Assignment/Problem Set 2

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1.1 Exercise 2

Assume having a square matrix A, which we want to maximize. $\max A$. We assume that A is a symmetric matrix. If A is non-zero, we can extract eigenvectors out of it: $Ax = \lambda x$. We get:

$$Ax = \lambda x$$
$$x^T Ax = x^T(Ax) = x^T(\lambda x) = \lambda x^T x = \lambda \sum_i^n |x_i|^2$$

To maximize the equation given, we need to maximize $|\lambda|$, since the summation in the second term adds up to one. So one can see that by maximizing λ, we maximize A.

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2.1 Exercise 5

A is called idempotent if $A^2 = A$. Show that each eigenvalue of an idempotent matrix is either 0 or 1. Using this property, we can show:

$$\begin{array}{rc} \lambda x = & Ax \\ = & A^2x \\ = & A(Ax) \\ = & A(\lambda x) \\ = & \lambda(Ax) \\ = & \lambda(\lambda x) \\ = & \lambda^2 x \end{array}$$

Since $\lambda^2 x$ equals to Ax, we can compute the eigenvalues:

$$Ax - \lambda x = 0$$
$$\lambda^2 x - \lambda x = 0$$
$$x(\lambda^2 - \lambda) = 0$$
$$\rightarrow \lambda_1 = 1 \ \lambda_2 = 0$$

So we can see that the eigenvalues are either zero or one, as required.

2.2 Exercise 6

Here we use the same procedure as in Exercise 5.

$$\lambda x = A^q x$$
$$\lambda x = \underbrace{AA...AA}_{q} x$$
$$\lambda x = \underbrace{AA...AA}_{q-1}(Ax)$$
$$\lambda x = \underbrace{AA...AA}_{q-1}(\lambda x)$$
$$\lambda x = \lambda \underbrace{AA...AA}_{q-2} Ax$$
$$\vdots$$
$$\lambda x = \lambda^q x$$

Now since $A^q = 0$, we can solve the equation $A^q x = \lambda^q x$, for any $x \neq 0$.

$$A^q x = \lambda^q x \rightarrow \lambda = 0$$

We have shown that all the eigenvalues in a positive nilpotent matrix are zero.

An example for a nilpotent matrix is the following:

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

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3.1 Exercise 4

Given that $A \in M_n$ and $A_i = \text{adj}(A)$, we want to show that equation 1 holds.

$$\frac{d}{dt} p_A(t) = \sum_i^n p_{A_t}(t) \tag{1}$$

Assuming our characteristic polynomial $B = (tI - A)$ and $b_{ii} = (t - a_i i)$, we get the following equations:

$$\begin{aligned} \det(B) = \quad & \textstyle\sum_i^n (-1)^{i+i} b_{ii} A_i \\ = \quad & \textstyle\sum_i^n b_{ii} A_i \\ \text{taking derivative} \quad \frac{d}{dt}\det(B) &= \frac{d}{dt}\sum_i^n (t - a_{ii}) A_i \\ \frac{d}{dt}\det(B) &= \sum_i^n A_i = \frac{d}{dt} p_A(t) \\ p_{A_i}(t) &= \sum (-1)^{2i} a_{ii} A_i^{'} = A_i \end{aligned}$$

This shows that the determinant of B, which is the characteristic polynomial of A, is $\sum_i^n p_{A_i}(t)$.

3.2 Exercise 6

We want to proof that $\text{rank}(A - \lambda I) = n - 1$. The root of the characteristic polynomial is 0. We can see that $\frac{d}{dt}p_A(t)$ at $t = \lambda$ is non-zero, since it only evolves calculating the principal sub matrix of A. From there on we can follow:

$$\frac{d}{dt}p_A(\lambda) = \sum_i^n A_i \neq 0$$
$$\rightarrow \sum_{i=1}^n p_{A_i}(\lambda) \neq 0$$
$$\sum_{i=1}^n p_{A_i}(\lambda) \neq 0$$
$$\exists A_i \neq 0 \rightarrow \text{rank}(A_i) = n - 1$$

The submatrix rank follows from the column/row rank independence theorem. If one removes a row and a column from an n rank matrix, the submatrix needs to have n-1 rank, because the columns and rows are independent.

From here on we can follow, that $\text{rank}(A - \lambda I) = n - 1$, since $\frac{d}{dt}p_A(\lambda) \neq 0$. The converse is of course not true, as seen in example 1.2.7b. Consider the matrix:

$$\begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \end{pmatrix} \tag{2}$$

As it can be seen, its characteristic polynomial is 1, yet it's rank is still n and not $n - 1$.

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4.1 Exercise 5

If $A \in M_n$ and has distinct eigenvalues, show that if $AB = BA$, where $B \in M_n$, B is a polynomial of degree at most $n-1$ Since $A^0v, A^1v, \dots, A^{n-1}v$ are linearly independent, they form a basis for $\mathcal{R}^n$. Thus:

$$Bv = c_0A^0v + c_1A^1v + \dots + c_{n-1}A^{n-1}v = p(A)v$$

where

$$p(x) = c_0 + c_1x + \dots + c_{n-1}x^{n-1}$$

for $r = 0, 1, \dots, n-1$. For some constants $c_0, c_1, \dots, c_{n`1}$. Since B commutes with A, it commutes with all powers of A, so $B(A^rv) = A^rBv = A^rp(A)v = p(A)(A^rv)$.

But again the vectors A^rv are a basis, so $B = p(A)$, which shows that the characteristic polynomial in B has $n - 1$ degrees.

4.2 Exercise 6

If A is diagonalizable, which means that $A^{'} = P^{-1}AP$, so that $A^{''}$ is diagonal. Since $A^{'}$ is similar to A, both share similar eigenvalues. That said, the characteristic polynomial of $A^{'}$ will have at least one eigenvalue, except A would be the zero matrix. We calculate:

$$A^{'} = \begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix}$$

$$p_A(A) = \det(A - A) = \begin{pmatrix} a_{11} - a_{11} & & & \\ & a_{22} - a_{22} & & \\ & & \ddots & \\ & & & a_{nn} - a_{nn} \end{pmatrix} = 0$$

We have shown that if a matrix A is diagonalizable, the characteristic polynomial with respect to itself is 0.

4.3 Exercise 7

We show that every diagonalizable matrix has a square root. We assume that a matrix A can be decomposed into its diagonal form D by using a matrix $Q \times Q = D$, which is again a square root.

$$AA = B$$
$$A = P^{-1}DP$$
$$QQ = D$$
$$(P^{-1}QP)(P^{-1}QP) = P^{-1}Q(PP^{-1})QP = P^{-1}QQP = P^{-1}DP = A$$

We have shown that every Matrix in M_n which is diagonalizable has a square root.

4.4 Exercise 12

Suppose having two matrices $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and B. We want to show that these two only commute if B is diagonal itself. We assume that B is an arbitrary matrix, while Λ is a diagonal one.

$$\Lambda = \begin{pmatrix} \lambda_1 I_{k_1} & 0 & \cdots & 0 \\ 0 & \lambda_2 I_{k_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m I_{k_n} \end{pmatrix}, B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{pmatrix}$$

$$\Lambda B = \begin{pmatrix} \lambda_1 B_{11} & \lambda_1 B_{12} & \cdots & \lambda_1 B_{1n} \\ \lambda_2 B_{21} & \lambda_2 B_{22} & \cdots & \lambda_2 B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n B_{n1} & \lambda_n B_{n2} & \cdots & \lambda_n B_{nn} \end{pmatrix}, B\Lambda = \begin{pmatrix} \lambda_1 B_{11} & \lambda_2 B_{12} & \cdots & \lambda_n B_{1n} \\ \lambda_1 B_{21} & \lambda_2 B_{22} & \cdots & \lambda_n B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 B_{n1} & \lambda_2 B_{n2} & \cdots & \lambda_n B_{nn} \end{pmatrix}$$

From the equation given in this example , we get:

$$\lambda_i B_{ij} = B_{ij} \lambda_j$$

Since all n eigenvalues of A are distinct, we know that the resulting equation $(\lambda_i - \lambda_j)B_{ij} = 0$ will tell us that the non-diagonal elements are zero.

$$\Lambda B_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ B_{ij} = B_{ii} & \text{else} \end{cases}$$

From here on we conclude, that both Λ and B are diagonal. This means that the commutation can only be done if B is diagonal with arbitrary entries within the diagonal.

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5.1 Exercise 1

Show that $A \in M_n$ has rank 1 if and only if there exist two non-zero vectors $x, y \in \mathcal{C}^n$ such that $A = xy*$

a Let $A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, where a_i represents a row in A. The rank can only be one, if

$$a_{i1} = \alpha_{i-1}a_{11}, a_{i2} = \alpha_{i-1}a_{11}, \ldots, a_{in} = \alpha_{i-1}a_{1n}$$

The vectors which would span A, can be defined as:

$$x = \begin{pmatrix} 1 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} y* = (a_{11}, \ldots, a_{1n})$$

From the definition of x, y we can now use the relation $A = xy^*$ to show that these vectors are both nonzero. Otherwise A would be the zero matrix, contradicting with A having rank 1.

The characteristic polynomial of A can be written as the sum of all eigenvalues times the .

$$p_A(t) = t^n - (\sum_{i=1}^{n} \lambda_i)t^{n-1} + \ldots + \det(A)$$

Since $\det(A) = 0$(rank 1) then $t = 0$ is a root of p_A. This means that 0 is an eigenvalue of A. Suppose that A has at least two non-zero distinct eigenvalues, which are defined as:

$$Ax_1 = \lambda_1 x_1, x_1 \neq 0 Ax_2 = \lambda_2 x_2, x_2 \neq 0$$

Assuming x_1 and x_2 are linearly independent vectors and $A = xy^*$, we get the following relations:

$$xy^*x_1 = \lambda_1 x_1$$
$$x = \frac{\lambda_1}{y^*x_1} x_1$$
$$xy^*x_2 = \lambda_2 x_2$$
$$x = \frac{\lambda_2}{y^*x_2} x_2$$

The quantities y^*x_1, y^*x_2 are scalars so this contradicts the assumption that x_1, x_2 are linearly independent. If this eigenvalue x exists, it has algebraic multiplicity 1. It is known that if an eigenvalue has multiplicity $k \geq 1$ then the rank of the matrix $A - \lambda I$ is $n - k$. The first already known eigenvalue (0) has at least multiplicity of $n - 1$ (since we have rank 1), therefore we would have at least $n - (n - 1) = 1$ eigenvalues.

b Here we proof that the eigenvalue can be rewritten as y^*x.

$$Av = \lambda v$$
$$xy^*v = \lambda v$$
$$y^*xy^*v = y^*\lambda v$$
$$y^*xy^*v = \lambda y^*v$$
$$(y^*x - \lambda)y^*v = 0$$
$$\rightarrow y^*x = \lambda$$

As required.

c The left eigenvector, using 5.1:

$$y^*A = (y^*x)y^* \tag{3}$$

The right eigenvector is:

$$Ax = (xy^*)x \tag{4}$$

The geometric multiplicity of the eigenvalue 0 is at least $n-1$.