
Assignment/Problem Set 1

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1 Exercise 1

1 I shorten the notation by using \sin as a replacement for $\sin(x)$ and \cos as a replacement for $\cos(x)$. We use the binet formula to get the power of A^n . We first estimate the eigenvectors, then concatenate them into matrix P and use this matrix to compute the diagonal matrix B , which can be used to directly take the power for the matrix A . Afterwards we need to reverse the operations, which we had done on achieving B to get A^n .

$$\begin{aligned} A^n &= \begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix}^n, v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, (A - \lambda I)v = 0 \\ &\begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \\ &(\cos^2 - \lambda) + \sin^2 = 0 \\ &\lambda_1 = i \sin + \cos \\ &\lambda_2 = -i \sin + \cos \end{aligned}$$

Now we achieved the eigenvalues, so we can estimate the eigenvectors. The eigenvector λ_1

$$\begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix} - \begin{pmatrix} i \sin + \cos & 0 \\ 0 & i \sin + \cos \end{pmatrix} = \begin{pmatrix} i & \\ & 1 \end{pmatrix}$$

Symmetrically the vector for λ_2 is:

$$\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

The matrix P is just a concatenation of λ_1 and λ_2 .

$$P = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \tag{1}$$

To calculate an diagonal matrix, we use the equation $B = P^{-1}AP$ to transform A into a diagonal matrix.

$$\begin{aligned}
P^{-1} &= \frac{1}{\det(P)} \text{adj}(P) = \\
&\frac{1}{2i} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} = \begin{pmatrix} \frac{1}{2i} & \frac{1}{2} \\ -\frac{1}{2i} & \frac{1}{2} \end{pmatrix} \\
P^{-1}AP &= \begin{pmatrix} \frac{1}{2i} & \frac{1}{2} \\ -\frac{1}{2i} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} = \\
&\begin{pmatrix} \left(\frac{\cos}{2i} - \frac{\sin}{2} \right) i + \frac{\sin}{2i} + \frac{\cos}{2} & \left(\frac{\cos}{2i} - \frac{\sin}{2} \right) (-i) + \frac{\sin}{2i} + \frac{\cos}{2} \\ \left(-\frac{\cos}{2i} - \frac{\sin}{2} \right) i + -\frac{\sin}{2i} + \frac{\cos}{2} & \left(-\frac{\cos}{2i} - \frac{\sin}{2} \right) (-i) - \frac{\sin}{2i} + \frac{\cos}{2} \end{pmatrix} = \\
&\begin{pmatrix} \cos - i \sin & 0 \\ 0 & \cos + i \sin \end{pmatrix}
\end{aligned}$$

Now we can apply the power operation on the matrix.

$$B^n = \begin{pmatrix} (\cos - i \sin)^n & 0 \\ 0 & (\cos + i \sin)^n \end{pmatrix}$$

Now we can apply the matrix exponential onto A, since $A = PB^nP^{-1}$.

$$\begin{aligned}
&\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (\cos - i \sin)^n & 0 \\ 0 & (\cos + i \sin)^n \end{pmatrix} \begin{pmatrix} \frac{1}{2i} & \frac{1}{2} \\ -\frac{1}{2i} & \frac{1}{2} \end{pmatrix} = \\
&\begin{pmatrix} (\cos - i \sin)^n i & (\cos + i \sin)^n (-i) \\ (\cos - i \sin)^n & (\cos + i \sin)^n \end{pmatrix} \begin{pmatrix} \frac{1}{2i} & \frac{1}{2} \\ -\frac{1}{2i} & \frac{1}{2} \end{pmatrix} = \\
&\frac{1}{2} \begin{pmatrix} (\cos - i \sin)^n + (\cos + i \sin)^n & (\cos - i \sin)^n i - (\cos + i \sin)^n i \\ i(\cos + i \sin)^n - i(\cos - i \sin)^n & (\cos - i \sin)^n + (\cos + i \sin)^n \end{pmatrix} = A^n
\end{aligned}$$

After some basic mathematics procedures we get the simplified matrix as shown below.

$$A^n = \begin{pmatrix} \cos(n \cdot x) & \sin(n \cdot x) \\ -\sin(n \cdot x) & \cos(n \cdot x) \end{pmatrix} \quad (2)$$

2 Like in exercise 1 we begin by getting the eigenvalues and eigenvectors and the matrix P.

$$\begin{aligned}
A &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \text{Eigenvalues of } A : \begin{pmatrix} 1 - \lambda & 1 \\ -1 & 1 - \lambda \end{pmatrix} = (1 - \lambda)^2 + 1 = 0 \\
&\lambda_1 = 1 + i \\
&\lambda_2 = 1 - i \\
\text{Eigenvector of } \lambda_1, \lambda_2 &\left[\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} 1 + i & 0 \\ 0 & 1 + i \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \\
&v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} \\
&v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}
\end{aligned}$$

Since we computed the eigenvectors v_1 and v_2 , we continue by calculating the diagonal matrix B .

$$\begin{aligned}
 P &= \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \Rightarrow P^{-1} = \frac{1}{\det(P)} \begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} = \frac{1}{-2i} \begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{pmatrix} \\
 B &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{pmatrix} \begin{pmatrix} 1+i & 1-i \\ -1+i & -1-i \end{pmatrix} = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix} \\
 B^n &= \begin{pmatrix} (1+i)^n & 0 \\ 0 & (1-i)^n \end{pmatrix} \\
 A^n &= PB^nP^{-1} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} (1+i)^n & 0 \\ 0 & (1-i)^n \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \frac{(1+i)^n}{2} & \frac{(1+i)^n}{2i} \\ \frac{(1-i)^n}{2} & -\frac{(1-i)^n}{2i} \end{pmatrix} = \\
 &\quad \frac{1}{2} \begin{pmatrix} (1+i)^n + (1-i)^n & (1+i)^n - (1-i)^n \\ i(1+i)^n - i(1-i)^n & (1+i)^n + (1-i)^n \end{pmatrix} = A^n
 \end{aligned}$$

3 To solve these we do not use the Methods as in Exercise 1 or 2 because the Eigenvector of these Matrix is:

$$V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

With these vector it is not possible to make the transformations as shown at the previous paragraphs. After calculating a few powers of the Matrix we got these series:

$$\begin{aligned}
 A^2 &= \begin{pmatrix} a^2 & 2*a & 1 & 0 & 0 \\ 0 & a^2 & 2*a & 1 & 0 \\ 0 & 0 & a^2 & 2*a & 1 \\ 0 & 0 & 0 & a^2 & 2*a \\ 0 & 0 & 0 & 0 & a^2 \end{pmatrix} & A^3 &= \begin{pmatrix} a^3 & 3*a^2 & 3*a & 1 & 0 \\ 0 & a^3 & 3*a^2 & 3*a & 1 \\ 0 & 0 & a^3 & 3*a^2 & 3*a \\ 0 & 0 & 0 & a^3 & 3*a^2 \\ 0 & 0 & 0 & 0 & a^3 \end{pmatrix} \\
 A^4 &= \begin{pmatrix} a^4 & 4*a^3 & 6*a^2 & 4*a & 1 \\ 0 & a^4 & 4*a^3 & 6*a^2 & 4*a \\ 0 & 0 & a^4 & 4*a^3 & 6*a^2 \\ 0 & 0 & 0 & a^4 & 4*a^3 \\ 0 & 0 & 0 & 0 & a^4 \end{pmatrix} & A^5 &= \begin{pmatrix} a^5 & 5*a^4 & 10*a^3 & 10*a^2 & 5*a \\ 0 & a^5 & 5*a^4 & 10*a^3 & 10*a^2 \\ 0 & 0 & a^5 & 5*a^4 & 10*a^3 \\ 0 & 0 & 0 & a^5 & 5*a^4 \\ 0 & 0 & 0 & 0 & a^5 \end{pmatrix}
 \end{aligned}$$

After looking for a longer time at these equations It is totally obvious the Matrix has to be:

$$A^n = \begin{pmatrix} a^n & a^{-1+n} \cdot n & \frac{1}{2}a^{-2+n} \cdot (-1+n) \cdot n & \frac{1}{6}a^{-3+n} \cdot (-2+n) \cdot (-1+n) \cdot n & \frac{1}{24}a^{-4+n} \cdot (-3+n) \cdot (-2+n) \cdot (-1+n) \cdot n \\ 0 & a^n & a^{-1+n} \cdot n & \frac{1}{2}a^{-2+n} \cdot (-1+n) \cdot n & \frac{1}{6} \cdot a^{-3+n} \cdot (-2+n) \cdot (-1+n) \cdot n \\ 0 & 0 & a^n & a^{-1+n} \cdot n & \frac{1}{2}a^{-2+n} \cdot (-1+n) \cdot n \\ 0 & 0 & 0 & a^n & a^{-1+n} \cdot n \\ 0 & 0 & 0 & 0 & a^n \end{pmatrix}$$

2 Exercise 2

1 Computing $A^{-1}B$.

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 4 & 5 & 0 \\ 2 & 3 & 1 \\ 2 & 7 & -3 \end{pmatrix} = -\frac{1}{24} \begin{pmatrix} -16 & 15 & 5 \\ 8 & -12 & -4 \\ 8 & -18 & 2 \end{pmatrix}$$

$$AB = \frac{1}{12} \begin{pmatrix} 12 & 0 & -30 & 95 \\ 0 & 12 & 24 & -52 \\ 0 & 0 & 0 & -46 \end{pmatrix}$$

2 Computing CA^{-1}

$$CA^{-1} = \begin{pmatrix} 4 & 5 & 0 \\ 2 & 3 & 1 \\ 2 & 7 & 9 \\ -2 & 3 & 7 \end{pmatrix} \left(-\frac{1}{24}\right) \begin{pmatrix} -16 & 15 & 5 \\ 8 & -12 & -4 \\ 8 & -18 & 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ -12 & 27 & 0 \\ -14 & 24 & 1 \end{pmatrix}$$

3 Exercise 3

We want to proof the following equation:

$$\text{adj}(AB) = \text{adj}(B)\text{adj}(A) \quad (3)$$

It is already known that $A\text{adj}(A) = \det(A)I$, so

$$\text{adj}(A) = A^{-1}\det(A)I \quad (4)$$

Also it is known that

$$\det(AB) = \det(A)\det(B) \quad (5)$$

We substitute 4 into 3, by using 5.

$$\begin{aligned} \text{adj}(B)\text{adj}(A) &= \det(B)B^{-1}I\det(A)A^{-1}I = \\ \det(A)\det(B)B^{-1}IA^{-1}I &= \det(AB)(AB)^{-1} \end{aligned}$$

Using $C = AB$ we get:

$$\det(C)C^{-1} = \text{adj}(C) \Rightarrow \text{adj}(AB) = \text{adj}(A)\text{adj}(B)$$

4 Exercise 4

Assuming having a matrix A with dimensions of $n \times m$.

We assume that there exists two ranks, $\text{rowrank}(A)$ and $\text{colrank}(A)$.

$$A = \begin{pmatrix} a_{11} & \dots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1m} & \dots & a_{nm} \end{pmatrix}$$

If we modify A to A' by doing Gaussian elimination, A' will have exactly r non-zero rows, which will be denoted as $\text{rowrank}(A) = r$. By doing the transpose A'^T , we can still observe that the rowrank is r times non-zero rows. Since the columns in the transpose are the rows of the non-transpose, it leads to:

$$\text{rowrank}(A') = \text{rowrank}(A'^T) = \text{colrank}(A') = \text{colrank}(A'^T) = \text{rowrank}(A) = \text{colrank}(A)$$

I show that this behaviour is working as well for the square AA^T matrix. Since it is already proofed that the column rank is the row rank, we can use the basic formula for ranks : $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$.

$$\text{rank}(AA^T) \leq \min(\text{rank}(A), \text{rank}(A^T)) \Rightarrow \text{rank}(A) = \text{rank}(A^T) = \text{rank}(AA^T) \quad (6)$$

5 Exercise 5

To proof $\text{rank}(A^n) = \text{rank}(A^{n+1})$ we use recursion.

$$\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B)) \quad (7)$$

By the use of 7 we get the following recursion:

$$\begin{aligned} \text{rank}(A^{n+1}) &= \text{rank}(A^n A) \leq \min(\text{rank}(A^n), \text{rank}(A)) \\ \text{rank}(A^n) &= \text{rank}(A^{n-1} A) \leq \min(\text{rank}(A^{n-1}), \text{rank}(A)) \\ &\vdots \\ \text{rank}(AA) &= \min(\text{rank}(A), \text{rank}(A)) = \text{rank}(A) \end{aligned}$$

So after this recursion, it can be seen that $\text{rank}(A^{n+1}) = \text{rank}(A^n)$

6 Exercise 6

We use the Leibnitz formula to determinate the determinate.

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D) \det(A - BD^{-1}C) \quad (8)$$

From equation 8, we can rewrite the matrix X given as:

$$\begin{aligned} &\begin{pmatrix} 0 & x_1 & \dots & x_n \\ -x_1 & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ -x_n & a_{n1} & \dots & a_{nn} \end{pmatrix} \\ &\det(A) \det(0 - xA^{-1}(-x^T)) = \\ &\det(A) \det(x \frac{1}{\det(A)} \text{adj}(A)x^T) = \\ &\det(x * \text{adj}(A) * x^T) = |x * \text{adj}(A) * x^T| \end{aligned}$$

7 Exercise 7

a The measure matrix as the inner product is given as:

$$\begin{pmatrix} \langle 1, 1 \rangle & \langle 1, t \rangle & \langle 1, t^2 \rangle \\ \langle t, 1 \rangle & \langle t, t \rangle & \langle t, t^2 \rangle \\ \langle t^2, 1 \rangle & \langle t^2, t \rangle & \langle t^2, t^2 \rangle \end{pmatrix}$$

To fill in the values, we use the given formula for the inner product, which is:

$$\int_{-1}^1 f(t)g(t)dt$$

Now we fill in the measure matrix with the corresponding values:

$$M = \begin{pmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} \end{pmatrix} \quad (9)$$

b Using the measure matrix 9, we use $h = (1, -1, 1)^T$ and $p = (1, -4, -5)^T$.

$$\begin{aligned} N &= h^T M p = \\ (1, -1, 1) &\begin{pmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix} = 0 \end{aligned}$$

As it can be seen, the vectors h and p are orthogonal, so that $hp = 0$.