Assignment/Problem Set 6

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1 Exercise 1

We need to find the Schur decomposition of the given matrix A.

a

$$A = \begin{pmatrix} 3 & 8 \\ -2 & 3 \end{pmatrix}$$
$$P^*AP = T$$

We first compute the eigenvalue and vectors of the given matrix A.

$$\det(\lambda I - A) = 0 = (\lambda - 3)^2 + 16 = 0$$
$$= \lambda^2 - 6\lambda + 25 = 0$$
$$\frac{6 \pm \sqrt{-64}}{2} \rightarrow \lambda_1 = 3 + 4i \;\; \lambda_2 = 3 - 4i$$

The following eigenvectors result from the eigenvalues:

$$v_1 = \begin{pmatrix} -2i \\ 1 \end{pmatrix} v_2 = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

We set $v_1 = w_1 = X_1$ and compute w_2 by using the given formula:

$$w_2 = X_2 - \frac{w_1 \cdot X_2}{\|w_1\|^2} w_1$$

$$w_2 = \begin{pmatrix} 2i \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} -2i \\ 1 \end{pmatrix}\begin{pmatrix} 2i \\ 1 \end{pmatrix}}{5} \begin{pmatrix} -2i \\ 1 \end{pmatrix} = \begin{pmatrix} 2i \\ 1 \end{pmatrix} + \frac{3}{5}\begin{pmatrix} -2i \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{4}{5}i \\ \frac{8}{5} \end{pmatrix}$$

Further we compute the normed set of $\left[\frac{w_1}{|w_1|}, \frac{w_2}{|w_2|}\right]$.

$$\frac{w_1}{||w_1||} = \begin{pmatrix} \frac{-2i}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}, \frac{w_2}{||w_2||} = \begin{pmatrix} \frac{i}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

The set represents our transformation vector P, which is :

$$P = \frac{1}{\sqrt{5}} \begin{pmatrix} -2i & i \\ 1 & 2 \end{pmatrix}$$
$$T = P^*AP = \begin{pmatrix} 3 + 4i & 6i \\ 0 & 3 - 4i \end{pmatrix}$$

So we can decompose A into PTP^*.

b

$$A = \begin{pmatrix} 4 & 7 \\ 1 & 12 \end{pmatrix}$$
$$P^*AP = T$$

We first compute the eigenvalue and vectors of the given matrix A.

$$\begin{aligned} \det(\lambda I - A) = 0 &= (\lambda - 4)(\lambda - 12) - 7 = 0 \\ &= \lambda^2 - 16\lambda + 41 = 0 \\ 8 \pm \sqrt{23} &\rightarrow \lambda_1 = 8 + \sqrt{23}\ \lambda_2 = 8 - \sqrt{23} \end{aligned}$$

The following eigenvectors result from the eigenvalues:

$$v_1 = \begin{pmatrix} -4 + \sqrt{23} \\ 1 \end{pmatrix} v_2 = \begin{pmatrix} -4 - \sqrt{23} \\ 1 \end{pmatrix}$$

We set $v_1 = w_1 = X_1$ and compute w_2 by using the given formula:

$$w_2 = X_2 - \frac{w_1 \cdot X_2}{\|w_1\|^2} w_1$$

$$w_2 = \begin{pmatrix} -4 - \sqrt{23} \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} -4 + \sqrt{23} \\ 1 \end{pmatrix}\begin{pmatrix} -4 - \sqrt{23} \\ 1 \end{pmatrix}}{1 + \left(\sqrt{23} - 4\right)^2} \begin{pmatrix} -4 + \sqrt{23} \\ 1 \end{pmatrix} =$$
$$\begin{pmatrix} -4 - \sqrt{23} \\ 1 \end{pmatrix} + \frac{6}{1 + \left(\sqrt{23} - 4\right)^2} \begin{pmatrix} -4 + \sqrt{23} \\ 1 \end{pmatrix}$$
$$= \text{ Something large}$$

Since Gram Schmitt gives us too large vectors here, we better use the simple:

$$w_1 = \begin{pmatrix} -4 + \sqrt{23} \\ 1 \end{pmatrix} w_2 = \begin{pmatrix} 1 \\ 4 - \sqrt{23} \end{pmatrix}$$

Further we compute the normed set of $\left[\frac{w_1}{|w_1|}, \frac{w_2}{|w_2|}\right]$.

$$\frac{w_1}{||w_1||} = \begin{pmatrix} \frac{\sqrt{(23)} - 4}{\sqrt{1 + \left(\sqrt{23} - 4\right)^2}} \\ \frac{1}{\sqrt{1 + \left(\sqrt{23} - 4\right)^2}} \end{pmatrix}, \frac{w_2}{||w_2||} = \begin{pmatrix} \frac{1}{\sqrt{1 + \left(\sqrt{23} - 4\right)^2}} \\ \frac{4 - \sqrt{23}}{\sqrt{1 + \left(\sqrt{23} - 4\right)^2}} \end{pmatrix}$$

The set represents our transformation vector P, which is :

$$P = \frac{1}{\sqrt{1 + \left(\sqrt{23} - 4\right)^2}} \begin{pmatrix} \sqrt{23} - 4 & 1 \\ 1 & 4 - \sqrt{23} \end{pmatrix}$$
$$T = P^*AP = \begin{pmatrix} \frac{2\left(1247 + 243\sqrt{23}\right)}{2401} & -\frac{12\left(107 + 17\sqrt{23}\right)}{2041} \\ 0 & \frac{2\left(465 + 29\sqrt{23}\right)}{2401} \end{pmatrix}$$

So we can decompose A into PTP^*.

2 Exercise 2

By applying Gershgorins Theorem to $D + E$, we see that the spectrum of $D + E$ is the union of the Gershgorin disks of $D + E$, that is:

$$z \in C : |z - a_{ii}| \leq \sum_{j=1, j\neq i}^{n} |a_{ij}|$$

We get:

$$D \cap E = \lambda - \lambda_i - a_{ii} \leq \sum_{j=1, j\neq i}^{n} |a_{ij} + a_{ij}| = 2 \sum_{j=1, j\neq i}^{n} |a_{ij}|$$

Which means that within the intersection, all elements of a are contained by $D \cap E$.

3 Exercise 4

We need to prove if A is Hermitian then the deflation procedure will produce a Hermitian matrix. The procedure can be broken down into:

$$V = BU$$
$$B = \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}$$
$$U = \text{ unitary}$$
$$Q = \text{ unitary } \rightarrow B^* = B$$
$$VAV^* = BUA(BU)^* = B(UAU^*)B^*$$
$$UAU^* = (UAU^*)^* = UAU^* = \text{ diagonal}$$
$$B(UAU^*)B^* = BDB^* = (BDB^*)^* = BD^*B^* = VAV^*$$

As we can see VAV^* is hermitian, since its conjugate transpose it itself again.

4 Exercise 12

We need to prove that if $(I - vv*)x = y$ then $< x, y >$ is real. We can show:

$$(I - vv*) = Q$$
$$Qx = y$$
$$x^*Qx = x^*y =< x, y >$$

As we can see, the problem is only to show that Q is real, so that x^*Qx is real. We see that if vv^* is real, it is obvious that x, y and Q are real. Assume that vv^* is complex, so the matrix which result will be real in the diagonal and complex on the off diagonal elements. Since vv^* is symmetric, the items $vv^*_{ij} = vv^*_{ji}$. We can moreover observe that Schurs decomposition only works if $I - vv^*$ is unitary, meaning that $||vv^*|| = 0$ or $||vv^*||^2 = 2$. The case of $||vv^*||^2 = 0$ is trivial, since the identity matrix is a non complex one, we can verify that x^*Qx is real. Otherwise, if $||vv^*||^2 = 2$, we can decompose Q into a diagonal form:

$$B = PQP^{-1}$$
$$Q \sim B$$

Since Q and B are similar in that case, finding a suitable orthogonalization leads then to the following form:

$$B = \begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix}$$

$$x * PQP^{-1}x = x * Bx =$$

$$(x_1, x_2, x_3, \ldots, x_n) \begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix} \begin{pmatrix} \bar{x_1} \\ \bar{x_2} \\ \bar{x_3} \\ \vdots \\ \bar{x_n} \end{pmatrix} =$$

$$a_{11}x_1\bar{x_1} + a_{22}x_2\bar{x_2} + \ldots + a_{nn}x_n\bar{x_n}$$

As we can see the summations which will be proceeded cancel the complex terms out.

$$< x, y >= x^*Qx = \text{ real} \tag{1}$$

5 Exercise 14

First we show that $||QA||_2 = ||A||_2$

$$||QA||_2 = (QA)^*(QA) =$$
$$A^*Q^*QA =$$
$$A^*A = ||A||_2$$

Now we show that $||AQ||_2 = ||A||_2$

$$||AQ||_2 = (AQ)^*(AQ) =$$
$$Q^*A^*AQ =$$
$$Q^*||A||_2Q =$$
$$\text{Since the norm is a number}$$
$$||A||_2Q^*Q = ||A||_2I = ||A||_2$$

6 Exercise 29

$$A = \begin{pmatrix} 6 & 2 & 1 \\ 1 & -5 & 0 \\ 2 & 1 & 4 \end{pmatrix}$$

The upper limit of the eigenvalues are bounded by Gershgorin's Theorem.

$$\lambda \in C : |\lambda| \leq ||A||_\infty$$

We can calculate $||A||_\infty$:

$$||A||_\infty = \max(6 + 2 + 1, 1 + |-5| + 0, 2 + 1 + 4) = 9$$
$$\rightarrow |\lambda| \leq 9$$

The lower bound can be found by using the following theorem:

$$\lambda - a_{ii} \leq \sum_{i} \neq j |a_{ij}|$$

We get the following equations:

$$\lambda - a_{11} \leq 2 + 1$$
$$\lambda - a_{22} \leq 1$$
$$\lambda - a_{33} \leq 2 + 1$$
$$\therefore \lambda \geq 1$$

Which concludes that $1 \leq \lambda \leq 9$.