Assignment/Problem Set 10

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1 Exercise 3

In this exercise we need to prove that if $x_0 \in (a, b)$ and if $x_1, x_2, \ldots, x_n$ all converge to x_0, then $f[x_0, x_1, \ldots, x_n]$ will converge to $\frac{f^n(x_0)}{n!}$.

Let P be the Lagrange interpolation polynomial for f at $x_0, ..., x_n$. Then it follows from the Newton form of P that the highest term of P is $f[x_0, \ldots, x_n]x^n$. Let g be the remainder of the interpolation, defined by $g = f - P$. Then g has $n+1$ zeros: $x_0, ..., x_n$. We get:

$$0 = g^{(n)}(\xi) = f^{(n)}(\xi) - f[x_0, \ldots, x_n]n!$$
$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

Where $\xi \in (\min\{x_0, \ldots, x_n\}, \max\{x_0, \ldots, x_n\})$. Since in our example all terms converge to x_0, the minimum and maximum of ξ is both x_0, so we get:

$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(x_0)}{n!}$$

2 Exercise 4

Suppose $p(x)$ is the interpolation polynomial of at most degree n for f, then

$$p(x_i) = f(x_i), i = 0, 1, \ldots, n$$

Let $q(x) = p(x) - f(x)$ be a polynomial of at most degree n. From above, we know that $q(x)$ has at least $n+1$ roots, hence:

$$q(x) = 0 \Rightarrow p(x) = f(x) = 0 \Leftrightarrow p(x) = f(x)$$

Which means that $p(x)$ is a polynomial of degree k . By divided difference, we have

$$p(x) = \sum_{i=0}^{n} f[x_0, x_1, \ldots, x_i] \prod_{j=0}^{i-1} (x - x_j)$$

Then $f[x_0, x_1, \ldots, x_i]$ is the coefficient of x^n. Hence $f[x_0, x_1, \ldots, x_i] = 0$ when $n > k$.

3 Exercise 5

As seen in this book, p is a polynomial of degree at most n

$$p(x) = \sum_{k=0}^{n} c_k q_k(x) = \sum_{k=0}^{n} f[x_0, \ldots, x_k] \prod_{j=0}^{k-1} (x - x_j)$$

4 Exercise 8

For any f, $p_n(x) = \sum_{i=0}^{n} f(x_i) l_i(x)$ is the interpolation polynomial interpolating $f(x)$ at $x_0, x_1, \ldots, x_n$.

Let $Q_n(x) = \sum_{i=0}^{n} f[x_0, x_1, \ldots, x_i] \prod_{j=0}^{i-1}(x - x_j)$ be the interpolation polynomial interpolation $f(x)$ at $x_0, x_1, \ldots, x_n$ too.

$$\therefore P_n(x) \equiv Q_n(x)$$

$$\sum_{i=0}^{n} f(x_i) l_i(x) = \sum_{i=0}^{n} f[x_0, x_1, \ldots x_i] \prod_{j=0}^{i-1}(x - x_j)$$

5 Exercise 9

Assuming having a divided difference in the following form:

$$f[x_\nu, \ldots, x_{\nu+j}] := \frac{f[x_{\nu+1}, \ldots, x_{\nu+j}] - f[x_\nu, \ldots, x_{\nu+j-1}]}{x_{\nu+j} - x_\nu}, \qquad \nu \in \{0, \ldots, k-j\},\ j \in \{1, \ldots, k\}.$$

We can expand the terms and show:

$$\begin{aligned}
f[x_0] &= f(x_0) \\
f[x_0, x_1] &= \frac{f(x_0)}{(x_0 - x_1)} + \frac{f(x_1)}{(x_1 - x_0)} \\
f[x_0, x_1, x_2] &= \frac{f(x_0)}{(x_0 - x_1)\cdot(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)\cdot(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)\cdot(x_2 - x_1)} \\
f[x_0, x_1, x_2, x_3] &= \frac{f(x_0)}{(x_0 - x_1)\cdot(x_0 - x_2)\cdot(x_0 - x_3)} + \frac{f(x_1)}{(x_1 - x_0)\cdot(x_1 - x_2)\cdot(x_1 - x_3)} \\
&\quad + \frac{f(x_2)}{(x_2 - x_0)\cdot(x_2 - x_1)\cdot(x_2 - x_3)} + \frac{f(x_3)}{(x_3 - x_0)\cdot(x_3 - x_1)\cdot(x_3 - x_2)} \\
&\ \vdots \\
f[x_0, \ldots, x_n] &= \sum_{j=0}^{n} \frac{f(x_j)}{\prod_{k\in\{0,\ldots,n\}\setminus\{j\}}(x_j - x_k)}
\end{aligned}$$

Which shows that $f[x_0, \ldots, x_n] = \sum_{i=0}^{n} f(x_i) \prod_{\substack{j=0 \\ j\neq i}} (x_i - x_j)^{-1}$ as required.

6 Exercise 12

Divided difference produces an interpolated polynom of degree n, with the following factors:

$$\begin{array}{llccc}
x_0 & y_0 = f[y_0] & & & \\
 & & f[y_0, y_1] & & \\
x_1 & y_1 = f[y_1] & & f[y_0, y_1, y_2] & \\
 & & f[y_1, y_2] & & f[y_0, y_1, y_2, y_3] \\
x_2 & y_2 = f[y_2] & & f[y_1, y_2, y_3] & \\
 & & f[y_2, y_3] & & \\
x_3 & y_3 = f[y_3] & & &
\end{array}$$

First of all we show that $m = n$ will lead the polynom being 1. Assume having an $n = 3$ polynom, so we need at first 3 equations.

$$\begin{aligned} f[x_0, x_1] &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\ f[x_1, x_2] &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ f[x_2, x_3] &= \frac{f(x_3) - f(x_2)}{x_3 - x_2} \end{aligned}$$

As we can easily see, in this first iteration step $n = 1$, which we wrote out, if we set $m = n$, all equations in this nth step will result in being 1.

$$\begin{gathered} \text{If } m = n \\ f[x_0, x_1] = 1 \\ f[x_1, x_2] = 1 \\ f[x_2, x_3] = 1 \end{gathered}$$

Since we calculate further on the divided differences, as soon as we calculated $m = n$ and calculate further on $n > m$, the difference in between the terms will be zero:

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_0 - x_2} = \frac{1 - 1}{x_0 - x_2} = 0$$

So in the end, if we have n recursions, with degree m, then at the mth recursion,all terms within this recursion will be 1, leading to the next recursion and the following ones being 0, as shown above.

7 Exercise 17

$$p(x) = 3 + \frac{1}{2}(x - 1) + \frac{1}{3}(x - 1)(x - \frac{3}{2}) - 2(x - 1)(x - \frac{3}{2})x$$

8 Exercise 21

We need to show that c_0 is the value of the cubic interpolation at x: We simply expand:

$$\begin{aligned} c_0 &= \frac{(x_3 - x)b_0 + ((x - x_0)b_1}{x_3 - x_0} \\ &= \frac{(x_3 - x)\left(\frac{(x_2-x)a_0+(x-x_0)a_1}{x_2-x_0}\right)}{x_2 - x_0} + \frac{(x - x_0)\left(\frac{(x_3-x)a_1+(x-x_1)a_2}{x_2-x_1}\right)}{x_3 - x_1} \\ &= (x_3 - x)\left(\frac{\left(\frac{(x_2-x)\left(\frac{(x_1-x)y_0+(x-x_0)y_1}{x_1-x_0}\right)}{x_2-x_0}\right)}{x_3 - x_0}\right. \end{aligned}$$

Not that we only expanded the left hand side, since we can easily show from here on that the central terms are only rewritten newton form polynomials:

$$
\begin{aligned}
&\frac{(x_1 - x)y_0 + (x - x_0)y_1}{x_1 - x_0} \\
&= \frac{(x_1 - x)y_0}{x_1 - x_0} + \frac{(x - x_0)y_1}{x_1 - x_0} \\
&= \frac{(x - x_1)y_0}{x_0 - x_1} + \frac{(x - x_0)y_1}{x_0 - x_1} \\
&= \sum_{j=0}^{k} y_j \ell_j(x) = L(x)
\end{aligned}
$$

We have shown that the above polynomial will be a 3rd order Lagrange interpolation.