Image Processing

Transforms of 2D signals

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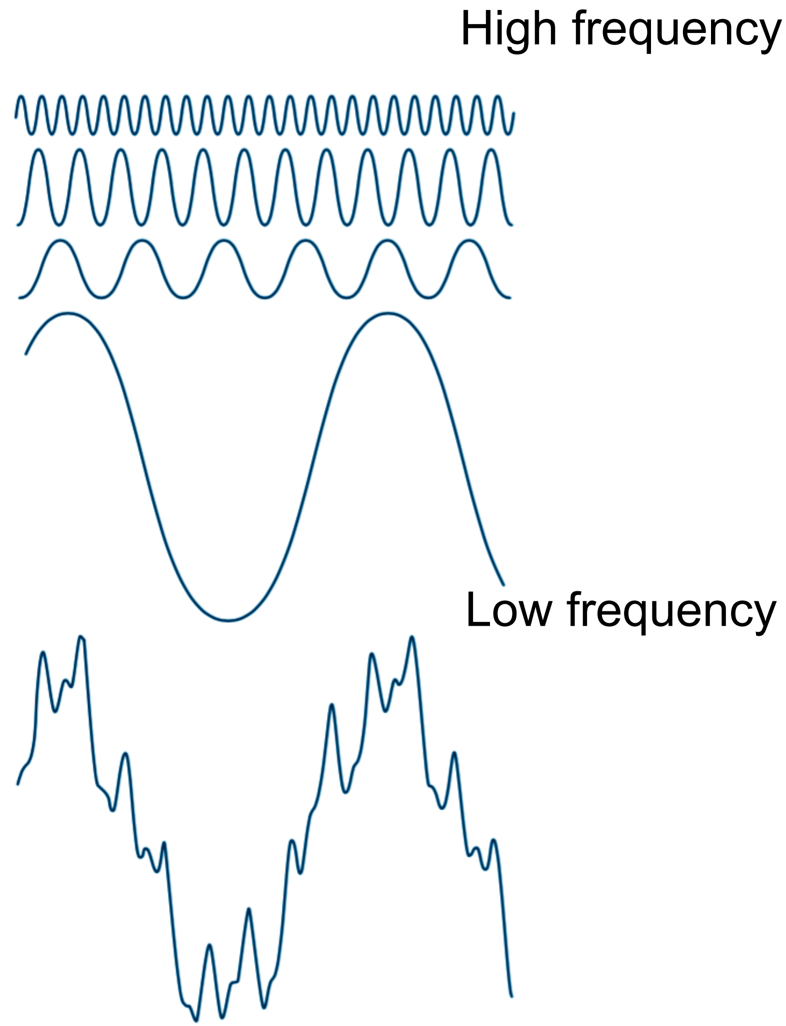
Slides modified from materials provided by Dr. Tien D Bui

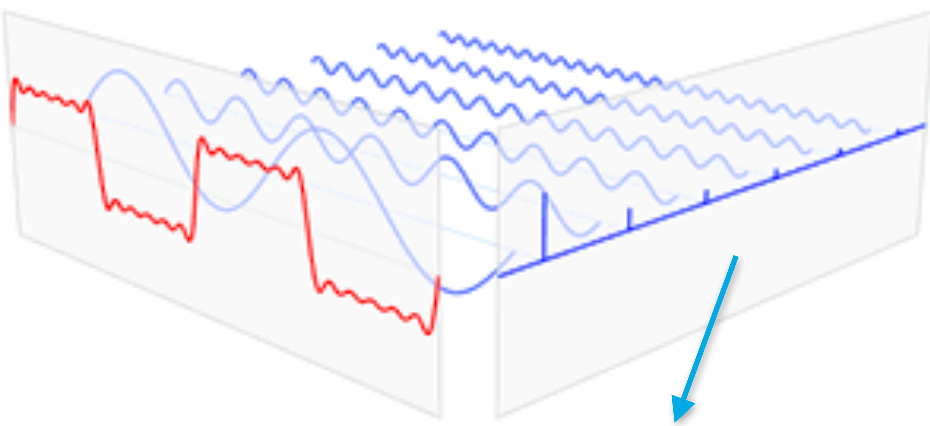
Frequency domain representation

- The **frequency domain analysis** refers to the analysis of signals wrt. their frequency components.
- The purpose of the **Fourier transform** is to represent a signal as a linear combination of sinusoidal signals of various frequencies.
- Mathematically easier to analyze the effects of signal properties (e.g., noise)



Frequency domain representation

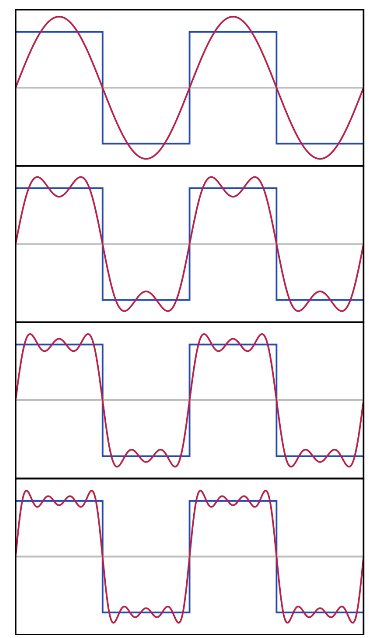
s(x)



S(f)

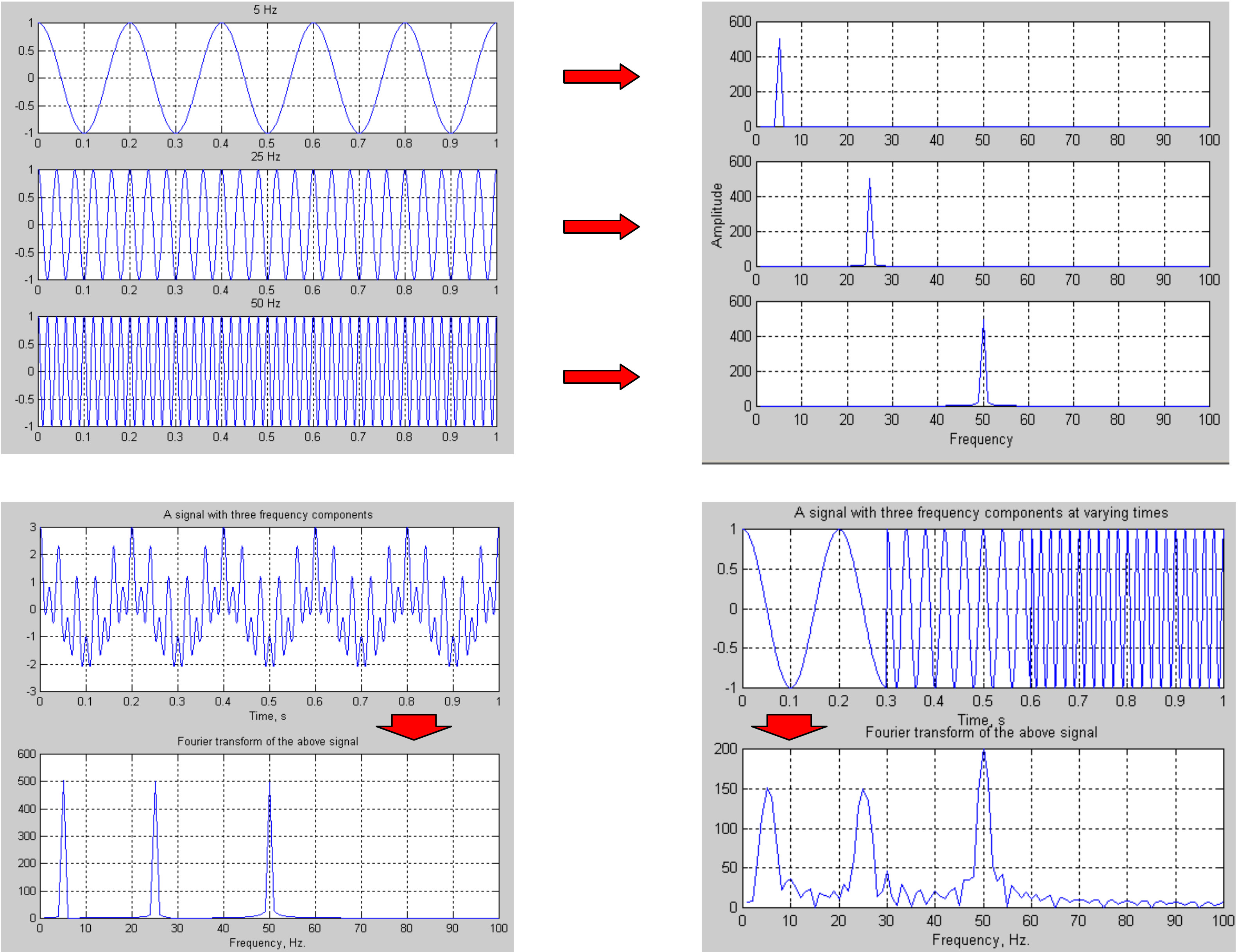
Just 6 scalars!

$a_n \cos(nx) + b_n \sin(nx)$

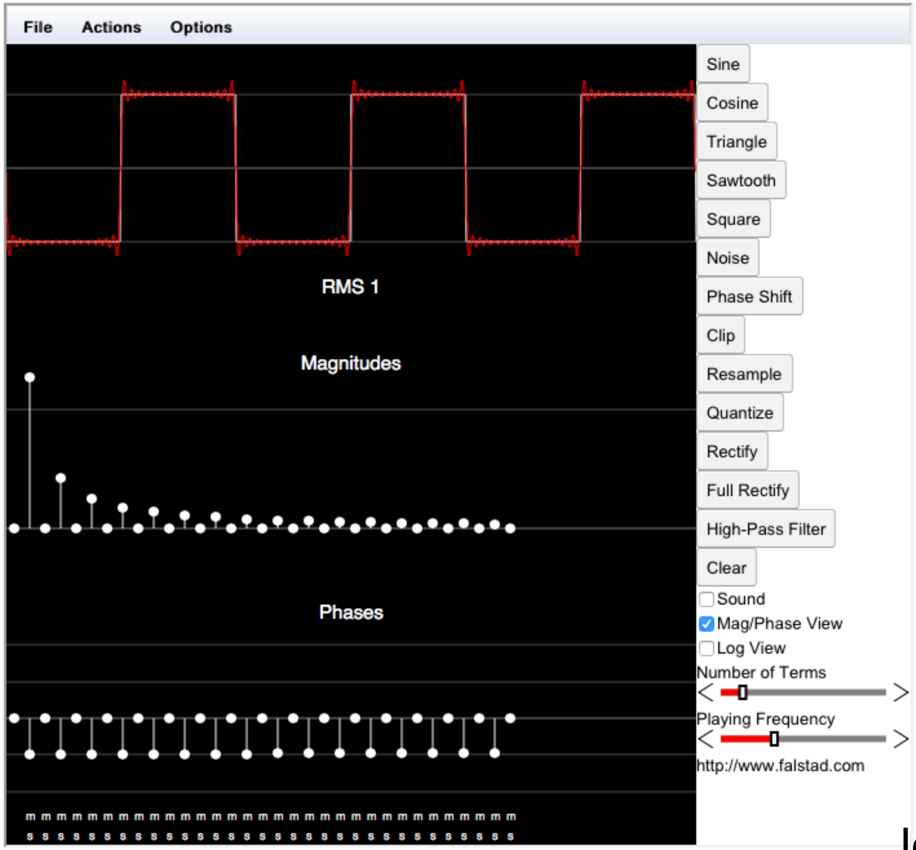


https://en.wikipedia.org/wiki/Fourier_series

Examples of Fourier Transform



Give this try yourself



http://www.falstad.com/fourier/

Jean-Baptiste Joseph Fourier (1768 – 1830)
French mathematician and physicist

Continuous Fourier Transform

1-D and continuous

Freq in rad or Hz?

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \cdot \left[\cos(\omega x) - \mathrm{i} \cdot \sin(\omega x)\right] \, \mathrm{d}x$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \cdot e^{-\mathrm{i}\omega x} \, \mathrm{d}x.$$

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-j2\pi\omega x} dx \longleftarrow$$ *Fourier transform*

$$f(x) = \int_{-\infty}^{\infty} F(\omega)\, e^{j2\pi\omega x} d\omega \longleftarrow$$ *Inverse Fourier transform*

$$F(\omega) = \int_{-\infty}^{\infty} \delta(x)\, e^{-j2\pi\omega x} dx = e^{-j2\pi\omega .0} = 1$$

$$F(\omega) = \int_{-\infty}^{\infty} \delta(x - x_0)\, e^{-j2\pi\omega x} dx = e^{-j2\pi\omega . x_0} = \cos(2\pi\omega . x_0) - j\sin(2\pi\omega . x_0)$$

$$H(\omega) = F[f(x-\alpha)] = F(\omega)\, e^{-j2\pi\omega .\alpha} \quad \text{where} \quad F(\omega) = F[f(x)]$$

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(u) e^{-j\omega u/a} d(u/a) = \frac{1}{a} X\left(\frac{\omega}{a}\right) \quad (a>0)$$ **scaling**

http://www.cs.cornell.edu/courses/cs5540/2010sp/lectures/Lec5.Transforms.pdf

Fourier transform: magnitude & phase

F(*u*) can be expressed in polar coordinates:

$$F(u) = |F(u)| e^{j\ \ (u)}$$

where $|F(u)| = \left[R^2(u) + I^2(u)\right]^{\frac{1}{2}}$ (magnitude or spectrum)

$$(u) = \tan^{\ 1} \frac{I(u)}{R(u)}$$ (phase angle or phase spectrum)

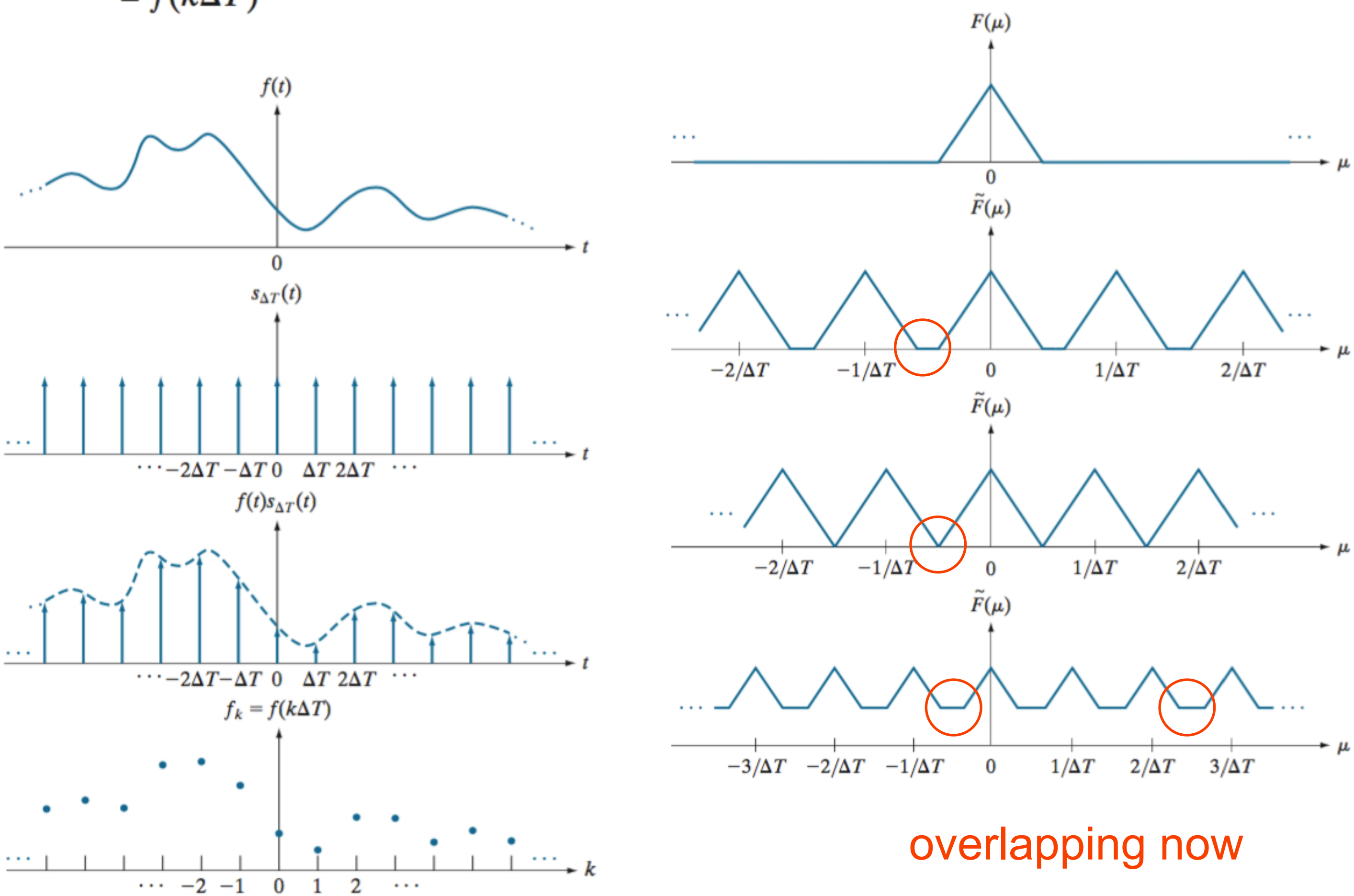
$R(u)$: the real part of $F(u)$
$I(u)$: the imaginary part of $F(u)$

Power spectrum:

$$P(u) = |F(u)|^2 = R^2(u) + I^2(u)$$

Sampling Theorem

$$f_k = \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)\,dt$$
$$= f(k\Delta T)$$

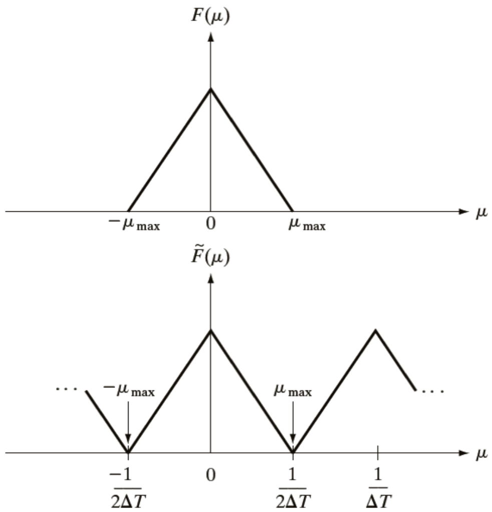


a b c d

FIGURE 4.6 (a) Illustrative sketch of the Fourier transform of a band-limited function. (b)–(d) Transforms of the corresponding sampled functions under the conditions of over-sampling, critically sampling, and under-sampling, respectively.

overlapping now

Sampling Theorem - Nyquist Theorem



a
b

FIGURE 4.7
(a) Transform of a band-limited function.
(b) Transform resulting from critically sampling the same function.

The period is ΔT; sufficient separation is guaranteed if $\frac{1}{\Delta T} > 2\omega_{max}$
i.e. a continuous, band-limited function can be recovered completely from sampling if the samples are at the rate exceeding twice the highest frequency content of the func. (*Sampling Theorem*)

Or: the max frequency can be captured by sampling at the rate of:

$$\frac{1}{\Delta T} \text{ that is } \omega_{max} = \frac{1}{2\Delta T} \Rightarrow \boxed{\text{Nyquist Rate}}$$

Aliasing

What happens if a band-limited function is sampled at a rate less than twice the highest frequency?

This is under-sampled. The inverse transform would yield a corrupted func. This effect of under-sampling is called **frequency aliasing**, This is because the high frequencies overlap with lower frequencies (*hence the name aliasing = false identity*)

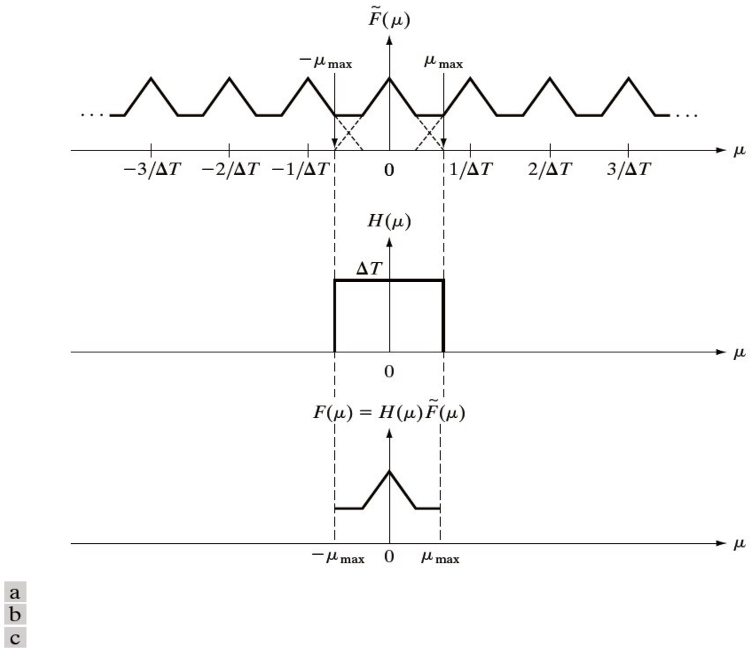


FIGURE 4.9 (a) Fourier transform of an under-sampled, band-limited function. (Interference from adjacent periods is shown dashed in this figure). (b) The same ideal lowpass filter used in Fig. 4.8(b). (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, therefore, of the original, band-limited continuous function. Compare with Fig. 4.8.

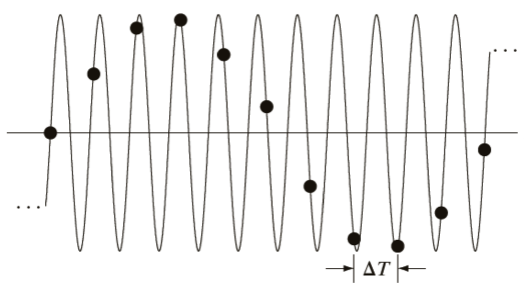


FIGURE 4.10 Illustration of aliasing. The under-sampled function (black dots) looks like a sine wave having a frequency much lower than the frequency of the continuous signal. The period of the sine wave is 2 s, so the zero crossings of the horizontal axis occur every second. ΔT is the separation between samples.

Discrete Fourier Transform (1-D)

Let us define a set of orthonormal basis vectors

$$\underbrace{\frac{1}{\sqrt{N}}\begin{bmatrix}1\\1\\.\\.\\1\end{bmatrix}}_{b_0},\quad \underbrace{\frac{1}{\sqrt{N}}\begin{bmatrix}1\\e^{j\frac{2\pi}{N}.1}\\e^{j\frac{2\pi}{N}.2}\\.\\.\\e^{j\frac{2\pi}{N}.(N-1)}\end{bmatrix}}_{b_1},\quad \underbrace{\frac{1}{\sqrt{N}}\begin{bmatrix}1\\e^{j\frac{2\pi}{N}.2.1}\\e^{j\frac{2\pi}{N}.2.2}\\.\\e^{j\frac{2\pi}{N}.2(N-1)}\end{bmatrix}}_{b_2},\quad \underbrace{\cdots}_{....}\quad \underbrace{\frac{1}{\sqrt{N}}\begin{bmatrix}1\\e^{j\frac{2\pi}{N}.(N-1).1}\\e^{j\frac{2\pi}{N}.(N-1).2}\\.\\e^{j\frac{2\pi}{N}.(N-1)(N-1)}\end{bmatrix}}_{b_{N-1}}$$

Any vector $\vec{x}$ can be written as :

$$\vec{x} = \sum_{k=0}^{N-1} X_k\, \vec{b}_k \quad \text{or}$$

$$x[n] = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} X_k\, e^{j\frac{2\pi}{N}kn}$$

Inverse Fourier transform

$$X_k = \frac{1}{\sqrt{N}}\sum_{n} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

Fourier transform

Discrete Fourier Transform pair

Discrete Fourier Transform (1-D)

DTFT : $x[n]$ is discrete, $X(\omega)$ is continuous

$$X(\omega) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$$

ω can take any value

$$x[n] = \frac{1}{2\pi} \sum_{-\pi}^{\pi} X(\omega) e^{j\omega n}$$

only discrete in time domain

DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$\omega = \frac{2\pi k}{N}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

Differences Between 1D and Multi-Dimensional Signal Processing

- **Much more data for M-D signal processing**

  1) 1-D signal: speech ⟹ 10K samples/sec

  2) 3-D signal: videos ⟹ 500x500 pixels/frame, 30 frames/sec, **hence 7.5 Mega samples/sec.**

  Again, this is why we need to do frequency domain analysis

- **Mathematics for M-D is not as complete as 1-D**

  1) 1-D systems are described by ODEs vs. M-D by PDEs.

  2) Fundamental results of algebra do not hold in M-D, e.g. factorization of polynomials in 1-D is known but not in M-D.

  3) This effects filter design, filter stability, reconstruction, etc…

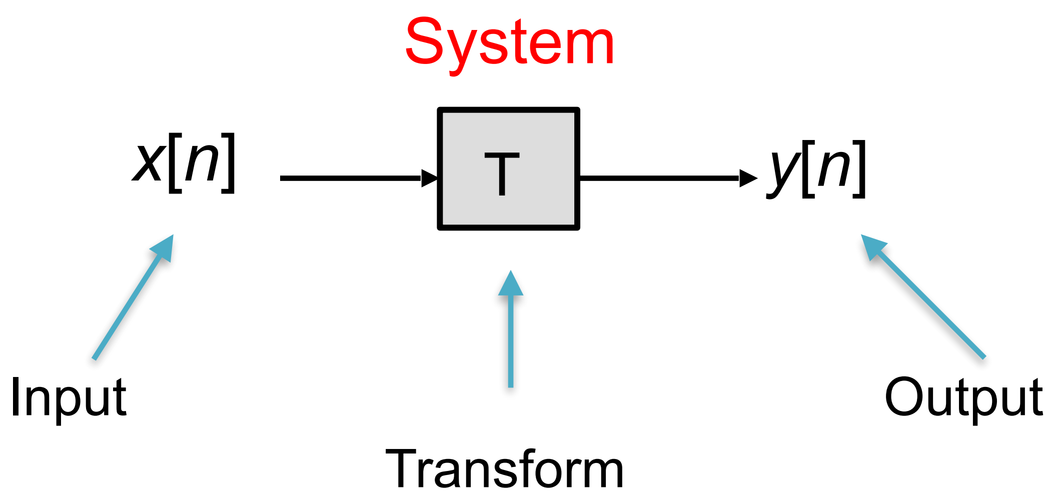
Signals, Systems and Transforms

Digital Signal Processing ➡ Digital Image Processing

An image is a signal in 2-D or *m*-D (e.g., colors)

Signal carries physical information like speech, music, images

- **continuous** $x(t)$
- **discrete** $x[n]$



Signals, Systems and Transforms

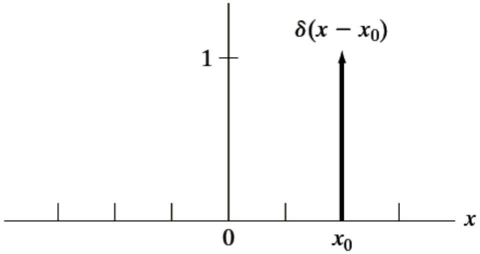
- Linear system $T(ax_1[n] + bx_2[n]) = T(ax_1[n])+T(bx_2[n]) = ay_1[n] + by_2[n]$
- Time (shift) invariant system if $T(x[n]) = y[n]$ *then* $T(x[n-n_0]) = y[n-n_0]$
- Linear Time Invariant system (LTI):

  Impulse response can entirely characterize an LTI system.

- Impulse response and convolution (e.g.,Filters)
- Separable LTI system: **computational efficiency**

- Transforms: *Bounded input bounded output (BIBO) stability*
- Finite impulse response (FIR): always stable (e.g, averaging)
- Infinite impulse response (IIR): not always stable (e.g., feedback loop)

Special signal: Impulse $\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$



$\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow f[n]$

Impulse response

Sifting property of an impulse function: $\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

(generalization)

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0)$$

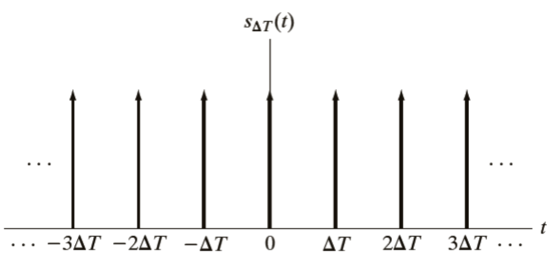
(discrete cases)

Given a LTI system T, we can easily obtain output signal for *all possible* input signals $x[n]$ from the Impulse response!

$$\int_{-\infty}^{\infty} \delta(x)d(x) = 1 \qquad \sum_{-\infty}^{\infty} \delta[n] = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta(x)d(x) = f(0) \qquad \sum_{-\infty}^{\infty} f[n]\delta[n] = f[0]$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)d(x) = f(x_0) \qquad \sum_{-\infty}^{\infty} f[n]\delta[n-n_0] = f[n_0]$$

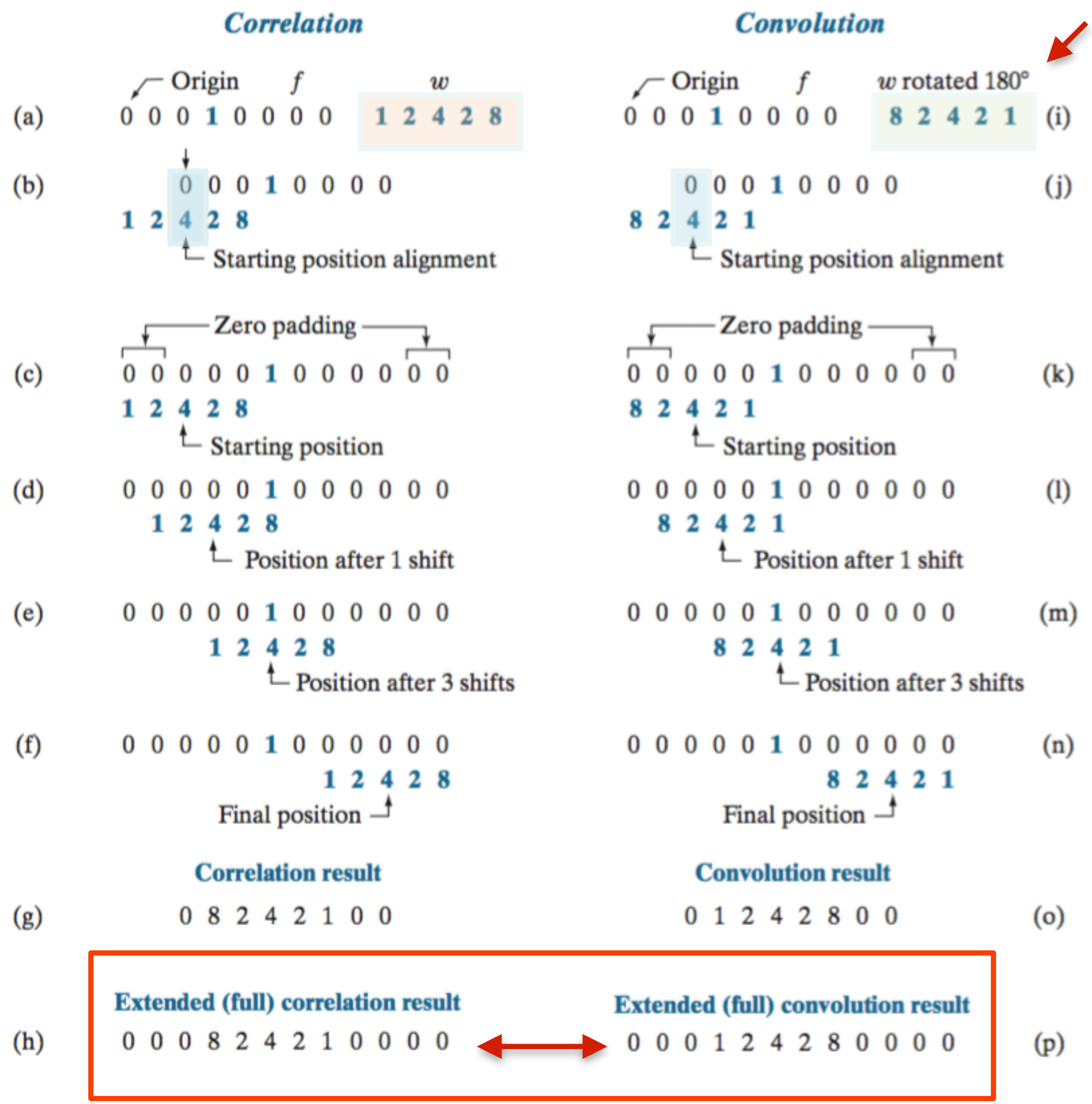


$$s_n(x) = \sum_{-\infty}^{\infty} \delta(x-n) \qquad \text{impulse train}$$

$$\boxed{x[n] = \sum_k x[k]\delta[n-k]}$$

Computation of Correlation and Convolution (1-D)

FIGURE 3.29 Illustration of 1-D correlation and convolution of a kernel, w, with a function f consisting of a discrete unit impulse. Note that correlation and convolution are functions of the variable x, which acts to *displace* one function with respect to the other. For the extended correlation and convolution results, the starting configuration places the right-most element of the kernel to be coincident with the origin of f. Additional padding must be used.



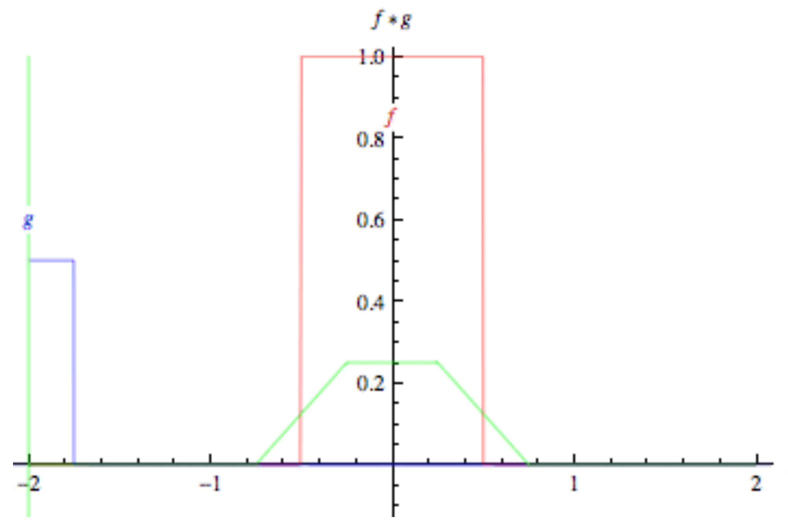
$$(f * g)(t) \triangleq \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau.$$

$$(f * g)[n] = \sum_{m=-\infty}^{\infty} f[m]g[n-m],$$

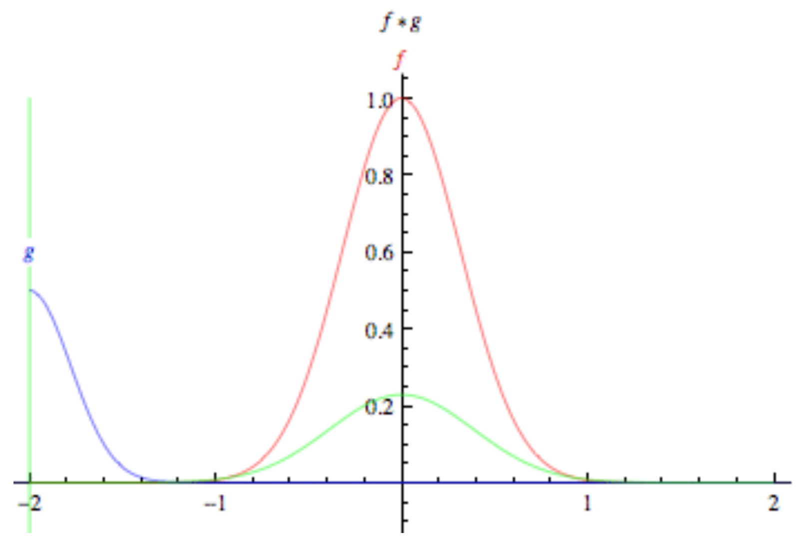
Examples of convolution operations between function f and g

f*g

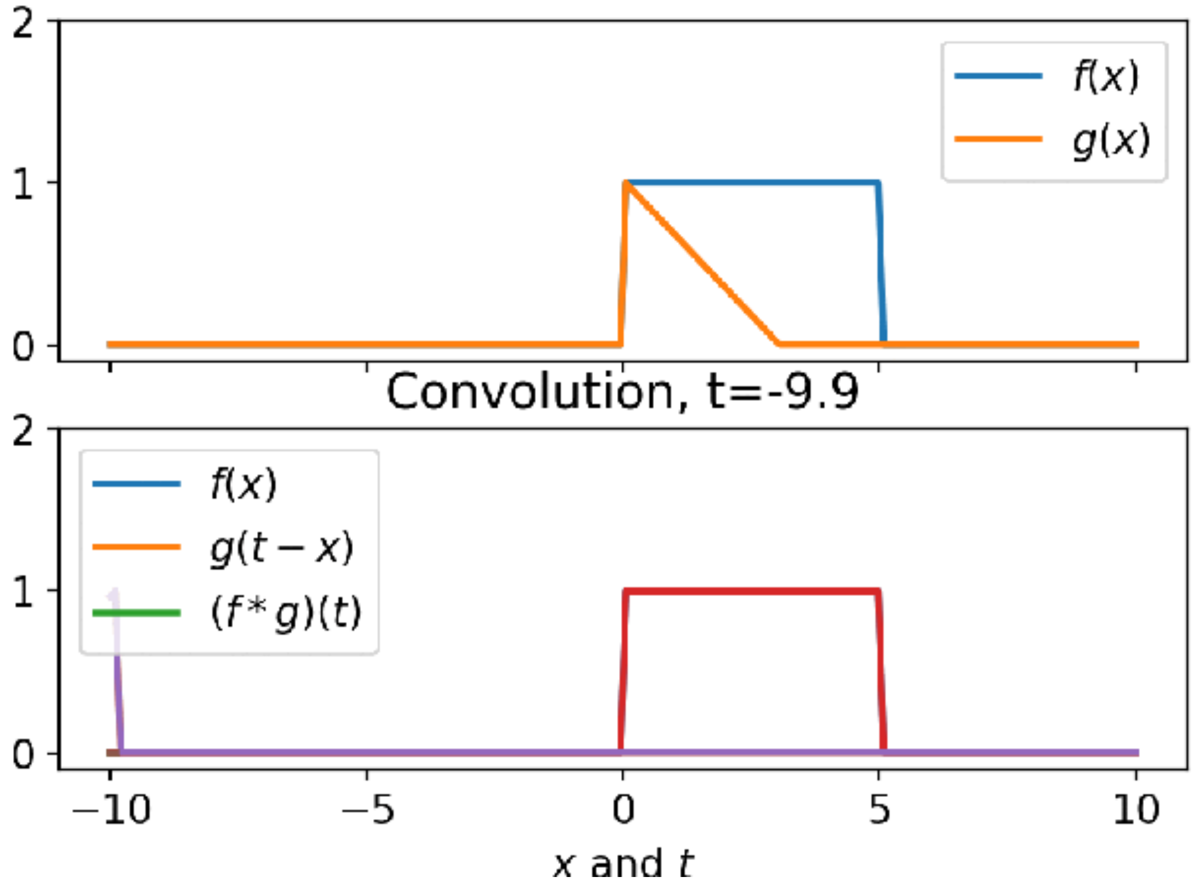
Square functions



Gaussian functions



Examples of convolution operations between function f and g



Convolution Theorem

$$Definition: f(x)*h(x)=\int_{-\infty}^{\infty} f(t)h(x-t)dt$$ continuous

$$F[f(x)*h(x)]=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)h(x-t)dt\right]e^{-j2\pi\omega x}dx$$

$$=\int_{-\infty}^{\infty} f(t)\underline{\left[\int_{-\infty}^{\infty} h(x-t)\,e^{-j2\pi\omega x}dx\right]}dt$$

$$F[h(x-t)]=H(\omega)\,e^{-j2\pi\omega .t}$$

$$F[f(x)*h(x)]=\int_{-\infty}^{\infty} f(t)\left[H(\omega)\,e^{-j2\pi\omega .t}\right]dt$$

$$=H(\omega)\int_{-\infty}^{\infty} f(t)\,e^{-j2\pi\omega .t}\,dt$$

$$=H(\omega).F(\omega)$$

Theorem: The FT of the convolution of two functions is equal to the product of the FTs of the two functions.

Reverse: The inverse FT of $H(\omega).F(\omega)$ gives the convolution of the two functions $f(x)*h(x)$ in spatial domain.

In General:

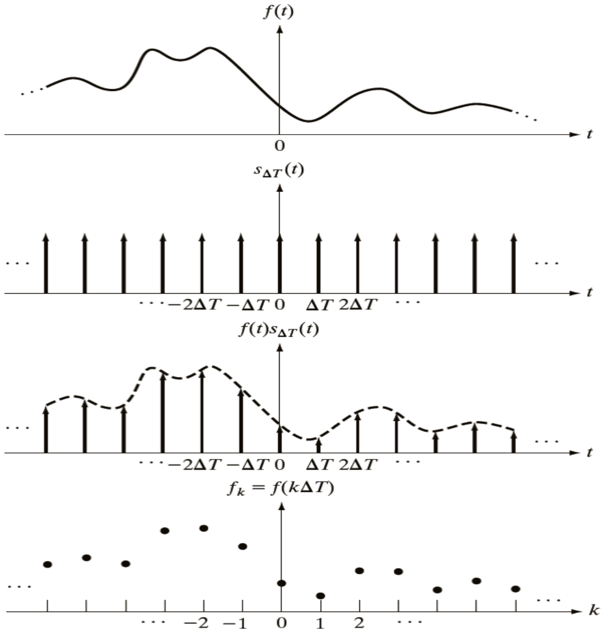
$$f(x)*h(x) \Leftrightarrow F(\omega).H(\omega)$$

$$f(x).h(x) \Leftrightarrow F(\omega)*H(\omega)$$

Discrete Fourier Transform (1-D)

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\; e^{j\frac{2\pi}{N}kn}$$



2-D D.F.T

$$D.F.T\ \{x(n_1,n_2)\} \triangleq X(k_1,k_2)$$

$$X(k_1,k_2) = \begin{cases} \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} x(n_1,n_2)\, e^{-j\frac{2\pi\, n_1k_1}{N_1}} e^{-j\frac{2\pi\, n_2k_2}{N_2}} & 0 \le k_1 < N_1 \\ & 0 \le k_2 < N_2 \\ 0 & Otherwise \end{cases}$$

$$I.D.F.T\ \{X(k_1,k_2)\} \triangleq x(n_1,n_2)$$

$$x(n_1,n_2) = \begin{cases} \frac{1}{N_1N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1} X(k_1,k_2)\, e^{j\frac{2\pi\, k_1n_1}{N_1}} e^{j\frac{2\pi\, k_2n_2}{N_2}} & 0 \le n_1 < N_1 \\ & 0 \le n_2 < N_2 \\ 0 & Otherwise \end{cases}$$

Computation of 2D-DFT

- Fourier transform matrix:

$$\mathbf{F}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}$$

where $W_N = e^{-j2\pi/N}$

Computation of 2D-DFT

- Inverse Fourier transform matrix:

$$\mathbf{F}_N^{-1} = \frac{1}{N}\mathbf{F}_N^*$$

where

$$\mathbf{F}_N^* = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{1-N} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & W_N^{1-N} & W_N^{2(1-N)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix}$$

Computation of 2D-DFT

- Example. N = 4:

$$\mathbf{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \qquad \mathbf{F}_4^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

- Matlab function: **`dftmtx`**

Computation of 2D-DFT

- 1D-DFT (of 1D signal **x**):

$$\tilde{\mathbf{x}} = \mathbf{F}_N \mathbf{x}$$

- Inverse 1D-DFT:

$$\mathbf{x} = \frac{1}{N} \mathbf{F}_N^* \tilde{\mathbf{x}}$$

- 2D-DFT (of 2D image **X**):

$$\tilde{\mathbf{X}} = \mathbf{F}_N \mathbf{X} \mathbf{F}_N$$

- Inverse 2D-DFT:

$$\mathbf{X} = \frac{1}{N^2} \mathbf{F}_N^* \tilde{\mathbf{X}} \mathbf{F}_N^*$$

Computation of 2D-DFT

- Verify with Matlab:

```
clear

A = dftmtx(4);
I = [1 2 3 4; 0 0 5 5; -2 -2 4 4; 5 4 2 1];

Adft = A*I*A
Afft = fft2(I)

Aidft = 1/16 * conj(A)*Adft*conj(A)
Aifft = ifft2(Afft)
```

Take a break!

How to compute 2D D.F.T ?

1. Direct Method
2. Row/Column Decomposition
3. Fast Fourier Transform (FFT)

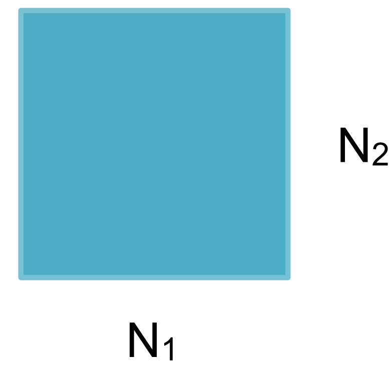
Factors effecting speed:

1. Number of arithmetic operations (*addition & multiplication*)
2. Complexity of program: memory usage
3. Language and platform

How to compute 2D D.F.T ?

Direct Method

$$X(k_1,k_2)=\begin{cases}\displaystyle\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}x(n_1,n_2)\,e^{-j\frac{2\pi\, n_1k_1}{N_1}}\,e^{-j\frac{2\pi\, n_2k_2}{N_2}} & \begin{matrix}0\le k_1<N_1\\ 0\le k_2<N_2\end{matrix}\\ 0 & \mathit{Otherwise}\end{cases}$$



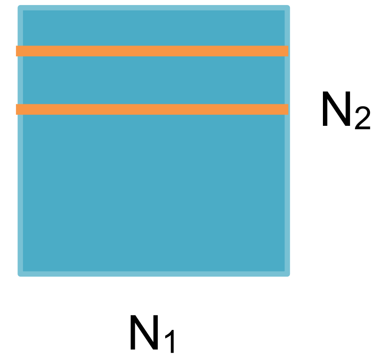
Assuming $N_1 = N_2 = N$,

For each (k_1,k_2), we need N^2 multiplications and N^2 adds

For all pairs of (k_1,k_2), we will need $\mathbf{N^4}$ multiplications and $\mathbf{N^4}$ adds

How to compute 2D D.F.T ?

Row/column decomposition



$$X(k_1,k_2)=\sum_{n_2=0}^{N-1}\underbrace{\left(\sum_{n_1=0}^{N-1}x(n_1,n_2)\,e^{-j\frac{2\pi\, n_1 k_1}{N}}\right)}_{f(k_1,n_2)}e^{-j\frac{2\pi\, n_2 k_2}{N}}$$

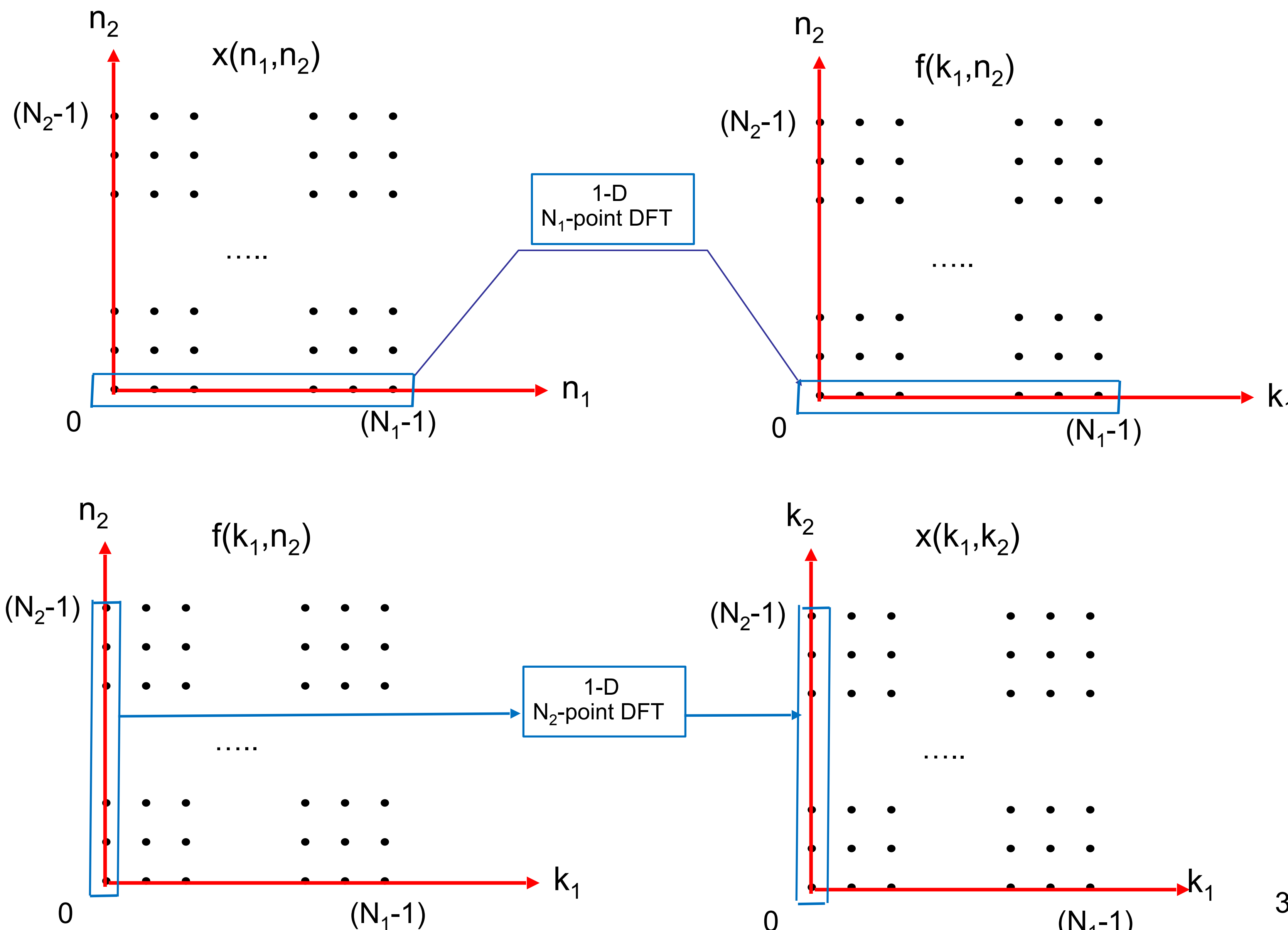
1-D DFT

- By varying n2 from 0 to N-1, we compute a set of 1-D DFTs for all rows of $f(n_1,n_2)$
- The 2D-DFT is just the 1-D transforms of columns of $f(k_1,n_2)$

So 2-D DFT of f can be obtained by computing the 1-D transform of each row of f and then computer the 1-D transform along each column of the result

we will need **$2N^3$** multiplications and **$2N^3$** adds

How to compute 2D D.F.T ?



After J.S.Lim: 2-D Signal and Image Processing

How to compute 2D D.F.T ?

Fast Fourier Transform (FFT)

- Divides original vector into 2
- Calculates FFT of each half recursively
- Merges results

$$F(u) = \sum_{x=0}^{M-1} f(x) W_M^{ux} \tag{4-159}$$

for $u = 0, 1, 2, \dots, M-1$, where

$$W_M = e^{-j2\pi/M} \tag{4-160}$$

and M is assumed to be of the form

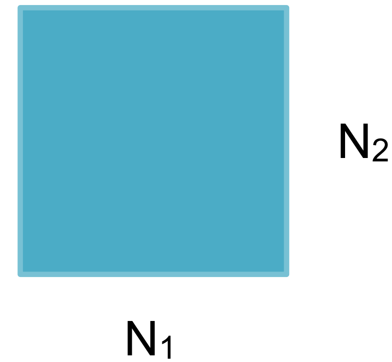
$$M = 2^p \tag{4-161}$$

where p is a positive integer. Then it follows that M can be expressed as

$$M = 2K \tag{4-162}$$

with K being a positive integer also. Substituting Eq. (4-162) into Eq. (4-159) yields

$$\begin{aligned} F(u) &= \sum_{x=0}^{2K-1} f(x) W_{2K}^{ux} \\ &= \sum_{x=0}^{K-1} f(2x) W_{2K}^{u(2x)} + \sum_{x=0}^{K-1} f(2x+1) W_{2K}^{u(2x+1)} \end{aligned} \tag{4-163}$$



For 1 DFT operation,

FFT requires

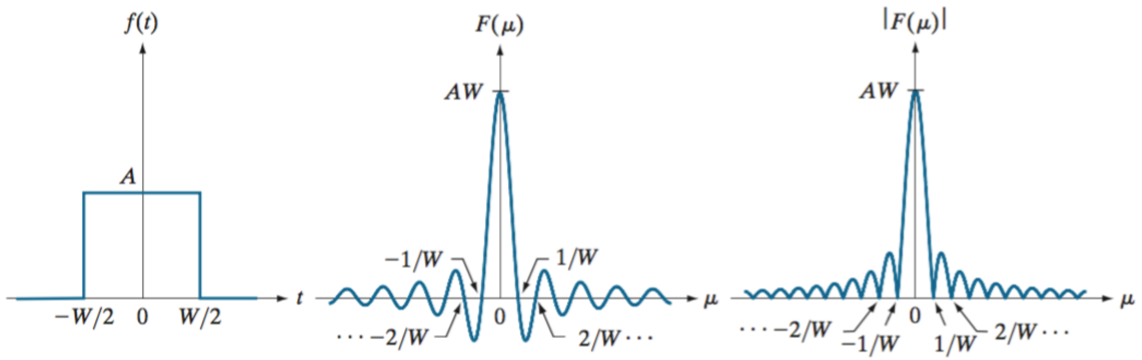
$(N/2)\log_2 N$ multiplications

$N=2048$

92 million mult for FFT
1 trillion for direct method

Using FFT in row/column decomposition,
we will need $\mathbf{N^2 \log_2 N}$ multiplications

2-D Fourier Transform

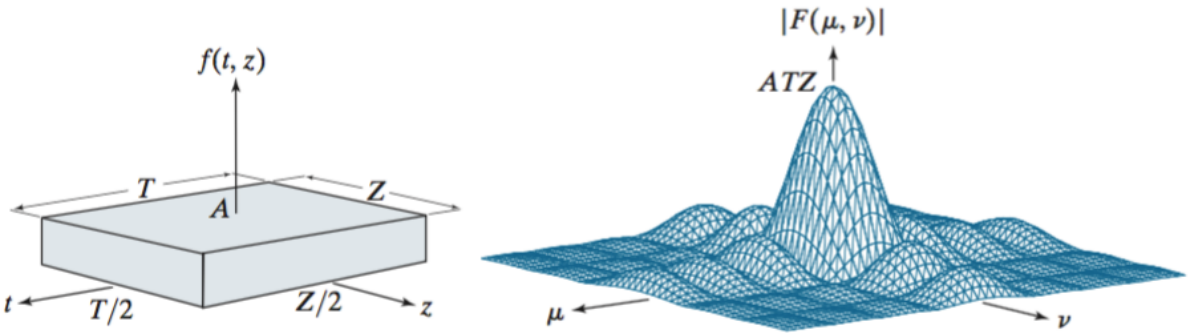


a b c

FIGURE 4.4 (a) A box function, (b) its Fourier transform, and (c) its spectrum. All functions extend to infinity in both directions. Note the inverse relationship between the width, W, of the function and the zeros of the transform.

a b

FIGURE 4.14 (a) A 2-D function and (b) a section of its spectrum. The box is longer along the t-axis, so the spectrum is more contracted along the μ-axis.



Deriving Fourier transforms of box functions

Do Example 4.1 (1-D) and 4.5 (2-D) of the Textbook

Aliasing in 2-D

a b

FIGURE 4.16 Two-dimensional Fourier transforms of (a) an over-sampled, and (b) an under-sampled, band-limited function.

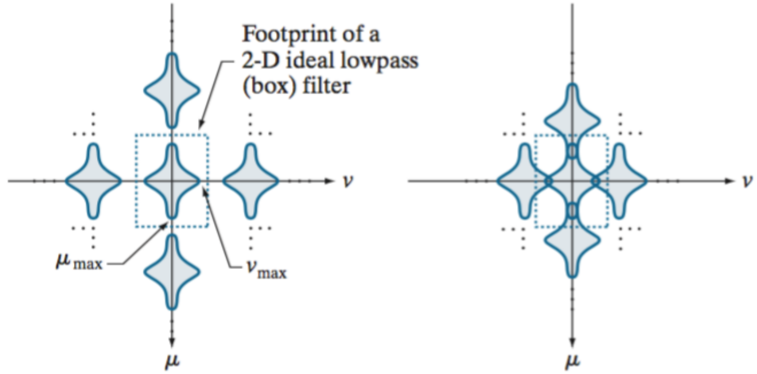
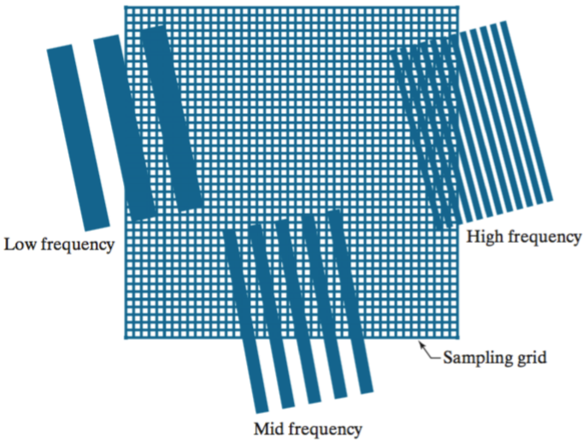
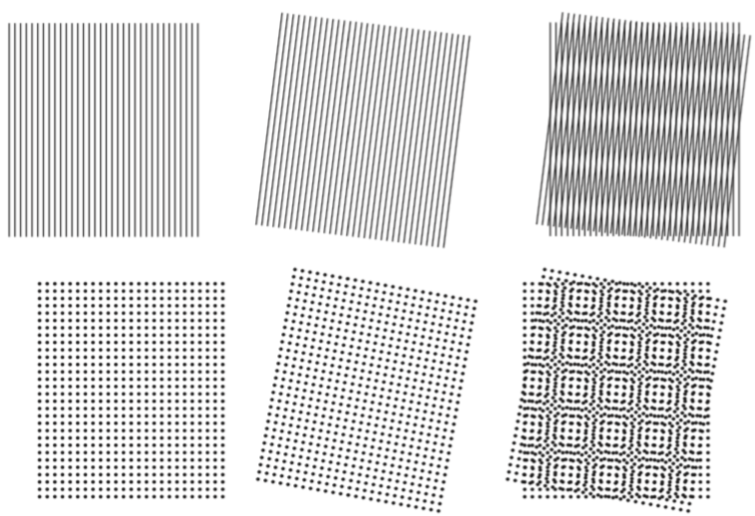


FIGURE 4.17 Various aliasing effects resulting from the interaction between the frequency of 2-D signals and the sampling rate used to digitize them. The regions outside the sampling grid are continuous and free of aliasing.

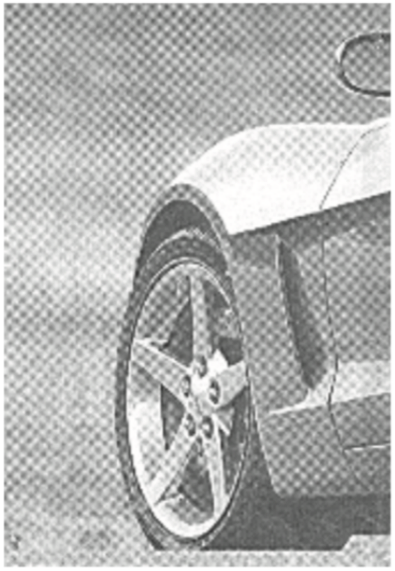


moiré-like patterns



Newspaper printing (75dpi)

FIGURE 4.21 A newspaper image digitized at 75 dpi. Note the moiré-like pattern resulting from the interaction between the ±45° orientation of the half-tone dots and the north-south orientation of the sampling elements used to digitized the image.



Systems and Transforms (2-D)

- Transform (Operator): input into output $T[X(n_1,n_2)] = Y(n_1,n_2)$
- Linear System (Operator)

$$T[aX_1(n_1,n_2)+bX_2(n_1,n_2)] = aT[X_1(n_1,n_2)]+bT[X_2(n_1,n_2)]$$

- Time Invariant System

$$If \;\; T[X(n_1,n_2)] = Y(n_1,n_2)$$

$$then \;\; T[X(n_1-k_1,n_2-k_2)] = Y(n_1-k_1,n_2-k_2)$$

- Impulse Response

$$Y(n_1,n_2) = T[X(n_1,n_2)] = T[\sum_{k_1}\sum_{k_2} X(k_1,k_2)\delta(n_1-k_1,n_2-k_2)]$$

$$= \sum_{k_1}\sum_{k_2} X(k_1,k_2)T[\delta(n_1-k_1,n_2-k_2)]$$

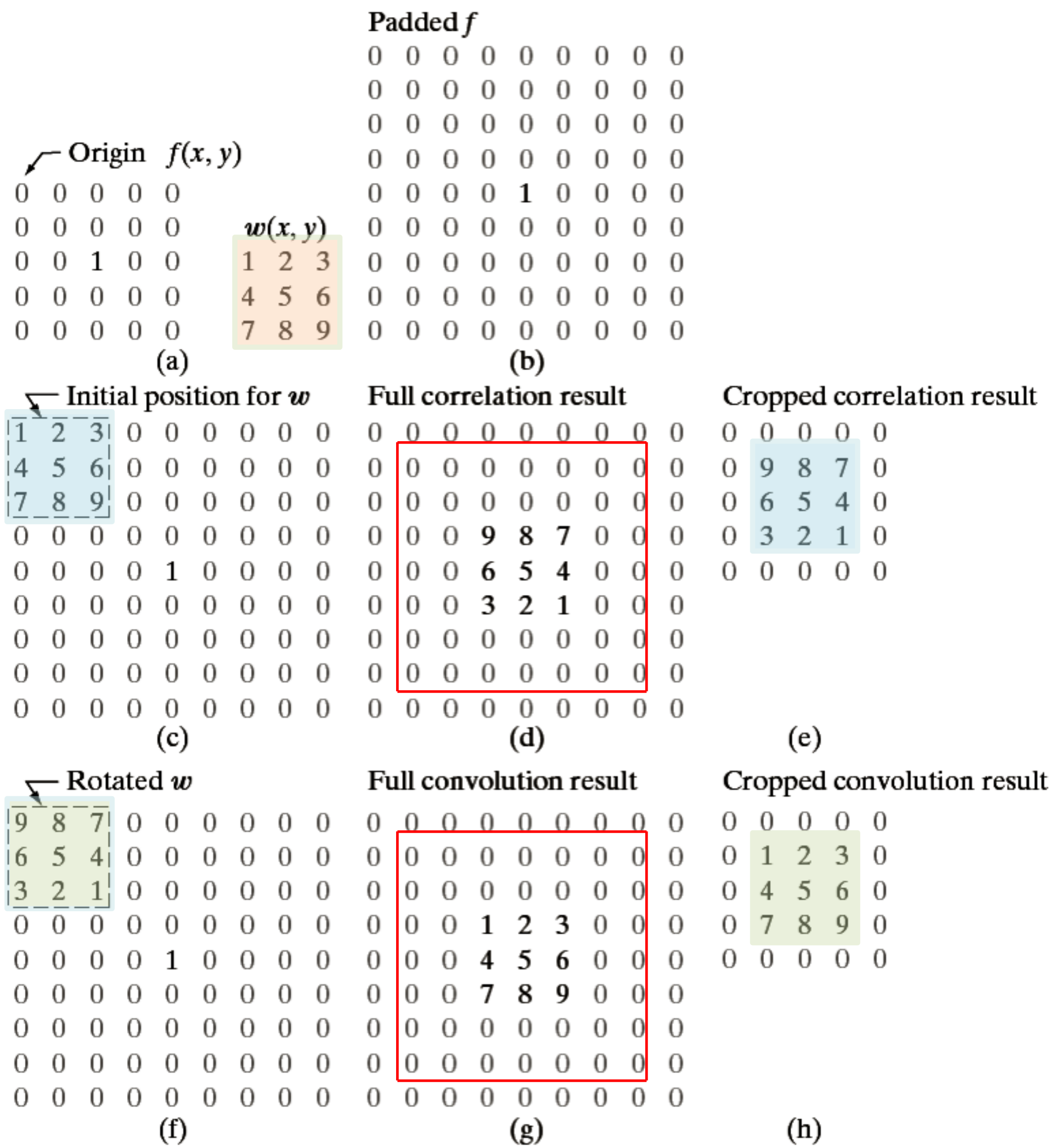
$$Y(n_1,n_2) = \sum_{k_1}\sum_{k_2} X(k_1,k_2)h(n_1-k_1,n_2-k_2)$$ (2-D convolution)

- LTI system can be completely characterized by its response to the impulse and its shift.

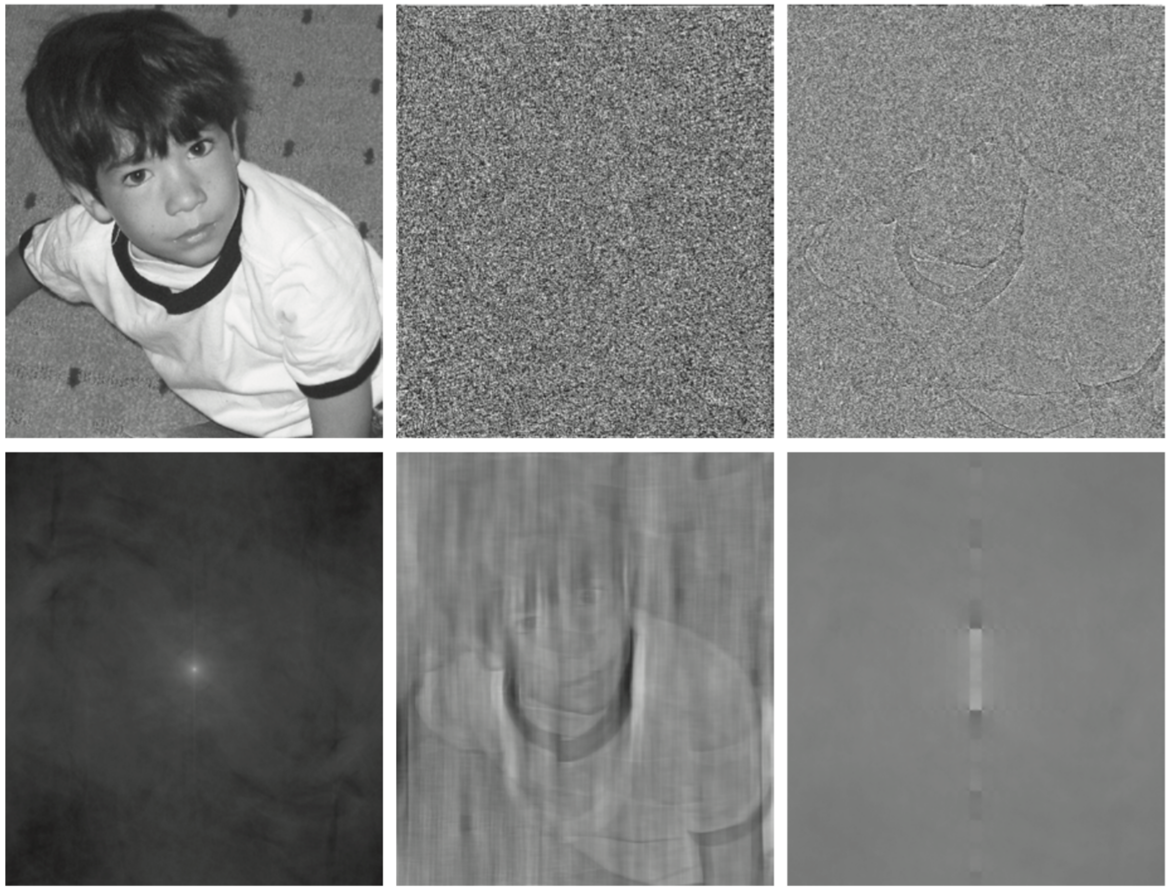
Computation of Correlation and Convolution (2-D)

Correlation

Convolution



Phase vs. Magnitude



a b c
d e f

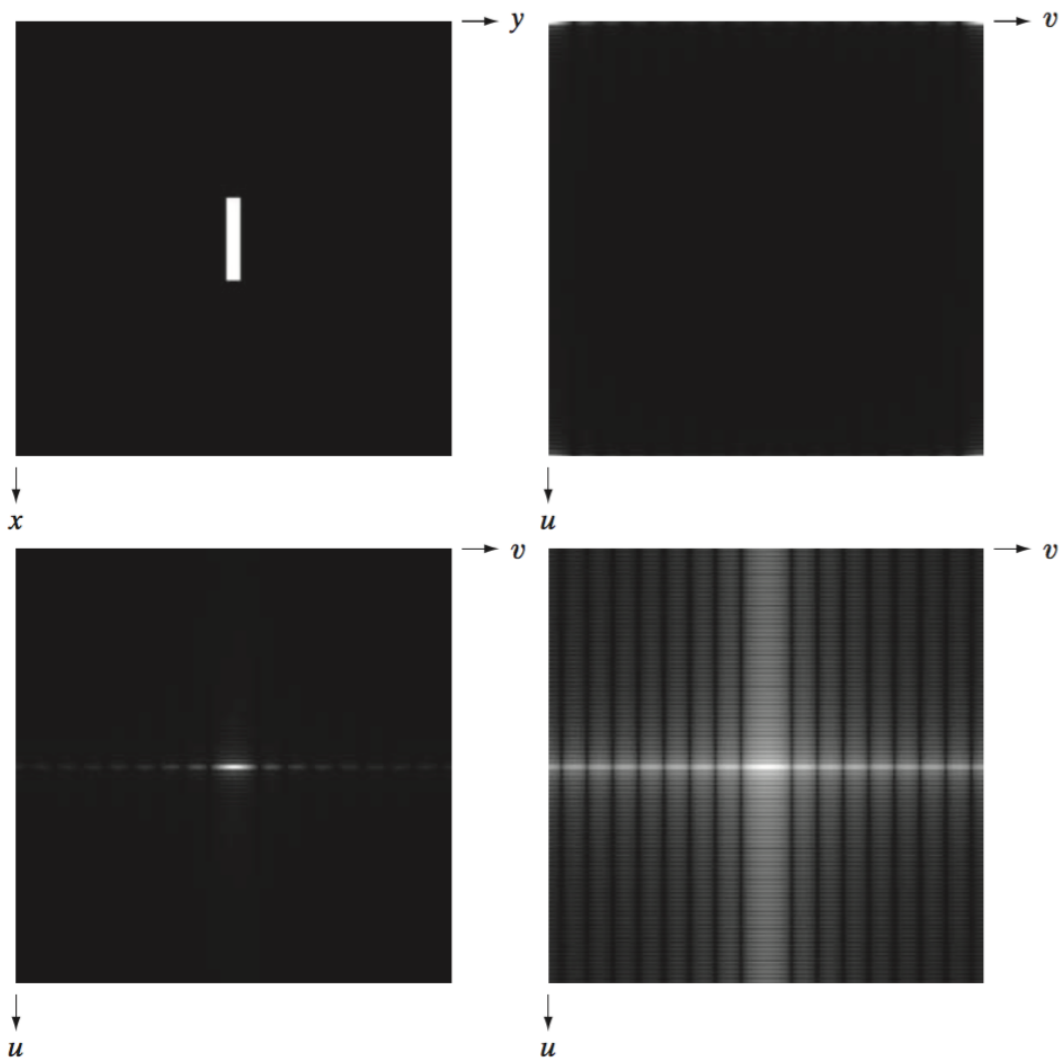
FIGURE 4.26 (a) Boy image. (b) Phase angle. (c) Boy image reconstructed using only its phase angle (all shape features are there, but the intensity information is missing because the spectrum was not used in the reconstruction). (d) Boy image reconstructed using only its spectrum. (e) Boy image reconstructed using its phase angle and the spectrum of the rectangle in Fig. 4.23(a). (f) Rectangle image reconstructed using its phase and the spectrum of the boy's image.

Properties of 2D Fourier Transform

Linearity	$af_1(x,y)+bf_2(x,y) \Leftrightarrow aF_1(u,v)+bF_2(u,v)$
Translation (general)	$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0,v-v_0)$ $f(x-x_0,y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$
Translation to center of the frequency rectangle, $(M/2,N/2)$	$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2,v-N/2)$ $f(x-M/2,y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$
Rotation	$f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\varphi+\theta_0)$ $r=\sqrt{x^2+y^2} \quad \theta=\tan^{-1}(y/x) \quad \omega=\sqrt{u^2+v^2} \quad \varphi=\tan^{-1}(v/u)$
Convolution theorem[†]	$f\star h)(x,y) \Leftrightarrow (F\bullet H)(u,v)$ $(f\bullet h)(x,y) \Leftrightarrow (1/MN)[(F\star H)(u,v)]$

Discrete unit impulse	$\delta(x,y) \Leftrightarrow 1$ $1 \Leftrightarrow MN\delta(u,v)$
Rectangle	$\text{rec}[a,b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
Sine	$\sin(2\pi u_0x/M+2\pi v_0y/N) \Leftrightarrow \dfrac{jMN}{2}[\delta(u+u_0,v+v_0)-\delta(u-u_0,v-v_0)]$
Cosine	$\cos(2\pi u_0x/M+2\pi v_0y/N) \Leftrightarrow \dfrac{1}{2}[\delta(u+u_0,v+v_0)+\delta(u-u_0,v-v_0)]$
Gaussian	$A2\pi\sigma^2e^{-2\pi^2\sigma^2(t^2+z^2)} \Leftrightarrow Ae^{-(\mu^2+\nu^2)/2\sigma^2}$ (A is a constant)

Intuitive experience with 2D FT



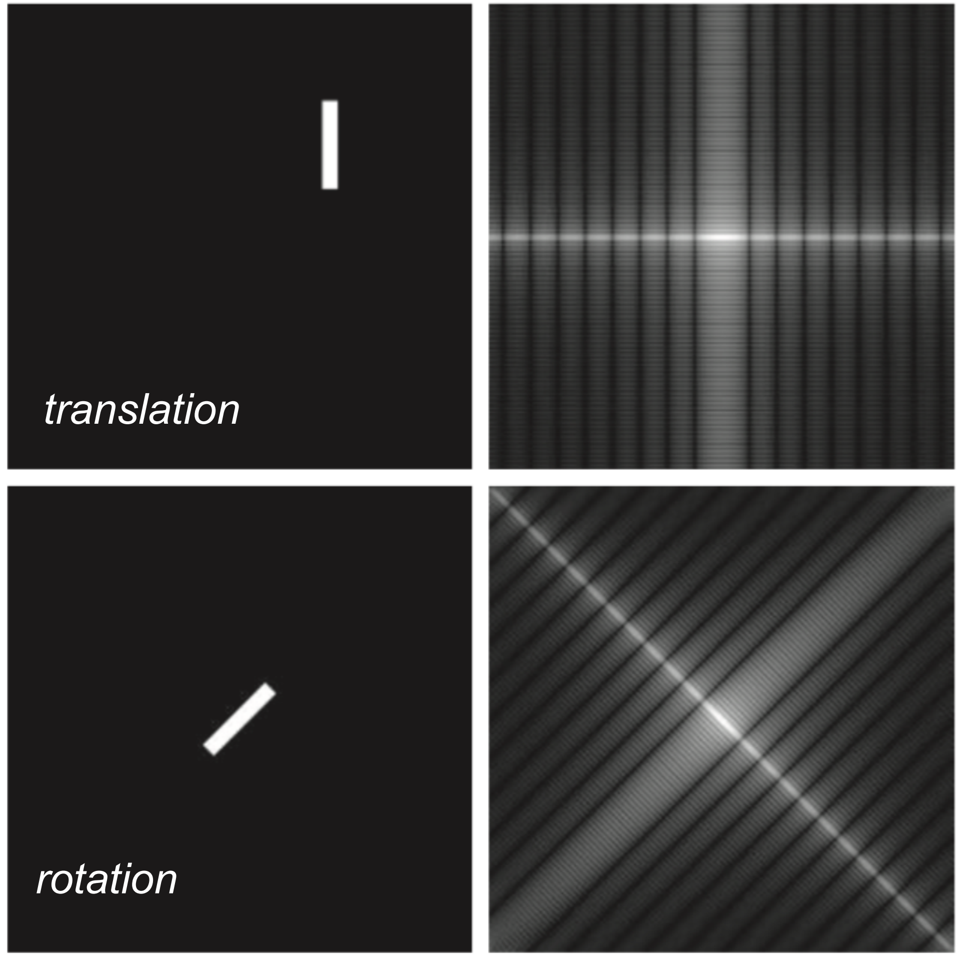
a	b
c	d

FIGURE 4.23
(a) Image.
(b) Spectrum, showing small, bright areas in the four corners (you have to look carefully to see them).
(c) Centered spectrum.
(d) Result after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The right-handed coordinate convention used in the book places the origin of the spatial and frequency domains at the top left (see Fig. 2.19).

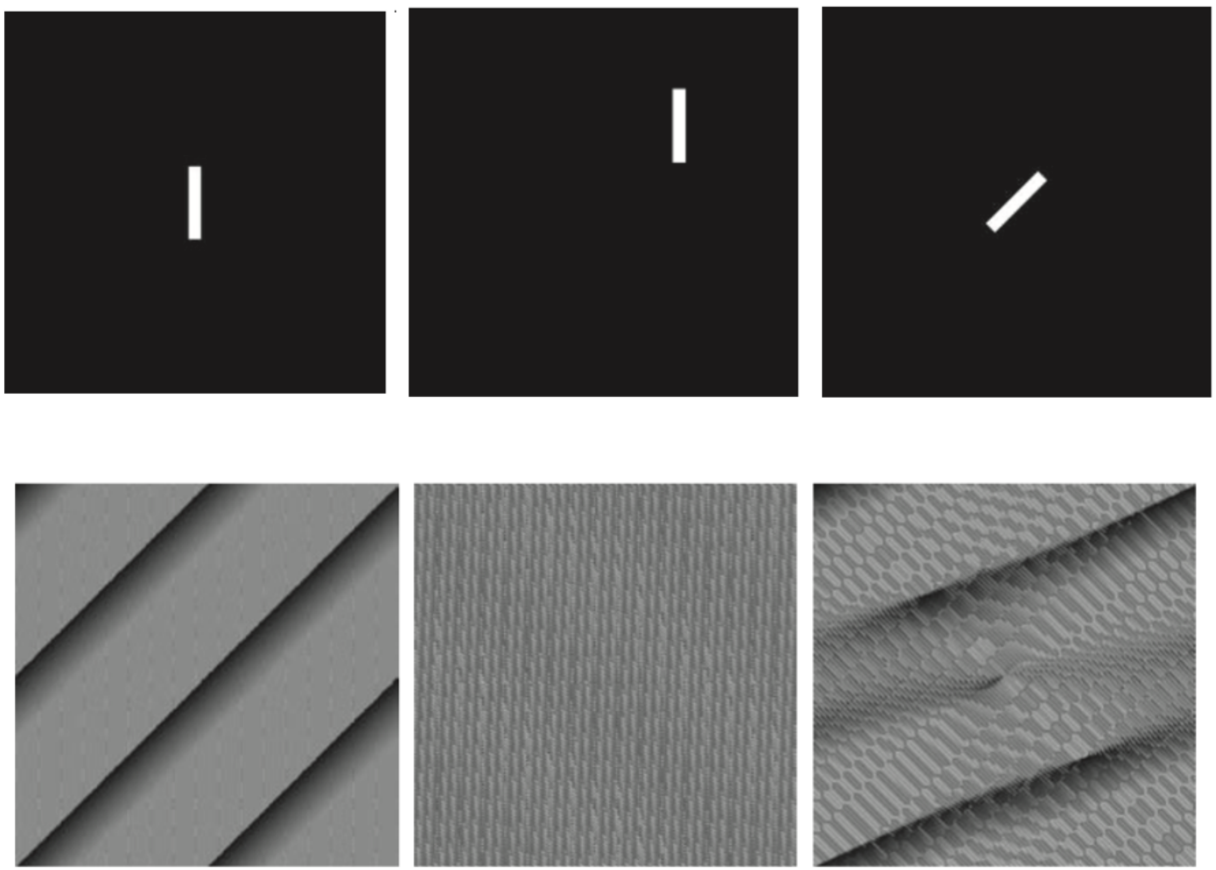
Change in magnitude

a b
c d

FIGURE 4.24
(a) The rectangle in Fig. 4.23(a) translated.
(b) Corresponding spectrum.
(c) Rotated rectangle.
(d) Corresponding spectrum.
The spectrum of the translated rectangle is identical to the spectrum of the original image in Fig. 4.23(a).

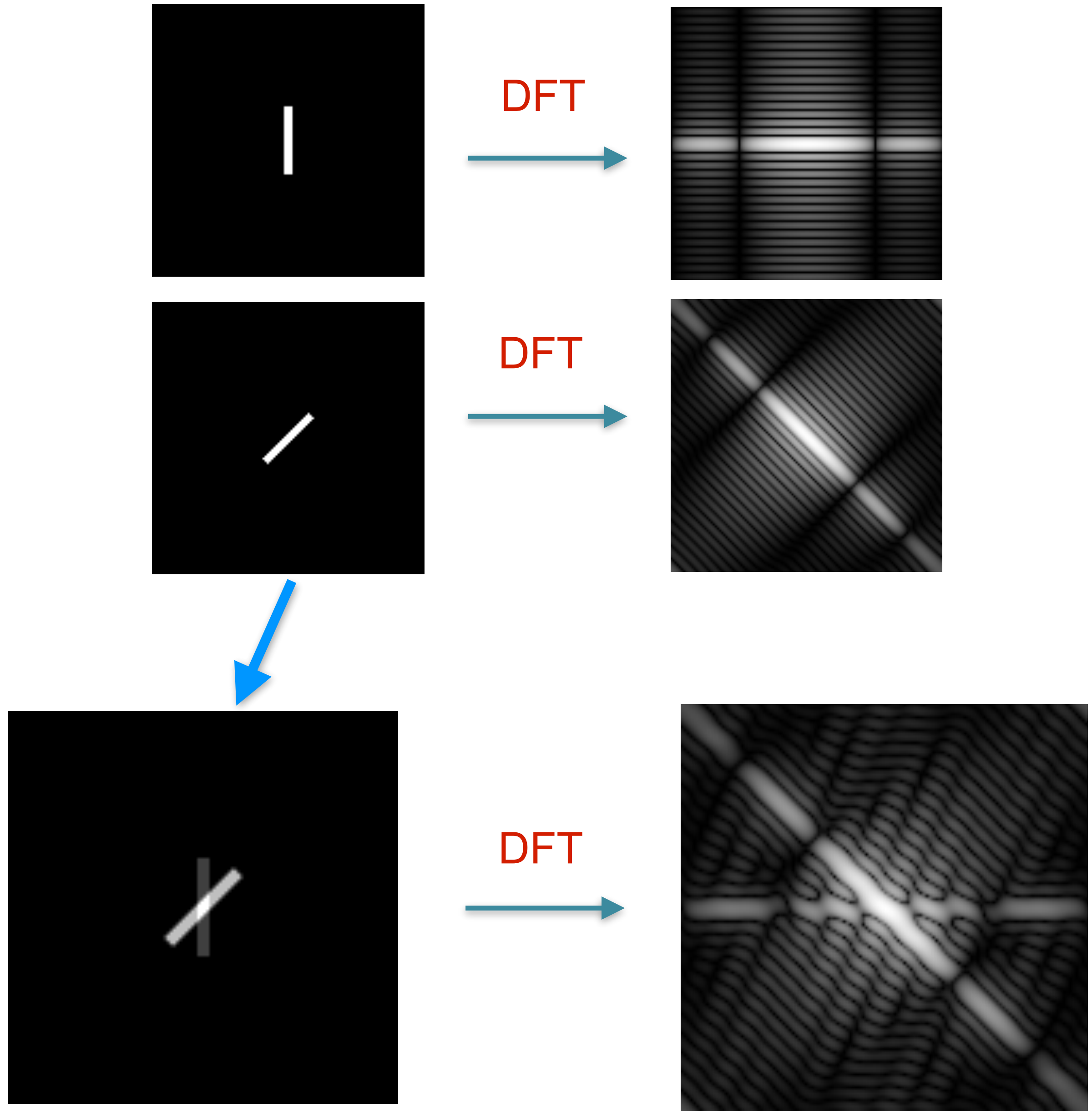


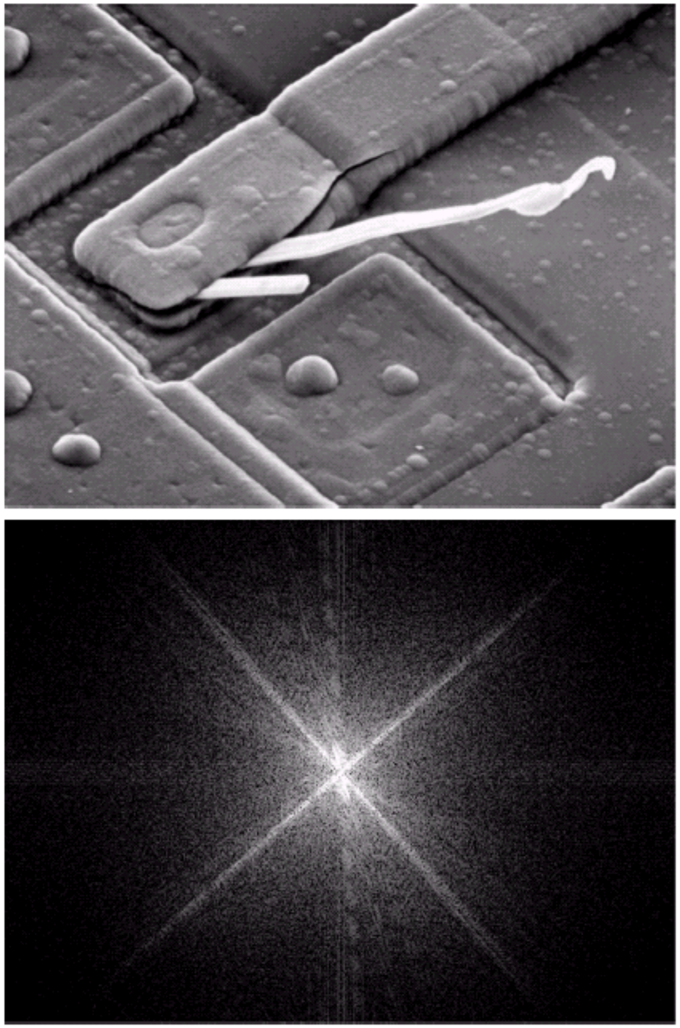
Change in phase maps



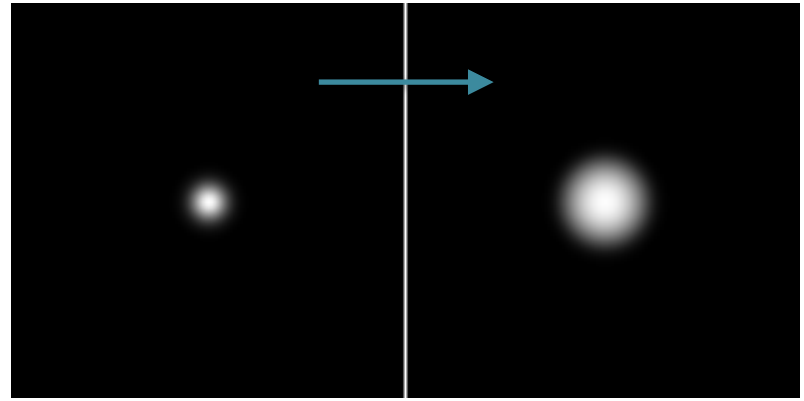
a b c

FIGURE 4.25 Phase angle images of (a) centered, (b) translated, and (c) rotated rectangles.


DFT
DFT
DFT

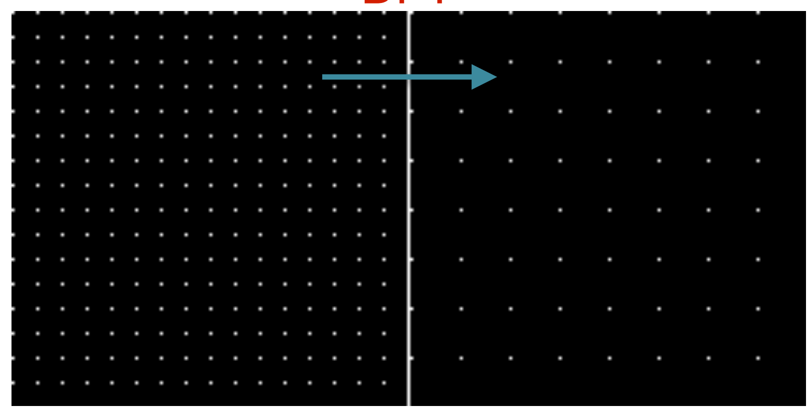


DFT



Gaussian

DFT



Dot grid

Think about other patterns,
what their FT look like?

Summary

- 1-D Fourier transform -> 2D Fourier Transform
- Aliasing: reasons and impacts
- Discrete Fourier Transform (DFT) and its computation (FT matrix, direct method, row/column decomposition, FFT, computational complexity)
- Convolution and properties
- Properties of FT: phase & magnitude
- Intuitive understanding of images vs. FT

Reading materials

- Textbook Chapter 4

Page 204-224, 230-240, 249-253, 260-261

Questions

- What is the Nyquist theorem? what is the type of signal fitted?
- Image reconstruction with inverse Fourier transform: phase vs. magnitude
- Moire-like patterns in old news-paper printing: what is the cause?
- What are the benefits of Fourier analysis?
- Is Fourier transform a linear operation?
- What is the “”sifting property” of the delta/impulse function?
- How would the magnitude and phase maps change if spatial transformations are applied to the image?
- Images vs. frequency representation
- Correlation vs. convolution