Assignment/Problem Set 3

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1 Exercise 1

We want to prove that $\rho(I - Q^{-1}A) < 1$. To do this, we will show that $\max \det(I - Q^{-1}A) < 1$, which means that the eigenvalues $\max_i(\lambda_i) < 1$. First we use the assumptions of having a Jacobian method to set $Q = D$, where $D = \operatorname{diag}(A)$ is diagonal. Moreover since Q is diagonal, Q^{-1} of element $q_{ij}^{-1} = \frac{1}{q_{ij}}$. Moreover we can see from the definition that $q_{ii} = \frac{1}{a_{ii}}$.

From that on follows:

$$\det(\lambda I - I - Q^{-1}A) = 0$$

$$= \det\left(\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{a_{11}} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{a_{nn}} \end{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}\right) =$$

$$\begin{pmatrix} \lambda_1 - 1 & \frac{1}{a_{11}} a_{1k} & \frac{1}{a_{11}} a_{1n} \\ \frac{1}{a_{kk}} a_{k1} & \ddots & \frac{1}{a_{kk}} a_{1n} \\ \frac{1}{a_{nn}} a_{n1} & \frac{1}{a_{nn}} a_{kk} & \lambda_n - 1 \end{pmatrix} = B$$

When calculating the determinant:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} M_{i,j} = 0$$

$$\begin{cases} a_{i,j} = 0, & \text{if } \lambda_i \leq 1 \\ M_{i,j} = 0 & \text{if } \lambda_i \leq 1 \end{cases}$$

Since the resulting matrix is still diagonally dominant, because we only subtracted and added values onto the diagonal, it must be that our resulting matrix B, still is strictly diagonally dominant. That means that $|b_{ii}| \geq \sum_{j \neq i} |b_{ij}| \quad \text{for all } i$, so that follows : $\max_i(\lambda_i) < 1$, which means that $\rho(I - Q^{-1}A) < 1$.

2 Exercise 2

The Richardson Iteration is stated as : $x^k = (I - A)x^{k-1} + b$. The Richardson iteration will be successful if $||I - A|| < 1$.We have:

$$I - A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & a_{1k} & a_{1n} \\ a_{k1} & \ddots & \vdots \\ a_{n1} & \dots & 1 \end{pmatrix} = \begin{pmatrix} 0 & -a_{1k} & -a_{1n} \\ -a_{k1} & \ddots & \vdots \\ -a_{n1} & \dots & 0 \end{pmatrix}$$

Since A is diagonally dominant, $\sum_i |a_{ij}| < 1$, after the subtraction of both matrices we get $0 - \sum_i |a_{ij}| < 1$, which only reverses the sign, but this fact doesn't matter, since we use the absolute value, so that still $||I - A|| < 1$.

3 Exercise 5

We need to prove that $||x||^{'} = ||Sx||$ is a norm.

1. Prove that $||x||^{'} > 0$. Since it is assumed that S is non-singular, we know that if we would modify S to be in Echelon form, we would not get any row r, which has at it's rth column entry a zero value, therefore when we multiply S with x, the resulting value would be > 0.

2. To show that $||\lambda Sx|| \to |\lambda|||Sx||$. Assuming having $\lambda > 0$ we can easily see that this equation holds, since the matrix multiplication with a scalar is commutative.

3. To show that the triangle inequality holds, we see that $||x + y|| = \sup ||S_1x + S_2y|| \leq \sup ||S_1x|| + \sup ||S_2y||$. It follows : $||x + y|| \leq ||x|| + ||y||$.

4 Exercise 7

We will prove that $||I - Q^{-1}A||_\infty < 1$, which differently interpreted means $\left\|I - Q^{-1}A\right\|_\infty = \max\limits_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|$. Q is a Gauss Seidel iteration variable. $Q = D - L$, $A = D - L - U$ and $D - L = A + U$ We expand:

$$I - (A + U)^{-1}A = \frac{(A + U)(I - (A + U)^{-1}A)}{A + U} = \frac{(A + U) - A}{A + U} = \frac{U}{A + U}$$

Now it is sufficient to show that $||U|| < ||A + U||$.

$$A + U = \begin{pmatrix} a_{11} & 2a_{1k} & 2a_{1k+1} & 2a_{1n} \\ \vdots & \ddots & & 2a_{kn} \\ \vdots & & \ddots & 2a_{k+1n} \\ a_{n1} & \dots & \dots & a_{nn} \end{pmatrix}$$

So it can be seen that the upper triangular elements in $A+U$ do double, whereas all other elements remain the same. This means that $\max_i \sum_i |u_{ij}| < \max_i \sum_i |ua_{ij}|$. Since $||U|| > 0$ and therefore $||A + U|| > 0 \to ||I - Q^{-1}A|| < 1$.

5 Exercise 8

We want to show that $\left(\lim_{k\to\infty} A^k = 0 \Rightarrow \rho(A) < 1\right)$. Moreover since ρ is defined as : $\rho(A) = \max_i(|\lambda_i|)$, we need only to show that all eigenvalues of $\left(\lim_{k\to\infty} A^k\right) < 1$. For any eigenvector v, we get via eigenvector definition:

$$A^k v = \lambda^k v$$

$$0 = \left(\lim_{k\to\infty} A^k\right) v = \lim_{k\to\infty} A^k v = \lim_{k\to\infty} \lambda^k v = v \lim_{k\to\infty} \lambda^k$$

Per definition $v \neq 0$, it is obvious that $\lim_{k\to\infty} \lambda^k = 0$. This fact already implies $|\lambda| < 1$.

6 Exercise 15

Let λ be an eigenvalue of $I - Q^{-1}A$ and x the corresponding eigenvectors with $||x||_\infty = 1$. We get:

$$(Q - A)\,x = \lambda Q x$$

$$\lambda a_{ii} x_i = -\lambda \sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} x_j$$

$$|x_i| = 1 \geq ||x_j||\forall j$$

$$|\lambda||a_{ii}| \leq |\lambda| \sum_{j=1}^{i-1} |a_{ij}| + \sum_{j=i+1}^{n} |a_{ij}|$$

$$\text{knowing that } |a_{ii}| > \sum_{j,j\neq i}^{n} |a_{ij}| = \sum_{j=i+1}^{n} |a_{ij}| + \sum_{j=1}^{i-1} |a_{ij}|$$

$$|\lambda||a_{ii}| - |\lambda| \sum_{j=i+1}^{n} |a_{ij}| \leq \sum_{j=1}^{i-1} |a_{ij}|$$

$$|\lambda|(|a_{ii}| - \sum_{j=1}^{i-1} |a_{ij}|) \leq \sum_{j=i+1}^{n} |a_{ij}|$$

$$|\lambda| \leq r_i < 1, \text{ where } \ r_i = \frac{\sum\limits_{j=i+1}^{n} |a_{ij}|}{|a_{ii}| - \sum\limits_{j=1}^{i-1} |a_{ij}|}$$

This shows that $\rho(I - Q^{-1}A)$ is not greater than r_i.

7 Exercise 20

We need to prove that if $\rho(A) < 1$, $(I - A)^{-1}$ exists and $\sum\limits_{k}^{\infty} A^k = (I - A)^{-1}$. First we know that the formula to calculate the eigenvalues is $\det(\lambda I - A) = 0$. In other words, we seek for these λ_is ,which will lead the term $\lambda I - A$ to become not invertible. Since we know that $\rho(A) < 1$, which means that all eigenvalues λ_i are less than 1. We come to the conclusion, that $(\beta I - A)$ is invertible, if $\beta_i \neq \lambda_i$. In other words, if we choose β as I, we can guarantee that $(I - A)$ is invertible, since all the necessary eigenvalues in the diagonal of I need to be less than 1.

Now we know that $(I - A)$ is invertible, so we can use this knowledge to modify the equations and get:

$$(I - A)^{-1} = \sum_{k}^{\infty} A^k I = \sum_{k}^{\infty} A^k (I - A) = (I - A) \sum_{k}^{\infty} A^k \sum_{k}^{\infty} \left(A^k - A^{k+1}\right) \tag{1}$$

Since all the terms in the equation 1 cancel itself out, except the first and the last one, we get:

$$I - A^{m+1} \text{ where } \ m \to \infty$$

Again by using our basic assumption that $\rho(A) < 1$, we show that the matrix norm is less than 1. For an eigenvector v, we have the following equation based on the eigenvalue extraction.

$$|\lambda|^k \|v\| = \|\lambda^k \mathbf{v}\| = \|A^k v\| \leq \|A^k\| \cdot \|v\| \to |\lambda|^k \leq \|A^k\| \tag{2}$$

Since our maximum eigenvalue is less than 1, $|\lambda|^k$ is surely less than 1 and therefore $||A^k||$ is too. This fact leads to that A^{m+1} in equation 2 will converge to zero, so that I will be the result of the series. This proofs that $(I - A)$ is invertible and has a series expansion.

8 Exercise 30

All matrices follow a general approach, that they seek a splitting matrix Q, so that $Qx = (Q - A)x + b$ can iteratively be solved. This follows into the basic iteration equations.

$$Qx^k = (Q - A)x^{k-1} + b \tag{3}$$

$$x^k = Gx^{k-1} + c \tag{4}$$

Richardson For the Richardson iteration, we choose Q as I, so that we get:

$$\begin{aligned} Q &= I \\ x^k &= (I - A)x^{k-1} + b \\ \mathcal{R} &= (I - A) = G \end{aligned}$$

The matrix R is the iteration matrix and uniquely also G.

Jacobi For the Jacobi iteration, we choose Q as D, so that we get for G:

$$\begin{aligned} Dx^k &= (D - A)x^{k-1} + b \\ Dx^k &= (D - (D - L - U))x^{k-1} + b \\ Dx^k &= (L + U)x^{k-1} + b \\ x^k &= D^{-1}(L + U)x^{k-1} + D^{-1}b \\ &\to G = D^{-1}(L + U) \end{aligned}$$

And to show the iteration matrix:

$$\begin{aligned} Dx^k &= (D - A)x^{k-1} + b \\ x^k &= D^{-1}(D - A)x^{k-1} + D^{-1}b \\ x^k &= I - D^{-1}Ax^{k-1} + D^{-1}b \\ &\to \mathcal{J} = I - D^{-1}A \end{aligned}$$

Gauss Seidel For the Gauss Seidel iteration, we choose Q as $D - L$, so that we get:

$$\begin{aligned} (D - L)x^k &= (D - L - A)x^{k-1} + b \\ (D - L)x^k &= (D - L - (D - L - U))x^{k-1} + b \\ x^k &= (D - L)^{-1}Ux^{k-1} + (D - L)^{-1}b \\ &\to G = (D - L)^{-1}U \end{aligned}$$

To show that the iteration matrix is correct:

$$\begin{aligned} (D - L)x^k &= (D - L - A)x^{k-1} + b \\ x^k &= (D - L)^{-1}(D - L - A)x^{k-1} + (D - L)^{-1}b \\ x^k &= (D - L)^{-1}(D - L) - (D - L)^{-1}Ax^{k-1} + (D - L)^{-1}b \\ x^k &= I - (D - L)^{-1}Ax^{k-1} + (D - L)^{-1}b \\ &\to \mathcal{G} = I - (D - C_L)^{-1}A, \text{where } L = C_L \end{aligned}$$

Forward SOR For the forward SOR method, we choose Q as $\omega^{-1}(D-\omega C_L)$, so we get:

$$\omega^{-1}(D-\omega C_L)x^k = (\omega^{-1}(D-\omega C_L)-A)x^{k-1}+b$$
$$\omega^{-1}x^k = (D-\omega C_L)^{-1}(\omega^{-1}(D-\omega C_L)-A)x^{k-1}+(D-\omega C_L)^{-1}b$$
$$\omega^{-1}x^k = \omega^{-1}(D-\omega C_L)^{-1}(D-\omega C_L)-(D-\omega C_L)^{-1}Ax^{k-1}+((D-\omega C_L))^{-1}b$$
$$\omega^{-1}x^k = \omega^{-1}I-(D-\omega C_L)^{-1}Ax^{k-1}+((D-\omega C_L))^{-1}b$$
$$x^k = \omega\omega^{-1}I-\omega(D-\omega C_L)^{-1}Ax^{k-1}+\omega((D-\omega C_L))^{-1}b$$
$$\rightarrow \mathcal{L}_\omega = I-\omega(D-\omega C_L)^{-1}A$$

To find out G, we get:

$$D(x^k-x^{k-1}) = \omega b-\omega Dx^{k-1}+\omega Lx^k+\omega Ux^{k-1}$$
$$(D-\omega L)x^k = Dx^{k-1}-\omega Dx^{k-1}+\omega Ux^{k-1}+\omega b$$
$$(D-\omega L)x^k = (D-\omega D+\omega U)x^{k-1}+\omega b$$
$$x^k = (D-\omega L)^{-1}(D-\omega D+\omega U)x^{k-1}+(D-\omega L)^{-1}\omega b$$
$$\rightarrow G = (D-\omega L)^{-1}(D-\omega D+\omega U)$$

Backward SOR Since Backward and Forward SOR are only a manipulation of $C_L = C_U$, replace the variables and it can be easily seen that the result is the same, except for this variable.

$$G = (D-\omega C_R)^{-1}(D-\omega D+\omega U)$$
$$\mathcal{U}_\omega = I-\omega(D-\omega C_R)^{-1}A$$

SSOR In SSOR, our splitting matrix $Q = (\omega(2-\omega)^{-1}(D-\omega C_L)D^{-1}(D-\omega C_U)$

$$Qx^k = (Q-A)x^{k-1}+b$$
$$x^k = Q^{-1}(Q-A)x^{k-1}+Q^{-1}b$$
$$x^k = (\omega(2-\omega)(D-\omega C_L)^{-1}D(D-\omega C_U)^{-1}(\omega(2-\omega)^{-1}(D-\omega C_L)D^{-1}(D-\omega C_U)-$$
$$(\omega(2-\omega)(D-\omega C_L)^{-1}D(D-\omega C_U)^{-1}Ax^{k-1}+Q^{-1}b$$
$$x^k = I-(\omega(2-\omega)(D-\omega C_L)^{-1}D(D-\omega C_U)^{-1}Ax^{k-1}+Q^{-1}b$$
$$\rightarrow \mathcal{S}_\omega = I-(\omega(2-\omega)(D-\omega C_L)^{-1}D(D-\omega C_U)^{-1}A$$

By expanding the terms A and C_U, like in the SOR iteration, we get:

$$x^k = I-(\omega(2-\omega)(D-\omega C_L)^{-1}D(D-\omega C_U)^{-1}Ax^{k-1}$$
$$x^k = (D-\omega C_U)^{-1}(\omega C_L+(1-\omega)D)(D-\omega C_L)^{-1}(\omega C_U+(1-\omega D)x^{k-1}$$
$$\rightarrow G = (D-\omega C_U)^{-1}(\omega C_L+(1-\omega)D)(D-\omega C_L)^{-1}(\omega C_U+(1-\omega D)x^{k-1}$$

9 Exercise 31

In this exercise we need to find the explicit form of $I-Q^{-1}A$.

$$A = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} Q = \begin{pmatrix} 2 & 0 & & & & \\ 1 & 2 & 0 & & & \\ & 1 & 2 & 0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 2 & 0 \\ & & & & 1 & 2 \end{pmatrix}$$

Since Q is lower triagonal, the determinant is easy to find. To get the adjugate matrix, we can show that it will be 2^{n-1} for every diagonal entry and will decrease in for each step farer away from the diagonal.

$$Q^{-1} = \begin{pmatrix} 2^n & 0 & & & & \\ -2^{n-1} & 2^n & 0 & & & \\ 2^{n-2} & -2^{n-1} & 2^n & 0 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ 2^{n-4} & -2^{n-3} & 2^{n-2} & -2^{n-1} & 2^n & 0 \\ & 2^{n-4} & -2^{n-3} & 2^{n-2} & -2^{n-1} & 2^n \end{pmatrix}$$

$$Q^{-1}A = \begin{pmatrix} 2^{n+1}+2^{n-1} & -2^n & & & \\ -2^n-2^n & 2^{n+1}+2^{n-1} & -2^n & & \\ 2^{n-1}+2^{n-1} & -2^n-2^n & 2^{n+1}+2^{n-1} & -2^n & \\ -2^{n-2}-2^{n-2} & \ddots & \ddots & \ddots & \\ & & -2^n-2^n & 2^{n+1}+2^{n-1} & -2^n \\ & -2^{n-2}-2^{n-2} & 2^{n-1}+2^{n-1} & -2^n-2^n & 2^{n+1}+2^{n-1} \end{pmatrix}$$

$$I - Q^{-1}A = \begin{pmatrix} 1-2^{n+1}+2^{n-1} & 2^n & & & \\ 2^n+2^n & 1-2^{n+1}+2^{n-1} & 2^n & & \\ -2^{n-1}-2^{n-1} & 2^n+2^n & 1-2^{n+1}+2^{n-1} & 2^n & \\ 2^{n-2}+2^{n-2} & \ddots & \ddots & \ddots & \\ & & 2^n+2^n & 1-2^{n+1}+2^{n-1} & 2^n \\ & 2^{n-2}+2^{n-2} & -2^{n-1}-2^{n-1} & 2^n+2^n & 1-2^{n+1}+2^{n-1} \end{pmatrix}$$

Which is a closed form solution.

10 Exercise 35

We want to show that $||x^k - x|| = \dfrac{\delta}{1-\delta}||x^k - x^{k-1}||$. We set $d(x_m, x_n) = ||x^k - x||$, so that x_n is a fixed point and x_m is an iteration. Assuming $m > n$:

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq q^{m-1}d(x_1, x_0) + q^{m-2}d(x_1, x_0) + \cdots + q^n d(x_1, x_0) \\ &= q^n d(x_1, x_0) \sum_{k=0}^{m-n-1} q^k \\ &\leq q^n d(x_1, x_0) \sum_{k=0}^{\infty} q^k \quad \text{since } q < 1 \\ &= q^n d(x_1, x_0) \left(\frac{1}{1-q}\right) \end{aligned}$$

If we choose n as being equal to m, this equation simplifies to $||x^k - x|| = \dfrac{\delta}{1-\delta}||x^k - x^{k-1}||$, as required.