Assignment/Problem Set 5

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1.1 Exercise 3

Two diagonalizable matrices are similar if all their eigenvalues are equal, meaning that they have the exact same diagonal form. We firstly show that the eigenvalues of two similar matrices A and B need to be the same.

$$A = S^{-1}BS, B = Q\Lambda Q^{-1}, \Lambda \text{ is diagonal}$$
$$A = S^{-1}Q\Lambda Q^{-1}S = R\Lambda R^{-1}, R = S^{-1}Q$$
$$\Lambda = Q^{-1}BQ$$
$$B = Q\Lambda Q^{-1} \Rightarrow A = R\Lambda R^{-1} = RQ^{-1}BQR^{-1} = S^{-1}BS, S = QR^{-1}$$

As we have shown, A and B can both be diagonalized to a form Λ, in which they both have the same eigenvalues, which means they both can be decomposed into their diagonal form, using two different matrices U, V. Since Λ is a representation of the eigenvalues of A, B respectively, the eigenvalues are real. From here on we pick U, V so that both are unitary, meaning that:

$$A^* = A$$
$$A^* = (U\Lambda U^{-1})^* = (U^{-1*}\Lambda U^*) \neq A$$

The same goes for B, if U, V are unitary, $A = A^*$ and therefore A, B are unitary similar.

$$A = U\Lambda U^{-1}, B = V\Lambda V^{-1}$$
$$\Lambda = U^{-1}AU = V^{-1}BV = UAU^* = VBV^*$$

So we have shown that if we pick U, V to be unitary, A, B are unitary equivalent.

1.2 Exercise 13

First of all we verify the cauchy-schwarz inequality, let's assume having a map $(A, B) \mapsto tr(AB^*)$, the cauchy-schwarz inequality states that $\langle u, v\rangle^2 \leq \langle u, u\rangle \langle v, v\rangle$, so we get:

$$tr(AB^*)^2 \leq tr(AA^*)tr(BB^*)$$

Now since A is in our case hermitian and B is the identity, we get:

$$tr(A)^2 \leq tr(AA^*)tr(I_r I_r^*) = tr(AA^*)tr(I_r) = r * tr(AA^*)$$
$$\therefore \left(\sum_i^r \lambda_i\right)^2 \leq r\sum_i^r \lambda_i^2$$

Since r is the amount of nonzero eigenvalues in the diagonalized form of A, it is equivalent to the rank of A. We can rearrange the above equation to get:

$$r \geq \frac{tr(A)^2}{tr(A^2)}$$

This equation will resolve into an equality if $A = aI = aU^*U$, since:

$$tr(A)^2 \leq tr(AA^*) * tr(I_r I_r^*)$$
$$a * tr(I)^2 = a * tr(I) * tr(I_r I_r^*) =$$
$$ar^2 = arr = ar^2$$

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2.1 Exercise 1

Weyls theorem states that:

$$\lambda_1(B) + \lambda_k(A) \leq \lambda_k(A+B) \leq \lambda_n(B) + \lambda_k(A)$$

From here on we can simply subtract by $\lambda_k(A)$ and get:

$$\lambda_1(B) \leq \lambda_k(A+B) - \lambda_k(A) \leq \lambda_n(B)$$

Moreover we can extend this inequality by using $a \leq b \leq c = |b| \leq \max(a, c)$. In our case, this can be represented as:

$$|\lambda_k(A+B) - \lambda_k(A)| \leq \max(\lambda_1(B), \lambda_n(B))$$
$$= \rho(B)$$

2.2 Exercise 7

Asked already Miss Wang Fang, she didn't had an Idea how to solve this either.

2.3 Exercise 14

Once again we use Weyls theorem:

$$\lambda_i(A) = \lambda_i(B + A - B)$$
$$\lambda_i(B) + \lambda_1(A - B) \leq \lambda_i(B + A - B) = \lambda_i(A)$$
$$\text{Since } A - B > 0$$
$$\lambda_i(B) \leq \lambda_i(B) + \lambda_1(A - B) \leq \lambda_i(B + A - B) = \lambda_i(A)$$

So we can finally see that $\lambda_i(A) \geq \lambda_i(B)$ as required.

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3.1 Exercise 1

Let $A \in M_n$, positive semi definite (it is Hermitian)
Consider $p(t) = (x + ty)^* A(x + ty), t \in R$ Assume that $x^*Ax = 0$ We get:

$$p(t) = x^*Ax + t^2y^*Ay + t(y^*Ax + x^8Ay) =$$
$$0 + \underbrace{t^2y^*Ay}_{\geq 0} + t(y^*Ax + x^*Ay)$$

We can see that the term $t(y^*Ax + x^*Ay) \geq 0$, since A is positive definite and so $z^*Az \geq 0$, for any z.

Clearly $p(0) = 0$, therefore p has a minimum at $t = 0$.

$$\frac{\delta p}{\delta t} = 2ty^*Ay + y^*Ax + x^*Ay$$
$$\text{at } t = 0 : y^*Ax + x^*Ay = 0$$
$$\text{let } z = Ax$$
$$= y^*z + z^*y = 0$$

We choose $y = z$:

$$\frac{\delta p}{\delta t} = z^*z + z^*z = 2|z|^2 = 0 \Rightarrow z = 0, y = 0, Ax = 0$$
$$\Rightarrow y^*Ax = 0$$

3.2 Exercise 2

Assume any 2×2 principal submatrix of A:

$$B = \begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix}$$

Since A is positive semidefininte, it's determinate is ≥ 0. Furthermore all its principal sub-matrices are also positive semi definite so that $\det(B) \geq 0$. Recall that since B is positive definite it is also hermitian, so :

$$B = \begin{pmatrix} a_{ii} & a_{ij} \\ \overline{a_{ij}} & a_{jj} \end{pmatrix}$$
$$\det(B) = a_{ii}a_{jj} - |a_{ij}|^2 \geq 0 \Rightarrow a_{ii}a_{jj} \geq |a_{ij}|^2$$

Let the i th diagonal element be zero , $a_{ii} = 0$:

$$|a_{ij}|^2 \leq 0 \Rightarrow |a_{ij}| = 0 \qquad \forall 1 \leq j \leq n$$

Hence ith row of $A = 0$. Since A is hermitian, the ith column is also 0.

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4.1 Exercise 10

We know that any matrix can be decomposed into a $PLUQ$ form. In this form, we can easily see that the rank r are the first r rows which are having a non zero diagonal entry in the L matrix (or U, depending on how we define L and U). If we have the LU decomposition, we generally need to swap rows by using the matrix P and swap columns using the matrix Q. If we apply these two rotations onto A, we can simply pick the first r rows and r columns of A and use this submatrix, which is definitely invertible.

$$A = PLUQ$$
$$P^{-1}AQ^{-1} = LU$$

Moreover since we have principal sub matrices, we can set $P = Q$.

4.2 Exercise 11

In the case of having a classical adjoint, where $A > 0$ we get:

$$A\text{adj}(A) = |A|$$

Here we can already see that if A is invertible it is positive definite. For the more strict case that $A \geq 0$, we can assume having a small epsilon ϵ, which leads to having $A_\epsilon = A + \epsilon I$. By the same conclusion from above we have:

$$A_\epsilon \text{adj}(A_\epsilon) = |A_\epsilon|$$

Once again we implicitly assume that A is invertible. We have shown that if A_ϵ is positive semidefinite $adj(A_\epsilon)$ is semi-definite too and $\det A_\epsilon \geq 0$

4.3 Exercise 12

a The matrix A looks as the following:

$$A = \begin{pmatrix} 1 & r & r^2 & r^3 & \dots & r^{n-1} \\ r & 1 & r & r^2 & \dots & r^{n-2} \\ r^2 & r & \ddots & & \ddots & \vdots \\ r^3 & r^2 & \ddots & \ddots & r & r^2 \\ \vdots & & \ddots & r & 1 & r \\ r^{n-1} & r^{n-2} & \dots & r^2 & r & 1 \end{pmatrix}$$

From here on we can see that if we calculate any Minor A_{ij}, get for $|i - j| < 2$ a matrix of this form (we simply use any minor at the super or sub diagonal):

$$A_{1,2} = \begin{pmatrix} r & r & \dots \\ r^2 & 1 & \dots \\ r^3 & r & \dots \end{pmatrix}$$

Here we can see, that the rows and columns are still linearly independent from each other, but if we use any $|i, j| \geq 2$, we get:

$$A_{1,3} = \begin{pmatrix} r & 1 & \dots \\ r^2 & r & \dots \\ r^3 & r^2 & \dots \\ \vdots & \vdots & \vdots \\ r^{n-1} & r^{n-2} & \end{pmatrix}$$

As it can be seen any sub matrix which is generated using $|i - j| \geq 2$ will result in having a linear dependency between at least two columns, which would result in having non full rank, meaning that determinate is also zero.

b Assuming we calculate the determinant by the general formula, which is:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} M_{i,j} = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} M_{i,j}.$$

We choose to expand over the rows, beginning with $i = 1$:

$$D_{n+1} = (-1)^{1+1} D_n + (-1)^{1+2} r * det \begin{pmatrix} r & r & r^2 & r^3 & \dots & r^{n-2} \\ r^2 & 1 & r & r^2 & \dots & r^{n-3} \\ r^3 & r & 1 & & \ddots & \vdots \\ r^4 & r^2 & \ddots & \ddots & r & r^2 \\ \vdots & & \ddots & r & 1 & r \\ r^{n-1} & r^{n-3} & \dots & r^2 & r & 1 \end{pmatrix}$$

It can be seen that the determinant on the RHS has in it's first column a common factor, which is r, removing this factor leads to:

$$D_{n+1} = (-1)^{1+1} D_n + (-1)^{1+2} r * r * det \begin{pmatrix} 1 & r & r^2 & r^3 & \dots & r^{n-2} \\ r & 1 & r & r^2 & \dots & r^{n-3} \\ r^2 & r & 1 & & \ddots & \vdots \\ r^3 & r^2 & \ddots & \ddots & r & r^2 \\ \vdots & & \ddots & r & 1 & r \\ r^{n-2} & r^{n-3} & \dots & r^2 & r & 1 \end{pmatrix} = D_n - r^2 D_n$$

All other terms are not necessary since we have shown already in exercise a, that the minors if $|i - j| \geq 2$ are 0. Since the formula $D_{n+1} = D_n - r^2 D_n$ is recursive, we can simplify it into: $D_{n+1} = (1 - r^2) D_n$, where we see that again $D_n = (1 - r^2) D_{n-1}$ and so on until $D_1 = 1$, which means we apply $(1 - r^2)$ n times, resulting in $D_{n+1} = (1 - r^2)^n$ So we have shown that $D_{n+1} = D_n - r^2 D_n = (1 - r^2) D_n = (1 - r^2)^n$

c As already shown in paragraph b, we can calculate the prinipal minors $D_1, D_2, \dots, D_n$. As we can see in the matrix, the diagonal entries are all 1, meaning that $D_1 = 1$. From here on we calculate $D_2 = (1 - r^2) D_1$, which is $D_2 = (1 - r^2)$. Since $r \in (0, 1)$, $D_2 > 0$, which means that all the following determinants $D_3, \dots, D_n$ will be strictly larger than 0. We can conclude that A only has positive minors, therefore is positive definite.

4.4 Exercise 13

In Problem 12 we already have shown that the matrix A is symmetric, hence it is automatically triagonal and also hermitian (is real), thus it has only real eigenvalues, thus having only real entries. The inverse of the matrix A is also symmetric :

$$(A^T)^{-1} = (A^{-1})^T$$
$$A^T (A^{-1})^T = (A^{-1} A)^T = I$$

So we can determine the values for A^{-1}, using the property that if we use $A^{-1} = \frac{adj(A)}{det(A)}$, we know that the elements where $|i - j| \geq 2$ are zero, so we can easily calculate the entries.

$$AA^{-1} = I$$

$$A^{-1} = \begin{pmatrix} \frac{1}{1-r^2} & \frac{r}{r^2-1} & 0 & 0 & 0 \\ \frac{r}{r^2-1} & \frac{r^2+1}{1-r^2} & \frac{r}{r^2-1} & 0 & 0 \\ 0 & \frac{r}{r^2-1} & \frac{r^2+1}{1-r^2} & \frac{r}{r^2-1} & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \frac{r}{r^2-1} & \frac{1}{1-r^2} \end{pmatrix}$$

If we now multiply A^{-1} with $(1-r^2)$, we obtain the matrix which is searched for:

$$(1-r^2)A^{-1} = \begin{pmatrix} 1 & -r & 0 & 0 & 0 \\ -r & 1+r & -r & 0 & 0 \\ 0 & -r & \ddots & -r & 0 \\ 0 & 0 & -r & 1+r & -r \\ 0 & 0 & 0 & -r & 1 \end{pmatrix}$$

5 Gershgorin Exercise

We have given the following matrix and need to find out if it's diagonalizable:

$$A = \begin{pmatrix} 2 & 2^{-1} & 2^{-2} & \dots & 2^{-(n-1)} \\ \frac{2}{3} & 4 & \frac{2}{3^2} & \dots & \frac{2}{3^{n-1}} \\ \frac{3}{4} & \frac{3}{4^2} & 6 & \dots & \frac{3}{4^{n-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{n}{n+1} & \frac{n}{(n+1)^2} & \frac{n}{(n+1)^3} & \dots & 2n \end{pmatrix}$$

We firstly generally calculate the sum for the off diagonal elements, for a given row $p = 1, \dots, n$

$$\begin{aligned} \left(\sum_{i=1}^{n-1} \frac{p}{(p+1)^i} \right) &= \\ s_n &= \frac{p}{(p+1)} + \frac{p}{(p+1)^2} + \dots \frac{p}{(p+1)^{n-1}} \\ (p+1)s_n &= \frac{p(p+1)}{(p+1)} + \frac{p(p+1)}{(p+1)^2} + \dots \frac{p(p+1)}{(p+1)^{n-1}} \\ (p+1)s_n &= p + \frac{p}{(p+1)} + \frac{p}{(p+1)^2} + \dots \frac{p}{(p+1)^{n-2}} \\ (p+1)s_n &= p + s_n - \frac{p}{(p+1)^{n-1}} \\ (p+1)s_n - s_n &= p - \frac{p}{(p+1)^{n-1}} \\ ps_n &= p - \frac{p}{(p+1)^{n-1}} \\ s_n &= 1 - \frac{1}{(p+1)^{n-1}} \end{aligned}$$

We can see that all rows are bounded by 1. So we can assume that we have n distinct disks, which are all centered at $2p$ with radius 1, meaning they lie within $(2p-1; 2p+1), p = 1, \dots, n$. Moreover we can say that since the matrix A is clearly diagonally dominant and no disk encloses zero, it is invertible. We can see that since no disks intersect each other, we get n distinct eigenvalues, moreover since all circles are disjoint, we can only have real eigenvalues. Therefore the Jordan normal form has n distinct blocks, or in other words, it is diagonal. We have proven that the matrix A is diagonalizable.