Assignment/Problem Set 13

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1 Exercise 4

We have the following newton iteration formula given:

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)}$$
$$g(x) = \frac{f(x + f(x)) - f(x)}{f(x)}$$

Let r be a solution such that $f(r) = 0$ and $f'(r) \neq 0$. Moreover assume $|f''|$ is bounded in a neighborhood of r. We shall show that Newtons method is quadratically convergent provided that x_0 is sufficiently close to r. Consider the function:

$$k(x, \epsilon, \eta) = \frac{f''(\epsilon)(f'(x) - \frac{1}{2}f''(\eta)(x - r)) + f''(\eta))}{2f'(x) + f''(\epsilon)f(x)}$$

By assumption the limit supremum of $|k(x, \epsilon, \eta)|$ exists as $x \to r$, $\epsilon \to r$ and $\eta \to r$. Let C denote this limit and choose $\delta_1 > 0$ so small that $|k(x, \epsilon, \eta)| \leq 2C$ for every x, ϵ, η such that $|x - r| < \delta_1, |\epsilon - r| < \delta_1, |\eta - r| < \delta_1$. Choose $\delta_2 > 0$ so small that $|x - r| < \delta_2$ implies $2|f(x)| \leq \delta_1$. Let:

$$\delta = \min\left\{\frac{\delta}{2}, \delta_2, \frac{1}{2C}\right\}$$

Claim that if $x_o \in (r - \delta, r + \delta)$ then x_n defined according to Newtons method satisfied $x_n \in (r - \delta, r + \delta)$ for all $n \in N$. By Taylors theorem there exists ϵ_n between x_n and $x_n + f(x_n)$ such that:

$$f(x_n + f(x_n)) = f(x_n) + f'(x_n)f(x_n) + \frac{1}{2}f''(\epsilon_n)f(x_n)^2$$
$$\therefore g(x_n) = f'(x_n) + \frac{1}{2}f''(\epsilon_n)f(x_n)$$

It follows that:

$$e_{n+1} = x_{n+1} - r = e_n - \frac{f(x_n)}{f'(x_n) + \frac{1}{2}f''(\epsilon_n)f(x_n)}$$

where $e_n = x_n - r$. Using Taylor again, yields:

$$0 = f(r) = f(x_n - e_n) = f(x_n) = f'(x_n)e_n + \frac{1}{2}f''(\eta_n)e_n^2$$

Fro some choice of η_n between r and x_n, Therefore:

$$e_{n+1} = e_n - e_n \frac{f'(x_n) + \frac{1}{2}f''(\eta_n)e_n}{f'(x_n) + \frac{1}{2}f''(\epsilon_n)f(x_n)}$$
$$= e_n - e_n \frac{f'(x_n) - \frac{1}{2}f''(\epsilon_n)f(x_n) + \frac{1}{2}f''(\epsilon_n)f(x_n) + \frac{1}{2}f''(\eta_n)e_n}{f'(x_n) + \frac{1}{2}f''(\epsilon_n)f(x_n)}$$
$$= -e_n \frac{f''(\epsilon_n)f(x_n) + f''(\eta_n)e_n}{2f'(x_n) + f''(\epsilon_n)f(x_n)}$$
$$= -e_n \frac{f''(\epsilon_n)(f'(x_n)e_n - \frac{1}{2}f''(\eta_n)e_n^2) + f''(\eta_n)e_n}{2f'(x_n) + f''(\epsilon_n)f(x_n)}$$
$$= -e_n^2 \frac{f''(\epsilon_n)(f'(x_n) - \frac{1}{2}f''(\eta_n)e_n) + f''(\eta_n)}{2f'(x_n) + f''(\epsilon_n)f(x_n)}$$
$$= -\epsilon_n^2 k(x_n, \epsilon_n, \eta_n)$$

Claim that $x_n \in (r-\delta, r+\delta)$. Since ϵ_n is between x_n and $x_n + f(x_n)$ then:

$$|\epsilon_n - r| \le |\epsilon_n - x_n| + |x_n - r| \le |f(x_n)| + |x_n - r| < \delta_2 + \delta \le \delta_1$$

Since η_n is between x and x_n:

$$|\eta_n - r| \le \delta < \delta_1$$

Thus x_n, ϵ_n, η_n satisfy $|x_n - r| < \delta_1, |\epsilon_n - r| < \delta_1, |\eta_n - r| < \delta_1$, so consequently $|k(x_n, \epsilon_n, \eta_n)| < 2C$, therefore:

$$|x_{n+1} - r| = |e_{n+1}| \le 2Ce_n^2 = 2C|x_n - r|^2 \le 2C\delta^2 < \delta$$

This shows that $x_{n+1} \in (r-\delta, r+\delta)$ and completes the induction. Moreover we have shown that x_n is quadratically convergent.

2 Exercise 7

We need to find out the corresponding function for the newton iteration, for:

$$x_{n+1} = 2x_n - x_n^2 y$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$2x_n - x_n^2 y = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$f(x_n) = (-x_n + x_n^2 y)f'(x_n)$$

3 Exercise 37

a We use the non-linear gauss method to compute two iterations of the function:

$$\begin{cases} f_1(x_1, x_2) = 4x_1^2 - x_2^2 \\ f_2(x_1, x_2) = 4x_1x_2^2 - x_1 - 1 \end{cases}$$

We compute the general Jacobi matrix:

$$J = \begin{pmatrix} 8x_1 & 4x_2 \\ 4x_2^2 & 8x_1x_2 \end{pmatrix}, J^{-1} = \frac{1}{16x_1^2 - 2x_2^2} \begin{pmatrix} 2x_1 & -\frac{1}{2} \\ -x_2 & \frac{2x_1}{x_2} \end{pmatrix}$$

So we can calculate our first iteration, using $x_1 = 0, x_2 = 1$:

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{4} \\ -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} \Rightarrow h_1 = \frac{1}{4}, h_2 = \frac{1}{2}$$

$$\begin{pmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ \frac{3}{2} \end{pmatrix} \Rightarrow x_1^{(1)} = \frac{1}{4}, x_2^{(1)} = \frac{3}{2}$$

Now for the second iteration we still use the matrix J^{-1}, but we set $x_1 = \frac{1}{4}, x_2 = \frac{3}{2}$, so we get:

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{14} & -\frac{2}{14} \\ \frac{3}{14} & \frac{2}{21} \end{pmatrix} \begin{pmatrix} -2 \\ \frac{13}{4} \end{pmatrix} \Rightarrow h_1 = -\frac{3}{4}, h_2 = -\frac{5}{42}$$

$$\begin{pmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{2} \end{pmatrix} - \begin{pmatrix} \frac{3}{4} \\ \frac{5}{42} \end{pmatrix} \Rightarrow x_1^{(1)} = -\frac{1}{2}, x_2^{(1)} = \frac{57}{42}$$

b We use the non-linear gauss method to compute two iterations of the function:

$$\begin{cases} f_1(x,y) = xy^2 + x^2 + x^4 - 3 \\ f_2(x,y) = x^3y^5 - 2x^5y - x^2 + 2 \end{cases}$$

We compute the general Jacobi matrix:

$$J = \begin{pmatrix} y^2 + 2xy + 4x^3 & 2xy + x^2 \\ 2x^2y^5 - 10x^4y - 2x & 5x^3y^4 - 2x^5 \end{pmatrix},$$

$$J^{-1} = \frac{1}{-8x^6 + x^4(20y^4 + 6y) + 18x^3y^2 + 8x^2y^5 + x(y^6 + 2) + 4y} \begin{pmatrix} 5xy^4 - 2x^3 & \frac{-x-2y}{x} \\ -\frac{2(xy^5 - 5x^3y - 1)}{x} & -\frac{-4x^3 - 2yx - y^2}{x^2} \end{pmatrix}$$

When we plug in the numbers, we get for J^{-1}:

$$\frac{1}{51} \begin{pmatrix} 3 & -3 \\ 10 & 7 \end{pmatrix}$$

Calculating the next iteration step:

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{51} & \frac{3}{51} \\ -\frac{10}{51} & -\frac{7}{51} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow h_1 = 0, h_2 = 0$$

Our calculations stop here, since we can see that the initial guess of $x = 1, y = 1$ is indeed the intersection between the two functions.