This is the SOLUTION of our final exam. ASK ME for help if you are confused on any part of it.

**Problem 1.** (15 points) Evaluate the following integrals. Make sure your notation is perfectly correct.

(a) $\displaystyle\int (\sec(x)+\tan(x))^2\,dx$

(b) $\displaystyle\int_0^{1/2} \frac{x^3}{\sqrt{1-x^2}}\,dx$

*Solution.* (a) We can expand the integrand:

$$(\sec(x)+\tan(x))^2 = \sec^2(x)+2\sec(x)\tan(x)+\tan^2(x)$$

Then we obtain

$$\begin{aligned}
\int (\sec(x)+\tan(x))^2\,dx &= \int \left(\sec^2(x)+2\sec(x)\tan(x)+\tan^2(x)\right)\,dx\\
&= \int \sec^2(x)\,dx + 2\int \sec(x)\tan(x)\,dx + \int \tan^2(x)\,dx\\
&= \tan(x)+2\sec(x)+\tan(x)-x+C\\
&= 2\sec(x)+2\tan(x)-x+C
\end{aligned}$$

(b) Observe that the denominator of the integrand consists of $1-x^2$, which is in the form of $a^2-x^2$. We can use a trigonometric substitution and let $x=\sin(\theta)$. Then we obtain $x^3=\sin^3(\theta)$ and

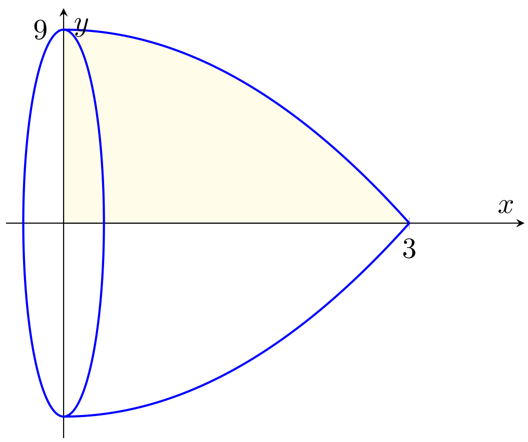
$$\sqrt{1-x^2}=\sqrt{1-\sin^2(\theta)}=\sqrt{\cos^2(\theta)}=\cos(\theta)$$

Also, $\frac{dx}{d\theta}=\cos(\theta)$, which implies that $dx=\cos(\theta)\,d\theta$. When $x=0$, $\theta=\sin^{-1}(0)=0$; when $x=\frac{1}{2}$, $\theta=\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$. Using this substitution, we obtain

$$\begin{aligned}
\int_0^{1/2}\frac{x^3}{\sqrt{1-x^2}}\,dx &= \int_0^{\frac{\pi}{6}}\frac{\sin^3(\theta)}{\cos(\theta)}\cdot\cos(\theta)\,d\theta\\
&= \int_0^{\frac{\pi}{6}}\sin^3(\theta)\,d\theta\\
&= \left(-\cos(\theta)+\frac{\cos^3(\theta)}{3}\right)\Bigg|_0^{\frac{\pi}{6}}\\
&= \left(-\cos\left(\frac{\pi}{6}\right)+\frac{\cos^3\left(\frac{\pi}{6}\right)}{3}\right)-\left(-\cos(0)+\frac{\cos^3(0)}{3}\right)\\
&= -\frac{\sqrt{3}}{2}+\frac{3\sqrt{3}}{24}+1-\frac{1}{3}\\
&= \frac{2}{3}-\frac{3\sqrt{3}}{8}\\
&\approx 0.01715
\end{aligned}$$

$\square$

**Problem 2.** (14 points) Let $\mathcal{R}$ be the region under $x^2 + y = 9$ in the first quadrant. The graph is shown below. Let $\mathcal{S}$ be the solid obtained by rotating $\mathcal{R}$ about the $x$-axis.

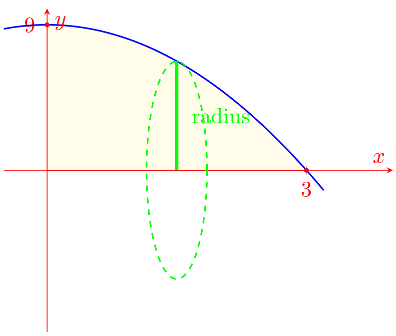



(a) Set up an expression to find the volume of $\mathcal{S}$ using the **Disk Method**. You do NOT need to evaluate this integral.

(b) Set up an expression to find the volume of $\mathcal{S}$ using the **Shell Method**. You do NOT need to evaluate this integral.

*Solution.* A sketch of the solid $\mathcal{S}$ is shown on the graph above.

(a) A typical disk obtained from the rotation is shown below.




The radius of a disk, marked as radius on the graph, is

$$r = 9 - x^2$$

Hence, the volume of the solid $\mathcal{S}$ is

$$V = \pi \int_0^3 \left(9 - x^2\right)^2 \, dx$$

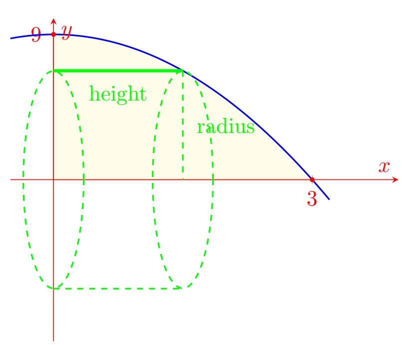
(b) A typical cylindrical shell obtained from the rotation is shown below.

The radius of a cylindrical shell, marked as radius on the graph, is $r = y$ and the height of a cylindrical shell, marked as height on the graph, is

$$h = \sqrt{9 - y}$$

Hence, the volume of the solid $\mathcal{S}$ is

$$V = 2\pi \int_0^9 y\sqrt{9 - y}\, dy$$

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**Problem 3.** (14 points) Evaluate the following integrals. Also, check the box that best describes the solution.

(a) $\displaystyle\int_1^\infty \frac{\ln(x)}{x^3}\,dx$ ☑ the integral **converges**. ☐ the integral **diverges**.

(b) $\displaystyle\int_0^3 \frac{5x}{(x+2)(x-3)}$ ☐ the integral **converges**. ☑ the integral **diverges**.

*Solution.* (a) We can evaluate this integral using the integration by parts. Let

$$u = \ln x \qquad dv = \frac{1}{x^3}\,dx$$
$$du = \frac{1}{x}\,dx \qquad v = -\frac{1}{2x^2}$$

Using the formula for the Integration by parts, we obtain

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{x^3}\,dx &= \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^3}\,dx \\
&= \lim_{b\to\infty}\left(\ln x \cdot -\frac{1}{2x^2}\Big|_1^b - \int_1^b -\frac{1}{2x^2}\cdot\frac{1}{x}\,dx\right) \\
&= \lim_{b\to\infty}\left(-\frac{\ln x}{2x^2} - \frac{1}{4x^2}\right)\Big|_1^b \\
&= \lim_{b\to\infty}\left(-\frac{2\ln b + 1}{4b^2}\right) + \frac{1}{4} \\
&= \lim_{b\to\infty}\left(-\frac{1}{4b^2}\right) + \frac{1}{4} && \text{by the L'Hôpital's rule} \\
&= \frac{1}{4}
\end{aligned}$$

(b) Notice that $f(x) = \dfrac{5x}{(x+2)(x-3)}$ is NOT a continuous function on $[0, 3]$. More specifically, $f$ is NOT continuous at $x = 3$. So we will need to swap out $x = 3$ with a variable (and then take the limit, of course).

$$\int_0^3 \frac{5x}{(x+2)(x-3)}\,dx = \lim_{b\to 3^-} \int_0^b \frac{5x}{(x+2)(x-3)}\,dx$$

We can decompose the fraction $\dfrac{5x}{(x+2)(x-3)}$ using partial fraction decomposition. That is

$$\frac{5x}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}$$

$$5x = A(x-3) + B(x+2)$$

If $x = -2$, then $-10 = A(-2-3) + B(-2+2) = -5A$. Hence, $A = 2$.

If $x = 3$, then $15 = A(3-3) + B(3+2) = 5B$. Hence, $B = 3$.

Hence, we have $\dfrac{5x}{(x+2)(x-3)} = \dfrac{2}{x+2} + \dfrac{3}{x-3}$. So we obtain

$$\begin{aligned}
\int_0^3 \frac{5x}{(x+2)(x-3)}\,dx &= \lim_{b\to 3^-} \int_0^b \frac{5x}{(x+2)(x-3)}\,dx \\
&= \lim_{b\to 3^-} \left( \int_0^b \frac{2}{x+2}\,dx + \int_0^b \frac{3}{x-3}\,dx \right) \\
&= \lim_{b\to 3^-} \left(2\ln|x+2| + 3\ln|x-3|\right)\Big|_0^b \\
&= \lim_{b\to 3^-} \left(2\ln|b+2| + 3\ln|b-3| - 2\ln(2) - 3\ln(3)\right) \\
&= 2\ln(5) + 3\lim_{b\to 3^-} \ln|b-3| - 2\ln(2) - 3\ln(3) \\
&= -\infty
\end{aligned}$$

This shows that the integral $\displaystyle\int_0^3 \frac{5x}{x^2-x-6}\,dx$ **diverges**.

$\square$

**Problem 4.** (15 points) Approximate the integral $\int_0^{2\pi} \frac{dx}{(5+3\sin(x))^2}$ using the following approximation methods:

(a) $M_4$ (b) $T_4$ (c) $S_4$

*Solution.* (a) Given that $N = 4$, we know that $\Delta x = \frac{2\pi - 0}{4} = \frac{\pi}{2}$. Using the formula for the *Midpoint rule*, we obtain

$$\begin{aligned} M_4 &= \Delta x \left( f\left(\frac{0+\frac{\pi}{2}}{2}\right) + f\left(\frac{\frac{\pi}{2}+\pi}{2}\right) + f\left(\frac{\pi+\frac{3\pi}{2}}{2}\right) + f\left(\frac{\frac{3\pi}{2}+2\pi}{2}\right)\right) \\ &= \frac{\pi}{2}\left( f\left(\frac{\pi}{4}\right) + f\left(\frac{3\pi}{4}\right) + f\left(\frac{5\pi}{4}\right) + f\left(\frac{7\pi}{4}\right)\right) \\ &= \frac{\pi}{2}(0.019719 + 0.019719 + 0.120674 + 0.120674) \\ &\approx 0.4411 \end{aligned}$$

(b) Given that $N = 4$, we know that $\Delta x = \frac{2\pi - 0}{4} = \frac{\pi}{2}$. Using the formula for the *Trapezoid rule*, we obtain

$$\begin{aligned} T_4 &= \frac{\Delta x}{2}\left( f(0) + 2f\left(\frac{\pi}{2}\right) + 2f(\pi) + 2f\left(\frac{3\pi}{2}\right) + f(2\pi)\right) \\ &= \frac{\frac{\pi}{2}}{2}\left(\frac{1}{25} + 2\cdot\frac{1}{64} + 2\cdot\frac{1}{25} + 2\cdot\frac{1}{4} + \frac{1}{25}\right) \\ &\approx 0.5429 \end{aligned}$$

(c) Given that $N = 4$, we know that $\Delta x = \frac{2\pi - 0}{4} = \frac{\pi}{2}$. Using the formula for the *Simpson's rule*, we obtain

$$\begin{aligned} S_4 &= \frac{\Delta x}{3}\left( f(0) + 4f\left(\frac{\pi}{2}\right) + 2f(\pi) + 4f\left(\frac{3\pi}{2}\right) + f(2\pi)\right) \\ &= \frac{\frac{\pi}{2}}{3}\left(\frac{1}{25} + 4\cdot\frac{1}{64} + 2\cdot\frac{1}{25} + 4\cdot\frac{1}{4} + \frac{1}{25}\right) \\ &\approx 0.6401 \end{aligned}$$

$\square$

**Problem 5.** (14 points) Let $f(x) = xe^{-x}$ if $x \geq 0$ and $f(x) = 0$ if $x < 0$.

(a) *Verify* that $f$ is a probability density function.

> **Hint:** Recall the two conditions for a function to be a probability density function based on the definition. You will need to argue why $f(x)$ satisfies BOTH conditions.

(b) Find $P(1 \leq X \leq 2)$. Round your answer to six decimal places.

*Solution.* (a) Notice that $e^{-x} > 0$ for all real number $x$. That is, $f(x) = xe^{-x} \geq 0$ for all $x \geq 0$. Since $f(x) = 0$ for all $x < 0$, then $f(x) \geq 0$ for all real values of $x$. Also,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} xe^{-x}\,dx = \int_{0}^{\infty} xe^{-x}\,dx$$

So we want to verify that $\int_0^\infty xe^{-x}\,dx = 1$, using the method of integration by parts with the following substitution:

$$\begin{aligned} u &= x \qquad & dv &= e^{-x}\,dx \\ du &= dx \qquad & v &= -e^{-x} \end{aligned}$$

We obtain

$$\begin{aligned} \int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty} \int_0^b xe^{-x}\,dx &= \lim_{b\to\infty} \left( -xe^{-x}\Big|_0^b - \int_0^b -e^{-x}\,dx \right) \\ &= \lim_{b\to\infty} \left( -xe^{-x}\Big|_0^b + \int_0^b e^{-x}\,dx \right) \\ &= \lim_{b\to\infty} \left( -xe^{-x}\Big|_0^b - e^{-x}\Big|_0^b \right) \\ &= \lim_{b\to\infty} \left( \left((-be^{-b}) - (-0\cdot e^0)\right) - \left((-e^{-b}) - (e^0)\right) \right) = 1 \end{aligned}$$

Hence, $f$ is a probability density function.

(b) We know that $P(1 \leq X \leq 2) = \int_1^2 xe^{-x}\,dx$, and we can also use the antiderivative we came up with in part (a). That is,

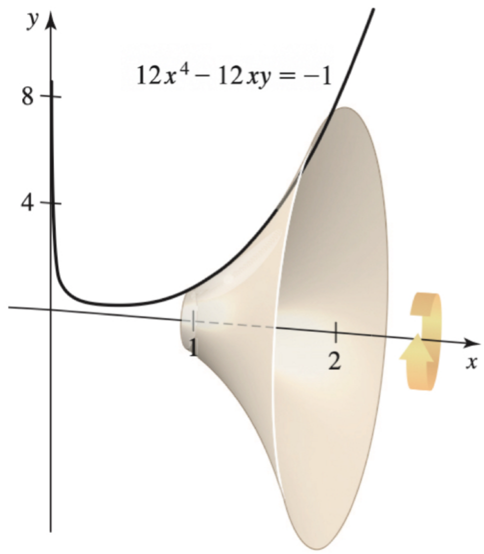
$$\begin{aligned} P(1 \leq X \leq 2) = \int_1^2 xe^{-x}\,dx &= \left(-xe^{-x} - e^{-x}\right)\Big|_1^2 \qquad \text{from Part (a)} \\ &= \left(-2e^{-2} - e^{-2}\right) - \left(-e^{-1} - e^{-1}\right) \\ &\approx 0.329753 \end{aligned}$$

□

**Problem 6.** (14 points) The curved surface of a funnel is generated by revolving the graph of

$$12x^4 - 12xy = -1$$

on the interval $[1, 2]$ about the $x$-axis (see the figure below). Set up an expression to find the **surface area** of the funnel. You do NOT need to evaluate this integral.




*Solution.* Recall that the formula to find the surface area of a solid obtained by rotating a region around the $x$-axis is

$$SA = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx$$

We can obtain $f(x)$ by isolating $y$ in the equation given. Then $f(x) = x^3 + \dfrac{1}{12x}$. This implies that

$$f'(x) = 3x^2 - \frac{1}{12x^2}$$

Hence, the surface area of this funnel is

$$SA = 2\pi \int_1^2 \left(x^3 + \frac{1}{12x}\right)\sqrt{1 + \left(3x^2 - \frac{1}{12x^2}\right)^2}\, dx$$

$\square$

**Problem 7.** (14 points) Approximate the integral $\int_0^1 x\cos\left(x^3\right)\,dx$ using a Maclaurin polynomial in the following steps.

(a) Find the 4th-degree Maclaurin polynomial for $\cos(x)$ using the definition of Maclaurin polynomial.

(b) Find the 13th-degree Maclaurin polynomial for $x\cos\left(x^3\right)$ using the result you obtained in part (a).

(c) Approximate the integral using the Maclaurin polynomial you found in part (b). Round your answer to six decimal places.

*Solution.* (a) Let $f(x) = \cos(x)$. Notice that

$$\begin{aligned} f(x) &= \cos(x) & f(0) &= \cos(0) = 1 \\ f'(x) &= -\sin(x) & f'(0) &= -\sin(0) = 0 \\ f''(x) &= -\cos(x) & f''(0) &= -\cos(0) = -1 \\ f'''(x) &= \sin(x) & f'''(0) &= \sin(0) = 0 \\ f^{(4)}(x) &= \cos(x) & f^{(4)}(0) &= \cos(0) = 1 \end{aligned}$$

Using the Maclaurin polynomial formula, we obtain

$$\begin{aligned} f(x) = \cos(x) &\approx f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f''''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{24} \end{aligned}$$

(b) Observe that $x\cos\left(x^3\right) = xf\left(x^3\right)$. To obtain the Maclaurin polynomial for $x\cos\left(x^3\right)$, we can plug in $x^3$ to the Maclaurin polynomial we obtained in part (a) and then multiply it by $x$. That is,

$$\begin{aligned} x\cos\left(x^3\right) = xf\left(x^3\right) &\approx x\left(1 - \frac{\left(x^3\right)^2}{2} + \frac{\left(x^3\right)^4}{24}\right) \\ &= x - \frac{x^7}{2} + \frac{x^{13}}{24} \end{aligned}$$

(c) We can approximate this integral using the Maclaurin polynomial we got in part (b).

$$\begin{aligned} \int_0^1 x\cos\left(x^3\right)\,dx &\approx \int_0^1 \left(x - \frac{x^7}{2} + \frac{x^{13}}{24}\right)dx \\ &= \left(\frac{x^2}{2} - \frac{x^8}{16} + \frac{x^{14}}{336}\right)\Bigg|_0^1 \\ &= \frac{1}{2} - \frac{1}{16} + \frac{1}{336} \\ &= \frac{37}{84} \\ &\approx 0.440476 \end{aligned}$$

$\square$

**Bonus.** (10 points) Determine whether the following statements are true. If so, justify it; if not, provide a counterexample to show what breaks down (for example, what value for $m$ does not work).

(a) If $m$ is a positive integer, then $\int_0^\pi \sin^m(x)\,dx = 0$.

(b) If $m$ is a positive integer, then $\int_0^\pi \cos^{2m+1}(x)\,dx = 0$.

**Hint:** Notice that the power of $2m+1$ on cosine is always an odd number since $2m$ is always divisible by 2 and we added 1 to it. What do we do if we see an odd power on cosine?

*Solution.* (a) This is a **FALSE** statement. You can verify using a calculator easily that no $m$ value will make the statement true. A counterexample will be let $m = 1$. Then

$$\int_0^\pi \sin(x)\,dx = 2 \neq 0$$

(b) This is a **TRUE** statement. You should be able to guess this statement is true by plugging in some values for $m$ using a calculator. I will provide a brief justification on why the statement is true.

Notice that

$$\int_0^\pi \cos^{2m+1}(x)\,dx = \int_0^\pi \cos^{2m}(x)\cos(x)\,dx = \int_0^\pi \left(\cos^2(x)\right)^m \cos(x)\,dx = \int_0^\pi \left(1-\sin^2(x)\right)^m \cos(x)\,dx$$

Let $du = \cos(x)\,dx$. Then $u = \sin(x)$. Using $u$-sub, the upper limit becomes $u(\pi) = \sin(\pi) = 0$ and the lower limit becomes $u(0) = \sin(0) = 0$. So we obtain

$$\int_0^\pi \left(1-\sin^2(x)\right)^m \cos(x)\,dx = \int_0^0 \left(1-u^2\right)^m\,du = 0$$

since the upper limit and the lower limit are the same.

□