Assignment/Problem Set 7

Alexander Schmitt
ID: 1140339105
E-mail: info@a-schmitt.de

1 Exercise 1

Let

$$H = I - 2uu^*$$
$$||u|| = 1$$
$$Hx = -\alpha e_1$$
$$Hx = x - 2(u^*x)u = -\alpha e_1$$

Since H is an orthonormal matrix:

$$||x|| = ||Hx|| = |\alpha|$$
$$\alpha = \pm||x||$$
$$x^H x = ||x||2 - 2(u^*x)^2 = -\alpha e_1^T x$$

So we can get:

$$u^*x = \sqrt{||x||^2 \pm ||x||e_1^*x}$$

And set $\alpha = \text{sgn}(e_1^*x)||x||$. Thus:

$$u^*x = \sqrt{\frac{1}{2}||x||(||x|| + ||x_1||}$$
$$u = \frac{x + \alpha e_1}{2(u^*x)}$$

This method is better for constructing householder matrices because we dont have any special case for complex numbers since the inner product is real.

2 Exercise 9

We simply want to minimize our given formula for this exercise. The task is to maximize t for $f(t) = ||u - tx||_2$.

$$\frac{\partial}{\partial x_k}\|\mathbf{x}\|_2 = \frac{x_k}{\|\mathbf{x}\|_2}$$
$$||u - tx||^2 \frac{\partial}{\partial t} = -2x||u - tx||Norm^{'}(u - tx) =$$
$$-2x||u - tx||\frac{u - tx}{||u - tx||} = -2x(u - tx)$$

Searching for a minimum:

$$-2x(u-tx)=0 \Rightarrow t=ux^*$$

Now we need to verify that this is a minimum.

$$f^{''}(t)=\frac{\partial}{\partial t}-2x(u-tx)=2x^2 \Rightarrow 2x^2>0 \Rightarrow \text{ Minimum}$$

As we can see since the second derivative is always >0, we found a minimum for $f(t)$ at ux^*.

3 Exercise 16

We need to find the QR of $\begin{pmatrix} 0 & -4 \\ 0 & 0 \\ -5 & -2 \end{pmatrix}$. We recall the algorithm:

$$\begin{gathered}
\alpha=-\mathrm{e}^{\mathrm{i}\arg x_k}\|\mathbf{x}\| \\
\mathbf{u}=\mathbf{x}-\alpha\mathbf{e}_1 \\
e_1=(1,\ldots) \\
\mathbf{v}=\frac{\mathbf{u}}{\|\mathbf{u}\|} \\
Q=I-2\mathbf{v}\mathbf{v}^T
\end{gathered}$$

$$a_1=\begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix}, ||a_1||e_1=\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}, \alpha=-5$$

$$u=\begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix}+(-5)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} -5 \\ 0 \\ -5 \end{pmatrix}$$

$$v=\frac{u}{||u||}=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, H=I-2v^*v=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}-\frac{2}{\sqrt{2}\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$H=Q_1=\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$Q_1A=\begin{pmatrix} 5 & 2 \\ 0 & 0 \\ 0 & 4 \end{pmatrix},$$

Now we do the second iteration:

$$a_2=\begin{pmatrix} 0 \\ 4 \end{pmatrix}, \alpha=-4, u=\begin{pmatrix} -4 \\ 4 \end{pmatrix}$$

$$v=\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}, Q_2=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

So we can compute Q and R :

$$Q=Q_1^*Q_2^*=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, R=Q^*A=\begin{pmatrix} -5 & 2 \\ 0 & -4 \\ 0 & 0 \end{pmatrix}$$

4 Exercise 17

We need to proof that $||x||_2 = ||Qx||_2$ and $\langle x, y\rangle = \langle Qx, Qy\rangle$. We can proof the inner product:

$$\begin{aligned}\langle Qx, Qy\rangle &= (Qx)^* Qy \\ &= x^* Q^* Q y \\ &= x^* y \\ &= \langle x, y\rangle\end{aligned}$$

Which means that Q preserves the length of the vectors.

$$\begin{aligned}||Qx||_2^2 &= (Qx)^* Qx \\ &= x^* Q^* Q x \\ &= x^* x \\ &= ||x||_2^2 \\ \Rightarrow ||Qx||_2 &= ||x||_2\end{aligned}$$

5 Exercise 19

We can compute the derivative of the following function, to show that $F(x) = ||Ax - b||_2^2 + \alpha||x||_2^2$ minimizes $(A^T A + \alpha I)x = A^T b$.

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$\lim_{h\to 0} \frac{F(x) + (Ah)^T Ah + \alpha h^T h - F(x)}{h}$$

$$\lim_{h\to 0} \frac{(Ah)^T Ah + \alpha h^T h}{h}$$

$$\lim_{h\to 0} \frac{||Ah||_2^2 + \alpha||h||_2^2}{h}$$

Since the limit goes to zero, we can see that the function above minimizes $(A^T A + \alpha I)x = A^T b, \forall x$.

6 Exercise 33

From the matrices we can write out the equation system

$$\begin{aligned}3x + 2y &= 3 \\ 2x + 3y &= 0 \\ x + 2y &= 0\end{aligned}$$

We define a function $S(x, y)$ which is out least squares function.

$$S(x,y) = (3 - (3x + 2y))^2 + (0 - (2x + 3y))^2 + (1 - (x + 2y))^2 = 14x^2 + 17y^2 + 28xy - 20x - 16y + 10$$

$$\begin{aligned}\frac{\partial S}{\partial x} &= 28x + 28y - 20 = 0 = S_1 \\ \frac{\partial S}{\partial y} &= 34y + 28x - 16 = 0 = S_2 \\ S_2 - S_1 &= 6y + 4 = 0 \\ y &= -\frac{2}{3}, \\ x &= \frac{29}{21}\end{aligned}$$