

作业 1

YOUR NAME

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- 1.1. 设 X 和 Y 是各有均值 m_x, m_y , 方差为 σ_x^2, σ_y^2 , 且相互独立的高斯随机变量, 已知 $U = X + Y, V = X - Y$ 。试求 $I(U; V)$ 。

解. U, V 的联合分布是均值为 $[\mu_x + \mu_y, \mu_x - \mu_y]$, 协方差矩阵为

$$\Lambda_{U,V} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^T = \begin{bmatrix} \sigma_x^2 + \sigma_y^2 & \sigma_x^2 - \sigma_y^2 \\ \sigma_x^2 - \sigma_y^2 & \sigma_x^2 + \sigma_y^2 \end{bmatrix}$$

由多元高斯分布微分熵的公式

$$h(U) = \frac{1}{2} \log((2\pi e)^2 |\Lambda_{U,V}|) = \frac{1}{2} \log(16\pi^2 e^2 \sigma_x^2 \sigma_y^2)$$

$U|V = v$ 也是高斯分布, 方差为 $\frac{4\sigma_x^2 \sigma_y^2}{\sigma_x^2 + \sigma_y^2}$, 与 v 无关, 因此

$$h(U|V) = \mathbb{E}_V[h(U|V = v)] = \frac{1}{2} \log(2\pi e \frac{4\sigma_x^2 \sigma_y^2}{\sigma_x^2 + \sigma_y^2}) \Rightarrow$$

$$\begin{aligned} I(U; V) &= h(U) - h(U|V) \\ &= \frac{1}{2} \log(16\pi^2 e^2 \sigma_x^2 \sigma_y^2) - \frac{1}{2} \log(2\pi e \frac{4\sigma_x^2 \sigma_y^2}{\sigma_x^2 + \sigma_y^2}) \\ &= \frac{1}{2} \log(2\pi e (\sigma_x^2 + \sigma_y^2)) \end{aligned}$$

- 1.2. 设有随机变量 X, Y, Z 均取值于 $\{0, 1\}$, 已知

$$I(X; Y) = 0, I(X; Y|Z) = 1. \text{ 求证 } H(Z) = 1, H(X, Y, Z) = 2$$

证明.

$$I(X; Y|Z) = H(X|Z) - H(X|Y, Z) \leq H(X|Z) \leq H(X) \leq \log(2) = 1$$

所以等号全都成立 $\Rightarrow X \sim B(\frac{1}{2})$ 。同理可知 $Y \sim B(\frac{1}{2})$ 。另外

$$H(Y|Z) = H(Y) \Rightarrow I(Y; Z) = 0 \Rightarrow H(Z|Y) = H(Z)$$

$$H(X|Y, Z) = 0$$

$$\Leftrightarrow H(X, Y, Z) = H(Y, Z)$$

$$\Leftrightarrow H(X, Y) + H(Z|X, Y) = H(Y) + H(Z|Y)$$

$$\Leftrightarrow 2 + H(Z|X, Y) = 1 + H(Z)$$

$$\Leftrightarrow H(Z) = 1 + H(Z|X, Y)$$

由上式推出 $H(Z) \geq 1$, 又

$$H(Z) \leq 1 \Rightarrow H(Z) = 1 \Rightarrow H(X, Y, Z) = 2 \quad \square$$

- 1.3. 设有信号 X 经过处理器 A 后获输出 Y , Y 再经处理器 B 后获输出 Z 。
已知处理器 A 和 B 分别独立处理 X 和 Y 。试证: $I(X; Z) \leq I(X; Y)$

证明. $I(X; Z) = H(Z) - H(Z|X) = H(Z)$; $I(Y; Z) = H(Y)$ 因为 Z 是 Y 的函数 $\Rightarrow H(Z) \leq H(Y) \Rightarrow I(X; Z) \leq I(X; Y) \quad \square$

- 1.4. 已知随机变量 X 和 Y 的联合概率密度 $p(a_k, b_j)$ 满足

$$p(a_1) = \frac{1}{2}, p(a_2) = p(a_3) = \frac{1}{4}, p(b_1) = \frac{2}{3}, p(b_2) = p(b_3) = \frac{1}{6}$$

试求能使 $H(X, Y)$ 取得最大值的联合概率密度分布。

解. $H(X, Y) = H(X) + H(Y) - I(X; Y) \leq H(X) + H(Y) = \frac{7}{6} + \log 3$
等号成立当且仅当 X, Y 相互独立 $\Rightarrow p(x, y) = p(x)p(y)$

- 1.5. 设随机变量 X, Y, Z 满足 $p(x, y, z) = p(x)p(y|x)p(z|y)$ 。求证
 $I(X; Y) \geq I(X; Y|Z)$

证明. 因为 $p(x, y, z) = p(x)p(y|x)p(z|y, x) \Rightarrow p(z|y, x) = p(z|x) \Rightarrow x$
与 z 关于 y 条件独立 $\Rightarrow I(X; Y|Z) = H(X|Z) - H(X|Y, Z) =$
 $H(X|Z) - H(X|Y) \leq H(X) - H(X|Y) = I(X; Y) \quad \square$

- 1.6. 求证 $I(X; Y; Z) =$
 $H(X, Y, Z) - H(X) - H(Y) - H(Z) + I(X; Y) + I(Y; Z) + I(Z; X)$,
其中 $I(X; Y; Z) \triangleq I(X; Y) - I(X; Y|Z)$

证明.

$$\begin{aligned} I(X; Y; Z) &= I(X; Y) - I(X; Y|Z) \\ &= H(X) + H(Y) - H(X, Y) - (H(X|Z) - H(X|Y, Z)) \\ &= H(X) + H(Y) - H(X, Y) - (H(X, Z) - H(Z)) + H(X, Y, Z) - H(Y, Z) \\ &= H(X, Y, Z) - H(X) - H(Y) - H(Z) + (H(X) + H(Y) - H(X, Y)) \\ &\quad + (H(Y) + H(Z) - H(Y, Z)) + (H(Z) + H(X) - H(X, Z)) \\ &= H(X, Y, Z) - H(X) - H(Y) - H(Z) + I(X; Y) + I(Y; Z) + I(Z; X) \end{aligned}$$

\square

- 1.7. 令 $p = (p_1, p_2, \dots, p_a)$ 是一个概率分布, 满足 $p_1 \geq p_2 \geq \dots p_a$, 假设 $\epsilon > 0$ 使得 $p_1 - \epsilon \geq p_2 + \epsilon$ 成立, 证明:

$$H(p_1, p_2, \dots, p_a) \leq H(p_1 - \epsilon, p_2 + \epsilon, p_3, \dots, p_a)$$

证明. 设 $f(\epsilon) = (p_1 - \epsilon) \log(p_1 - \epsilon) + (p_2 + \epsilon) \log(p_2 + \epsilon)$ 由已知

$$0 \leq \epsilon \frac{p_2 - p_1}{2} \quad f'(\epsilon) = \log \frac{p_2 + \epsilon}{p_1 - \epsilon} \leq 0$$

$$\Rightarrow f(\epsilon) \leq f(0) \Rightarrow H(p_1, p_2, \dots, p_a) \leq H(p_1 - \epsilon, p_2 + \epsilon, p_3, \dots, p_a) \quad \square$$

- 1.8. 设 $p_i(x) \sim N(\mu_i, \sigma_i^2)$, 试求相对熵 $D(p_2||p_1)$

解.

$$\begin{aligned} D(p_2||p_1) &= \int_{\mathbb{R}} p_2(x) \log \frac{p_2(x)}{p_1(x)} dx \\ &= \int_{\mathbb{R}} p_2(x) \left(\log \frac{\sigma_1^2}{\sigma_2^2} + \frac{1}{2} ((x - \mu_1)^2 - (x - \mu_2)^2) \log e \right) dx \\ &= 2 \log \frac{\sigma_1}{\sigma_2} + \frac{1}{2} (\mu_1^2 - \mu_2^2) \log e + (\mu_2 - \mu_1) \mu_2 \log e \\ &= 2 \log \frac{\sigma_1}{\sigma_2} + \frac{1}{2} (\mu_1 - \mu_2)^2 \log e \end{aligned}$$

- 1.9. 若 $f(x)$ 分别是区间 $(0, 0.01), (0, 0.5), (0, 1), (0, 2), (0, 5)$ 上均匀分布的分布函数, 计算 $f(x)$ 的微分熵。

解. 设 U_t 是 $(0, t)$ 上的均匀分布, 则 $h(U_t) = \log t$

- $h(U_{0.01}) = \log 0.01$
- $h(U_{0.5}) = -1$
- $h(U_1) = 0$
- $h(U_2) = 1$
- $h(U_5) = \log 5$

- 1.10. 设

$$\begin{aligned} p_1(x, y) &= \frac{1}{2\pi\sigma_x\sigma_y} \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)\right] \\ p_2(x, y) &= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2} - 2\rho\frac{xy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)\right] \end{aligned}$$

试求 $D(p_2||p_1)$ 和 $I(X; Y)$, 其中 $X, Y \sim p_2$

解.

$$\begin{aligned}
 D(p_2||p_1) &= \iint_{\mathbb{R}^2} p_2(x, y) \log \frac{p_2(x, y)}{p_1(x, y)} dx dy \\
 &\quad - \frac{1}{2} \log(1 - \rho^2) \\
 &\quad - \frac{1}{2} (\log e) \iint_{\mathbb{R}^2} p_2(x, y) \left[\frac{\rho^2 x^2}{\sigma_x^2 (1 - \rho^2)} + \frac{\rho^2 y^2}{\sigma_y^2 (1 - \rho^2)} - \frac{2\rho xy}{(1 - \rho^2) \sigma_x \sigma_y} \right] dx dy \\
 &= -\frac{1}{2} \log(1 - \rho^2)
 \end{aligned}$$

$X|Y = y$ 服从高斯分布，方差为 $(1 - \rho^2)\sigma_x^2$

$$\begin{aligned}
 I(X; Y) &= h(X) - h(X|Y) \\
 &= \frac{1}{2} \log(2\pi e \sigma_x^2) - \frac{1}{2} \log(2\pi e \sigma_x^2 (1 - \rho^2)) \\
 &= \frac{1}{2} \log\left(\frac{2\pi e}{1 - \rho^2}\right)
 \end{aligned}$$