Assignment/Problem Set 1

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1 Exercise 1

1 I shorten the notation by using sin as a replacement for sin(x) and cos as a replacement for cos(x). We use the binet formula to get the power of A^n. We first estimate the eigenvectors, then concatenate them into matrix P and use this matrix to compute the diagonal matrix B, which can be used to directly take the power for the matrix A. Afterwards we need to reverse the operations, which we had done on achieving B to get A^n.

$$\begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix}^n, v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, (A - \lambda I)v = 0$$
$$\begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} =$$
$$(\cos^2 - \lambda) + \sin^2 = 0$$
$$\lambda_1 = \imath \sin + \cos$$
$$\lambda_2 = -\imath \sin + \cos$$

Now we achieved the eigenvalues, so we can estimate the eigenvectors. The eigenvector λ_1

$$\begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix} - \begin{pmatrix} \imath \sin + \cos & 0 \\ 0 & \imath \sin + \cos \end{pmatrix} = \begin{pmatrix} \imath \\ 1 \end{pmatrix}$$

Symmetrically the vector for λ_2 is:

$$\begin{pmatrix} -\imath \\ 1 \end{pmatrix}$$

The matrix P is just a concatenation of λ_1 and λ_2.

$$P = \begin{pmatrix} \imath & -\imath \\ 1 & 1 \end{pmatrix} \tag{1}$$

To calculate an diagonal matrix, we use the equation $B = P^{-1}AP$ to transform A into a diagonal matrix.

$$P^{-1} = \frac{1}{\det(P)}\text{adj}(P) =$$

$$\frac{1}{2\imath}\begin{pmatrix} 1 & \imath \\ -1 & \imath \end{pmatrix} = \begin{pmatrix} \frac{1}{2\imath} & \frac{1}{2} \\ -\frac{1}{2\imath} & \frac{1}{2} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} \frac{1}{2\imath} & \frac{1}{2} \\ -\frac{1}{2\imath} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix}\begin{pmatrix} \imath & -\imath \\ 1 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \left(\frac{cos}{2\imath} - \frac{\sin}{2}\right)\imath + \frac{\sin}{2\imath} + \frac{\cos}{2} & \left(\frac{cos}{2\imath} - \frac{\sin}{2}\right)(-\imath) + \frac{\sin}{2\imath} + \frac{\cos}{2} \\ \left(-\frac{\cos}{2\imath} - \frac{\sin}{2}\right)\imath + -\frac{\sin}{2\imath} + \frac{\cos}{2} & \left(-\frac{\cos}{2\imath} - \frac{\sin}{2}\right)(-\imath) - \frac{\sin}{2\imath} + \frac{\cos}{2} \end{pmatrix} =$$

$$\begin{pmatrix} \cos - \imath\sin & 0 \\ 0 & \cos + \imath\sin \end{pmatrix}$$

Now we can apply the power operation on the matrix.

$$B^n = \begin{pmatrix} (\cos - \imath\sin)^n & 0 \\ 0 & (\cos + \imath\sin)^n \end{pmatrix}$$

Now we can apply the matrix exponential onto A, since $A = PB^nP^{-1}$.

$$\begin{pmatrix} \imath & -\imath \\ 1 & 1 \end{pmatrix}\begin{pmatrix} (\cos - \sin\imath)^n & 0 \\ 0 & (\cos + (\imath\sin)^n \end{pmatrix}\begin{pmatrix} \frac{1}{2\imath} & \frac{1}{2} \\ -\frac{1}{2\imath} & \frac{1}{2} \end{pmatrix} =$$

$$\begin{pmatrix} (\cos - \sin\imath)^n\imath & (\cos + \imath\sin)^n)(-\imath) \\ (\cos - \sin\imath)^n & (\cos + \imath\sin)^n \end{pmatrix}\begin{pmatrix} \frac{1}{2\imath} & \frac{1}{2} \\ -\frac{1}{2\imath} & \frac{1}{2} \end{pmatrix} =$$

$$\frac{1}{2}\begin{pmatrix} (\cos - \sin\imath)^n + (\cos + \sin\imath)^n & (\cos - \sin)^n\imath - (\cos + \sin\imath)^n\imath \\ \imath(\cos + \sin)^n - \imath(\cos - \sin\imath)^n & (\cos - \sin\imath)^n + (\cos + \sin\imath)^n \end{pmatrix} = A^n$$

2 Exercise 2

Like in exercise 1 we begin by getting the eigenvalues and eigenvectors and the matrix P.

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \text{Eigenvalues of } A: \begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 1 = 0$$

$$\lambda_1 = 1 + \imath$$

$$\lambda_2 = 1 - \imath$$

$$\text{Eigenvector of } \lambda_1 \left[\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} 1+\imath & 0 \\ 0 & 1+\imath \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$v_1 = \begin{pmatrix} 1\imath \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 1 \\ -\imath \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 \\ \imath & -\imath \end{pmatrix} \Rightarrow P^{-1} = \frac{1}{\det(P)} \begin{pmatrix} -\imath & -1 \\ -\imath & 1 \end{pmatrix} = \frac{1}{-2\imath} \begin{pmatrix} -\imath & -1 \\ -\imath & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\imath} \\ \frac{1}{2} & -\frac{1}{2\imath} \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\imath} \\ \frac{1}{2} & -\frac{1}{2\imath} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \imath & -\imath \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\imath} \\ \frac{1}{2} & -\frac{1}{2\imath} \end{pmatrix} \begin{pmatrix} 1+\imath & 1-\imath \\ -1+\imath & -1-\imath \end{pmatrix} = \begin{pmatrix} 1+\imath & 0 \\ 0 & 1-\imath \end{pmatrix}$$

$$B^n = \begin{pmatrix} (1+\imath)^n & 0 \\ 0 & (1-\imath)^n \end{pmatrix}$$

$$A^n = PB^nP^{-1} = \begin{pmatrix} 1 & 1 \\ \imath & -\imath \end{pmatrix} \begin{pmatrix} (1+\imath)^n & 0 \\ 0 & (1-\imath)^n \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2\imath} \\ \frac{1}{2} & -\frac{1}{2\imath} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \imath & -\imath \end{pmatrix} \begin{pmatrix} \frac{(1+\imath)^n}{2} & \frac{(1+\imath)^n}{2\imath} \\ \frac{(1-\imath)^n}{2} & -\frac{(1-\imath)^n}{2\imath} \end{pmatrix} =$$

$$\frac{1}{2} \begin{pmatrix} (1+\imath)^n + (1-\imath)^n & (1+\imath)^n - (1-\imath)^n \\ \imath(1+\imath)^n - \imath(1-\imath)^n & (1+\imath)^n + (1-\imath)^n \end{pmatrix} = A^n$$