1 Second law of thermodynamics for viscous fluids

We have introduced the new variable s – entropy. Also we've got Gibs-Duhems relation:

$$T\frac{Ds}{Dt} = \frac{De}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) \implies \frac{Ds}{Dt} = \frac{r}{T} \quad \text{(using Euler equations)}$$

From the first equation from Navier-Stokes equations, obtain:

$$\frac{De}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) = \frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt}$$

$$\frac{D}{Dt}f = (\partial_t + u\nabla)f = \partial_t f + u\nabla f, \quad \text{f is scalar function}$$

Consider $f = \frac{1}{\rho}$

$$\partial\frac{1}{\rho} = -\frac{1}{\rho^2}\partial\rho$$

$$\partial_{x_i} = -\frac{1}{\rho^2}\partial_{x_i}\rho, \quad i = 1, 2, 3$$

So,

$$\frac{D}{Dt}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\frac{D}{Dt}\rho$$

$$\frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = \frac{De}{Dt} + \frac{p}{\rho}\, div\, u$$

From the third equation from Navier-Stokes equations:

$$\frac{De}{Dt} + \frac{p}{\rho}\operatorname{div} u = -\frac{p}{\rho}\operatorname{div} u + \frac{p}{\rho}\, div\, u + \frac{1}{\rho}\left(\kappa \operatorname{div}(\nabla u) + 2\mu \operatorname{Tr}(D^2) + \lambda(\operatorname{div} u)^2 + \rho r\right)$$

$$= -\frac{\operatorname{div} q}{\rho} + \frac{2\mu}{\rho}\operatorname{Tr}(D^2) + \frac{\lambda}{\rho}(\operatorname{div} u)^2 + r$$

Multiply left and right side by ρ and divide by T:

$$\rho\frac{Ds}{Dt} = -\frac{1}{T}\operatorname{div} q + \frac{2\mu}{T}\operatorname{Tr}(D^2) + \frac{\lambda}{\mu}(\operatorname{div} u)^2 + \frac{r\rho}{T}$$

Observe

$$-\frac{1}{T}\operatorname{div} q = \frac{\kappa}{T}\operatorname{div}(\nabla T) = \kappa \operatorname{div}\left(\frac{\nabla T}{T}\right) + \kappa\frac{(\nabla T)^2}{T^2} = \operatorname{div}\frac{q}{T} + \kappa\frac{(\nabla T)^2}{T^2}$$

Hence, we can rewrite last equation:

$$\rho \frac{Ds}{Dt} = -\operatorname{div}\left(\frac{q}{T}\right) + \frac{\rho r}{T} + \frac{1}{T}\left(2\mu \operatorname{Tr}(D^2) + \lambda(\operatorname{div} u)^2 + \kappa(\nabla\sqrt{T})^2\right)$$

$$\partial_{x_i}\sqrt{T} = \partial_{x_i}(T^{\frac{1}{2}}) = \frac{1}{2}T^{-\frac{1}{2}}\partial_{x_i}T$$

$$\kappa(\nabla\sqrt{T}) = \frac{1}{2}\frac{1}{T^{\frac{1}{2}}}\nabla T$$

$$\kappa(\nabla\sqrt{T})^2 = \frac{1}{4}\frac{1}{T}(\nabla T)^2$$

Let us denote

$$\pi := \frac{1}{T}\left(2\mu \operatorname{Tr}(D^2) + \lambda(\operatorname{div} u)^2 + \kappa(\nabla\sqrt{T})^2\right)$$

$$\rho \frac{Ds}{Dt} = -\operatorname{div}\left(\frac{q}{T}\right) + \frac{\rho r}{T} + \pi$$

Observe $\pi \geq 0$

$$\rho \frac{Ds}{Dt} \geq -\operatorname{div}\left(\frac{q}{T}\right) + \frac{\rho r}{T} \quad \textbf{Clausis-Duhem relation}$$

Remark 1 *The entropy can not be constant*

2 Incompressible Fluids

The volume for any quantity of fluid remains constant in time along the evoulution. That means, at time t

$$|A(t)| = \int_{A(t)} dx$$

Thus, we can obtain

$$0 = \frac{d}{dt}|A(t)| = \frac{d}{dt}\int_{A(t)} dx = (\text{by transport theorem}) = \int_{A(t)} \operatorname{div} u \, dx \implies \operatorname{div} u = 0$$

And because of first equation in Navier-Stokes

$$\operatorname{div} u = 0 \implies \frac{D\rho}{Dt} = 0$$

This means that along the trajectory ρ is constant. So, in our case, we also assume that $\rho = const$.

$$\rho = \rho_0 \in \mathbb{R} \implies \frac{D\rho}{Dt} = 0$$

From Navier-Stokes we can write:

- $\operatorname{div} u = 0$
- 3 equations for u
- energy

So, consider second equation from Navier-Stokes. Now, $b = 0$, $\rho = const$ and applying divergence operator we get

$$\rho(\partial_t u + u\nabla u) + \nabla p = \mu\Delta u + (\lambda + \mu)\nabla \operatorname{div} u + b\rho$$

$$-\Delta p = \operatorname{div}(u\nabla u) = \sum_{i,j=1}^{3} \partial_{ij}(u_j u_i)$$

Under assumptions $r = b = 0$ (means we don't have external forces).

Remark 2 *p can be used as a Lagrange multiplier*

$$\begin{cases} -\Delta u + \nabla p = f \\ \operatorname{div} u = 0 \end{cases} \quad \Leftrightarrow \inf_{\operatorname{div} u = 0} (||\nabla u||_i - < f_i, u >)$$

At the end of the day:

$$\begin{cases} \operatorname{div} u = 0 \\ \partial_t u + u\nabla u + \nabla p = \mu\Delta u + b \\ \partial_t e + u\nabla e = \kappa\Delta T + 2\mu \operatorname{Tr}(D^2) + r \end{cases}$$

And in Euiler fluids, because of no viscousity, equivalently obtain:

$$\begin{cases} \operatorname{div} u = 0 \\ \partial_t u + u\nabla u + \nabla p = b \\ \partial_t e + u\nabla e = r \end{cases}$$

Remark 3 *Incompressibility is not an absolute property of a fluid.*

3 Boundary condition

$\Omega \subseteq \mathbb{R}^3$, $\Omega = \Omega(t)$ is called free boundary problem (boundary depends on time). In our study we consider Ω fixed:

1. $\Omega = \mathbb{R}^3$

2. $\Omega = \mathbb{T}^3$ (torus) (our domain is periodic and periodic means $f(t,x) = f(t,x+L) \quad \forall L$)

3. $\Omega \subset \mathbb{R}^3$

By $\partial\omega$ we denote the boundary of ω.

$$u|_{\partial\omega} = u_B$$

$$u|_{\partial\omega=0}(\implies u_B = 0) \text{ - impermeable domain (we can not exit from } \omega)$$

This $u|_{\partial\omega} = 0$ called no slip boundary condition; the fluid is adherent to the boundary of ω. Note that we also assign density on the boundary

$$\rho|_{\partial\omega} = \rho_B$$

We want to relax no slip condition. For Euler Fluid the conditions in the following $u * n|_{\partial\omega} = 0$ (means vector field is orthogonal to the other normal vectors to the surface).

For viscous fluids we impose the following **Navier's slip condition**

$$\beta u_\tau + [\mathbb{S} \cdot n]_\tau = 0, \quad \text{where } \beta = const > 0$$

If we remember Stokes law: $\lambda = -\frac{2}{3}\mu$ (μ is viscousity)

$$\mathbb{S} = \mu(\operatorname{div} u + \operatorname{div} u^T) - \frac{2\mu}{3} \operatorname{div} u \, \mathrm{Id}$$

$$N = 2\mu \left(D - \frac{1}{3} \operatorname{div} u \, \mathrm{Id} \right)$$

$n-$ normal vetor and $\tau-$ tangent vector to $\partial\omega \implies$:

$$u_\tau = u \cdot \tau$$

$$[\mathbb{S} \cdot n]_\tau = (\mathbb{S} \cdot n)\, \tau$$