Assignment/Problem Set 2

Heinrich Dinkel
ID: 1140339107
E-mail: heinrich.dinkel@sjtu.edu.cn

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1.1 Exercise 2

Assume having a square matrix A, which we want to maximize. $\max A$. We assume that A is a symmetric matrix. If A is non-zero, we can extract eigenvectors out of it: $Ax = \lambda x$. We get:

$$Ax = \lambda x$$
$$x^T Ax = x^T(Ax) = x^T(\lambda x) = \lambda x^T x = \lambda \sum_i^n |x_i|^2$$

To maximize the equation given, we need to maximize $|\lambda|$, since the summation in the second term adds up to one. So one can see that by maximizing λ, we maximize A.

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2.1 Exercise 5

A is called idempotent if $A^2 = A$. Show that each eigenvalue of an idempotent matrix is either 0 or 1. Using this property, we can show:

$$\begin{aligned} \lambda x = & \quad Ax \\ = & \quad A^2x \\ = & \quad A(Ax) \\ = & \quad A(\lambda x) \\ = & \quad \lambda(Ax) \\ = & \quad \lambda(\lambda x) \\ = & \quad \lambda^2 x \end{aligned}$$

Since $\lambda^2 x$ equals to Ax, we can compute the eigenvalues:

$$Ax - \lambda x = 0$$
$$\lambda^2 x - \lambda x = 0$$
$$x(\lambda^2 - \lambda) = 0$$
$$\rightarrow \lambda_1 = 1 \ \lambda_2 = 0$$

So we can see that the eigenvalues are either zero or one, as required.

2.2 Exercise 6

Here we use the same procedure as in Exercise 5.

$$\lambda x = A^q x$$
$$\lambda x = \underbrace{AA...AA}_{q} x$$
$$\lambda x = \underbrace{AA...AA}_{q-1}(Ax)$$
$$\lambda x = \underbrace{AA...AA}_{q-1}(\lambda x)$$
$$\lambda x = \lambda \underbrace{AA...AA}_{q-2} Ax$$
$$\vdots$$
$$\lambda x = \lambda^q x$$

Now since $A^q = 0$, we can solve the equation $A^q x = \lambda^q x$, for any $x \neq 0$.

$$A^q x = \lambda^q x \rightarrow \lambda = 0$$

We have shown that all the eigenvalues in a positive nilpotent matrix are zero.

An example for a nilpotent matrix is the following:

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

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3.1 Exercise 4

Given that $A \in M_n$ and $A_i = \text{adj}(A)$, we want to show that equation 1 holds.

$$\frac{d}{dt} p_A(t) = \sum_i^n p_{A_t}(t) \tag{1}$$

Assuming our characteristic polynomial $B = (tI - A)$ and $b_{ii} = (t - a_i i)$, we get the following equations:

$$\det(B) = \sum_i^n (-1)^{i+i} b_{ii} A_i$$
$$= \sum_i^n b_{ii} A_i$$
$$\text{taking derivative} \quad \frac{d}{dt}\det(B) = \frac{d}{dt}\sum_i^n (t - a_{ii}) A_i$$
$$\frac{d}{dt}\det(B) = \sum_i^n A_i = \frac{d}{dt} p_A(t)$$
$$p_{A_i}(t) = \sum (-1)^{2i} a_{ii} A_i^{'} = A_i$$

This shows that the determinant of B, which is the characteristic polynomial of A, is $\sum_i^n p_{A_i}(t)$.

3.2 Exercise 6

We want to proof that $\text{rank}(A - \lambda I) = n - 1$. The root of the characteristic polynomial is 0. We can see that $\frac{d}{dt}p_A(t)$ at $t = \lambda$ is non-zero, since it only evolves calculating the principal sub matrix of A. From there on we can follow:

$$\frac{d}{dt}p_A(\lambda) = \sum_{i}^{n} A_i \neq 0$$
$$\rightarrow \sum_{i=1}^{n} p_{A_i}(\lambda) \neq 0$$
$$\sum_{i=1}^{n} p_{A_i}(\lambda) \neq 0$$
$$\exists A_i \neq 0 \rightarrow \text{rank}(A_i) = n - 1$$

The submatrix rank follows from the column/row rank independence theorem. If one removes a row and a column from an n rank matrix, the submatrix needs to have n-1 rank, because the columns and rows are independent.

From here on we can follow, that $\text{rank}(A - \lambda I) = n - 1$, since $\frac{d}{dt}p_A(\lambda) \neq 0$. The converse is of course not true, as seen in example 1.2.7b. Consider the matrix:

$$\begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \end{pmatrix} \tag{2}$$

As it can be seen, its characteristic polynomial is 1, yet it's rank is still n and not $n - 1$.

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4.1 Exercise 5

If $A \in M_n$ and has distinct eigenvalues, show that if $AB = BA$, where $B \in M_n$, B is a polynomial of degree at most $n - 1$.

4.2 Exercise 6

If A is diagonalizable, which means that $A^{'} = P^{-1}AP$, so that A'' is diagonal. Since $A^{'}$ is similar to A, both share similar eigenvalues. That said, the characteristic polynomial of $A^{'}$ will have at least one eigenvalue, except A would be the zero matrix. We calculate:

$$A^{'} = \begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix}$$
$$p_A(A) = \det(A - A) = \begin{pmatrix} a_{11} - a_{11} & & & \\ & a_{22} - a_{22} & & \\ & & \ddots & \\ & & & a_{nn} - a_{nn} \end{pmatrix} = 0$$

We have shown that if a matrix A is diagonalizable, the characteristic polynomial with respect to itself is 0.

4.3 Exercise 7

We show that every diagonalizable matrix has a square root. We assume that a matrix A can be decomposed into its diagonal form D by using a matrix $Q \times Q = D$, which is again a square root.

$$AA = B$$
$$A = P^{-1}DP$$
$$QQ = D$$
$$(P^{-1}QP)(P^{-1}QP) = P^{-1}Q(PP^{-1})QP = P^{-1}QQP = P^{-1}DP = A$$

We have shown that every Matrix in M_n which is diagonalizable has a square root.

4.4 Exercise 12

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5.1 Exercise 1