**Assignment/Problem Set 9**

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# 1 Exercise 1

We use here the standard lagrange formula to calculate the polynomial.

**a**

$$p(x) = y_0 \left( \frac{x - x_1}{x_0 - x_1} \right) + y_1 \left( \frac{x - x_0}{x_1 - x_0} \right)$$
$$p(x) = 7 \left( \frac{x-3}{3-7} \right) + (-1) \left( \frac{x-7}{7-3} \right)$$
$$= -\frac{7}{4}x + \frac{3}{4} - \frac{1}{4}x + \frac{7}{4} = -2x + \frac{5}{2}$$

**b** The 3 data points formula for the Lagrange interpolation is the following:

$$p(x) = y_0 \left( \frac{x - x_1}{x_0 - x_1} \right) \left( \frac{x - x_2}{x_1 - x_2} \right) + y_1 \left( \frac{x - x_0}{x_1 - x_0} \right) \left( \frac{x - x_2}{x_1 - x_2} \right) + y_2 \left( \frac{x - x_0}{x_2 - x_0} \right) \left( \frac{x - x_1}{x_2 - x_1} \right)$$

We plug in the numbers and get:

$$p(x) = 146 \left( \frac{x-1}{7-1} \right) \left( \frac{x-2}{1-2} \right) + 2 \left( \frac{x-7}{1-7} \right) \left( \frac{x-2}{1-2} \right) + \left( \frac{x-7}{2-7} \right) \left( \frac{x-1}{2-1} \right)$$

Which leads to the expanded polynomial:

$$p(x) = \frac{1}{3}(2-x)(7-x) + \frac{1}{5}(x-1)(7-x) + \frac{73}{3}(2-x)(x-1)$$

# 2 Exercise 6

Assume having a polynominal $q$, which can be written as a sum of two polynomials, namely from $i < k, k < n$:

$$q = q_1 + q_2$$
$$q_1 = \sum_{i=0}^{k} q_1(x_i) l_{x_i}$$
$$q_2 = \sum_{i=k+1}^{n} q_2(x_i) l_{x_i}$$

Using the linearity property from exercise 2, we can rewrite:

$$Lq = L(q_1 + q_2) = Lq_1 + Lq_2 = \sum_{i=0}^{k} q_1(x_i) l_{x_i} + \sum_{i=k+1}^{n} q_2(x_i) l_{x_i} = q_1 + q_2 = q$$

So we can see that $Lq = q$.

## 3 Exercise 7

We need to show that:

$$\sum_{i=0}^{n} l_i(x) = 1$$

We can easily proof that by rewriting:

$$\sum_{i=0}^{n} = l_i(x) = \sum_{i=0}^{n} l_i(x_i) = \prod_{m \neq i} \frac{x_i - x_m}{x_i - x_m} = 1$$

As required.

## 4 Exercise 9

Let $F(x) = g(x) + \frac{x_0 - x}{x_n - x_0}[g(x) - h(x)]$.We know that $g(x_0) = f(x_0), h(x_n) = f(x_n)$ and $g(x_i) = h(x_i) = f(x_i)$ for $1 \leq i \leq n-1$. Hence:

$$F(x_0) = g(x_0) + \frac{x_0 - x_0}{x_n - x_0}[g(x_0) - h(x_0)] = g(x_0) + \frac{0}{x_n - x_0}[g(x_0) - h(x_0)] = g(x_0) = h(x_0)$$
$$F(x_i) = g(x_i) + \frac{x_0 - x_i}{x_n - x_0}[g(x_i) - h(x_i)] = g(x_i) + \frac{x_0 - x_i}{x_n - x_0}[0] = g(x_i) = h(x_i) \ 1 \leq i \leq n-1$$
$$F(x_n) = g(x_n) + \frac{x_0 - x_n}{x_n - x_0}[g(x_n) - h(x_n)] = g(x_n) - [g(x_n) - h(x_n)] = h(x_n) = f(x_n)$$

## 5 Exercise 10

We simply expand:

$$p(x) = \sum_{i=0}^{k} y_k l_i(x) = \sum_{i=0}^{k} y_k \prod_{j \neq i} \frac{x - x_j}{x_j - x_m}$$
$$= \sum_{i=0}^{k} y_k \prod_{j \neq i} (x - x_j) \prod_{j \neq i} (x_j - x_m)^{-1} =$$
$$\sum_{i=0}^{k} y_k \prod_{j \neq i} (x_j - x_m)^{-1} \prod_{j \neq i} (x - x_j) =$$
$$c \prod_{j \neq i} (x - x_j), \ c = \sum_{i=0}^{k} y_k \prod_{j \neq i} (x_j - x_m)^{-1}$$

We have shown that the factor before the polynomial $x^n$ is $c$.

## 6 Exercise 11

We need to proof that any polynomial $q$ of degree $\leq n-1$

$$\sum_{i=0}^{n} q(x_i) \prod_{j \neq i}^{n} (x_i - x_j)^{-1} = 0$$

We can see that we have $n-1$ degrees, but $n+1$ equations:

$$\sum_{i=0}^{n} q(x_i) \prod_{j \neq i}^{n} (x_i - x_j)^{-1} = q(x) \prod_{x_j \neq X}^{n} (x - x_j)^{-1}$$

So we can follow that we will end in a zero termed solution.

# 7 Exercise 21

We need to calculate the interpolated polynomial via Lagrange and newton.
Lagrange:

$$p(x) = 11\left(\frac{x}{2-0}\right)\left(\frac{x-3}{0-3}\right) + 7\left(\frac{x-2}{0-2}\right)\left(\frac{x-3}{0-3}\right) + 28\left(\frac{x-2}{3-2}\right)\left(\frac{x-0}{3-0}\right) =$$
$$5x^2 - 8x + 7$$

Newton:

$$a_0 = y_0 = 11$$
$$a_1 = f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = 2$$
$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 7$$
$$a_2 = f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} = 5$$
$$p(x) = 11 + 2(x-2) + 5(x-2)(x-0) = 5x^2 - 8x + 7$$