Homework 1: Dmytrenko Dmytrenko

Theorem 1 *Denote by* $\delta = \det A$, $\tau = \operatorname{Tr} A$ *and* $\dot{x} = Ax$ *then:*

- *saddle if* $\delta < 0$
- *node (stable for* $\tau < 0$ *and unstable for* $\tau > 0$*) if* $\begin{cases} \delta > 0 \\ \tau^2 - 4\delta \geq 0 \end{cases}$
- *focus (stable for* $\tau < 0$ *and unstable for* $\tau > 0$*) if* $\begin{cases} \delta > 0 \\ \tau^2 - 4\delta < 0 \\ \tau \neq 0 \end{cases}$
- *center if* $\begin{cases} \delta > 0 \\ \tau = 0 \end{cases}$

Exercise 1

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

First of all, we should solve the corresponding characteristic equation

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = (\lambda - 4)(\lambda - 2)$$

So, $\lambda_1 = 2$ and $\lambda_1 = 4$. Corresponding eigenvectors are $v_1 = (-1, 1)^T$ and $v_2 = (1, 1)^T$. Then

$$B = PAP^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

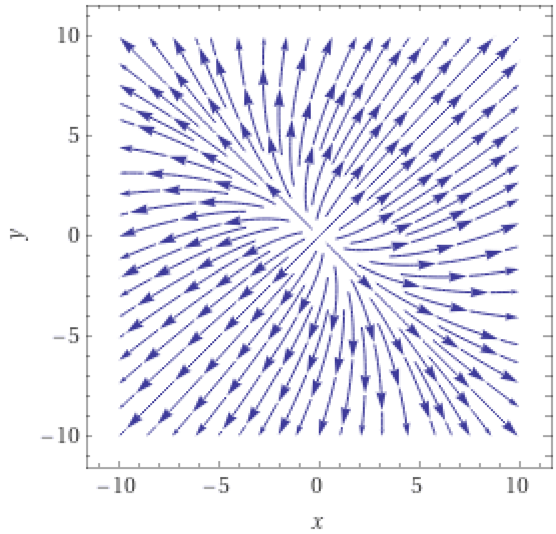
$$x(t) = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{pmatrix} x_0$$

After that, observe

$$\delta = 8, \quad \tau = 6$$

$$\delta > 0, \quad \tau > 0, \quad \tau^2 - 4\delta > 0$$

Thus, using theorem, we have *unstable node* and phase portrait consists of trajectories $x_1 = x_2$, $x_1 = -x_2$ and set of parabols.



Finally,

$$E^u = span\left\{\begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}, \quad E^s = \emptyset$$

Exercise 2

$$A = \begin{pmatrix}2 & 4\\-1 & 2\end{pmatrix}$$

First of all, we should solve the corresponding characteristic equation

$$\det(A-\lambda I) = \begin{vmatrix}2-\lambda & 4\\-1 & 2-\lambda\end{vmatrix} = (2-\lambda)^2+4 = \lambda^2-4\lambda+8$$

So, $\lambda_1 = 2-2i$ and $\lambda_1 = 2+2i$. Then

$$B = PAP^{-1} = \begin{pmatrix}2 & -2\\2 & 2\end{pmatrix}$$

$$x(t) = e^{2t}\begin{pmatrix}\cos 2t & -\sin 2t\\ \sin 2t & \cos 2t\end{pmatrix}x_0$$

After that, observe

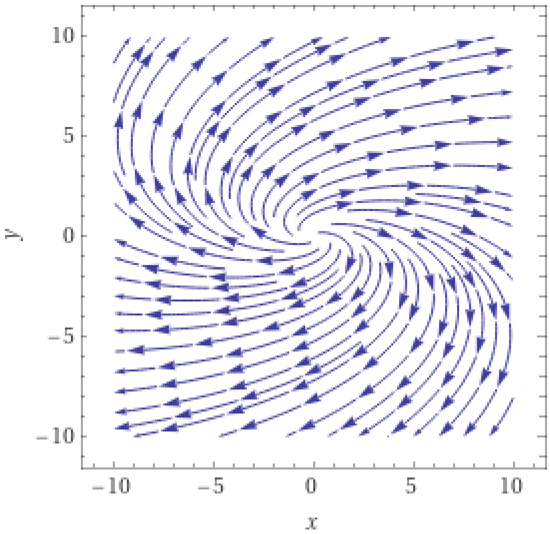
$$\delta = 8, \quad \tau = 4$$

$$\delta > 0, \quad \tau > 0, \quad \tau^2 - 4\delta > 0$$

Thus, using theorem, we have *unstable node*. Computing velocity vector at arbitrary point (1,0)

$$\begin{pmatrix}\dot{x_1}\\\dot{x_2}\end{pmatrix}\Bigg|_{(1,0)} = \begin{pmatrix}2\\-1\end{pmatrix}$$

our phase portrait looks like



Finally, $E^u = x_1x_2$ plane and $E^s = \emptyset$

Exercise 3

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

First of all, we should solve the corresponding characteristic equation

$$\det(A - \lambda I) = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 1 & 0 & -1-\lambda \end{pmatrix} = -\lambda^3 + 2\lambda^2 + \lambda - 2$$

So, $\lambda_1 = 2$, $\lambda_2 = -1$ and $\lambda_3 = 1$. Corresponding eigenvectors are $v_1 = (0,1,0)^T$, $v_2 = (0,0,1)^T$, $v_3 = (2,-2,1)^T$ and . Then

$$P = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}$$

$$B = PAP^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

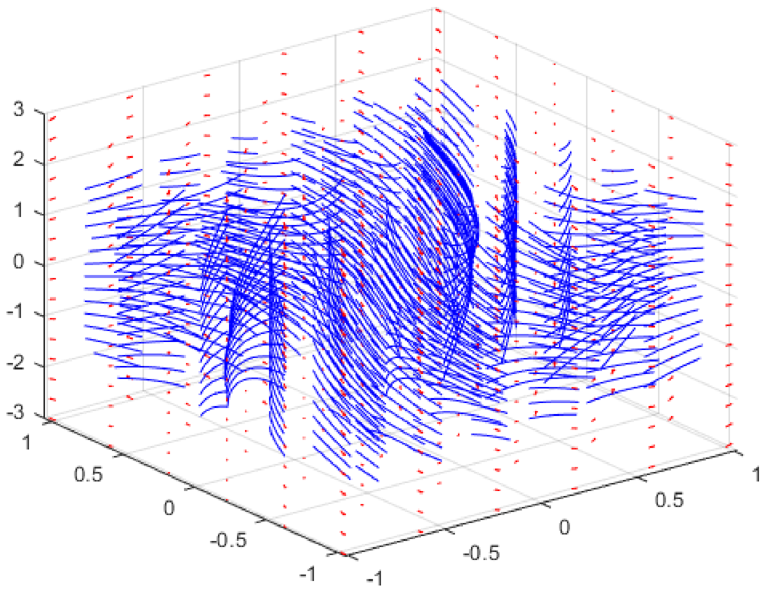
$$x(t) = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix} x_0$$

After that, observe

$$\delta = -2, \quad \tau = 0$$

$$\delta < 0, \quad \tau = 0, \quad \tau^2 - 4\delta > 0$$

Thus, using theorem, we have *saddle*.



Finally,

$$E^u = span\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}, \quad E^s = \emptyset$$

Exercise 4

$$A = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

First of all, we should solve the corresponding characteristic equation

$$\det(A-\lambda I) = \begin{pmatrix} -\lambda & -2 & 0 \\ 1 & 2-\lambda & 0 \\ 1 & 0 & -2-\lambda \end{pmatrix} = -\lambda^3 + 2\lambda^2 + \lambda - 2$$

So, $\lambda_1 = -2$, $\lambda_2 = 1 - i$ and $\lambda_3 = 1 + i$. Corresponding eigenvectors are $v_1 = (0,0,1)^T$, $v_{2,3} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then

$$P = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$B = PAP^{-1} = P \cdot \begin{pmatrix} 0 & e^t \cos t & -e^t \sin t \\ 0 & e^t \sin t & e^t \cos t \\ e^t & 0 & 0 \end{pmatrix} \cdot P^{-1}$$

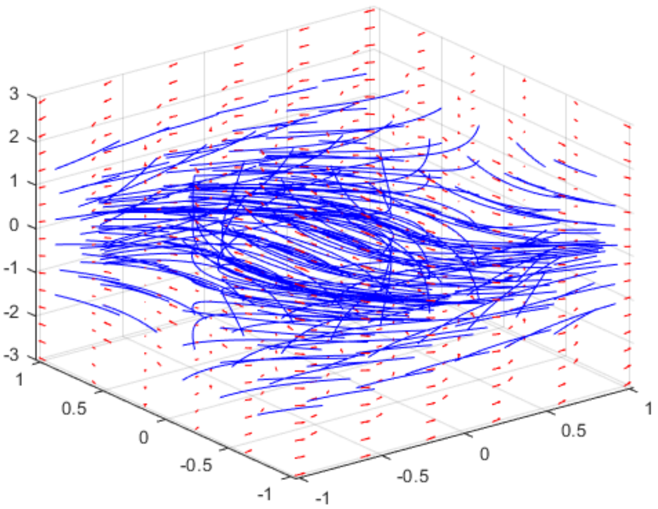
$$x(t) = B \cdot x_0$$

After that, observe

$$\delta = -4, \quad \tau = 0$$

$$\delta < 0, \quad \tau = 0, \quad \tau^2 - 4\delta > 0$$

Thus, using theorem, we have *saddle*.



Finally, $Re\lambda_{1,2,3} < 0$.

$$E^u = span\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}, \quad E^s = \emptyset$$

Exercise 5

$$A = \begin{pmatrix}1 & 0 & 0\\-1 & 2 & 0\\1 & 1 & 2\end{pmatrix}$$

First of all, we should solve the corresponding characteristic equation

$$\det(A - \lambda I) = \begin{pmatrix}1-\lambda & 0 & 0\\-1 & 2-\lambda & 0\\1 & 1 & 2-\lambda\end{pmatrix} = -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$

So, $\lambda_1 = 1$ and $\lambda_{2,3} = 2$. Corresponding eigenvectors are $v_1 = \begin{pmatrix}-1\\-1\\2\end{pmatrix}$, $v_{2,3} = \begin{pmatrix}0\\0\\1\end{pmatrix}$.

Because of $\lambda_{2,3} = 2$ has multiplicity 2, then we need to find extra generalized eigenvector of A when $\lambda = 2$ and $k = 2$

$$(A - \lambda I)^k \cdot v = \begin{pmatrix}-1 & 0 & 0\\-1 & 0 & 0\\1 & 1 & 0\end{pmatrix}^2 \cdot v = \begin{pmatrix}-1 & 0 & 0\\-1 & 0 & 0\\1 & 1 & 0\end{pmatrix} \cdot \begin{pmatrix}-1 & 0 & 0\\-1 & 0 & 0\\1 & 1 & 0\end{pmatrix} \cdot \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} =$$

$$\begin{pmatrix}1 & 0 & 0\\1 & 0 & 0\\0 & 0 & 0\end{pmatrix} \cdot \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} \implies \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}$$

So, now we obtain:

$$P = \begin{pmatrix}-1 & 0 & 0\\-1 & 0 & 1\\2 & 1 & 1\end{pmatrix}$$

$$P^{-1} = \begin{pmatrix}-1 & 0 & 0\\3 & -1 & 1\\-1 & 1 & 0\end{pmatrix}$$

Then

$$S = \begin{pmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{pmatrix}$$

$$N = A - S = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$$N^2 = N \cdot N = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$N^3 = N^2 \cdot N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, N is nilpotent of order k=3.
Thus,

$$x(t) = P \cdot diag\left[e^{\lambda_1 t} e^{\lambda_2 t} e^{\lambda_3 t}\right] \cdot P^{-1} \cdot \left[I + Nt + \frac{N^2 t^2}{2}\right] x_0$$

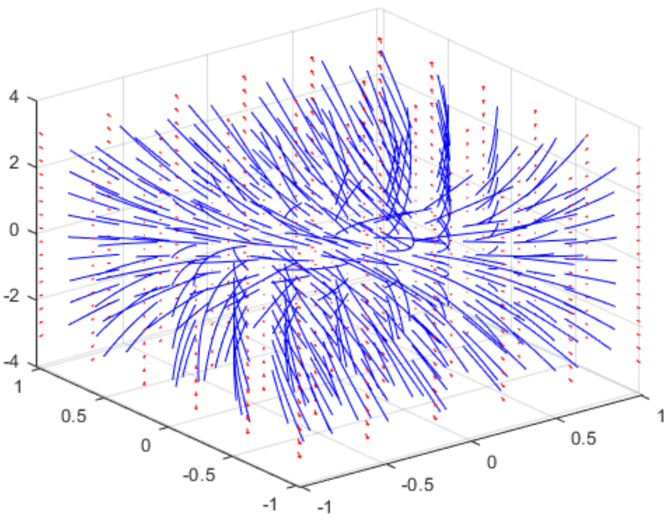
$$x(t) = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^t & 0 & 0 \\ -1 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 0 \\ 3 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ t - \frac{t^2}{2} & t & 1 \end{pmatrix} x_0$$

After that, observe

$$\delta = 4, \quad \tau = 5$$

$$\delta > 0, \quad \tau > 0, \quad \tau^2 - 4\delta > 0$$

Thus, using theorem, we have *unstable node.*



Finally, $\lambda_{1,2,3} > 0$.

$$E^u = \text{our space}\, x_1, x_2, x_3; \quad E^s = \emptyset$$