
Assignment/Problem Set 3

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Since this Problemset was heavily time consuming, I decided to split the work with my classmate Alexander Schmitt.

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1.1 Exercise 2

Assume having an inner product of two eigenvectors v :

$$\begin{aligned}\langle v, v \rangle &= \langle Uv, Uv \rangle = \langle \lambda v, \lambda v \rangle = \lambda \bar{\lambda} \langle v, v \rangle \\ (1 - \lambda \bar{\lambda} \langle v, v \rangle) &= 0 \text{ since } \langle v, v \rangle \neq 0 \Rightarrow \lambda \bar{\lambda} = 1 \Rightarrow |\lambda|^2 = 1 \Rightarrow \lambda = 1\end{aligned}$$

Which shows that the eigenvalues of a given unitary matrix are 1.

1.2 Exercise 7

We need to show that $x^*x = y^*y, Ux = y \Rightarrow U$ is unitary.

$$\begin{aligned}0 &= 0 \\ 0 &= x0 \\ 0x^* &= x0x^* \\ 0x^* &= x(I - I)x^* \\ 0x^* &= x(U^{-1}U - I)x^*\end{aligned}$$

We can now observe that $H = U^{-1}U - I, H^* = H$.

$$\begin{aligned}H &= U^{-1}U - I = I - I \\ H^* &= I - I = U^{-1}U - I = U^*U - I\end{aligned}$$

We consider :

$$\begin{aligned}e^{i\theta}y &= q \\ (x+q)^* H (x+q) & \\ (x^*+q^*) H (x+q) & \\ = \overbrace{x^* H x}^{\text{Constant}} + x^* H q + q^* H x + \overbrace{q^* H q}^{\text{Constant}} &= 0 \\ &= x^* H q + q^* H x = 0 \\ &= x^* H e^{i\theta}y + (e^{i\theta}y)^* H x \\ x^* H e^{i\theta}y + (e^{i\theta}y)^* H x &= \begin{cases} \theta = 0 & x^* H y + (x^* H y)^* = x^* H y = 0 \text{ since it is real} \\ \theta = \frac{\pi}{2} & x^* H i y + (x^* H i y)^* = 2i (x^* H y) \\ \theta = \theta & e^{i\theta} (x^* H y) \end{cases}\end{aligned}$$

We can see that this identity will be zero for every $x, y \in C^n$ and all $\theta \in R$. We can choose for the vectors x, y at first $x = y \neq 0$, so we get:

$$x^* H x = 0 \Rightarrow H = 0$$

Moreover the case where x, y are not equal needs to be shown too $x \neq y \neq 0$. From the formula we assume having two different statements, using the hermitian property of H

$$x^* H y = 0 \tag{1}$$

$$x^* H^* y = 0 \tag{2}$$

If we combine the equations in 1, we get:

$$x^* H y = x^* H^* y$$

$$x^* H y - x^* H^* y = 0$$

$$x^* (H y - H^* y) = 0$$

$$x^* (H y - H y) = 0$$

$$x^* (H (y - y)) = 0 \Rightarrow H = 0$$

We can therefore conclude that H must be zero, independent of x, y .

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2.1 Exercise 4

We need to show that a householder transformation does preserve the vector length r , but can transform any vector $x \in R^n$ to $y \in R^n$, so that $x \neq y$.

$$r = ||x||$$

$$w = x - y$$

$$y = U_w x$$

$$y = (I - 2(x - y)(x - y)^T) x$$

$$y = \left(I - 2 \frac{(x - y)(x - y)^T}{||x - y||_2^2} \right) x$$

From here on we can observe, that this equation is an equality if $I - 2 \frac{(x - y)(x - y)^T}{||x - y||_2^2} = 1$. This means that $2 \frac{(x - y)(x - y)^T}{||x - y||_2^2} = 0$. We can follow that this equation is only zero if and only if $x = y$. So we can conclude that the householder transformation does map a vector x with length r to a vector y with the same length. We need to show that the length still will be preserved:

$$||Ux||^2 = |x|^2$$

$$||Ux||^2 = (Ux)^*(Ux) = x^* U^* U x = x^* x = ||x||^2$$

Which means that the transformation of U does preserve the length, which means that $Ux = y$ will preserve r , so that $|x| = |y| = r$.

2.2 Exercise 8

If A and B are unitary equivalent then :

$$\exists U : U^* = U^{-1}, U^* A U = B$$

We need only to focus on these properties. Indeed it is easy to find two matrices that satisfy

$$\sum_{i,j=1}^n |b_{ij}|^2 = \sum_{i,j=1}^n |a_{ij}|^2.$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$$

We can see that $(U^*AU)^* = U^*AU$, but $B^* \neq B$, so we can see that these two matrices are holding the theorem $\sum_{i,j=1}^n |b_{ij}|^2 = \sum_{i,j=1}^n |a_{ij}|^2$, but fail to be unitary equivalent.

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3.1 Exercise 7

In this exercise we need to show that not every square matrix in the complex plane can be decomposed into an upper triangular form. In this exercise we need only to show that there exists one eigenvector x so that $x^T x \neq 0$. Assuming we can decompose the matrix into a triangular form of $Q^T T Q$

$$T = \begin{pmatrix} \lambda_1 & x & x & x \\ & \lambda_2 & x & x \\ & & \ddots & x \\ & & & \lambda_n \end{pmatrix}$$

We pick a y , which is :

$$y = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

We get:

$$\begin{aligned} Ax &= \lambda x \\ Q^T T Q x &= \lambda x \\ T Q x &= \lambda Q x \\ T y &= \lambda y \\ Q x &= y y^T y = (Q x)^T Q x = x^T x \neq 0 \end{aligned}$$

As we have shown, there exists one nonzero vector y , so that $x^T x \neq 0$

3.2 Exercise 8

We need to show that given a complex orthogonal matrix Q , having an eigenvector x , with corresponding eigenvalues $\lambda \neq \pm 1$, then we can follow that $x^T x = 0$.

$$\begin{aligned} Qx &= \lambda x \\ (Qx)^T Qx &= (Qx)^T x = \\ x^T Q^T Qx &= (Qx)^T \lambda x = \\ x^T x &= \lambda^2 x^T x = \\ x^T x (\lambda^2 - 1) &= \\ 0 &= x^T x (\lambda^2 - 1) \end{aligned}$$

As we can see since $\lambda \neq \pm 1$, we can conclude that $x^T x = 0$.

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4.1 Exercise 1

a We can see that if two matrices commute they can be simultaneously diagonalized. So we try to diagonalize $C = T^{-1}ABT$, by applying the diagonalization with one known matrix T .

$$\begin{aligned} C &= AB = BA \\ C &= T^{-1}ATT^{-1}BT = \\ C &= T^{-1}ABT = \end{aligned}$$

As we can see we can diagonalize AB by using C , so if we successfully diagonalize both matrices, we can read out the eigenvalues at the diagonal. Moreover, since $\det(AB) = \det(A)\det(B)$, which means that our eigenvalues can be simply multiplied. The only difference in the notation of this exercise is the permutation. Assume having a geometric multiplicity $G(\lambda) > 1$, so that we have multiple blocks within the matrix C . Then we can write out the multiplication of the eigenvalues, if both A and B are diagonalized:

$$C = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \alpha_n \end{pmatrix} \begin{pmatrix} \beta_{i_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \beta_{i_n} \end{pmatrix}$$

Since we have at least a block diagonal matrix, we can permute the indices of $T^{-1}BT$, since the geometric multiplicity only describes how many blocks we have and what size, but not how they are organized. Having said that, we could also say that $T^{-1}AT$, has permuted blocks, which it surely can have, but in this exercise it is not required.

b A polynomial in two variables can be written in its bilinear as:

$$p(A, B) = \begin{pmatrix} 1 & A & A^2 \\ B & BA & BA^2 \\ B^2 & B^2A & B^2A^2 \end{pmatrix}$$

From this point on, if we want to calculate the eigenvalues, we need to calculate the eigenvalues of each sub block in matrices A, B respectively. Since we assume having already the eigenvalues of A, B , we can rewrite the eigenvectors in diagonal form for A, B :

$$P^{-1}AP = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \alpha_n \end{pmatrix} P^{-1}BP = \begin{pmatrix} \beta_{i_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \beta_{i_n} \end{pmatrix}$$

So when calculating A^2, B^2 , we can simply square the eigenvalue respectively, giving us the following form:

$$p(A, B) = \begin{pmatrix} 1 & p_A(\alpha) & p_A(\alpha^2) \\ p_B(\beta) & p_{BA}(\alpha\beta) & p_{BA}(\alpha^2\beta) \\ p_B(\beta^2) & p_{BA}(\alpha\beta^2) & p_{BA}(\alpha^2\beta^2) \end{pmatrix}$$

We can therefore write out the characteristic polynomials:

$$p(A, B) = \begin{pmatrix} 1 & \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} & \begin{pmatrix} \alpha_1^2 \\ \vdots \\ \alpha_n^2 \end{pmatrix} \\ \begin{pmatrix} \beta_{i_1} \\ \vdots \\ \beta_{i_n} \end{pmatrix} & \begin{pmatrix} \alpha_1 \beta_{i_1} \\ \vdots \\ \alpha_n \beta_{i_n} \end{pmatrix} & \begin{pmatrix} \alpha_1^2 \beta_{i_1} \\ \vdots \\ \alpha_n^2 \beta_{i_n} \end{pmatrix} \\ \begin{pmatrix} \beta_{i_1}^2 \\ \vdots \\ \beta_{i_n}^2 \end{pmatrix} & \begin{pmatrix} \alpha_1 \beta_{i_1}^2 \\ \vdots \\ \alpha_n \beta_{i_n}^2 \end{pmatrix} & \begin{pmatrix} \alpha_1^2 \beta_{i_1}^2 \\ \vdots \\ \alpha_n^2 \beta_{i_n}^2 \end{pmatrix} \end{pmatrix} = p(\alpha_1, \beta_{i_1}), \dots, p(\alpha_n, \beta_{i_n})$$

As we can see, using the property that both B and A are diagonalizable, we can show that $p(A, B) = p(\alpha_1, \beta_{i_1}), \dots, p(\alpha_n, \beta_{i_n})$.

c As already seen in the first two paragraphs, the assumption of being commutative is not necessary to make these theorems hold. For example:

$$\begin{aligned} & P^{-1}APP^{-1}BP \\ &= \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \alpha_n \end{pmatrix} \begin{pmatrix} \beta_{i_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \beta_{i_n} \end{pmatrix} \end{aligned}$$

Which will conclude even if $BA \neq AB$, since the multiplication between the two matrices is commutative ($\alpha_1 \beta_{i_1} = \beta_{i_1} \alpha_1$).

4.2 Exercise 2

If we assume having a matrix T which is similar to A , we can write out T as:

$$T = \begin{pmatrix} t_{11} & x & x & x \\ & t_{22} & x & x \\ & & \ddots & x \\ & & & t_{nn} \end{pmatrix}$$

We can directly see that the rank of T is equal to its nonzero diagonal components, which means that $r(T) = r(A)$. Since $A \sim T$, the eigenvalues of A are equally these of T , so $\sigma(A) = \sigma(T)$. We can see that these eigenvalues are stored in T , where the diagonal elements are so :

$$T = \begin{pmatrix} \lambda_{11} & x & x & x \\ & \lambda_{22} & x & x \\ & & \ddots & x \\ & & & \lambda_{nn} \end{pmatrix}$$

So we can see that the rank is not less than the number of nonzero eigenvalues, since in this case, the rank and the eigenvalues are both stored in the diagonal. Having a zero in the diagonal, will result in having an eigenvalue of zero and a rank reduction.

4.3 Exercise 5

In this exercise we can assume that A is full rank, since it has n distinct eigenvalues and dimensionality of $n \times n$. First we show that the trace is equal to the sum of eigenvalues:

$$\begin{aligned} A &= Q^* T Q \\ Q^* A Q &= T \\ \text{tr}(Q^* A Q) &= \text{tr}(T) \\ \text{tr}(Q^* A Q) &= \sum_i^n \lambda_i \Rightarrow \text{tr}(A) = \sum_i^n \lambda_i \end{aligned}$$

Now when we apply matrix power for the equation, we can see that since the matrix T is upper triangular, the power operation affects the diagonal, we get the following results:

$$\begin{aligned} A^k &= (Q^* T Q)^k \\ \text{tr}(A^k) &= \text{tr}((Q^* T Q)^k) \\ &= \text{tr}(Q^* T^k Q) \\ &= \text{tr}(T^k Q^* Q) \\ &= \text{tr}(T^k) \\ &= \sum_i^n \lambda_i^k \end{aligned}$$

4.4 Exercise 6

The LU decomposition of the resulting matrix $aA + bB = C$ are:

$$C = \begin{pmatrix} a - 2b & b & 2b \\ -b & 2a - 2b & -b \\ b & b & 3a + b \end{pmatrix}$$

So we calculated the eigenvalues (since it is very messy calculation from here on, we dropped this step to show that the calculation is working out) by using $\det(C - \lambda I) = 0$ and get:

$$\sigma(C) = \{a - 2b, 2a - 2b, 3a + b\}$$

The eigenvalues of $\sigma(AB)$ are the following:

$$\begin{aligned} \lambda_1 &= -3 \\ \lambda_2 &= -2\sqrt{2} \\ \lambda_3 &= 2\sqrt{2} \end{aligned}$$

4.5 Exercise 9

In this exercise we need to use induction to proof the following for the case $k = 1$:

$$\begin{aligned} AX &= XB \\ A(AX) &= A(XB) \\ A(AX) &= (AX)B \\ A^2X &= XBB \\ A^2X &= XB^2 \end{aligned}$$

Moreover we can see in the case of $k = n$:

$$\begin{aligned}
AX &= XB \\
A^{n-1}(AX) &= A^{n-1}(XB) \\
A^n X &= A^{n-2}(AX)B \\
A^n X &= A^{n-2}XB^2 \\
&\vdots \\
A^n X &= XB^n
\end{aligned}$$

And lastly we show for the case of $k = n + 1$.

$$\begin{aligned}
A^n(AX) &= A^n(XB) \\
A^{n+1}X &= A^{n-1}A(XB) \\
&\vdots \\
A^{n+1}X &= XB^{n+1}
\end{aligned}$$

Next up we need to proof that the characteristic polynomials are equal:

$$\begin{aligned}
p(A)X &= \sum_{k=0}^n a_k A^k X = \\
\sum_{k=0}^n a_k B^k X &= X \sum_{k=0}^n a_k B^k = Xp(B)
\end{aligned}$$

We use Caylay-Hamilton to see that, $p_A(A) = 0$, so that we can conclude $p_A(A)X = 0$, which is equal to our previous calculations:

$$\begin{aligned}
p_A(A)X &= 0 = Xp_A(B) \\
p_A(B) &= (B - \lambda_1 I) \dots (B - \lambda_n I)
\end{aligned}$$

Since $p_A(B)$, will be zero if and only if the eigenvalues of A and B intersect, which won't happen since our assumption tells us that $\sigma(A) \cap \sigma(B) = \emptyset$. We have shown the required proof.

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5.1 Exercise 8

We need to prove that the matrices $H(A)$ and $S(A)$ commute if the matrix A is normal.

$$\begin{aligned}
AA^* &= (H(A) + S(A))((H(A) + S(A)))^* \\
&= H(A) + (S(A)H(A))^* + S(A)^* \\
&= H(A) + (H(A)^*S(A)) + S(A)^* \\
&= (H(A)^* + H(A))(S(A)S(A)^*) \\
&= H(A)^* + (S(A)H(A)) + S(A)^* \\
&= (H(A)^* + S(A))(S(A)^* + H(A)) \\
&= H(A)^* + (S(A)^*S(A)) + H(A) \\
&= (H(A) + S(A))^* + (H(A) + S(A)) \\
&= A^*A
\end{aligned}$$

We have shown that A is normal if $S(A)$ and $H(A)$ commute.

5.2 Exercise 10

We need to show that A symmetric $\Leftrightarrow A$ is normal. To show that if A is symmetric it is normal we can point out:

$$\begin{aligned} A &= A^T = \\ AA &= AA^T \\ A^T A &= AA^T \end{aligned}$$

To show the other side, we use the property that $H(A)$ is hermitian.

$$\begin{aligned} H(A)x &= \lambda_1 x \\ S(A)x &= \lambda_2 x \end{aligned}$$

We can decompose $H(A)$ into being diagonal, since it is hermitian:

$$\begin{aligned} (\alpha_1, \dots, \alpha_2) &\text{ Is the eigenvector for } H(A) \\ H(A)\alpha_i &= \hat{\lambda}_i \alpha_i \\ S(A)\alpha_i &= \hat{\lambda}_i \alpha_i \end{aligned}$$

The hat operator denotes that the eigenvalue can also occur multiple times. We get the decomposition:

$$\begin{aligned} U &= (\alpha_1, \dots, \alpha_2) \\ U^*AU &= U^*HU + U^*SU = \Phi \end{aligned}$$

Which only shows that Φ is diagonal.

$$\begin{aligned} A &= U^*\Phi U \\ A^* &= U^*\Phi U \Rightarrow A^*A = AA^* \end{aligned}$$

5.3 Exercise 14

$$\begin{aligned} A^*A &= AA^* \\ A &= U^*\Lambda U \\ A^* &= U^*\Lambda U \\ A^* &= A \Rightarrow A = A^T \end{aligned}$$

5.4 Exercise 31

Here we simply assume having the two similar matrices A, B as:

$$\begin{aligned} A &= U^*\Phi U^* \\ B &= V^*\Omega V \\ A \sim B &\Leftrightarrow \Phi \sim \Omega \end{aligned}$$

As both are diagonally similar matrices, we can easily pick a permutation matrix P , so that $\Phi = \Omega$.