# HOMEWORK 3–CSC 320-SOLUTIONS

(1) Show that the perfect shuffle of two regular languages is regular.

Let $M_A$ and $M_B$ be DFA's which accept $A$ and $B$. Construct a new machine M whose states are the cross-product of the states of $M_A$ and $M_B$ with an additional flag to remember which machine it is currently in. Thus each state in the new machine tells which machine it is currently in and which last state of the other machine it last saw.

Given $M_A = (Q_A, \Sigma, \delta_A, q_{0,A} F_A)$ and $M_B = (Q_B, \Sigma, \delta_B, q_{0,B}, F_B)$, build $M = (Q, \Sigma, \delta, q_0, F)$ such that $Q = Q_A \times Q_B \times \{A, B\}$. Then depending on the value of the flag ($A$ or $B$), the next state is determined by $\delta_A$ and the first state or $\delta_B$ and the second state. E.g., $\delta((q, q', A), a) = \delta_A(q, a)$ and $\delta((q, q', B), a) = \delta_B(q', a)$. The final states are $\{(q, q', B) \mid s.t.\ q \in F_A,\ q' \in F_B\}$.

(2) Let $\Sigma = \{0, 1, +, =\}$

$$ADD = \{x = y + z \mid x, y, z \text{ are binary integers and } x \text{is the sum of } y\ and\ z\}$$

Prove $ADD$ is not regular using the pumping lemma.

Assume $ADD$ is regular. Let $p$ be the pumping length of $ADD$.
Let $w$ be $1^p = 1^p + 0$. We see $w \in ADD$ since $1^p + 0 = 1^p$.
Then $w = xyz$
$x = 1^i,\ for\ i \geq 0$
$y = 1^j\ for\ j > 0$
$z = 1^l = 1^p + 0$ where $i + j + l = p$.

Pump up $y$ twice. We get a string $w'$ which is $1^{p+j} = 1^p + 0$ which is not in $ADD$ since $1^p + 0$ doesn't equal $1^{p+j}$ where $j > 0$. By the pumping lemma if $ADD$ was regular $w'$ would be in ADD, but it isn't, giving a contradiction.

(3) Let $L_1 = \{0^k w 0^k \mid k \geq 1\ and\ w \in \{0, 1\}^*\}$.
Let $L_2 = \{0^k 1 w 0^k \mid k \geq 1\ and\ w \in \{0, 1\}^*\}$.

Show $L_1$ is regular. Show that $L_2$ is not regular (using the pumping lemma).

$L_1$ is regular, because it is the language $0(0 \cup 1)^*0$ which is a regular expression.

$L_2$ is not regular.
Let $w = 0^p 1 0^p$. $w \in L_2$.

$w = xyz$ where
$x = 0^i, \; for \; i \geq 0$
$y = 0^j \; for \; j > 0$
$z = 0^l 10^p$ where $i + j + l = p$.

Pump $y$ up twice and we get $w' = 0^{p+j}10^p$. If $w' \in L_2$ it must be the case that $k = p+j$, since this is the size of the prefix before the first 1. But then $w' \neq 0^k 1 w 0^k$ since there fewer than $k$ 0's after the 1. Since $w' \notin L_2$, $L_2$ can't be regular.

(4) Show $\equiv_L$ is an equivalence relation.
Reflexive, since for any strings $x, z$, $xz \in L$ iff $xz$ is in $L$.
Symmetric since for any strings $x, y$, if $x$ and $y$ are indistinguishable, then by definition for all $z$, $xz \in L$ iff $yz \in L$, so since the implication (iff) is in both directions, $yz \in L$ iff $xz \in L$, so $y$ is indistinguishable from $x$.
Transitive: consider $x, y, w$. If for all $z$, $xz \in L$ implies $yz \in L$ and if for all $z$, $yz \in L$ implies $wz \in L$ then $xz \in L$ implies $wz \in L$. And in the opposite direction, if for all $z$ $yz \in L$ implies $xz \in L$ and if for all $z$, $wz \in L$ implies $yz \in L$ then $wz \in L$ implies $xz \in L$. Hence $w$ and $x$ are indistinguishable.

(5) What is the minimum number of states in a DFA which accepts the language $L = 01 \cup 0*1*$?
Prove your answer by giving a DFA and proving it is minimal using the Myhill Nerode Lemma.

First, there is a A DFA which accepts this with 2 states: $A = (\{q_0, q_1, q_2\}, \{0, 1\}, \delta, q_0, \{q_0, q_1\})$ where $\delta(q_0, 0) = q_0$; $\delta(q_0, 1) = q_1$; $\delta(q_1, 1) = q_1$ $\delta(q_1, 0) = q_2$,$\delta(q_2, 0) = q_2$, $\delta(q_2, 1) = q_2$.

To show this is minimum, show that the strings to get to the each state are pairwise indistinguishable by demonstrating strings $z$ which distinguish each pair.

for $q_0$: $\epsilon$
for $q_1$: 1
for $q_2$: 10

Claim: these strings are pairwise indistinguishable.
$(\epsilon, 1)$: $z = 0$ since $0 \in L$ and $10 \notin L$
$(\epsilon, 10)$: $z = 0$ since $0 \in L$ and $100 \notin L$
$(1, 10)$: $z = 1$ since $11 \in L$ and $101 \notin L$

Index = 3.

(6) Give a context free grammar for $L = \{w \in \{a, b\}^* \mid w$ starts and ends with the same symbol$\}$.
$S \rightarrow aWa \mid bWb \mid \epsilon \mid$
$W \rightarrow aW \mid bW \mid \epsilon$

$L = (a(a \cup b)^*a) \cup (b(a \cup b)^*b)$ is regular since we can write it as a regular expression.

(7) Give a context free grammar for the language of regular expressions over $\Sigma = \{a, b\}$.
$G = (E, \{a, b, \cup, (, ), \emptyset, \epsilon, *\}, P, E)$ where $P$ is:

$E \to (E) \mid EE \mid E^* \mid E \cup E$
$E \to a \mid b \mid \epsilon \mid \emptyset$