Lecture 2
Vector analysis & PDE

GEOL 4397: Electromagnetic Methods for Exploration
GEOL 6398: Special Problems

Jiajia Sun, Ph.D.

August 23rd, 2018

UNIVERSITY of HOUSTON
YOU ARE THE PRIDE
EARTH AND ATMOSPHERIC SCIENCES

Take attendance on CourseKey

Agenda

- Vector analysis
- PDE

Maxwell equations

$$\nabla \cdot \boldsymbol{d} = \rho_f$$

Gauss's law for electric fields

$$\nabla \cdot \boldsymbol{b} = 0$$

Gauss's law for magnetic fields

$$\nabla \times \boldsymbol{e} = -\frac{\partial \boldsymbol{b}}{\partial t}$$

Faraday's law

$$\nabla \times \mathbf{h} = \boldsymbol{j}_f + \frac{\partial \mathbf{d}}{\partial t}$$

Ampere-Maxwell equation

Constitutive relationships

$$\boldsymbol{j}_f = \sigma \boldsymbol{e} \qquad \boldsymbol{b} = \mu \boldsymbol{h} \qquad \boldsymbol{d} = \varepsilon \boldsymbol{e}$$

Maxwell equations

$$\nabla \cdot \boldsymbol{d} = \rho_f$$

Gauss's law for electric fields

$$\nabla \cdot \boldsymbol{b} = 0$$

Gauss's law for magnetic fields

$$\nabla \times \boldsymbol{e} = -\frac{\partial \boldsymbol{b}}{\partial t}$$

Faraday's law

$$\nabla \times \mathbf{h} = \boldsymbol{j}_f + \frac{\partial \mathbf{d}}{\partial t}$$

Ampere-Maxwell equation

Maxwell equations

$$\nabla \cdot \boldsymbol{d} = \rho_f$$

Gauss's law for electric fields

$$\nabla \cdot \boldsymbol{b} = 0$$

Gauss's law for magnetic fields

$$\nabla \times \boldsymbol{e} = -\frac{\partial \boldsymbol{b}}{\partial t}$$

Faraday's law

$$\nabla \times \mathbf{h} = \boldsymbol{j}_f + \frac{\partial \mathbf{d}}{\partial t}$$

Ampere-Maxwell equation

Several things that you might have noticed:

- ∇
- $\cdot$
- $\times$
- vectors

Matrix

- Rectangular array of numbers

$$\boldsymbol{A} = \begin{bmatrix} 1402 & 25 \\ 1650 & 46 \\ 2058 & 57 \end{bmatrix}_{3X2}$$

Dimension of a matrix: # of rows X # of columns

Vector: An N X 1 matrix

$$\boldsymbol{y} = \begin{bmatrix} 604 \\ 731 \\ 172 \\ 495 \end{bmatrix}$$

- N = 4, therefore, $\boldsymbol{y}$ is a 4-dimensional vector

Vector: direction and magnitude

$$\boldsymbol{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Dot product

- Two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$
- Their dot product is defined as

$$\boldsymbol{a} \cdot \boldsymbol{b} = |a||b|cos\theta$$

Dot product

- Two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$
- Their dot product is defined as

$$\boldsymbol{a} \cdot \boldsymbol{b} = |a||b|cos\theta$$

- The component form

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Cross product

- Two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$
- Their cross product is defined as

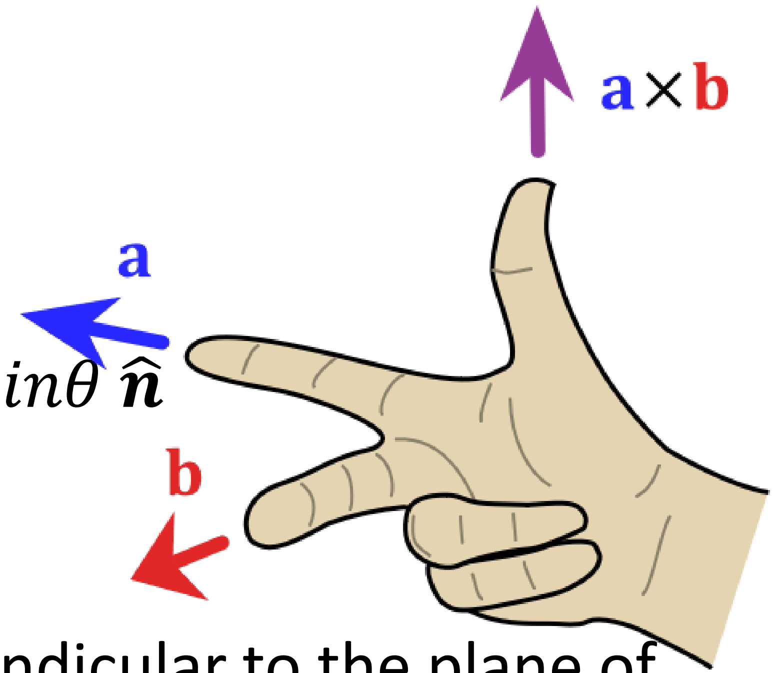
$$\boldsymbol{a} \times \boldsymbol{b} = |a||b| sin\theta \, \widehat{\boldsymbol{n}}$$

- Note that $\boldsymbol{a} \times \boldsymbol{b}$ is itself a vector
- $\widehat{\boldsymbol{n}}$ is a unit vector pointing perpendicular to the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$.
- There are actually two such directions
- The ambiguity is resolved by the right-hand rule.

Cross product

- Two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$
- Their cross product is defined as

$$\boldsymbol{a} \times \boldsymbol{b} = |a||b|sin\theta\ \widehat{\boldsymbol{n}}$$



- Note that $\boldsymbol{a} \times \boldsymbol{b}$ is itself a vector
- $\widehat{\boldsymbol{n}}$ is a unit vector pointing perpendicular to the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$.
- There are actually two such directions
- The ambiguity is resolved by the right-hand rule.

Cross product

- Two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$
- Their cross product is defined as

$$\boldsymbol{a} \times \boldsymbol{b} = |a||b| sin\theta\, \widehat{\boldsymbol{n}}$$

- The component form

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \widehat{\boldsymbol{x}} & \widehat{\boldsymbol{y}} & \widehat{\boldsymbol{z}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Determinant (optional material)

- In the case of a 2 X 2 matrix,

$$|\boldsymbol{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- For a 3 X 3 matrix A,

$$|\boldsymbol{A}| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

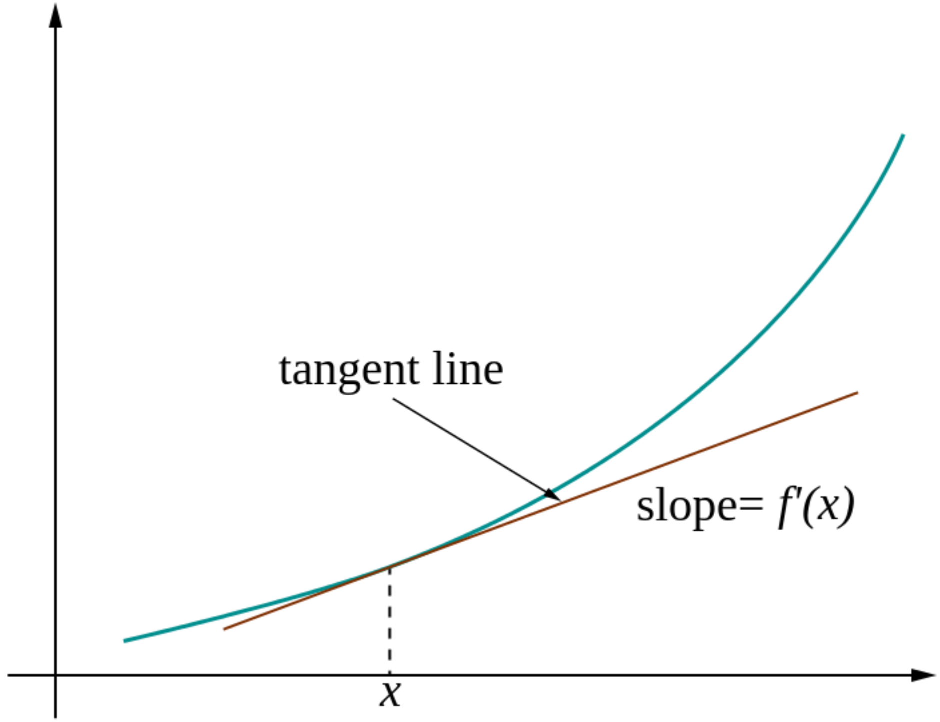
- To learn more about determinant, https://en.wikipedia.org/wiki/Determinant

Useful resources

- https://en.wikipedia.org/wiki/Dot_product
- https://en.wikipedia.org/wiki/Cross_product

What is gradient?

- Let us first recall what is derivative.



Picture taken from https://en.wikipedia.org/wiki/Derivative

From derivative to gradient

- Let us consider a function $f(x, y)$
- Two partial derivatives

$$\frac{\partial f}{\partial x}$$

$$\frac{\partial f}{\partial y}$$

Gradient

- Gradient of a function $f(x, y)$ is defined as

$$\nabla f(x, y) = \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \\ \dfrac{\partial f}{\partial y} \end{bmatrix}$$

Gradient in 3D

- Gradient of a function $f(x, y, z)$ is defined as

$$\nabla f(x, y, z) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \\ \frac{\partial f}{\partial y} \\ \\ \frac{\partial f}{\partial z} \end{bmatrix}$$

More on gradient

- It is a vector
- Therefore, it has direction and magnitude
- Its direction points in the direction of the greatest rate of increase (i.e., direction of maximum increase) of the function
- Its magnitude is the slope of the graph of the function (i.e., the rate of increase) in that direction

Put gradient in context

- Imagine you are standing on a hillside. Look all around you, and find the direction of steepest ascent.

Put gradient in context

- Imagine you are standing on a hillside. Look all around you, and find the direction of steepest ascent.
- That is the direction of the gradient.

Put gradient in context

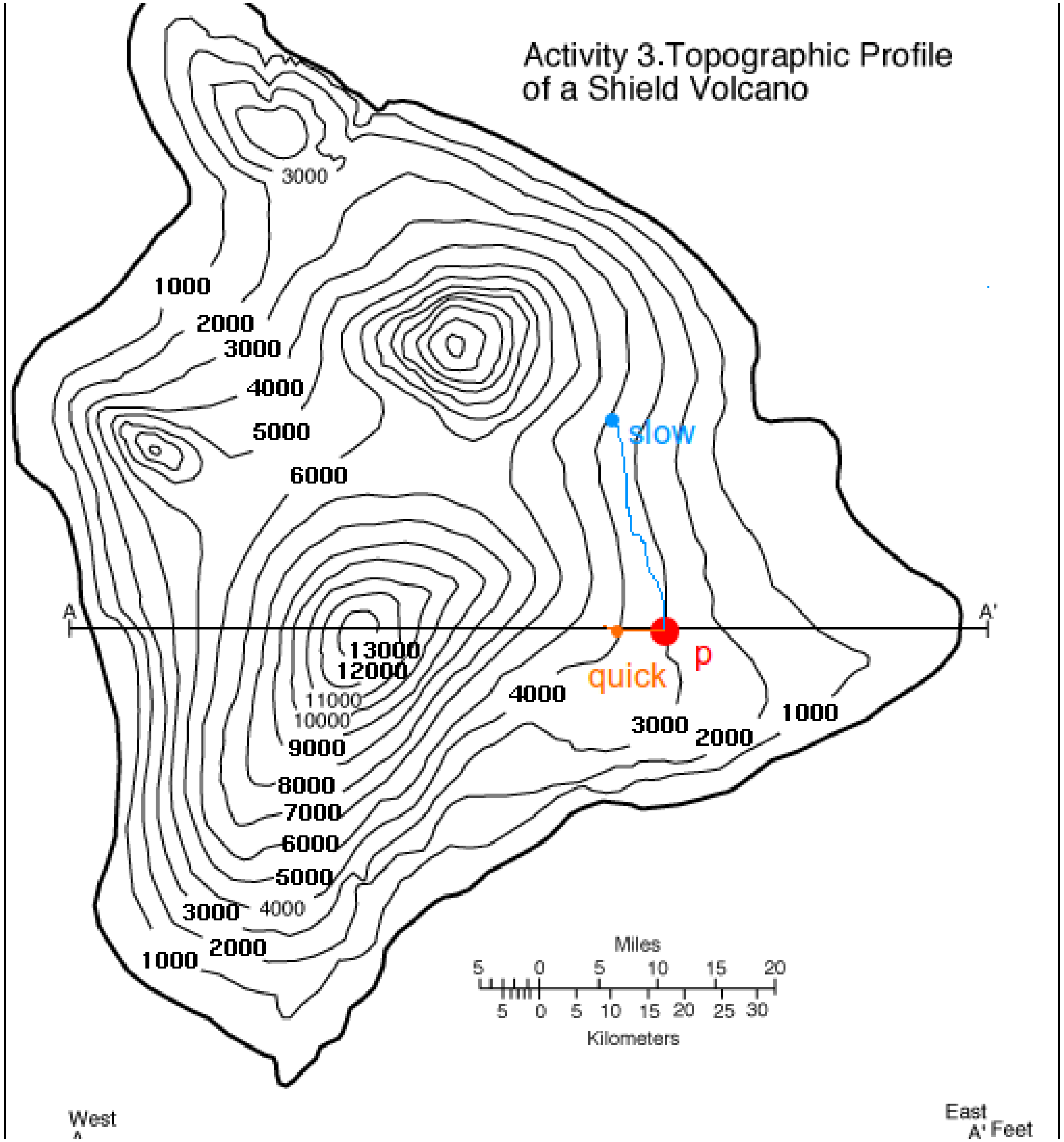
- Imagine you are standing on a hillside. Look all around you, and find the direction of steepest ascent.
- That is the direction of the gradient.
- Now measure the slope in that direction (rise over run)

Put gradient in context

- Imagine you are standing on a hillside. Look all around you, and find the direction of steepest ascent.
- That is the direction of the gradient.
- Now measure the slope in that direction (rise over run)
- That is the magnitude of the gradient.
- Here, the function is the height of hill (as a function of positions).

Understanding gradient

- Consider the topography as a 2D function $f(x, y)$
- The gradient direction tells you the fastest way up



Picture taken from https://mathoverflow.net/questions/1977/why-is-the-gradient-normal

Gradient in 3D

- Gradient of a function $f(x, y, z)$ is defined as

$$\nabla f(x, y, z) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}$$

Gradient in 3D

- Gradient of a function $f(x, y, z)$ is defined as

$$\nabla f(x,y,z) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \\ \frac{\partial f}{\partial y} \\ \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \\ \frac{\partial}{\partial y} \\ \\ \frac{\partial}{\partial z} \end{bmatrix} f$$

Gradient in 3D

- Gradient of a function $f(x, y, z)$ is defined as

$$\nabla f(x, y, z) = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} f$$

Gradient in 3D

- Gradient of a function $f(x, y, z)$ is defined as

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

The Del operator

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

The Del operator

- It looks like a vector

- But it doesn't mean much until we provide it with a function to act upon

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

Griffiths, 4th edition, pp 16

The Del operator

- It looks like a vector
- But it doesn't mean much until we provide it with a function to act upon
- To be precise, we say that ∇ is a vector operator that acts upon $f(x, y, z)$

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

Griffiths, 4th edition, pp 16

The Del operator: why bother?

- ∇ mimics the behavior of an ordinary vector in virtually every way
- Almost anything that can be done with other vectors can also be done with ∇

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

Griffiths, 4th edition, pp 16

The Del operator: why bother?

An ordinary vector $\boldsymbol{x}$ can multiply in three ways:

- By a scalar a: $a\boldsymbol{x}$
- By a vector $\boldsymbol{y}$, via dot product: $\boldsymbol{x} \cdot \boldsymbol{y}$
- By a vector $\boldsymbol{y}$, via cross product: $\boldsymbol{x} \times \boldsymbol{y}$

Griffiths, 4^{th} edition, pp 16

The Del operator: why bother?

An ordinary vector $\boldsymbol{x}$ can multiply in three ways:

- By a scalar a: $\boldsymbol{x}a$
- By a vector $\boldsymbol{y}$, via dot product: $\boldsymbol{x} \cdot \boldsymbol{y}$
- By a vector $\boldsymbol{y}$, via cross product: $\boldsymbol{x} \times \boldsymbol{y}$

Correspondingly, there are three ways ∇ can act:

- On a scalar function f: ∇f
- On a vector function $\boldsymbol{v}$, via dot product: $\nabla \cdot \boldsymbol{v}$
- On a vectorfunction $\boldsymbol{v}$, via cross product: $\nabla \times \boldsymbol{v}$

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

Griffiths, 4th edition, pp 16

The Del operator: why bother?

- It is a marvelous piece of notational simplification.

- By all means, take the vector appearance of ∇ seriously.

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

Griffiths, 4th edition, pp 16

Maxwell equations

$$\nabla \cdot \boldsymbol{d} = \rho_f$$ Gauss's law for electric fields

$$\nabla \cdot \boldsymbol{b} = 0$$ Gauss's law for magnetic fields

$$\nabla \times \boldsymbol{e} = -\frac{\partial \boldsymbol{b}}{\partial t}$$ Faraday's law

$$\nabla \times \mathbf{h} = \boldsymbol{j}_f + \frac{\partial \mathbf{d}}{\partial t}$$ Ampere-Maxwell equation

Several things that you might have noticed:

- ∇
- $\cdot$
- $\times$
- vectors

Divergence of a vector field

- Dot product $\nabla \cdot \boldsymbol{v}$
- From the definition of ∇, write out the component form

Divergence of a vector field

- Dot product $\nabla \cdot \boldsymbol{v}$
- From the definition of ∇, write out the component form

- Note that the divergence of a vector function is a scalar.

Understanding divergence

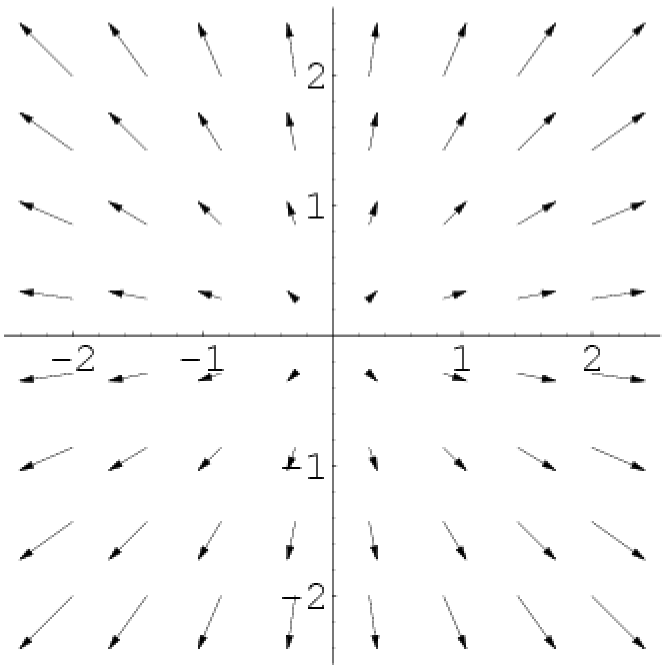
- Divergence $\nabla \cdot \boldsymbol{v}$ measures how much a vector $\boldsymbol{v}$ spreads out (diverges) from the point in question.

- Divergence is a local measure of its 'outgoingness'- the extent to which there is more of some quantity exiting an infinitesimal region than entering it.
- If the divergence is nonzero at some point, there is compression or expansion at that point.

Griffiths, 4th edition, pp 17 https://en.wikipedia.org/wiki/Divergence

Understanding divergence

- Imagine that a vector field $\boldsymbol{v}$ gives velocity of some fluid flow. It appears that the fluid is exploding outward from origin.



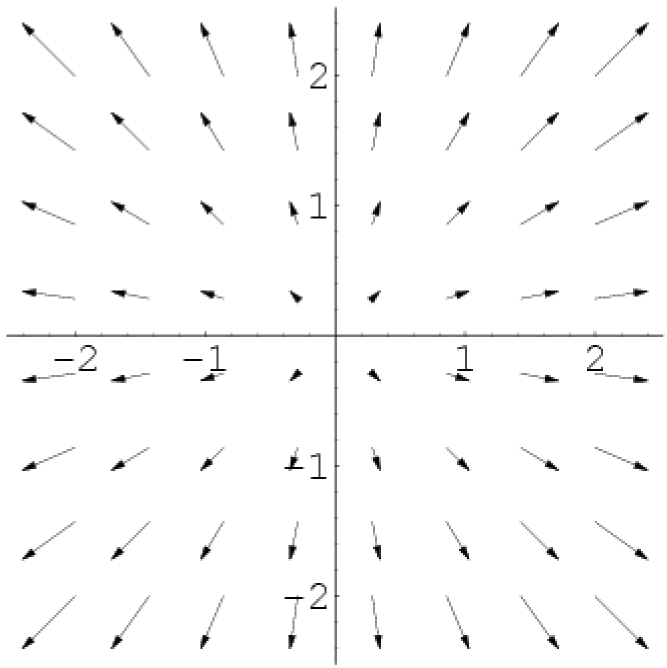
The expansion of the fluid flowing with velocity field $\boldsymbol{v}$ is captured by the divergence $\nabla \cdot \boldsymbol{v}$

The divergence here is positive since the flow is expanding.

https://mathinsight.org/divergence_idea

Understanding divergence

- Imagine that a vector field $\boldsymbol{v}$ gives velocity of some fluid flow. It appears that the fluid is exploding outward from origin.



The expansion of the fluid flowing with velocity field $\boldsymbol{v}$ is captured by the divergence $\nabla \cdot \boldsymbol{v}$

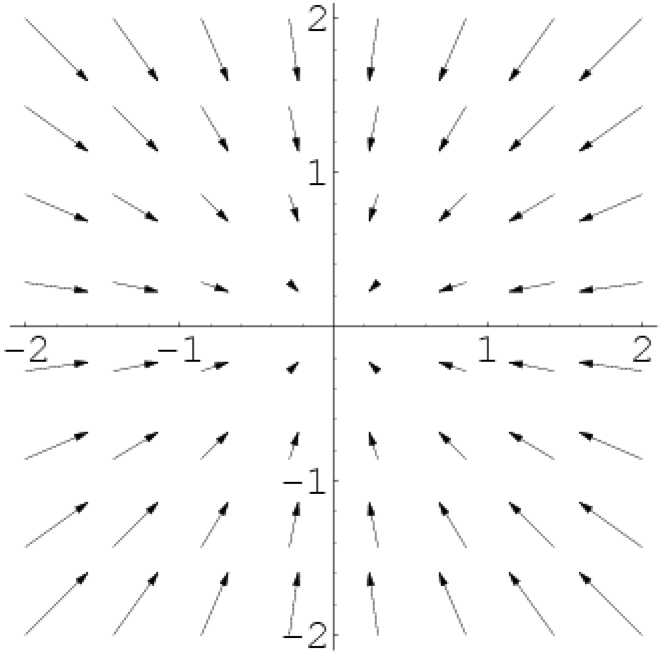
The divergence here is positive since the flow is expanding.

Regions of positive divergences are sources

https://mathinsight.org/divergence_idea

Understanding divergence

- The picture below shows the fluid is moving toward the origin.



The divergence here is negative.

Regions of negative divergence are sinks.

https://mathinsight.org/divergence_idea

The curl of a vector field

- The curl of a vector field $\boldsymbol{v}$: $\nabla \times \boldsymbol{v}$
- From the definition of ∇

The curl of a vector field

- The curl of a vector field $\boldsymbol{v}$: $\nabla \times \boldsymbol{v}$
- From the definition of ∇

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

Griffiths, 4th edition, pp 18

The curl of a vector field

- The curl of a vector field $\boldsymbol{v}$: $\nabla \times \boldsymbol{v}$
- From the definition of ∇

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

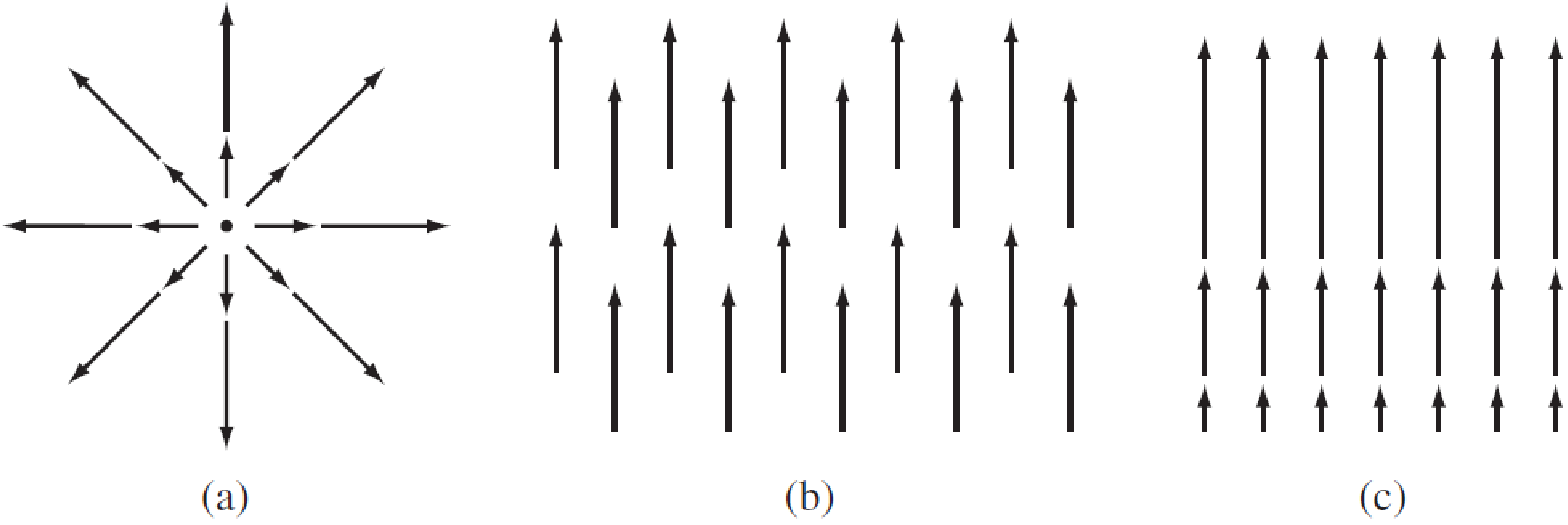
$$= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

- Note that the curl of a vector field is a vector.

Griffiths, 4th edition, pp 18

Understanding curl

- $\nabla \times \boldsymbol{v}$ is a measure of how much a vector field $\boldsymbol{v}$ swirls around the point in question.

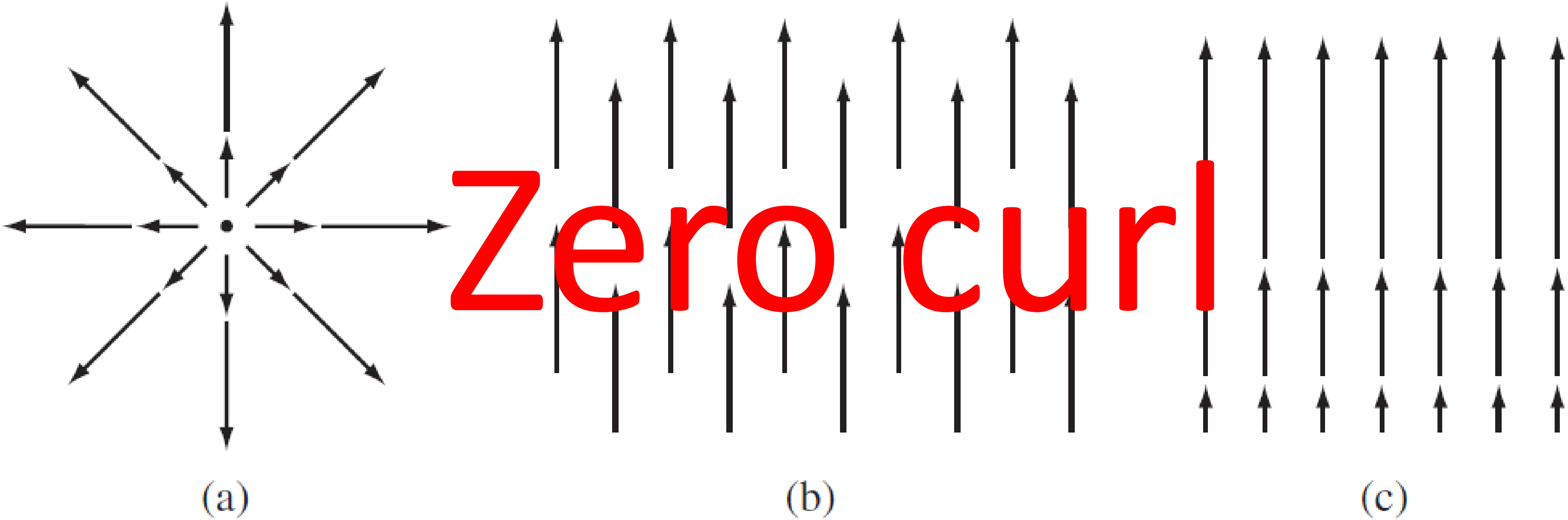


Griffiths, 4th edition, pp 17

Send the first assessment from CourseKey

Understanding curl

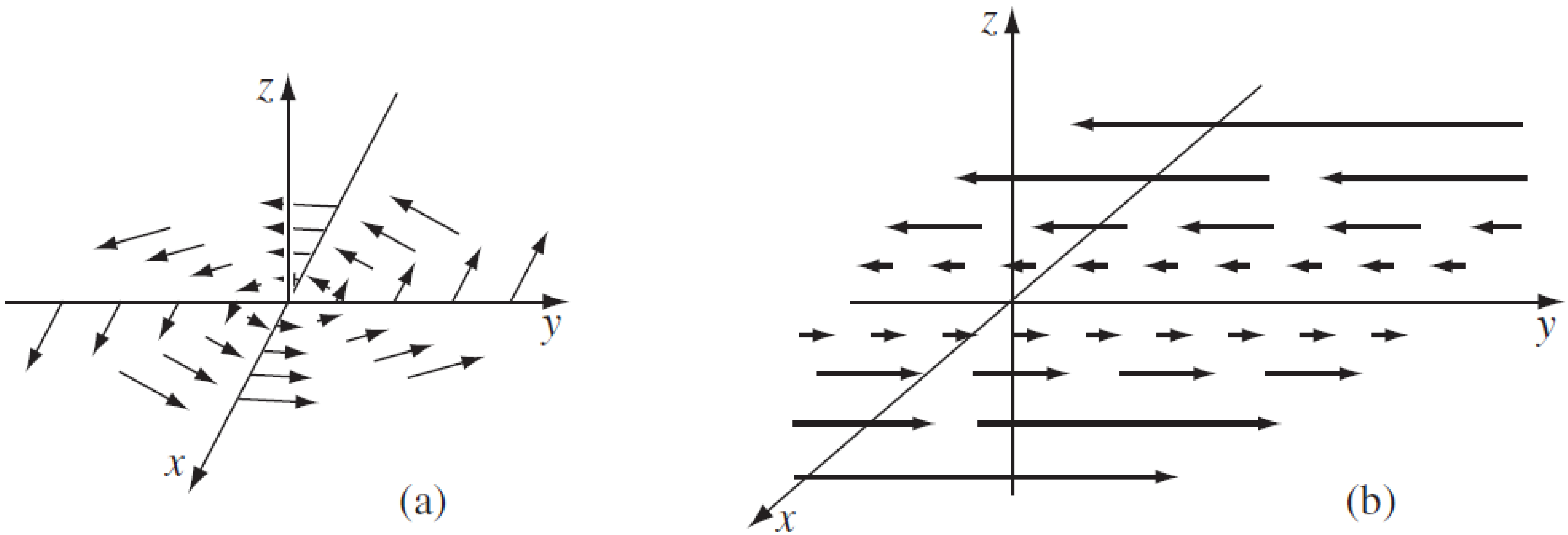
- $\nabla \times \boldsymbol{v}$ is a measure of how much a vector field $\boldsymbol{v}$ swirls around the point in question.



Griffiths, 4th edition, pp 17

Understanding curl

- $\nabla \times \boldsymbol{v}$ is a measure of how much a vector field $\boldsymbol{v}$ swirls around the point in question.

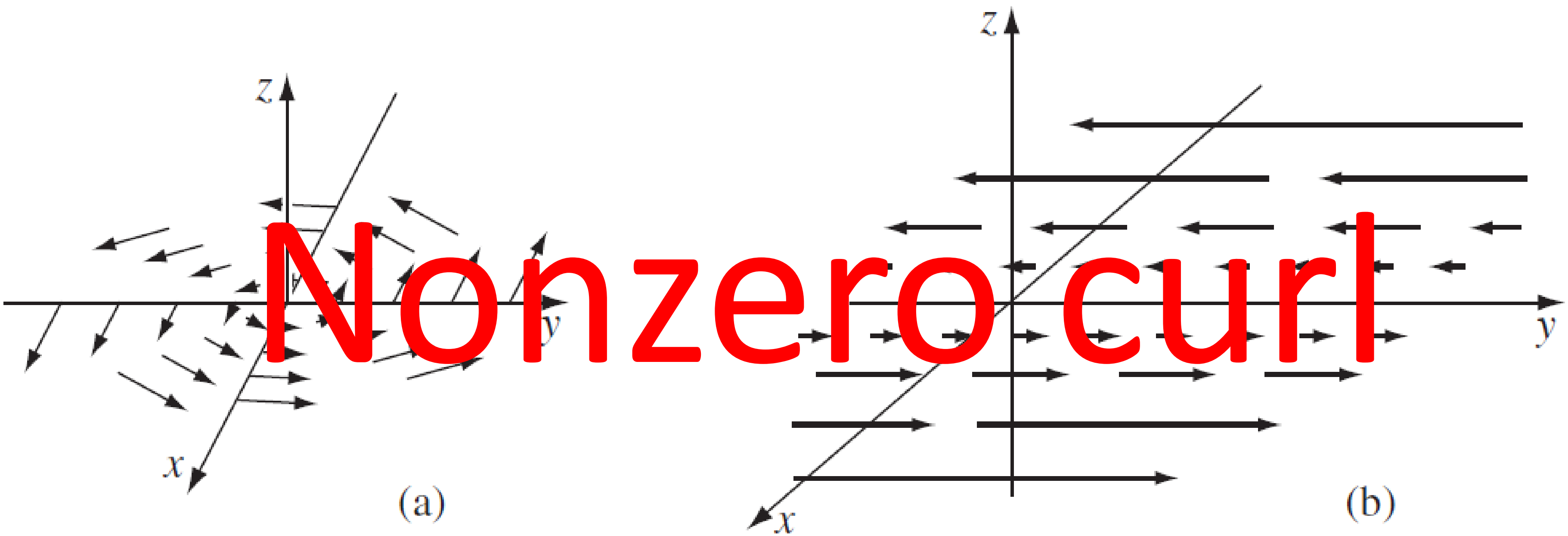


Imagine it as the velocity field of water. Now float a small paddlewheel. If it starts to rotate, then you placed it at a point of nonzero curl. Griffiths, 4th edition, pp 19

- Send the second assessment from CourseKey

Understanding curl

- $\nabla \times \boldsymbol{v}$ is a measure of how much a vector field $\boldsymbol{v}$ swirls around the point in question.



Griffiths, 4th edition, pp 19

Understanding curl

- $\nabla \times \boldsymbol{v}$ is a measure of how much a vector field $\boldsymbol{v}$ swirls (or, rotates) around the point in question.

Galaxy swirl

Whirlpool

https://cassmob.files.wordpress.com/2011/03/whirlpool-weirdness-2.jpg

Two identities

- The curl of a gradient is always zero

$$\nabla \times \nabla f = \mathbf{0}$$

- The divergence of a curl is always zero.

$$\nabla \cdot (\nabla \times \boldsymbol{v}) = 0$$

Two theorems

- Divergence theorem

$$\iiint_V (\nabla \cdot \mathbf{F})\, dV = \oiint_S (\mathbf{F} \cdot \mathbf{n})\, dS.$$

- The outward flux of a vector field through a closed surface is equal to the volume integral of the divergence over the region inside the surface.

- Intuitively, it states that the sum of all sources (with sinks regarded as negative sources) gives the net flux out of a region.

https://en.wikipedia.org/wiki/Divergence_theorem

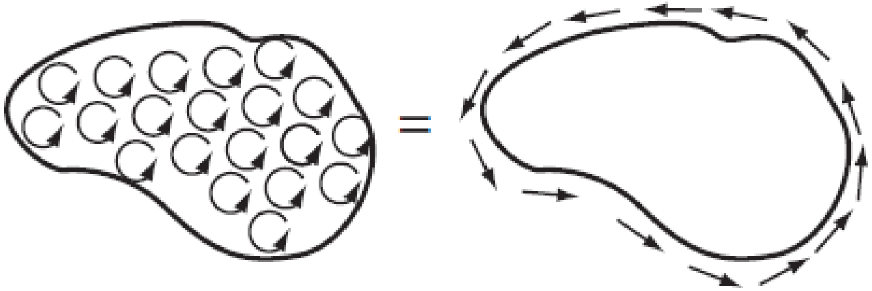
Two theorems

- Stokes' theorem

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_\Gamma \mathbf{F} \cdot d\mathbf{\Gamma}$$

Integral of curl over some surface → total amount of swirl

Circulation of the vector field along the boundary



Recall that curl measures the twist of a vector field; a region of high curl is a whirlpool. If you put a tiny paddle wheel, it will rotate.

Griffiths, 4th edition, pp 35

Revisiting Maxwell equations

$$\nabla \cdot \boldsymbol{d} = \rho_f$$

Gauss's law for electric fields

$$\nabla \cdot \boldsymbol{b} = 0$$

Gauss's law for magnetic fields

$$\nabla \times \boldsymbol{e} = -\frac{\partial \boldsymbol{b}}{\partial t}$$

Faraday's law

$$\nabla \times \mathbf{h} = \boldsymbol{j}_f + \frac{\partial \mathbf{d}}{\partial t}$$

Ampere-Maxwell equation

Differential equations

- A differential equation is a mathematical equation that relates some function with its derivatives

$$\varepsilon - L\frac{dI}{dt} = IR$$

$$I(t = 0) = 0$$

Solving differential equations

$$\varepsilon - L\frac{dI}{dt} = IR$$

$$L\frac{dI}{dt} = \varepsilon - IR$$

$$\frac{dI}{dt} = \frac{\varepsilon - IR}{L}$$

$$\frac{dI}{\varepsilon - IR} = \frac{dt}{L}$$

$$\int_I \frac{dI}{\varepsilon - IR} = \int_t \frac{dt}{L}$$

$$-\frac{\ln(\varepsilon - IR)}{R} = \frac{t}{L} + const.$$

$$\ln(\varepsilon - IR) = -\frac{R}{L}t + const.$$

$$\varepsilon - IR = ke^{-\frac{R}{L}t}$$

where k is again some constant

$$I(t) = \frac{1}{R}(\varepsilon - ke^{-\frac{R}{L}t})$$

Remember that $I(t=0) = 0$

Therefore, $k = \varepsilon$

Therefore, $I(t) = \frac{\varepsilon}{R}(1 - e^{-\frac{R}{L}t})$

If still have time,

- Show one of the EM apps
- Show em.geosci.xyz
- Show slack