Optimal feedback control of gaze

This project is based on the material described in chapters 12.4-12.6.

In many laboratory experiments on eye movements, the head of the subject is kept at rest and only the eyes are allowed to move. But in more natural, unconstrained conditions the head participates by redirecting the gaze. When we look around a room, searching for our keys, our eyes and head move in fairly complex and coordinated patterns. As we shift the gaze from one point to another, the eyes tend to start the movement with a saccade (Fig. 12.5A). During the saccade, the head starts rotating. While the head is still moving, the saccade ends and the eyes roll back in the head. If the motion of the head is perturbed, the saccade is altered in mid-flight (Fig. 12.5B). For example, if a brake is applied to the head for 100ms at saccade onset, the saccade is longer in duration and amplitude as compared to when the head is free to rotate, implying that the motion of the eyes during the saccade is affected by the state of the head.

In this project, we will write a simulation to show that all of these behaviors are consistent with a very simple goal: keep the target on the fovea, and keep the eyes centered in the head. We will express this goal as a cost function, and then use optimal feedback control theory to produce movements that best achieve the goal, i.e., minimize the cost. We will perturb the motion of the simulated system and test whether it produces movements that resemble the recorded data.

Our model of the eye dynamics is as follows. We have a third order system where x_e represents eye position in the orbit, and f_e is the torque on the eye. We also assume that when $x_e = 0$ the eye is in the central neutral position in its orbit. The third order system has three states $x_1 \equiv x_e$, $x_2 \equiv x_e$, and $x_3 \equiv f_e$. We have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{k_e}{m_e} & -\frac{b_e}{m_e} & \frac{1}{m_e} \\ 0 & 0 & -\frac{\alpha_2}{\alpha_1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\alpha_1} \end{bmatrix} u_e \tag{12.40}$$

The parameters of the eye plant are set so that the resulting system has time constants of 224, 13, and 4 ms. So we set $k_e = 1$, $b_e = \tau_1 + \tau_2$, $m_e = \tau_1 \tau_2$, $\alpha_2 = 1$, and $\alpha_1 = 0.004$, where $\tau_1 = 0.224$ and $\tau_2 = 0.013$. We represent Eq. (12.40) as:

$$\mathbf{x}_e = A_e \mathbf{x}_e + \mathbf{b}_e u_e \tag{12.41}$$

Our head model is similar to the eye model, but with time constants of 270, 150, and 10 ms. The head position, x_h, is an angle that we measure with respect to a stationary frame in the environment. Therefore, the direction of gaze in the same stationary frame is $x_e + x_h$. The noise-free version of our dynamical system in continuous time is:

$$\begin{gathered} A_c \equiv \begin{bmatrix} A_e & \mathbf{0} \\ \mathbf{0} & A_h \end{bmatrix} \\ B_c = \begin{bmatrix} \mathbf{b}_e & \mathbf{0} \\ \mathbf{0} & \mathbf{b}_h \end{bmatrix} \\ \begin{bmatrix} \mathbf{x}_e \\ \mathbf{x}_h \end{bmatrix} = A_c \begin{bmatrix} \mathbf{x}_e \\ \mathbf{x}_h \end{bmatrix} + B_c \begin{bmatrix} u_e \\ u_h \end{bmatrix} \end{gathered} \tag{12.42}$$

Suppose that g represents the position of our target. We translate our continuous time model into discrete time with time step Δ as follows. Set the state of the system to be:

$$\mathbf{x} \equiv \begin{bmatrix} \mathbf{x}_e & \mathbf{x}_h & g \end{bmatrix}^T \tag{12.43}$$

In Eq. (12.43), g represents the goal location, described as an angle in the same coordinate system in which we measure position of our head. Define the following matrices using matrix exponentials:

$$\begin{gathered} A \equiv \begin{bmatrix} \exp(A_c \Delta) & \mathbf{0}_{6\times1} \\ 0 & 1 \end{bmatrix} \\ B \equiv \begin{bmatrix} A_c^{-1}\left(\exp(A_c \Delta) - I\right) B_c \\ \mathbf{0}_{1x2} \end{bmatrix} \end{gathered} \tag{12.44}$$

1. (5 points) We will simulate the system using a step size $\Delta = 0.002$ sec. Print the matrices A and B.

2. (40 points) The discrete version of our model is:

$$
\begin{aligned}
\mathbf{x}^{(k+1)} &= A\mathbf{x}^{(k)} + B\mathbf{u}^{(k)} + \boldsymbol{\varepsilon}_x^{(k)} + B\sum\nolimits_i C_i \mathbf{u}^{(k)}\phi_i^{(k)} \\
\mathbf{y}^{(k)} &= H\mathbf{x}^{(k)} + \boldsymbol{\varepsilon}_y^{(k)} + H\sum\nolimits_i D_i \mathbf{x}^{(k)}\mu_i^{(k)} \\
H &\equiv \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
\boldsymbol{\varepsilon}_x &\quad N(\mathbf{0}, Q_x) \quad \boldsymbol{\varepsilon}_y \quad N(\mathbf{0}, Q_y) \\
\phi &\quad N(0,1) \qquad \mu \quad N(0,1)
\end{aligned}
\tag{12.45}
$$

The *H* matrix in Eq. (12.45) implies that we can sense the position of the eye x_e, as well as the position of the target on the retina. The position of the target on the retina is the difference between the goal location and the sum of the eye and head positions: $g-(x_e+x_h)$. What we want is to place the target at the center of the fovea, so we assume a cost per step that penalizes the distance of the target to the fovea. We also want to keep the eyes centered in their orbit, and so we will penalize eccentricity of the eye. Finally, we want to minimize the motor commands to eye and head. Our cost per step is:

$$
\begin{aligned}
\alpha^{(k)} &= \mathbf{y}^{(k)T} T^{(k)} \mathbf{y}^{(k)} + \mathbf{u}^{(k)T} L \mathbf{u}^{(k)} \\
&= \mathbf{x}^{(k)T} H^T T^{(k)} H \mathbf{x}^{(k)} + \mathbf{u}^{(k)T} L \mathbf{u}^{(k)}
\end{aligned}
\tag{12.46}
$$

Suppose that we want gaze position $x_e + x_h$ to arrive at target at time step k_1. The state cost matrix T is zero until time step k_1, and then is kept constant until end of simulation period p. The motor cost L is diagonal with equal costs for eye and head motor commands. In these simulations, we will assume zero signal-dependent sensory noise $D_i = 0$.

Let us start by simulating a movement to a target at 40°. Use the following parameter values. Assume zero signal dependent noise associated with motor commands, $C_i = 0$. $p = 0.3$sec. $k_1 = 0.1$sec. The state cost matrix T is zero until time step k_1, and then after that time point is $T = \begin{pmatrix} 100000 & 0 \\ 0 & 300 \end{pmatrix}$.

The motor cost is $L = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$. State noise $Q_x = 10^{-5} I_{7\times 7}$, which means a 7x7 identity matrix times 10^{-5}. Observation noise $Q_y = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix}$. State uncertainty matrix at time point 0: a 7x7 zero matrix. State at time zero is $\mathbf{x}^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & g \end{bmatrix}^T$, where g represents goal location. In this simulation, use $g = 40\dfrac{\pi}{180}$, that is, 40 degrees, represented in radians. Set $\hat{\mathbf{x}}^{(0)} = \mathbf{x}^{(0)}$.

Using these values, find the Kalman gains from the first time point until time point p. With these Kalman gains, working from the last time point to the first, find the feedback control gains. Now starting at the first time point, apply the feedback control gain to the estimate of state and generate a motor command. Apply the motor command to the noisy model of the system and update the state and measurement values. Using the Kalman gain, update estimate of state by comparing the observed state and predicted state (the observed state will have noise in it, as will the state update equation, simulate these noises in your equations). Continue this until time p and plot the position of eye and head as a function of time, as well as gaze (sum of position of eye and head).

Your simulation results should show that the gaze arrives at target at around 100ms, as our cost function had implied, and this is accomplished through a cooperation of the eye and head systems. The gaze change begins with motion of the eye, making saccade, and then is accompanied with motion of the head. Upon saccade completion the eyes roll back in the head as the head continues to move toward the target. The gaze change begins with motion of the eye because the dynamics of the eye present a lighter system to move, and therefore it costs less in terms of effort (squared motor commands) than moving the head. When the gaze is at the target, i.e., target is on the fovea, the eyes roll back. This is because we incur a cost for not having the eyes centered with respect to the head. These two costs, gaze on target, eyes centered, are sufficient to produce the coordinated motion of a natural gaze change.

3. (40 points) Now let us consider a situation in which we have a small amount of signal dependent noise on the motor commands: $C_1 = \begin{pmatrix} 0.01 & 0 \\ 0 & 0 \end{pmatrix}$ and $C_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0.01 \end{pmatrix}$. To start, find the sequence of Kalman gains assuming that there is no signal dependent noise. Next, using these Kalman gains find the sequence of feedback gains. Now re-compute the Kalman gains using the derivation in Eq. (12.39). Next, recompute the feedback gain based on these new sequence of Kalman gains. A few iterations like this should be sufficient to make the Kalman and feedback gains converge. Simulate the system and plot the eye, head, and gaze positions for one trial. Repeat this 50 times and plot the average trajectories.

Repeat this process for large signal dependent noise: $C_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $C_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$. Compare the resulting trajectories in these two simulations and give a rationale for why the system behaves differently with larger signal dependent noise.

4. (15 points) Using the parameters for small signal dependent noise, simulate motion of the system for a condition where you hold the head and prevent it from moving for the first 50ms of the simulation. You can do this by updating your state equation for the first 50ms in such a way as to prevent the motion of the head (set state vector elements 4 and 5 to zero after each update, preventing the head from moving). Repeat this for a situation where you hold the head for the first 100ms.