Assignment/Problem Set 4

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1 Exercise 2

We prove that $x_n = (1 + \frac{1}{n})^n$ converges to e in its limits. The proof assumes that some real function exists, e^x, which is monotonic, so that it's inverse $\log(x)$ also has a monotonic growth.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(e^{\log(1+\frac{1}{n})^n}\right) = e^{\lim_{n\to\infty}\left(\log(1+\frac{1}{n}\right)^n}$$
$$= e^{\lim_{n\to\infty}\left(n\log(1+\frac{1}{n}\right)} = e^{\lim_{n\to\infty}\left(\frac{\log\left(1+\frac{1}{n}\right)}{\frac{1}{n}}\right)}$$
$$= e^{\lim_{n\to\infty}\left(\frac{\frac{1}{1+\frac{1}{n}}\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}}\right)} = e^{\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)} = e^1 = e < 3$$

We have shown that the series $(1 + \frac{1}{n})$ converges to a value lesser than 3.

2 Exercise 6

We need to show that $x_n = O(\alpha_n) \to cx_n = O(\alpha_n)$.
We use the definition of big O, so we get:

$$cx_n = O(\alpha_n) = \lim_{n\to\infty} c\left|\frac{x_n}{\alpha_n}\right| < \infty$$
$$= c\lim_{n\to\infty}\left|\frac{x_n}{\alpha_n}\right| < \infty$$
$$= \lim_{n\to\infty}\left|\frac{x_n}{\alpha_n}\right| < \infty$$
$$= x_n = O(\alpha_n)$$

It is shown that this equation holds, since constants in cases of infinite calculations cancel out.

3 Exercise 7

We want to show that from $x_n = O(\alpha_n)$ follows $\frac{x_n}{\log(n)} = o(\alpha_n)$.

$$\frac{x_n}{ln(n)} = o(\alpha_n) = \lim_{n\to\infty}\left|\frac{x_n}{\log(n)\alpha_n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{1}{\log(n)}\right|\left|\frac{x_n}{\alpha_n}\right|$$
$$\lim_{n\to\infty}\left|\frac{1}{\log(n)}\right|\lim_{n\to\infty}\left|\frac{x_n}{\alpha_n}\right| \leq C \to O(\alpha_n)$$

Since the equation $\frac{x_n}{\alpha_n} \leq C$ is bounded by C, yet converges to 0, if $n \to \infty$, we can see that the factor $\log(n)$ only increases its convergence.

4 Exercise 8

We need to find the best value for k in the term $\cos(x) - 1 + \frac{x^2}{2} = O(x^k)$. We use the Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ here to estimate $\cos(x)$. We use for convenience $a = 0$.

$$
\begin{aligned}
\cos(x) - 1 + \frac{x^2}{2} &= O(x^k) \\
1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 \ldots - 1 + \frac{x^2}{2} \\
= \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 \ldots &= O(x^k) \to k = 4
\end{aligned}
$$

For the value of $k =$, we can see that, as long as $x \to 0$, the terms after the x^4 are always smaller than Cx^4, so that x^4 is an upper bound.

5 Exercise 12

We show that for any $r > 0, x^r = O(x^r)$.

$$
\begin{aligned}
\lim_{x\to\infty} \frac{x^r}{e^x} = \lim_{x\to\infty} \frac{r\log(x)}{x} = r \lim_{x\to\infty} \frac{\log(x)}{x} \\
\left| \lim_{x\to\infty} \frac{\log(x)}{x} \right| \leq \frac{C}{r}
\end{aligned}
$$

Which shows that this equation holds for every $r > 0$.

6 Exercise 13

We show that for any $r > 0, \log(x) = O(x^r)$.

$$
\lim_{x\to\infty} \left| \frac{\log(x)}{x^r} \right|
$$

we use Taylor series at a = 1

$$
\log(x) = -\frac{1}{x}(x-1) + \frac{1}{x^2}(x-1)^2 + \underbrace{\ldots}_{\to 0}
$$

If the denominator is always larger than the nominator, the fraction will converge to zero

$$
\begin{aligned}
\log(x) = -1 + \frac{1}{x} + 1 - \frac{2}{x} + \frac{1}{x^2} \\
\lim_{x\to\infty} \underbrace{x^r}_{\to\infty} > \log(x) = \underbrace{-\frac{1}{x} + \frac{1}{x^2}}_{\to 0}
\end{aligned}
$$

So the fraction $\frac{\log(x)}{x^r}$ is bounded and will converge to zero.

7 Exercise 16

We need to determinate the best value for k in the equation $\tan^{-1}(x) = x + O(x^k)$, alternatively : $O(x^k) = \tan^{-1}(x) - x$

$$\begin{aligned}
\tan^{-1}(x) &= 1 - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 \dots \\
\tan^{-1}(x) - x &= 1 - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 \dots - x \\
&1 + x\left(-\frac{1}{3}x^2 + \frac{1}{5}x^5 - \frac{1}{7}x^7 \dots - 1\right)
\end{aligned}$$

As we can see the whole expression converges to one, if $x \to 0$. So we choose our C so that $1 \leq Cx^k$. If we choose $k = 0$, C can be chosen ≥ 1, so that the equation holds.

8 Exercise 22

In each $n \to \infty$.

a This assertion does not hold:

$$\frac{n+1}{n^2} = o(\frac{1}{n})$$

$$\lim_{n\to\infty} \frac{\frac{n+1}{n^2}}{\frac{1}{n}} \widehat{=} 0$$

$$= \lim_{n\to\infty} \frac{n^2+n}{n^2} = \lim_{n\to\infty} 1 + \frac{1}{n} = 1 \neq 0$$

The value of this equation is not bounded by 0, so it could be big O, but not a small o.

b This assertion does not hold either.

$$\frac{n+1}{\sqrt[2]{n}} = o(1)$$

$$\lim_{n\to\infty} \frac{n+1}{\sqrt[2]{n}} = \lim_{n\to\infty} \sqrt[2]{n} + 1 = \infty \neq 0$$

c

$$\frac{1}{ln(n)} = O(\frac{1}{n})$$

$$\lim_{n\to\infty} \frac{e^n}{n} = \infty$$

This term should approach any constant L, which is smaller than ∞, which it doesn't.

d This term holds the assertion

$$\frac{1}{n\log n} = o(\frac{1}{n})$$

$$\lim_{n\to\infty} \frac{n}{n\log(n)} = \lim_{n\to\infty} \frac{1}{\log n} = \lim_{n\to\infty} \frac{e}{n} = 0$$

This term holds.

e This assertion does not hold

$$\frac{e^n}{n^5} = O(\frac{1}{n})$$

$$\lim_{n\to\infty} \frac{ne^n}{n^5} = \lim_{n\to\infty} \frac{e^n}{n^4}$$

$$= \lim_{n\to\infty} \frac{n}{4\log(n)}$$

Using L'hospital

$$= \lim_{n\to\infty} \frac{1}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} n = \infty$$

As it can be seen this assertion does not hold either.