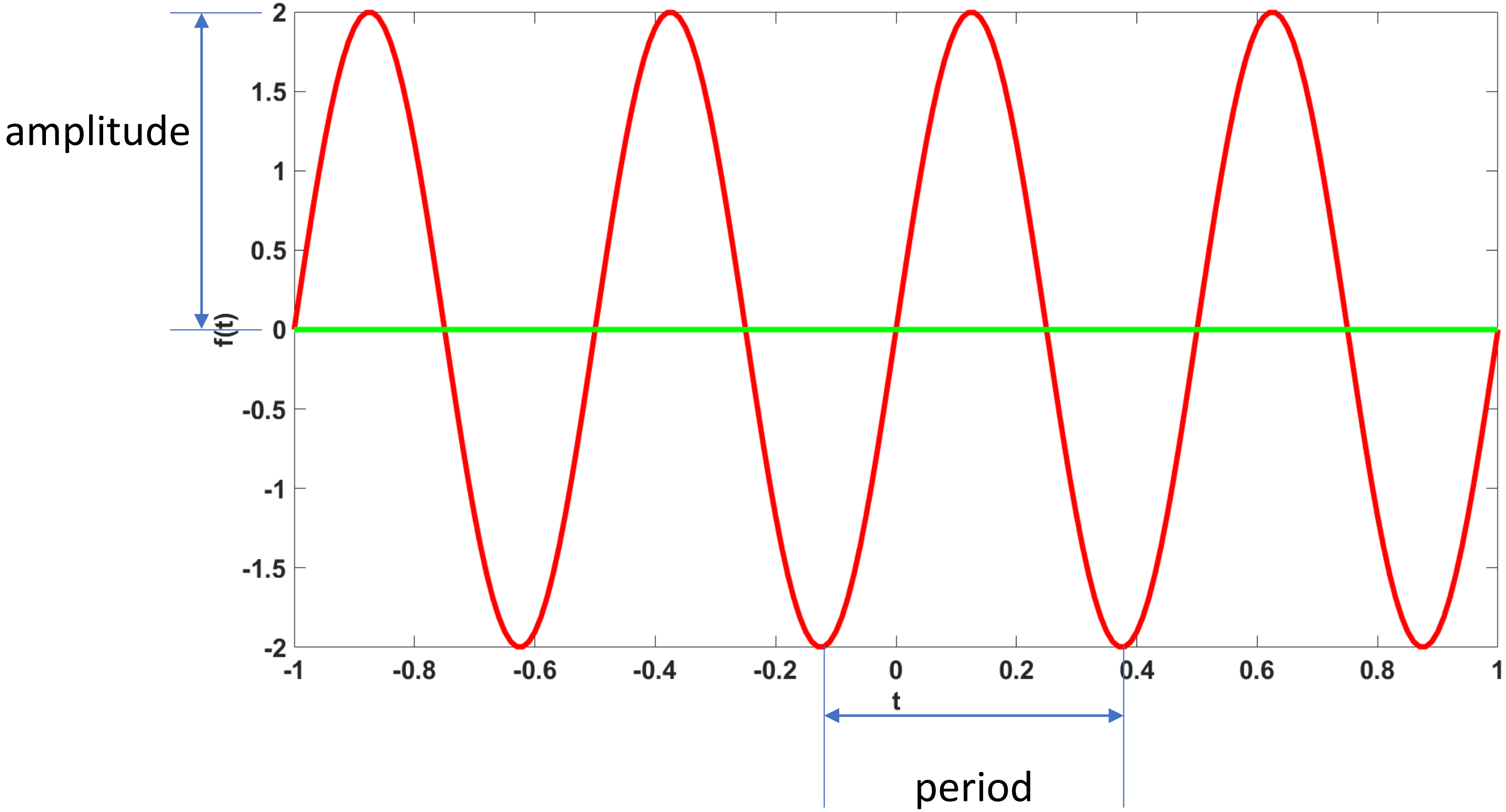
$$f(t) = 2.0 * \sin(2 * \pi * 2 * (t - 0.2))$$



Two waveforms have the same amplitude, the same frequency. But they look different! What makes them look different is their phase.

Phase

$$f(t) = 2.0 * \sin(\boxed{2 * \pi * 2 * t})$$



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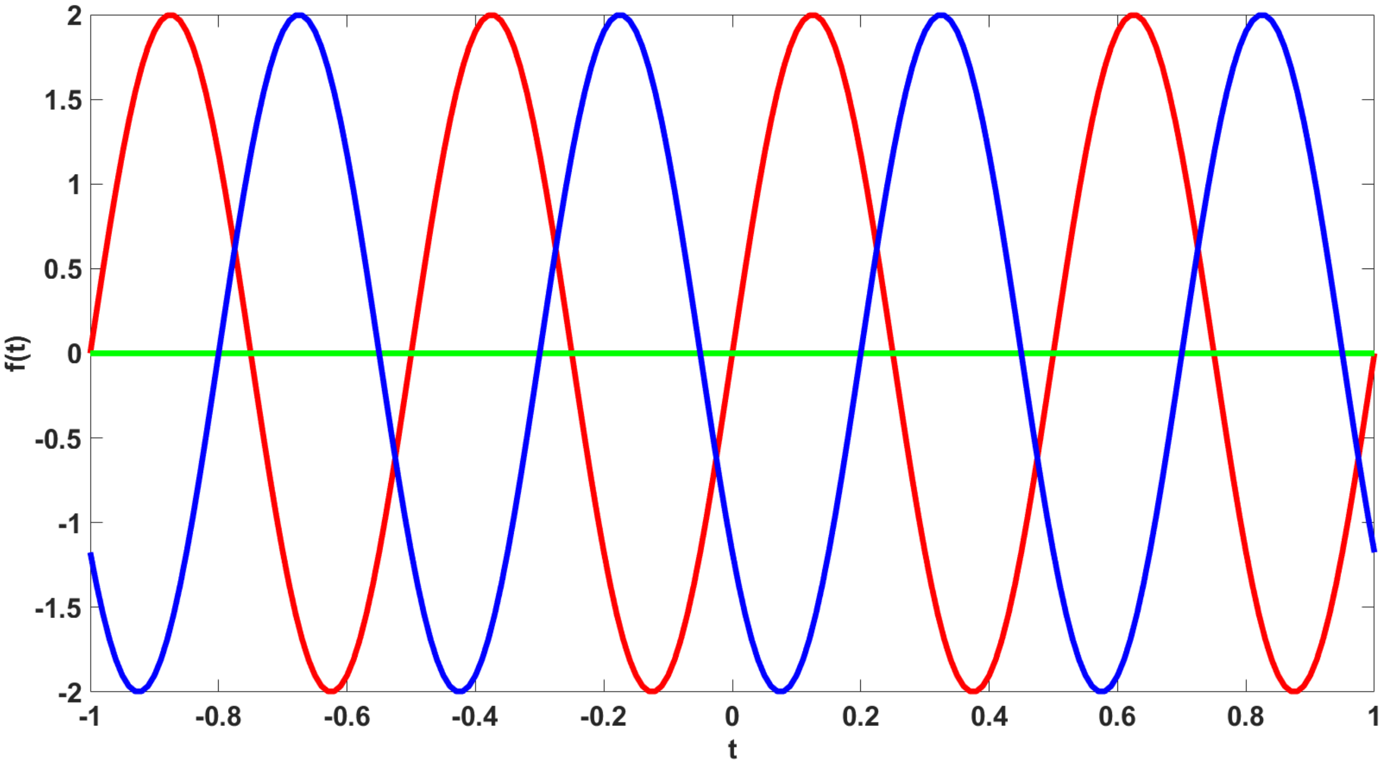

Two waveforms have the same amplitude, the same frequency. But they look different! What makes them look different is their phase.

So, what is phase?

- It is the argument of the sine (or cosine) function
- It is difficult to explain what is phase
- Given a fixed amplitude and frequency, phase determines when the peaks (or troughs) occur,
- Or simply, it specifies where in its cycle the oscillation is at time 0.
- It is measured in degrees or radians

Summary

- Any periodically oscillating sinusoidal wave can be characterized by three fundamental properties:
  - Amplitude
  - Frequency
  - Phase

A more general notation

$$f(t) = A \cdot sin(\omega t + \varphi)$$

Or

$$f(t) = A \cdot cos(\omega t + \varphi)$$

where ω is angular frequency $\omega = \frac{2\pi}{T} = 2\pi f$ in the unit of radians per second.

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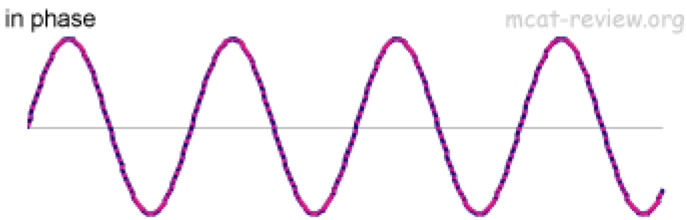
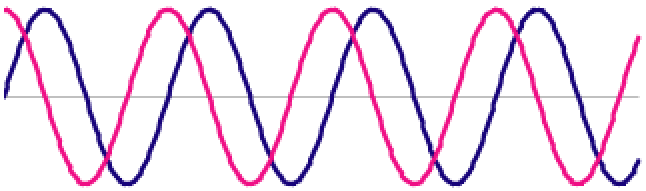
- In-phase vs. out-of-phase
- Complex variables
- Fourier Transform

In-phase vs out-of-phase

- If the peaks of two signals with the same frequency are in exact alignment at the same time, they are said to be in phase



- if the peaks of two signals with the same frequency are not in exact alignment at the same time, they are said to be out of phase



http://mcat-review.org/

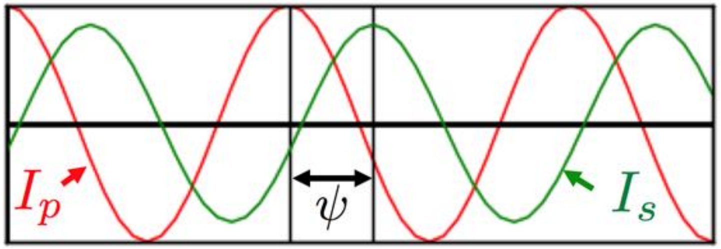
- A sine wave and a cosine wave are 90^0 out of phase with each other

In-phase vs out-of-phase

- Suppose we have two waveforms

$$I_p(t) = I_p \cos \omega t$$

$$I_s(t) = I_s \cos(\omega t - \psi)$$



- Let us decompose $I_s(t)$ into two parts

$$\begin{aligned} I_s(t) &= I_s \cos(\omega t - \psi) \\ &= \underbrace{I_s \cos \psi}_{\substack{\text{In-Phase} \\ \text{Real}}} \cos \omega t + \underbrace{I_s \sin \psi}_{\substack{\text{Out-of-Phase} \\ \text{Quadrature} \\ \text{Imaginary}}} \sin \omega t \end{aligned}$$

In-phase vs out-of-phase

- Suppose we have two waveforms

$$I_p(t) = I_p \cos \omega t$$

$$I_s(t) = I_s \cos(\omega t - \psi)$$

- Let us decompose $I_s(t)$ into two parts

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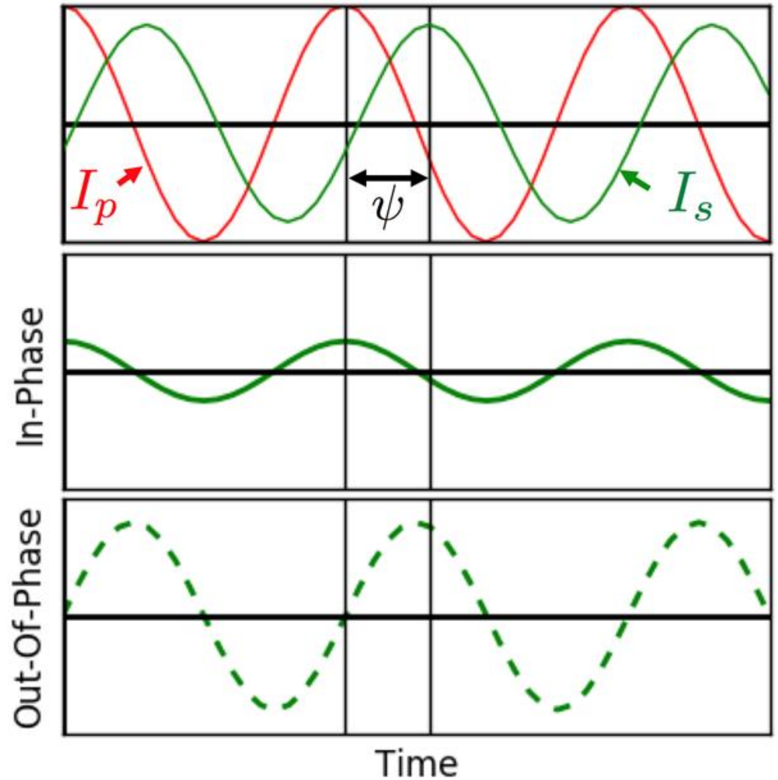
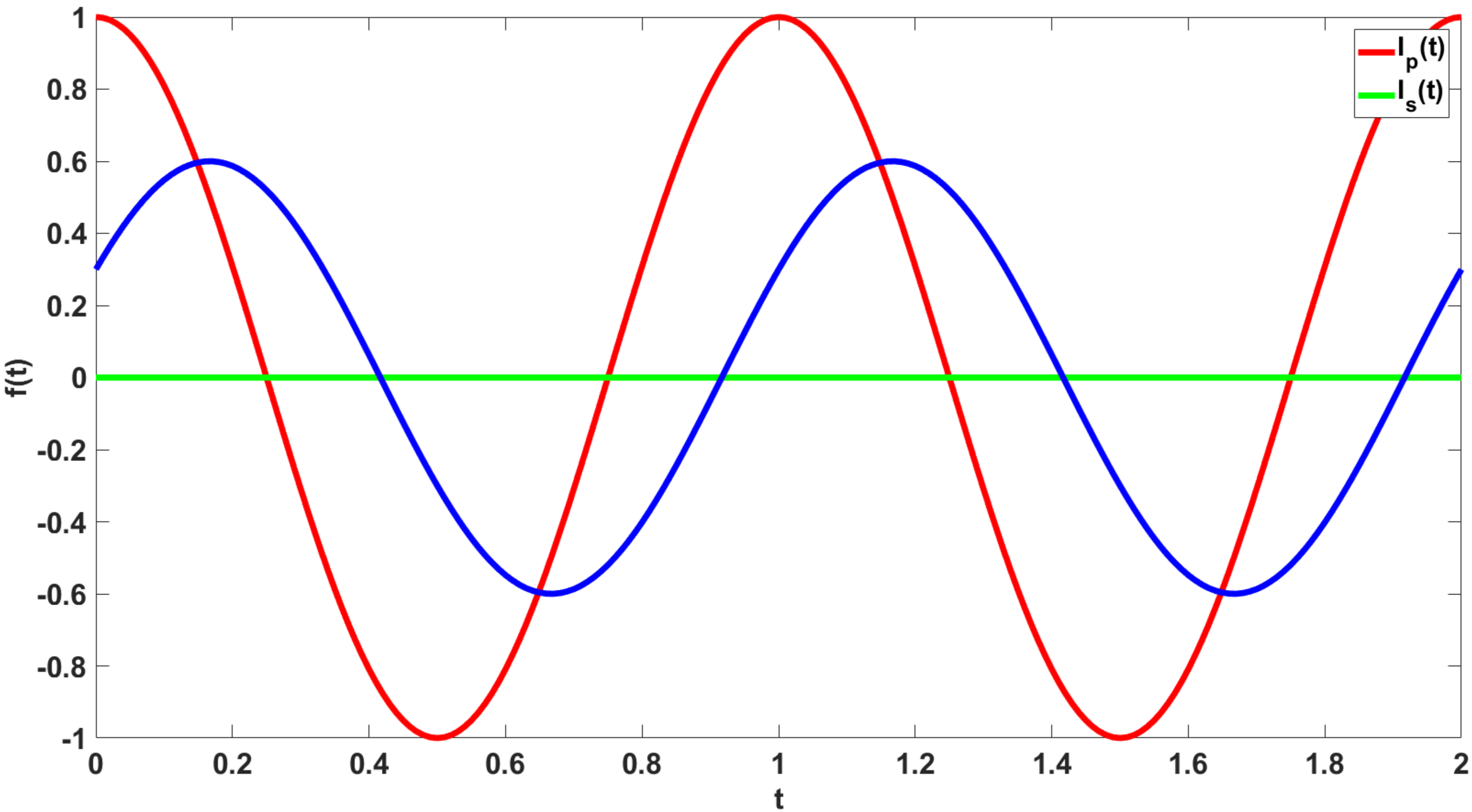


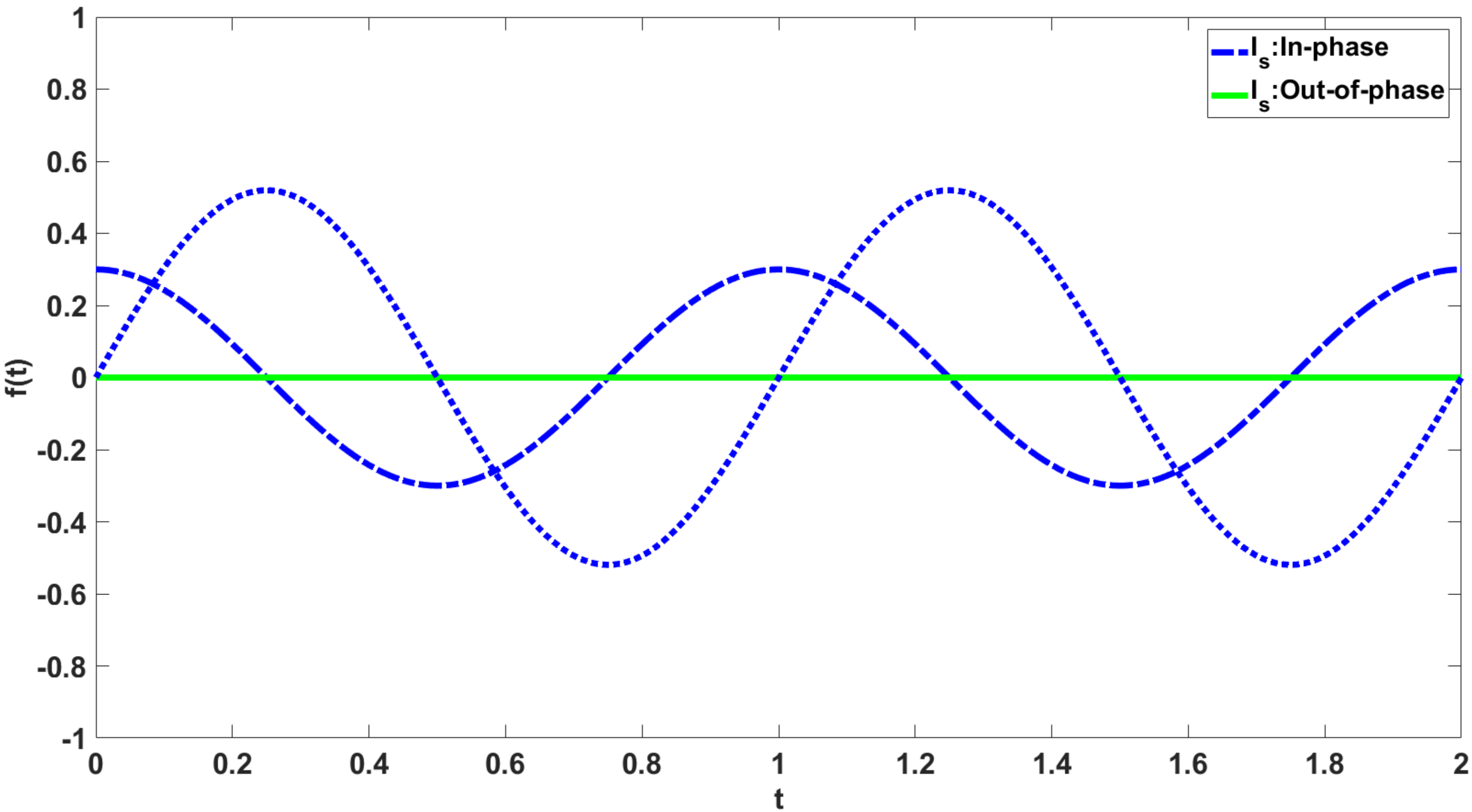
Therefore, $I_s(t)$, the green curve in upper panel, is the sum of in-phase component in the middle panel and the out-of-phase component in the bottom panel.

In-phase vs out-of-phase: an example



$$I_p(t) = 1.0 * \cos(2\pi * 1.0 * t)$$
$$I_s(t) = 0.6 * \cos(2\pi * 1.0 * t - \pi/3)$$

In-phase vs out-of-phase: an example



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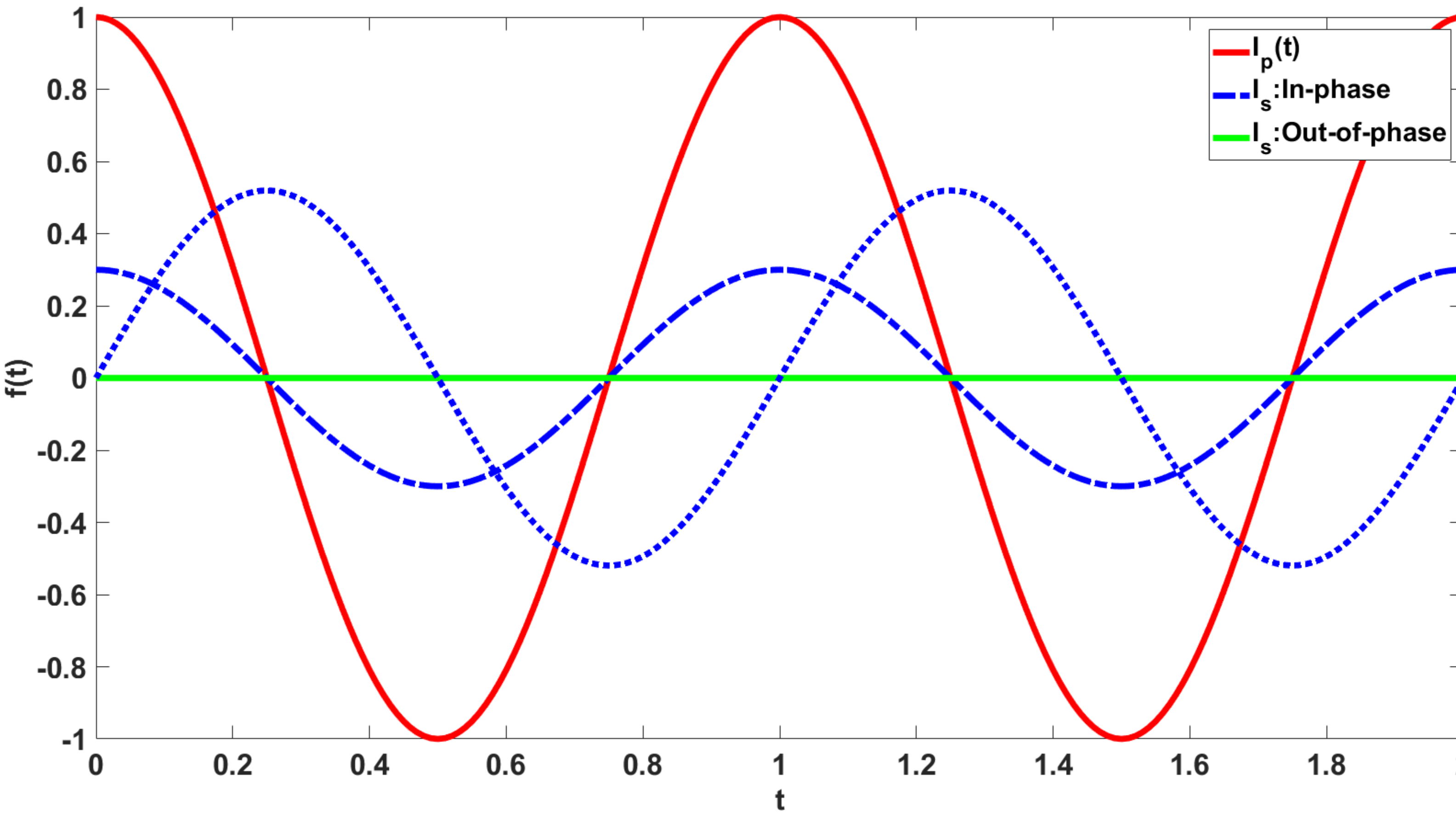

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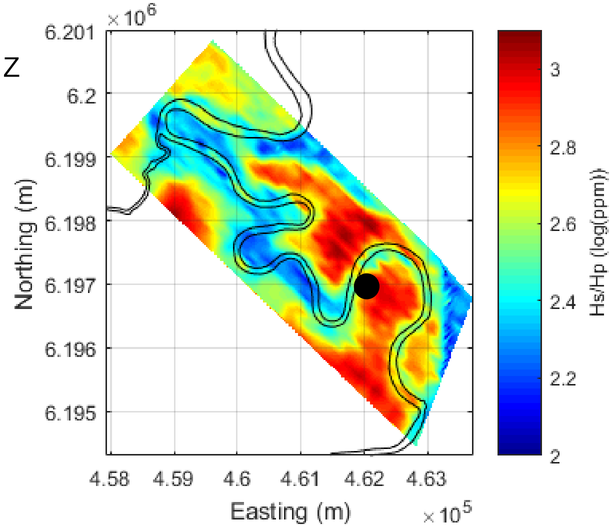
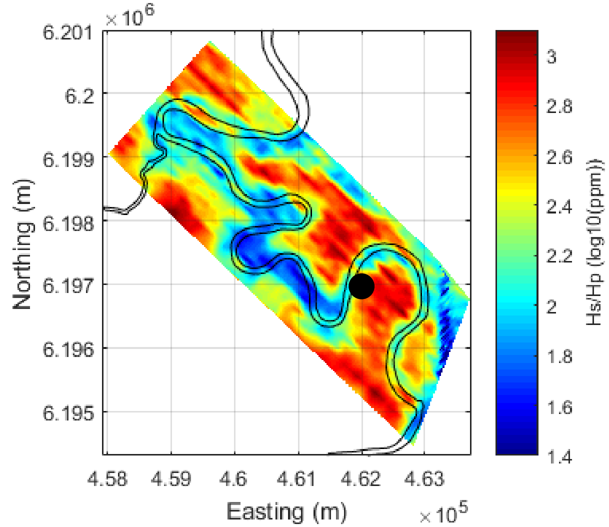
Horizontal Co-planar (HCP) data

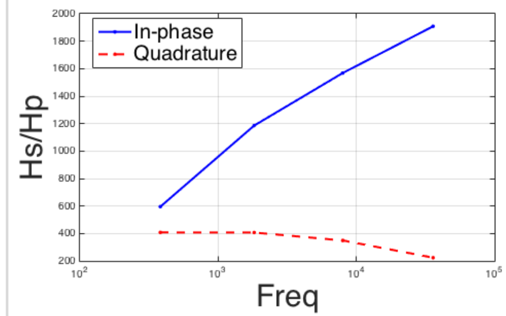
In-Phase (Real) | Quadrature (Imaginary)

382 Hz

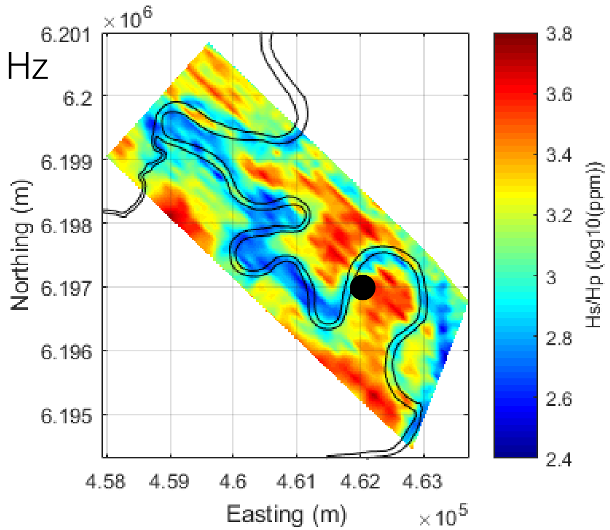


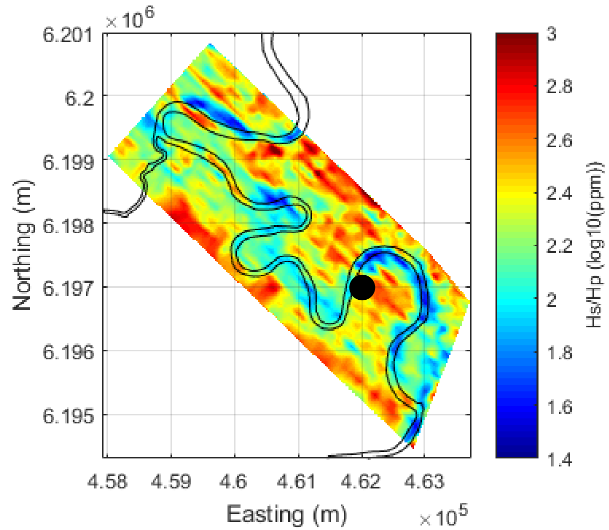


Sounding curve

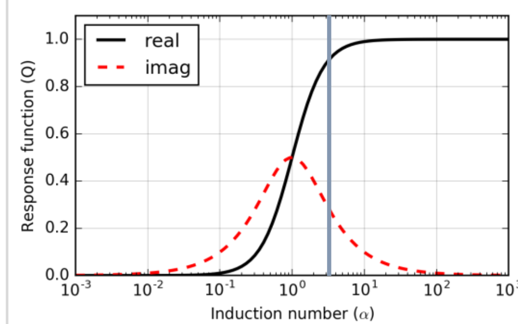


35920 Hz





Response curve



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- Complex variables
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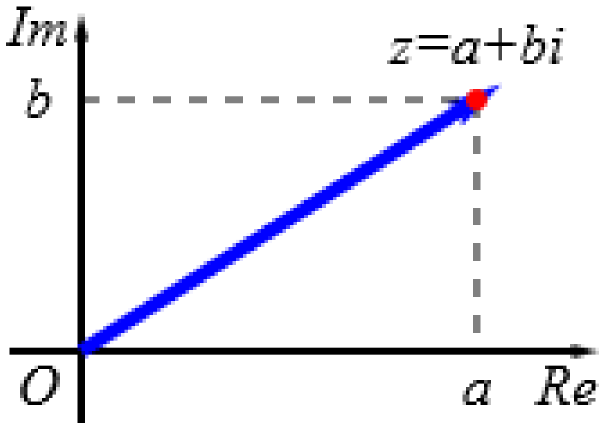
Complex number

$$z = a + i\,b$$

where a and b are real numbers, and i is the imaginary unit equal to $\sqrt{-1}$, i.e., $i^2 = -1$

a: real part b: imaginary part

A complex number can be viewed as a point in a complex plane



https://en.wikipedia.org/wiki/Complex_number

Absolute value and argument

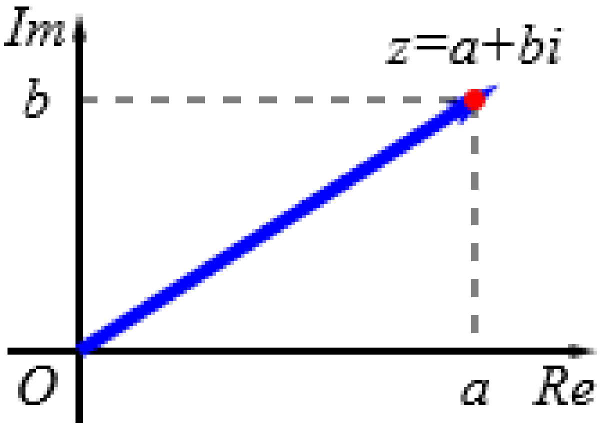
$$z = a + i\,b$$

Absolute value $r = |z| = \sqrt{a^2 + b^2}$

Also called modulus or magnitude

The argument (or phase) is the angle between the vector and the positive real axis.

$$\arg(z) = \theta = atan2(y, x)$$



Therefore, $a = rcos(\theta), b = rsin(\theta)$

Therefore, $z = rcos(\theta) + i\,rsin(\theta)$

To learn more about atan2(y,x), https://en.wikipedia.org/wiki/Argument_(complex_analysis)

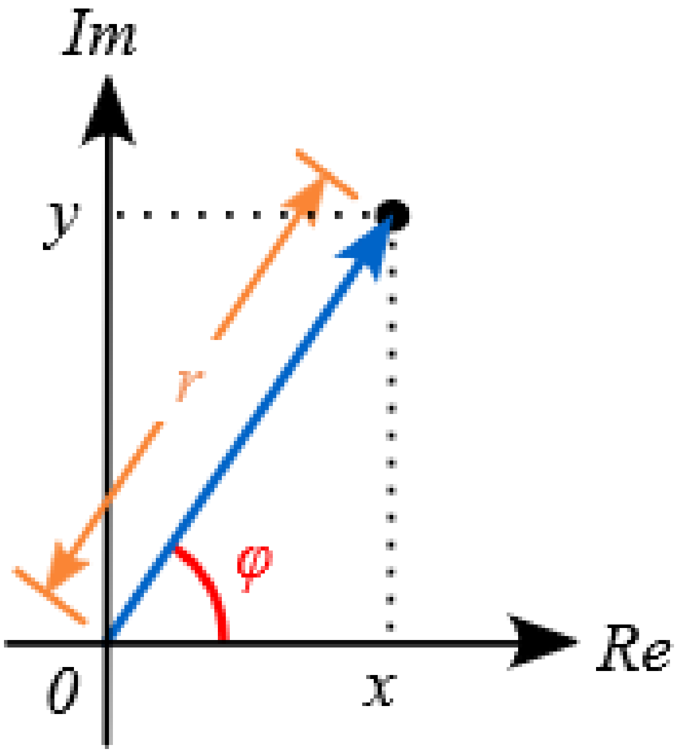
https://en.wikipedia.org/wiki/Complex_number

In-class quiz

Given a complex number

$$z = 1 + i\sqrt{3}$$

Calculate its modulus and argument



https://en.wikipedia.org/wiki/Complex_number

Euler's formula

- Fundamental relationship between the trigonometric functions and the complex exponential function

$$e^{i\theta} = \cos(\theta) + i\, sin(\theta)$$

where e is the base of the natural logarithm.

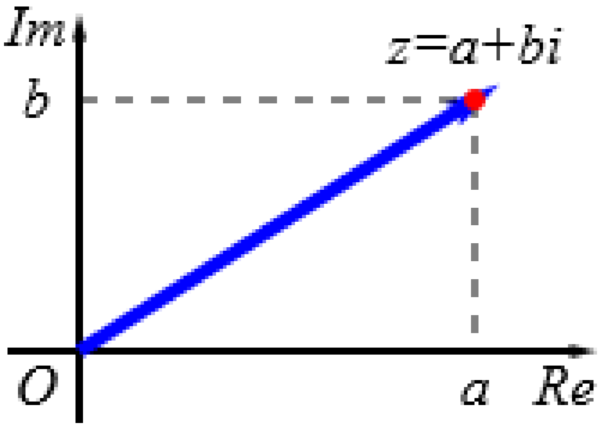
- Therefore,

$$z = rcos(\theta) + i\, rsin(\theta) = re^{i\theta}$$

Also remember $z = a + i\, b$

- Therefore,

$$z = a + i\, b = rcos(\theta) + i\, rsin(\theta) = re^{i\theta}$$



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Euler’s formula

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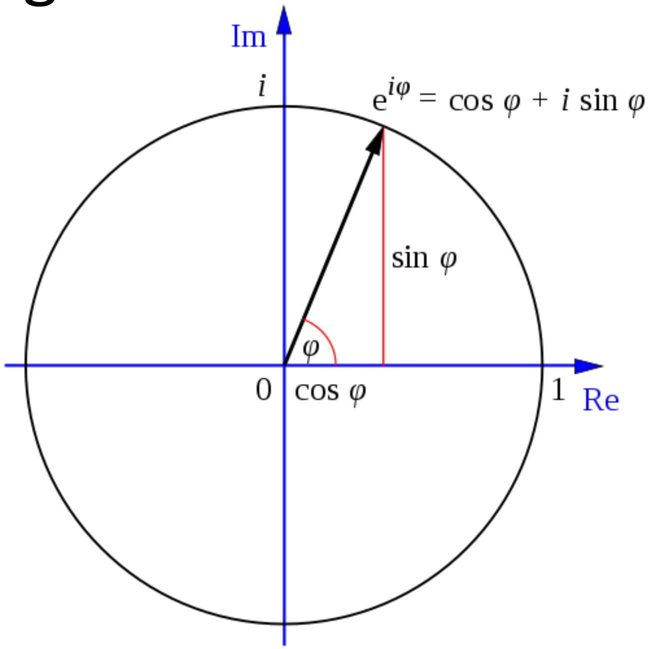
where e is the base of the natural logarithm.

Quiz:

Let $\theta = \frac{\pi}{2}$

Calculate $e^{i\theta}$



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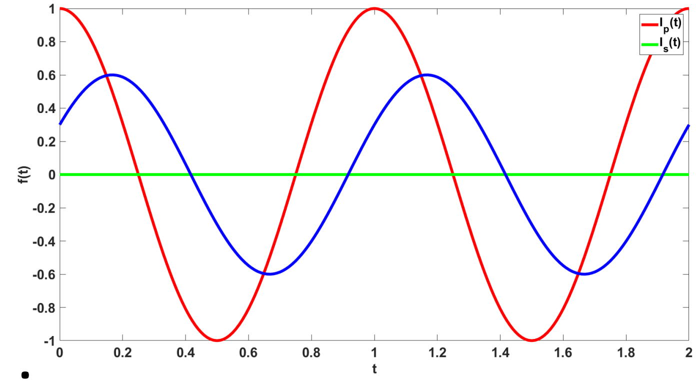


- A mathematical notation for cosine waves

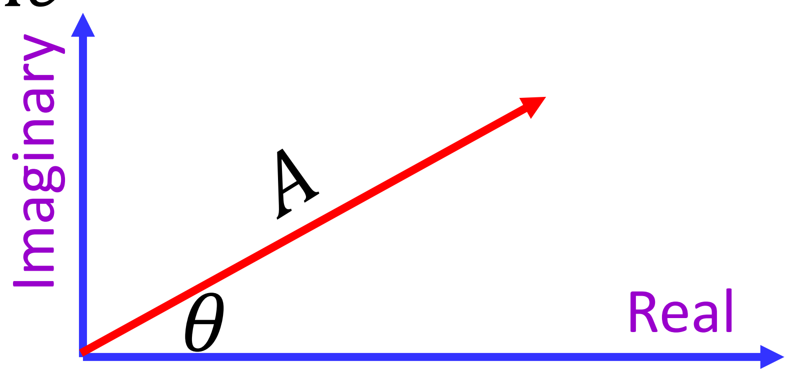
$$f(t) = A \cdot cos(\theta)$$

where A is amplitude and $\theta = \omega t + \varphi$ is phase.

- A different notation

$$f(t) = Ae^{i\theta}$$

where $i^2 = -1$



Why bother?

- Everyone is using it, unfortunately …
  - The whole Fourier theory was built upon complex variables

- It offers a lot of mathematical convenience
  - You do not have to deal with trigonometric functions any more.
  - Just exponentials, which are much more easier to manipulate!
  - Some problems can be solved much more easily using complex numbers and complex analysis, e.g., quantum physics, conformal transformations, AC circuits, etc.

Optional reading materials on complex numbers

- Complex Analysis Made Simple on youtube.
- MIT OpenCourseWare: Development of the complex numbers
- https://physics.stackexchange.com/questions/100553/what-are-functions-of-a-complex-variable-used-for-in-physics
- https://betterexplained.com/articles/intuitive-arithmetic-with-complex-numbers/
- https://betterexplained.com/articles/a-visual-intuitive-guide-to-imaginary-numbers/

In future,

- If you see something similar to $Ae^{i\theta}$, you now know it represents a sinusoidal wave with amplitude A and phase θ

One more thing

$$\frac{\partial e^{iwt}}{\partial t} ??$$

Tip: treat i the same as any other real numbers

One more thing

$$\frac{\partial e^{i\omega t}}{\partial t} ??$$

Tip: treat i the same as any other real numbers

$$\frac{\partial e^{i\omega t}}{\partial t} = i\omega e^{i\omega t}$$

Remember

$$i = e^{i\frac{\pi}{2}}$$

Therefore,

$$\frac{\partial e^{i\omega t}}{\partial t} = i\omega e^{i\omega t} = e^{i\frac{\pi}{2}}\omega e^{i\omega t}$$

$$\frac{\partial e^{i\omega t}}{\partial t} = \omega e^{i(\omega t+\frac{\pi}{2})}$$

Conclusion: When you take the time derivative, the phase will change by $\frac{\pi}{2}$

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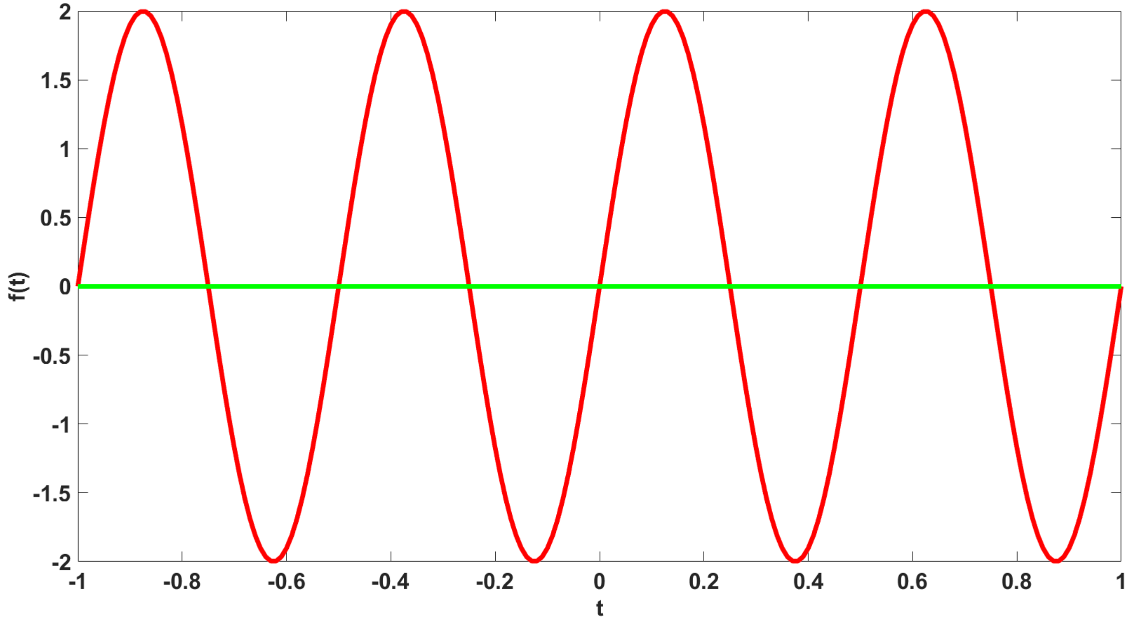
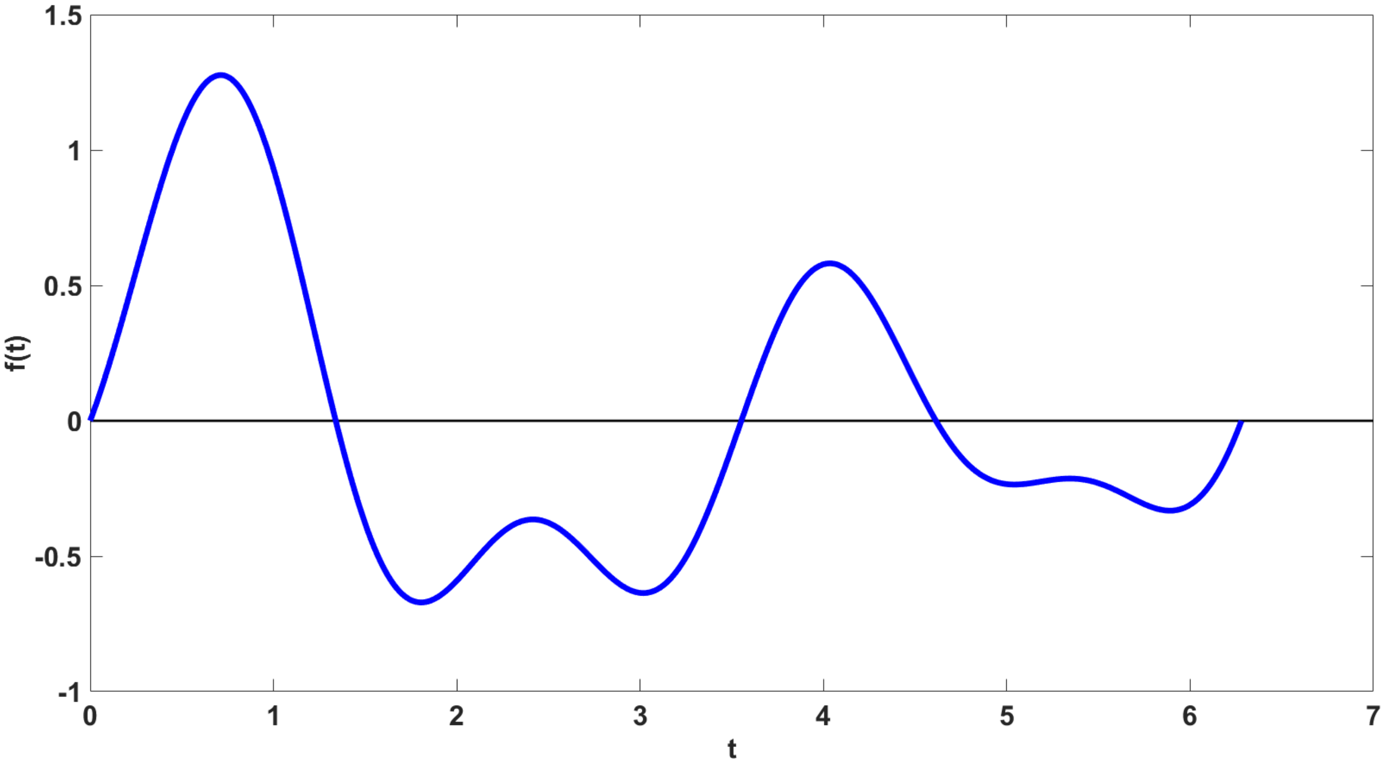
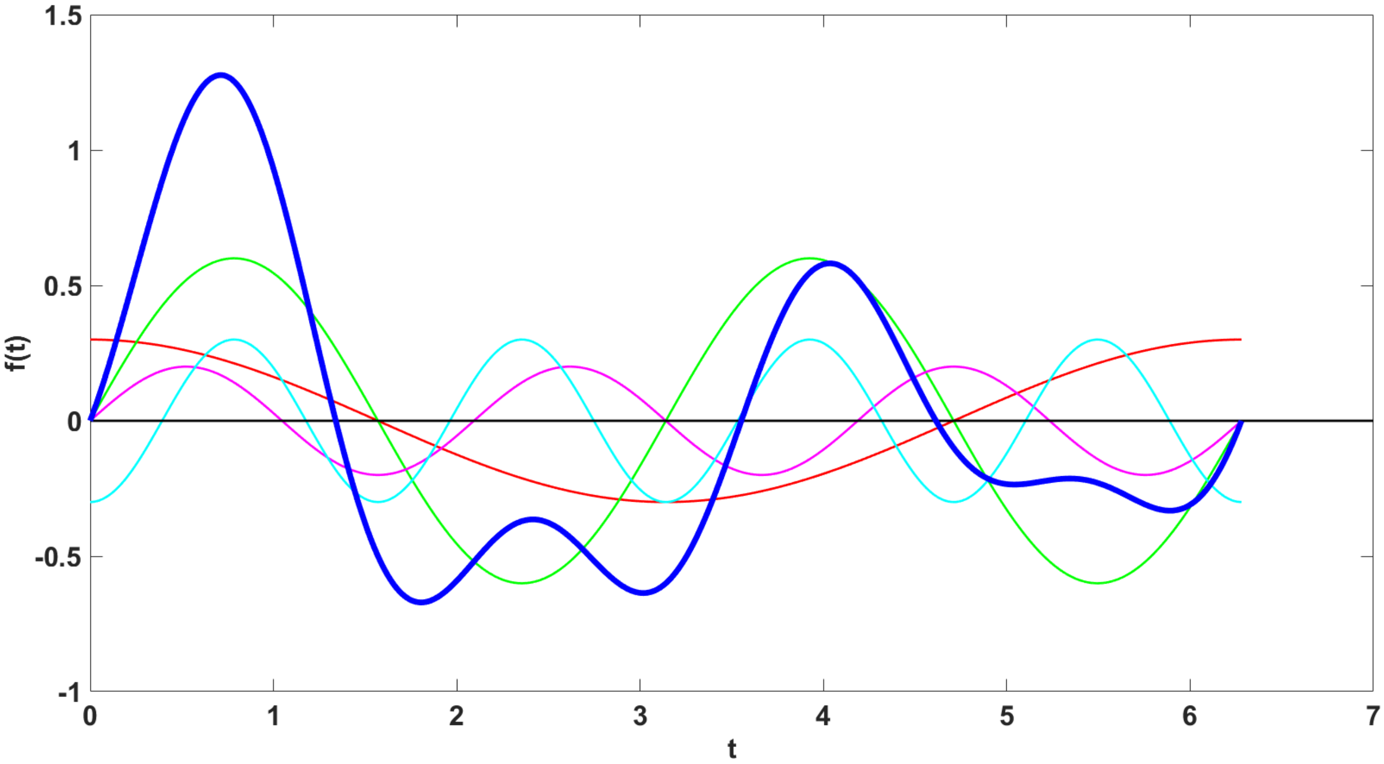
Virtually any real world waveforms can be represented as a sum of sins, no matter how complicated they may look.



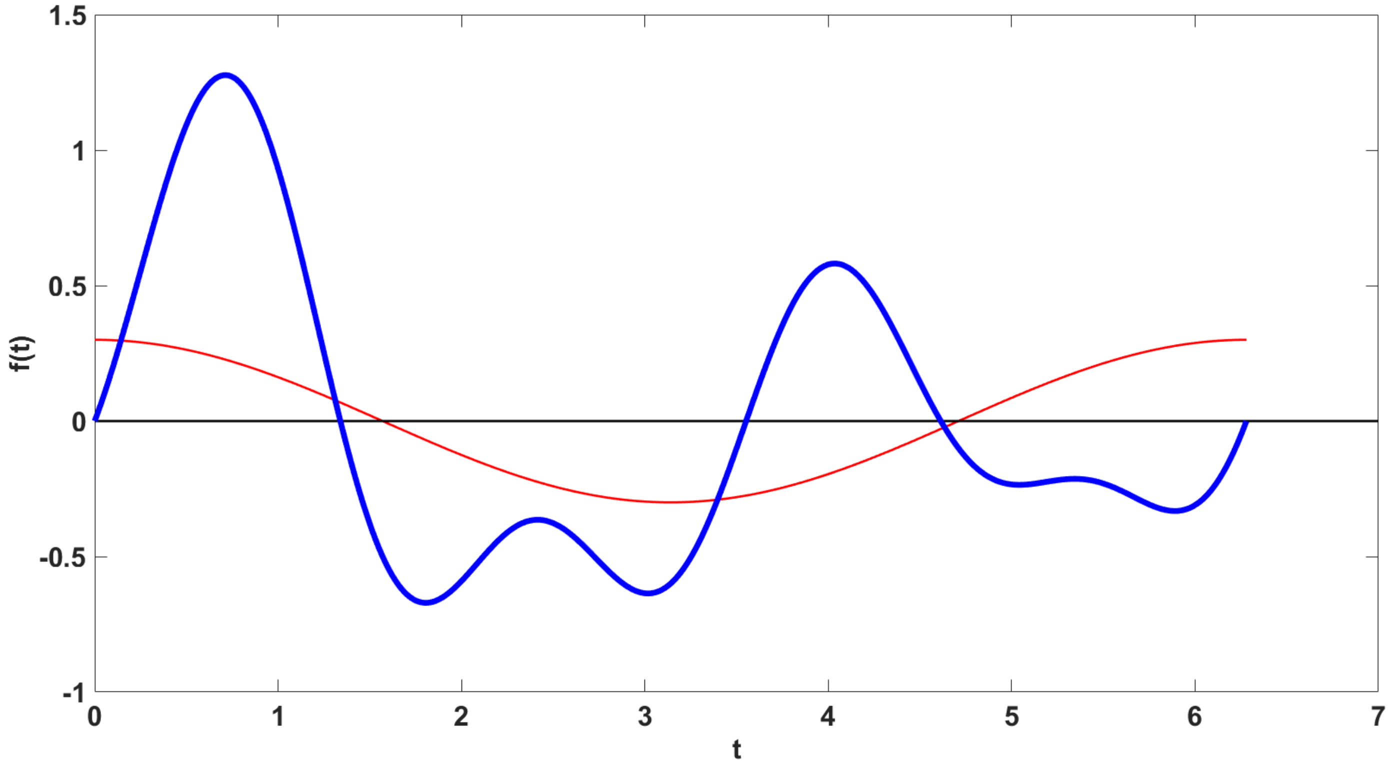
An illustrative example



Four components

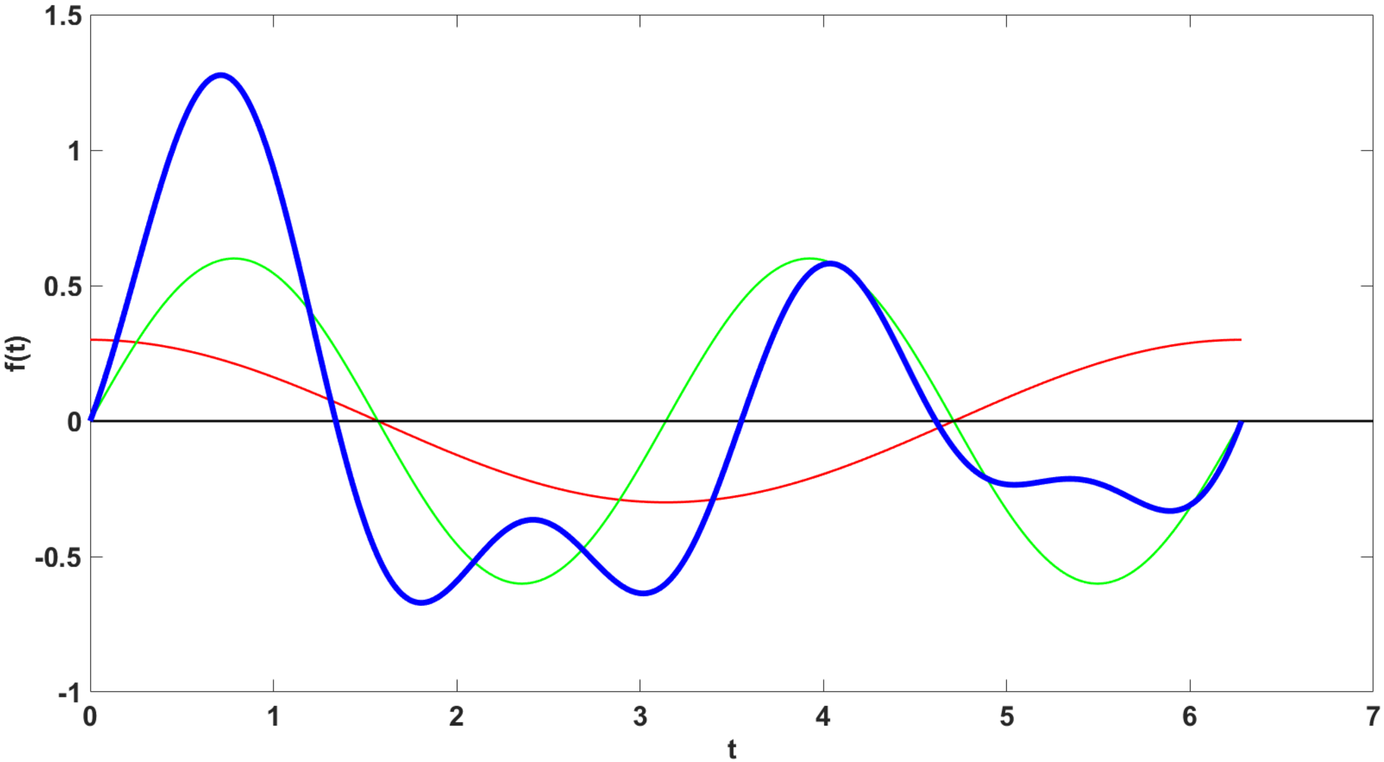


First componet



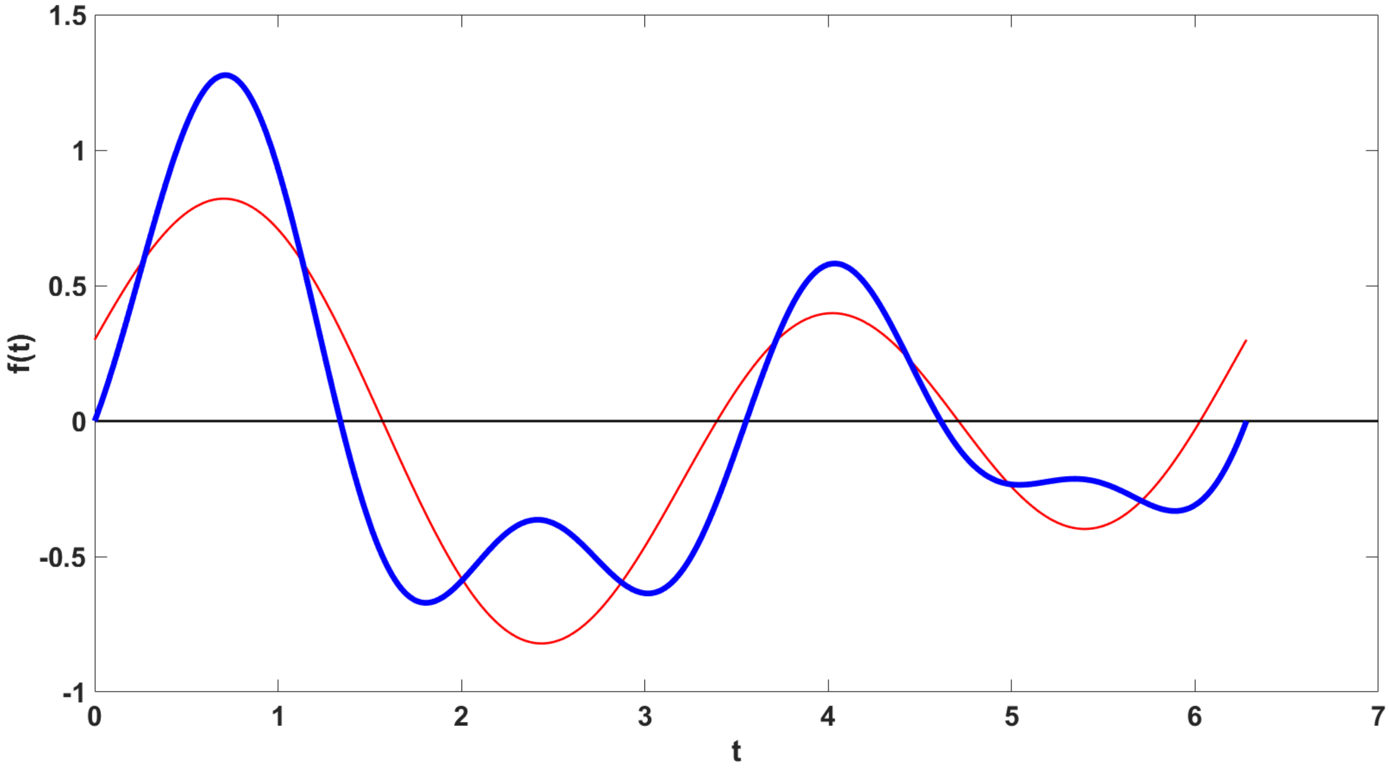
First component: $f(t) = 0.3\sin(\frac{2\pi}{T} * t + \frac{\pi}{2})$, where T = 6.28 s, t = 0:0.01:T

Second component

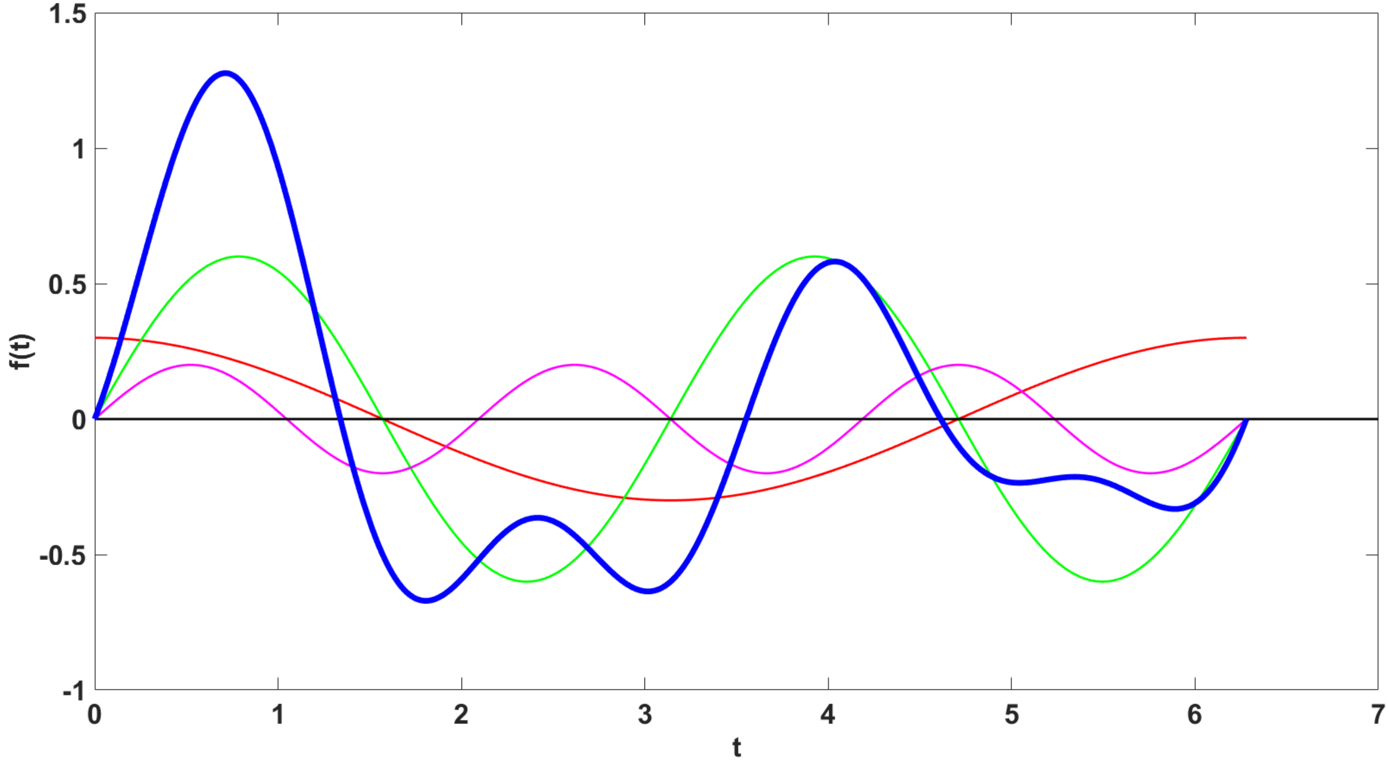


Second component: $f(t) = 0.6\sin(\frac{4\pi}{T} * t)$, where T = 6.28 s, t = 0:0.01:T

Sum of first two components

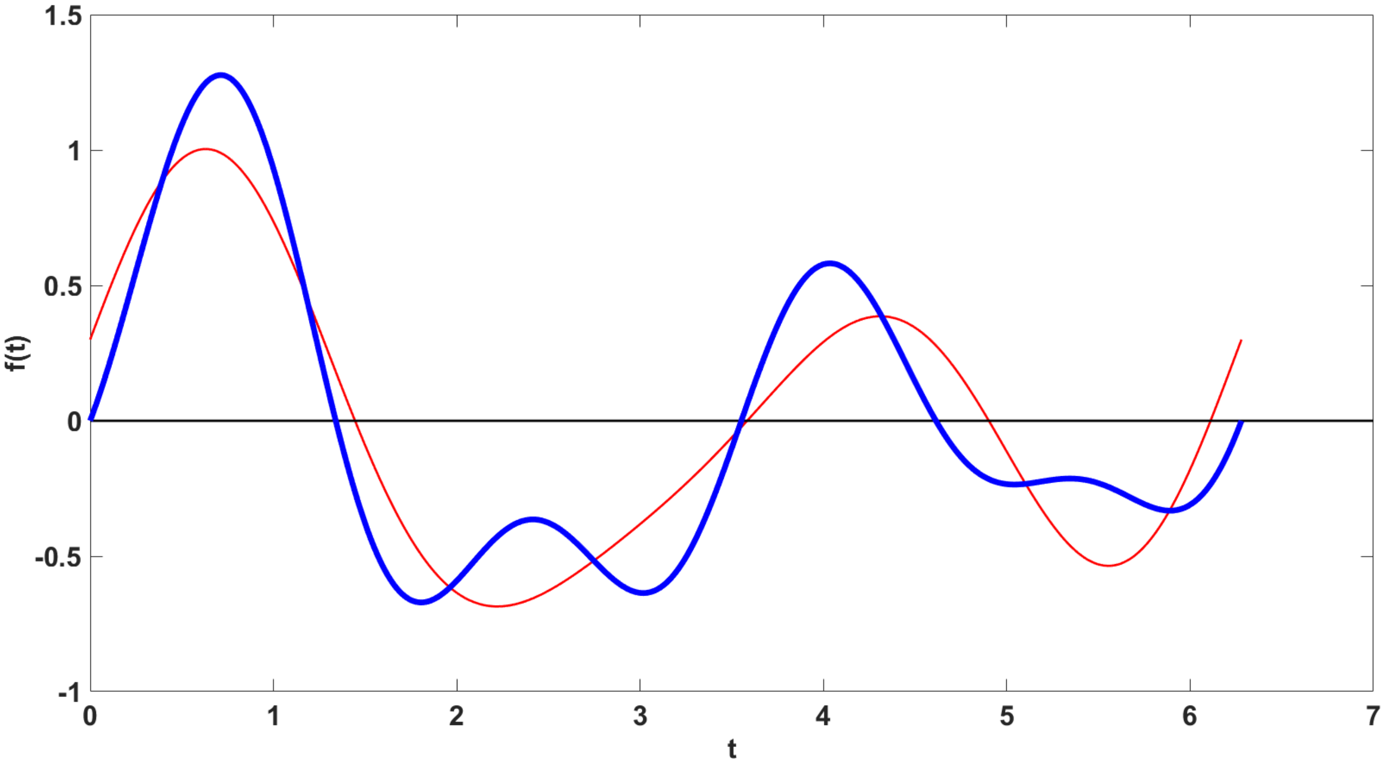


First three components

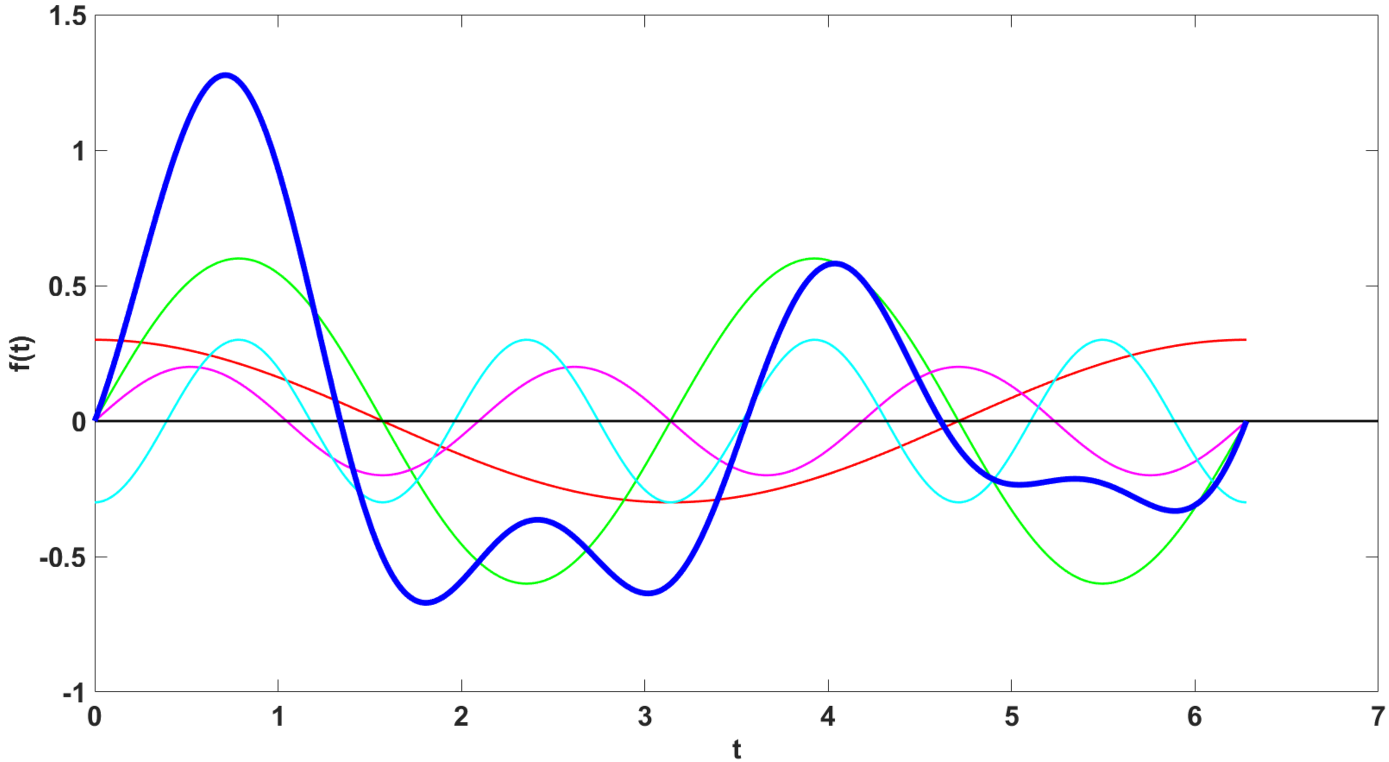


Third component: $f(t) = 0.2\sin(\frac{6\pi}{T} * t)$, where T = 6.28 s, t = 0:0.01:T

Sum of first three components

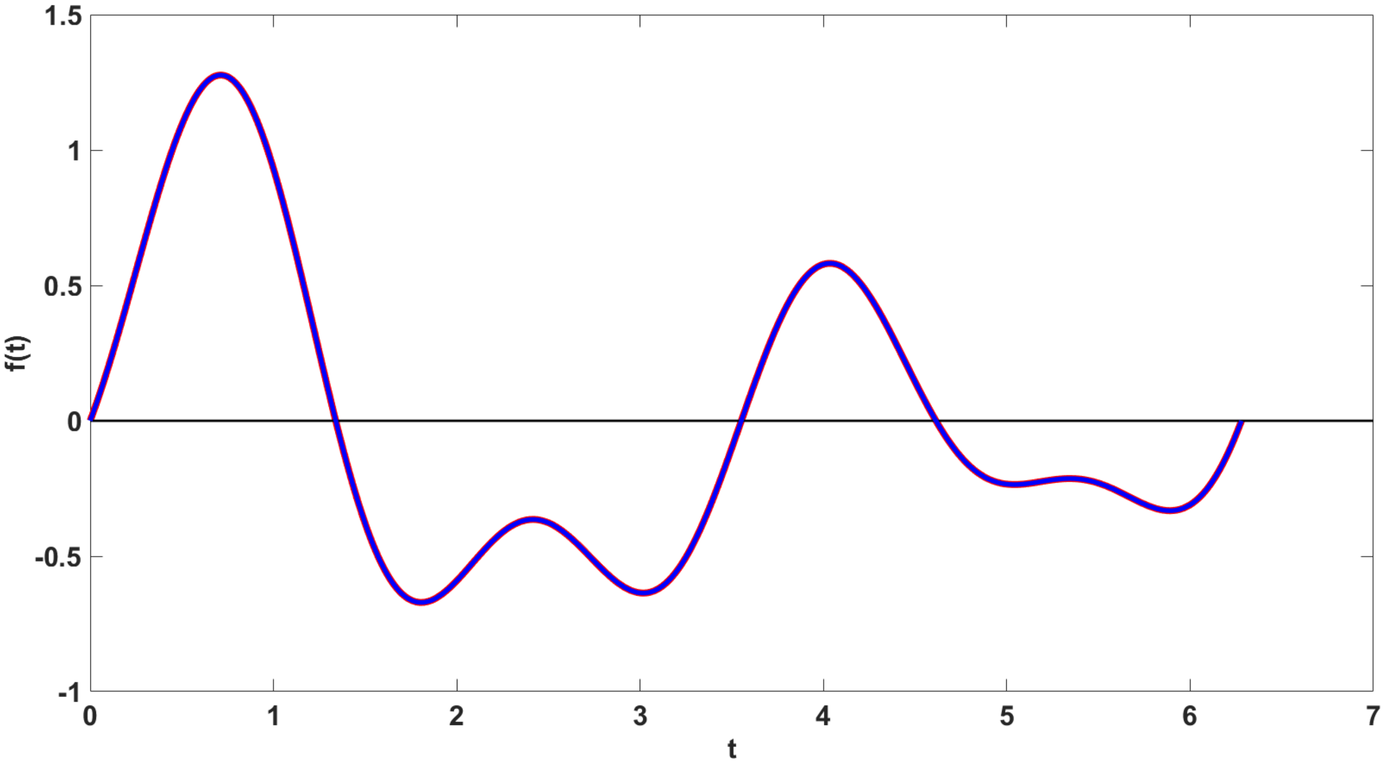


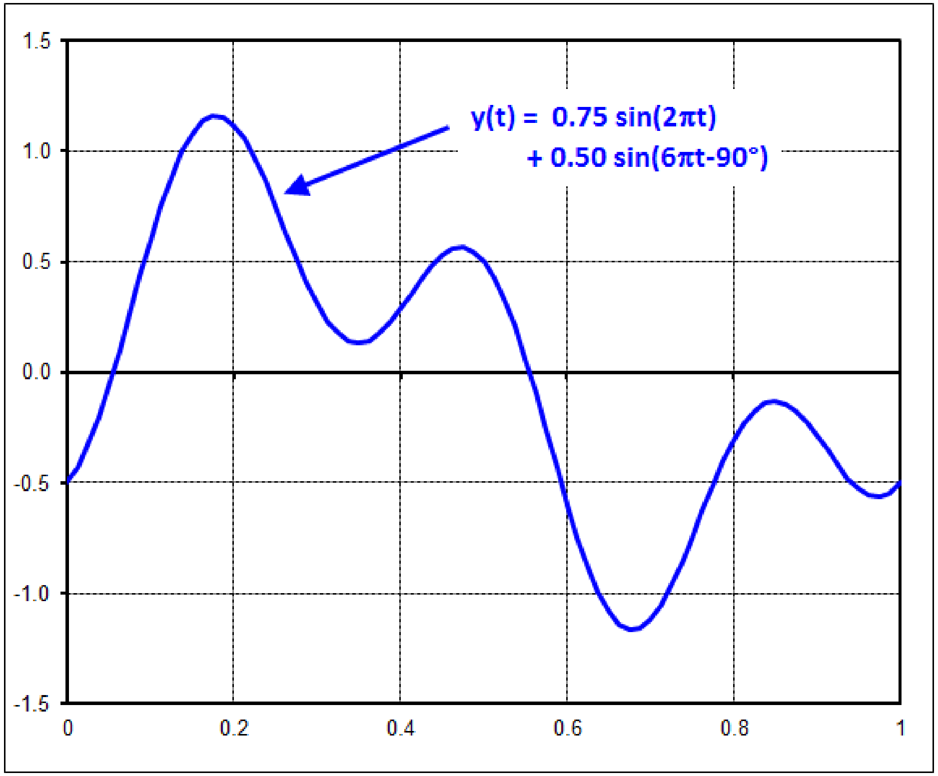
All four components

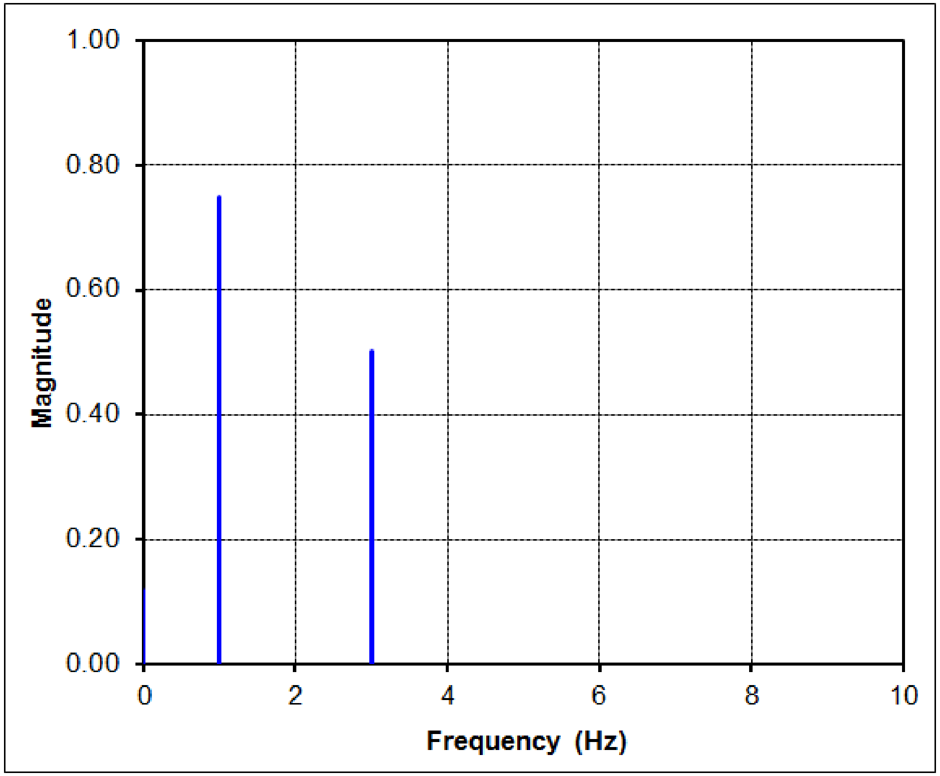


Fourth component: $f(t) = 0.3\ \sin(\frac{8\pi}{T} * t - \frac{\pi}{2})$, where T = 6.28 s, t = 0:0.01:T

Sum of all four components

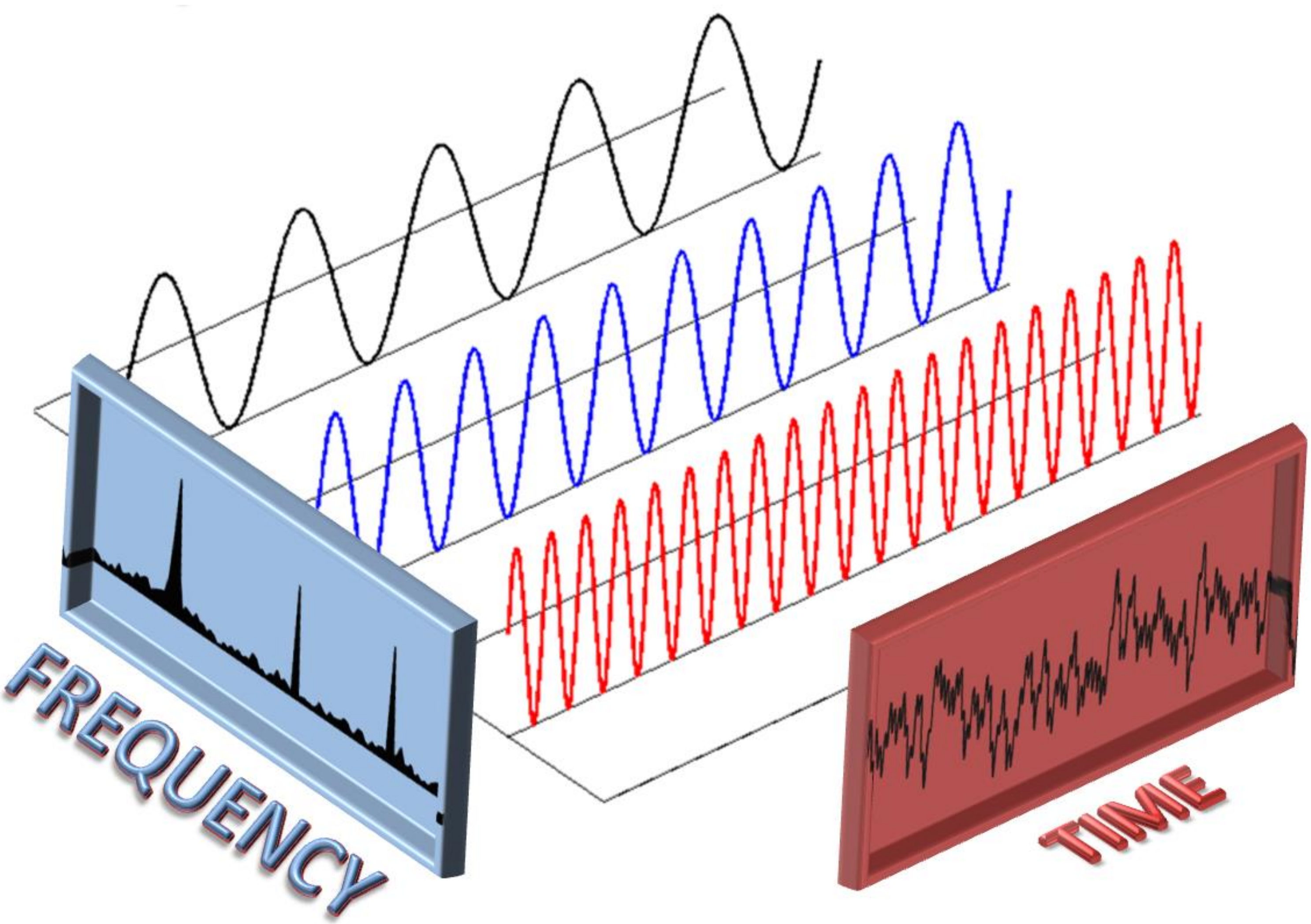






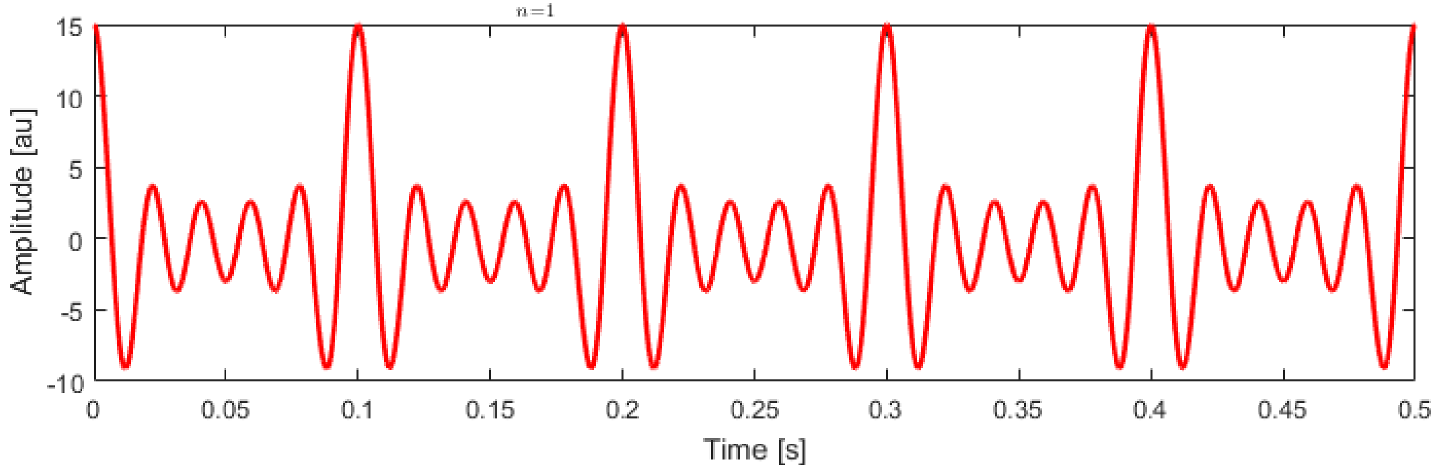
- The signal in blue results from the sum of two sine waves.
- These two sine waves have different amplitudes, frequencies, and phases.

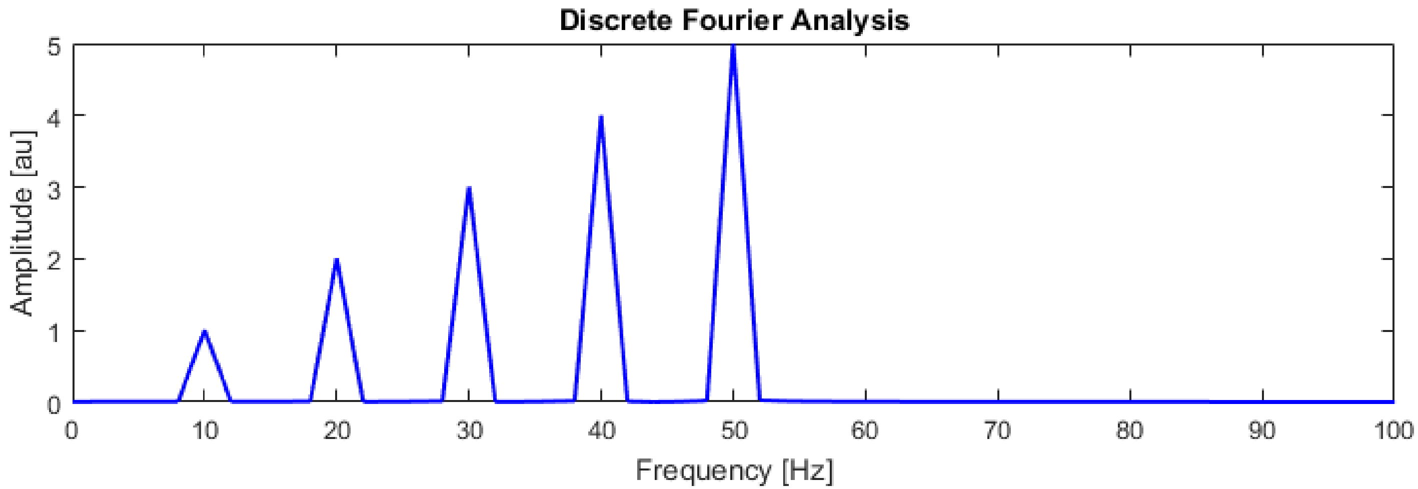
http://www.continuummechanics.org/fourierxforms.html



http://visualizingmathsandphysics.blogspot.com/2015/06/fourier-transforms-intuitively.html

$$\sum_{n=1}^{5} n \times cos(n \times \omega \times t), \qquad \omega = 10 \times 2\pi$$





https://en.wikipedia.org/wiki/Fast_Fourier_transform

Mathematically,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{F}(\omega)\, e^{iwt} dw$$

Mathematically,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \boxed{\mathcal{F}(\omega)\, e^{iwt}} dw$$

This is nothing but a sinusoidal wave!

Mathematically,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{F}(\omega)\, e^{iwt} dw$$

A waveform $f(t)$ can be expressed as the sum of many sinusoidal waveforms of different amplitudes, phases and frequencies.

Mathematically,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathcal{F}(\omega)\, e^{iwt} dw$$

A waveform $f(t)$ can be decomposed into many sinusoidal waveforms of different amplitudes, phases and frequencies.

Fourier transform

$$\mathcal{F}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-iwt} dt$$

- Decompose a signal into a series of sinusoidal waves of different amplitudes, frequencies and phases.
- Helps identify what sine and cosine components make up the signal.

Inverse Fourier transform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{F}(\omega)\, e^{iwt} dw$$

A waveform $f(t)$ can be decomposed into many sinusoidal waveforms of different amplitudes, phases and frequencies.

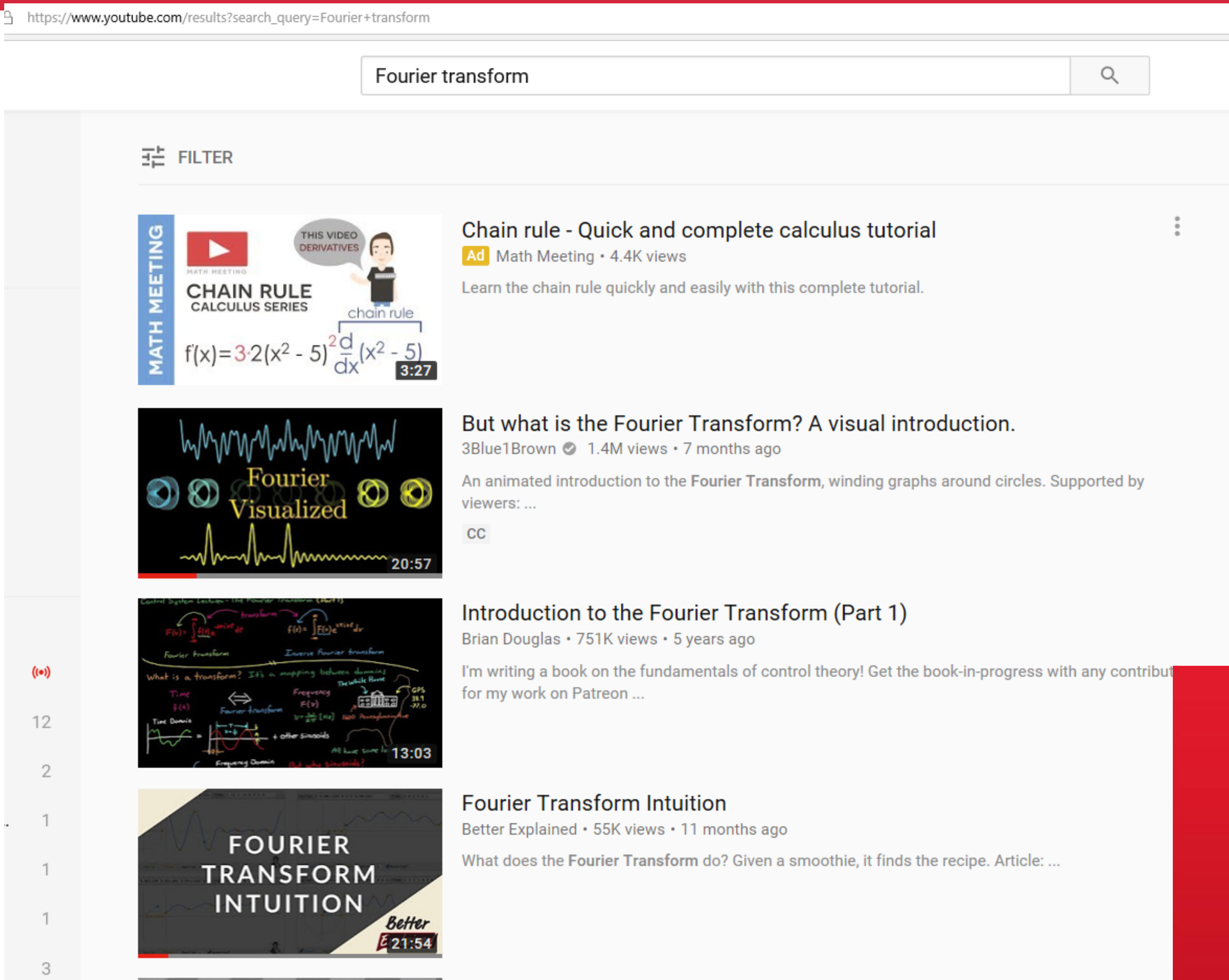
Notation

Fourier transform: $\mathcal{F}[f(t)] = \mathcal{F}(\omega)$

Inverse Fourier transform: $\mathcal{F}^{-1}[\mathcal{F}(\omega)] = f(t)$

Optional reading materials on FT

- https://betterexplained.com/articles/an-interactive-guide-to-the-fourier-transform/
- http://visualizingmathsandphysics.blogspot.com/2015/06/fourier-transforms-intuitively.html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