Lecture 10

Circuit Model & Plane Waves

GEOL 4397: Electromagnetic Methods for Exploration
GEOL 6398: Special Problems

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UNIVERSITY of HOUSTON
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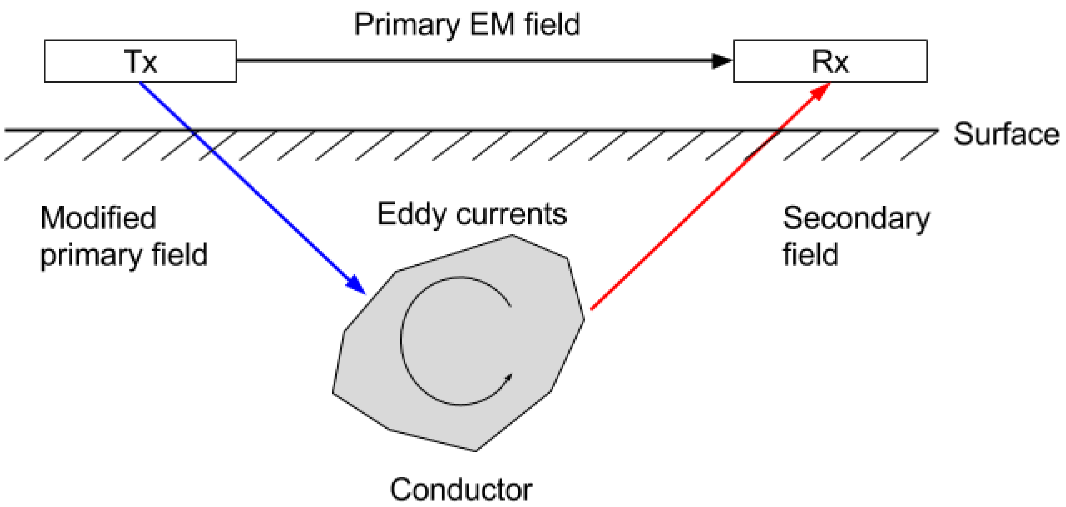
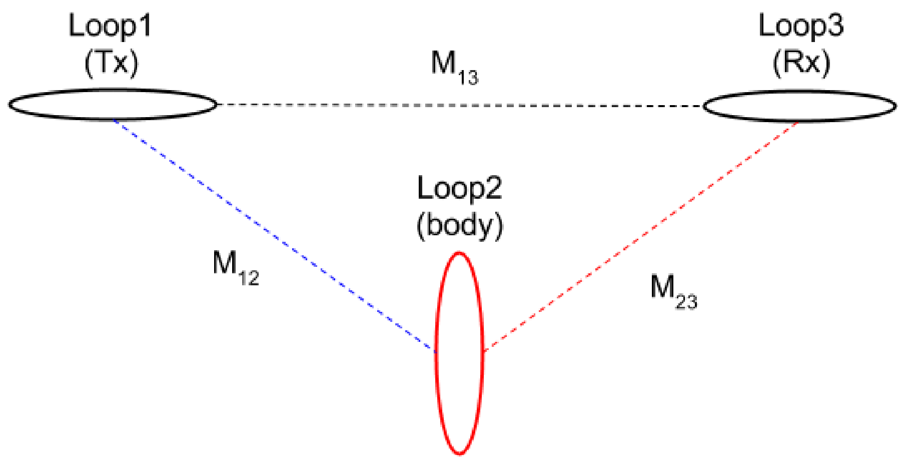
EARTH AND ATMOSPHERIC SCIENCES

Outline

- Circuit model
  - Understanding EM response
- Plane waves in a homogeneous media
  - Quasi-static approximation
  - Skin depth
  - Diffusion distance

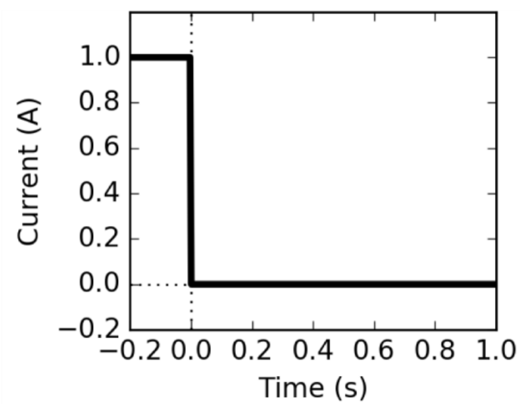




https://em.geosci.xyz/content/maxwell3_fdem/circuitmodel_for_eminduction/index.html

Time-domain EM response

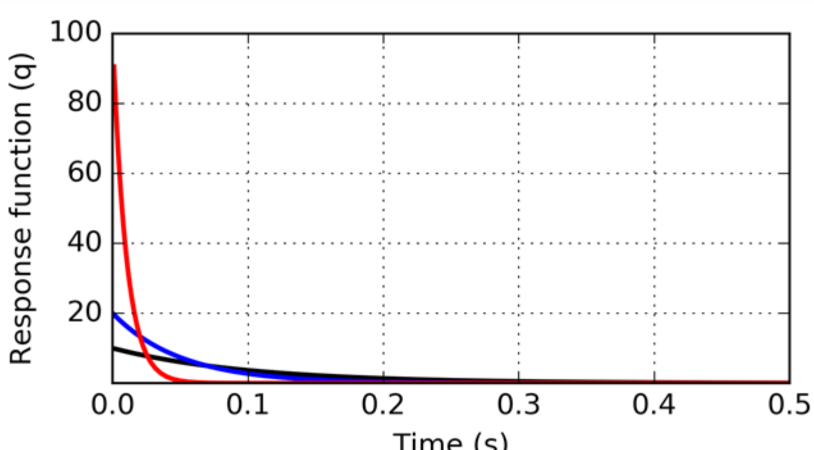
Step-off current in Tx

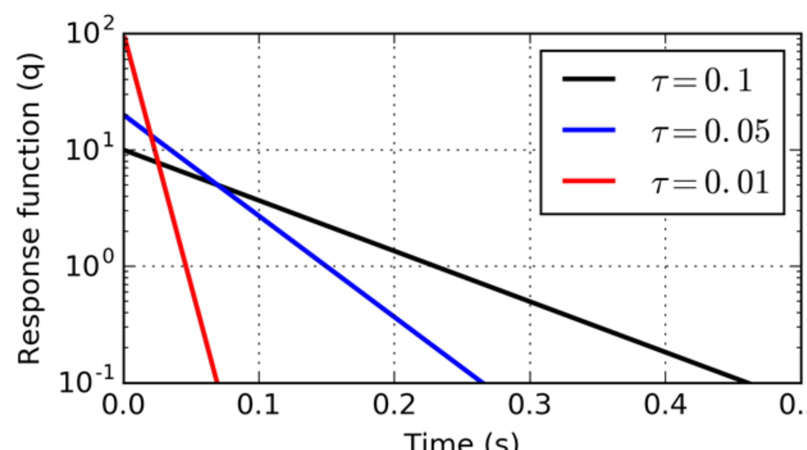


$$\varepsilon_3^s = \frac{M_{12}M_{23}}{L}\frac{I_1}{\tau}e^{-\frac{t}{\tau}}$$

where $\tau = \frac{L}{R}$, and $t > 0$

The larger the τ, the slower it decays. Therefore, the more conductive the subsurface, the slower it decays.



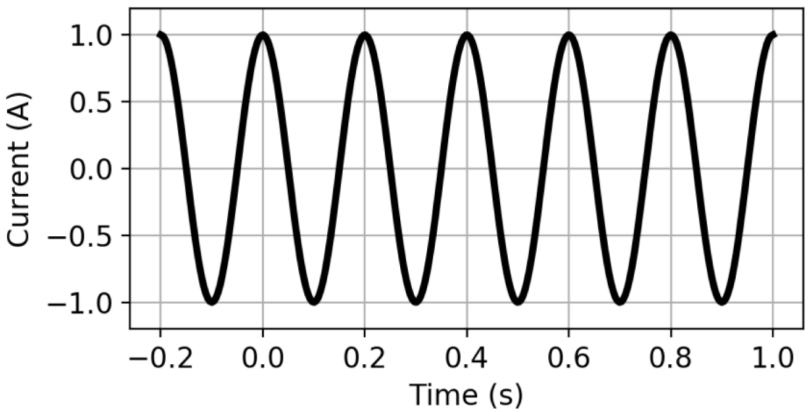


$$q = e^{-t/\tau}/\tau$$

Image credit: Doug Oldenburg, Seogi Kang and Linsey Heagy from UBC-GIF

Frequency-domain EM response

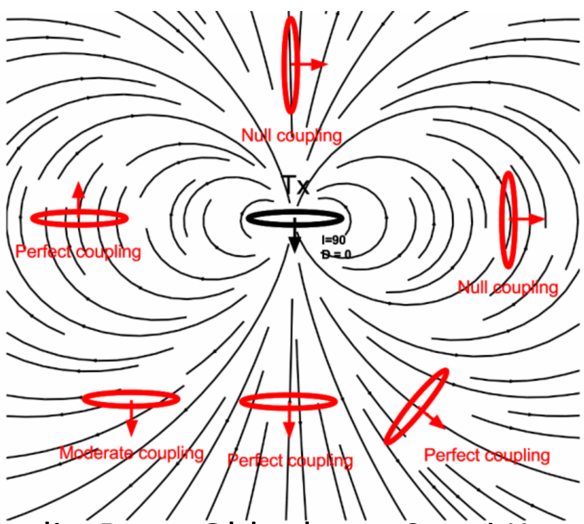
AC current in Tx



$$\frac{H_s^3}{H_p^3} = \frac{\varepsilon_3^s}{\varepsilon_3^p} = \boxed{-\frac{M_{12}M_{23}}{M_{13}L}}\boxed{\frac{\alpha^2 + i\alpha}{1+\alpha^2}}$$

Coupling coefficient

Response function



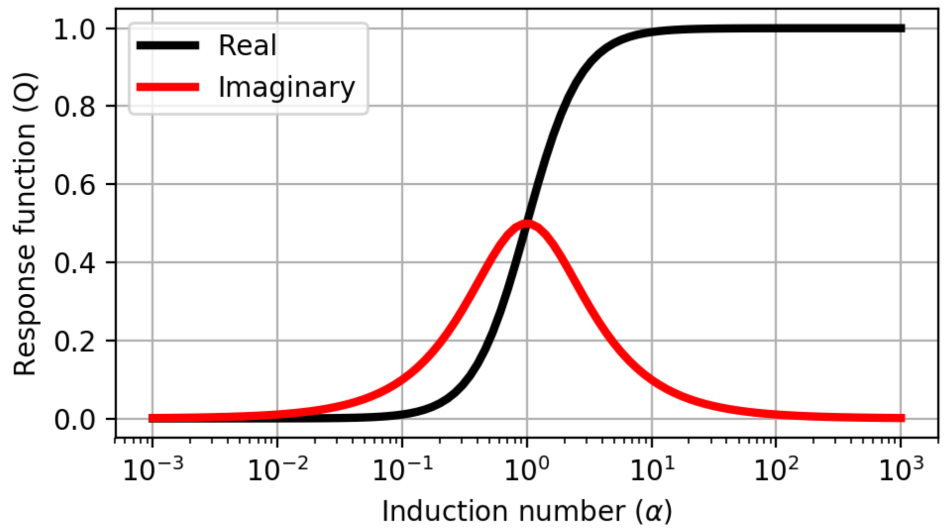
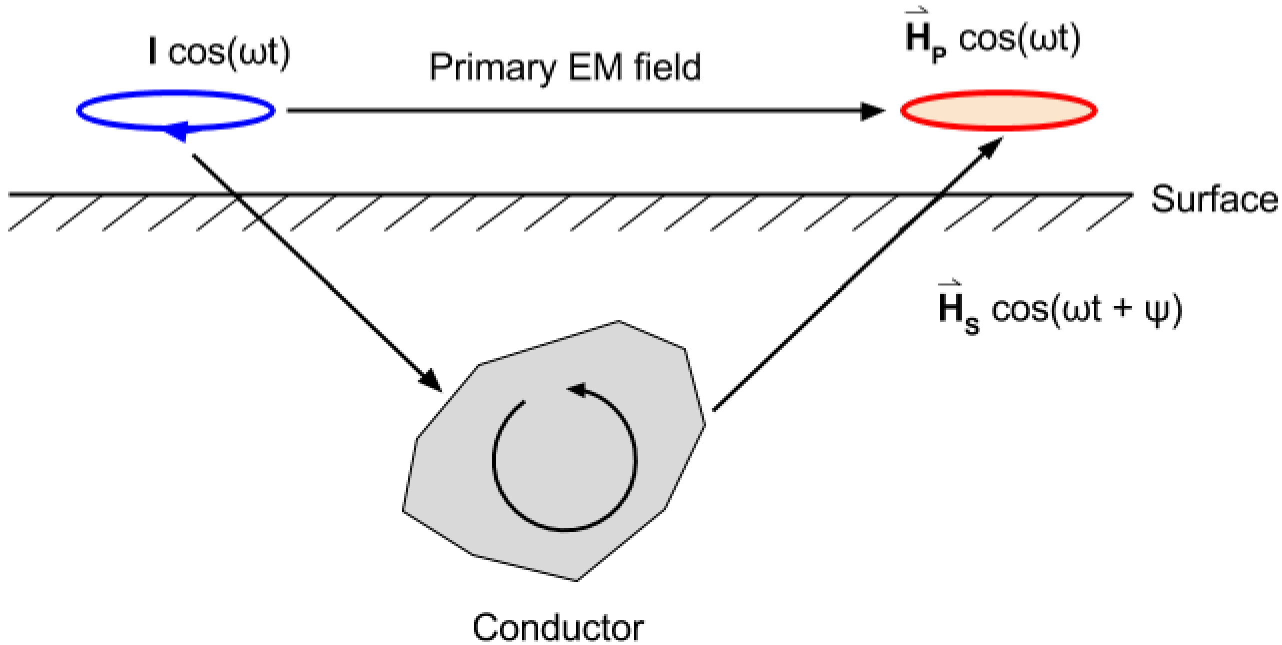


Image credit: Doug Oldenburg, Seogi Kang and Linsey Heagy from UBC-GIF

Frequency-domain EM response



The phase lag, $\psi = \frac{\pi}{2} + tan^{-1}\left(\frac{wL}{R}\right)$, is diagnostic of the conductivity of the subsurface body.

- Very resistivity body: 90^0
- Very conductive body: 180^0

Image credit: https://gpg.geosci.xyz/content/electromagnetics/electromagnetic_data.html

FEM survey

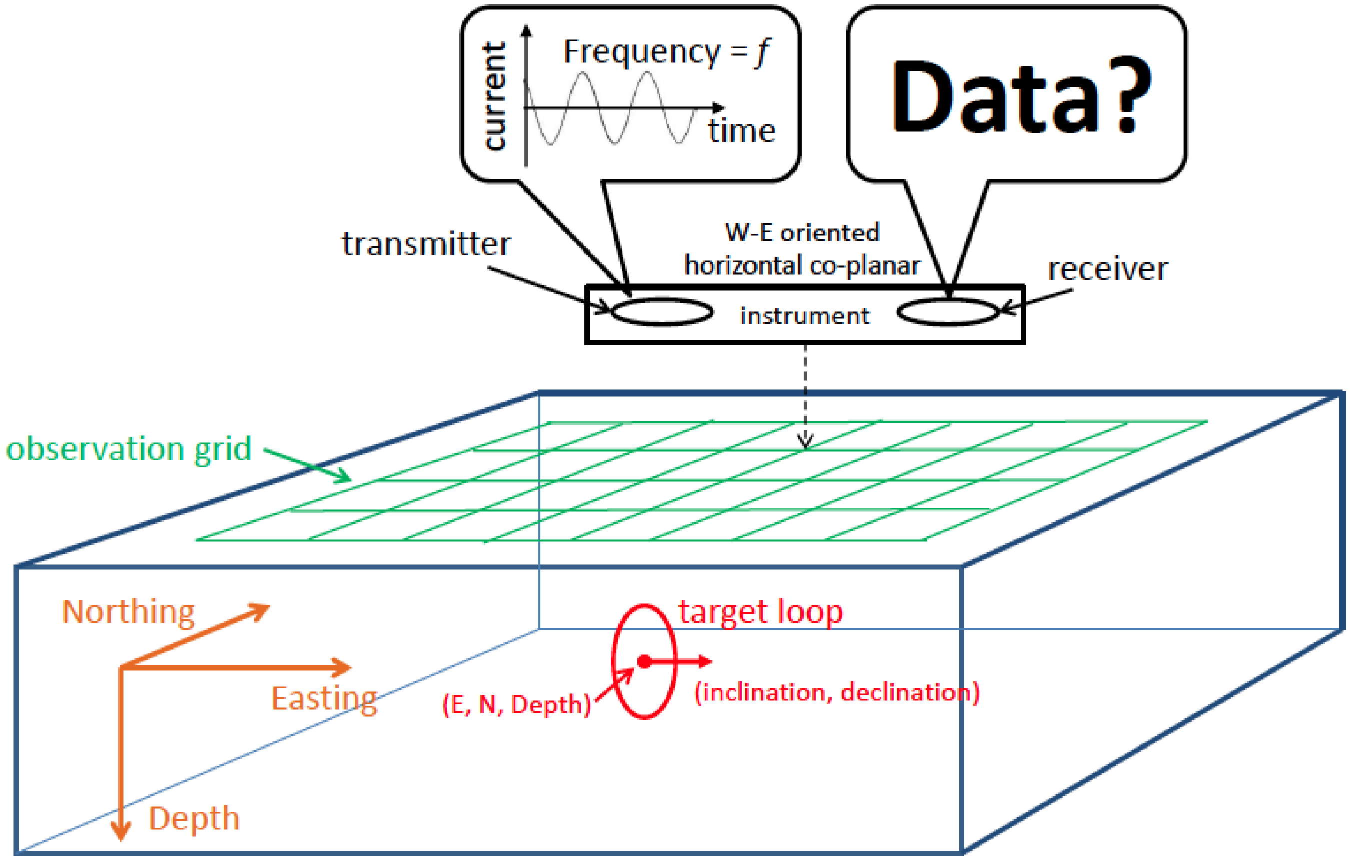


Image credit: Doug Oldenburg, Seogi Kang and Linsey Heagy from UBC-GIF

FEM data at 10000 Hz

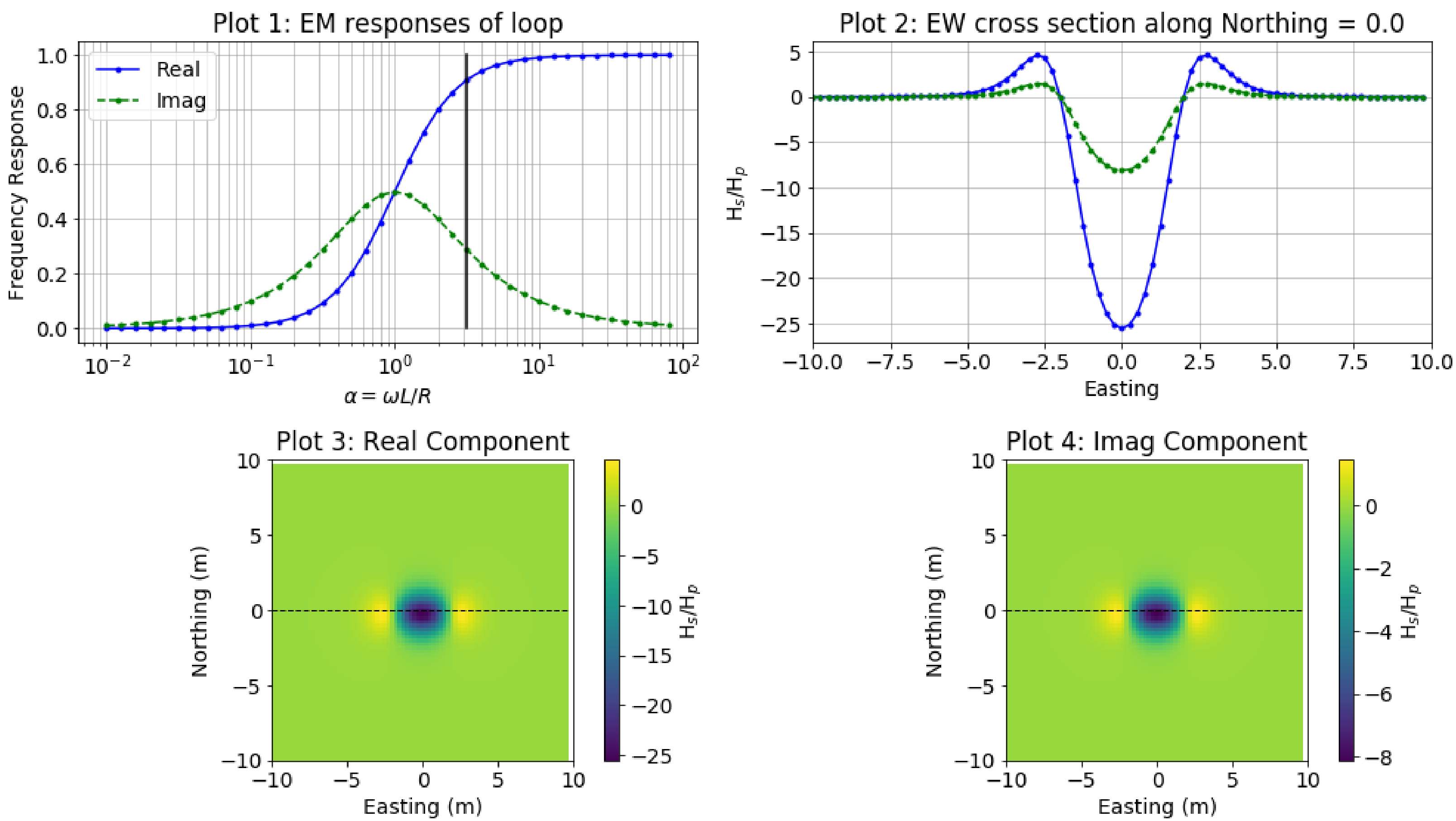
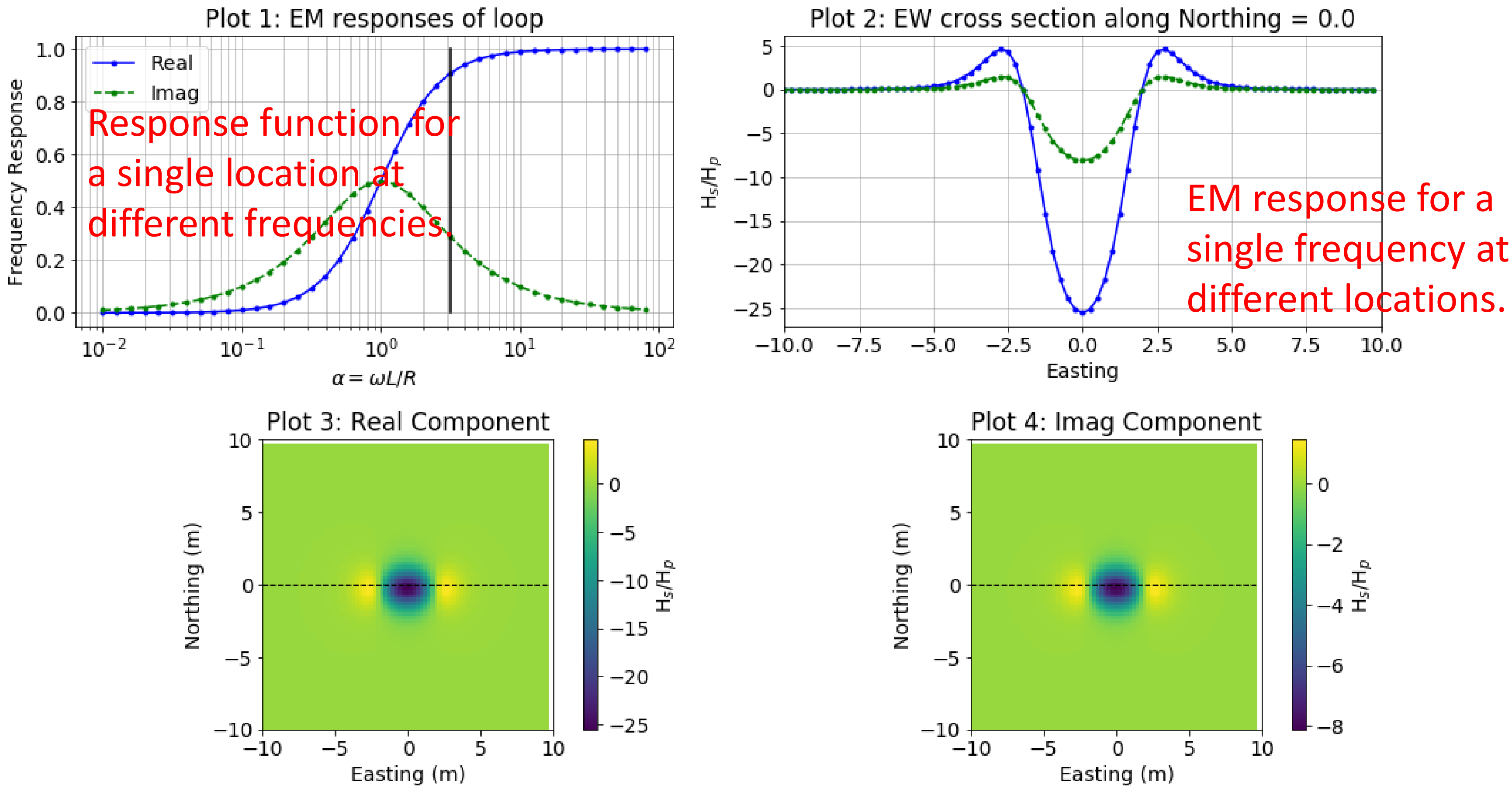


Image created using FDEM_ThreeLoopModel.ipynb. Inductance L = 0.1, Resistance R = 2000, xc = 0, yc = 0, zc = 1, dincl = 0, ddecl = 90, frequency = 10000 Hz, sampling spacing dx = 0.25

FEM data at 10000 Hz



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FEM data at 3180 Hz

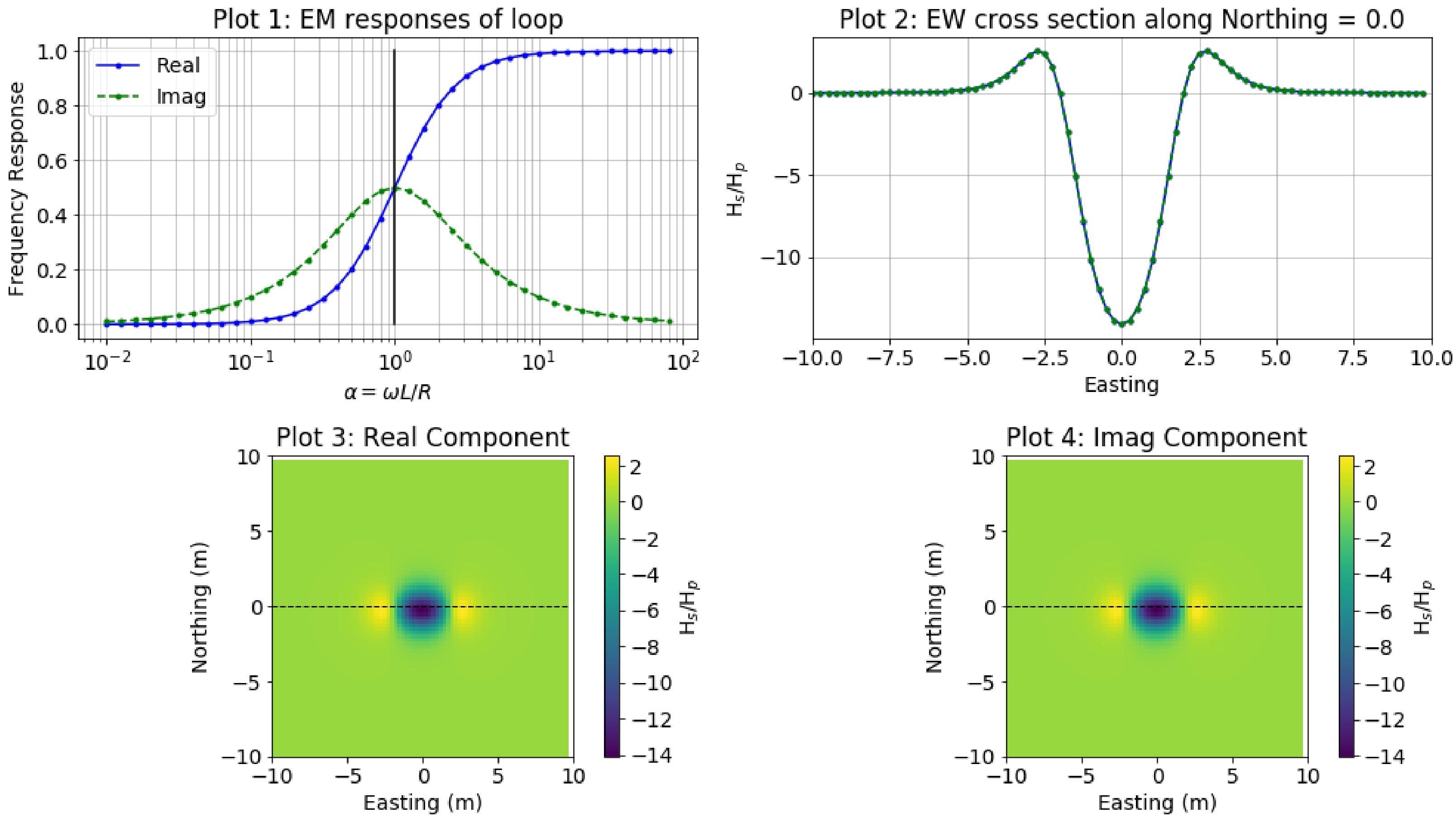
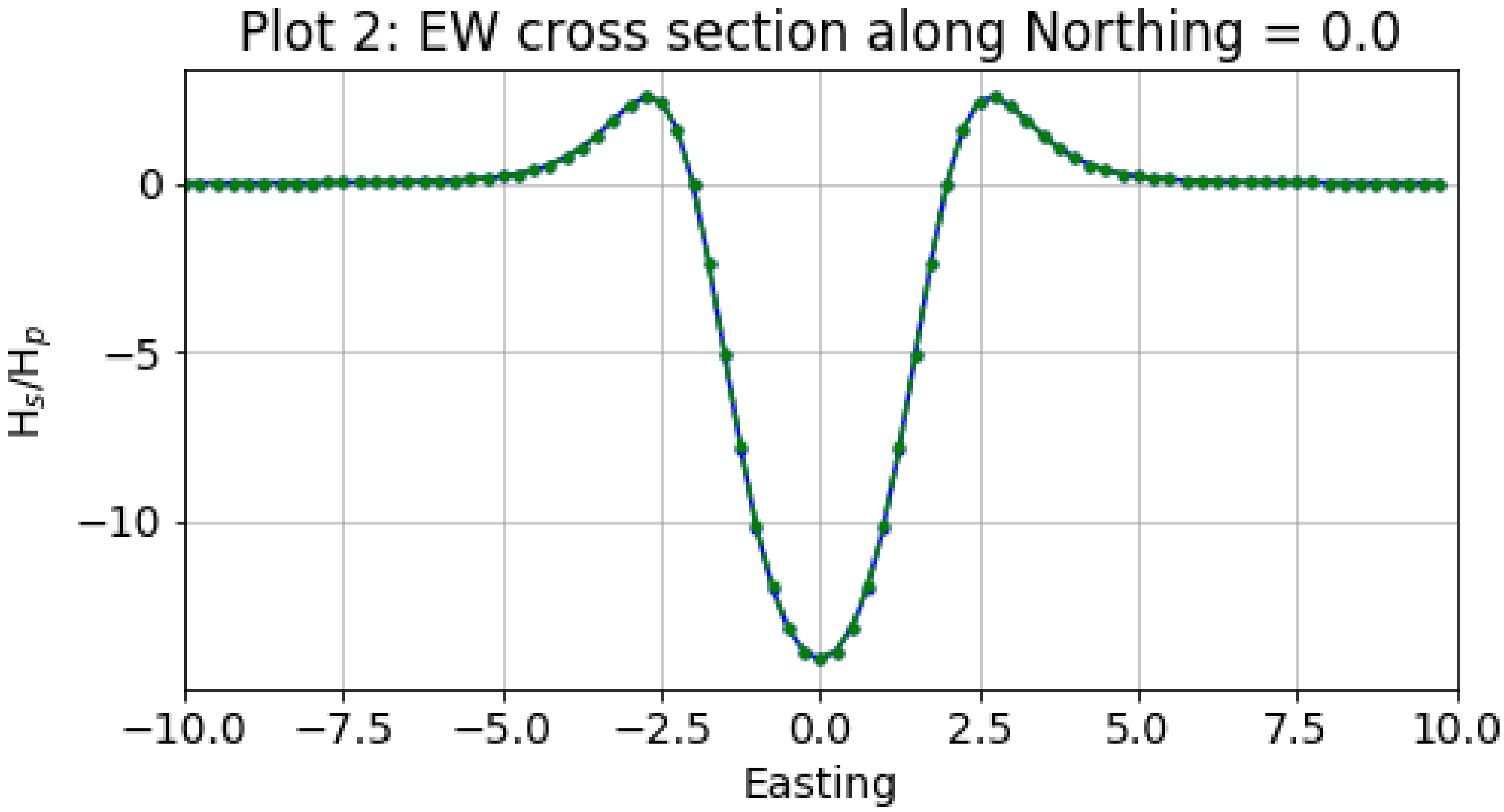
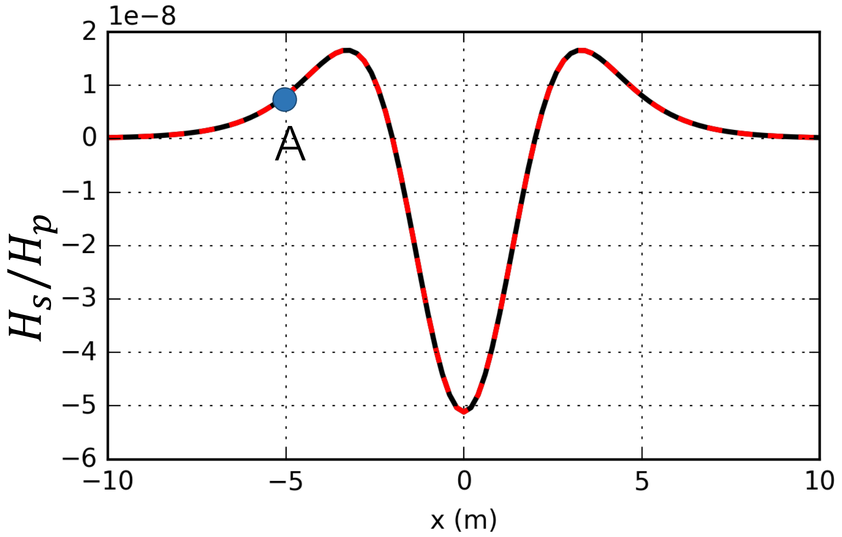


Image created using FDEM_ThreeLoopModel.ipynb. Inductance L = 0.1, Resistance R = 2000, xc = 0, yc = 0, zc = 1, dincl = 0, ddecl = 90, frequency = 3180 Hz, sampling spacing dx = 0.25

How to understand this?





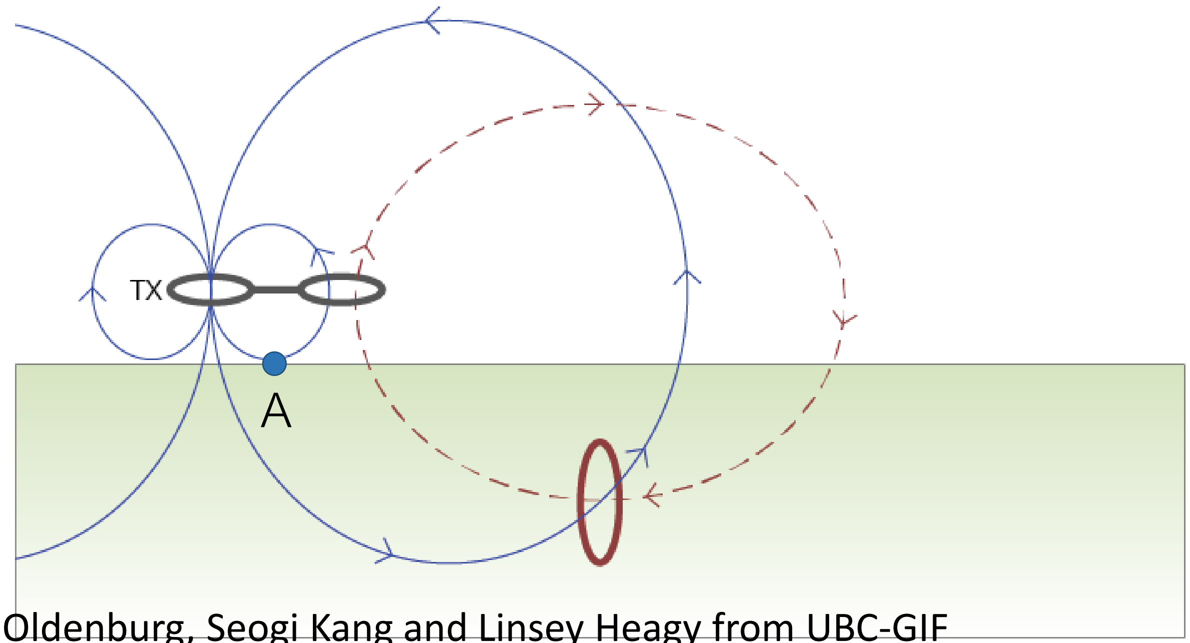
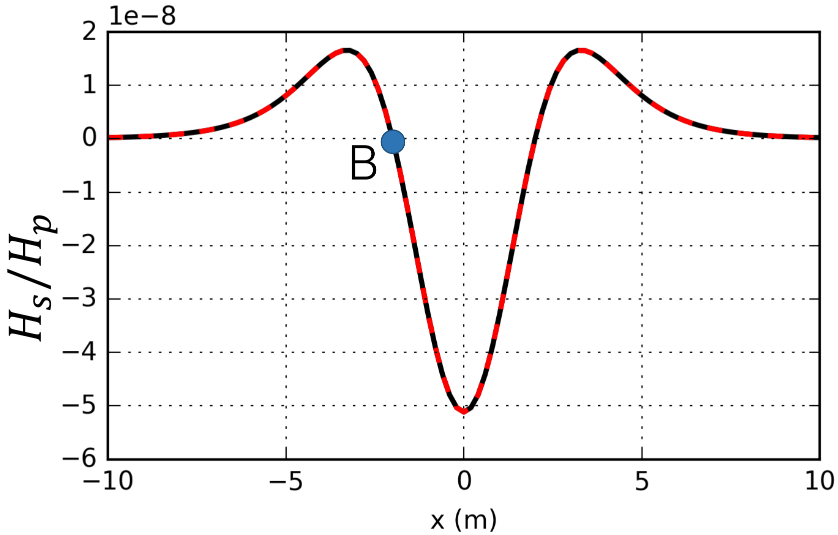
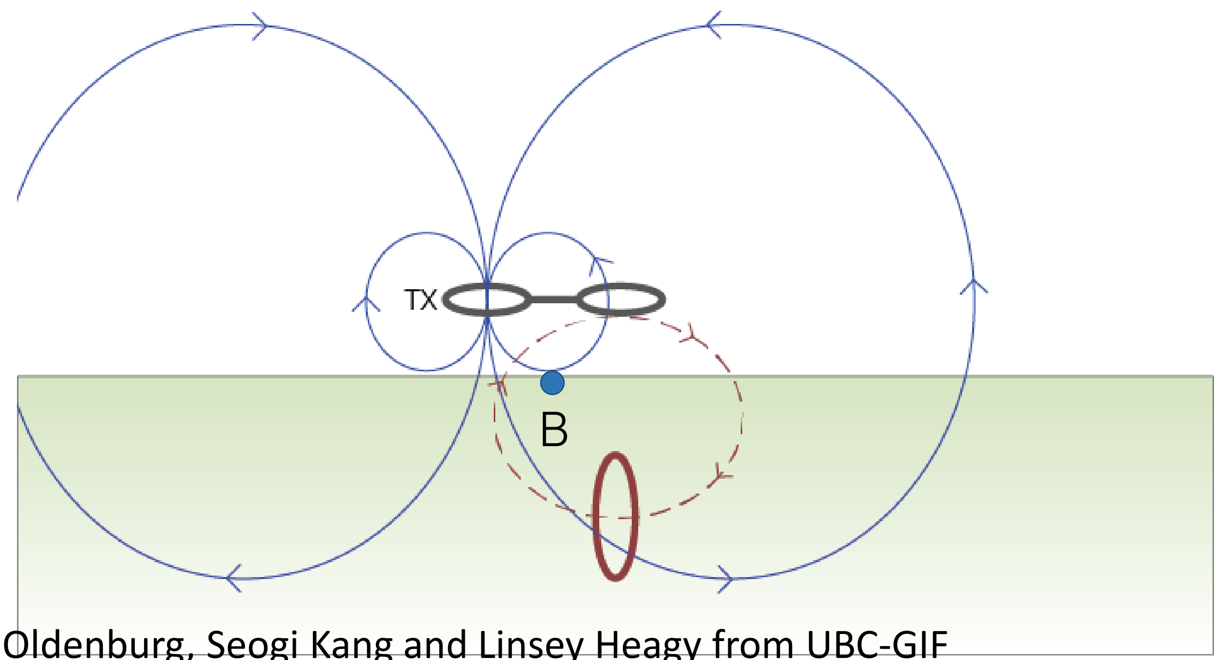


Image credit: Doug Oldenburg, Seogi Kang and Linsey Heagy from UBC-GIF





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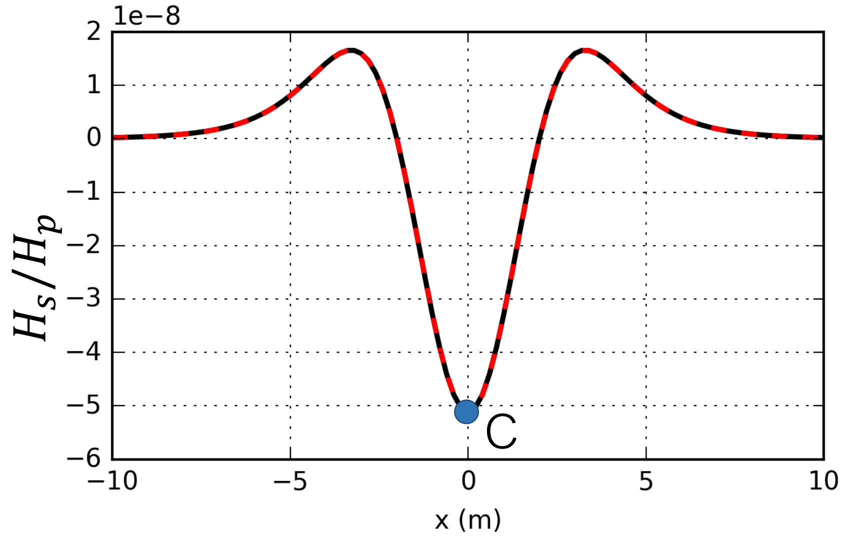

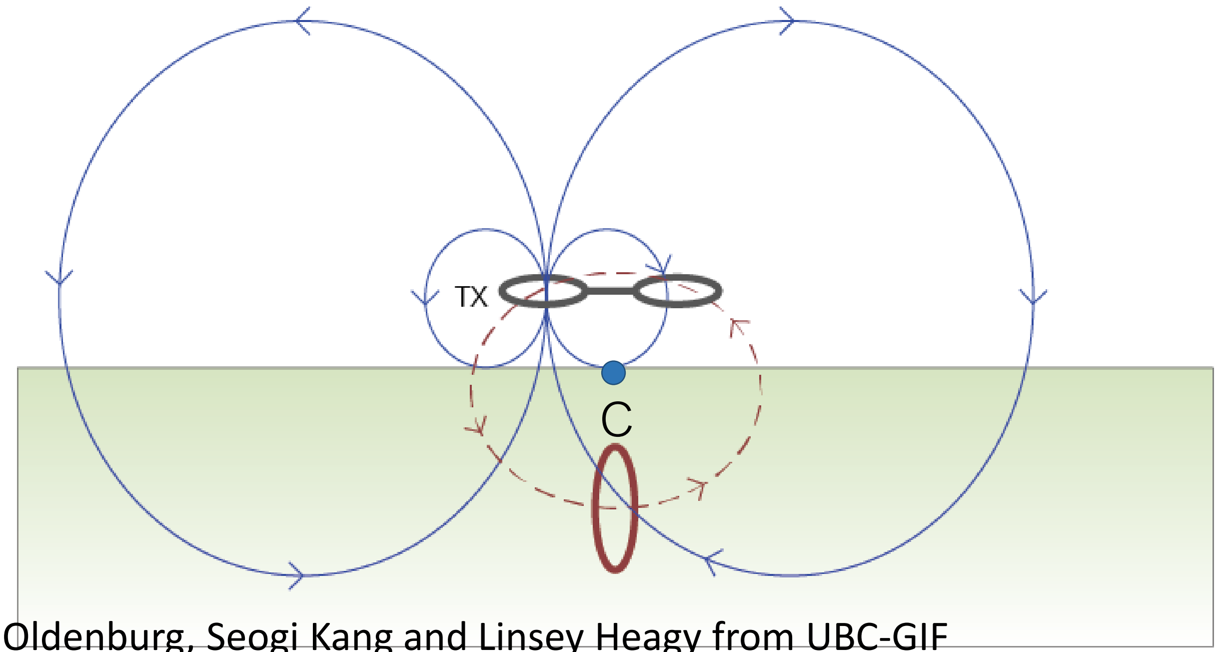

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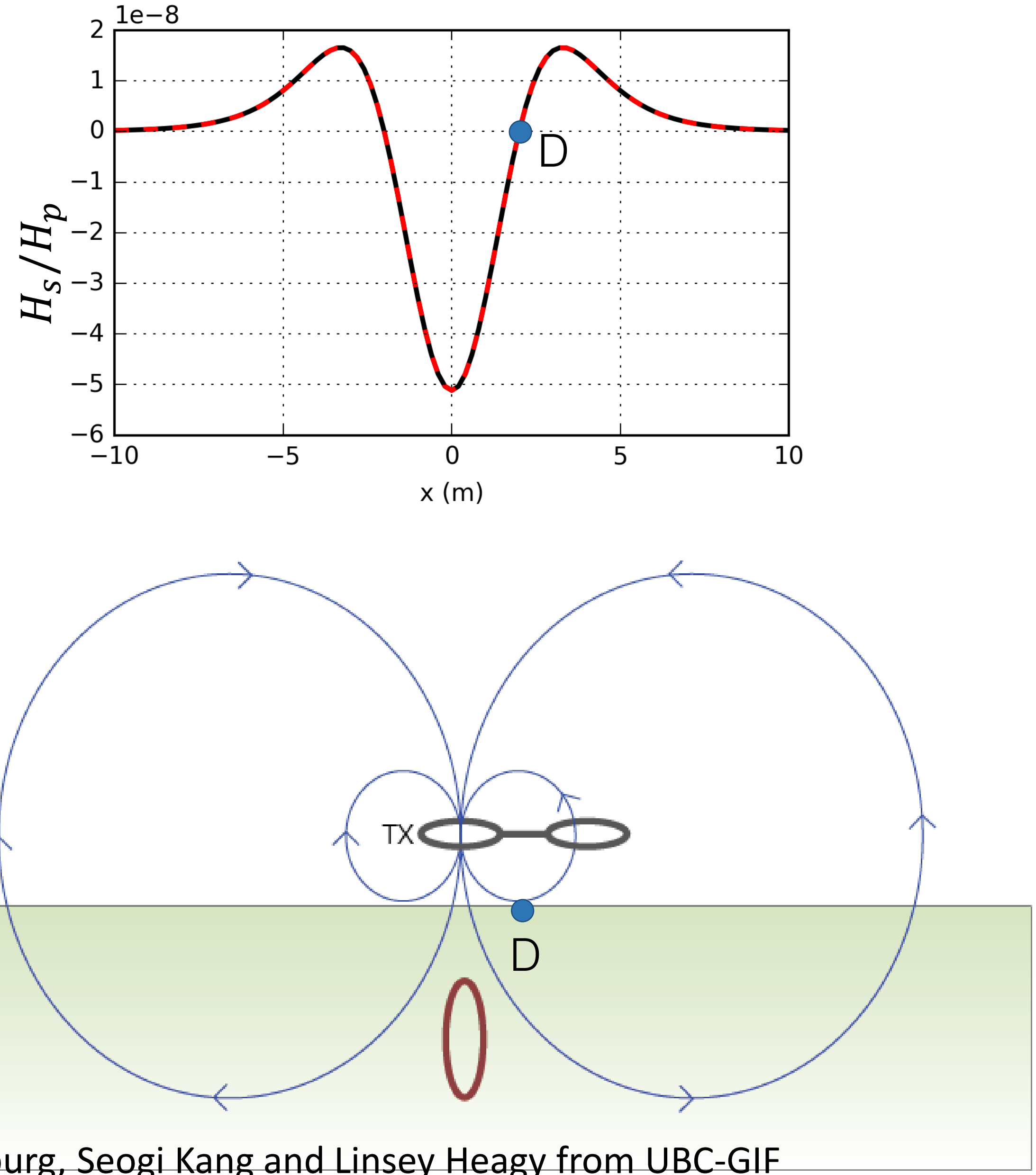

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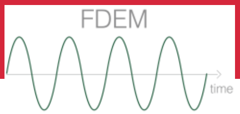
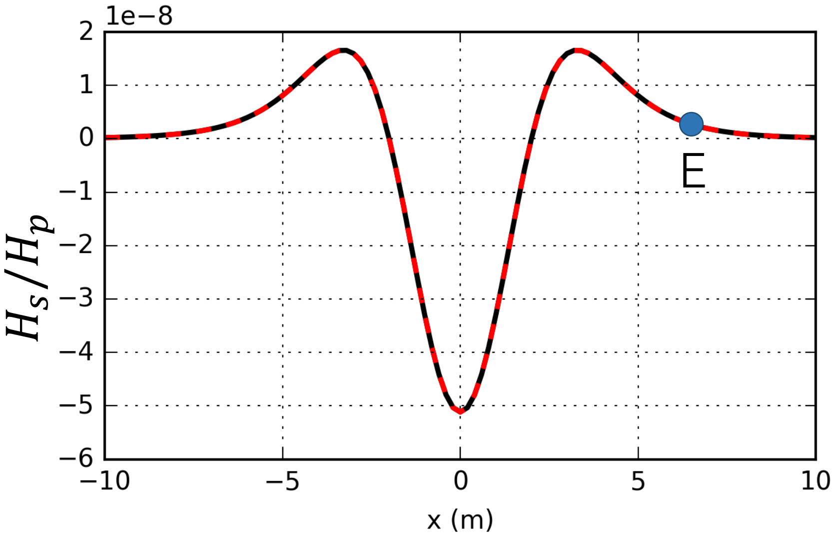




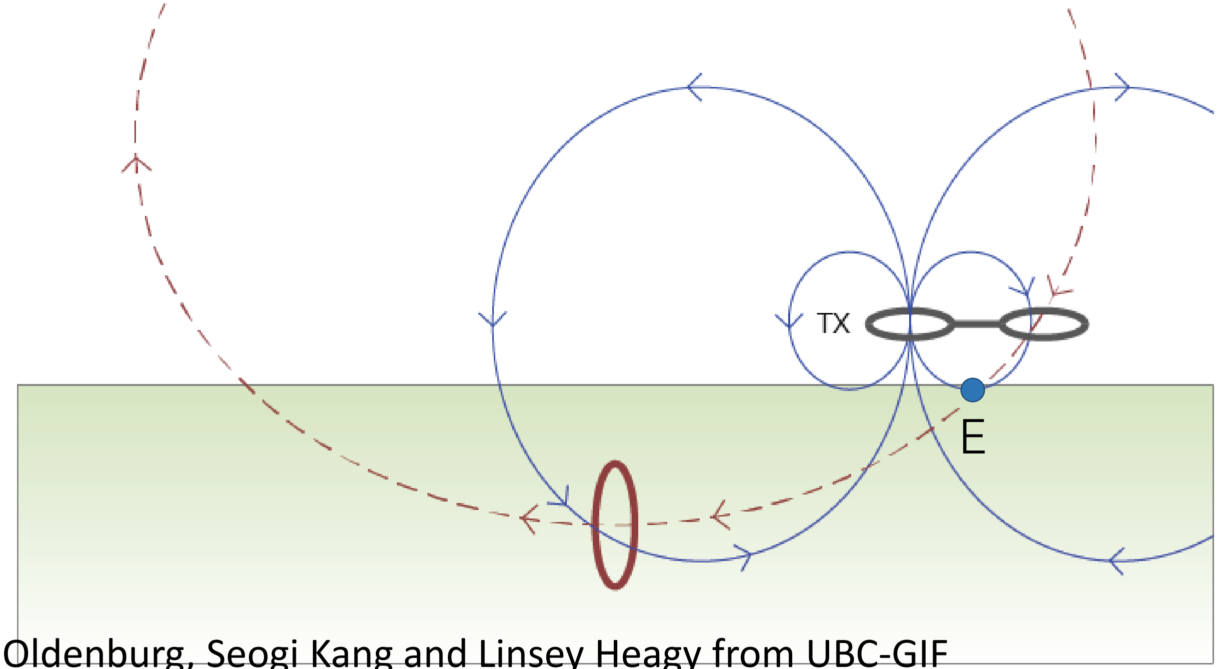
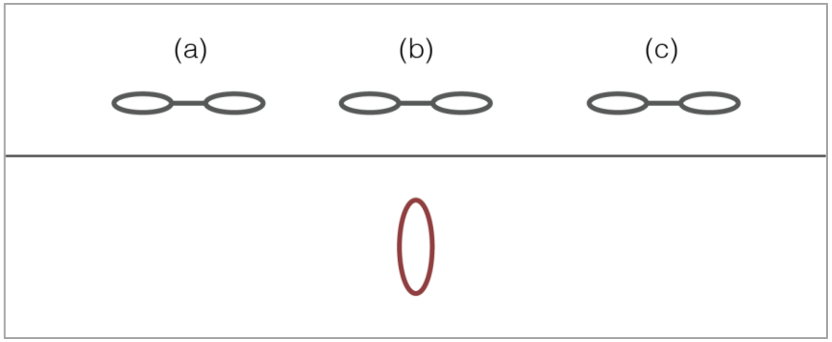
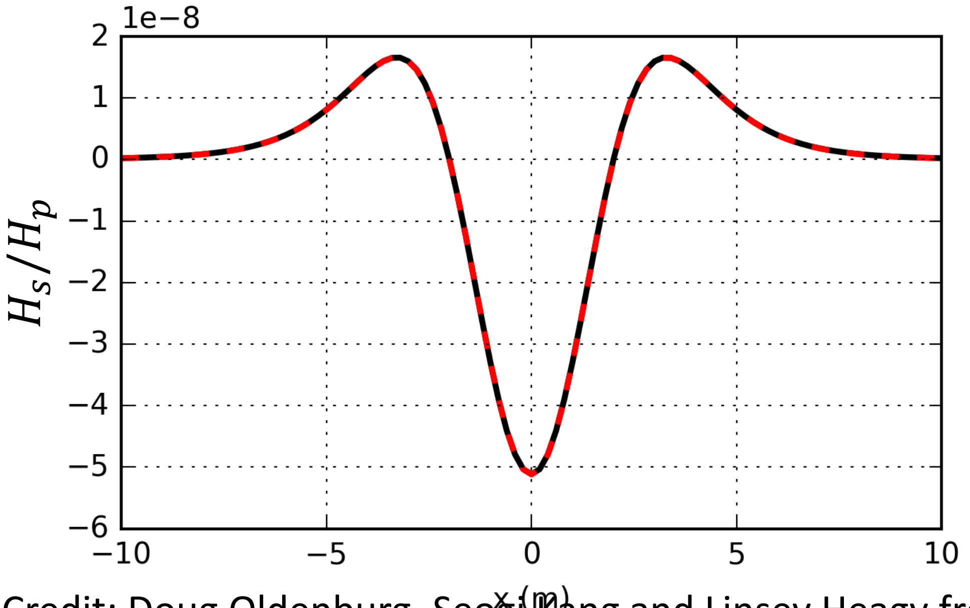


Image credit: Doug Oldenburg, Seogi Kang and Linsey Heagy from UBC-GIF

Conductor in a resistive earth: Frequency

Profile over the loop



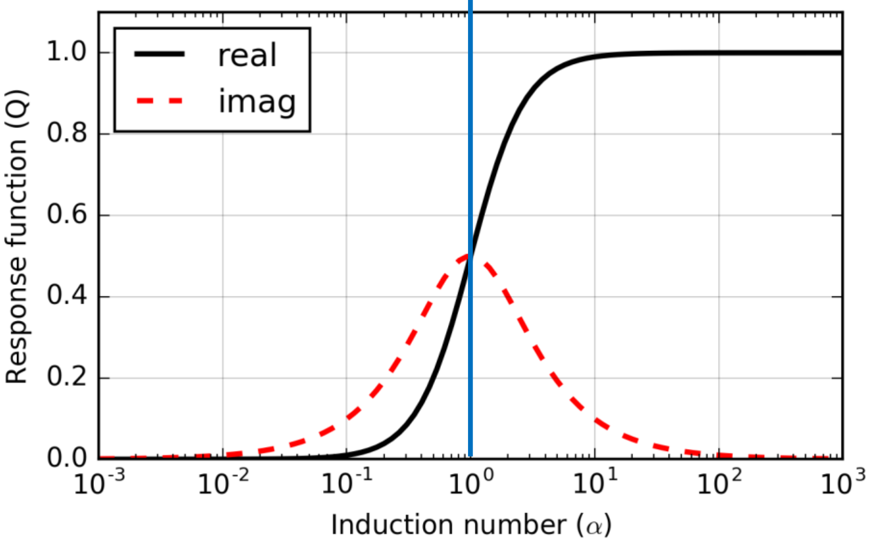


- Induction number

$$\alpha = \frac{\omega L}{R}$$

- When $\alpha = 1$
  - Real = Imag

$\alpha = 1$



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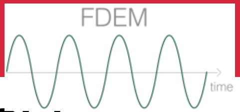
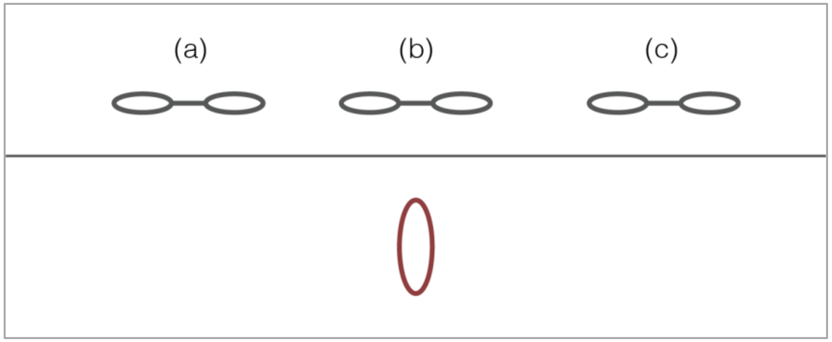
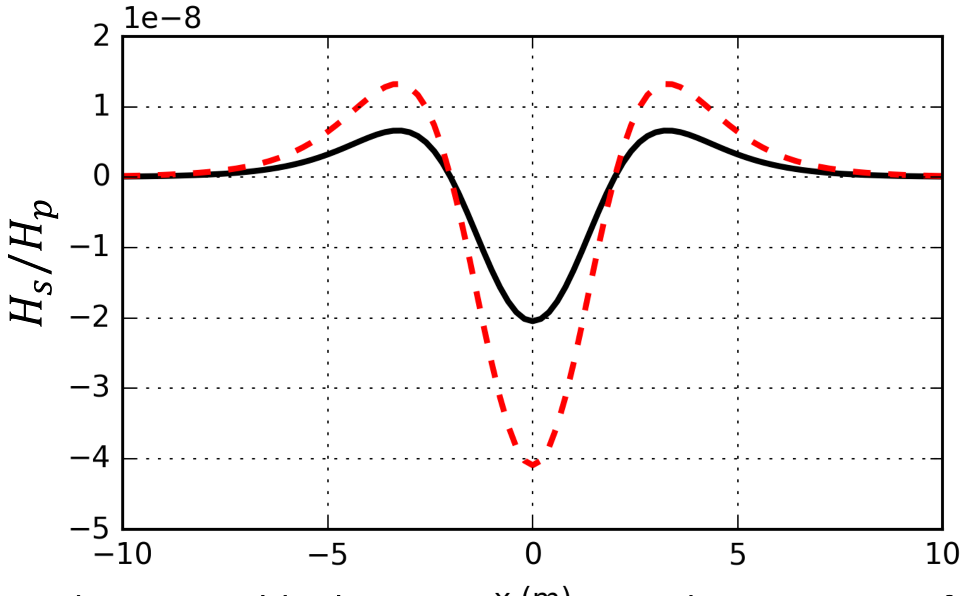
Conductor in a resistive earth: Frequency



Profile over the loop



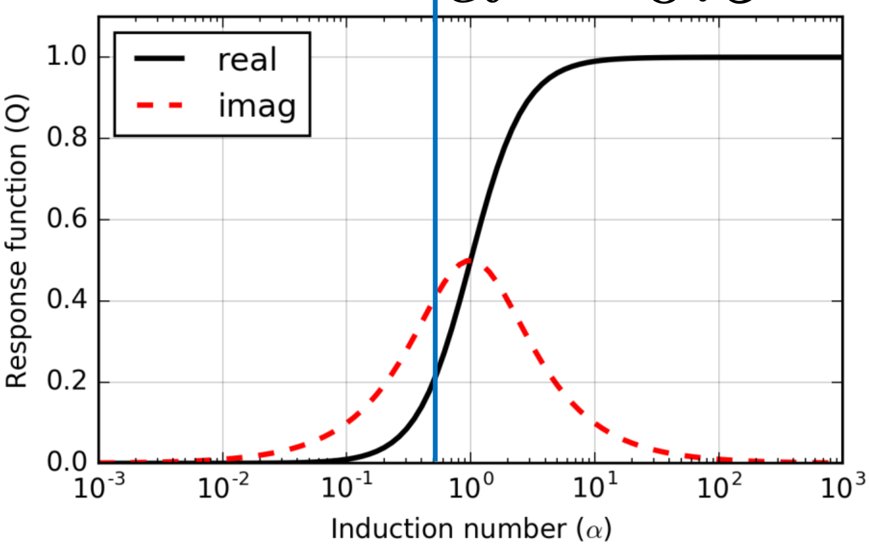


- **Induction number**

$$\alpha = \frac{\omega L}{R}$$

- When $\alpha = 1$
  - Real < Imag

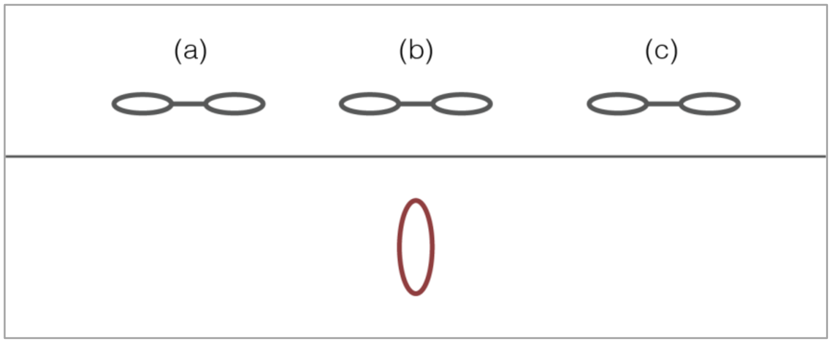
$$\alpha = 0.5$$

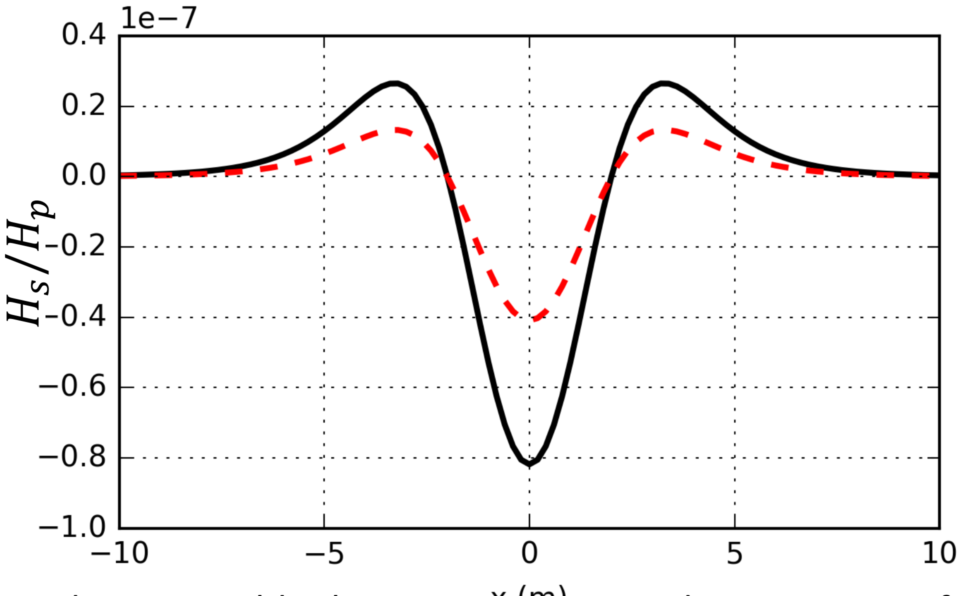


Credit: Doug Oldenburg, Seogi Kang and Linsey Heagy from UBC-GIF

Conductor in a resistive earth: Frequency

Profile over the loop

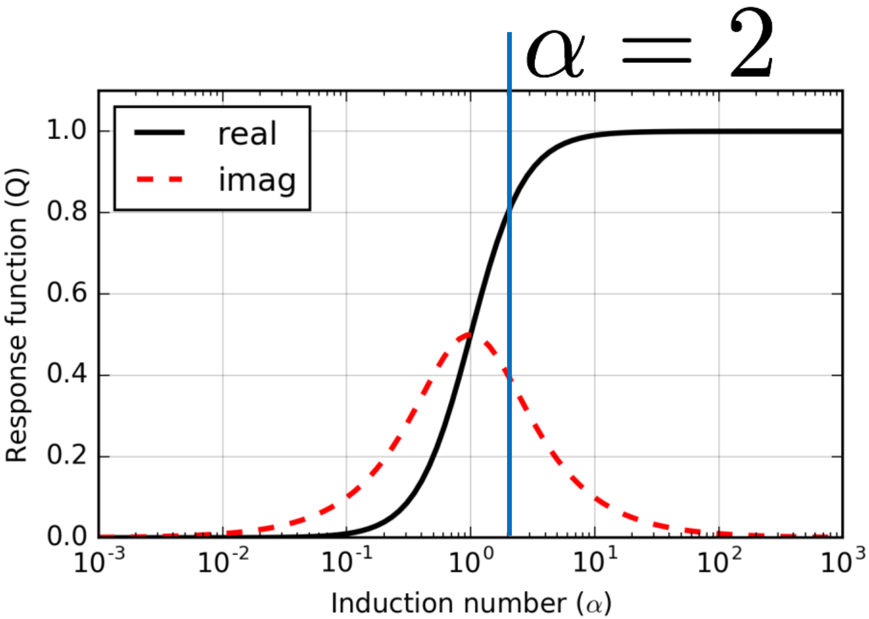




- Induction number

$$\alpha = \frac{\omega L}{R}$$

- When $\alpha = 1$
  - Real > Imag



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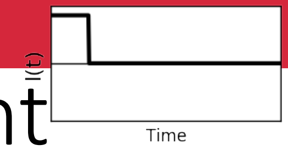
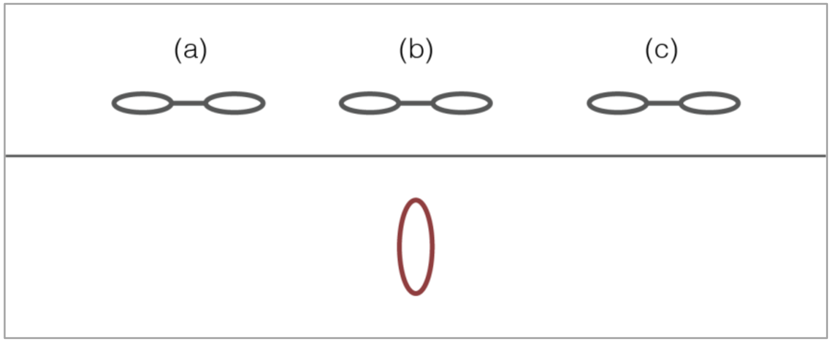
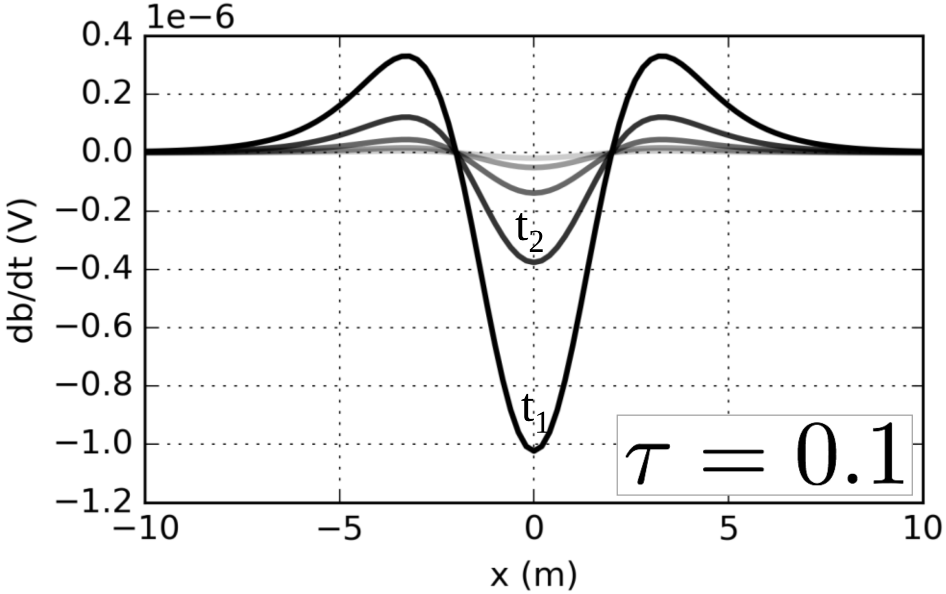


Conductor in a resistive earth: Transient

Profile over the loop



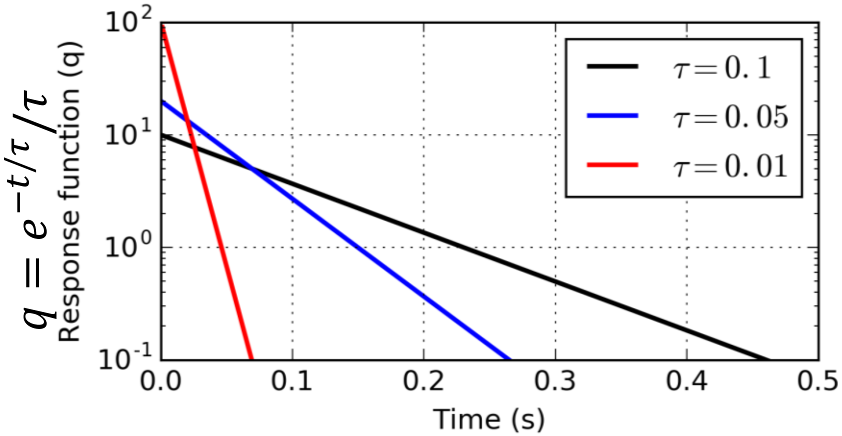


- Time constant

$$\tau = L/R$$

- Response depends upon time constant

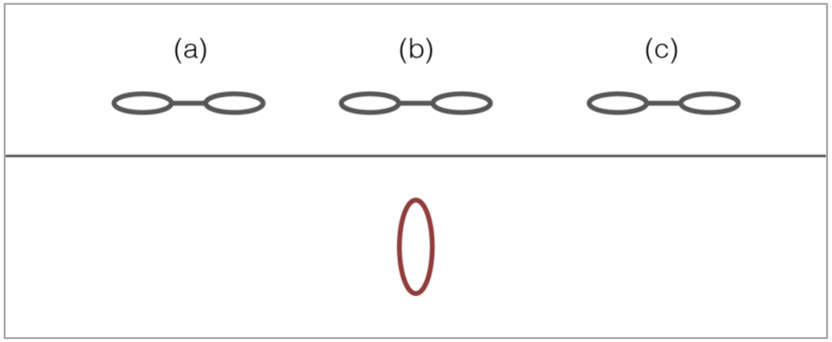
$$\varepsilon_3^s = \frac{M_{12}M_{23}}{L}\frac{I_1}{\tau}e^{-\frac{t}{\tau}}$$

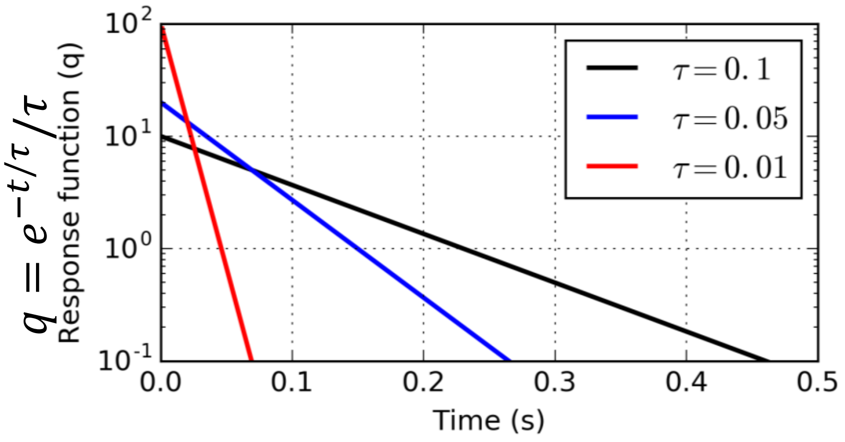


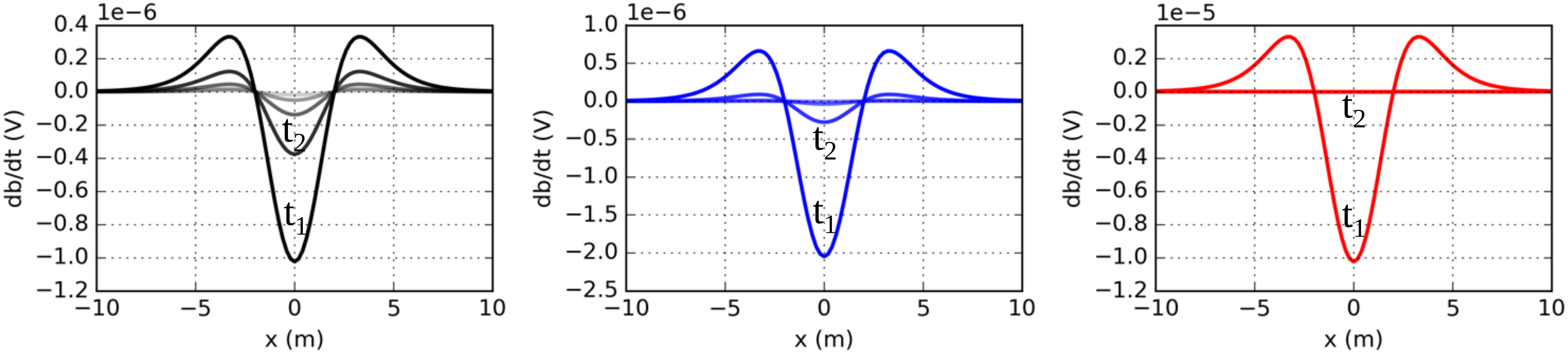
Credit: Doug Oldenburg, Seogi Kang and Linsey Heagy from UBC-GIF

Conductor in a resistive earth: Transient

Profile over the loop







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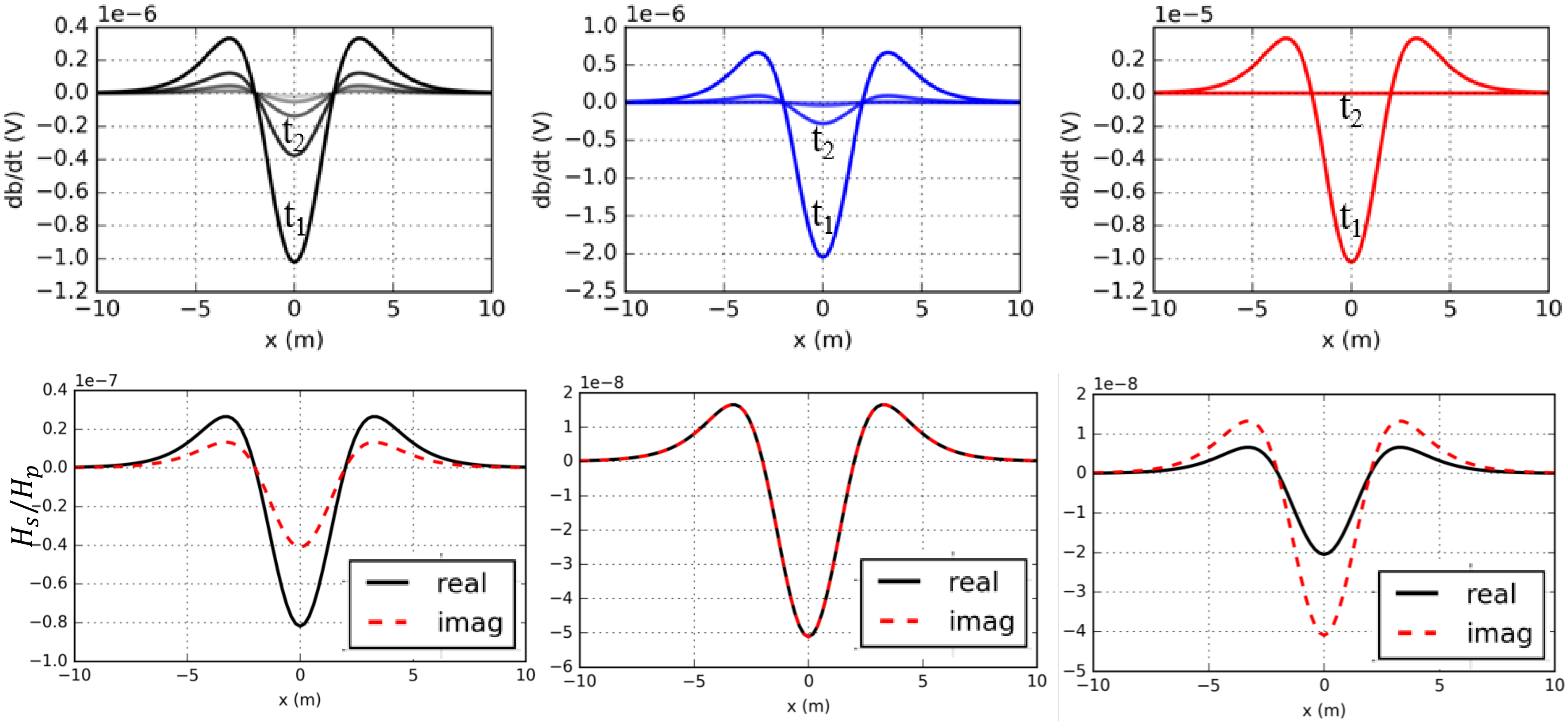
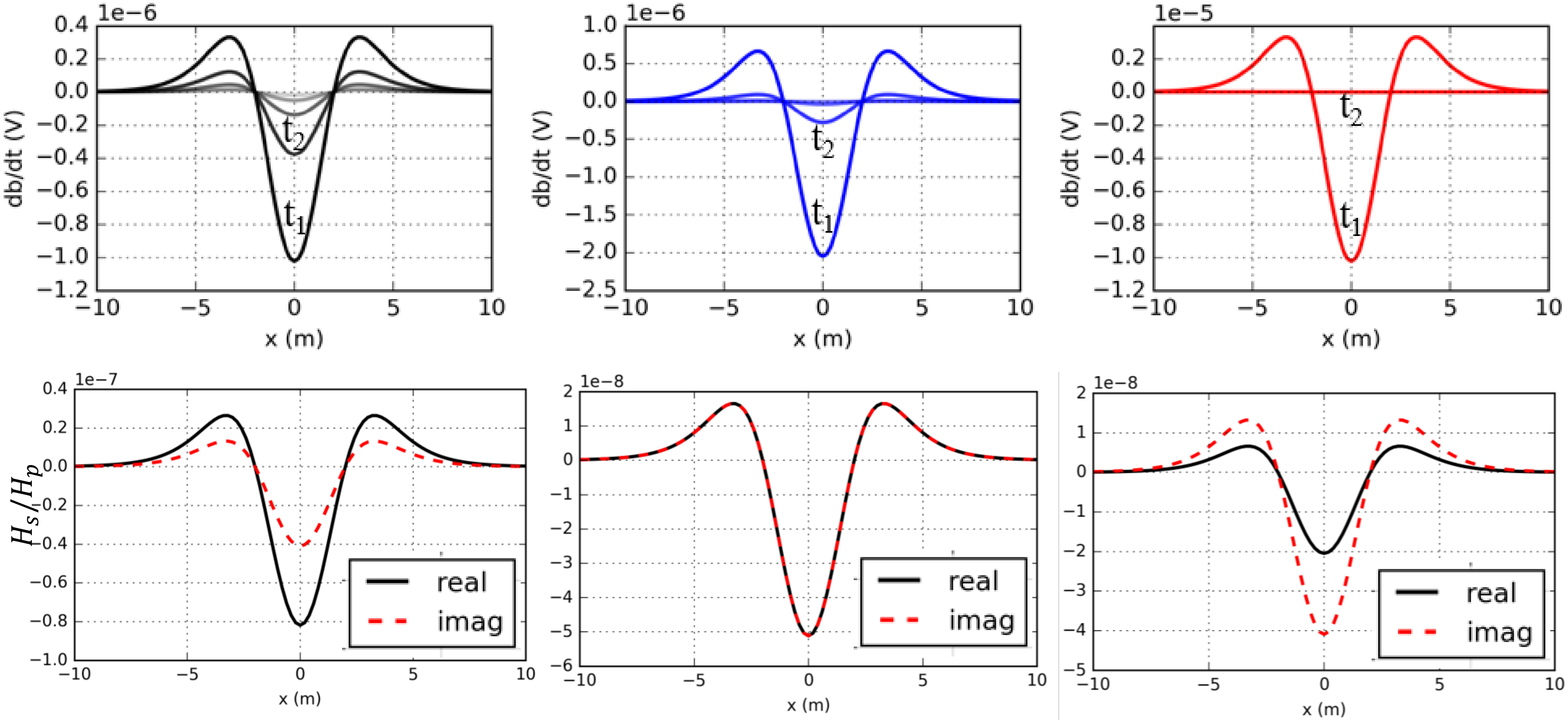
A summary



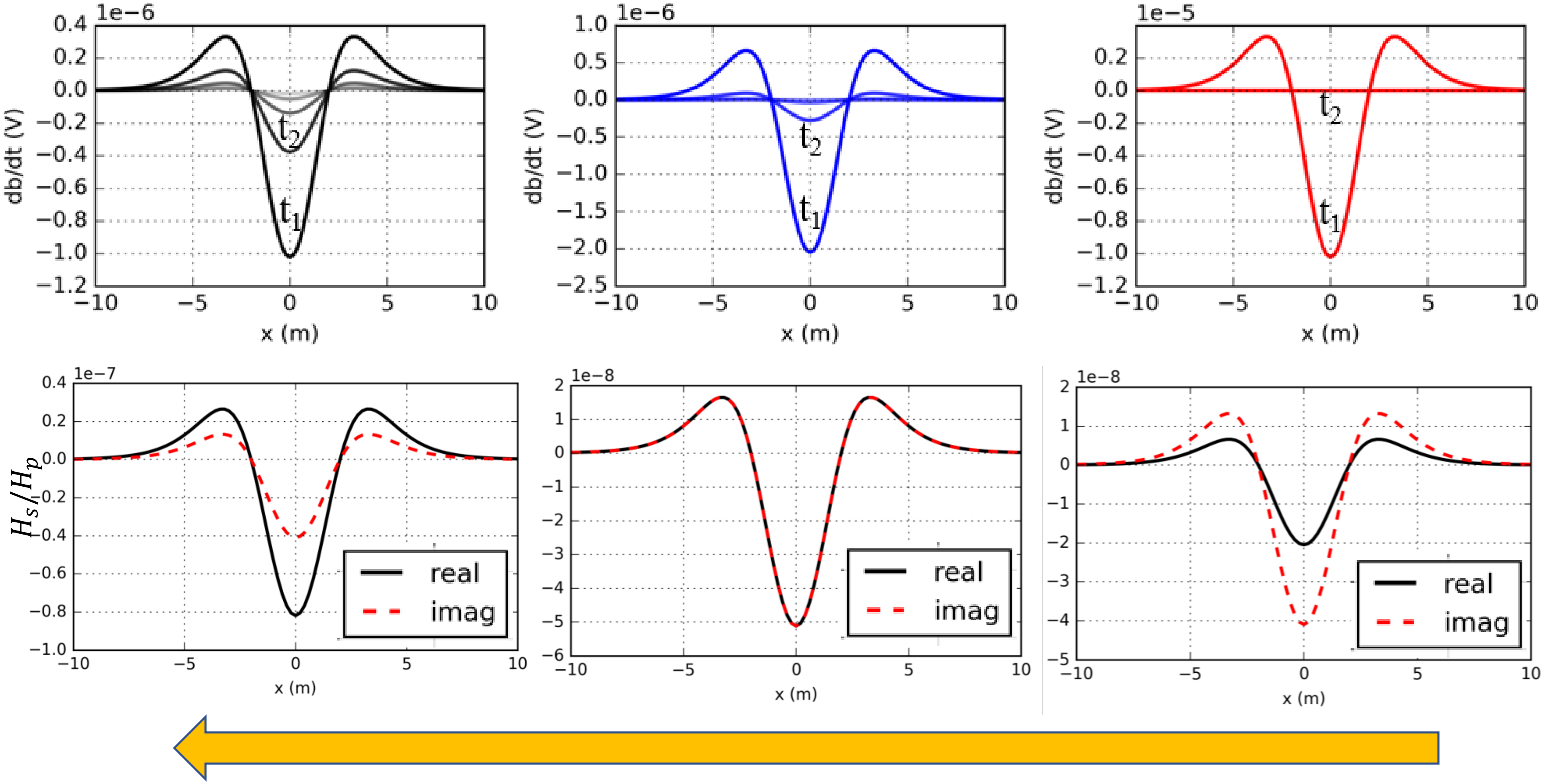
Much of what we see (e.g., the shape of the curve, the zero crossings) is from survey geometry (i.e., coupling, or the relative position of target w.r.t Tx & Rx).

A summary



But what we are really interested in is conductivity. In time domain, the decay rate tells you how conductive a target is. In frequency domain, the relative amplitudes of the real and imaginary curves tell about conductivity.

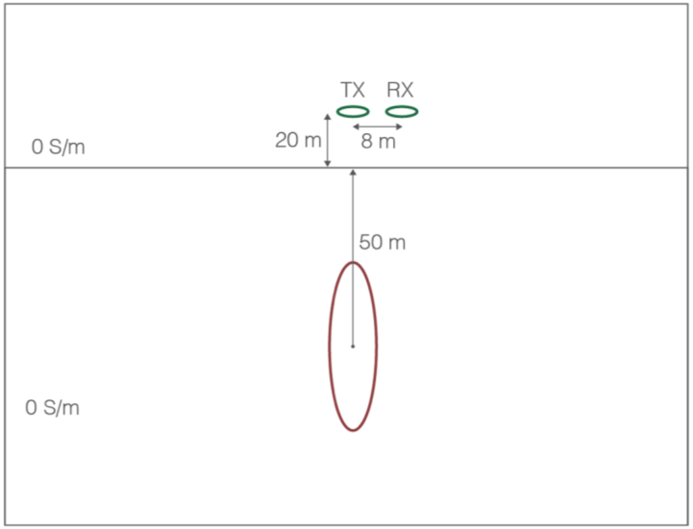
A summary

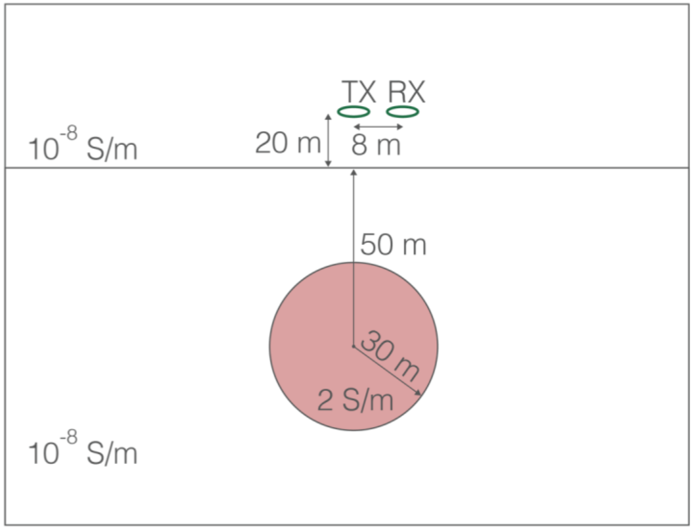


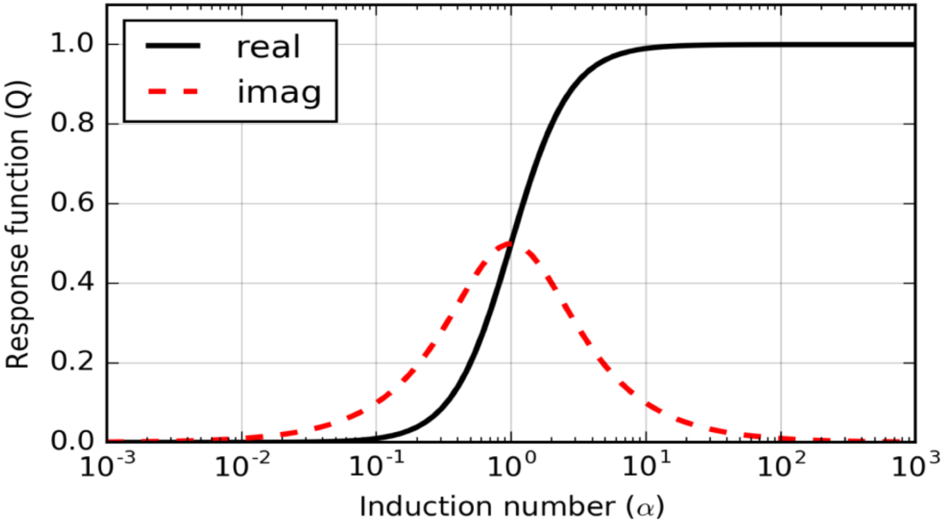
Strong conductor

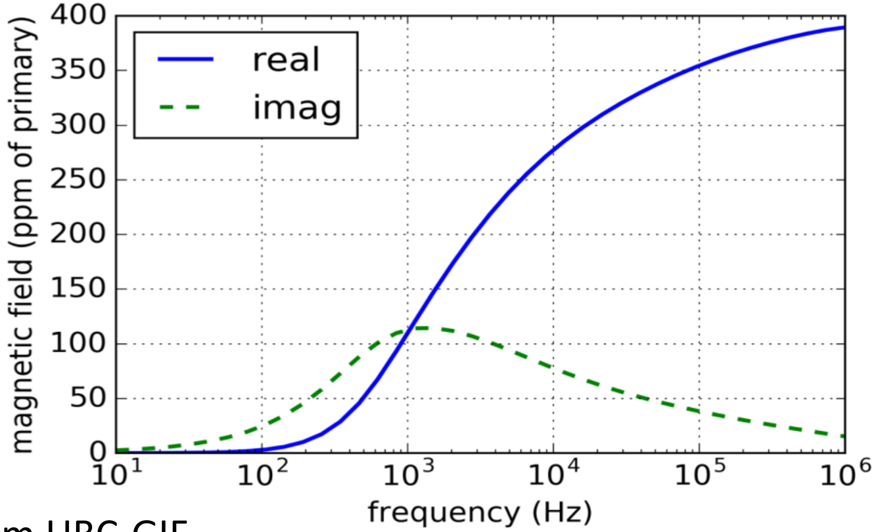
Weak conductor

How representative is a circuit model?









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Outline

- Circuit model
  - Understanding EM response
- Plane waves in a homogeneous media
  - Quasi-static approximation
  - Skin depth
  - Diffusion distance

Revisit Maxwell equations

$$\nabla \cdot \boldsymbol{d} = \rho_f$$ Gauss's law for electric fields

$$\nabla \cdot \boldsymbol{b} = 0$$ Gauss's law for magnetic fields

$$\nabla \times \boldsymbol{e} = -\frac{\partial \boldsymbol{b}}{\partial t}$$ Faraday's law

$$\nabla \times \mathbf{h} = \boldsymbol{j}_f + \frac{\partial \mathbf{d}}{\partial t}$$ Ampere-Maxwell equation

Constitutive relationships

$$\boldsymbol{j}_f = \sigma \boldsymbol{e} \qquad \boldsymbol{b} = \mu \boldsymbol{h} \qquad \boldsymbol{d} = \varepsilon \boldsymbol{e}$$

σ: electrical conductivity μ: magnetic permeability ε: dielectric permittivity

Revisit Maxwell equations

$$\nabla \cdot \boldsymbol{d} = \rho_f$$
Gauss's law for electric fields

$$\nabla \cdot \boldsymbol{b} = 0$$
Gauss's law for magnetic fields

$$\boxed{\begin{aligned} \nabla \times \boldsymbol{e} &= -\frac{\partial \boldsymbol{b}}{\partial t} \\ \nabla \times \mathbf{h} &= \boldsymbol{j}_f + \frac{\partial \mathbf{d}}{\partial t} \end{aligned}}$$

Faraday's law

Ampere-Maxwell equation

First order equations

Constitutive relationships

$$\boldsymbol{j}_f = \sigma \boldsymbol{e} \qquad \boldsymbol{b} = \mu \boldsymbol{h} \qquad \boldsymbol{d} = \varepsilon \boldsymbol{e}$$

σ: electrical conductivity μ: magnetic permeability ε: dielectric permittivity

Revisit Maxwell equations

- First order equations

$$\nabla \times \mathbf{e} = -\frac{\partial \mathbf{b}}{\partial t}$$

$$\nabla \times \mathbf{h} = \mathbf{j} + \frac{\partial \mathbf{d}}{\partial t}$$

$$\mathbf{j} = \sigma \mathbf{e}$$

$$\mathbf{b} = \mu \mathbf{h}$$

$$\mathbf{d} = \varepsilon \mathbf{e}$$

- Second order equations

$$\nabla^2 \mathbf{h} - \underbrace{\mu\sigma \frac{\partial \mathbf{h}}{\partial t}}_{\text{diffusion}} - \underbrace{\mu\epsilon \frac{\partial^2 \mathbf{h}}{\partial t^2}}_{\text{wave propagation}} = 0$$

* Same equation holds for E

From first order equations to second order equations (Optional)

Start from $\nabla \times \mathbf{h} = \boldsymbol{j}_f + \frac{\partial \mathbf{d}}{\partial t}$

Perform curl operation on both sides:

$$\nabla \times (\nabla \times \mathbf{h}) = \nabla \times (\boldsymbol{j}_f + \frac{\partial \mathbf{d}}{\partial t})$$

The left hand side: $\nabla \times (\nabla \times \mathbf{h}) = \nabla(\nabla \cdot \boldsymbol{h}) - \nabla^2 \boldsymbol{h} = -\nabla^2 \boldsymbol{h}$

The RHS term: $\nabla \times \left(\boldsymbol{j}_f + \frac{\partial \mathbf{d}}{\partial t}\right) = \nabla \times \boldsymbol{j}_f + \nabla \times \left(\frac{\partial \boldsymbol{d}}{\partial t}\right) = \nabla \times \boldsymbol{j}_f + \frac{\partial (\nabla \times \boldsymbol{d})}{\partial t}$

Recall $\boldsymbol{j}_f = \sigma \boldsymbol{e}$

Thus,

$$\nabla \times \boldsymbol{j}_f = \sigma \nabla \times \boldsymbol{e} = -\sigma \frac{\partial \boldsymbol{b}}{\partial t} = -\mu\sigma \frac{\partial \boldsymbol{h}}{\partial t}$$

$$\nabla \times \boldsymbol{d} = \varepsilon \nabla \times \boldsymbol{e} = -\varepsilon \frac{\partial \boldsymbol{b}}{\partial t} = -\mu\varepsilon \frac{\partial \boldsymbol{h}}{\partial t}$$

Thus, $\nabla^2 \boldsymbol{h} - \mu\sigma \frac{\partial \boldsymbol{h}}{\partial t} - \mu\varepsilon \frac{\partial^2 \boldsymbol{h}}{\partial t^2} = 0$

From time domain to frequency domain

$$\nabla^2 \mathbf{h} - \underbrace{\mu\sigma \frac{\partial \mathbf{h}}{\partial t}}_{\text{diffusion}} - \underbrace{\mu\epsilon \frac{\partial^2 \mathbf{h}}{\partial t^2}}_{\text{wave propagation}} = 0$$

Apply Fourier transform

$$\nabla^2 \boldsymbol{H} - i\omega\mu\sigma\boldsymbol{H} + \omega^2\mu\epsilon\boldsymbol{H} = 0$$

$$\nabla^2 \boldsymbol{H} + k^2 \boldsymbol{H} = 0$$

where $k^2 = \omega^2\mu\epsilon - i\omega\mu\sigma$

Quasi-static approximation

$$\nabla^2 \boldsymbol{H} + k^2 \boldsymbol{H} = 0$$

$$k^2 = \omega^2 \mu \epsilon - i\omega\mu\sigma$$

$$k^2 = \omega\mu(\omega\epsilon - i\sigma)$$

Quasi-static approximation

$$\nabla^2 \boldsymbol{H} + k^2 \boldsymbol{H} = 0$$

$$k^2 = \omega^2 \mu\epsilon - i\omega\mu\sigma$$

$$k^2 = \omega\mu(\omega\epsilon - i\sigma)$$

Let us compare $\omega\epsilon$ and σ

By computing $\frac{\omega\epsilon}{\sigma}$

Quasi-static approximation

$$\nabla^2 \boldsymbol{H} + k^2 \boldsymbol{H} = 0$$

$$k^2 = \omega^2 \mu \epsilon - i\omega\mu\sigma$$

$$k^2 = \omega\mu(\omega\epsilon - i\sigma)$$

Let us compare $\omega\epsilon$ and σ

By computing $\frac{\omega\epsilon}{\sigma}$

In-class exercise

Compute $\frac{\omega\epsilon}{\sigma}$

$$\sigma = 10^{-4} S/m$$

$$f = 10^4 Hz$$

$$\epsilon = 8.85 \times 10^{-12} F/m$$

Quasi-static approximation

$$\nabla^2 \boldsymbol{H} + k^2 \boldsymbol{H} = 0$$

$$k^2 = \omega^2 \mu \epsilon - i\omega\mu\sigma$$

$$k^2 = \omega\mu(\omega\epsilon - i\sigma)$$

Let us compare $\omega\epsilon$ and σ

By computing $\frac{\omega\epsilon}{\sigma}$

In-class exercise

Compute $\frac{\omega\epsilon}{\sigma}$

$$\sigma = 10^{-4} S/m$$

$$f = 10^4 Hz$$

$$\epsilon = 8.85 \times 10^{-12} F/m$$

Even with the above parameter values, $\frac{\omega\epsilon}{\sigma} \ll 1$

Therefore, $k^2 = -i\omega\mu\sigma$

Plane waves in a homogeneous media: frequency domain

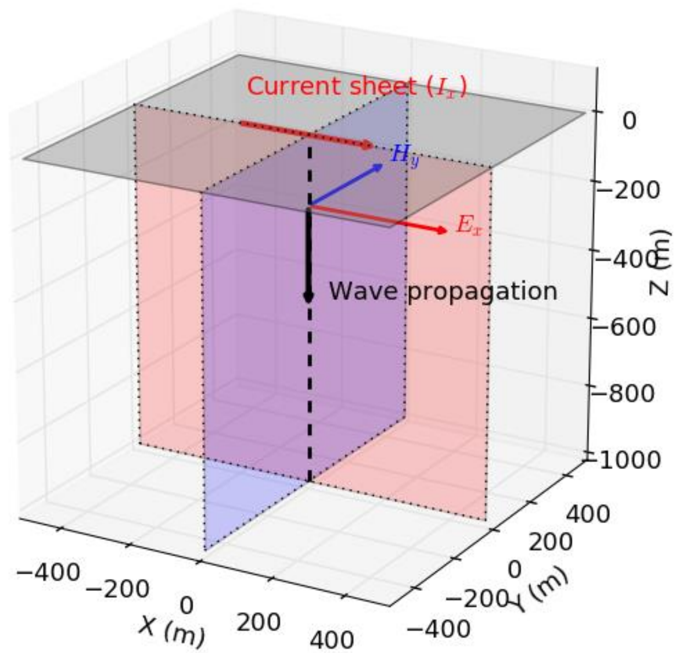
$$\nabla^2 \boldsymbol{H} + k^2 \boldsymbol{H} = 0$$

$$k^2 = -i\omega\mu\sigma$$

$$k = \sqrt{-i\omega\mu\sigma} = (1-i)\sqrt{\frac{\omega\mu\sigma}{2}}$$

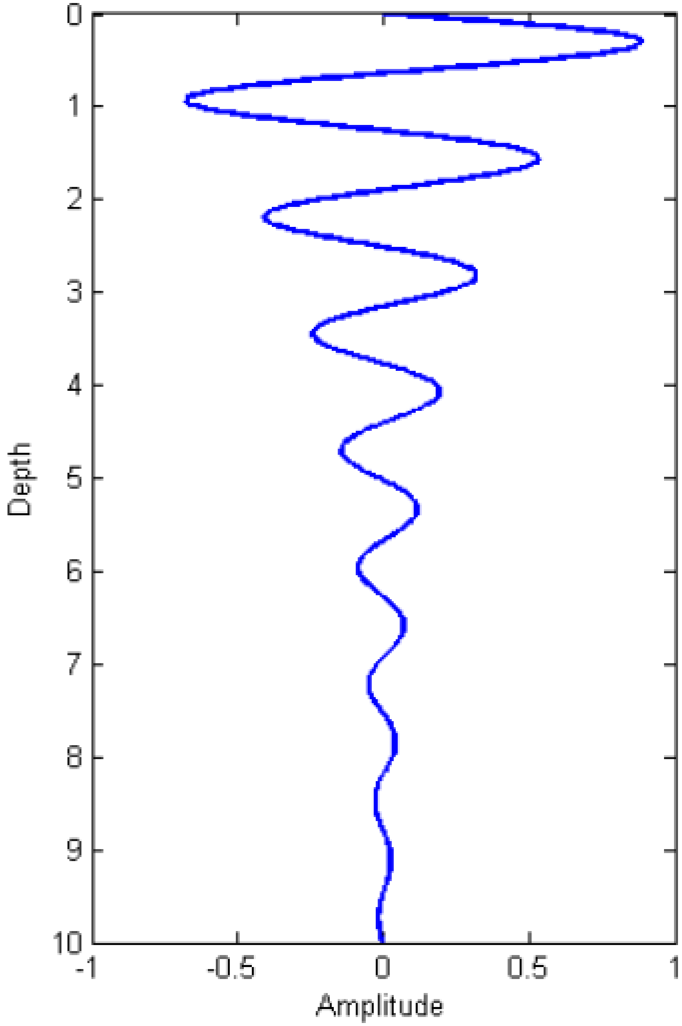
$$\equiv \alpha - i\beta$$

Plane wave solution

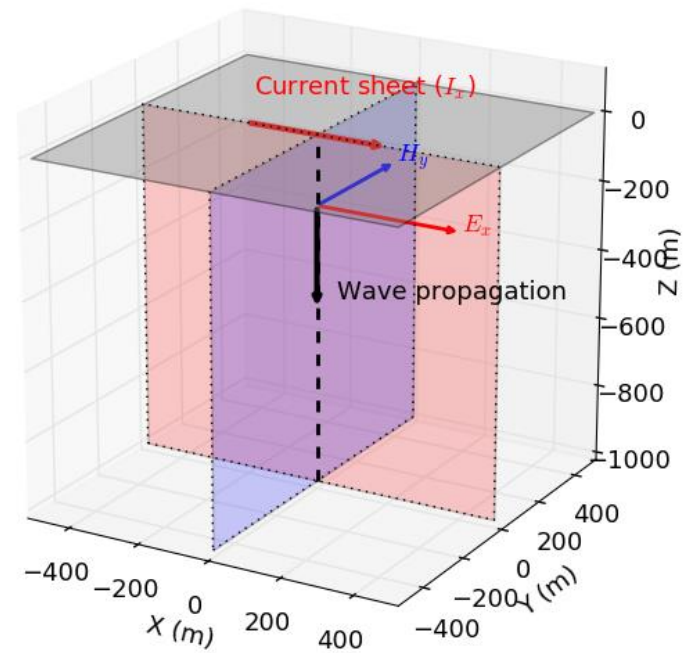


$$\mathbf{H} = \mathbf{H_0} \underbrace{e^{-\alpha z}}_{\text{attenuation}} \underbrace{e^{-i(\beta z - \omega t)}}_{\text{phase}}$$

Plane waves in a homogeneous media: frequency domain



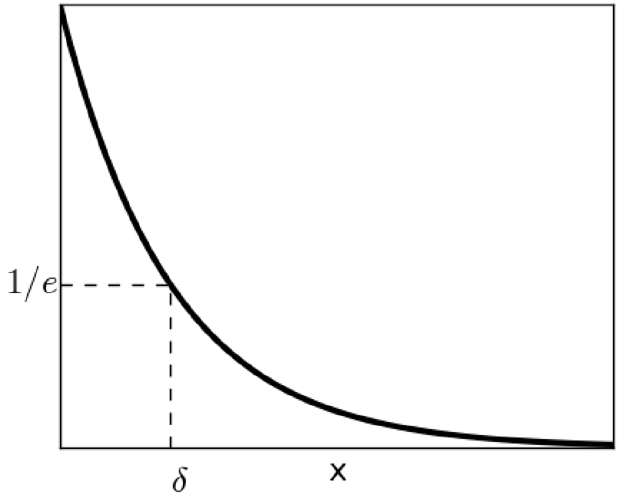
Plane wave solution



$$\mathbf{H} = \mathbf{H_0} \underbrace{e^{-\alpha z}}_{\text{attenuation}} e^{-i\underbrace{(\beta z - \omega t)}_{\text{phase}}}$$

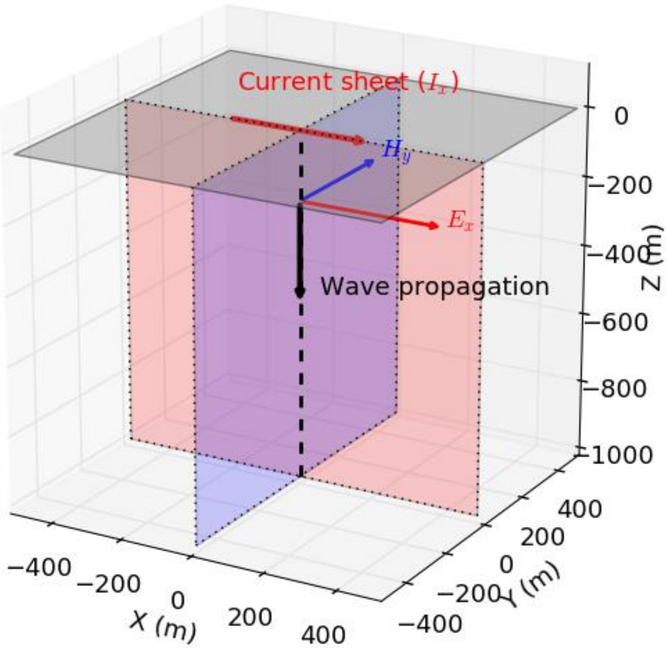
Skin depth

Plane wave solution



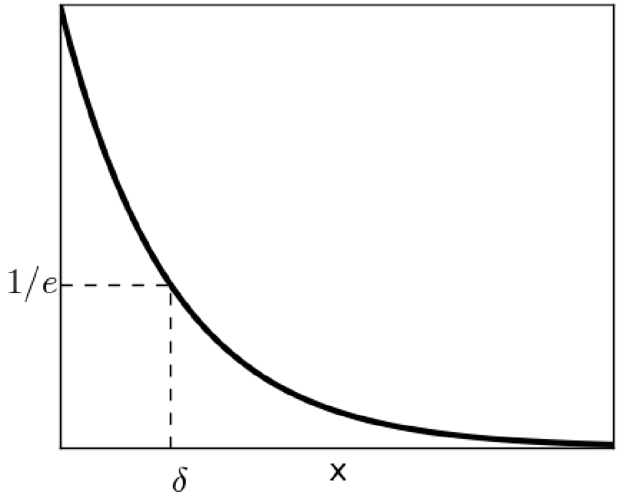
δ : skin depth

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = 503\sqrt{\frac{1}{\sigma f}}$$



$$\mathbf{H} = \mathbf{H_0}\underbrace{e^{-\alpha z}}_{\text{attenuation}} e^{-i\underbrace{(\beta z - \omega t)}_{\text{phase}}}$$

Skin depth



δ : skin depth

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = 503\sqrt{\frac{1}{\sigma f}}$$

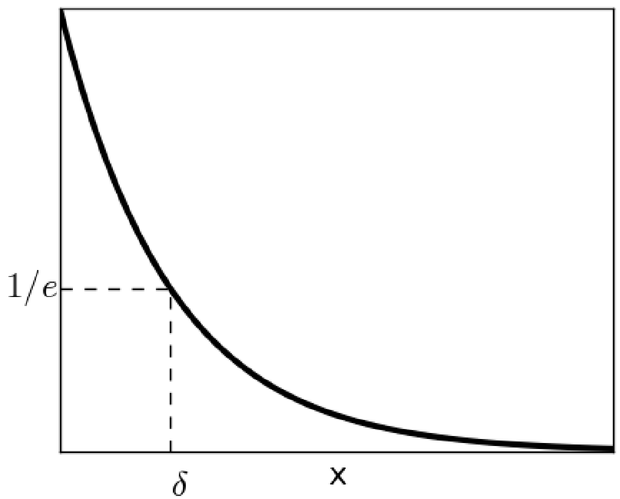
$$\mu = 4\pi \times 10^{-7} H/m$$

In-class exercise:

Calculate the skin depths

Type	σ (S/m)	δ(1 Hz)	δ(1 kHz)	δ(1 MHz)
Air	0			
Sea water	3.3			
Igneous	10^{-4}			
Sedimen ts	10^{-2}			

Skin depth



δ : skin depth

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = 503\sqrt{\frac{1}{\sigma f}}$$

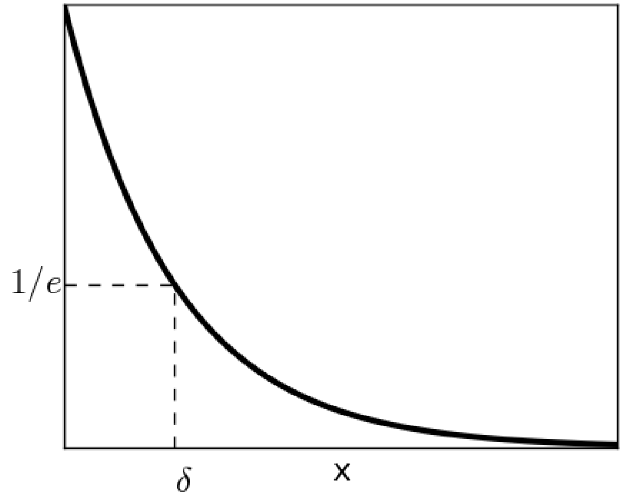
$$\mu = 4\pi \times 10^{-7} H/m$$

In-class exercise:

Calculate the skin depths

Type	σ (S/m)	δ(1 Hz)	δ(1 kHz)	δ(1 MHz)
Air	0	∞	∞	∞
Sea water	3.3	277	8.76	0.277
Igneous	10^{-4}	50300	1590	50.3
Sedimen ts	10^{-2}	5030	159	5.03

Skin depth



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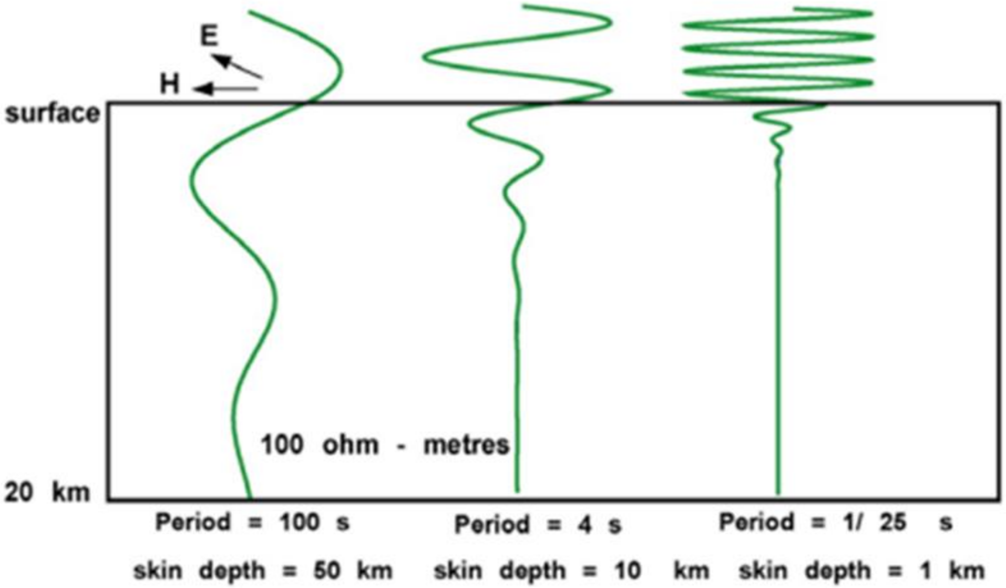
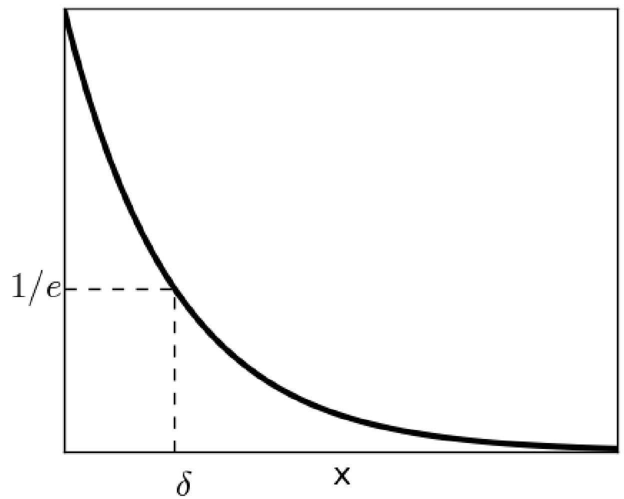


Skin depth



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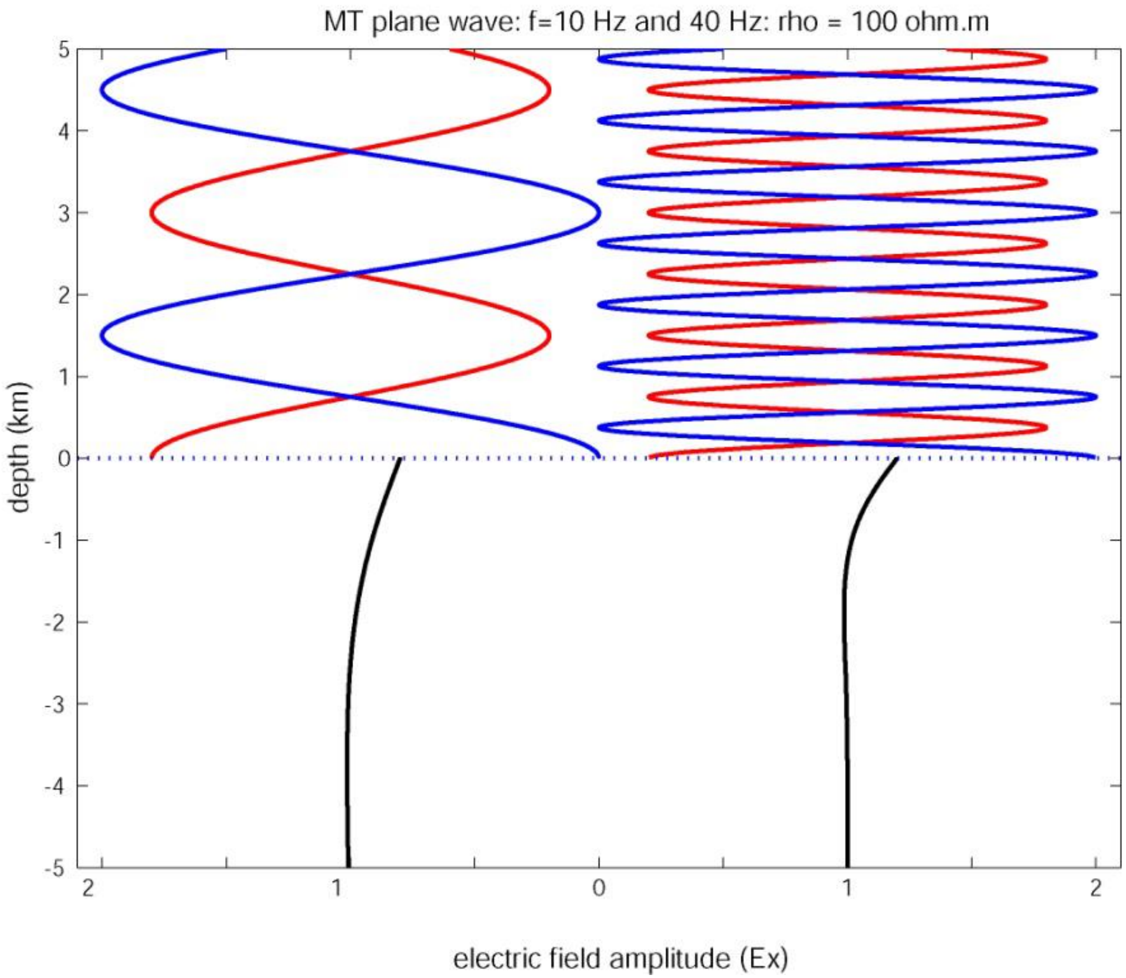


Plane waves in a homogeneous media: time domain

$$\nabla^2 \mathbf{h} - \mu\epsilon \frac{\partial^2 \mathbf{h}}{\partial t^2} - \mu\sigma \frac{\partial \mathbf{h}}{\partial t} = 0$$

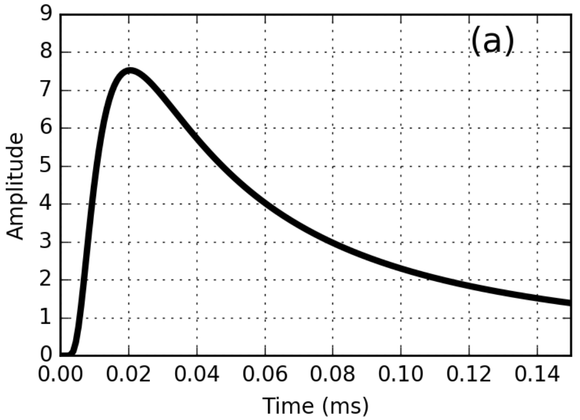
$$\mathbf{h}(t=0) = \mathbf{h}_0 \delta(t)$$

Solution for quasi-static

$$\mathbf{h}(t) = -\frac{(\mu\sigma)^{1/2} z}{2\pi^{1/2} t^{3/2}} e^{-\mu\sigma z^2/(4t)}$$

$$z: \text{ depth (m)}$$

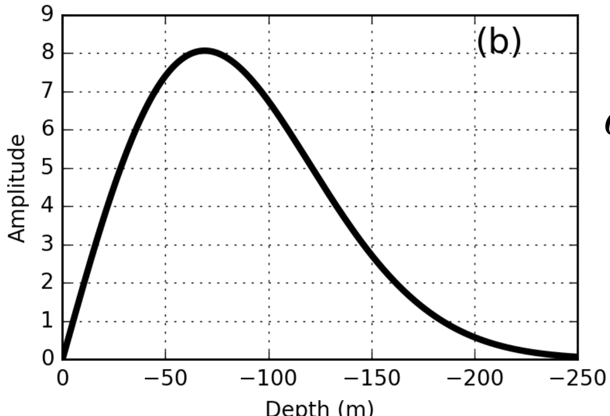
Peak time:



$$t_{max} = \frac{\mu\sigma z^2}{6}$$

Electric field as a function of time 100 m from a 1D impulse in the field in a 0.01 S/m whole space

Diffusion distance



$$d = \sqrt{\frac{2t}{\mu\sigma}}$$

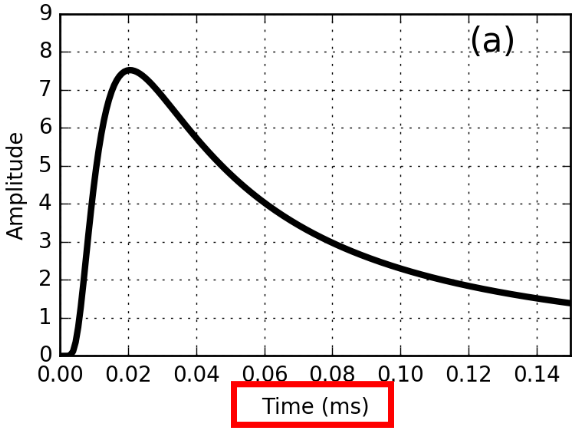
$$\approx 1260\sqrt{\frac{t}{\sigma}}$$

Electric field at t = 0.03 ms as a function of distance

Plane waves in a homogeneous media: time domain

Peak time:

At any given depth z, when does the maximum field (e.g., magnetic field) occur?

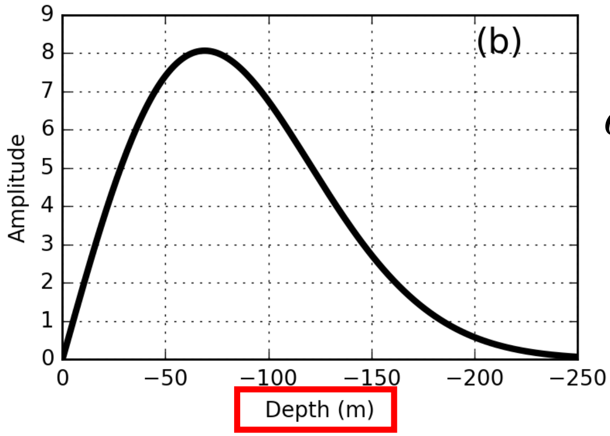


$$t_{max} = \frac{\mu \sigma z^2}{6}$$

Diffusion distance

At any given time t , where does the maximum field (e.g., magnetic field) occur?

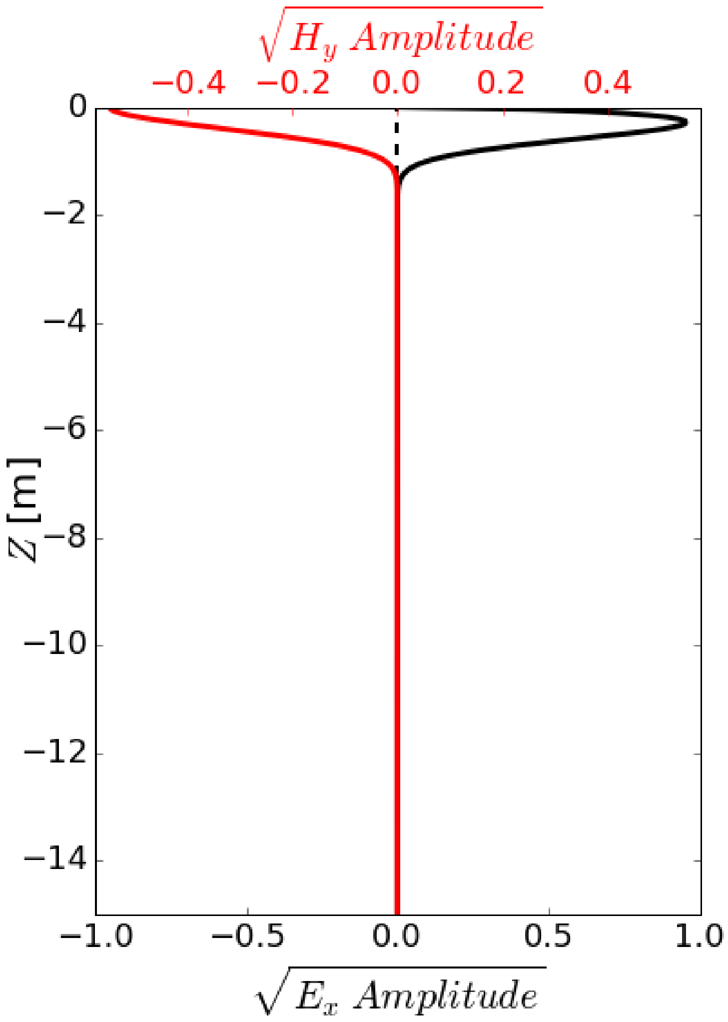
Also called peak distance



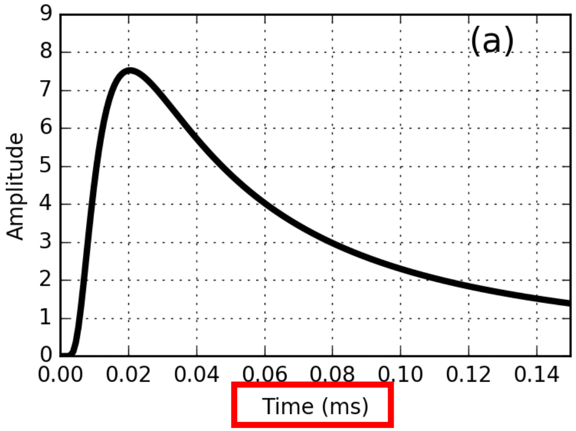
$$d = \sqrt{\frac{2t}{\mu \sigma}}$$

$$\approx 1260 \sqrt{\frac{t}{\sigma}}$$

Plane waves in a homogeneous media: time domain

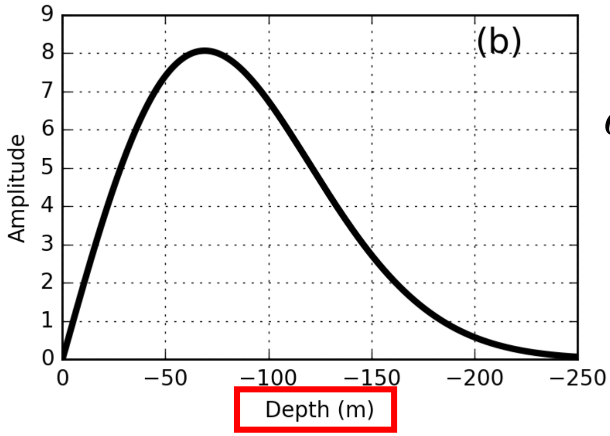


Peak time:



$$t_{max} = \frac{\mu \sigma z^2}{6}$$

Diffusion distance



$$d = \sqrt{\frac{2t}{\mu\sigma}}$$

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