Assignment/Problem Set 3

Heinrich Dinkel
ID: 1140339107
E-mail: heinrich.dinkel@sjtu.edu.cn

1 Exercise 2

We need to show that if $||B - A|| < ||A^{-1}||^{-1}$ exists,then B is also invertible.

$$||B - A|| < \frac{1}{||A^{-1}||} \leq \frac{1}{||A||^{-1}} = ||A||$$

From here on follows that:

$$||B - A|| < ||A||$$
$$A^{-1} \text{ exists } \rightarrow ||A|| < 1 \rightarrow ||B - A|| < 1 \rightarrow ||B||^{-1} \text{ exists}$$

We have shown that if A is invertible, B is too.

2 Exercise 3

Assuming that $||A|| < 1$, we want to prove that $||I - A||^{-1} \geq \frac{1}{1 + ||A||}$. First it needs to be shown that $||I - A||^{-1}$ is invertible. If it is singular, then a vector exists $||x|| = 1$, so that $(I - A)x = 0$. It follows that:

$$||x|| = 1 = ||Ax|| \leq ||A|| ||x|| = ||A||$$

So it can be seen, that if $||x|| = 1$, $(I - A)$ is invertible. Furthermore we use the Neumann series to show that the series converges towards $\frac{1}{1 + ||A||}$.

$$(I - A) \sum_{k}^{p} A^k = \sum_{k}^{p} (A^k - A^{k+1}) = (I - A^{p+1}) \rightarrow I$$

It should be noted that in 2, the elements within the sum cancel out, except in the case $k = 0$ and $k = p$, so that in an limit case, where $p \rightarrow \infty$, $A^{p+1} \rightarrow 0$. Since $||A^{p+1}|| \leq ||A||^{p+1}$, we get:

$$||(I - A)^{-1}|| \leq \sum_{k=0}^{\infty} ||A^k|| \leq \sum_{k=0}^{\infty} ||A||^k = \frac{1}{1 - ||A||}$$

As it can be seen, our assumption was wrong.

3 Exercise 4