Assignment/Problem Set 1

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1 Exercise 1

Prove that these facts, needed in the proof of Theorem 2:

a If U is invertible and upper triagonal, then $U\,adj(U) = I\det(U)$. It is now necessary to show that $adj(U)$ does not change the order of elements within the matrix. Via definition $adj(U) = C^T$, where $C_{ji} = (-1)^{i+j} A_{ij}$. A defines the sub matrix which will be generated by removing the ith row and the jth column. Moreover U has zero entries for every $i > j$. Since the entries on the diagonal of C are the same as the ones of U except that at the ith row. Suppose we remove the ith row, so that the new entry at that spot has to be $u_{i+1,i}$. But since A is upper triagonal and $i+1 > i$, this guarantees to have at least one 0 entry on the diagonal of A. This results in $det(A_{ij}) = 0$ for every $i > j$. Thus C^T is upper triagonal, thus U^{-1} is upper triagonal.

b The logic of (a) can be applied in this proof. Just some adjustment for the variables need to be done. Determinates of submatrices are also here zero, which leads to a zero entry above the diagonal where $j > i$.

c Assume having an upper triagonal matrix U. Since the general matrix multiplication of row r and column c between one and the same matrix U multiplies the $u_{ik}u_{kj}$ element, it needs to be shown that this result is zero for every $i > j$.

- If $i > k$ then $u_{ik} = 0$ and therefore $u_{ik}u_{kj} = 0$
- If $k > j$ then $u_{kj} = 0$ and therefore $u_{ik}u_{kj} = 0$

To conclude the proof, the resulting matrix will be zero if $i > k$ or the other case if $k > j$. Both cases are covered if $i > j$.

2 Exercise 2

If a matrix exists that the LU decomposition is not unique, then this means that $A = LU = \hat{L}\hat{U}$ Now that would mean that $L^{-1}\hat{L} = U^{-1}\hat{U}$. Since L and U are lower and upper triagonal, there can only be one possibility, that this equation holds, if both generate the identity matrix I. This is the case where $L = \hat{L}$ and $U = \hat{U}$.

3 Exercise 3

If A is singular, it would lead to linear dependence, which means that a row will nullify or cancel out another row. In the formulations, the variable $a_{pi}i$ will result in being zero, if linear dependence occurs. In any other case where $a_{pi}i \neq 0$, we will get a valid result.

4 Exercise 4

As already shown before, if L is lower triagonal, then $adj(L)$ is also lower triagonal. The equation $det(L)I = Ladj(L)$ holds, which means that L is nonsingular, if $det(L)$ is nonzero, whereas otherwise, the diagonal would result in being zero and therefore being singular.

5 Exercise 6

The problem displayed in this task is that the matrix A does not have any LU decomposition. One can show that there is no possibility that both, a lower triagonal and an upper triagonal matrix will be generated.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \tag{1}$$

To display this problem, one could try to find any possibility in which the $1, 1$ entry will be generated. Namely there are:

$$L = \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} U = \begin{pmatrix} 0 & b \\ 1 & d \end{pmatrix} \tag{2}$$

Here one can clearly see that this is the only valid $A = LU$ decomposition, but this leads to a contradiction, since the Upper Triagonal matrix U has $u_{11} = 0$, which means that this matrix is not a upper triagonal one.

6 Exercise 7

In this exercise we need to write down the row and the column version of the doolittle algorithm.

a Assuming having an matrix, which can be LU decomposed and $a_{i,i} \neq 0$. For the k-th row, we obtain the L and U elements by:

$$l_{k,n} \leftarrow -\frac{a_{k,n}^{(n-1)}}{a_{n,n}^{(n-1)}} \text{ if } k \geq n$$

$$u_{k,n} \leftarrow a_{k,n} - l_{k,n} \text{ if } k < n$$

b The same decomposition as in 6 can be done, but by using the inverse indices.

$$l_{n,k} \leftarrow -\frac{a_{n,k}^{(n-1)}}{a_{n,n}^{(n-1)}} \text{ if } k \leq n$$

$$u_{n,k} \leftarrow a_{n,k} + l_{n,k} \text{ if } k > n$$

7 Exercise 8

In this exercise an algorithm was written to solve equations in the form of $UU^{-1} = I$. I use the nice property of the upper triagonal matrix, that the dot products when calculating the k, k elements is cancelling every term out except the diagonal ones.

$$(u_{k,k})^{-1} = \frac{1}{u_{k,k}} \tag{3}$$

The diagonals can be straight forwardly calculated. The whole algorithm will then calculate the next diagonal elements. To calculate these we need to use the already obtained k, k elements.

The algorithm works as follows, assuming we iterate over k iterations.

1. Calculate the diagonal $(u_{k,k})^{-1} = \dfrac{1}{u_{k,k}}$
2. Calculate the next level diagonal $(u_{k,p})^{-1} = -\dfrac{\sum_{i=k+1}^{p}\left(u_{k,i}(u_{i,p})^{-1}\right)}{u_{k,k}}$
3. Repeat 1. and 2. until $(u_{1,k})^{-1}$ is calculated

The algorithm finds a result in $O(k^2)$

8 Exercise 12

We need to proof that A has an LU decomposition.

$$A = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} = LU = \begin{pmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} =$$
$$l_{11}u_{11} = 0$$
$$l_{11}u_{12} = a$$
$$l_{21}u_{11} = 0$$
$$l_{21}u_{12} + l_{22}u_{22} = b$$

We can see that the equation system cannot be directly solved. Since there is only one fixed parameter, namely $u_{11} = 0$, I did plug in some reasonable numbers into the equations to get an intuition about the final matrices.

$$l_{11}u_{12} = a \rightarrow u_{11} = 0$$
$$l_{21}u_{12} + \ldots = b \rightarrow u_{12} \wedge l_{11} = a, \rightarrow u_{12} = a, l_{11} = 1$$
$$l_{21} = 0$$
$$u_{22} \wedge l_{22} = b \rightarrow u_{22} = b, l_{22} = 1$$
$$L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} U = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$$

As it can be seen, this factorization is not unique e.g. one can interchange the L and U elements in l_{22} with u_{22} and get a new matrix. Now I will proof that assuming a unit matrix, it doesn't change the behaviour of the system.

$$A = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} = LU = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} =$$
$$u_{11} = 0$$
$$u_{12} = a$$
$$l_{21}u_{11} = 0$$
$$l_{21}u_{12} + u_{22} = b \rightarrow u_{22} = b - l_{21}a \rightarrow l_{21} = 0$$

We can see that assuming a unit lower triagonal matrix, will make all the values within the matrix unique and we obtain only one solution. It should be noted that as long $b \neq na$, we can even obtain any solution, since in the case of $b = na$, the matrix would be singular.

9 Exercise 13

We need to proof that A has an LU decomposition.

$$A = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} = LU = \begin{pmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}$$

Which leads to following linear equations:

$$\begin{aligned} l_{11}u_{11} &= 0 \\ l_{11}u_{12} &= 0 \\ l_{21}u_{11} &= a \rightarrow l_{11} = 0, l_{21} \wedge u_{11} = a \\ l_{21}u_{12} + l_{22}u_{22} &= b \rightarrow l_{22} \wedge u_{22} = b \end{aligned}$$

Again in this equation system it is under determined, so we cannot obtain a solution for every variable. If we now set L to be unit triagonal, we get:

$$A = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} = LU = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}$$

$$\begin{aligned} u_{11} &= 0 \\ u_{12} &= 0 \\ l_{21}u_{11} &= a \\ l_{21}u_{12} + u_{22} &= b \end{aligned}$$

Yet again in this case, the equations will follow to a contradiction, so that A has no LU factorization. We can see that $l_{21}u_{11} = a$, where u_{11}, so that this equation system has no LU factorization.

10 Exercise 15

Find all factorizations of A which are unit lower triagonal.

$$A = \begin{pmatrix} 1 & 5 \\ 3 & 15 \end{pmatrix} = LU = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}$$

$$\begin{aligned} u_{11} &= 1 \\ u_{12} &= 5 \\ l_{21}u_{11} &= 3 \rightarrow l_{21} = 3 \\ u_{12}l_{21} + u_{22} &= 15 \rightarrow u_{22} = 0 \end{aligned}$$

It can be seen that A has only one LU factorization.

$$LU = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 0 & 0 \end{pmatrix}$$

11 Exercise 16

If A is invertible and has an LU decomposition then all principal minors of A are non-singular. Given $a_{11} \neq 0$ in A so that we can transform this matrix A with Gaussian elimination into a LU form so that $l_{i,i} \neq 0$ and $\det(A) = \sum_i u_{ii}$. After k steps of Gaussian elimination A^k, we can see that the sub matrix $1 : k-1, 1 : k-1$ is unit lower triagonal, so per definition its determinate is non-zero: $det(A1 : k-1, 1 : k-1) = \sum_i a_{ii} = 1 \neq 0$. Therefore the k-th pivot is non-zero, so we can proceed to find a LU decomposition.

If L is unit lower triagonal $A = LU$ then $\det(A) = \det(LU) = \det(L)\det(U) = \sum_i u_{ii}$.

12 Exercise 19

If the unit triangular LU decomposition exists so that:

$$LU = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}$$

We want to know if this equation can also state that a unit upper triangular matrix can be generated. To distinguish the two different matrices, I use a superscript to denote if it's the lower unit triangular (superscript 1) or 2 for the upper triangular.

$$LU = \begin{pmatrix} 1 & 0 \\ l^1_{21} & 1 \end{pmatrix} \begin{pmatrix} u^1_{11} & u^1_{12} \\ 0 & u^1_{22} \end{pmatrix} = \begin{pmatrix} l^2_{11} & 0 \\ l^2_{21} & l^2_{22} \end{pmatrix} \begin{pmatrix} 1 & u^2_{12} \\ 0 & 1 \end{pmatrix}$$

This gives us following equations:

$$\begin{gathered} u^1_{11} = l^2_{11} \\ u^1_{12} = l^2_{11} u^2_{12} \\ l^1_{21} u^1_{11} = l^2_{21} \\ l^1_{12} u^1_{12} + u^1_{22} = l^2_{21} u^2_{12} + l^2_{22} \end{gathered}$$

This leads to the following terms, in respect to the 2 superscript terms:

$$\begin{gathered} l^2_{11} = u^1_{11} u^2_{12} = \frac{u^1_{12}}{u^1_{11}} \\ l^2_{21} = l^1_{21} u^1_{11} \\ l^2_{22} = \frac{l^1_{21} u^1_{12} + u^1_{22}}{l^2_{21} u^2_{12}} = 1 + \frac{u^1_{22}}{l^1_{21} u^1_{12}} \end{gathered}$$

That is, if we write down (the terms denoted by superscript 2) in the matrix form:

$$\begin{pmatrix} u^1_{11} & 0 \\ l^1_{21} u^1_{11} & 1 + \dfrac{u^1_{22}}{l^1_{21} u^1_{12}} \end{pmatrix} \begin{pmatrix} 1 & \dfrac{u^1_{12}}{u^1_{11}} \\ 0 & 1 \end{pmatrix}$$

So it is seen that an unit lower triangular can be transformed into an upper triangular matrix.

13 Exercise 24

We need to show that an invertible matrix A can be factorized into a LDU decomposition.

Assume that having an transformed matrix $U^{'}$, which has $u^{'}_{ii} = 1$. In that way we can use a diagonal matrix D to restore the missing values, by stetting $d_{ii} = u_{ii}$.

$$A = (LDU^{'})^{T^T} = (U^{'^T} D L^T)^T$$

As we can see, the matrix $U^{'T}$ is a lower triangular one and the matrix L^T is a upper triangonal one. This means that $LDU^{'}$ is just another LU decomposition of A. We can set $L = U^{'T}$ so that we can see that this decomposition is unique:

$$A = (LDL^T)^T$$

This shows that the LDU decomposition of A can be found.

14 Exercise 29

In this exercise a LDL^T transformation needs to be done.

$$A = \begin{pmatrix} 2 & 6 & -4 \\ 6 & 17 & -17 \\ -4 & -17 & -20 \end{pmatrix} = LDL^T = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \begin{pmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{pmatrix}$$

We will write out the resulting equations.

$$\begin{aligned} d_1 &= 2 \\ l_{21}d_1 &= 6 \to l_{21} = 3 \\ d_2 &= 17 \\ l_{31}d_1 &= -4 \to l_{31} = -2 \\ l_{32}d_2 &= -17 \to l_{32} = -1 \\ d_3 &= -20 \end{aligned}$$

Which will result in the following matrices:

$$A = \begin{pmatrix} 2 & 6 & -4 \\ 6 & 17 & -17 \\ -4 & -17 & -20 \end{pmatrix} = LDL^T = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & -20 \end{pmatrix} \begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

15 Exercise 31

In this exercise I use scaled row pivoting to compute the LU decomposition.

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 3 \\ \frac{1}{3} & 3 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 0 & -\frac{1}{3} & \frac{26}{9} \\ \frac{1}{3} & 3 & -\frac{1}{3} \end{pmatrix}$$

Matrices here are already permutated by P:

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & -\frac{1}{3} \\ 0 & 0 & -\frac{26}{9} \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$