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ON PEIRCE'S CRITERION FOR THE REJECTION OF DOUBTFUL OBSERVATIONS, WITH TABLES FOR FACILITATING ITS APPLICATION.

A REPORT TO PROFESSOR A. D. BACHE, SUPERINTENDENT OF THE U. S. COAST-SURVEY.

By B. A. GOULD, Jr.

[Communicated by Authority of the Hon. Secretary of the Treasury.]

Dear Sir,—In the *Astronomical Journal*, No. 45, Professor Peirce has given the results of the successful investigation of a singular problem, and one unquestionably among the most important of any which could be proposed in its relations to all those exact sciences to which quantitative research or measurement may be applied. This problem was nothing less than the attainment of a formula which should be legitimately derived from the fundamental principles of the Calculus of Probabilities, and furnish an exact criterion for the recognition of those observations which differ so much from the average of a series as to indicate some abnormal source of error, which would vitiate the result. The delicate task of discriminating between such observations, and those whose discordance, although great, ought not to be deemed abnormal, has hitherto been left to the arbitrary judgement of individuals; and the present introduction of a rigorous mathematical ordeal for testing the extent of tolerable discrepancy cannot fail to exercise a highly beneficial influence.

But important as is the use of this criterion, and simple as is its practical application, I am not aware of its adoption in the published discussion of any extended series of observations. In the criticism, however, of various series with which I have had to do in the study of the telegraphic results of the Coast-Survey, I have applied Peirce's Criterion with signal advantage, and have been led to the preparation of tables for facilitating its use. Such tables form a part of my Report upon the Difference of Longitude between the Seaton Station and Raleigh. Subsequently, having in mind its great importance for the discussion of other groups of observations, possibly extremely dissimilar in their character, it has seemed desirable to extend the limits of these tables, and to submit them in an independent form.

The discussion of the formulas is unnecessary, after the reference to the original memoir. But it may not be amiss to repeat them concisely; and perhaps the slightly different form in which I have arranged them, with a view solely to facility in their practical employment, may not be deemed ill-judged.

Let, then,

$m =$ the number of unknown quantities contained in the observations.

$N =$ the whole number of observations.

$\varepsilon =$ the mean error of the series.

$x\,\varepsilon =$ the limit of error which demands the rejection of n observations.

$y =$ the probability that an observation ought to be rejected on account of its discordance.

$\lambda\,\varepsilon =$ the mean error after the n observations have been rejected.

$\psi\,x = \frac{2}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{1}{2}x^2}$, e being the hyperbolic base.

Then, to authorize and require the rejection of n observations, we must have

$$\lambda^{1-y}\; e^{\frac{1}{2}y\,(x^2-1)}\; (\psi\,x)^y < Q$$

or

$$\lambda^{N-n}\; e^{\frac{1}{2}n\,(x^2-1)}\; (\psi\,x)^n < Q^N \qquad (A.)$$

the value of Q being determined by the formula

$$Q^N = \frac{n^n\,(N-n)^{N-n}}{N^N}\,. \qquad (B.)$$

The assumption that the excess of the sum of the squares of the residual errors above the corresponding sum in the series remaining after the n observations have been excluded is only equal to the sum of the squares of the rejected residuals, gives the approximate equation,

$$\lambda^2 = \frac{N-m-nx^2}{N-m-n},$$

whence

$$x^2 = 1 + \frac{N-m-n}{n}(1-\lambda^2). \qquad (C.)$$

Then if

$$R = e^{\frac{1}{2}(x^2-1)}\,\psi x \qquad (D.)$$

the equation (A) becomes

$$\lambda^{N-n} R^n = Q^N. \qquad (A'.)$$

In the practical employment of these formulas, it will be found most convenient to determine first the value of Q from (B). Next, assuming any approximate R, the corresponding λ is determined by means of (A'), and by substitution in (C) furnishes a value of x from which R may be obtained with greatly increased precision. The computation is then repeated if necessary.

Professor Peirce proposed three tables. The first, containing values of x for the double argument $m' = \frac{m}{N}$ and $y = \frac{n}{N}$, would include the values corresponding to any given number of unknown quantities. The second, a table of single entry, would give log (Br.) Q for different values of y, and the third contain log R, for argument x, being deduced from the solution of equation (D).

This arrangement is unquestionably the neatest and most compendious; still I have concluded to give the tables a greater expansion by employing both n and N as arguments. This demands a separate table for x, corresponding to each value of m; but as the number of unknown quantities implicitly involved will be large in very rare cases only, I have felt less hesitation upon this account,—and the extreme facility of its use will more than compensate for a sacrifice of theoretical elegance. The most frequent case of a number of unknown quantities greater than two is probably that presented by the elements of a celestial body,—all of these elements being supposed unknown. Yet, even here, it will in general be found possible to consider two, if indeed not three, of these as sufficiently known for the purpose of ordinary investigations.

The following Tables I. and II. have therefore been computed by means of approximations through equations (A'), (B), and (C) for every number of observations up to 60 inclusive, and for the several hypotheses corresponding to the rejection of any number less than ten. The former table is calculated for the case of *one* unknown quantity; the latter, for *two*. The function obtained by entering at the side with N, and at the top with n is x^2, which when multiplied by ε^2 gives the limit of Δ^2, the square of the residual of any observation which ought to be tolerated. The value of ε^2 is of course deduced from the formula

$$\Sigma \Delta^2 = (N-m)\,\varepsilon^2.$$

It will be carefully borne in mind, that, in the application of these tables, each hypothesis must be tried in succession, beginning with $n = 1$. After it is found that one observation is excluded upon this supposition, the same criterion for two observations may be applied, using the column for $n = 2$, and so on in succession.

No farther explanation seems to be required, but if examples be wished, they may be easily obtained by comparison with the two problems given by Peirce in his memoir (*A. J.*, II. 162). For the first, where $N = 30$, $n = 1$, $m = 2$, our Table II. shows $x^2 = 5.54$; the value obtained in the place cited was after two approximations $x^2 = 5.51$. For the second example, $N = 15$, $m = 2$, $n = 1$; the corresponding limit which the criterion furnishes for the rejection of one observation is $x^2 = 4.080$, and inasmuch as one observation was thus excluded, the limit for the rejection of two is found to be $x^2 = 2.991$.

For those cases especially which fall beyond the limits of Tables I. and II., the tables III. and IV. are added, the former giving by double entry with the arguments n and N the values of log. Q^N deduced from equation (B) for values of N up to 90, and of n to 9 inclusive. The table of single entry proposed by Professor Peirce, and giving the values of log. Q for y, may, like the tables for x, be thus condensed into much smaller compass; but as the values of N increase, the influence of the neglected decimals becomes more and more important, and in the determination of $N \log Q$ we must always apprehend an error amounting to $\frac{1}{2} N$ in the last decimal place. Embarrassment from analogous reasons attends the computation of Q itself, and since many interesting problems in the theory of numbers lead to the use of analogous tables, it has appeared quite desirable to me, while engaged in its preparation, that it should be extended to a sufficient number of decimal places, to permit of its convenient use in still other investigations. To attain the requisite precision, therefore, at least ten, and sometimes more, decimals have been employed, and the seventh place as given may thus be considered trustworthy. For application in the Criterion more than five places will seldom be required, except when it is desired to push the ordeal to a refinement of criticism which will not ordinarily be exercised in a problem of probabilities, and especially when its solution depends upon an approximate assumption like that in equation (C).

The last Table, IV., is that for log. R to argument x, which has been computed by equation (D) to seven, and is here given to six decimal places. To aid in its calculation the table of $\int_0^t \frac{2e^{-t^2}dt}{\sqrt{\pi}}$ given by Encke in the *Berl. Astr. Jahrbuch* for 1834, has been employed, as advised by Professor Peirce. This table was itself deduced from Bessel's table of $e^{t^2}\int_t^\infty e^{-t^2}\,dt$ (*Fundam. Astron.*, pp. 36, 37). The limits are sufficient for all practical purposes; since $x = 3$, the highest value of the argument corresponds to the case of $N = 156$, for one observation deserving exclusion. When the number of observations exceeds this, the series may conveniently be divided into groups of 150 observations each.

I append one of the examples adduced by PEIRCE, and already cited, to illustrate the practical application of Tables III. and IV., for those who have not the original memoir at hand. The computation is greatly facilitated by the use of the Gaussian logarithms, which are now habitually inserted in the five-figure logarithm tables. For as will readily be perceived on inspecting the figures in the example, we have but to enter column B with the arithmetical complement of log λ^2, to obtain in column C the arithmetical component of log $(1-\lambda^2)$, and, similarly, the argument log (x^2-1) in column A gives log x^2 in column C.

Example. "To determine the limit for rejection of observations in the case of fifteen observations of the vertical semidiameter of *Venus*."

Professor PEIRCE assumed two unknown quantities, one of them being constant, and the other proportional to the horizontal parallax, and then found for the residual errors,

$-0''.30$	$+0''.48$	$+0''.63$	$-0''.22$	$+0''.18$
-0.44	-0.24	-0.13	-0.05	$+0.39$
$+1.01$	$+0.06$	-1.40	$+0.20$	$+0.10$

We have then $N=15$, $m=2$, $\log \varepsilon^2 = 9.53050$.

For the rejection of *one* observation, $n=1$, and by Table III. $\log Q^N = 8.40442$. Assume for a first hypothesis $\log R = 9.3$.

		I. Approximation.	II.
By (A')	$\log \lambda^{14}$	9.104	9.097
	$\log \lambda^2$	9.872	9.871
	$\log (1-\lambda^2)$	9.407	9.410
By (C)	$\log 12$	1.079	
	$\log (x^2-1)$	0.486	0.489
	$\log x^2$	0.6087	0.6111
	$\log x$	0.3044	0.3056
	x	2.015	2.021
By Table IV.	$\log R$	9.307	

A third approximation would give $x = 2.019$, the same as would be afforded by the more natural assumption of $\log R = 9.305$ instead of 9.307 for the second hypothesis.

The first approximation would, in fact, have been sufficient to indicate that the observation corresponding to the residual $-1''.40$ ought to be rejected, and we pass to the criterion for the rejection of *two* observations as follows:—

$$n = 2, \quad \log Q^N = 7.44195.$$

		I.	II.	III.
Assume	$\log R$	9.3	9.362	9.3544
	$\log R^2$	8.6	8.724	8.7088
	$\log \lambda^{13}$	8.842	8.7179	8.7332
	$\log \lambda^2$	9.822	9.8028	9.8051
	$\log (1-\lambda^2)$	9.522	9.5622	9.5581
	$\log \frac{11}{2}$	0.740	0.7404	
	$\log (x^2-1)$	0.267	0.3026	0.2986
	$\log x^2$	0.455	0.4782	0.4755
	$\log x$	0.228	0.2391	0.2378
	x	1.689	1.734	1.729
	$\log R$	9.362	9.3544	9.3553

The third approximation and the increase in the number of decimals are rendered necessary in this especial case by the close vicinity of the residual $1''.01$ to the limit of exclusion. We find $\log x^2 \varepsilon^2 = 0.005$, which *forbids* the rejection of this observation.

The precise value of x^2 obtained by still farther approximation, and the use of a higher number of decimal figures, would have been, as shown by Table II.,

$$x^2 = 2.9914, \quad \log x^2 = 0.47587, \quad \log \varepsilon^2 x^2 = 0.0064,$$

corresponding to a residual 1.015.

It is needless to remark that, in the actual application of these approximative formulas, the most convenient assumptions for the successive values of log R would not be precisely those yielded by the preceding hypothesis. Moreover, if, owing to the magnitude of any excluded observation, apprehensions be entertained as to the allowableness of equation (C), the most that need be feared is the necessity of a new determination of ε^2 and the corresponding series of residuals.

It is but proper to state that I have been aided in the necessary computations by Mr. J. N. STOCKWELL of Brecksville, Ohio.

In the hope that these tables may not be without their value in the discussion of the results of the Coast-Survey, I remain, as ever, most respectfully and truly yours,

B. A. GOULD, JR.

Cambridge, 1854, *November* 15.

To PROFESSOR A. D. BACHE,
SUPERINTENDENT UNITED STATES COAST SURVEY.

In Tables I., II., and III. the argument N will be found at the side, and n at the top of the table.

TABLE I.

VALUES OF x^2 FOR $m = 1$.

	1	2	3	4	5	6	7	8	9
3	1.480								
4	1.912	1.163							
5	2.278	1.439							
6	2.592	1.687	1.208						
7	2.866	1.910	1.409	1.045					
8	3.109	2.112	1.588	1.229					
9	3.327	2.295	1.753	1.386	1.091				
10	3.526	2.464	1.904	1.531	1.238				
11	3.707	2.621	2.046	1.662	1.373	1.122			
12	3.875	2.766	2.176	1.785	1.492	1.249	1.018		
13	4.029	2.902	2.299	1.901	1.603	1.362	1.145		
14	4.173	3.030	2.416	2.009	1.709	1.465	1.255	1.053	
15	4.309	3.151	2.526	2.111	1.807	1.561	1.354	1.163	
16	4.436	3.264	2.630	2.207	1.898	1.651	1.445	1.259	1.080
17	4.555	3.371	2.729	2.300	1.985	1.736	1.529	1.347	1.176
18	4.668	3.475	2.824	2.389	2.069	1.817	1.609	1.428	1.261
19	4.776	3.571	2.914	2.474	2.150	1.895	1.685	1.504	1.341
20	4.878	3.664	3.001	2.556	2.227	1.970	1.757	1.576	1.415
21	4.975	3.755	3.084	2.634	2.301	2.041	1.827	1.644	1.483
22	5.068	3.840	3.164	2.709	2.373	2.111	1.893	1.710	1.549
23	5.157	3.923	3.240	2.782	2.442	2.176	1.957	1.773	1.612
24	5.242	4.002	3.315	2.852	2.509	2.240	2.019	1.833	1.671
25	5.324	4.078	3.387	2.920	2.573	2.302	2.079	1.892	1.729
26	5.403	4.151	3.456	2.986	2.636	2.362	2.137	1.948	1.784
27	5.479	4.222	3.523	3.049	2.697	2.420	2.194	2.003	1.838
28	5.552	4.291	3.588	3.111	2.756	2.477	2.249	2.056	1.891
29	5.622	4.358	3.651	3.171	2.813	2.532	2.302	2.108	1.941
30	5.690	4.422	3.712	3.229	2.869	2.586	2.354	2.158	1.990
31	5.756	4.484	3.772	3.285	2.923	2.638	2.404	2.207	2.038
32	5.820	4.545	3.829	3.340	2.976	2.689	2.454	2.255	2.085
33	5.882	4.604	3.884	3.394	3.028	2.738	2.502	2.302	2.130
34	5.942	4.661	3.939	3.446	3.078	2.787	2.549	2.347	2.175
35	6.001	4.717	3.992	3.497	3.127	2.834	2.594	2.392	2.218
36	6.058	4.771	4.044	3.547	3.174	2.880	2.639	2.436	2.261
37	6.113	4.823	4.095	3.595	3.221	2.926	2.683	2.478	2.302
38	6.167	4.874	4.144	3.643	3.267	2.970	2.726	2.520	2.343
39	6.219	4.925	4.192	3.689	3.312	3.013	2.768	2.561	2.383
40	6.270	4.974	4.239	3.734	3.356	3.055	2.809	2.601	2.421
41	6.320	5.022	4.285	3.779	3.398	3.097	2.849	2.640	2.460
42	6.369	5.069	4.331	3.822	3.440	3.138	2.888	2.678	2.497
43	6.416	5.114	4.375	3.865	3.481	3.178	2.927	2.716	2.534
44	6.463	5.159	4.418	3.906	3.520	3.217	2.965	2.753	2.570
45	6.508	5.199	4.460	3.947	3.559	3.255	3.002	2.789	2.606
46	6.552	5.245	4.501	3.987	3.597	3.293	3.039	2.825	2.641
47	6.596	5.287	4.542	4.026	3.637	3.330	3.075	2.860	2.675
48	6.639	5.328	4.581	4.065	3.675	3.366	3.110	2.894	2.708
49	6.681	5.368	4.620	4.103	3.712	3.401	3.145	2.928	2.741
50	6.720	5.408	4.657	4.140	3.748	3.436	3.179	2.962	2.774
51	6.760	5.447	4.695	4.176	3.784	3.471	3.213	2.994	2.806
52	6.800	5.484	4.732	4.212	3.819	3.505	3.246	3.027	2.838
53	6.838	5.522	4.768	4.247	3.853	3.538	3.279	3.059	2.869
54	6.876	5.559	4.804	4.282	3.887	3.571	3.311	3.090	2.899
55	6.913	5.595	4.839	4.316	3.920	3.603	3.342	3.121	2.929
56	6.950	5.630	4.873	4.349	3.952	3.635	3.373	3.151	2.959
57	6.986	5.665	4.907	4.382	3.984	3.666	3.404	3.181	2.988
58	7.021	5.699	4.941	4.415	4.016	3.697	3.434	3.210	3.007
59	7.056	5.733	4.974	4.447	4.047	3.728	3.463	3.239	3.046
60	7.090	5.766	5.006	4.478	4.078	3.758	3.492	3.268	3.074

TABLE II.

VALUES OF x^2 FOR $m = 2$.

	1	2	3	4	5	6	7	8	9
4	1.484								
5	1.887	1.235							
6	2.230	1.479	1.114						
7	2.528	1.705	1.288	1.025					
8	2.793	1.913	1.459	1.163					
9	3.029	2.102	1.620	1.304	1.066				
10	3.243	2.277	1.771	1.439	1.191				
11	3.437	2.440	1.913	1.566	1.310	1.098			
12	3.616	2.592	2.046	1.687	1.423	1.208	1.015		
13	3.781	2.734	2.171	1.802	1.529	1.310	1.122		
14	3.936	2.867	2.290	1.910	1.631	1.409	1.220	1.045	
15	4.080	2.991	2.403	2.014	1.727	1.501	1.312	1.141	
16	4.215	3.109	2.510	2.112	1.819	1.589	1.398	1.229	1.070
17	4.342	3.221	2.611	2.206	1.907	1.673	1.480	1.311	1.157
18	4.462	3.328	2.708	2.295	1.991	1.753	1.557	1.388	1.236
19	4.576	3.429	2.801	2.382	2.072	1.830	1.631	1.461	1.310
20	4.684	3.526	2.890	2.465	2.150	1.904	1.703	1.531	1.380
21	4.787	3.619	2.975	2.544	2.225	1.976	1.772	1.598	1.447
22	4.885	3.707	3.057	2.621	2.298	2.045	1.838	1.663	1.511
23	4.979	3.792	3.136	2.695	2.368	2.112	1.902	1.725	1.572
24	5.069	3.874	3.212	2.766	2.435	2.176	1.964	1.785	1.631
25	5.155	3.953	3.286	2.835	2.501	2.239	2.024	1.843	1.688
26	5.238	4.029	3.357	2.902	2.565	2.299	2.082	1.900	1.743
27	5.317	4.103	3.426	2.967	2.626	2.358	2.139	1.955	1.796
28	5.394	4.174	3.492	3.030	2.686	2.415	2.194	2.008	1.848
29	5.468	4.242	3.556	3.091	2.744	2.471	2.248	2.060	1.898
30	5.539	4.309	3.619	3.150	2.801	2.525	2.300	2.111	1.948
31	5.608	4.373	3.680	3.208	2.856	2.578	2.351	2.160	1.996
32	5.675	4.435	3.739	3.264	2.909	2.630	2.401	2.208	2.042
33	5.740	4.496	3.796	3.319	2.961	2.680	2.449	2.255	2.088
34	5.803	4.555	3.852	3.372	3.012	2.729	2.496	2.301	2.132
35	5.864	4.613	3.906	3.424	3.062	2.777	2.543	2.345	2.176
36	5.924	4.669	3.959	3.474	3.111	2.824	2.588	2.389	2.219
37	5.981	4.723	4.011	3.523	3.158	2.870	2.632	2.432	2.260
38	6.037	4.776	4.061	3.572	3.205	2.914	2.675	2.474	2.301
39	6.092	4.827	4.111	3.619	3.250	2.958	2.717	2.515	2.341
40	6.145	4.878	4.159	3.665	3.294	3.001	2.759	2.555	2.380
41	6.197	4.927	4.206	3.710	3.338	3.043	2.800	2.595	2.419
42	6.247	4.975	4.252	3.755	3.381	3.084	2.840	2.634	2.457
43	6.297	5.022	4.297	3.798	3.422	3.124	2.879	2.672	2.494
44	6.345	5.068	4.341	3.840	3.463	3.164	2.917	2.709	2.530
45	6.392	5.113	4.384	3.882	3.503	3.203	2.955	2.746	2.566
46	6.438	5.157	4.426	3.923	3.543	3.241	2.992	2.782	2.601
47	6.483	5.200	4.468	3.963	3.581	3.278	3.029	2.817	2.635
48	6.527	5.242	4.508	4.002	3.619	3.315	3.064	2.852	2.669
49	6.570	5.283	4.548	4.040	3.656	3.351	3.099	2.886	2.703
50	6.612	5.323	4.587	4.078	3.693	3.386	3.134	2.920	2.736
51	6.653	5.362	4.626	4.115	3.728	3.421	3.168	2.953	2.768
52	6.694	5.401	4.663	4.151	3.764	3.456	3.201	2.986	2.800
53	6.734	5.440	4.700	4.187	3.798	3.489	3.234	3.018	2.831
54	6.773	5.478	4.736	4.222	3.833	3.523	3.266	3.049	2.862
55	6.811	5.515	4.772	4.257	3.867	3.555	3.298	3.080	2.892
56	6.848	5.551	4.807	4.291	3.900	3.588	3.329	3.111	2.922
57	6.885	5.587	4.842	4.325	3.932	3.619	3.360	3.141	2.951
58	6.921	5.622	4.876	4.357	3.964	3.650	3.390	3.171	2.980
59	6.957	5.656	4.909	4.390	3.996	3.681	3.419	3.200	3.009
60	6.993	5.690	4.942	4.421	4.027	3.711	3.448	3.229	3.037

TABLE III.

Values of N Log Q.

	1	2	3	4	5	6	7	8	9
2	9.39794 00								
3	9.17069 62	9.17069 62							
4	9.02312 38	8.79588 00	9.02312 38						
5	8.91338 99	8.53857 37	8.53857 37	8.91338 99					
6	8.82594 25	8.34139 25	8.19382 00	8.34139 25	8.82594 25				
7	8.75322 12	8.18122 37	7.92391 74	7.92391 74	8.18122 37	8.75322 12			
8	8.69096 64	8.04624 76	7.70149 39	7.59176 00	7.70149 39	8.04624 76	8.69086 64		
9	8.63653 73	7.92956 37	7.51208 87	7.31490 74	7.31490 74	7.51208 87	7.92956 37	8.63653 73	
10	8.58818 26	7.82677 99	7.34705 00	7.07714 75	6.98970 00	7.07714 75	7.34705 00	7.82677 99	8.58818 26
11	8.54468 05	7.73492 30	7.20076 41	6.96860 67	6.70843 80	6.70843 80	6.86860 67	7.20076 41	7.73492 30
12	8.50514 46	7.65188 50	7.06937 14	6.68278 49	6.46036 13	6.38764 01	6.46036 13	6.68278 49	7.06937 14
13	8.46891 14	7.57611 59	6.95010 02	6.51515 90	6.23830 63	6.10333 02	6.10333 02	6.23830 63	6.51515 90
14	8.43547 11	7.50644 24	6.84089 08	6.36244 75	6.03724 01	5.84783 49	5.78558 01	5.84783 49	6.03724 01
15	8.40442 36	7.44195 47	6.74016 98	6.22219 06	5.85348 11	5.61572 12	5.49903 73	5.49903 73	5.61572 12
16	8.37544 92	7.38193 28	6.64670 76	6.09249 52	5.68424 98	5.40298 78	5.23794 91	5.18352 01	5.23794 91
17	8.34828 81	7.32579 72	6.55952 46	5.97187 19	5.52739 33	5.20659 54	4.99805 46	4.89527 08	4.89527 08
18	8.32272 66	7.27307 46	6.47782 76	5.85912 74	5.38120 85	5.02417 74	4.77610 07	4.62981 48	4.58146 01
19	8.29858 67	7.22337 32	6.40096 51	5.75329 04	5.24432 41	4.85385 27	4.56954 28	4.38372 10	4.29186 42
20	8.27571 85	7.17636 52	6.32839 55	5.65355 98	5.11561 90	4.69410 01	4.37634 99	4.15429 49	4.02290 22
21	8.25399 47	7.13177 32	6.25966 37	5.55926 64	4.99416 46	4.54367 12	4.19487 36	3.93937 83	3.77175 23
22	8.23330 62	7.08936 09	6.19438 32	5.46984 61	4.87918 27	4.40152 82	4.02375 62	3.73721 34	3.53614 72
23	8.21355 87	7.04892 50	6.13222 34	5.38481 82	4.77001 49	4.26679 89	3.86186 58	3.54634 86	3.31423 49
24	8.19467 04	7.01028 92	6.07289 92	5.30377 01	4.66609 86	4.13874 28	3.70824 81	3.36556 98	3.10448 17
25	8.17656 96	6.97330 00	6.01616 25	5.22634 49	4.56694 97	4.01672 57	3.56209 12	3.19385 13	2.90560 21
26	8.15919 32	6.93782 27	5.96179 69	5.15223 19	4.47214 82	3.90020 04	3.42269 77	3.03031 79	2.71650 72
27	8.14248 54	6.90373 86	5.90961 19	5.08115 86	4.38132 74	3.78869 11	3.28946 46	2.87421 67	2.53626 60
28	8.12639 68	6.87094 22	5.85943 91	5.01288 49	4.29416 54	3.68178 16	3.16186 66	2.72489 49	2.36407 61
29	8.11088 29	6.83933 97	5.81112 89	4.94719 82	4.21037 79	3.57910 58	3.03944 33	2.58178 31	2.19924 06
30	8.09590 43	6.80884 72	5.76454 78	4.88390 94	4.12971 26	3.48033 97	2.92178 89	2.44438 12	2.04115 01
31	8.08142 51	6.77938 94	5.71957 61	4.82284 91	4.05194 46	3.38519 52	2.80854 36	2.31224 76	1.88926 91
32	8.06741 32	6.75089 83	5.67610 64	4.76386 55	3.97687 23	3.29341 52	2.69938 72	2.18499 04	1.74312 35
33	8.05383 93	6.72331 25	5.63404 14	4.70682 19	3.90431 49	3.20476 91	2.59403 33	2.06226 01	1.60229 24
34	8.04067 68	6.69657 61	5.59329 31	4.65159 44	3.83410 86	3.11904 92	2.49222 47	1.94374 38	1.46639 96
35	8.02790 16	6.67063 85	5.55378 15	4.55807 09	3.76610 61	3.03606 79	2.39372 96	1.82916 00	1.33510 81
36	8.01549 15	6.64545 31	5.51543 37	4.54614 92	3.70017 25	2.95565 51	2.29833 82	1.71825 47	1.20811 42
37	8.00342 62	6.62097 78	5.47818 31	4.49573 62	3.63618 55	2.87765 62	2.20586 01	1.61079 80	1.08514 37
38	7.99168 71	6.59717 33	5.44196 86	4.44674 65	3.57403 34	2.80193 01	2.11612 21	1.50658 09	0.96594 79
39	7.98025 70	6.57400 41	5.40673 41	4.39910 18	3.51361 35	2.72834 78	2.02896 59	1.40541 27	0.85030 06
40	7.96912 00	6.55143 70	5.37242 79	4.35273 03	3.45483 19	2.65679 10	1.94424 66	1.30711 96	0.73799 54
41	7.95826 15	6.52944 15	5.33900 23	4.30756 56	3.39760 19	2.58715 09	1.86183 13	1.21154 19	0.62884 38
42	7.94766 79	6.50798 94	5.30641 32	4.26354 64	3.34184 36	2.51932 73	1.78159 76	1.11853 29	0.52267 24
43	7.93732 66	6.48705 45	5.27461 98	4.22061 60	3.28748 31	2.45322 77	1.70343 27	1.02795 79	0.41932 22
44	7.92722 58	6.46661 24	5.24358 41	4.17872 19	3.23445 19	2.38876 64	1.62723 23	0.93969 22	0.31864 64
45	7.91735 46	6.44664 05	5.21327 08	4.13781 50	3.18268 65	2.32586 40	1.55289 98	0.85362 06	0.22050 95
46	7.90770 29	6.42711 75	5.18364 71	4.09784 99	3.13212 79	2.26444 69	1.48034 57	0.76963 63	0.12478 61
47	7.89826 09	6.40802 38	5.15468 22	4.05878 42	3.08272 09	2.20444 63	1.40948 66	0.68764 02	90.03135 99
48	7.88901 99	6.38934 09	5.12634 75	4.02057 83	3.03441 42	2.14579 83	1.34024 50	0.60754 02	89.94012 29
49	7.87997 15	6.37105 14	5.09861 61	3.98319 52	2.98715 99	2.08844 32	1.27254 86	0.52925 01	9.85097 43
50	7.87110 77	6.35313 92	5.07146 29	3.94660 00	2.94091 29	2.03232 50	1.20632 97	0.45268 99	9.76382 05
51	7.86242 12	6.33558 89	5.04486 42	3.91076 03	2.89563 13	1.97739 16	1.14152 51	0.37778 45	9.67857 38
52	7.85390 51	6.31838 64	5.01879 78	3.87564 55	2.85127 55	1.92359 39	1.07807 55	0.30446 38	9.59515 23
53	7.84555 28	6.30151 79	4.99324 29	3.84122 68	2.80780 83	1.87088 57	1.01592 54	0.23266 19	9.51347 93
54	7.83735 80	6.28497 08	4.96817 97	3.80747 71	2.76519 49	1.81922 38	0.95502 55	0.16231 71	9.43348 26
55	7.82931 51	6.26873 32	4.94358 97	3.77437 10	2.72340 23	1.76856 75	0.89531 77	0.09337 13	9.35509 49
56	7.82141 84	6.25279 35	4.91945 53	3.74188 43	2.68239 95	1.71887 82	0.83676 47	90.02576 98	9.27825 24
57	7.81366 27	6.23714 11	4.89576 00	3.70999 43	2.64215 71	1.67011 97	0.77931 97	89.95946 10	9.20289 52
58	7.80604 31	6.22176 59	4.87248 80	3.67867 94	2.60264 75	1.62225 77	0.72294 16	9.89439 65	9.12896 69

TABLE III. (*Continued.*)

	1	2	3	4	5	6	7	8	9
59	7.79855 49	6.20665 81	4.84962 46	3.64791 92	2.56384 44	1.57525 99	0.66759 15	9.83053 02	9.05641 41
60	7.79119 37	6.19180 86	4.82715 55	3.61769 45	2.52572 29	1.52909 55	0.61323 23	9.76781 85	8.98518 65
61	7.78395 51	6.17720 87	4.80506 75	3.58798 68	2.48825 96	1.48373 55	0.55982 94	9.70622 10	8.91523 65
62	7.77683 52	6.16285 03	4.78334 77	3.55877 89	2.45143 20	1.43915 23	0.50734 94	9.64569 82	8.84651 89
63	7.76983 01	6.14872 53	4.76198 42	3.53005 40	2.41521 90	1.39531 96	0.45576 12	9.58621 32	8.77899 10
64	7.76293 63	6.13482 64	4.74096 54	3.50179 67	2.37960 04	1.35221 28	0.40503 47	9.52773 11	8.71261 22
65	7.75615 01	6.12114 64	4.72028 03	3.47399 17	2.34455 69	1.30980 80	0.35514 17	9.47021 85	8.64734 39
66	7.74946 84	6.10767 86	4.69991 86	3.44662 50	2.31007 02	1.26808 28	0.30605 52	9.41364 38	8.58314 96
67	7.74288 80	6.09441 64	4.67987 03	3.41968 28	2.27612 30	1.22701 57	0.25774 95	9.35797 68	8.51999 44
68	7.73640 57	6.08135 37	4.66012 59	3.39315 22	2.24269 86	1.18658 62	0.21020 02	9.30318 89	8.45784 52
69	7.73001 88	6.06848 45	4.64067 62	3.36702 09	2.20978 11	1.14677 49	0.16338 38	9.24925 26	8.39667 03
70	7.72372 45	6.05580 32	4.62151 27	3.34127 69	2.17735 54	1.10756 30	0.11727 81	9.19614 19	8.33643 97
71	7.71752 00	6.04330 45	4.60262 71	3.31590 90	2.14540 70	1.06893 29	0.07186 18	9.14383 18	8.27712 46
72	7.71140 30	6.03098 31	4.58401 13	3.29090 63	2.11392 21	1.03086 75	90.02711 47	9.09229 85	8.21869 75
73	7.70537 09	6.01883 40	4.56565 78	3.26625 84	2.08288 73	0.99335 05	89.98301 72	9.04151 93	8.16113 21
74	7.69942 15	6.00685 25	4.54755 93	3.24195 55	2.05229 00	0.95636 63	9.93955 08	8.99147 24	8.10440 35
75	7.69355 25	5.99503 40	4.52970 87	3.21798 80	2.02211 81	0.91990 00	9.89669 76	8.94213 69	8.04848 76
76	7.68776 17	5.98337 42	4.51209 95	3.19434 67	1.99235 98	0.88393 73	9.85444 05	8.89349 29	7.99336 14
77	7.68204 72	5.97186 89	4.49472 52	3.17102 29	1.96300 39	0.84846 44	9.81276 32	8.84552 13	7.93900 28
78	7.67640 68	5.96051 40	4.47757 95	3.14800 82	1.93403 98	0.81346 82	9.77165 00	8.79820 37	7.88539 08
79	7.67083 88	5.94930 56	4.46065 66	3.12529 45	1.90545 71	0.77893 61	9.73108 58	8.75152 25	7.83250 52
80	7.66534 12	5.93824 00	4.44395 06	3.10287 40	1.87724 58	0.74485 58	9.69105 61	8.70546 07	7.78032 64
81	7.65991 24	5.92731 37	4.42745 62	3.08073 93	1.84939 65	0.71121 57	9.65154 70	8.66000 22	7.72883 58
82	7.65455 06	5.91652 31	4.41116 81	3.05888 31	1.82190 00	0.67800 46	9.61254 51	8.61513 13	7.67801 55
83	7.64925 42	5.90586 48	4.39508 11	3.03729 85	1.79474 74	0.64521 17	9.57403 76	8.57083 30	7.62784 82
84	7.64402 16	5.89533 59	4.37919 03	3.01597 89	1.76793 02	0.61282 65	9.53601 21	8.52709 29	7.57831 73
85	7.63885 13	5.88493 30	4.36349 10	2.99491 78	1.74144 03	0.58083 90	9.49845 66	8.48389 71	7.52940 69
86	7.63374 19	5.87465 32	4.34797 86	2.97410 91	1.71526 97	0.54923 96	9.46135 97	8.44123 21	7.48110 16
87	7.62869 18	5.86449 37	4.33264 88	2.95354 67	1.68941 09	0.51801 90	9.42471 03	8.39908 51	7.43338 66
88	7.62369 98	5.85445 16	4.31749 73	2.93322 48	1.66385 65	0.48716 82	9.38849 77	8.35744 37	7.38624 76
89	7.61876 46	5.84452 44	4.30252 00	2.91313 81	1.63859 95	0.45667 86	9.35271 16	8.31629 58	7.33967 09
90	7.61388 47	5.83470 93	4.28771 29	2.89328 09	1.61363 29	0.42654 17	89.31734 21	88.27562 99	87.29364 33

TABLE IV.

VALUES OF LOG R FOR ARGUMENT x.

x	0	1	2	3	4	5	6	7	8	9
1.0	9.5014 85	9.4992 09	9.4969 41	9.4946 82	9.4924 31	9.4901 88	9.4879 54	9.4857 29	9.4835 12	9.4813 03
1.1	.4791 02	.4769 10	.4747 25	.4725 49	.4703 80	.4682 20	.4660 67	.4639 23	.4617 86	.4596 57
1.2	.4575 36	.4554 22	.4533 16	.4512 18	.4491 27	.4470 44	.4449 68	.4429 00	.4408 39	.4387 85
1.3	.4367 39	.4347 00	.4326 69	.4306 44	.4286 27	.4266 16	.4246 13	.4226 17	.4206 27	.4186 45
1.4	.4166 70	.4147 01	.4127 39	.4107 85	.4088 36	.4068 95	.4049 60	.4030 32	.4011 11	.3991 96
1.5	.3972 87	.3953 85	.3934 90	.3916 01	.3897 18	.3878 42	.3859 72	.3841 08	.3822 51	.3804 00
1.6	.3785 55	.3767 16	.3748 83	.3730 56	.3712 36	.3694 21	.3676 12	.3658 09	.3640 13	.3622 22
1.7	.3604 37	.3586 57	.3568 84	.3551 16	.3533 54	.3515 98	.3498 47	.3481 02	.3463 63	.3446 29
1.8	.3429 01	.3411 78	.3394 61	.3377 49	.3360 42	.3343 41	.3326 46	.3309 56	.3292 71	.3275 91
1.9	.3259 17	.3242 47	.3225 83	.3209 25	.3192 71	.3176 22	.3159 79	.3143 40	.3127 07	.3110 78
2.0	.3094 55	.3078 37	.3062 23	.3046 15	.3030 11	.3014 13	.2998 19	.2982 29	.2966 45	.2950 65
2.1	.2934 91	.2919 20	.2903 55	.2887 94	.2872 38	.2856 86	.2841 39	.2825 97	.2810 59	.2795 26
2.2	.2779 98	.2764 74	.2749 54	.2734 38	.2719 27	.2704 20	.2689 17	.2674 19	.2659 25	.2644 37
2.3	.2629 52	.2614 72	.2599 96	.2585 24	.2570 57	.2555 93	.2541 33	.2526 78	.2512 26	.2497 79
2.4	.2483 35	.2468 95	.2454 59	.2440 27	.2426 00	.2411 76	.2397 57	.2383 42	.2369 31	.2355 24
2.5	.2341 21	.2327 21	.2313 26	.2299 35	.2285 48	.2271 64	.2257 84	.2244 06	.2230 32	.2216 62
2.6	.2202 96	.2189 33	.2175 75	.2162 19	.2148 68	.2135 20	.2121 76	.2108 36	.2094 99	.2081 66
2.7	.2068 37	.2055 11	.2041 88	.2028 68	.2015 51	.2002 39	.1989 29	.1976 25	.1963 24	.1950 27
2.8	.1937 33	.1924 43	.1911 57	.1898 74	.1885 95	.1873 19	.1860 46	.1847 77	.1835 11	.1822 53
2.9	.1809 98	9.1797 46	9.1784 98	9.1772 52	9.1760 09	9.1747 70	9.1735 33	9.1723 00	9.1710 69	9.1698 42
3.0	9.1686 17									