CIS 5020 Analysis of Algorithms Fall 2024

Assignment 4

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1 Question 1

1. Let c be the number of connected components in G. Since we know the multiplicity of the eigenvalue zero of the Laplacian matrix L_G equals the number of connected componets in G, we can deduce that the multiplicity of the eigenvalue zero is c. This implies the smallest c eigenvalues are all zero:

$$\lambda_1 = \lambda_2 = \cdots = \lambda_c = 0$$

It obvious that $c \geq k$, so $\lambda_k = 0$. **Therefore, if G has at least k connected components, $\lambda_k = 0$.**

Secondly, let us assume $\lambda_k = 0$, this implies the multiplicity of the zero eigenvalue is at least k. Since we also know the multiplicity of the zero eigenvalue equals the number of connected components, G should have at least k connected components. **Therefore, if $\lambda_k = 0$, G should have at least k connected components.**

Thus, **$\lambda_k = 0$ iff the Graph G has at least k connected components.**

2 Question 2

1. Consider the expression $(PAP^{-1})(Pv)$, we first regroup this expression:

$$(PAP^{-1})(Pv) = (PA)(P^{-1}Pv)$$

Notice that $P^{-1}P = I$, where I is the identity matrix:

$$P^{-1}Pv = v$$

Thus, the expression simplifies to:

$$(PAP^{-1})(Pv) = (PA)v = P(Av)$$

Notice that v is the eigenvector of A:

$$Av = \lambda v$$

Substitute it to the former expression. Notice that λ is a scalar:

$$P(Av) = P(\lambda v) = \lambda(Pv)$$

Therefore, we have show that:

$$(PAP^{-1})(Pv) = \lambda(Pv)$$

This means Pv is an eigenvector of PAP^{-1} with eigenvalue λ

2. Consider the property of the inverse P^{-1}:

$$I(i,j) = \sum_{k=1}^{n} P(i,k)P^{-1}(k,j)$$

$P(i,k) = 1$ if and only if $k = \sigma(i)$. $I(i,j) = 1$ if and only if $i = j$. Thus, the variable k collapses to one term:

$$1 = P(i,\sigma(i))P^{-1}(\sigma(i),i)$$

Which means for every i:

$$P^{-1}(\sigma(i),i) = 1$$

Now consider the (i,j)th entry of PAP^{-1}:

$$PAP^{-1}(i,j) = \sum_{k=1}^{n}\sum_{l=1}^{n} P(i,k)A(k,l)P^{-1}(l,j)$$

$P(i,k) = 1$ if and only if $k = \sigma(i)$. Thus, the sum over k collapses to the term where $k = \sigma(i)$:

$$PAP^{-1}(i,j) = \sum_{l=1}^{n} A(\sigma(i),l)P^{-1}(l,j)$$

$P^{-1}(l,j) = 1$ if and only if $l = \sigma(j)$. Thus, the sum over l collapses to the term where $l = \sigma(j)$:

$$PAP^{-1}(i,j) = A(\sigma(i),\sigma(j))$$

Therefore the (i,j)th entry of PAP^{-1} is $A_{\sigma(i),\sigma(j)}$

3 Question 3

1. We know $\{u_i\}$ and $\{v_i\}$ form orthonormal bases, so unit vectors x and y can be expressed as follows:

$$x = \sum_i \alpha_i u_i, \quad y = \sum_i \beta_i v_i$$

We also know that $\sum_i \alpha_i^2 = 1$ and $\sum_i \beta_i^2 = 1$.

Then, we want to compute $x^T Ay$.

$$x^T Ay = (\sum_i \alpha_i u_i^T)\left(\sum_j \sigma_j u_j v_j^T\right)(\sum_k \beta_k v_k) = \sum_i \sigma_i \alpha_i \beta_i.$$

To maximize $x^T Ay$, we need to maximize $\sum_i \sigma_i \alpha_i \beta_i$ under the constraints we mentioned above.

By applying Cauchy-Schwarz inequality ($|\sum_i a_i b_i| \leq (\sum_i a_i^2)^{1/2}(\sum_i b_i^2)^{1/2}$) and set $a_i = \sigma_i^{1/2}\alpha_i$ and $b_i = \sigma_i^{1/2}\beta_i$, we get:

$$\sum_i \sigma_i \alpha_i \beta_i \leq \left(\sum_i \sigma_i \alpha_i^2\right)^{1/2} \left(\sum_i \sigma_i \beta_i^2\right)^{1/2}$$

Since σ_i is in decreasing order, $\sigma_1 = \max_i \sigma_i$ because it is the largest. The maximum value is achieved when $\alpha_1 = 1$, $\beta_1 = 1$, and all other α_i, β_i are zero.

Therefore:

$$\max_{\substack{x\in\mathbb{R}^m,\, \|x\|_2=1 \\ y\in\mathbb{R}^n,\, \|y\|_2=1}} x^T Ay = \sigma_1 = \max_i \sigma_i$$

$$\max_{\substack{x\in\mathbb{R}^m,\, \|x\|_2=1 \\ y\in\mathbb{R}^n,\, \|y\|_2=1}} x^T Ay = \max_i \sigma_i$$

2. (a) Operrator norm is as follows:

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2$$

Based on triangle inequality, we can deduce that for any x with $\|x\|_2 = 1$:

$$\|(A+B)x\|_2 \leq \|Ax\|_2 + \|Bx\|_2$$

If we divide both sides by $\|x\|_2$, the equation remains the same as $\|x\|_2 = 1$. Take the maximum over all x, we get:

$$\|A+B\|_2 = \max_{\|x\|_2=1} \|(A+B)x\|_2 \leq \max_{\|x\|_2=1} (\|Ax\|_2 + \|Bx\|_2) \leq \|A\|_2 + \|B\|_2$$

Therefore:

$$\|A+B\|_2 \leq \|A\|_2 + \|B\|_2$$

(b) We start by recalling the definition of the operator norm for a product of matrices:

$$\|AB\|_2 = \max_{\|x\|_2=1} \|ABx\|_2$$

Using the property of norms, we deduce that:

$$\|ABx\|_2 \leq \|A\|_2 \|Bx\|_2$$

Next, we bound $\|Bx\|_2$ by applying the operator norm of B. We also know $\|x\|_2 = 1$:

$$\|Bx\|_2 \leq \|B\|_2 \|x\|_2 = \|B\|_2$$

Combine these inequalities, we get this:

$$\|ABx\|_2 \leq \|A\|_2 \|B\|_2$$

Finally, taking the maximum over all x with $\|x\|_2 = 1$, we deduce:

$$\|AB\|_2 = \max_{\|x\|_2=1} \|ABx\|_2 \leq \|A\|_2 \|B\|_2$$

Thus,

$$\|AB\|_2 \leq \|A\|_2 \cdot \|B\|_2$$

4 Question 4

1. Given that $\|v-u\|_2 \leq \epsilon$. This means:

$$\sum_{i=1}^{n} (v_i - u_i)^2 \leq \epsilon^2$$

$|S \Delta S'|$ denote the size of the symmetric difference between sets S and S', which is:

$$|S \Delta S'| = |(S \setminus S') \cup (S' \setminus S)|$$

Notice that $S = \{i : u_i \geq 0\}, S' = \{i : v_i \geq 0\}$. By definition, the size of the symmetric difference increases when:

$$u_i \geq 0, v_i < 0 \text{ or } u_i < 0, v_i \geq 0$$

1. when $u_i \geq 0, v_i < 0$ then $u_i = \frac{1}{\sqrt{n}}, v_i < 0$. The squared difference is:

$$(v_i - u_i)^2 \geq (\frac{1}{\sqrt{n}})^2 = \frac{1}{n}$$

2. when $u_i < 0, v_i \geq 0$ then $u_i = -\frac{1}{\sqrt{n}}, v_i \geq 0$. The squared difference is:

$$(v_i - u_i)^2 \geq (\frac{1}{\sqrt{n}})^2 = \frac{1}{n}$$

Let d be the size of the symmetric difference, i.e. $d = |S \Delta S'|$, then:

$$d \cdot \frac{1}{n} \leq \sum_{i \in S \Delta S'} (v_i - u_i)^2 \leq \sum_{i=1}^{n} (v_i - u_i)^2 \leq \epsilon^2$$

Thus:

$$d \leq \epsilon^2 n$$

This proves that $|S \Delta S'| = O(\epsilon^2 n)$