Assignment/Problem Set 8

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1 Exercise 1

As we can see, the $n \times n$ upper hessenberg matrix, we can see that the spectra of A is given as $\sigma(A)$, which are the eigenvalues of A. The two submatrices at any given index k, have the following form:

$$A_{ij}(1) = (1 \leq i, j < k)$$
$$A_{ij}(2) = (k < i, j \leq n)$$

The second matrix is always non existing as long $k < i$, which means if we choose $k = n$, we only generate the first matrix, which is in that case the:

$$A_{ij}(1) = (1 \leq i, j < n)$$
$$A_{ij}(2) = (n < i, j \leq n)$$

In the other extreme case where $k = 1$, we get:

$$A_{ij}(1) = (1 \leq i, j < 1)$$
$$A_{ij}(2) = (1 < i, j \leq n)$$

Here we can see that the second matrix plays the only role and is the only submatrix, thus the $\sigma(A_{ij}(2))$ is equal to $\sigma(A)$. So we can see that in every case in between, the two sub matrices are only occupying the relevant diagonal terms plus the upper diagonal terms, so that for every i, j the two spectra sum to to be the spectrum of A. $\sigma(A_{ij}(1)) + \sigma(A_{ij}(2)) = \sigma(A)$.

2 Exercise 2

We need firstly to show that $A_{k+1} = Q_k^* A_k Q_k$.

$$A_k = Q_k R_k$$
$$A_{k+1} = R_k Q_k = (Q_k^* Q_k)(R_k Q_k) = Q_k^* A_k Q_k$$

Moreover we need to prove the specific form of A^k:

$$Q_1^* A_1 Q_1 = A_2$$
$$(Q_1^* A_1 Q_1)^k = A_2^k$$
$$Q_1^* A_1^k Q_1 = A_2^k$$
$$A_1^k = A^k = Q_1 A_2^k Q_1^*$$
$$= Q_1 (A_2^{k-1} A_2) Q_1^*$$
$$= Q_1 (A_2^{k-1} (A_2 Q_1^*)$$
$$= Q_1 A_2^{k-1} (R_1 Q_1 Q_1^*)$$
$$= Q_1 A_2^{k-1} (R_1)$$

Now we apply the same procedure recursively for all k, so we get:

$$
\begin{aligned}
(A_3)^{k-1} &= (Q_2^* A_2 Q_2)^{k-1} \\
A_2^{k-1} &= Q_2 A_3^{k-1} Q_2 \\
&\vdots \\
&Q_2 A_3^{k-2} R_2
\end{aligned}
$$

If we insert all our solutions into each other, we get the final result as:

$$
(Q_1, \ldots, Q_k)(R_k, \ldots, R_1) = A^k
$$

3 Exercise 6

We need to find the eigenvalues of A:

$$
\begin{pmatrix} -1 & -4 & 1 \\ -1 & -2 & -5 \\ 5 & 4 & 3 \end{pmatrix}
$$

We compute the eigenvalues via the usual determinant formula: $\det(A - \lambda I)$:

$$
\begin{aligned}
p_A(\lambda) &= -\lambda^3 - 4\lambda + 80 \\
\lambda_1 &= 4 \\
\lambda_2 &= 4i - 2 \\
\lambda_3 &= -4i - 2
\end{aligned}
$$

4 Exercise 7

Unitary similar means that $AP = PB$, where P is unitary. As we already have seen in Exercise 2:

$$
\begin{aligned}
A_k &= Q_k R_k \\
A_{k+1} &= R_k Q_k = (Q_k^* Q_k)(R_k Q_k) = Q_k^* A_k Q_k
\end{aligned}
$$

This already shows that A_{k+1} and A_k are similar. Since via definition Q is unitary, the similarity extends to unitary similarity.

5 Exercise 11

Since the matrix A is already upper triangular, even with it's sub matrices , we can easily compute the eigenvalues by only considering the diagonal terms, which are A_{ii}. We say that we have another matrix with the form of A, which we denote as Λ, which has the following properties:

$$
\Lambda = \begin{pmatrix} \Lambda_{11} & & & \\ & \Lambda_{22} & & \\ & & \ddots & \\ & & & \Lambda_{nn} \end{pmatrix}
$$

$$
\Lambda_{ii} = \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}
$$

Then we can calculate the eigenvalues as usual:

$$\det(\Lambda I - A) = \begin{pmatrix} \Lambda_{11} - A_{11} & -A_{12} & \dots & -A_{1n} \\ & \Lambda_{22} - A_{22} & \ddots & \vdots \\ & & \ddots & -A_{n-1,n} \\ & & & \Lambda_{nn} - A_{nn} \end{pmatrix}$$

Which is only again the calculation for each eigenvalue of $\Lambda_{ii} - A_{ii}$. For each of these matrices we use the following formula to calculate the eigenvalues:

$$\lambda_1 = \frac{t}{2} + \sqrt{\frac{t^2}{4 - d}}$$
$$\lambda_2 = \frac{t}{2} - \sqrt{\frac{t^2}{4 - d}}$$
$$t = tr(A_{ii})$$

Finally we can see that we get at most $2n$ eigenvalues, since for every block, we get at most 2 eigenvalues.