Assignment/Problem Set 1

Heinrich Dinkel
ID: 1140339107
E-mail: heinrich.dinkel@sjtu.edu.com.cn

1 Exercise 1

1 I shorten the notation by using sin as a replacement for sin(x) and cos as a replacement for cos(x). We use the binet formula to get the power of A^n. We first estimate the eigenvectors, then concatenate them into matrix P and use this matrix to compute the diagonal matrix B, which can be used to directly take the power for the matrix A. Afterwards we need to reverse the operations, which we had done on achieving B to get A^n.

$$A^n = \begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix}^n, v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, (A - \lambda I)v = 0$$
$$\begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} =$$
$$(\cos^2 - \lambda) + \sin^2 = 0$$
$$\lambda_1 = \imath \sin + \cos$$
$$\lambda_2 = -\imath \sin + \cos$$

Now we achieved the eigenvalues, so we can estimate the eigenvectors. The eigenvector λ_1

$$\begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix} - \begin{pmatrix} \imath \sin + \cos & 0 \\ 0 & \imath \sin + \cos \end{pmatrix} = \begin{pmatrix} \imath \\ 1 \end{pmatrix}$$

Symmetrically the vector for λ_2 is:

$$\begin{pmatrix} -\imath \\ 1 \end{pmatrix}$$

The matrix P is just a concatenation of λ_1 and λ_2.

$$P = \begin{pmatrix} \imath & -\imath \\ 1 & 1 \end{pmatrix} \tag{1}$$

To calculate an diagonal matrix, we use the equation $B = P^{-1}AP$ to transform A into a diagonal matrix.

$$P^{-1} = \frac{1}{\det(P)} \operatorname{adj}(P) =$$

$$\frac{1}{2\imath} \begin{pmatrix} 1 & \imath \\ -1 & \imath \end{pmatrix} = \begin{pmatrix} \frac{1}{2\imath} & \frac{1}{2} \\ -\frac{1}{2\imath} & \frac{1}{2} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} \frac{1}{2\imath} & \frac{1}{2} \\ -\frac{1}{2\imath} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix} \begin{pmatrix} \imath & -\imath \\ 1 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \left(\frac{cos}{2\imath} - \frac{\sin}{2}\right)\imath + \frac{\sin}{2\imath} + \frac{\cos}{2} & \left(\frac{cos}{2\imath} - \frac{\sin}{2}\right)(-\imath) + \frac{\sin}{2\imath} + \frac{\cos}{2} \\ \left(-\frac{\cos}{2\imath} - \frac{\sin}{2}\right)\imath + -\frac{\sin}{2\imath} + \frac{\cos}{2} & \left(-\frac{\cos}{2\imath} - \frac{\sin}{2}\right)(-\imath) - \frac{\sin}{2\imath} + \frac{\cos}{2} \end{pmatrix} =$$

$$\begin{pmatrix} \cos - \imath \sin & 0 \\ 0 & \cos + \imath \sin \end{pmatrix}$$

Now we can apply the power operation on the matrix.

$$B^n = \begin{pmatrix} (\cos - \imath \sin)^n & 0 \\ 0 & (\cos + \imath \sin)^n \end{pmatrix}$$

Now we can apply the matrix exponential onto A, since $A = PB^nP^{-1}$.

$$\begin{pmatrix} \imath & -\imath \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (\cos - \sin \imath)^n & 0 \\ 0 & (\cos + (\imath \sin)^n \end{pmatrix} \begin{pmatrix} \frac{1}{2\imath} & \frac{1}{2} \\ -\frac{1}{2\imath} & \frac{1}{2} \end{pmatrix} =$$

$$\begin{pmatrix} (\cos - \sin \imath)^n \imath & (\cos + \imath \sin)^n)(-\imath) \\ (\cos - \sin \imath)^n & (\cos + \imath \sin)^n \end{pmatrix} \begin{pmatrix} \frac{1}{2\imath} & \frac{1}{2} \\ -\frac{1}{2\imath} & \frac{1}{2} \end{pmatrix} =$$

$$\frac{1}{2} \begin{pmatrix} (\cos - \sin \imath)^n + (\cos + \sin \imath)^n & (\cos - \sin)^n \imath - (\cos + \sin \imath)^n \imath \\ \imath(\cos + \sin)^n - \imath(\cos - \sin \imath)^n & (\cos - \sin \imath)^n + (\cos + \sin \imath)^n \end{pmatrix} = A^n$$

After some basic mathematics procedures we get the simplified matrix as shown below.

$$A^n = \begin{pmatrix} \cos(n \cdot x) & \sin(n \cdot x) \\ -\sin(n \cdot x) & \cos(n \cdot x) \end{pmatrix} \tag{2}$$

2 Like in exercise 1 we begin by getting the eigenvalues and eigenvectors and the matrix P.

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \text{Eigenvalues of } A : \begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 1 = 0$$

$$\lambda_1 = 1 + \imath$$

$$\lambda_2 = 1 - \imath$$

$$\text{Eigenvector of } \lambda_1, \lambda_2 \left[\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} 1+\imath & 0 \\ 0 & 1+\imath \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$v_1 = \begin{pmatrix} 1\imath \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 1 \\ -\imath \end{pmatrix}$$

Since we computed the eigenvectors v_1 and v_2, we continue by calculating the diagonal matrix B.

$$P = \begin{pmatrix} 1 & 1 \\ \imath & -\imath \end{pmatrix} \Rightarrow P^{-1} = \frac{1}{\det(P)} \begin{pmatrix} -\imath & -1 \\ -\imath & 1 \end{pmatrix} = \frac{1}{-2\imath} \begin{pmatrix} -\imath & -1 \\ -\imath & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\imath} \\ \frac{1}{2} & -\frac{1}{2\imath} \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\imath} \\ \frac{1}{2} & -\frac{1}{2\imath} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \imath & -\imath \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\imath} \\ \frac{1}{2} & -\frac{1}{2\imath} \end{pmatrix} \begin{pmatrix} 1+\imath & 1-\imath \\ -1+\imath & -1-\imath \end{pmatrix} = \begin{pmatrix} 1+\imath & 0 \\ 0 & 1-\imath \end{pmatrix}$$

$$B^n = \begin{pmatrix} (1+\imath)^n & 0 \\ 0 & (1-\imath)^n \end{pmatrix}$$

$$A^n = PB^nP^{-1} = \begin{pmatrix} 1 & 1 \\ \imath & -\imath \end{pmatrix} \begin{pmatrix} (1+\imath)^n & 0 \\ 0 & (1-\imath)^n \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2\imath} \\ \frac{1}{2} & -\frac{1}{2\imath} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \imath & -\imath \end{pmatrix} \begin{pmatrix} \frac{(1+\imath)^n}{2} & \frac{(1+\imath)^n}{2\imath} \\ \frac{(1-\imath)^n}{2} & -\frac{(1-\imath)^n}{2\imath} \end{pmatrix} =$$

$$\frac{1}{2} \begin{pmatrix} (1+\imath)^n + (1-\imath)^n & (1+\imath)^n - (1-\imath)^n \\ \imath(1+\imath)^n - \imath(1-\imath)^n & (1+\imath)^n + (1-\imath)^n \end{pmatrix} = A^n$$

3 To solve these we do not use the Methods as in Exercise 1 or 2 because the Eigenvector of these Matrix is:

$$V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

With these vector it is not possible to make the transformations as shown at the previous paragraphs. After calculating a few powers of the Matrix we got these series:

$$A^2 = \begin{pmatrix} a^2 & 2*a & 1 & 0 & 0 \\ 0 & a^2 & 2*a & 1 & 0 \\ 0 & 0 & a^2 & 2*a & 1 \\ 0 & 0 & 0 & a^2 & 2*a \\ 0 & 0 & 0 & 0 & a^2 \end{pmatrix} \qquad A^3 = \begin{pmatrix} a^3 & 3*a^2 & 3*a & 1 & 0 \\ 0 & a^3 & 3*a^2 & 3*a & 1 \\ 0 & 0 & a^3 & 3*a^2 & 3*a \\ 0 & 0 & 0 & a^3 & 3*a^2 \\ 0 & 0 & 0 & 0 & a^3 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} a^4 & 4*a^3 & 6*a^2 & 4*a & 1 \\ 0 & a^4 & 4*a^3 & 6*a^2 & 4*a \\ 0 & 0 & a^4 & 4*a^3 & 6*a^2 \\ 0 & 0 & 0 & a^4 & 4*a^3 \\ 0 & 0 & 0 & 0 & a^4 \end{pmatrix} \qquad A^5 = \begin{pmatrix} a^5 & 5*a^4 & 10*a^3 & 10*a^2 & 5*a \\ 0 & a^5 & 5*a^4 & 10*a^3 & 10*a^2 \\ 0 & 0 & a^5 & 5*a^4 & 10*a^3 \\ 0 & 0 & 0 & a^5 & 5*a^4 \\ 0 & 0 & 0 & 0 & a^5 \end{pmatrix}$$

After looking for a longer time at these equations It is totally obvious the Matrix has to be:

$$A^n = \begin{pmatrix} a^n & a^{-1+n}\cdot n & \frac{1}{2}a^{-2+n}\cdot(-1+n)\cdot n & \frac{1}{6}a^{-3+n}\cdot(-2+n)\cdot(-1+n)\cdot n & \frac{1}{24}a^{-4+n}\cdot(-3+n)\cdot(-2+n)\cdot(-1+n)\cdot n \\ 0 & a^n & a^{-1+n}\cdot n & \frac{1}{2}a^{-2+n}\cdot(-1+n)\cdot n & \frac{1}{6}\cdot a^{-3+n}\cdot(-2+n)\cdot(-1+n)\cdot n \\ 0 & 0 & a^n & a^{-1+n}\cdot n & \frac{1}{2}a^{-2+n}\cdot(-1+n)\cdot n \\ 0 & 0 & 0 & a^n & a^{-1+n}\cdot n \\ 0 & 0 & 0 & 0 & a^n \end{pmatrix}$$

2 Exercise 2

1 Computing $A^{-1}B$.

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 4 & 5 & 0 \\ 2 & 3 & 1 \\ 2 & 7 & -3 \end{pmatrix} = -\frac{1}{24} \begin{pmatrix} -16 & 15 & 5 \\ 8 & -12 & -4 \\ 8 & -18 & 2 \end{pmatrix}$$

$$AB = \frac{1}{12} \begin{pmatrix} 12 & 0 & -30 & 95 \\ 0 & 12 & 24 & -52 \\ 0 & 0 & 0 & -46 \end{pmatrix}$$

2 Computing CA^{-1}

$$CA^{-1} = \begin{pmatrix} 4 & 5 & 0 \\ 2 & 3 & 1 \\ 2 & 7 & 9 \\ -2 & 3 & 7 \end{pmatrix} (-\frac{1}{24}) \begin{pmatrix} -16 & 15 & 5 \\ 8 & -12 & -4 \\ 8 & -18 & 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ -12 & 27 & 0 \\ -14 & 24 & 1 \end{pmatrix}$$

3 Exercise 3

We want to proof the following equation:

$$\text{adj}(AB) = \text{adj}(B)\text{adj}(A) \tag{3}$$

It is already known that $A\text{adj}(A) = \det(A)I$, so

$$\text{adj}(A) = A^{-1}\det(A)I \tag{4}$$

Also it is known that

$$\det(AB) = \det(A)\det(B) \tag{5}$$

We substitute 4 into 3, by using 5.

$$\text{adj}(B)\text{adj}(A) = \det(B)B^{-1}I\det(A)A^{-1}I =$$
$$\det(A)\det(B)B^{-1}IA^{-1}I = \det(AB)(AB)^{-1}$$

Using $C = AB$ we get:

$$\det(C)C^{-1} = \text{adj}(C) \Rightarrow \text{adj}(AB) = \text{adj}(A)\text{adj}(B)$$

4 Exercise 4

Assuming having a matrix A with dimensions of $n \times m$ with r different row vectors spanning the vector space. Since the transpose of A will transform rows into columns, we can say that the column rank of A^T is also r.

$$A = \begin{pmatrix} a_{11} & \dots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1m} & hdots & a_{nm} \end{pmatrix}$$

If we modify A to get $A^{'}$, by deleting one row of A, we can get the following inequality:

$$\text{rank}(A^{'}) \leq \text{rank}(A)$$

If we use now the transpose of $A^{'}$, we can see that the column rank will hold the following inequality:

$$\text{rank}(A^{'}) \leq$$

5 Exercise 5

To proof $\text{rank}(A^n) = \text{rank}(A^{n+1})$