Assignment/Problem Set 1

Heinrich Dinkel
ID: 1140339107
E-mail: heinrich.dinkel@sjtu.edu.com.cn

1 Exercise 1

1 I shorten the notation by using sin as a replacement for sin(x) and cos as a replacement for cos(x). We use the binet formula to get the power of A^n. We first estimate the eigenvectors, then concatenate them into matrix P and use this matrix to compute the diagonal matrix B, which can be used to directly take the power for the matrix A. Afterwards we need to reverse the operations, which we had done on achieving B to get A^n.

$$A^n = \begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix}^n, v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, (A - \lambda I)v = 0$$
$$\begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} =$$
$$(\cos^2 - \lambda) + \sin^2 = 0$$
$$\lambda_1 = \imath \sin + \cos$$
$$\lambda_2 = -\imath \sin + \cos$$

Now we achieved the eigenvalues, so we can estimate the eigenvectors. The eigenvector λ_1

$$\begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix} - \begin{pmatrix} \imath \sin + \cos & 0 \\ 0 & \imath \sin + \cos \end{pmatrix} = \begin{pmatrix} \imath \\ 1 \end{pmatrix}$$

Symmetrically the vector for λ_2 is:

$$\begin{pmatrix} -\imath \\ 1 \end{pmatrix}$$

The matrix P is just a concatenation of λ_1 and λ_2.

$$P = \begin{pmatrix} \imath & -\imath \\ 1 & 1 \end{pmatrix} \tag{1}$$

To calculate an diagonal matrix, we use the equation $B = P^{-1}AP$ to transform A into a diagonal matrix.

$$P^{-1} = \frac{1}{\det(P)} \text{adj}(P) =$$

$$\frac{1}{2\imath} \begin{pmatrix} 1 & \imath \\ -1 & \imath \end{pmatrix} = \begin{pmatrix} \frac{1}{2\imath} & \frac{1}{2} \\ -\frac{1}{2\imath} & \frac{1}{2} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} \frac{1}{2\imath} & \frac{1}{2} \\ -\frac{1}{2\imath} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix} \begin{pmatrix} \imath & -\imath \\ 1 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \left(\frac{cos}{2\imath} - \frac{\sin}{2}\right)\imath + \frac{\sin}{2\imath} + \frac{\cos}{2} & \left(\frac{cos}{2\imath} - \frac{\sin}{2}\right)(-\imath) + \frac{\sin}{2\imath} + \frac{\cos}{2} \\ \left(-\frac{\cos}{2\imath} - \frac{\sin}{2}\right)\imath + -\frac{\sin}{2\imath} + \frac{\cos}{2} & \left(-\frac{\cos}{2\imath} - \frac{\sin}{2}\right)(-\imath) - \frac{\sin}{2\imath} + \frac{\cos}{2} \end{pmatrix} =$$

$$\begin{pmatrix} \cos - \imath \sin & 0 \\ 0 & \cos + \imath \sin \end{pmatrix}$$

Now we can apply the power operation on the matrix.

$$B^n = \begin{pmatrix} (\cos - \imath \sin)^n & 0 \\ 0 & (\cos + \imath \sin)^n \end{pmatrix}$$

Now we can apply the matrix exponential onto A, since $A = PB^nP^{-1}$.

$$\begin{pmatrix} \imath & -\imath \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (\cos - \sin \imath)^n & 0 \\ 0 & (\cos + (\imath \sin)^n \end{pmatrix} \begin{pmatrix} \frac{1}{2\imath} & \frac{1}{2} \\ -\frac{1}{2\imath} & \frac{1}{2} \end{pmatrix} =$$

$$\begin{pmatrix} (\cos - \sin \imath)^n \imath & (\cos + \imath \sin)^n)(-\imath) \\ (\cos - \sin \imath)^n & (\cos + \imath \sin)^n \end{pmatrix} \begin{pmatrix} \frac{1}{2\imath} & \frac{1}{2} \\ -\frac{1}{2\imath} & \frac{1}{2} \end{pmatrix} =$$

$$\frac{1}{2} \begin{pmatrix} (\cos - \sin \imath)^n + (\cos + \sin \imath)^n & (\cos - \sin)^n \imath - (\cos + \sin \imath)^n \imath \\ \imath(\cos + \sin)^n - \imath(\cos - \sin \imath)^n & (\cos - \sin \imath)^n + (\cos + \sin \imath)^n \end{pmatrix} = A^n$$

After some basic mathematics procedures we get the simplified matrix as shown below.

$$A^n = \begin{pmatrix} \cos(n \cdot x) & \sin(n \cdot x) \\ -\sin(n \cdot x) & \cos(n \cdot x) \end{pmatrix} \tag{2}$$

2 Like in exercise 1 we begin by getting the eigenvalues and eigenvectors and the matrix P.

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \text{Eigenvalues of } A : \begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 1 = 0$$

$$\lambda_1 = 1 + \imath$$

$$\lambda_2 = 1 - \imath$$

$$\text{Eigenvector of } \lambda_1, \lambda_2 \left[\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} 1+\imath & 0 \\ 0 & 1+\imath \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$v_1 = \begin{pmatrix} 1\imath \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 1 \\ -\imath \end{pmatrix}$$

Since we computed the eigenvectors v_1 and v_2, we continue by calculating the diagonal matrix B.

$$P = \begin{pmatrix} 1 & 1 \\ \imath & -\imath \end{pmatrix} \Rightarrow P^{-1} = \frac{1}{\det(P)} \begin{pmatrix} -\imath & -1 \\ -\imath & 1 \end{pmatrix} = \frac{1}{-2\imath} \begin{pmatrix} -\imath & -1 \\ -\imath & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\imath} \\ \frac{1}{2} & -\frac{1}{2\imath} \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\imath} \\ \frac{1}{2} & -\frac{1}{2\imath} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \imath & -\imath \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\imath} \\ \frac{1}{2} & -\frac{1}{2\imath} \end{pmatrix} \begin{pmatrix} 1+\imath & 1-\imath \\ -1+\imath & -1-\imath \end{pmatrix} = \begin{pmatrix} 1+\imath & 0 \\ 0 & 1-\imath \end{pmatrix}$$

$$B^n = \begin{pmatrix} (1+\imath)^n & 0 \\ 0 & (1-\imath)^n \end{pmatrix}$$

$$A^n = PB^nP^{-1} = \begin{pmatrix} 1 & 1 \\ \imath & -\imath \end{pmatrix} \begin{pmatrix} (1+\imath)^n & 0 \\ 0 & (1-\imath)^n \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2\imath} \\ \frac{1}{2} & -\frac{1}{2\imath} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \imath & -\imath \end{pmatrix} \begin{pmatrix} \frac{(1+\imath)^n}{2} & \frac{(1+\imath)^n}{2\imath} \\ \frac{(1-\imath)^n}{2} & -\frac{(1-\imath)^n}{2\imath} \end{pmatrix} =$$

$$\frac{1}{2} \begin{pmatrix} (1+\imath)^n + (1-\imath)^n & (1+\imath)^n - (1-\imath)^n \\ \imath(1+\imath)^n - \imath(1-\imath)^n & (1+\imath)^n + (1-\imath)^n \end{pmatrix} = A^n$$

3 To solve these we do not use the Methods as in Exercise 1 or 2 because the Eigenvector of these Matrix is:

$$V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

With these vector it is not possible to make the transformations as shown at the previous paragraphs. After calculating a few powers of the Matrix we got these series:

$$A^2 = \begin{pmatrix} a^2 & 2*a & 1 & 0 & 0 \\ 0 & a^2 & 2*a & 1 & 0 \\ 0 & 0 & a^2 & 2*a & 1 \\ 0 & 0 & 0 & a^2 & 2*a \\ 0 & 0 & 0 & 0 & a^2 \end{pmatrix} \qquad A^3 = \begin{pmatrix} a^3 & 3*a^2 & 3*a & 1 & 0 \\ 0 & a^3 & 3*a^2 & 3*a & 1 \\ 0 & 0 & a^3 & 3*a^2 & 3*a \\ 0 & 0 & 0 & a^3 & 3*a^2 \\ 0 & 0 & 0 & 0 & a^3 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} a^4 & 4*a^3 & 6*a^2 & 4*a & 1 \\ 0 & a^4 & 4*a^3 & 6*a^2 & 4*a \\ 0 & 0 & a^4 & 4*a^3 & 6*a^2 \\ 0 & 0 & 0 & a^4 & 4*a^3 \\ 0 & 0 & 0 & 0 & a^4 \end{pmatrix} \qquad A^5 = \begin{pmatrix} a^5 & 5*a^4 & 10*a^3 & 10*a^2 & 5*a \\ 0 & a^5 & 5*a^4 & 10*a^3 & 10*a^2 \\ 0 & 0 & a^5 & 5*a^4 & 10*a^3 \\ 0 & 0 & 0 & a^5 & 5*a^4 \\ 0 & 0 & 0 & 0 & a^5 \end{pmatrix}$$

After looking for a longer time at these equations It is totally obvious the Matrix has to be:

$$A^n = \begin{pmatrix} a^n & a^{-1+n}\cdot n & \frac{1}{2}a^{-2+n}\cdot(-1+n)\cdot n & \frac{1}{6}a^{-3+n}\cdot(-2+n)\cdot(-1+n)\cdot n & \frac{1}{24}a^{-4+n}\cdot(-3+n)\cdot(-2+n)\cdot(-1+n)\cdot n \\ 0 & a^n & a^{-1+n}\cdot n & \frac{1}{2}a^{-2+n}\cdot(-1+n)\cdot n & \frac{1}{6}\cdot a^{-3+n}\cdot(-2+n)\cdot(-1+n)\cdot n \\ 0 & 0 & a^n & a^{-1+n}\cdot n & \frac{1}{2}a^{-2+n}\cdot(-1+n)\cdot n \\ 0 & 0 & 0 & a^n & a^{-1+n}\cdot n \\ 0 & 0 & 0 & 0 & a^n \end{pmatrix}$$

2 Exercise 2

1 Computing $A^{-1}B$.

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 4 & 5 & 0 \\ 2 & 3 & 1 \\ 2 & 7 & -3 \end{pmatrix} = -\frac{1}{24} \begin{pmatrix} -16 & 15 & 5 \\ 8 & -12 & -4 \\ 8 & -18 & 2 \end{pmatrix}$$

$$AB = \frac{1}{12} \begin{pmatrix} 12 & 0 & -30 & 95 \\ 0 & 12 & 24 & -52 \\ 0 & 0 & 0 & -46 \end{pmatrix}$$

2 Computing CA^{-1}

$$CA^{-1} = \begin{pmatrix} 4 & 5 & 0 \\ 2 & 3 & 1 \\ 2 & 7 & 9 \\ -2 & 3 & 7 \end{pmatrix} (-\frac{1}{24}) \begin{pmatrix} -16 & 15 & 5 \\ 8 & -12 & -4 \\ 8 & -18 & 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ -12 & 27 & 0 \\ -14 & 24 & 1 \end{pmatrix}$$

3 Exercise 3

We want to proof the following equation:

$$\text{adj}(AB) = \text{adj}(B)\text{adj}(A) \tag{3}$$

It is already known that $A\text{adj}(A) = \det(A)I$, so

$$\text{adj}(A) = A^{-1}\det(A)I \tag{4}$$

Also it is known that

$$\det(AB) = \det(A)\det(B) \tag{5}$$

We substitute 4 into 3, by using 5.

$$\begin{gathered} \text{adj}(B)\text{adj}(A) = \det(B)B^{-1}I\det(A)A^{-1}I = \\ \det(A)\det(B)B^{-1}IA^{-1}I = \det(AB)(AB)^{-1} \end{gathered}$$

Using $C = AB$ we get:

$$\det(C)C^{-1} = \text{adj}(C) \Rightarrow \text{adj}(AB) = \text{adj}(A)\text{adj}(B)$$

4 Exercise 4

Assuming having a matrix A with dimensions of $n \times m$.
We assume that there exists two ranks, rowrank(A) and colrank(A).

$$A = \begin{pmatrix} a_{11} & \dots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1m} & \dots & a_{nm} \end{pmatrix}$$

If we modify A to $A^{'}$ by doing Gaussian elimination, $A^{'}$ will have exactly r non-zero rows, which will be denoted as $\text{rowrank}(A) = r$. By doing the transpose $A^{'T}$, we can still observe that the rowrank is r times non-zero rows. Since the columns in the transpose are the rows of the non-transpose, it leads to:

$$\text{rowrank}(A^{'}) = \text{rowrank}(A^{'T}) = \text{colrank}(A^{'}) = \text{colrank}(A^{'T}) = \text{rowrank}(A) = \text{colrank}(A)$$

I show that this behaviour is working as well for the square AA^T matrix. Since it is already proofed that the column rank is the row rank, we can use the basic formula for ranks :$\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$.

$$\text{rank}(AA^T) \leq \min(\text{rank}(A), \text{rank}(A^T)) \Rightarrow \text{rank}(A) = \text{rank}(A^T) = \text{rank}(AA^T) \tag{6}$$

5 Exercise 5

To proof $\text{rank}(A^n) = \text{rank}(A^{n+1})$ we use recursion.

$$\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B)) \tag{7}$$

By the use of 7 we get the following recursion:

$$\begin{gathered} \text{rank}(A^{n+1}) = \\ \text{rank}(A^n A) \leq \min(\text{rank}(A^n), \text{rank}(A)) \\ \text{rank}(A^n) = \text{rank}(A^{n-1}A) \leq \min(\text{rank}(A^{n-1}), \text{rank}(A)) \\ \vdots \\ \text{rank}(AA) = \min(\text{rank}(A), \text{rank}(A)) = \text{rank}(A) \end{gathered}$$

So after this recursion, it can be seen that $\text{rank}(A^{n+1)}) = \text{rank}(A^n)$

6 Exercise 6

We use the Leibnitz formula to determinate the determinate.

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D) \det(A - BD^{-1}C) \tag{8}$$

From equation 8, we can rewrite the matrix X given as:

$$\begin{gathered} \begin{pmatrix} 0 & x_1 & \dots & x_n \\ -x_1 & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ -x_n & a_{n1} & \dots & a_{nn} \end{pmatrix} \\ \det(A) \det(0 - xA^{-1}(-x^T)) = \\ \det(A) \det(x \frac{1}{\det(A)} \text{adj}(A) x^T) = \\ \det(x * \text{adj}(A) * x^T) = |x * \text{adj}(A) * x^T| \end{gathered}$$

7 Exercise 7

a The measure matrix as the inner product is given as:

$$\begin{gathered} \{f(t) = a_0 + a_1 t + a_2 t^2 : a_0, a_1, a_2 \in R\} \\ 1 \rightarrow a_0 - a_1 = 1 - t^2 \\ t \rightarrow a_1 - a_2 = t - 1 \\ t^2 \rightarrow a_2 - a_0 = t^2 - t \end{gathered}$$

To get the measure matrix we concatenate:

$$1 \to \left(1, t, t^2\right) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$t \to \left(1, t, t^2\right) \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$t^2 \to \left(1, t, t^2\right) \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

The measure matrix:

$$\left(1, t, t^2\right) \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

b