**Assignment/Problem Set 1**

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# 1 Exercise 1

**c** LU decomposition

$$A = \begin{pmatrix} -1 & 1 & 0 & -3 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 3 & 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & 3 \\ -1 & 1 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ -3 & 3 & 1 & -7 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & 3 \\ -1 & 1 & 3 & -2 \\ 0 & 1 & -4 & 1 \\ -3 & 3 & -8 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & 3 \\ -1 & 1 & 3 & -2 \\ 0 & 1 & -4 & 1 \\ -3 & 3 & 2 & -3 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -3 & 3 & 2 & 1 \end{pmatrix} U = \begin{pmatrix} -1 & 1 & 0 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -3 \end{pmatrix} \tag{1}$$

When using scaled row pivoting:

$$\begin{pmatrix} -1 & 1 & 0 & -3 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 3 & 0 & 1 & 2 \end{pmatrix}, p = (1, 2, 3, 4)$$

$$\begin{pmatrix} -\frac{1}{3} & 1 & \frac{1}{3} & -\frac{7}{3} \\ \frac{1}{3} & 0 & \frac{8}{3} & \frac{1}{3} \\ 0 & 1 & -1 & -1 \\ 3 & 0 & 1 & 2 \end{pmatrix}, p = (4, 2, 3, 1)$$

$$\begin{pmatrix} -\frac{1}{3} & 1 & \frac{4}{3} & -\frac{4}{3} \\ \frac{1}{3} & 0 & \frac{8}{3} & \frac{1}{3} \\ 0 & 1 & -1 & -1 \\ 3 & 0 & 1 & 2 \end{pmatrix}, p = (4, 3, 2, 1)$$

$$\begin{pmatrix} -\frac{1}{3} & 1 & \frac{1}{2} & -\frac{3}{2} \\ \frac{1}{3} & 0 & \frac{8}{3} & \frac{1}{3} \\ 0 & 1 & -1 & -1 \\ 3 & 0 & 1 & 2 \end{pmatrix}, p = (4, 3, 2, 1)$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & 1 & 0 \\ -\frac{1}{3} & 1 & \frac{1}{2} & 1 \end{pmatrix}, U = \begin{pmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & \frac{8}{3} & \frac{1}{3} \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}, P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

## 2 Exercise 3

The Matrix $PA$ is a row lever permutation of $A$. The column $c$ of the 1 entry in row $r$ within $P$, will swap the $r$th row with the $c$th one. The Matrix $AP$ will only switch elements in every row of $A$. The row order stays the same, yet the elements within the rows are interchanged. Since the matrix $P$ is linear independent, an inverse does exist. In this case, we can solve for $PP^{-1} = \mathbf{I}$. In this case $P^{-1}$ is equal to $P^T$, since the only possible combination to achieve the invertible is by multiplying every row with itself as column. This results in $P = P^{-1} = P^T$. The matrix $PAP^{-1}$ swaps the rows like as $PA$ and then swaps the elements within every row like $AP$.

## 3 Exercise 8

Assuming that the matrix $A$ is linearly independent. The forward elimination matrix $B$ then consists of the cofactors and values of $L$ and $U$ respectively.

$$B = \begin{pmatrix} b_{1,1} & \dots & b_{1,n-1} & b_{1,n} \\ \vdots & \ddots & & \vdots \\ b_{n-1,1} & & \ddots & \\ b_{n,1} & \dots & \dots & b_{n,n} \end{pmatrix}$$

Since the matrix $B$ is not ordered, it cannot be seen which item $b_{ij}$ is either a cofactor for $L$ or value for $U$. If we apply $PB$, we get rotate the matrix rows into correct order, so that all entries below $b_{ii}$ are cofactors of $L$ and all above (including $b_{ii}$) are the upper triangular values $U$.

## 4 Exercise 10

$$\begin{pmatrix} 2 & -2 & -4 \\ 1 & 1 & -1 \\ 3 & 7 & 5 \end{pmatrix} s = (4, 1, 7) p = (1, 2, 3)$$
$$\begin{pmatrix} 2 & -4 & -2 \\ 1 & 1 & -1 \\ 3 & 4 & 8 \end{pmatrix} s = (1, 4, 7) p = (2, 1, 3)$$
$$\begin{pmatrix} 2 & -4 & -2 \\ 1 & 1 & -1 \\ 3 & -1 & 6 \end{pmatrix} s = (1, 4, 7) p = (2, 1, 3)$$
$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix}, U = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -4 & -2 \\ 0 & 0 & 6 \end{pmatrix}, P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

## 5 Exercise 12

To proof this inequality, one can use induction. Since the matrix is mirrored at the points $i = 1$ and $i = n$, it is convenient to use either side to begin with. We want to show that $|d_1| > |a_1| + |c_0|$. Since per definition $c_0 = 0$ and therefore we can need to show that $|d1| > |a_1|$. The elimination step can be viewed as: $d_1' = d_1 - c_0 \frac{a_1}{d_1}$ In this case it can be clearly seen that $d_1' > a_1$. In the next case, we will display the specific case of $n = 2$. $d_2' = d_2 - c_1 \frac{a_2}{d_2}$ That means we need to show that $|d_2| \geq |c_1 \frac{a_2}{d_2}|$. Given the inequality of $|d_i| \geq |c_{i-1} + a_i$, it is obvious that this inequality holds, so that the produced by Gaussian elimination is still non-singular and the columnise dominance is preserved.

# 6 Exercise 17

$$\begin{pmatrix} -9 & 1 & 17 \\ 3 & 2 & -1 \\ 6 & 8 & 1 \end{pmatrix} p = (1,2,3)$$
$$\begin{pmatrix} -3 & 7 & 14 \\ 3 & 2 & -1 \\ 2 & 4 & 3 \end{pmatrix} p = (2,1,3)$$
$$\begin{pmatrix} -3 & \frac{7}{4} & \frac{35}{4} \\ 3 & 2 & -1 \\ 2 & 4 & 3 \end{pmatrix} p = (2,3,1)$$
$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, L = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ -3 & \frac{7}{4} & 1 \end{pmatrix}, U = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & \frac{35}{4} \end{pmatrix}$$

Since this matrix is not diagonally dominant, because the first row has overall two elements which could be diagonally the maximum ( $-9$ and $17$ ). This results in a false LU decomposition.

# 7 Exercise 30

$$A = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix} p = (1,2,3,4)$$
$$\begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix} p = (4,2,3,1)$$
$$\begin{pmatrix} 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} & -1 \\ 2 & 0 & 1 & 0 \end{pmatrix} p = (4,2,3,1)$$
$$\begin{pmatrix} 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} & -1 \\ 2 & 0 & 1 & 0 \end{pmatrix} p = (4,2,3,1)$$
$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, U = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \frac{3}{2} & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

The definition $\det(AB) = \det(A)\det(B)$ can be used in conjunction to the definition of determinant of a triangular matrix : $\det(T) = \sum_i a_{ii}$. Since the lower triangular Matrix will have $\det(L) = 1$, only the upper triangular matrix $U$ will be considered. $\det(A) = \underline{6}$

# 8 Exercise 41

To show that diagonal dominance will be preserved, I show that after eliminating element $a_{i,j}$, the equation $|a_{i,i}| \geq \sum_{j=1 j\neq i} |a_{i,j}|$ still holds. After one iteration of gaussian elemination, the elements of $a_{i,j}, j > 1$ are all zero. To show that the new obtained matrix is still diagonal dominant we use:

$$\sum_{i=2,i\neq j} |a_{i,j}| = \sum_{i=2,i\neq j} |a_{i,j} - \frac{a_{1,j}a_{i,1}}{a_{11}} \leq \sum_{i=2,i\neq j} |a_{i,j}| + \sum_{i=2,i\neq j} \frac{a_{1,j}a_{i,1}}{a_{11}}$$

Since A is diagonally dominant, $|a_{j,j}||a_{1,1}| > |a1,j||a_{j,1}|$, which leads to following inequality:

$$\sum_{i=2,i\neq j} |a_{i,j}| + \sum_{i=2,i\neq j} \frac{a_{1,j}a_{i,1}}{a_{11}} < |a_{j,j}| - |a_{1,j}| + \frac{|a_{1,j}|(|a_{1,1}| - |a_{j,1}|)}{|a_{1,1}|} = |a_{j,j} - \frac{a_{j,1}a_{1,j}}{a_{1,1}}| = |a_{j,j}|$$