CIS 5020 Analysis of Algorithms | Fall 2024

Assignment 5

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1 Question 1

1. We want to show D_t is convex, so based on the definition of convex set, we need to prove: for any $m, n \in D_t$ and $\lambda \in [0,1]$, point $z = \lambda m + (1-\lambda)n$ also belongs to D_t.

   We know $m, n \in D_t$, and for any $x \in D_t$, $f(x) \leq t$, so we can deduce: $f(m) \leq t$, and $f(n) \leq t$.

   We know f is a convex function, so based on the convexity of the function, we have:

   $$f(z) = f(\lambda m + (1-\lambda)n) \leq \lambda f(m) + (1-\lambda)f(n) \leq \lambda t + (1-\lambda)t = t$$

   So,

   $$f(z) \leq t$$

   This implies:

   $$z \in D_t$$

   Thus, D_t is indeed a convex set.

2. We first want to prove function $f(x) = x^\top Ax$ is convex. That's to say, we want to prove:

   For any $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$:

   $$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

   $f(\lambda x+(1-\lambda)y) = (\lambda x+(1-\lambda)y)^\top A(\lambda x+(1-\lambda)y) = \lambda^2 x^\top Ax+2\lambda(1-\lambda)x^\top Ay+(1-\lambda)^2 y^\top Ay$

   Then, we compute $\lambda f(x) + (1-\lambda)f(y)$:

   $$\lambda f(x) + (1-\lambda)f(y) = \lambda x^\top Ax + (1-\lambda)y^\top Ay$$

   Now, we take the difference of the two (which is $f(\lambda x+(1-\lambda)y)-[\lambda f(x)+(1-\lambda)f(y)]$):

   $$\begin{aligned}
   \text{difference} &= [\lambda^2 x^\top Ax + 2\lambda(1-\lambda)x^\top Ay + (1-\lambda)^2 y^\top Ay] - [\lambda x^\top Ax + (1-\lambda)y^\top Ay] \\
   &= \lambda^2 x^\top Ax - \lambda x^\top Ax + 2\lambda(1-\lambda)x^\top Ay - (1-\lambda)y^\top Ay \\
   &= -\lambda(1-\lambda)x^\top Ax + 2\lambda(1-\lambda)x^\top Ay - \lambda(1-\lambda)y^\top Ay \\
   &= \lambda(1-\lambda)\left[-x^\top Ax + 2x^\top Ay - y^\top Ay\right]
   \end{aligned}$$

We notice:

$$\text{difference} = \lambda(1-\lambda)\left[-x^\top Ax + 2x^\top Ay - y^\top Ay\right] = \lambda(1-\lambda)\left[-(x-y)^\top A(x-y)\right]$$

Because we know A is psd and symmetric, $(x-y)^\top A(x-y) \geq 0$. Thus:

$$\text{difference} \leq 0$$

Thus:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

Therefore, $\boldsymbol{f(x) = x^\top Ax}$ is convex.

Secondly, we want to prove $Z_{A,\theta}$ is a convex set:
For $m, n \in Z_{A,\theta}$, we have $m^T Am \leq \theta$ and $n^T An \leq \theta$.
For $\lambda \in [0,1]$, we have:

$$(\lambda m + (1-\lambda)n)^T A(\lambda m + (1-\lambda)n) \leq \lambda m^T Am + (1-\lambda)n^T An \leq \lambda\theta + (1-\lambda)\theta = \theta$$

This implies:

$$\lambda m + (1-\lambda)n \in Z_{A,\theta}$$

Thus, $\boldsymbol{Z_{A,\theta}}$ is a convex set.

2 Question 2

1. $OPT(I)$ means the max size of the union of any k sets, given the instance I.

$$|A| = k, \quad Maximize \quad |\cup_{i\in A} S_i|$$

We know that $OPT(I)$ is an integer solution to $LP(I)$ because compared to $OPT(I)$, the contraints in $LP(I)$ are relaxed. In thius way, x_i and y_i can have fractional values in $[0,1]$, and the goal becomes:

$$\text{Maximize} \quad \sum_{j=1}^{n} y_j$$

Intuitively, we can say $\mathrm{OPT}(I) \leq \mathrm{LP}(I)$ because all the feasible solutions in $OPT(I)$ are also feasible for $LP(I)$ because the contraints of $LP(I)$ include those of $OPT(I)$. Additionally, x_i and y_i can have fractional values, as we mentioned before. Therefore, because $LP(I)$ expands the constraints, it expands the feasible region too (by having fractional values). This implies $LP(I)$ cannot be less than that of $OPT(I)$. Thus, $\mathrm{OPT}(I) \leq \mathrm{LP}(I)$.

Alternatively, we can construct an integer feasible solution and show that $LP(I)$'s optimal value is at least as large as $OPT(I)$.

Set L to be the optimal solution to $OPT(I)$. $|L| = k$. We have:

$x_i = 1$ if $i \in L$, and $x_i = 0$ otherwise.

$y_j = 1$ if j is covered by some S_i with $i \in L$, and $y_j = 0$ otherwise.

Then, we notice:

For any x_i and y_i, $0 \le x_i \le 1$ and $0 \le y_j \le 1$ because $x_i, y_j \in \{0, 1\}$.

$\sum_{i=1}^{m} x_i = k$ because $|L| = k$.

Thus, we conclude this integer solution is feasible for $LP(I)$ and achieves an objective value of $OPT(I)$. Since $LP(I)$'s feasible region include those integer solutions, $LP(I)$'s optimal value is at least as large as $OPT(I)$.

Thus, $\mathrm{OPT}(I) \le \mathrm{LP}(I)$.

2. By definition, the probability that j not in S is:

$$\Pr[j \notin S] = \sum_{i:j \notin S_i} z_i = 1 - \sum_{i:j \in S_i} z_i$$

From the LP constrain:

$$\sum_{i:j \in S_i} x_i \ge y_j$$

Therefore, we have:

$$\sum_{i:j \in S_i} z_i = \frac{1}{k} \sum_{i:j \in S_i} x_i \ge \frac{y_j}{k}$$

Thus, the probability is:

$$\Pr[j \notin S] = 1 - \sum_{i:j \in S_i} z_i \le 1 - \frac{y_j}{k}$$

3. Note that each sample is independent. Let M_j be the event: j is covered after k samples. The probability of counter event $\bar{M_j}$ is,

$$\Pr[\bar{M_j}] \le \left(1 - \frac{y_j}{k}\right)^k$$

Therefore the probability that j is covered:

$$\Pr[M_j] = 1 - \Pr[\bar{M_j}] \ge 1 - \left(1 - \frac{y_j}{k}\right)^k$$

Using the inequality $1 - \left(1 - \frac{x}{k}\right)^k \geq \left(1 - \frac{1}{e}\right) x$ without proof:

$$\Pr[M_j] \geq \left(1 - \frac{1}{e}\right) y_j$$

4. The expected number of elements covered by C (let it be N) is:

$$\mathbb{E}[N] = \sum_{j=1}^{n} \Pr[M_j] \geq \left(1 - \frac{1}{e}\right) \sum_{j=1}^{n} y_j = \left(1 - \frac{1}{e}\right) L(I)$$

Since the expected number of elements covered is at least $\left(1 - \frac{1}{e}\right) L(I)$, there must exist some specific collection C that covers at least this many elements.

The maximum number of elements that can be covered by any k sets is $\mathrm{OPT}(I)$, thus:

$$\mathrm{OPT}(I) \geq \left(1 - \frac{1}{e}\right) L(I)$$