Assignment/Problem Set 5

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1 Exercise 2

We want to know if a polynomial p of degree m has a solution space of equal dimension m.

$$p(E)x = 0$$

$$p(E) = \sum_{i}^{m} c_i E^i = c_0 + c_1 E + c_2 E^2 + \ldots + c_m E^m$$

From the definition we get:

$$p(E)u = \sum_{i}^{m} c_i (E^i u) = \sum_{i}^{m} c_i \lambda^i u = p(\lambda)u = 0$$

We can see from that definition, that if $Eu = \lambda u$, the resulting solution space spanned by u will be of dimension m. It is sufficient to show that $Eu = \lambda u$, or differently that an eigenvalue in E exists.

$$p_E(t) = \det(tI - A)$$

$$= (tI - E) = \begin{pmatrix} (t-1) & 1 & 0 & 0 & 0 \\ & (t-1) & 1 & 0 & 0 \\ & & \ddots & \ddots & 0 \\ & & & (t-1) & 1 \\ & & & & (t-1) \end{pmatrix}$$

$$t_1 = 1$$

As we can see, there exists the eigenvalue $\lambda_1 = t_1 = 1$ with multiplicity m, so that $Eu = \lambda u$, where $u = x$ holds.

2 Exercise 3

Let p be a polynomial of degree m with $p(0) \neq 0$. If a sequence x contains m consecutive zeros and $p(E)x = 0$, then $x = 0$. We can rewrite this statement as $\sum_{i=0}^{m} c_i \lambda^i x = 0$. To make sense in this expression x must have dimension m, since otherwise the equal statement $\sum_{i=0}^{m} c_i E^i x = 0$ would not make sense. Therefore if x needs to have m consecutive zeros and has dimension m, it must be zero $x = 0$.

3 Exercise 5

We define the operator E as:

$$E^0 = I$$

$$E^1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 1 & 0 & 0 \\ & & \ddots & \ddots & 0 \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

We calculate the eigenvalues:

$$\det(\lambda I - E)x = 0$$

$$= \begin{pmatrix} (\lambda-1) & -1 & 0 & 0 & 0 \\ & (\lambda-1) & -1 & 0 & 0 \\ & & \ddots & \ddots & 0 \\ & & & (\lambda-1) & -1 \\ & & & & (\lambda-1) \end{pmatrix}$$

$$(\lambda - 1)^n = 0 \to \lambda = 0$$

So what is left over to calculate the eigenvectors is the negative matrix E.

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} - Ev = 0$$

$$(-E)v = \begin{pmatrix} -v_2 \\ -v_3 \\ \vdots \\ -v_n \\ 0 \end{pmatrix}$$

So the eigenvectors are the negative of the applied vectors of E.

4 Exercise 11

a Give bases consisting of real sequences for the solution space.

$$p(E) = (4E^0 - 3E^2 + E^3)x = 0$$

$$p(\lambda) = 4 - 3\lambda^2 + \lambda^3$$

$$p(\lambda) = (\lambda + 1)(\lambda - 2)^2$$

$$\lambda_1 = -1, \lambda_2 = 2$$

The solution space spanned by $u_1 = (\lambda_1, \lambda_1^2, \lambda_1^3, \ldots)$ and $u_2 = (\lambda_2, \lambda_2^2, \lambda_2^3, \ldots)$. Since we find multiple zeros, we need to check if there are more. Indeed we can find one by using the derivation:

$$u_3 = \lambda_2^{'} = (1, 2\lambda_2, 3\lambda_2, \ldots)$$

If the determinate is independent, this means that the solution also is.

$$\det \begin{pmatrix} \lambda_1 & \lambda_2 & 1 \\ \lambda_1^2 & \lambda_2^2 & 2\lambda_2 \\ \lambda_1^3 & \lambda_2^3 & 3\lambda_2^2 \end{pmatrix} \det \begin{pmatrix} -1 & 2 & 1 \\ 1 & 4 & 4 \\ -1 & 6 & 12 \end{pmatrix} = -46$$

The third solution u_3 follows from:

$$p(E)u_3 = p(\lambda_2)u_3^{'} + p^{'}(\lambda_2)u_2 = 0$$

The basis for the solution space is then $\{u_1, u_2, u_3\}$.

b Give bases consisting of real sequences for the solution space.

$$p(E) = (3E^0 - 2E^2 + E^3)x = 0$$
$$p(\lambda) = 3 - 2\lambda^2 + \lambda^3$$
$$p(\lambda) = (\lambda + 1)(\lambda^2 - 3\lambda + 3) = 0$$
$$\lambda_1 = -1$$

Since the second term is always > 0, we cannot use it to calculate the eigenvalues. So we get as eigenvalues $\lambda_1 = -1$

5 Exercise 14

We define an operator $\triangle$, which is:

$$\triangle x = (x_2 - x_1, x_3 - x_2, x_4 - x_3, \ldots)$$

Let $E = I + \triangle$.

$$(I + \triangle)x = x + \triangle x = (x_1, x_2, x_3, \ldots) + (x_2 - x_1, x_3 - x_2, x_4 - x_3, \ldots) = (x_2, x_3, \ldots) = Ex$$

Let $\pi_n : R^\infty \to R^\infty$ be the projection onto the first n coordinated defined by :

$$\pi_n(x_1, x_2, x_3, \ldots) = (x_1, x_2, x_n, 0, 0, \ldots)$$

Suppose $K : R^\infty \to R^\infty$ and $L : R^\infty \to R^\infty$ are linear operators such that $Kx_i = Lx_i$ for every $i \in R$ where $x_i = (\lambda, \lambda^2, \ldots)$.If there exists $r \in N$ such that $\pi_n Kx = \pi_n K\pi_{n+r}x$ and $\pi_n Lx = \pi_n L\pi_{n+r}x$ holds for every $x \in R^\infty$ and $n \in N$, then $K = L$.

To show that we define $u_i = \pi_{n+r}x\lambda_i$ where $\lambda_i = i, i = 1, 2, \ldots n + r$.

$$M = \begin{pmatrix} u_{1,1} & \ldots & u_{n+r,1} \\ \vdots & \ddots & \vdots \\ u_{1,n+r} & \ldots & u_{n+r,n+r} \end{pmatrix} = \begin{pmatrix} 1 & 2 & \ldots & n+r \\ 1^2 & 2^2 & \ldots & (n+r)^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1^{n+r} & 2^{n+r} & \ldots & (n+r)^{n+r} \end{pmatrix}$$

The matrix M is therefore non -singular. Now given any $x \in R^\infty$ there is a unique $\beta \in R^{n+r}$ such that $\pi_{n+r}x = \beta_1 u_1 + \beta_2 u_2 + \ldots + \beta_{n+r}u_{n+r}$. Let $z = \beta_1 x_{\lambda_1} + \beta_2 x_{\lambda_2} + \ldots + \beta_{n+r}x_{\lambda_{n+r}}$. It follows that:

$$\pi_{n+r}x = \pi_{n+r}z$$

Moreover:

$$\pi_n Ku_i = \pi_n K\pi_{n+r}x_{\lambda_i} = \pi_n Kx_{\lambda_i} = \pi_n Lx_{\lambda_i} = \pi_n L\pi_{n+r}x_{\lambda_i} = \pi_n Lu_i$$
$$\pi_n Kx = \pi_n K\pi_{n+r}x = \pi_n K\pi_{n+r}z$$
$$= \beta_1 Ku_1 + \beta_2 Ku_2 + \ldots + \beta_{n+r}Ku_{n+r} = \beta_1 Lu_1 + \beta_2 Lu_2 + \ldots + \beta_{n+r}Lu_{n+r} = \pi_n Lx$$

Since $\pi_n Kx = \pi_n Lx$ it follows that $\pi_n K = \pi_n L$. So $K = L$.

Let

$$K = p(E) = \sum_{i=0}^{m} c_i E^i$$
$$L = p(I) + p^{'}(I) \triangle + \frac{1}{2} p^{''}(I) \triangle^2$$

From what on we proved we can follow that :

$$\pi_n K x = \pi_n K \pi_{n+m} x$$
$$pi_n L x = \pi_n L \pi_{n+m} x$$

We note that $Ex_\lambda = \lambda x_\lambda$.

$$\triangle x_\lambda = (E - I)x_\lambda = \lambda x_\lambda - x_\lambda = (\lambda - 1)x_\lambda$$

Since the eigenvalues are one, Taylor series expansion gives us:

$$p(\lambda) = \sum_{i}^{m} c_i \lambda^i = p(1) + p^{'}(1)(\lambda - 1) + \frac{1}{2!} p^{''}(1)(\lambda - 1)^2 + \ldots + \frac{1}{m!}(1)(\lambda - 1)^m$$
$$L x_\lambda = p(1)x_\lambda + p^{'}(1)(\lambda - 1)x_\lambda + \frac{1}{2!} p^{''}(1)(\lambda - 1)^2 x_\lambda + \ldots + \frac{1}{m!}(1)(\lambda - 1)^m x_\lambda$$
$$= x_\lambda \left(p(1) + p^{'}(1)(\lambda - 1) + \frac{1}{2!} p^{''}(1)(\lambda - 1)^2 + \ldots + \frac{1}{m!}(1)(\lambda - 1)^m \right) = p(\lambda)x_\lambda = p(E)x_\lambda = K x_\lambda$$

As it can be seen, it is analogous in its matrix form, by using $\lambda = E$, $1 = I$, so that the Taylor terms can be expressed as :

$$1 - \lambda = I - E = \triangle$$

Therefore we can satisfy our hypothesis with $r = m$. The result $K = L$ follows.

6 Exercise 15

We need to prove that if $x = (\lambda, \lambda^2, \lambda^3, \ldots)$ and p is a polynomial, then $p(\triangle)x = p(\lambda - 1)x$. We use induction to prove that :

$$p(\triangle)x_\lambda = p(\lambda - 1)x_\lambda$$

Let $p_0 \in P_0$. Then $p_0(\mu) = c$ for some $c \in C$. Therefore

$$p_0(\triangle)x_\lambda = c x_\lambda = p_0(\lambda - 1)x_\lambda$$

We begin the induction by supposing that $p_n(\triangle)x_n = p_n(\lambda - 1)x_\lambda$ for every $p_n \in P_n$. Let $p_{n+1} \in P_{n+1}$. We can write $p_{n+1}(\mu) = \mu p_n(\mu) + c$ for some $p_n \in P_n$ and $c \in C$. To complete the induction we need to show that $p_{n+1}(\triangle)x_\lambda = p_{n+1}(\lambda - 1)x_\lambda$. The hypothesis gives us $p_n(\triangle)x_\lambda = p_n(\lambda - 1)x_\lambda$.

$$p_{n+1}(\triangle)x_\lambda = (\triangle p_n(\triangle) + cI)x_\lambda = \triangle p_n(\triangle)x_\lambda + c x_\lambda = \triangle(p_n(\lambda - 1)x_\lambda) + c x_\lambda$$
$$= (p_n(\lambda - 1)) \triangle x_\lambda + c x_\lambda = (p_n(\lambda - 1))(E - 1)x_\lambda$$
$$= (p_n(\lambda - 1))(\lambda - 1)x_\lambda + c x_\lambda = ((\lambda - 1)p_n(\lambda - 1) + c)x_\lambda$$
$$= p_{n+1}(\lambda - 1)x_\lambda$$

We have shown that the induction holds.

To solve the difference equation $p(\triangle) = 0$, we define $q(\lambda) = p(\lambda - 1)$. Taylors series expansion gives us:

$$q(\lambda) = \sum_{i=0}^{m} \frac{q^{(i)}(1)}{i!}(\lambda - 1)^i$$
$$p(\mu) = \sum_{i=0}^{m} \frac{q^{(i)}(1)}{i!}\mu^i$$
$$q(E) = \sum_{i=0}^{m} \frac{q^{(i)}(1)}{i!}\triangle^i$$

Since $q^i(I) = q^i(1)I$, we get

$$q(E) = \sum_{i=0}^{m} \frac{q^{(i)}(1)}{i!}\triangle^i = p(\triangle)$$

We can solve $p(\triangle)x = 0$ by solving $q(E)x = 0$, which was already discussed in the lessons.

7 Exercise 16

We define $p(a) = a^n$ and $q(\lambda) = (\lambda - 1)$.

$$q(\lambda) = (\lambda - 1)^n = \sum_{i=0}^{n} \binom{n}{i} \lambda^i (-1)^{n-i} = (-1)^n \sum_{i=0}^{n} \binom{n}{i} \lambda^i (-1)^i$$
$$= (-1)^n \left(\lambda^0 - n\lambda + \frac{1}{2} n\,(n-1)\,\lambda^2 - \frac{1}{3!} n\,(n-1)\,(n-2)\,\lambda^3 + \ldots + (-1)^n\,\lambda^n \right)$$

Since $\triangle^n = p(\triangle) = q(E)$, we get:

$$\triangle^n = \left(E^0 - nE + \frac{1}{2} n(n-1)E^2 - \frac{1}{3!} n(n-1)(n-2)E^3 + \ldots + (-1)^n E^n \right)$$