§6. Appendix: Differential Geometry

This material is used most recently by Dzyaloshinski and Volovick (1980), de Wit (1981), Kröner (1981,1992), Venkataraman and Sahoo (1986), Dereli and Vercin (1987), and Kröner and Lagoudas (1992), to describe dislocations and disclinations in lattices.

The crystallographic coordinate systems used to define undeformed and deformed crystal lattices are examples of three dimensional "manifolds." A *manifold* is a "generalized" version of a parameterization of a surface or line (Choquet-Bruhat *et al.* 1982). The "natural coordinate system" used to describe a manifold will not generally be orthogonal, and the tangent and cotangent basis vectors will not generally be orthonormal (Misner *et al.* 1973). The *differential geometry* of a general, n-dimensional manifold is introduced here. Clearly, the case $n = 3$ represents the geometry of a crystal lattice. Note however, that this mathematics is also for solving systems of partial differential equations, *e.g.* Schouten and Kulk (1969), Levi-Cevita (1977), Edelen (1985), and Bryant *et al.* (1991), and this is its most important application.

Let χ^α denote the "natural coordinates" of an n-dimensional manifold and $x^a = x,y,z$ denote coordinates of an orthonormal reference frame: $\chi^\alpha = \chi^\alpha(x,y,z) = \chi^\alpha(x^a)$. So from the chain rule,

$$\mathbf{e}^\alpha \equiv \vec{\mathbf{d}}\chi^\alpha = (\partial\chi^\alpha/\partial x^a)\vec{\mathbf{d}}x^a \equiv \lambda^\alpha{}_a\vec{\mathbf{d}}x^a,\ \vec{\mathbf{d}}x^a = (\partial x^a/\partial\chi^\alpha)\vec{\mathbf{d}}\chi^\alpha \equiv \lambda_\alpha{}^a\vec{\mathbf{d}}\chi^\alpha;$$

$$\mathbf{e}_\alpha \equiv \vec{\boldsymbol{\partial}}_\alpha \equiv \partial/\partial\chi^\alpha = (\partial x^a/\partial\chi^\alpha)\partial/\partial x^a \equiv (\partial x^a/\partial\chi^\alpha)\vec{\boldsymbol{\partial}}_a = \lambda_\alpha{}^a\vec{\boldsymbol{\partial}}_a,\ \vec{\boldsymbol{\partial}}_a = \lambda^\alpha{}_a\vec{\boldsymbol{\partial}}_\alpha;$$

natural tangent and cotangent basis vectors $\vec{\boldsymbol{\partial}}_\alpha \equiv \partial/\partial\chi^\alpha \equiv \mathbf{e}_\alpha$ and $\vec{\mathbf{d}}\chi^\alpha \equiv \mathbf{e}^\alpha$, respectively, are obtained. They satisfy $\mathbf{e}^\alpha\bullet\mathbf{e}_\beta \equiv \langle\vec{\mathbf{d}}\chi^\alpha,\vec{\boldsymbol{\partial}}_\beta\rangle \equiv \vec{\mathbf{d}}\chi^\alpha(\vec{\boldsymbol{\partial}}_\beta) \equiv \partial\chi^\alpha/\partial\chi^\beta = \delta^\alpha{}_\beta$. These tangent basis vectors *commute*,

$$[\vec{\partial}_\alpha, \vec{\partial}_\beta] \equiv \vec{\partial}_\alpha[\vec{\partial}_\beta] - \vec{\partial}_\beta[\vec{\partial}_\alpha] \equiv \partial\vec{\partial}_\beta/\partial\chi^\alpha - \partial\vec{\partial}_\alpha/\partial\chi^\beta \equiv \partial^2/\partial\chi^\alpha\partial\chi^\beta - \partial^2/\partial\chi^\beta\partial\chi^\alpha = \mathbf{0},$$

for all $\alpha, \beta = 1,\ldots,n$, so the $\vec{\partial}_\alpha$ are a *coordinate, holonomic, or natural basis*. If some set of basis vectors do not commute, then they are a *anholonomic basis*, and

$$[\vec{\partial}_\alpha, \vec{\partial}_\beta] \equiv \vec{\partial}_\alpha[\vec{\partial}_\beta] - \vec{\partial}_\beta[\vec{\partial}_\alpha] \equiv \Omega_{\alpha\beta}{}^\gamma\vec{\partial}_\gamma \neq \partial^2/\partial\chi^\alpha\partial\chi^\beta - \partial^2/\partial\chi^\beta\partial\chi^\alpha,$$

where $\Omega_{\alpha\beta}{}^\gamma$ is the *anholonomic object* (Schouten 1954, 1989), $\Omega_{\alpha\beta}{}^\gamma = 0$ for a coordinate basis (Misner *et al.* 1973)[65] like those used in the text for crystallographic coordinate systems and, of course, the reference frame. The coefficients $\Omega_{\alpha\beta}{}^\gamma$ are antisymmetric $\Omega_{\alpha\beta}{}^\gamma = -\Omega_{\beta\alpha}{}^\gamma$, and vanish when $\alpha = \beta$. See Marcinkowski (1977,1979) for applications of anholonomic

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[65]: For example, consider the well known spherical coordinates r,θ,ϕ: $\mathbf{e}_r \equiv \partial/\partial r = \hat{\mathbf{r}} = \mathbf{e}^r \equiv \vec{\mathbf{d}}r \equiv {}^\flat\nabla r$; $\mathbf{e}_\theta \equiv \partial/\partial\theta \equiv r\hat{\boldsymbol{\theta}}$, $\mathbf{e}^\theta \equiv \vec{\mathbf{d}}\theta \equiv {}^\flat\nabla\theta = \hat{\boldsymbol{\theta}}/r$; $\mathbf{e}_\phi \equiv \partial/\partial\phi \equiv (r\sin\theta)\hat{\boldsymbol{\phi}}$, $\mathbf{e}^\phi \equiv \vec{\mathbf{d}}\phi \equiv {}^\flat\nabla\phi = \hat{\boldsymbol{\phi}}/(r\sin\theta)$ (Misner *et al.* 1973). Here, $\hat{\mathbf{r}},\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\phi}}$ are unit vectors in the respective directions; these are orthonormal and form a anholonomic basis for the r,θ,ϕ coordinate system. The tangent and cotangent basis vectors, $\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_\phi$ and $\mathbf{e}^r,\mathbf{e}^\theta,\mathbf{e}^\phi$, respectively, are functions of position, $\mathbf{e}_\alpha = \mathbf{e}_\alpha(x^a)$ and $\mathbf{e}^\alpha = \mathbf{e}^\alpha(x^a)$, as are the directions of the unit vectors $\hat{\mathbf{r}},\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\phi}}$. There are two singularities for these coordinates: all meridians (ϕ = *constant*) pass through the north and south poles (Misner and Wheeler 1957).

coordinates to dislocation theory.

The *Lie derivative* of a vector $\mathbf{v}$ in the direction of another vector $\mathbf{u}$, $\pounds_{\mathbf{u}}\mathbf{v}$, is given by the *Poisson bracket*,[66] or *commutator* of the vectors, $[\mathbf{u},\mathbf{v}]$:

$$\begin{aligned}
\pounds_{\mathbf{u}}\mathbf{v} \equiv [\mathbf{u},\mathbf{v}] \equiv \mathbf{u}[\mathbf{v}] - \mathbf{v}[\mathbf{u}] &\equiv u^{\alpha}\{\partial(v^{\beta}\mathbf{e}_{\beta})/\partial\chi^{\alpha}\} - v^{\alpha}\{\partial(u^{\beta}\mathbf{e}_{\beta})/\partial\chi^{\alpha}\},\\
&\equiv (u^{\alpha}v^{\beta}{}_{,\alpha} - v^{\alpha}u^{\beta}{}_{,\alpha})\mathbf{e}_{\beta} + (u^{\alpha}v^{\beta} - v^{\alpha}u^{\beta})\partial\mathbf{e}_{\beta}/\partial\chi^{\alpha},\\
&= (u^{\alpha}v^{\beta}{}_{,\alpha} - v^{\alpha}u^{\beta}{}_{,\alpha})\mathbf{e}_{\beta} + u^{\alpha}v^{\beta}(\partial\mathbf{e}_{\beta}/\partial\chi^{\alpha} - \partial\mathbf{e}_{\alpha}/\partial\chi^{\beta}),\\
&\equiv (u^{\alpha}v^{\beta}{}_{,\alpha} - v^{\alpha}u^{\beta}{}_{,\alpha})\mathbf{e}_{\beta} + u^{\alpha}v^{\beta}(\mathbf{e}_{\alpha}[\mathbf{e}_{\beta}] - \mathbf{e}_{\beta}[\mathbf{e}_{\alpha}]),\\
&\equiv (u^{\alpha}v^{\beta}{}_{,\alpha} - v^{\alpha}u^{\beta}{}_{,\alpha})\mathbf{e}_{\beta} + u^{\alpha}v^{\beta}[\mathbf{e}_{\alpha},\mathbf{e}_{\beta}],\\
&\equiv (u^{\alpha}v^{\beta}{}_{,\alpha} - v^{\alpha}u^{\beta}{}_{,\alpha})\mathbf{e}_{\beta} + u^{\alpha}v^{\beta}\Omega_{\alpha\beta}{}^{\gamma}\,\mathbf{e}_{\gamma},\\
&\equiv (u^{\alpha}v^{\beta}{}_{,\alpha} - v^{\alpha}u^{\beta}{}_{,\alpha})\mathbf{e}_{\beta}. \quad \text{(For a holonomic basis.)}
\end{aligned}$$

Here, $\mathbf{e}_{\alpha}[\mathbf{e}_{\beta}] \equiv \partial\mathbf{e}_{\beta}/\partial\chi^{\alpha}$ is the *directional derivative* of $\mathbf{e}_{\beta}$ along $\mathbf{e}_{\alpha}$ and $[\mathbf{e}_{\alpha},\mathbf{e}_{\beta}] \equiv \mathbf{e}_{\alpha}[\mathbf{e}_{\beta}] - \mathbf{e}_{\beta}[\mathbf{e}_{\alpha}] \equiv \Omega_{\alpha\beta}{}^{\gamma}\,\mathbf{e}_{\gamma}$. For example, the *linear elastic strain tensor* $\boldsymbol{\varepsilon}$ can be defined as the Lie derivative of the metric tensor $\mathbf{g}$ (for the deformed coordinates) with respect to the displacement vector $\mathbf{u}$ (Marsden and Hughes 1983): $\boldsymbol{\varepsilon} = \tfrac{1}{2}\pounds_{\mathbf{u}}\mathbf{g}$. If $\pounds_{\mathbf{u}}\mathbf{v} = \mathbf{0} \Rightarrow u^{\alpha}v^{\beta}{}_{,\alpha} = v^{\alpha}u^{\beta}{}_{,\alpha}$, then $\mathbf{v}$ is *Lie transported*, or *dragged* over the vector field $\mathbf{u}$ (Burke 1985), and these vector fields therefore commute.

If $\mathbf{S} = S^{\alpha\beta\ldots}{}_{\gamma\delta\ldots}\,\mathbf{e}_{\alpha}\otimes\mathbf{e}_{\beta}\otimes\ldots\otimes\mathbf{e}^{\gamma}\otimes\mathbf{e}^{\delta}\otimes\ldots$ is any tensor, then its Lie derivative with respect to a vector $\mathbf{v}$ is (Burke 1985)

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[66]: See Dzyaloshinskii and Volovick (1980) for several applications of Poisson brackets in condensed matter physics.

$$\pounds_{\mathbf{v}}(S^{\alpha\beta\ldots}{}_{\gamma\delta\ldots}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes\ldots\otimes \mathbf{e}^\gamma \otimes \mathbf{e}^\delta \otimes\ldots) = [S^{\alpha\beta\ldots}{}_{\gamma\delta\ldots,\sigma}v^\sigma - S^{\sigma\beta\ldots}{}_{\gamma\delta\ldots}v^\alpha{}_{,\sigma} - S^{\alpha\sigma\ldots}{}_{\gamma\delta\ldots}v^\beta{}_{,\sigma} - \ldots$$
$$+ S^{\alpha\beta\ldots}{}_{\sigma\delta\ldots}v^\sigma{}_{,\gamma} + S^{\alpha\beta\ldots}{}_{\gamma\sigma\ldots}v^\sigma{}_{,\delta} + \ldots]\mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes\ldots\otimes \mathbf{e}^\gamma \otimes \mathbf{e}^\delta \otimes\ldots\ .$$

Here again, $S^{\alpha\beta\ldots}{}_{\gamma\delta\ldots,\sigma} \equiv \partial S^{\alpha\beta\ldots}{}_{\gamma\delta\ldots}/\partial\chi^\sigma$ and $v^\alpha{}_{,\sigma} \equiv \partial v^\alpha/\partial\chi^\sigma$.

The Lie bracket obeys the *Jacobi identity* (Misner *et al.* 1973, Burke 1985)

$$[\mathbf{u},[\mathbf{v},\mathbf{w}]] + [\mathbf{v},[\mathbf{w},\mathbf{u}]] + [\mathbf{w},[\mathbf{u},\mathbf{v}]] = \mathbf{0},$$

which means that the vector fields **u**,**v**,**w** fill space with "blocks" that do not leave any gaps (Misner *et al.* 1973). For anholonomic basis vectors the Jacobi identity is (Edelen 1985):

$$[\mathbf{a}_\alpha,[\mathbf{a}_\beta,\mathbf{a}_\gamma]] + [\mathbf{a}_\beta,[\mathbf{a}_\gamma,\mathbf{a}_\alpha]] + [\mathbf{a}_\gamma,[\mathbf{a}_\alpha,\mathbf{a}_\beta]] = [\mathbf{a}_\alpha,\Omega_{\beta\gamma}{}^\epsilon\mathbf{a}_\epsilon] + [\mathbf{a}_\beta,\Omega_{\gamma\alpha}{}^\epsilon\mathbf{a}_\epsilon] + [\mathbf{a}_\gamma,\Omega_{\alpha\beta}{}^\epsilon\mathbf{a}_\epsilon],$$
$$= (\partial\Omega_{\beta\gamma}{}^\delta/\partial\chi^\alpha + \partial\Omega_{\gamma\alpha}{}^\delta/\partial\chi^\beta + \partial\Omega_{\alpha\beta}{}^\delta/\partial\chi^\gamma + \Omega_{\beta\gamma}{}^\epsilon\Omega_{\alpha\epsilon}{}^\delta + \Omega_{\gamma\alpha}{}^\epsilon\Omega_{\beta\epsilon}{}^\delta + \Omega_{\alpha\beta}{}^\epsilon\Omega_{\gamma\epsilon}{}^\delta)\mathbf{a}_\delta,$$
$$= \mathbf{0}.$$

Differential forms are antisymmetric tensors expanded with respect to the cotangent basis vectors. Any p-form $\boldsymbol{\omega}$ is

$$\boldsymbol{\omega} \equiv \frac{1}{p!}\omega_{\alpha_1\ldots\alpha_p}\vec{\mathbf{d}}\chi^{\alpha_1}\wedge\ldots\wedge\vec{\mathbf{d}}\chi^{\alpha_p} \equiv \omega_{|\alpha_1\ldots\alpha_p|}\vec{\mathbf{d}}\chi^{\alpha_1}\wedge\ldots\wedge\vec{\mathbf{d}}\chi^{\alpha_p},$$

vertical bars denoting summation over the restricted range $\alpha_1 < \ldots < \alpha_p$, which has components that satisfy the relationship

$$\omega_{\alpha_1\ldots\alpha_p} = \omega_{[\alpha_1\ldots\alpha_p]},$$

where

$$\omega_{[\alpha_1 \ldots \alpha_p]} \equiv \frac{1}{p} \sum_{\pi} \delta_\pi \omega_{\alpha_{\pi(1)} \ldots \alpha_{\pi(p)}},$$

with the sum being taken over all permutations, π, of $1,\ldots,p$ and δ_π being 1 for even permutations and —1 for odd permutations (Wald 1984). For example, the components of a 2-form $\boldsymbol{\omega}$ are

$$\omega_{\alpha\beta} = \omega_{[\alpha,\beta]} \equiv \tfrac{1}{2}(\omega_{\alpha\beta} - \omega_{\beta\alpha}) \Rightarrow \omega_{\alpha\beta} = -\omega_{\beta\alpha}.$$

Any p-form defines the integral (Misner *et al.* 1973, Abraham *et al.* 1988)

$$\int \boldsymbol{\omega} \equiv \frac{1}{p!} \int \omega_{\alpha_1 \ldots \alpha_p} \vec{\mathbf{d}}\chi^{\alpha_1} \ldots \vec{\mathbf{d}}\chi^{\alpha_p} \equiv \int \omega_{|\alpha_1 \ldots \alpha_p|} \vec{\mathbf{d}}\chi^{\alpha_1} \ldots \vec{\mathbf{d}}\chi^{\alpha_p}.$$

Let $\partial\mathbf{S}$ represent a closed p-dimensional boundary of a $(p + 1)$-dimensional surface $\mathbf{S}$. Then

$$\int_{\mathbf{S}} \vec{\mathbf{d}}\boldsymbol{\omega} = \int_{\partial\mathbf{S}} \boldsymbol{\omega},$$

where $\boldsymbol{\omega}$ is a p-form defined throughout $\mathbf{S}$, is the *generalized Stokes' theorem* (Misner *et al.* 1973).

The *exterior derivative operator* $\vec{\mathbf{d}}$ produces a $(p + 1)$-form from a p-form $\boldsymbol{\omega}$:

$$\vec{\mathbf{d}}\boldsymbol{\omega} = \vec{\mathbf{d}}(\omega_{|\alpha_1 \ldots \alpha_p|})\vec{\mathbf{d}}\chi^{\alpha_1} \wedge \ldots \wedge \vec{\mathbf{d}}\chi^{\alpha_p} = (\partial\omega_{|\alpha_1 \ldots \alpha_p|}/\partial\chi^{\beta})\vec{\mathbf{d}}\chi^{\beta} \wedge \vec{\mathbf{d}}\chi^{\alpha_1} \wedge \ldots \wedge \vec{\mathbf{d}}\chi^{\alpha_p}.$$

Schouten (1954,1989) calls this the *natural derivative* of $\boldsymbol{\omega}$.

For functions f the exterior derivative acts like $\vec{\mathbf{d}} f = (\partial f/\partial \chi^\alpha)\vec{\mathbf{d}}\chi^\alpha \equiv \nabla f$. Here, ∇ is the *covariant derivative operator*, which is analogous to the gradient operator $\boldsymbol{\nabla}$. The *directional derivative* of f in the direction of vector $\mathbf{u}$ is (Misner *et al.* 1973)

$$\langle \vec{\mathbf{d}} f, \mathbf{u} \rangle \equiv \partial_{\mathbf{u}} f \equiv \mathbf{u}[\,f\,] \equiv u^\alpha(\partial f/\partial x^\alpha).$$

The exterior derivative can be similarly "generalized" (Misner *et al.* 1973):

$$\vec{\mathbf{d}}\mathbf{v} \equiv \nabla \mathbf{v},\ \langle \vec{\mathbf{d}}\mathbf{v}, \mathbf{u} \rangle \equiv \nabla_{\mathbf{u}} \mathbf{v} \equiv u^\beta (v^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\gamma\beta} v^\gamma)\mathbf{e}_\alpha \equiv u^\beta v^\alpha{}_{;\beta}\mathbf{e}_\alpha,$$

where $\langle \vec{\mathbf{d}}\mathbf{v}, \mathbf{u} \rangle \equiv \nabla_{\mathbf{u}} \mathbf{v} \equiv u^\beta(\partial v^\alpha/\partial \chi^\beta + \Gamma^\alpha{}_{\gamma\beta} v^\gamma)\mathbf{e}_\alpha$ is the *covariant derivative* of vector $\mathbf{v}$ in the direction of vector $\mathbf{u}$, $\partial v^\alpha/\partial \chi^\beta \equiv v^\alpha{}_{,\beta}$, and $\Gamma^\alpha{}_{\gamma\beta}$ denotes the *connection coefficients*. The expression $\vec{\mathbf{d}}\mathbf{v}$ is a *"vector valued" 1-form*: <u>here, $\mathbf{v}$ is a contravariant vector, not a 1-form</u>. Any vector $\mathbf{v}$ can be written as

$$\langle \vec{\mathbf{d}}\wp, \mathbf{v} \rangle \equiv \mathbf{v}(\wp)$$

at a point $\wp$, where

$$\vec{\mathbf{d}}\wp \equiv \mathbf{e}_\alpha \otimes \mathbf{e}^\alpha$$

is a "unit tensor" with one slot for a vector, $\vec{\mathbf{d}}\wp = \vec{\mathbf{d}}\wp(\ldots)$: $\vec{\mathbf{d}}\wp(\mathbf{v}) \equiv \mathbf{v}(\wp)$. In general the basis vectors $\mathbf{e}_\alpha \equiv \vec{\partial}_\alpha$ and $\mathbf{e}^\alpha \equiv \vec{\mathbf{d}}\chi^\alpha$ take on different values for different points on the manifold. The ***"generalized exterior derivative operator"*** (Misner *et al.* 1973), like the covariant derivative operator, acts on both the components and basis vectors of a tensor; for the basis vectors:

Procedure for Relating Exterior Derivative and Covariant Derivative Operators

$$\vec{\mathbf{d}}\mathbf{e}_\alpha \equiv \nabla\mathbf{e}_\alpha \equiv \mathbf{e}_\beta\Gamma^\beta{}_{\alpha\gamma} \otimes \mathbf{e}^\gamma,\ \vec{\mathbf{d}}\mathbf{e}^\beta \equiv \nabla\mathbf{e}^\beta \equiv -\mathbf{e}^\alpha\Gamma^\beta{}_{\alpha\gamma} \otimes \mathbf{e}^\gamma;$$

$$\langle\vec{\mathbf{d}}\mathbf{e}_\alpha,\mathbf{e}_\gamma\rangle \equiv \nabla\mathbf{e}_\alpha(\ldots,\mathbf{e}_\gamma) \equiv \nabla_{\mathbf{e}_\gamma}\mathbf{e}_\alpha \equiv \mathbf{e}_\beta\Gamma^\beta{}_{\alpha\gamma};\ (\textit{how } \mathbf{e}_\alpha \textit{ changes in the direction of } \mathbf{e}_\gamma)$$

$$\langle\vec{\mathbf{d}}\mathbf{e}^\beta,\mathbf{e}_\gamma\rangle \equiv \nabla\mathbf{e}^\beta(\ldots,\mathbf{e}_\gamma) \equiv \nabla_{\mathbf{e}_\gamma}\mathbf{e}^\beta \equiv -\mathbf{e}^\alpha\Gamma^\beta{}_{\alpha\gamma};\ (\textit{how } \mathbf{e}^\beta \textit{ changes in the direction of } \mathbf{e}_\gamma)$$

$$\Gamma^\beta{}_{\alpha\gamma} \equiv \langle\mathbf{e}^\beta, \nabla_{\mathbf{e}_\gamma}\mathbf{e}_\alpha\rangle = -\langle\nabla_{\mathbf{e}_\gamma}\mathbf{e}^\beta,\mathbf{e}_\alpha\rangle.$$

Note that I, after Misner *et al.* (1973), am not assigning a different symbol for the "generalized exterior derivative operator." If the connection coefficients vanish, $\Gamma^\beta{}_{\alpha\gamma} = 0$, then $\nabla \rightarrow {}^\flat\nabla$. Finally, the covariant derivative of a covariant vector ${}^\flat\mathbf{v} = v_\alpha\vec{\mathbf{d}}\chi^\alpha \equiv v_\alpha\mathbf{e}^\alpha$ is

$$\nabla_{\mathbf{e}_\beta}{}^\flat\mathbf{v} = \nabla_{\mathbf{e}_\beta}(v_\alpha\mathbf{e}^\alpha) = [\partial v_\alpha/\partial\chi^\beta - \Gamma^\gamma{}_{\alpha\beta}\,v_\gamma]\mathbf{e}^\alpha \equiv v_{\alpha;\beta}\mathbf{e}^\alpha.$$

The "generalized exterior derivative operator" is used throughout the remainder of this section. When the basis vectors are constants, there is no difference between the "generalized" and "ordinary" exterior derivative operators.

For any p-form $\boldsymbol{\omega} = (1/p!)\,\omega_{\alpha_1\ldots\alpha_p}\vec{\mathbf{d}}\chi^{\alpha_1} \wedge \ldots \wedge \vec{\mathbf{d}}\chi^{\alpha_p} = \omega_{|\alpha_1\ldots\alpha_p|}\vec{\mathbf{d}}\chi^{\alpha_1} \wedge \ldots \wedge \vec{\mathbf{d}}\chi^{\alpha_p}$, the components of $\vec{\mathbf{d}}\boldsymbol{\omega}$, $(\vec{\mathbf{d}}\boldsymbol{\omega})_{\beta\alpha_1\ldots\alpha_p}$, can be expressed as (Wald 1984):

$$(\vec{\mathbf{d}}\boldsymbol{\omega})_{\beta\alpha_1\ldots\alpha_p} = (p+1)\,\nabla_{[\beta}\,\omega_{\alpha_1\ldots\alpha_p]},$$

where ∇_β denotes the components of $\nabla_{\mathbf{e}_\beta}$, *e.g.* $\nabla_{\mathbf{e}_\beta}(\omega_\alpha \mathbf{e}^\alpha) = [\partial\omega_\alpha/\partial\chi^\beta - \Gamma^\gamma{}_{\alpha\beta}\,\omega_\gamma]\mathbf{e}^\alpha$ implies that

$$\nabla_\beta\,\omega_\alpha = \partial\omega_\alpha/\partial\chi^\beta - \Gamma^\gamma{}_{\alpha\beta}\,\omega_\gamma \equiv \omega_{\alpha,\beta} - \Gamma^\gamma{}_{\alpha\beta}\,\omega_\gamma \equiv \omega_{\alpha;\beta}.$$

Thus, the $(p+1)$-form $\vec{\mathbf{d}}\boldsymbol{\omega}$ is:

$$\vec{\mathbf{d}}\boldsymbol{\omega} = \frac{1}{(p+1)}(\vec{\mathbf{d}}\boldsymbol{\omega})_{\beta\alpha_1\ldots\alpha_p}\mathbf{e}^\beta \wedge \mathbf{e}^{\alpha_1} \wedge \ldots \wedge \mathbf{e}^{\alpha_p} = (\vec{\mathbf{d}}\boldsymbol{\omega})_{|\beta\alpha_1\ldots\alpha_p|}\mathbf{e}^\beta \wedge \mathbf{e}^{\alpha_1} \wedge \ldots \wedge \mathbf{e}^{\alpha_p},$$

or

$$\vec{\mathbf{d}}\boldsymbol{\omega} = \nabla_{[\beta}\,\omega_{\alpha_1\ldots\alpha_p]}\mathbf{e}^\beta \wedge \mathbf{e}^{\alpha_1} \wedge \ldots \wedge \mathbf{e}^{\alpha_p}.$$

The term $\nabla_{[\beta}\,\omega_{\alpha_1\ldots\alpha_p]}$ contains the information about how the (cotangent) basis vectors change with position. So for a 1-form $\boldsymbol{\omega} = \omega_\alpha\,\vec{\mathbf{d}}\chi^\alpha \equiv \omega_\alpha\,\mathbf{e}^\alpha$:

$$\begin{aligned}\boldsymbol{\omega} = \omega_\alpha\mathbf{e}^\alpha \Rightarrow (\vec{\mathbf{d}}\boldsymbol{\omega})_{\beta\alpha} = 2\,\nabla_{[\beta}\,\omega_{\alpha]} &= (\omega_{\alpha,\beta} - \Gamma^\gamma{}_{\alpha\beta}\,\omega_\gamma) - (\omega_{\beta,\alpha} - \Gamma^\gamma{}_{\beta\alpha}\,\omega_\gamma),\\ &= 2\,\omega_{[\alpha,\beta]} + 2\Gamma^\gamma{}_{[\beta\alpha]}\,\omega_\gamma,\\ &= 2\,\omega_{[\alpha;\beta]};\end{aligned}$$

and

$$\begin{aligned}\vec{\mathbf{d}}\boldsymbol{\omega} = \omega_{[\alpha;\beta]}\mathbf{e}^\beta \wedge \mathbf{e}^\alpha = \omega_{\alpha;\beta}\mathbf{e}^\beta \wedge \mathbf{e}^\alpha,\\ = [(\omega_{z,y} - \omega_{y,z}) + (\Gamma^\gamma{}_{yz} - \Gamma^\gamma{}_{zy})\,\omega_\gamma]\mathbf{e}^y \wedge \mathbf{e}^z\\ + [(\omega_{x,z} - \omega_{z,x}) + (\Gamma^\gamma{}_{zx} - \Gamma^\gamma{}_{xz})\,\omega_\gamma]\mathbf{e}^z \wedge \mathbf{e}^x\\ + [(\omega_{y,x} - \omega_{x,y}) + (\Gamma^\gamma{}_{xy} - \Gamma^\gamma{}_{yx})\,\omega_\gamma]\mathbf{e}^x \wedge \mathbf{e}^y.\end{aligned}$$

When the connection ∇ is *symmetric*, $\Gamma^{\beta}{}_{\alpha\gamma} = \Gamma^{\beta}{}_{\gamma\alpha}$, and, for example, $\vec{\mathbf{d}}\boldsymbol{\omega} = \omega_{[\alpha,\beta]}\mathbf{e}^{\beta} \wedge \mathbf{e}^{\alpha}$. In this case

$$\vec{\mathbf{d}}\vec{\mathbf{d}}\wp \equiv \vec{\mathbf{d}}^2\wp = \mathbf{0}, \text{ (for \textit{symmetric connection})}$$

and the geometry is *Riemannian*, and *free of "torsion"*: closed, "infinitesimal parallelograms" can be defined by parallel transporting vectors along each other (Misner *et al.* 1973, Wasserman 1992). So for a p-form

$$(\vec{\mathbf{d}}^2\boldsymbol{\omega})_{\beta\gamma\alpha_1\ldots\alpha_p} = (p + 2)(p + 1)\partial_{[\beta}\partial_{\gamma}\ \omega_{\alpha_1\ldots\alpha_p]} = 0$$

is the Poincaré Lemma (Wald 1984). Since $\vec{\mathbf{d}}\wp \equiv \mathbf{e}_{\alpha} \otimes \mathbf{e}^{\alpha}$ (Nash and Sen 1983, Flanders 1989),

$$\begin{aligned}
\vec{\mathbf{d}}^2\wp &= \vec{\mathbf{d}}(\mathbf{e}_{\alpha} \otimes \mathbf{e}^{\alpha}), \\
&= \vec{\mathbf{d}}\mathbf{e}_{\alpha} \wedge \mathbf{e}^{\alpha} + \mathbf{e}_{\alpha} \otimes \vec{\mathbf{d}}\mathbf{e}^{\alpha}, \\
&= \mathbf{e}_{\beta}\Gamma^{\beta}{}_{\alpha\gamma} \otimes \mathbf{e}^{\gamma} \wedge \mathbf{e}^{\alpha} + \mathbf{e}_{\alpha} \otimes \vec{\mathbf{d}}\mathbf{e}^{\alpha}, \ (\vec{\mathbf{d}}\mathbf{e}_{\alpha} = \mathbf{e}_{\beta}\Gamma^{\beta}{}_{\alpha\gamma} \otimes \mathbf{e}^{\gamma}) \\
&= \mathbf{e}_{\beta} \otimes (\Gamma^{\beta}{}_{\alpha\gamma}\mathbf{e}^{\gamma} \wedge \mathbf{e}^{\alpha} + \vec{\mathbf{d}}\mathbf{e}^{\beta}), \ (\alpha \rightarrow \beta \text{ in second term}) \\
&= \mathbf{e}_{\beta} \otimes [\tfrac{1}{2}(\Gamma^{\beta}{}_{\alpha\gamma} - \Gamma^{\beta}{}_{\gamma\alpha})\mathbf{e}^{\gamma} \wedge \mathbf{e}^{\alpha} + \vec{\mathbf{d}}\mathbf{e}^{\beta}], \\
&\qquad (\Gamma^{\beta}{}_{\alpha\gamma}\mathbf{e}^{\gamma} \wedge \mathbf{e}^{\alpha} = -\Gamma^{\beta}{}_{\alpha\gamma}\mathbf{e}^{\alpha} \wedge \mathbf{e}^{\gamma} = -\Gamma^{\beta}{}_{\gamma\alpha}\mathbf{e}^{\gamma} \wedge \mathbf{e}^{\alpha}) \\
&= \mathbf{e}_{\beta} \otimes [\tfrac{1}{2}(\Gamma^{\beta}{}_{\alpha\gamma} - \Gamma^{\beta}{}_{\gamma\alpha})\mathbf{e}^{\gamma} \wedge \mathbf{e}^{\alpha} - \Gamma^{\beta}{}_{\alpha\gamma}\mathbf{e}^{\alpha} \otimes \mathbf{e}^{\gamma}], \\
&\qquad (\vec{\mathbf{d}}\mathbf{e}^{\beta} = -\mathbf{e}^{\alpha}\Gamma^{\beta}{}_{\alpha\gamma} \otimes \mathbf{e}^{\gamma}) \\
&\equiv \mathbf{T}(\ldots,\ldots,\ldots).
\end{aligned}$$

The *torsion tensor*, $\mathbf{T}(\ldots,\ldots,\ldots)$, is often defined with the relationship

$$\mathbf{T}(\ldots,\mathbf{u},\mathbf{v}) \equiv \langle \vec{\mathbf{d}}^{2}\wp,\mathbf{u} \wedge \mathbf{v} \rangle \equiv \nabla_{\mathbf{u}}\mathbf{v} - \nabla_{\mathbf{v}}\mathbf{u} - [\mathbf{u},\mathbf{v}],$$ [67]

where

$$[\mathbf{u},\mathbf{v}] \equiv \mathbf{u}[\mathbf{v}] - \mathbf{v}[\mathbf{u}] \equiv \partial_{\mathbf{u}}\partial_{\mathbf{v}} - \partial_{\mathbf{v}}\partial_{\mathbf{u}} = (u^{\beta} v^{\alpha}{}_{,\beta} - v^{\beta} u^{\alpha}{}_{,\beta})\vec{\partial}_{\alpha}$$

is again the commutator of **u** and **v** (Misner *et al.* 1973, Dodson and Poston 1979, Choquet-Bruhat 1982, Marsden and Hughes 1983, Nash and Sen 1983, Burke 1985, Wasserman 1992). The terms $\nabla_{\mathbf{u}}\mathbf{v} - \nabla_{\mathbf{v}}\mathbf{u}$ can also be expressed as a commutator, $[\nabla_{\mathbf{u}}, \nabla_{\mathbf{v}}] \equiv \nabla_{\mathbf{u}}\mathbf{v} - \nabla_{\mathbf{v}}\mathbf{u}$. Thus,

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[67]: Note that the torsion tensor defined as

$$\mathbf{T}(\ldots,\mathbf{u},\mathbf{v}) \equiv \langle \vec{\mathbf{d}}^{2}\wp,\mathbf{u} \wedge \mathbf{v} \rangle$$

resembles a flux: The magnetic flux Φ through an area $\mathbf{u} \wedge \mathbf{v}$ is $\Phi = \langle {}^{B}\mathbf{F},\mathbf{u} \wedge \mathbf{v} \rangle$, where ${}^{B}\mathbf{F} = B_x \vec{\mathbf{d}}y \wedge \vec{\mathbf{d}}z + B_y \vec{\mathbf{d}}z \wedge \vec{\mathbf{d}}x + B_z \vec{\mathbf{d}}x \wedge \vec{\mathbf{d}}y$ is the non-temporal part of the Faraday tensor **F**, **B** denoting the magnetic field intensity (Misner *et al.* 1973).

$$
\begin{aligned}
\mathbf{T}(\ldots,\mathbf{u},\mathbf{v}) &\equiv [\nabla_{\mathbf{u}}, \nabla_{\mathbf{v}}] - [\mathbf{u},\mathbf{v}],\\
&\equiv \nabla_{\mathbf{u}}\mathbf{v} - \nabla_{\mathbf{v}}\mathbf{u} - [\mathbf{u},\mathbf{v}],\\
&= u^{\beta}\nabla_{\mathbf{e}_\beta}(v^{\alpha}\mathbf{e}_{\alpha}) - v^{\gamma}\nabla_{\mathbf{e}_\gamma}(u^{\alpha}\mathbf{e}_{\alpha}) - [\mathbf{u},\mathbf{v}],\ (\nabla_{f\mathbf{e}_\alpha} = f\nabla_{\mathbf{e}_\alpha})\\
&= u^{\beta}(\partial v^{\alpha}/\partial\chi^{\beta} + v^{\gamma}\Gamma^{\alpha}{}_{\gamma\beta})\mathbf{e}_{\alpha} - v^{\gamma}(\partial u^{\alpha}/\partial\chi^{\gamma} + u^{\beta}\Gamma^{\alpha}{}_{\beta\gamma})\mathbf{e}_{\alpha} - [\mathbf{u},\mathbf{v}],\\
&= [u^{\beta}(\partial v^{\alpha}/\partial\chi^{\beta}) - v^{\gamma}(\partial u^{\alpha}/\partial\chi^{\gamma})]\mathbf{e}_{\alpha} + [u^{\beta}v^{\gamma}\Gamma^{\alpha}{}_{\gamma\beta} - v^{\gamma}u^{\beta}\Gamma^{\alpha}{}_{\beta\gamma}]\mathbf{e}_{\alpha} - [\mathbf{u},\mathbf{v}],\\
&= [u^{\beta}v^{\gamma}\Gamma^{\alpha}{}_{\gamma\beta} - v^{\gamma}u^{\beta}\Gamma^{\alpha}{}_{\beta\gamma}]\mathbf{e}_{\alpha},\ ([\mathbf{u},\mathbf{v}] = (u^{\beta}v^{\alpha}{}_{,\beta} - v^{\beta}u^{\alpha}{}_{,\beta})\mathbf{e}_{\alpha})\\
&= u^{\beta}v^{\gamma}(\Gamma^{\alpha}{}_{\gamma\beta} - \Gamma^{\alpha}{}_{\beta\gamma})\mathbf{e}_{\alpha},\\
&\equiv 2u^{\beta}v^{\gamma}\Gamma^{\alpha}{}_{[\gamma\beta]}\mathbf{e}_{\alpha},\ (\Gamma^{\alpha}{}_{[\gamma\beta]} = \tfrac{1}{2}[\Gamma^{\alpha}{}_{\gamma\beta} - \Gamma^{\alpha}{}_{\beta\gamma}])\\
&\equiv u^{\beta}v^{\gamma}T^{\alpha}{}_{\gamma\beta}\mathbf{e}_{\alpha}.
\end{aligned}
$$

The components of the torsion tensor $\mathbf{T}$ are

$$
\begin{aligned}
\mathbf{T}(\ldots,\mathbf{e}_\beta,\mathbf{e}_\gamma) \equiv [\nabla_{\mathbf{e}_\beta}, \nabla_{\mathbf{e}_\gamma}] - [\mathbf{e}_\beta,\mathbf{e}_\gamma] &= \nabla_{\mathbf{e}_\beta}\mathbf{e}_\gamma - \nabla_{\mathbf{e}_\gamma}\mathbf{e}_\beta - [\mathbf{e}_\beta,\mathbf{e}_\gamma],\\
&= (\Gamma^{\alpha}{}_{\gamma\beta} - \Gamma^{\alpha}{}_{\beta\gamma})\mathbf{e}_{\alpha} - \Omega_{\beta\gamma}{}^{\alpha}\mathbf{e}_{\alpha},\ \text{(for anholonomic basis)}\\
&\equiv 2\Gamma^{\alpha}{}_{[\gamma\beta]}\mathbf{e}_{\alpha},\ (\text{for holonomic basis, } \Omega_{\beta\gamma}{}^{\alpha} \equiv 0)\\
&\equiv T^{\alpha}{}_{\gamma\beta}\mathbf{e}_{\alpha};
\end{aligned}
$$

$$
T^{\alpha}{}_{\gamma\beta} \equiv \mathbf{T}(\mathbf{e}^{\alpha},\mathbf{e}_\beta,\mathbf{e}_\gamma) \equiv \langle\mathbf{e}^{\alpha},\mathbf{T}(\ldots,\mathbf{e}_\beta,\mathbf{e}_\gamma)\rangle = \Gamma^{\alpha}{}_{\gamma\beta} - \Gamma^{\alpha}{}_{\beta\gamma} - \Omega_{\beta\gamma}{}^{\alpha} \equiv 2\Gamma^{\alpha}{}_{[\gamma\beta]} - \Omega_{\beta\gamma}{}^{\alpha};
$$

$$
\begin{aligned}
\mathbf{T}(\ldots,\ldots,\ldots) &\equiv T^{\alpha}{}_{\gamma\beta}\mathbf{e}_{\alpha}\otimes\mathbf{e}^{\beta}\otimes\mathbf{e}^{\gamma},\\
&\equiv (2\Gamma^{\alpha}{}_{[\gamma\beta]} - \Omega_{\beta\gamma}{}^{\alpha})\mathbf{e}_{\alpha}\otimes\mathbf{e}^{\beta}\otimes\mathbf{e}^{\gamma},\ \text{(for anholonomic basis)}\\
&\equiv 2\Gamma^{\alpha}{}_{[\gamma\beta]}\mathbf{e}_{\alpha}\otimes\mathbf{e}^{\beta}\otimes\mathbf{e}^{\gamma}.\ \text{(For holonomic basis.)}
\end{aligned}
$$

The torsion tensor measures the antisymmetric part of the connection $\Gamma^{\alpha}{}_{\beta\gamma}$ (Nash and Sen 1983). It is proportional to the dislocation density when the manifold describes a crystal lattice, *e.g.* Kröner (1959/60,1981,1992), Bilby (1960), Kondo (1964), Nabarro (1967), Marcinkowski (1977,1979), Gairola (1979), de Wit (1981), Rivier (1983), and Venkataraman and Sahoo (1986). It is also applicable to the case where the lattice is non-crystalline, or amorphous (Rivier 1983, Dereli and Vercin 1987, Kröner and Lagoudas 1992).

The torsion tensor is antisymmetric. Thus, for holonomic coordinates:

$$
\begin{aligned}
\mathbf{T}(\ldots,\ldots,\ldots) &\equiv T^{\alpha}{}_{\beta\gamma}\,\mathbf{e}_{\alpha}\otimes\mathbf{e}^{\gamma}\otimes\mathbf{e}^{\beta} \equiv 2\Gamma^{\alpha}{}_{[\beta\gamma]}\mathbf{e}_{\alpha}\otimes\mathbf{e}^{\gamma}\otimes\mathbf{e}^{\beta} \equiv (\Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\gamma\beta})\mathbf{e}_{\alpha}\otimes\mathbf{e}^{\gamma}\otimes\mathbf{e}^{\beta},\\
&= \tfrac{1}{2}[T^{\alpha}{}_{\beta\gamma}\,\mathbf{e}_{\alpha}\otimes\mathbf{e}^{\gamma}\otimes\mathbf{e}^{\beta} - T^{\alpha}{}_{\gamma\beta}\,\mathbf{e}_{\alpha}\otimes\mathbf{e}^{\gamma}\otimes\mathbf{e}^{\beta}],\ (T^{\alpha}{}_{\beta\gamma} = -T^{\alpha}{}_{\gamma\beta})\\
&= \tfrac{1}{2}[T^{\alpha}{}_{\beta\gamma}\,\mathbf{e}_{\alpha}\otimes\mathbf{e}^{\gamma}\otimes\mathbf{e}^{\beta} - T^{\alpha}{}_{\beta\gamma}\,\mathbf{e}_{\alpha}\otimes\mathbf{e}^{\beta}\otimes\mathbf{e}^{\gamma}],\ \text{(index exchange)}\\
&= \tfrac{1}{2}T^{\alpha}{}_{\beta\gamma}\,\mathbf{e}_{\alpha}\otimes(\mathbf{e}^{\gamma}\otimes\mathbf{e}^{\beta} - \mathbf{e}^{\beta}\otimes\mathbf{e}^{\gamma}),\\
&\equiv \tfrac{1}{2}T^{\alpha}{}_{\beta\gamma}\,\mathbf{e}_{\alpha}\otimes\mathbf{e}^{\gamma}\wedge\mathbf{e}^{\beta},\ (\mathbf{e}^{\gamma}\wedge\mathbf{e}^{\beta} \equiv \mathbf{e}^{\gamma}\otimes\mathbf{e}^{\beta} - \mathbf{e}^{\beta}\otimes\mathbf{e}^{\gamma})\\
&\equiv \Gamma^{\alpha}{}_{[\beta\gamma]}\mathbf{e}_{\alpha}\otimes\mathbf{e}^{\gamma}\wedge\mathbf{e}^{\beta}.\ \ (\Gamma^{\alpha}{}_{[\beta\gamma]} \equiv \tfrac{1}{2}(\Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\gamma\beta}) \equiv \tfrac{1}{2}T^{\alpha}{}_{\beta\gamma}.)
\end{aligned}
$$

So for the 1-form $\boldsymbol{\omega} = \omega_{\alpha}\vec{\mathbf{d}}\chi^{\alpha} \equiv \omega_{\alpha}\mathbf{e}^{\alpha}$, components $2\,\omega_{[\alpha;\beta]} = 2\,\omega_{[\alpha,\beta]} + 2\Gamma^{\gamma}{}_{[\beta\alpha]}\,\omega_{\gamma}$ for its exterior derivative, $\vec{\mathbf{d}}\boldsymbol{\omega} = \omega_{[\alpha;\beta]}\mathbf{e}^{\beta}\wedge\mathbf{e}^{\alpha}$, can be reexpressed as $2\,\omega_{[\alpha;\beta]} \equiv 2\,\omega_{[\alpha,\beta]} + T^{\gamma}{}_{\beta\alpha}\,\omega_{\gamma}$ with the components of the torsion tensor. Thus

$$
\begin{aligned}
\boldsymbol{\omega} \equiv \omega_{\alpha}\mathbf{e}^{\alpha} \Rightarrow \vec{\mathbf{d}}\boldsymbol{\omega} \equiv \omega_{[\alpha;\beta]}\mathbf{e}^{\beta}\wedge\mathbf{e}^{\alpha} &= \omega_{[\alpha,\beta]}\mathbf{e}^{\beta}\wedge\mathbf{e}^{\alpha} + \tfrac{1}{2}\mathbf{T}(\boldsymbol{\omega},\mathbf{e}^{\alpha},\mathbf{e}^{\beta})\mathbf{e}^{\beta}\wedge\mathbf{e}^{\alpha},\\
&\equiv \omega_{[\alpha,\beta]}\mathbf{e}^{\beta}\wedge\mathbf{e}^{\alpha} + \tfrac{1}{2}\,\omega_{\gamma}\,T^{\gamma}{}_{\beta\alpha}\,\mathbf{e}^{\beta}\wedge\mathbf{e}^{\alpha},
\end{aligned}
$$

where

$$\mathbf{T}(\boldsymbol{\omega},\mathbf{e}^{\alpha},\mathbf{e}^{\beta}) = \langle \boldsymbol{\omega},\mathbf{T}(\ldots,\mathbf{e}^{\alpha},\mathbf{e}^{\beta})\rangle = \langle \omega_{\gamma}\,\mathbf{e}^{\gamma}, T^{\delta}{}_{\beta\alpha}\,\mathbf{e}_{\delta}\rangle = \omega_{\gamma}\,T^{\delta}{}_{\beta\alpha}\langle \mathbf{e}^{\gamma},\mathbf{e}_{\delta}\rangle = \omega_{\gamma}\,T^{\delta}{}_{\beta\alpha}\,\delta^{\gamma}{}_{\delta} = \omega_{\gamma}\,T^{\gamma}{}_{\beta\alpha}.$$

Now $\vec{\mathbf{d}}\mathbf{e}_{\alpha} \equiv \mathbf{e}_{\beta} \otimes \Gamma^{\beta}{}_{\alpha\gamma}\,\mathbf{e}^{\gamma}$ and (Misner *et al.* 1973, Flanders 1989)

$$\begin{aligned}
\vec{\mathbf{d}}^{2}\mathbf{e}_{\alpha} &\equiv \vec{\mathbf{d}}\mathbf{e}_{\beta} \wedge \Gamma^{\beta}{}_{\alpha\gamma}\,\mathbf{e}^{\gamma} + \mathbf{e}_{\beta} \otimes \vec{\mathbf{d}}(\Gamma^{\beta}{}_{\alpha\gamma}\,\mathbf{e}^{\gamma}) \\
&\equiv (\mathbf{e}_{\delta} \otimes \Gamma^{\delta}{}_{\beta\epsilon}\,\mathbf{e}^{\epsilon}) \wedge \Gamma^{\beta}{}_{\alpha\gamma}\,\mathbf{e}^{\gamma} + \mathbf{e}_{\beta} \otimes \vec{\mathbf{d}}(\Gamma^{\beta}{}_{\alpha\gamma}\,\mathbf{e}^{\gamma}),\ (\vec{\mathbf{d}}\mathbf{e}_{\beta} \equiv \mathbf{e}_{\delta} \otimes \Gamma^{\delta}{}_{\beta\epsilon}\,\mathbf{e}^{\epsilon}) \\
&= (\mathbf{e}_{\delta} \otimes \Gamma^{\delta}{}_{\beta\epsilon}\,\mathbf{e}^{\epsilon}) \wedge \Gamma^{\beta}{}_{\alpha\gamma}\,\mathbf{e}^{\gamma} + \mathbf{e}_{\delta} \otimes \vec{\mathbf{d}}(\Gamma^{\delta}{}_{\alpha\gamma}\,\mathbf{e}^{\gamma}),\ (\beta \rightarrow \delta \text{ in second term}) \\
&= \mathbf{e}_{\delta} \otimes [\Gamma^{\delta}{}_{\beta\epsilon}\,\mathbf{e}^{\epsilon} \wedge \Gamma^{\beta}{}_{\alpha\gamma}\,\mathbf{e}^{\gamma} + \vec{\mathbf{d}}(\Gamma^{\delta}{}_{\alpha\gamma}\,\mathbf{e}^{\gamma})], \\
&\equiv \mathbf{e}_{\delta} \otimes \Re^{\delta}{}_{\alpha},
\end{aligned}$$

where

$$\Re^{\delta}{}_{\alpha} \equiv \Gamma^{\delta}{}_{\beta\epsilon}\,\Gamma^{\beta}{}_{\alpha\gamma}\,\mathbf{e}^{\epsilon} \wedge \mathbf{e}^{\gamma} + \vec{\mathbf{d}}(\Gamma^{\delta}{}_{\alpha\gamma}\,\mathbf{e}^{\gamma})$$

are the *curvature 2-forms*. Let

$$\Re \equiv \mathbf{e}_{\delta} \otimes \mathbf{e}^{\alpha}\,\Re^{\delta}{}_{\alpha}$$

denote the *curvature 2-forms* (Misner *et al.* 1973). Then for any vector $\mathbf{w} = w^{\alpha}\,\mathbf{e}_{\alpha}$,

$$\vec{\mathbf{d}}^{2}\mathbf{w} \equiv \Re\mathbf{w}$$

is a function with two slots for other vectors, say $\mathbf{u}$ and $\mathbf{v}$:

$$\vec{\mathbf{d}}^{2}\mathbf{w}(\mathbf{u},\mathbf{v}) \equiv \Re(\mathbf{u},\mathbf{v})\mathbf{w} \equiv \langle \vec{\mathbf{d}}^{2}\mathbf{w},\mathbf{u} \wedge \mathbf{v}\rangle = \nabla_{\mathbf{u}}\nabla_{\mathbf{v}}\mathbf{w} - \nabla_{\mathbf{v}}\nabla_{\mathbf{u}}\mathbf{w} - \nabla_{[\mathbf{u},\mathbf{v}]}\mathbf{w},$$

where

$$\langle \mathfrak{R}, \mathbf{u} \wedge \mathbf{v} \rangle \equiv \mathfrak{R}(\mathbf{u},\mathbf{v})$$

and

$$\mathbf{e}_\delta \otimes \mathbf{e}^\alpha \langle \mathfrak{R}^\delta{}_\alpha, \mathbf{u} \wedge \mathbf{v} \rangle \equiv \mathbf{e}_\delta \otimes \mathbf{e}^\alpha R^\delta{}_{\alpha\beta\gamma} u^\beta v^\gamma \Rightarrow \mathfrak{R}^\delta{}_\alpha = R^\delta{}_{\alpha|\beta\gamma|} \mathbf{e}^\beta \wedge \mathbf{e}^\gamma = \tfrac{1}{2} R^\delta{}_{\alpha\beta\gamma} \mathbf{e}^\beta \wedge \mathbf{e}^\gamma .$$

Here, $\mathbf{R}$ is the *Riemann curvature tensor*, its components are

$$\begin{aligned}
R^\alpha{}_{\beta\gamma\delta} \equiv \mathbf{R}(\mathbf{e}^\alpha,\mathbf{e}_\beta,\mathbf{e}_\gamma,\mathbf{e}_\delta) &\equiv \langle \mathbf{e}^\alpha, \mathfrak{R}(\mathbf{e}_\gamma,\mathbf{e}_\delta)\mathbf{e}_\beta \rangle, \\
&= \langle \mathbf{e}^\alpha, (\nabla_{\mathbf{e}_\gamma} \nabla_{\mathbf{e}_\delta} - \nabla_{\mathbf{e}_\delta} \nabla_{\mathbf{e}_\gamma})\mathbf{e}_\beta \rangle, \\
&= \langle \mathbf{e}^\alpha, [\nabla_{\mathbf{e}_\gamma} \{ \nabla_{\mathbf{e}_\delta}(\mathbf{e}_\beta)\} - \nabla_{\mathbf{e}_\delta} \{ \nabla_{\mathbf{e}_\gamma}(\mathbf{e}_\beta)\}] \rangle, \\
&= \partial_\gamma \Gamma^\alpha{}_{\beta\delta} + \Gamma^\epsilon{}_{\beta\delta} \Gamma^\alpha{}_{\epsilon\gamma} - \partial_\delta \Gamma^\alpha{}_{\beta\gamma} - \Gamma^\epsilon{}_{\beta\gamma} \Gamma^\alpha{}_{\epsilon\delta} - \Gamma^\alpha{}_{\beta\epsilon} \Omega_{\gamma\delta}{}^\epsilon, \text{ (in general)} \\
&= \partial_\gamma \Gamma^\alpha{}_{\beta\delta} + \Gamma^\epsilon{}_{\beta\delta} \Gamma^\alpha{}_{\epsilon\gamma} - \partial_\delta \Gamma^\alpha{}_{\beta\gamma} - \Gamma^\epsilon{}_{\beta\gamma} \Gamma^\alpha{}_{\epsilon\delta}. \text{ (For holonomic basis.)}
\end{aligned}$$

Therefore $\mathbf{R} = R^\alpha{}_{\beta\gamma\delta} \mathbf{e}_\alpha \otimes \mathbf{e}^\beta \otimes \mathbf{e}^\gamma \otimes \mathbf{e}^\delta = \mathbf{R}(\ldots,\ldots,\ldots,\ldots)$. The last two indices are antisymmetric $R^\alpha{}_{\beta\gamma\delta} \equiv R^\alpha{}_{\beta[\gamma\delta]}$: $\mathbf{R} = \tfrac{1}{2} R^\alpha{}_{\beta\gamma\delta} \mathbf{e}_\alpha \otimes \mathbf{e}^\beta \otimes \mathbf{e}^\gamma \wedge \mathbf{e}^\delta$. For a manifold free of torsion, $R^\alpha{}_{[\beta\gamma\delta]} = 0$.

The curvature tensor measures the failure of covariant differentiation to commute: but $\mathbf{R}(\ldots,\mathbf{w},\mathbf{u},\mathbf{v}) = \nabla_{\mathbf{u}} \nabla_{\mathbf{v}} \mathbf{w} - \nabla_{\mathbf{v}} \nabla_{\mathbf{u}} \mathbf{w} - \nabla_{[\mathbf{u},\mathbf{v}]} \mathbf{w}$ (Misner *et al.* 1973) also implies that (Schouten 1954, Nash and Sen 1983)

$$2 \nabla_{[\gamma} \nabla_{\delta]} w^\alpha \equiv w^\alpha{}_{;\delta\gamma} - w^\alpha{}_{;\gamma\delta} = R^\alpha{}_{\beta\gamma\delta} w^\beta - T^\beta{}_{\delta\gamma} w^\alpha{}_{;\beta},$$

where $\nabla_{[\gamma} \nabla_{\delta]} w^\alpha \equiv \tfrac{1}{2}(\nabla_\gamma \nabla_\delta w^\alpha - \nabla_\delta \nabla_\gamma w^\alpha)$. In general therefore, the failure of covariant differentiation to commute is associated with both "curvature" and "torsion." Here (Misner *et al.* 1973)

$$w^{\alpha}{}_{;\gamma\delta} \equiv \nabla_{\delta}\nabla_{\gamma} w^{\alpha} \equiv \nabla\nabla\mathbf{w}(\mathbf{e}^{\alpha},\mathbf{e}_{\gamma},\mathbf{e}_{\delta}) \equiv \langle \mathbf{e}^{\alpha}, \nabla\nabla\mathbf{w}(\ldots,\mathbf{e}_{\gamma},\mathbf{e}_{\delta})\rangle;$$

and

$$\begin{aligned}\langle \mathbf{e}^{\alpha}, \nabla_{\mathbf{e}_{\delta}} \nabla_{\mathbf{e}_{\gamma}} \mathbf{w}\rangle &= \langle \mathbf{e}^{\alpha}, \nabla_{\mathbf{e}_{\delta}}(\mathbf{e}_{\gamma} \bullet \nabla\mathbf{w})\rangle, \\ &= \langle \mathbf{e}^{\alpha},(\nabla_{\mathbf{e}_{\delta}}\mathbf{e}_{\gamma}) \bullet \nabla\mathbf{w} + \mathbf{e}_{\gamma} \bullet (\nabla_{\mathbf{e}_{\delta}}\nabla\mathbf{w})\rangle, \\ &= \langle \mathbf{e}^{\alpha},\Gamma^{\beta}{}_{\gamma\delta}\,\mathbf{e}_{\beta} \bullet \nabla\mathbf{w} + \nabla\nabla\mathbf{w}(\ldots,\mathbf{e}_{\gamma},\mathbf{e}_{\delta})\rangle, \\ &= w^{\alpha}{}_{;\beta}\,\Gamma^{\beta}{}_{\gamma\delta} + w^{\alpha}{}_{;\gamma\delta}.\end{aligned}$$

Since $\mathbf{T}(\ldots,\mathbf{e}_{\gamma},\mathbf{e}_{\delta}) \equiv [\nabla_{\mathbf{e}_{\gamma}}, \nabla_{\mathbf{e}_{\delta}}] - [\mathbf{e}_{\gamma},\mathbf{e}_{\delta}]$ (or $T^{\beta}{}_{\delta\gamma} = (\Gamma^{\beta}{}_{\delta\gamma} - \Gamma^{\beta}{}_{\gamma\delta}) - [\mathbf{e}_{\gamma},\mathbf{e}_{\delta}]\bullet\mathbf{e}^{\beta}$, where $\mathbf{T}(\ldots,\mathbf{e}_{\gamma},\mathbf{e}_{\delta}) = T^{\beta}{}_{\delta\gamma}\,\mathbf{e}_{\beta}$):

$$\begin{aligned}\nabla_{\gamma}\nabla_{\delta} w^{\alpha} - \nabla_{\delta}\nabla_{\gamma} w^{\alpha} \equiv w^{\alpha}{}_{;\delta\gamma} - w^{\alpha}{}_{;\gamma\delta} &\equiv \langle \mathbf{e}^{\alpha},[\nabla_{\mathbf{e}_{\gamma}}, \nabla_{\mathbf{e}_{\delta}}]\mathbf{w}\rangle - w^{\alpha}{}_{;\beta}(\Gamma^{\beta}{}_{\delta\gamma} - \Gamma^{\beta}{}_{\gamma\delta}), \\ &= \langle \mathbf{e}^{\alpha},[\nabla_{\mathbf{e}_{\gamma}}, \nabla_{\mathbf{e}_{\delta}}]\mathbf{w}\rangle - \langle \mathbf{e}^{\alpha}, \nabla_{[\nabla_{\mathbf{e}_{\gamma}},\nabla_{\mathbf{e}_{\delta}}]}\mathbf{w}\rangle, \\ &= \langle \mathbf{e}^{\alpha},[\nabla_{\mathbf{e}_{\gamma}}, \nabla_{\mathbf{e}_{\delta}}]\mathbf{w}\rangle - \langle \mathbf{e}^{\alpha},(\nabla_{\mathbf{T}(\ldots,\mathbf{e}_{\gamma},\mathbf{e}_{\delta})} + \nabla_{[\mathbf{e}_{\gamma},\mathbf{e}_{\delta}]})\mathbf{w}\rangle, \\ &= \langle \mathbf{e}^{\alpha},([\nabla_{\mathbf{e}_{\gamma}}, \nabla_{\mathbf{e}_{\delta}}] - \nabla_{[\mathbf{e}_{\gamma},\mathbf{e}_{\delta}]})\mathbf{w}\rangle - \langle \mathbf{e}^{\alpha},T^{\beta}{}_{\delta\gamma}\nabla_{\mathbf{e}_{\beta}}\mathbf{w}\rangle, \\ &\equiv \langle \mathbf{e}^{\alpha},\Re(\mathbf{e}_{\gamma},\mathbf{e}_{\delta})\mathbf{w}\rangle - \langle \mathbf{e}^{\alpha},T^{\beta}{}_{\delta\gamma}\,w^{\epsilon}{}_{;\beta}\,\mathbf{e}_{\epsilon}\rangle, \\ &\equiv R^{\alpha}{}_{\beta\gamma\delta}\,w^{\beta} - w^{\alpha}{}_{;\beta}\,T^{\beta}{}_{\delta\gamma}, \\ &\equiv \mathbf{R}(\mathbf{e}^{\alpha},\mathbf{w},\mathbf{e}_{\gamma},\mathbf{e}_{\delta}) - \mathbf{T}(w^{\alpha}{}_{;\beta}\,\mathbf{e}_{\beta},\mathbf{e}_{\gamma},\mathbf{e}_{\delta}).\end{aligned}$$

Or for vectors **u** and **v** instead of tangent basis vectors $\mathbf{e}_{\gamma}$ and $\mathbf{e}_{\delta}$,

$$\{\nabla_{\mathbf{u}}\nabla_{\mathbf{v}} - \nabla_{\mathbf{v}}\nabla_{\mathbf{u}}\}\mathbf{w} = \mathbf{R}(\ldots,\mathbf{w},\mathbf{u},\mathbf{v}) - \mathbf{T}(\nabla\mathbf{w},\mathbf{u},\mathbf{v}).$$

When $\mathbf{R} = \mathbf{0}$ and $\mathbf{T} = \mathbf{0}$ there is no difference between the "ordinary" exterior derivative operation and the "generalized" one.

The *metric tensor*, $\mathbf{G}$, for the χ^α coordinate system is $\mathbf{G} \equiv G_{\alpha\beta}\,\vec{\mathbf{d}}\chi^\alpha \otimes \vec{\mathbf{d}}\chi^\beta$. It is a function for two vectors, *e.g.* $G_{\alpha\beta} \equiv \mathbf{G}(\vec{\partial}_\alpha,\vec{\partial}_\beta)$. It is symmetric, $G_{\alpha\beta} = G_{\beta\alpha}$, and $\nabla\mathbf{G} = \mathbf{0}$ (Schouten 1954). In general $\alpha \neq \beta$; the χ^α coordinates are then *non-orthogonal*. If $\mathbf{G}$ has only n non-zero terms, $G_{11}, G_{22}, \ldots, G_{nn}$, then the coordinates are *orthogonal*. Contravariant components of $\mathbf{G}$, $G^{\alpha\beta}$, are components of the matrix $[\mathbf{G}]^{\alpha\beta} = [\mathbf{G}]_{\alpha\beta}^{-1}$, $G^{\alpha\beta}G_{\beta\gamma} = \delta^\alpha{}_\gamma$. The determinant of $[\mathbf{G}]_{\alpha\beta}$, $|\mathbf{G}|$, is denoted by G. The determinant of $[\mathbf{G}]^{\alpha\beta} = [\mathbf{G}]_{\alpha\beta}^{-1}$ is $1/G$.

Sharp, '#', and *flat*, '$\flat$', operations, or index rasing and lowering, are $\#\vec{\mathbf{d}}\chi^\alpha = \vec{\partial}_\alpha$ and ${}^\flat\vec{\partial}_\alpha = \vec{\mathbf{d}}\chi^\alpha$: ${}^\flat v^\alpha \equiv v_\alpha = G_{\alpha\beta}v^\beta$ and $\#v_\alpha \equiv v^\alpha = G^{\alpha\beta}v_\beta$, (Abraham *et al.* 1988). Then for any 1-form $\boldsymbol{\omega} = \omega_\alpha\vec{\mathbf{d}}\chi^\alpha$ the corresponding vector is $\#\boldsymbol{\omega} \equiv (\#\omega_\alpha)(\#\vec{\mathbf{d}}\chi^\alpha) \equiv G^{\alpha\beta}\omega_\beta\vec{\partial}_\alpha$ and for any vector $\mathbf{v} = v^\alpha\vec{\partial}_\alpha$ the corresponding 1-form is ${}^\flat\mathbf{v} \equiv ({}^\flat v^\alpha)({}^\flat\vec{\partial}_\alpha) \equiv G_{\alpha\beta}v^\beta\vec{\mathbf{d}}\chi^\alpha \equiv v_\alpha\vec{\mathbf{d}}\chi^\alpha$.

The Levi-Cevita tensor, or *natural volume element* $\boldsymbol{\epsilon}$ is (Wald 1984)

$$*1 \equiv \boldsymbol{\epsilon} \equiv G^{1/2}\,\vec{\mathbf{d}}\chi^1 \wedge \vec{\mathbf{d}}\chi^2 \wedge \ldots \wedge \vec{\mathbf{d}}\chi^n \equiv \boldsymbol{\epsilon}_{\alpha_1\alpha_2\ldots\alpha_n}\vec{\mathbf{d}}\chi^{\alpha_1} \wedge \vec{\mathbf{d}}\chi^{\alpha_2} \wedge \ldots \wedge \vec{\mathbf{d}}\chi^{\alpha_n},$$

where the components of $\boldsymbol{\epsilon}$ are $\boldsymbol{\epsilon}_{\alpha_1\alpha_2\ldots\alpha_n} \equiv G^{1/2}e_{\alpha_1\alpha_2\ldots\alpha_n}$. If $\mathbf{v}$ is a vector field on the manifold, then the Lie derivative of $\boldsymbol{\epsilon}$ along $\mathbf{v}$ is equal to the divergence of $\mathbf{v}$ times $\boldsymbol{\epsilon}$ (Marsden and Hughes 1983): $\pounds_\mathbf{v}\boldsymbol{\epsilon} = (\nabla\!\bullet\mathbf{v})\boldsymbol{\epsilon}$. The covariant derivative of $\boldsymbol{\epsilon}$ vanishes: $\nabla\boldsymbol{\epsilon} = \mathbf{0}$ (Wald 1984). If the volume of the manifold is V and the area of the boundary of the manifold, ∂V, is A, then the *divergence theorem* is

$$\int_V \nabla\!\bullet\mathbf{v}\;\mathrm{d}V = \int_{\partial V}\mathbf{v}\bullet\mathbf{n}\;\mathrm{d}A,$$

$\mathbf{n}$ denoting the unit outward normal vector to ∂V.

Components of $\mathbf{G}$ and $\boldsymbol{\epsilon}$ will not generally be ones and zeros, but the latter obey the relationships

$$\boldsymbol{\epsilon}^{\alpha_1\ldots\alpha_n}\boldsymbol{\epsilon}_{\alpha_1\ldots\alpha_n} = e^{\alpha_1\ldots\alpha_n}e_{\alpha_1\ldots\alpha_n} = n!, \text{ and } \boldsymbol{\epsilon}^{\alpha_1\ldots\alpha_n}\boldsymbol{\epsilon}_{\beta_1\ldots\beta_n} = e^{\alpha_1\ldots\alpha_n}e_{\beta_1\ldots\beta_n} = n!\delta^{[\alpha_1}{}_{\beta_1}\ldots\delta^{\alpha_n]}{}_{\beta_n},$$

where $\boldsymbol{\epsilon}^{\alpha_1\ldots\alpha_n} \equiv G^{-1/2}e^{\alpha_1\ldots\alpha_n}$ are the contravariant components of $\boldsymbol{\epsilon}$. For example, in three dimensions:

$$\boldsymbol{\epsilon}^{\alpha\beta\gamma}\boldsymbol{\epsilon}_{\alpha\delta\epsilon} = e^{\alpha\beta\gamma}e_{\alpha\delta\epsilon} = \delta^{\beta}{}_{\delta}\delta^{\gamma}{}_{\epsilon} - \delta^{\beta}{}_{\epsilon}\delta^{\gamma}{}_{\delta} \text{ and } \boldsymbol{\epsilon}^{\alpha\beta\gamma}\boldsymbol{\epsilon}_{\alpha\beta\epsilon} = e^{\alpha\beta\gamma}e_{\alpha\beta\epsilon} = 2\delta^{\gamma}{}_{\epsilon}.$$

For any alternating, or completely antisymmetric tensor $\mathbf{A}$, with components $A_{\alpha_1\ldots\alpha_p} = A_{[\alpha_1\ldots\alpha_p]}$,

$$\frac{1}{p!}A_{\alpha_1\ldots\alpha_p}\delta^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_p} = A_{|\alpha_1\ldots\alpha_p|}\delta^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_p}.$$

Here,

$$\langle \vec{\mathbf{d}}\chi^{\alpha_1} \wedge \ldots \wedge \vec{\mathbf{d}}\chi^{\alpha_p}, \vec{\partial}_{\beta_1} \wedge \ldots \wedge \vec{\partial}_{\beta_p} \rangle \equiv \langle \mathbf{e}^{\alpha_1} \wedge \ldots \wedge \mathbf{e}^{\alpha_p}, \mathbf{e}_{\beta_1} \wedge \ldots \wedge \mathbf{e}_{\beta_p} \rangle \equiv \delta^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_p}$$

is the *"generalized Kronecker delta"*, or components of the *alternating tensor*, ***Alt*** (Misner *et al.* 1973):

$$\begin{aligned}(\boldsymbol{Alt})^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_p} = \delta^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_p} &= 1 \text{ if } \alpha_1,\ldots,\alpha_p \text{ is an even permutation of } \beta_1,\ldots,\beta_p,\\ &= -1 \text{ if } \alpha_1,\ldots,\alpha_p \text{ is an odd permutation of } \beta_1,\ldots,\beta_p,\\ &= 0 \text{ if any } \alpha_i \text{ or } \beta_j \text{ are the same, or if the } \alpha_i \text{ and } \beta_j \text{ are}\\ &\qquad \text{different sets of integers.}\end{aligned}$$

The *dual* of p-form $\boldsymbol{\omega}$ is the $(n - p)$-form, $^*\boldsymbol{\omega}$, with components

$$(^*\omega)_{\alpha_1 \ldots \alpha_{n-p}} \equiv \frac{1}{p!} \omega^{\beta_1 \ldots \beta_p} \boldsymbol{\epsilon}_{\beta_1 \ldots \beta_p \alpha_1 \ldots \alpha_{n-p}} \equiv \omega^{|\beta_1 \ldots \beta_p|} \boldsymbol{\epsilon}_{\beta_1 \ldots \beta_p \alpha_1 \ldots \alpha_{n-p}},$$

where $\omega^{\beta_1 \ldots \beta_p} \equiv G^{\beta_1 \ldots \beta_p} \omega_{\beta_1 \ldots \beta_p}$, with $G^{\beta_1 \ldots \beta_p} \equiv G^{\beta_1 \beta_2} \ldots G^{\beta_{p-1} \beta_p}$. Also (Schouten and Kulk 1969),

$$(n - p)!\, \omega^{\beta_1 \ldots \beta_p} = (^*\omega)_{\alpha_1 \ldots \alpha_n} \boldsymbol{\epsilon}^{\alpha_1 \ldots \alpha_n \beta_1 \ldots \beta_p};$$

$^*\boldsymbol{\omega}$ is a *tensor density*.

In three dimensions: The dual of a 0-form (function) f is a 3-form,

$$^*f \equiv f\boldsymbol{\epsilon} \equiv f\, \vec{\mathbf{d}}\chi^1 \wedge \vec{\mathbf{d}}\chi^2 \wedge \vec{\mathbf{d}}\chi^3, \text{ and } \int {}^*f \equiv \int_V f\, \mathrm{d}^3V \equiv \iiint G^{1/2} f\, \mathrm{d}\chi^1 \mathrm{d}\chi^2 \mathrm{d}\chi^3;$$

the dual of a 1-form is a 2-form; the dual of a 2-form is a 1-form; and the dual of a 3-form is a function. For example, if

$$\boldsymbol{\omega} \equiv \omega_{|\alpha\beta|} \vec{\mathbf{d}}\chi^\alpha \wedge \vec{\mathbf{d}}\chi^\beta \equiv \tfrac{1}{2}\, \omega_{\alpha\beta} \vec{\mathbf{d}}\chi^\alpha \wedge \vec{\mathbf{d}}\chi^\beta,$$

then

$$^*\boldsymbol{\omega} \equiv \boldsymbol{\epsilon}_{\alpha|\beta\gamma|} G^{\beta\delta} G^{\gamma\epsilon} \omega_{\delta\epsilon} \vec{\mathbf{d}}\chi^\alpha \equiv \tfrac{1}{2} \boldsymbol{\epsilon}_{\alpha\beta\gamma} G^{\beta\delta} G^{\gamma\epsilon} \omega_{\delta\epsilon} \vec{\mathbf{d}}\chi^\alpha,$$

where

$$^*\omega_\alpha \equiv \tfrac{1}{2} \boldsymbol{\epsilon}_{\alpha\beta\gamma} G^{\beta\delta} G^{\gamma\epsilon} \omega_{\delta\epsilon} \equiv \tfrac{1}{2} \boldsymbol{\epsilon}_{\alpha\beta\gamma}\, \omega^{\beta\gamma}.$$

This type of dual operation utilizes the *Hodge star operator*, *, which acts only on differential forms (Burke 1985). The dual of $\boldsymbol{\omega}$, $^*\boldsymbol{\omega}$, is perpendicular to $\boldsymbol{\omega}$ in the sense that if **u** is a vector lying in the intersection of surfaces of $\boldsymbol{\omega}$, and **v** is a vector lying in the intersection of surfaces of $*\boldsymbol{\omega}$, then **u** and **v** are perpendicular, $\mathbf{u}\bullet\mathbf{v} = \mathbf{0}$ (Misner *et al.* 1973). Also, $\boldsymbol{\omega} \wedge *\boldsymbol{\omega} = \|\boldsymbol{\omega}\|^2\boldsymbol{\epsilon}$, and $**\boldsymbol{\omega} = \boldsymbol{\omega}$.

Since $\mathbf{R} = R^{\alpha}{}_{\beta\gamma\delta}\mathbf{e}_{\alpha} \otimes \mathbf{e}^{\beta} \otimes \mathbf{e}^{\gamma} \otimes \mathbf{e}^{\delta}$, ${}^{\flat}\mathbf{R} = R_{\alpha\beta\gamma\delta}\mathbf{e}^{\alpha} \otimes \mathbf{e}^{\beta} \otimes \mathbf{e}^{\gamma} \otimes \mathbf{e}^{\delta}$, where $R_{\alpha\beta\gamma\delta} \equiv G_{\alpha\epsilon}R^{\epsilon}{}_{\beta\gamma\delta}$ is the completely covariant form of the curvature tensor (Misner *et al.* 1973). This tensor is antisymmetric on both the first and second sets of indices,

$$R_{\alpha\beta\gamma\delta} \equiv R_{[\alpha\beta]\gamma\delta} \equiv R_{\alpha\beta[\gamma\delta]} \equiv R_{[\alpha\beta][\gamma\delta]};$$

also

$$R_{[\alpha\beta\gamma\delta]} = 0, \text{ and } R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}.$$

The dual of ${}^{\flat}\mathbf{R}$ can therefore be performed on both these index sets, as follows for three dimensions:

$$\begin{aligned}
{}^{\flat}\mathbf{R} &\equiv (\tfrac{1}{2}\boldsymbol{\epsilon}_{\epsilon\theta\varkappa}G^{\theta\alpha}G^{\varkappa\beta})(\tfrac{1}{2}\boldsymbol{\epsilon}_{\eta\lambda\mu}G^{\lambda\gamma}G^{\mu\delta})R_{\alpha\beta\gamma\delta}\mathbf{e}^{\epsilon} \wedge \mathbf{e}^{\eta} = \tfrac{1}{4}\boldsymbol{\epsilon}_{\epsilon\theta\varkappa}\boldsymbol{\epsilon}_{\eta\lambda\mu}R^{\theta\varkappa\lambda\mu}\mathbf{e}^{\epsilon} \wedge \mathbf{e}^{\eta},\\
&= \tfrac{1}{4}(\boldsymbol{\epsilon}_{\epsilon\theta\varkappa}G^{\theta\alpha}G^{\varkappa\beta})(\boldsymbol{\epsilon}_{\eta\lambda\mu}G^{\lambda\gamma}G^{\mu\delta})R_{\alpha\beta\gamma\delta}(G^{\epsilon\nu}\mathbf{e}_{\nu}) \wedge (G^{\eta\rho}\mathbf{e}_{\rho}),\ (\mathbf{e}^{\epsilon} \equiv G^{\epsilon\nu}\mathbf{e}_{\nu},\ \mathbf{e}^{\eta} \equiv G^{\eta\rho}\mathbf{e}_{\rho})\\
&= \tfrac{1}{4}(\boldsymbol{\epsilon}_{\epsilon\theta\varkappa}G^{\epsilon\nu}G^{\theta\alpha}G^{\varkappa\beta})(\boldsymbol{\epsilon}_{\eta\lambda\mu}G^{\eta\rho}G^{\lambda\gamma}G^{\mu\delta})R_{\alpha\beta\gamma\delta}\mathbf{e}_{\nu} \wedge \mathbf{e}_{\rho},\\
&= \tfrac{1}{4}\boldsymbol{\epsilon}^{\nu\alpha\beta}\boldsymbol{\epsilon}^{\rho\gamma\delta}R_{\alpha\beta\gamma\delta}\mathbf{e}_{\nu} \wedge \mathbf{e}_{\rho},\ (\boldsymbol{\epsilon}^{\nu\alpha\beta} \equiv \boldsymbol{\epsilon}_{\epsilon\theta\varkappa}G^{\epsilon\nu}G^{\theta\alpha}G^{\varkappa\beta})\\
&= \tfrac{1}{4}\boldsymbol{\epsilon}^{\nu\alpha\beta}\boldsymbol{\epsilon}^{\rho\gamma\delta}R_{\alpha\beta\gamma\delta}\mathbf{e}_{\nu} \otimes \mathbf{e}_{\rho},\ \text{(McConnell 1957)}\\
&\qquad (\boldsymbol{\epsilon}^{\nu\alpha\beta}\boldsymbol{\epsilon}^{\rho\gamma\delta}\mathbf{e}_{\nu} \wedge \mathbf{e}_{\rho} = \boldsymbol{\epsilon}^{\nu\alpha\beta}\boldsymbol{\epsilon}^{\rho\gamma\delta}[\mathbf{e}_{\nu} \otimes \mathbf{e}_{\rho} - \mathbf{e}_{\rho} \otimes \mathbf{e}_{\nu}],\\
&\qquad\qquad = [\boldsymbol{\epsilon}^{\nu\alpha\beta}\boldsymbol{\epsilon}^{\rho\gamma\delta} - \boldsymbol{\epsilon}^{\rho\alpha\beta}\boldsymbol{\epsilon}^{\nu\gamma\delta}]\mathbf{e}_{\nu} \otimes \mathbf{e}_{\rho} = \boldsymbol{\epsilon}^{\nu\alpha\beta}\boldsymbol{\epsilon}^{\rho\gamma\delta}\mathbf{e}_{\nu} \otimes \mathbf{e}_{\rho} - \mathbf{0}.)\\
&\equiv N^{\nu\rho}\mathbf{e}_{\nu} \otimes \mathbf{e}_{\rho}.
\end{aligned}$$

Here, $N^{\nu\rho} \equiv \frac{1}{4}\boldsymbol{\epsilon}^{\nu\alpha\beta}\boldsymbol{\epsilon}^{\rho\gamma\delta}R_{\alpha\beta\gamma\delta}$ are components of the *incompatibility tensor*, $\mathbf{N} \equiv N^{\nu\rho}\mathbf{e}_\nu \otimes \mathbf{e}_\rho$; it is symmetrical, $N^{\nu\rho} = N^{\rho\nu}$ (Nabarro 1967, Kröner 1992):

$$N^{\alpha\beta} = G^{-1}\begin{Vmatrix} R_{yzyz} & R_{zxyz} & R_{xyyz} \\ R_{zxyz} & R_{zxzx} & R_{xyzx} \\ R_{xyyz} & R_{xyzx} & R_{xyxy} \end{Vmatrix}.$$

(The incompatibility tensor can also be written as the 2-form $\tilde{\mathbf{N}} \equiv \frac{1}{4}\boldsymbol{\epsilon}_{\epsilon\theta\chi}\boldsymbol{\epsilon}_{\eta\lambda\mu}R^{\theta\chi\lambda\mu}\mathbf{e}^\epsilon \wedge \mathbf{e}^\eta$.)

I will examine the properties of the incompatibility tensor for two types of manifold: **(1)** torsion free, $\mathbf{T} = \mathbf{0}$, and **(2)** with torsion $\mathbf{T} \neq \mathbf{0}$.

If the manifold is torsion-free, $\mathbf{T} = \mathbf{0}$, then

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{N} = \mathbf{0},$$

is the *"second Bianchi identity"*, which is $\nabla_\alpha N^{\alpha\beta} = 0 = \nabla_\beta N^{\alpha\beta}$ in components (Kröner 1992). The equation $\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{N} = \mathbf{0}$ means that **the "incompatibility" has no source**, and that **the "boundary of a boundary" vanishes** (Misner *et al.* 1973). Equivalently (Misner *et al.* 1973, Wasserman 1992),

<u>First Bianchi Identity</u>: $R^\alpha{}_{[\beta\gamma\delta]} = 0 \Rightarrow N^{\alpha\beta} = N^{\beta\alpha}$ (Kröner 1992);

<u>Second Bianchi Identity</u>: $\vec{\mathbf{d}}\,\boldsymbol{\Re} = \mathbf{0}$, or $\vec{\mathbf{d}}\,\boldsymbol{\Re}^\alpha{}_\beta = \mathbf{0} \Rightarrow R^\alpha{}_{\beta[\gamma\delta;\epsilon]} = 0$.

Misner *et al.* (1973) refer to the second Bianchi identity as the *"full Bianchi identity"*:

***full Bianchi identity* = *second Bianchi identity* = *THE Bianchi identity*.**

Integrability conditions from electromagnetism, a torsion free example, are summarized next. Then the Bianchi identities (integrability conditions) are given for a manifold with torsion.

<u>Outline of Electrodynamics Structure (Misner *et al.* 1973):</u>

<u>Duality (→) and Differentiation (↓)</u>

$\mathbf{A}$ (electromagnetic 4-vector potential)

$\downarrow$

Faraday $\equiv \mathbf{F} \equiv \mathbf{d}\mathbf{A}$ $\quad\longrightarrow\quad$ $^*\mathbf{F}$ (dual field; ***Maxwell***)

$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

$\mathbf{d}\mathbf{F} = \mathbf{d}\mathbf{d}\mathbf{A} = 0$ $\qquad\qquad$ $\mathbf{d}{}^*\mathbf{F} = 4\pi{}^*\mathbf{J}$ ($\mathbf{J}$ is the electromagnetic flux 4-vector)

$\downarrow$

$4\pi\mathbf{d}{}^*\mathbf{J} = \mathbf{d}\mathbf{d}{}^*\mathbf{F} = 0$

$\boxed{\mathbf{d}{}^*\mathbf{J} = 0 \text{ or equivalently } \nabla\cdot\mathbf{J} = 0}$

("automatic" conservation of source)

<u>Electromagnetism with Differential Forms</u>

Maxwell's equations for the *electric* **E** and *magnetic* **B** *fields* are

$$(1)\ \nabla \bullet \mathbf{E} = 4\pi\rho,\ (2)\ \nabla \bullet \mathbf{B} = 0,$$

$$(3)\ \nabla \times \mathbf{E} + \partial\mathbf{B}/\partial t = \mathbf{0} \text{ and, } (4)\ \nabla \times \mathbf{B} - \partial\mathbf{E}/\partial t = 4\pi\mathbf{J};$$

ρ is the *charge density* and **J** is the *current density* (Burke 1985). The *force* on a moving particle with charge q is $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ where $\mathbf{v} \equiv \partial\rho/\partial t$ is the particle velocity.

A force is a 1-form and so, therefore, is the electric field. The *electric field 1-form*, $^\flat\mathbf{E}$, is $^\flat\mathbf{E} = E_a \vec{\mathbf{d}}x^a$, where $E_a = -\partial V/\partial x^a$, V denoting the *potential* for the electric field. The magnetic field 2-form is $\mathbf{B} = B_x\vec{\mathbf{d}}y \wedge \vec{\mathbf{d}}z + B_y\vec{\mathbf{d}}z \wedge \vec{\mathbf{d}}x + B_z\vec{\mathbf{d}}x \wedge \vec{\mathbf{d}}y$. Since

$$\vec{\mathbf{d}}\,{}^\flat\mathbf{E} = (\partial E_a/\partial x^b)\vec{\mathbf{d}}x^b \wedge \vec{\mathbf{d}}x^a = \tfrac{1}{2}(\partial E_b/\partial x^a - \partial E_a/\partial x^b)\vec{\mathbf{d}}x^a \wedge \vec{\mathbf{d}}x^b,$$

$$= (\partial E_z/\partial y - \partial E_y/\partial z)\vec{\mathbf{d}}y \wedge \vec{\mathbf{d}}z + \ldots,$$

the <u>third</u> of Maxwell's equations corresponds to $\partial\mathbf{B}/\partial t = -\vec{\mathbf{d}}\,{}^\flat\mathbf{E}$, where time t is a parameter, not a coordinate. $\vec{\mathbf{d}}\mathbf{B} = (\partial B_x/\partial x + \partial B_y/\partial y + \partial B_z/\partial z)\vec{\mathbf{d}}x \wedge \vec{\mathbf{d}}y \wedge \vec{\mathbf{d}}z = \mathbf{0}$ corresponds to the <u>second</u> of Maxwell's equations.

The dual of $^\flat\mathbf{E}$ is the 2-form $*\,{}^\flat\mathbf{E} = E_x\vec{\mathbf{d}}y \wedge \vec{\mathbf{d}}z + E_y\vec{\mathbf{d}}z \wedge \vec{\mathbf{d}}x + E_z\vec{\mathbf{d}}x \wedge \vec{\mathbf{d}}y$; here $E_a \equiv E^a\epsilon_{abc}$, *e.g.* $E_x \equiv E^x\epsilon_{xyz}$. Thus, $\vec{\mathbf{d}}*{}^\flat\mathbf{E} = (\partial E_x/\partial x + \partial E_y/\partial y + \partial E_z/\partial z)\vec{\mathbf{d}}x \wedge \vec{\mathbf{d}}y \wedge \vec{\mathbf{d}}z \equiv 4\pi\rho\vec{\mathbf{d}}x \wedge \vec{\mathbf{d}}y \wedge \vec{\mathbf{d}}z \equiv 4\pi{*\rho}$ corresponds to the <u>first</u> of Maxwell's equations.

The dual of $\mathbf{B} = B_x\vec{\mathbf{d}}y \wedge \vec{\mathbf{d}}z + B_y\vec{\mathbf{d}}z \wedge \vec{\mathbf{d}}x + B_z\vec{\mathbf{d}}x \wedge \vec{\mathbf{d}}y$ is the 1-form $*\mathbf{B} = B_c\vec{\mathbf{d}}x^c$; here, $B_c \equiv \tfrac{1}{2}B^{ab}\epsilon_{abc}$, *e.g.* $B_x \equiv B^{yz}\epsilon_{yzx}$, where $-B^{zy} = B^{yz} \equiv B^x$. So $\vec{\mathbf{d}}*\mathbf{B} = (\partial B_a/\partial x^b)\vec{\mathbf{d}}x^b \wedge \vec{\mathbf{d}}x^a$. The current **J** is a 2-form: $\mathbf{J} = J_x\vec{\mathbf{d}}y\wedge\vec{\mathbf{d}}z + J_y\vec{\mathbf{d}}z\wedge\vec{\mathbf{d}}x + J_z\vec{\mathbf{d}}x\wedge\vec{\mathbf{d}}y$. Thus, the <u>fourth</u> of Maxwell's equations corresponds to $\vec{\mathbf{d}}*\mathbf{B} - \partial*{}^\flat\mathbf{E}/\partial t = 4\pi\mathbf{J}$.

Electromagnetism with Differential Forms

(Continued)

From the second and third equations:

$$\vec{\mathbf{d}}\mathbf{B} = \mathbf{0} \Rightarrow \partial\vec{\mathbf{d}}\mathbf{B}/\partial t = \mathbf{0} \Rightarrow \vec{\mathbf{d}}(\partial\vec{\mathbf{d}}\mathbf{B}/\partial t) = \mathbf{0} \Rightarrow -\vec{\mathbf{d}}\vec{\mathbf{d}}\,{}^{\flat}\mathbf{E} = \mathbf{0};$$

the last equation following because $\vec{\mathbf{d}}\vec{\mathbf{d}}(\ldots) \equiv \mathbf{0}$. From the first and fourth equations:

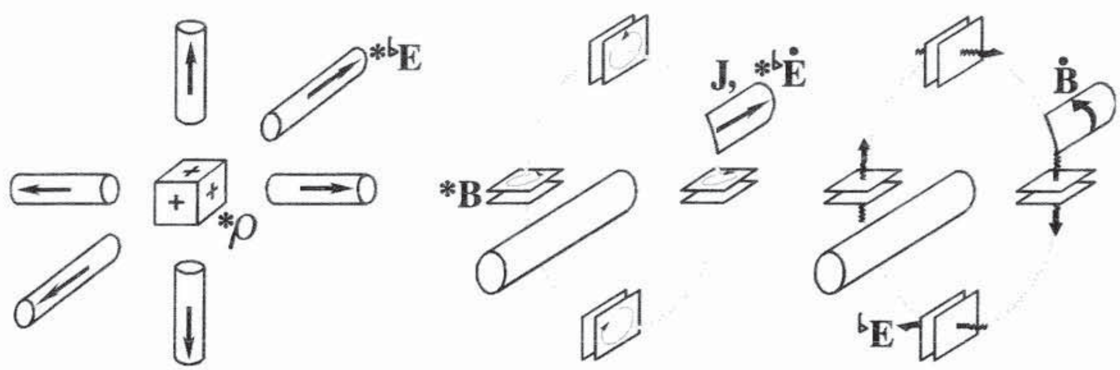
$$\partial[\vec{\mathbf{d}}*{}^{\flat}\mathbf{E} - 4\pi\rho\vec{\mathbf{d}}x\wedge\vec{\mathbf{d}}y\wedge\vec{\mathbf{d}}z]/\partial t = \vec{\mathbf{d}}[\vec{\mathbf{d}}*\mathbf{B} - 4\pi\mathbf{J}] - 4\pi(\partial\rho/\partial t)\vec{\mathbf{d}}x\wedge\vec{\mathbf{d}}y\wedge\vec{\mathbf{d}}z = \mathbf{0}.$$

This holds only if the integrability condition $(\partial\rho/\partial t)\vec{\mathbf{d}}x \wedge \vec{\mathbf{d}}y \wedge \vec{\mathbf{d}}z + \vec{\mathbf{d}}\mathbf{J} = \mathbf{0}$ is satisfied, which is the *conservation of charge*: *integrability condition* = *conservation of charge*.

An integrability condition for a field is associated with a conserved quantity.

The ***Faraday*** tensor is $$F_{\alpha\beta} \equiv \left\| \begin{matrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{matrix} \right\|.$$

The 4-vector $\mathbf{J}$ is $\mathbf{J} \equiv \rho\mathbf{e}_t + J^x\mathbf{e}_x + J^y\mathbf{e}_y + J^z\mathbf{e}_z$, where $\mathbf{e}_t \equiv \partial/\partial t$ and, for example, $\mathbf{e}_x \equiv \mathbf{i}$, $\mathbf{e}_y \equiv \mathbf{j}$, $\mathbf{e}_z \equiv \mathbf{k}$. The force 1-form is ${}^{\flat}\mathbf{F} = q({}^{\flat}\mathbf{E} - \mathbf{v} \lrcorner \mathbf{B}) = q(E_b - v^a B_{ab})\vec{\mathbf{d}}x^b$; $-B_{yx} = B_{xy} \equiv B_z$, $-B_{xz} = B_{zx} \equiv B_y$, $-B_{zy} = B_{yz} \equiv B_x$, $B_{xx} = B_{yy} = B_{zz} = 0$.



Electromagnetism with Differential Forms

(Continued)

Electrostatic flux "tubes" are described by the "twisted" 2-form $* {}^{\flat}\mathbf{E}$; their *source* is the "twisted" 3-form $*\rho \equiv \rho\boldsymbol{\epsilon}$. The source of the (*electric field intensity*) 1-form ${}^{\flat}\mathbf{E}$ is $\partial\mathbf{B}/\partial t$. The magnetic field $\mathbf{B}$ is a 2-form; $*\mathbf{B}$ is a "twisted" 1-form; the current density $\mathbf{J}$ is a "twisted" 2-form and is the source of $*\mathbf{B}$. See Burke (1985) for discussion of "twisted" differential forms, or Schouten (1954,1989), who calls them Weyl-tensors.

See Misner and Wheeler (1957), Misner *et al.* (1973), and Schouten (1989) for additional discussion about the geometry of electromagnetic fields.

Since components for the covariant derivative of a contravariant vector $\mathbf{v}$ are $\nabla_\gamma v^\alpha \equiv v^\alpha{}_{;\gamma} \equiv v^\alpha{}_{,\gamma} + \Gamma^\alpha{}_{\beta\gamma}\, v^\beta$, while for a covariant vector ${}^{\flat}\mathbf{v}$ they are $v_{\alpha;\gamma} \equiv v_{\alpha,\gamma} - \Gamma^\beta{}_{\alpha\gamma}\, v_\beta$, components for the covariant derivative of the torsion tensor $\mathbf{T}$ are

$$\nabla_\delta T^\alpha{}_{\beta\gamma} \equiv T^\alpha{}_{\beta\gamma;\delta} = T^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\epsilon\delta}\, T^\epsilon{}_{\beta\gamma} - \Gamma^\epsilon{}_{\beta\delta}\, T^\alpha{}_{\epsilon\gamma} - \Gamma^\epsilon{}_{\gamma\delta}\, T^\alpha{}_{\beta\epsilon},$$

$$= T^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\epsilon\delta}\, T^\epsilon{}_{\beta\gamma} + T^\epsilon{}_{\beta\gamma}\, T^\alpha{}_{\delta\epsilon} \text{ (Choquet-Bruhat } et\ al. \text{ 1982).}$$ [68]

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[68]: Choquet-Bruhat *et al.* (1982) define components for $\mathbf{R}$ as

$$R_\beta{}^\alpha{}_{\gamma\delta} \equiv \mathbf{R}(\mathbf{e}_\beta, \mathbf{e}^\alpha, \mathbf{e}_\gamma, \mathbf{e}_\delta),$$

with $R_\beta{}^\alpha{}_{\gamma\delta} = \partial_\gamma \Gamma^\alpha{}_{\delta\beta} + \Gamma^\epsilon{}_{\delta\beta}\Gamma^\alpha{}_{\gamma\epsilon} - \partial_\delta \Gamma^\alpha{}_{\gamma\beta} - \Gamma^\epsilon{}_{\gamma\beta}\Gamma^\alpha{}_{\delta\epsilon}$ and $\nabla_{\mathbf{e}_\gamma}\mathbf{e}_\alpha \equiv \mathbf{e}_\beta \Gamma^\beta{}_{\gamma\alpha}$. They define components for $\mathbf{T}$ as $T^\alpha{}_{\beta\gamma} \equiv \mathbf{T}(\mathbf{e}^\alpha, \mathbf{e}_\beta, \mathbf{e}_\gamma) \equiv \Gamma^\alpha{}_{\beta\gamma} - \Gamma^\alpha{}_{\gamma\beta}$. They write

$$\nabla_\gamma \nabla_\delta w^\alpha - \nabla_\delta \nabla_\gamma w^\alpha = w^\alpha{}_{;\delta\gamma} - w^\alpha{}_{;\gamma\delta} = R_\beta{}^\alpha{}_{\gamma\delta}\, w^\beta - w^\alpha{}_{;\beta}\, T^\beta{}_{\gamma\delta}.$$

Thus,

$$\underline{\textbf{First Bianchi Identity}}:\ R^{\alpha}{}_{[\beta\gamma\delta]} = \nabla_{[\gamma}T^{\alpha}{}_{\beta\delta]} - T^{\epsilon}{}_{[\beta\gamma}T^{\alpha}{}_{\delta]\epsilon}.$$

Therefore $R^{\alpha}{}_{[\beta\gamma\delta]} = 0$ if either $T^{\alpha}{}_{\beta\gamma} = 0$ or $\nabla_{[\gamma}T^{\alpha}{}_{\beta\delta]} = T^{\epsilon}{}_{[\beta\gamma}T^{\alpha}{}_{\delta]\epsilon}$.

Writing the torsion $\mathbf{T}$ as the set of 2-forms $\mathbf{T}^{\alpha} \equiv \frac{1}{2}T^{\alpha}{}_{\beta\gamma}\,\mathbf{e}^{\gamma} \wedge \mathbf{e}^{\beta}$, the superscript α denoting which 2-form, and recalling that the curvature 2-forms are $\Re^{\alpha}{}_{\beta} = \frac{1}{2}R^{\alpha}{}_{\beta\gamma\delta}\,\mathbf{e}^{\gamma} \wedge \mathbf{e}^{\delta}$, then *Cartan's structural equations of the affine connection* ∇ have been assembled (Schouten 1954, Choquet-Bruhat *et al.* 1982, Edelen 1985, Flanders 1989, Wasserman 1992):[69]

Cartan's Structural Equations

$$\mathbf{T}^{\alpha} = \tfrac{1}{2}T^{\alpha}{}_{\beta\gamma}\,\mathbf{e}^{\gamma} \wedge \mathbf{e}^{\beta};\quad \Re^{\alpha}{}_{\beta} = \tfrac{1}{2}R^{\alpha}{}_{\beta\gamma\delta}\,\mathbf{e}^{\gamma} \wedge \mathbf{e}^{\delta}.$$

Differentiating the first (Choquet-Bruhat *et al.* 1982),

Integrability Conditions (First Bianchi Identity)

$$\vec{\mathbf{d}}\mathbf{T}^{\alpha} = \nabla_{[\delta}T^{\alpha}{}_{\beta\gamma]}\mathbf{e}^{\delta} \wedge \mathbf{e}^{\gamma} \wedge \mathbf{e}^{\beta} = (R^{\alpha}{}_{[\beta\delta\gamma]} - \Gamma^{\alpha}{}_{\epsilon[\delta}T^{\epsilon}{}_{\beta\gamma]})\mathbf{e}^{\delta} \wedge \mathbf{e}^{\gamma} \wedge \mathbf{e}^{\beta},$$

Choquet-Bruhat *et al.*'s components for **R** can therefore be converted to Misner *et al.*'s (1973) by just moving the upper index all the way to the left. Similarly, to convert my torsion tensor components to their's, or *vice versa*, just switch the two lower indices.

[69]: Eli Cartan, French mathematician who invented exterior calculus.

gives the *integrability conditions* for the connection (Flanders 1989); a "gradient" in the torsion produces curvature. Differentiating the second (Choquet-Bruhat *et al.* 1982) produces the *Bianchi identities* (Schouten 1954)

Bianchi Identities

$$\vec{\mathbf{d}}\,\Re^{\alpha}{}_{\beta} = \nabla_{[\epsilon} R^{\alpha}{}_{|\beta|\gamma\delta]} \mathbf{e}^{\epsilon} \wedge \mathbf{e}^{\gamma} \wedge \mathbf{e}^{\delta} = T^{\varkappa}{}_{[\gamma\epsilon} R^{\alpha}{}_{|\beta|\delta]\varkappa} \mathbf{e}^{\epsilon} \wedge \mathbf{e}^{\gamma} \wedge \mathbf{e}^{\delta},$$

or "second Bianchi identity." So a "gradient" in curvature produces torsion. Flanders (1989) also refers to the Bianchi identities as integrability conditions.