Ångström bar

1 Introduction

A landmark of early 19th Century physics was the work of J. Fourier on the conduction of heat. Besides establishing a partial differential equation to describe heat conduction as a diffusion process, he introduced and developed the techniques of Fourier analysis/transforms which pervade many branches of modern science and engineering.

In this practical we will explore both topics when applying an elegant method devised by A. Ångström to determine the thermal conductivity of metals [1]. A sufficiently long bar of a given metal is alternately heated and cooled at one end (labeled O in the following), resulting in temperature changes along the bar. For periodic heat input at O it can be expected that the temperature changes at another point P on the bar will have the same period. Fig. 1 shows that this is indeed the case, when after a transient phase a steady state is reached. It is not surprising that the amplitude of the temperature
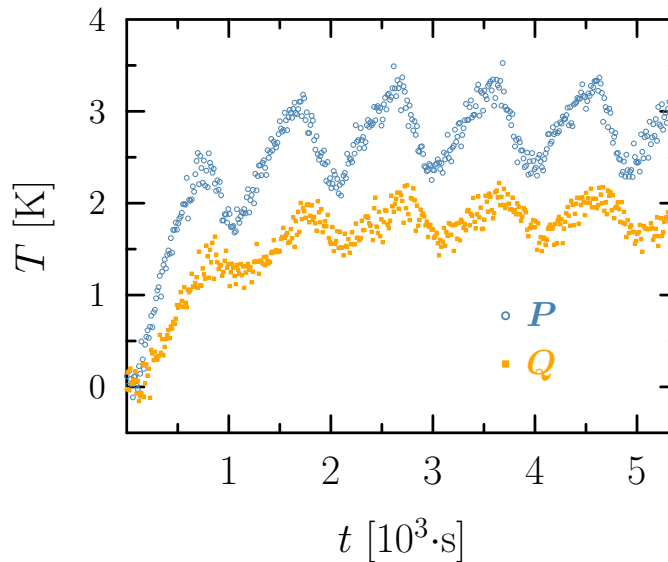

Figure 1: A typical temperature variation T measured at two points P and Q for periodic heat input: After an initial transient phase a steady state is approached where the temperature oscillates with the same period as the heat source.

changes is smaller at a more distant point Q. Note also that the maxima/minima occur slightly later at Q than at P. These observations can be understood in terms

of attenuated temperature waves propagating along the bar, as solutions of the heat equation.

2 Heat equation

To derive the heat equation in a simple setting, consider a uniform bar with cross-sectional area A and axis in the x-direction. Fourier's law states that the heat flux h
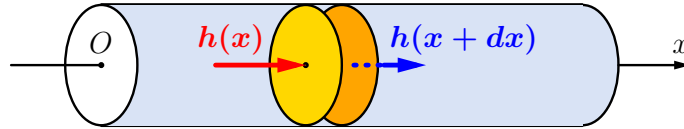
through a surface is proportional to the local temperature gradient; in our case $h(x) = -\alpha A\,\partial T/\partial x$, where α is the thermal conductivity. A slice of the bar has heat flux through its left and right boundary, thus the *net* heat flux into a thin slice of thickness dx is $h(x) - h(x + dx) = -(\partial h/\partial x)\,dx = \alpha A\,(\partial^2 T/\partial x^2)\,dx$.

The resulting heat transfer during a small time interval dt, $dQ = dt\,\alpha A\,(\partial^2 T/\partial x^2)\,dx$, causes a temperature change $dT = dQ/(c\,dm)$ in the slice with mass $dm = \rho dV = \rho A dx$, where c and ρ denote the specific heat capacity and the density of the bar. Hence

$$\frac{\partial T}{\partial t} = D\,\frac{\partial^2 T}{\partial x^2}\,, \tag{1}$$

which is the *heat (diffusion) equation* [2] in one dimension.[1] The coefficient $D = \alpha/(c\rho)$ is called thermal diffusivity.

We may generalize Eqn. (1) to also take into account the possibility of heat transfer from the perimeter of the bar to its environment at a certain (constant) temperature T_e, which will be proportional to the temperature difference $\tilde{T} = T - T_e$. This leads (since the derivatives of T and $\tilde{T}$ agree) to the generalized heat equation

$$\frac{\partial \tilde{T}}{\partial t} = D\,\frac{\partial^2 \tilde{T}}{\partial x^2} - \mu\,\tilde{T}\,, \tag{2}$$

where the constant μ parameterizes the thermal contact to the environment.

Eqs. (1) and (2) are *linear* partial differential equations (PDEs) because the temperature $T(t, x)$, the function to be determined, and its derivatives appear only as first powers. Obviously, any linear combination (superposition) of solutions is also a solution (which makes linear differential equations much easier than non-linear ones). Recall that for specific solutions of ordinary differential equations we need initial conditions; PDEs require in addition also boundary conditions (which come in different types). As the heat equation is a prototypical PDE (and relevant beyond Physics), this practical can be seen as a hands-on opportunity to refresh and extend your knowledge on PDEs.

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[1]In n dimensions, $\partial T/\partial t = D\,\Delta T$, where $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ is the Laplace operator.

In the following we will presume a basic understanding at the level of the informative reference [3] and gear towards what is needed for the analysis of this experiment.

Let us first consider the (unmodified) heat equation (1). As for *all* linear differential equations it is useful to make an 'exponential *Ansatz*',

$$T(t,x) \propto \exp[-i(\Omega t - Kx)] \,, \tag{3}$$

which in Physics terms is motivated by plane waves. Note that we do not have to specify the (dimensionful) prefactor on the right hand side of (3) thanks to the superposition principle. Substituting this *Ansatz* in Eq. (1) implies that Ω and K must be related by

$$i\Omega = -DK^2 \,. \tag{4}$$

Hence, Ω and K cannot both be real valued. Whether Ω or K is real (or neither) depends on the given situation, specified via the initial and boundary conditions. E. g., for a bar with an initial temperature profile $T(t_0, x)$ and isolated ends, the wavenumbers K are real (and discrete, to match the boundary conditions at its ends), thus $i\Omega$ in (4) is purely real and negative which, when inserted in (3), gives an exponential relaxation, $T(t, \cdot) \propto \exp(-|\Omega|t)$, towards equilibrium.

Of interest for our considerations here is another type of boundary condition, to describe the time-dependent heat input $h(t)$ at point O ($x = 0$), which according to our derivation of Eq. (1) reads $\partial T(t,x)/\partial x|_{x=0} \propto h(t)$. It will simplify our analysis when we notice that the temperature sufficiently far away from the heater remains virtually constant. We may therefore model the bar as semi-infinite, $x \in [0, +\infty]$; this 'open' right boundary will not put extra constraints on K.

For harmonic heat input, $h(t) \propto \sin(\omega t)$, Ω in our *Ansatz* (3) has to coincide in the steady state with the frequency ω of the heat source and is therefore real, thus K must be complex. Decomposing $K = k - i\kappa$ into real and imaginary parts, Eq. (4) implies that $k = \kappa = \sqrt{\omega/(2D)}$. Then our *Ansatz* describes temperature waves,

$$T(t,x) \propto e^{-\kappa x} \exp[-i(\omega t - kx)] \,, \tag{5}$$

with a strong spatial attenuation: over one wavelength $2\pi/k$ the amplitude decreases by a factor $\exp(\kappa \cdot 2\pi/k) = \exp(2\pi) \approx 535$. Note also that the waves (5) are *dispersive*: their phase velocity $v^{\mathrm{ph}} = \omega/k = \sqrt{2D\omega}$ depends on the frequency ω (as opposed to solutions of the wave equation).

Now, by the superposition principle we can add solutions of the form (5) with different frequencies, e. g.

$$\omega_m = m\,\omega \qquad \text{with integer } m \,,$$

the so-called harmonics of the fundamental frequency ω. One of Fourier's important insights (applied to our situation) is that the sum of these *partial waves*,

$$T(t,x) = \sum_m T_m \, e^{-\kappa_m x} \exp[-i(\omega_m t - k_m x)] \,, \tag{6}$$

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can describe the temperature along the bar with *general* $(2\pi/\omega)$-periodic heat input at $x = 0$. The amplitudes T_m are to be determined by Fourier analysis (see below); they are complex since $T(t,x)$ is, as an observable, manifestly real. Note that the wavenumbers and attenuation coefficients of the partial waves depend on the mode index m (via the frequency), $k_m = \kappa_m = \sqrt{\omega_m/(2D)}$. Hence the modes propagate with different velocities $v_m^{\mathrm{ph}} = \omega_m/k_m = \sqrt{2D\omega_m}$ as well as with different attenuation: waves with larger ω_m are faster but subject to stronger attenuation. Consequently, the time dependent temperature $T^X(t) = T(t,x)$ at point X with coordinate x changes not only amplitude but also shape with x (which is visible in Fig. 1, although these data follow the generalized heat equation (2)).

This feature of dispersion is crucial for Ångström's method to determine thermal conductivities. Therefore it is useful for the preparation and planning of this practical to develop some intuition for the magnitude of the relevant wavenumbers and attenuation lengths, which will help in particular to make suitable choices for the time-dependence of the heat input. A basis of these considerations should be some expectation on the value of the thermal diffusivity D of the metal bar. Noting from the heat equation that its physical dimension is $[D] = \mathrm{length}^2/\mathrm{time}$, it should not be difficult to give an order-of-magnitude estimate of the diffusivity of typical metals. In case you don't know yet how to do that: Take a break, have a cup of hot tea and think about D of the spoon's material while stirring.

Turning now to the modified heat eq. (2) as a more realistic model for the Ångström-bar experiment, we can expect similar attenuation and dispersion effects (at least for sufficiently small values of the parameter μ for the thermal contact to the environment). Substituting the *Ansatz* (3) into (2) implies now $K^2 = -(i\omega + \mu)/D$, thus the real and imaginary parts of K are related by $\kappa^2 = k^2 + \mu/D$, and they also have to satisfy

$$D = \frac{\omega}{2k\kappa}\,. \tag{7}$$

While it is not difficult to work out explicitly how k and κ depend on ω, D and μ, relation (7) will allow us to determine D from the wavenumbers and attenuation coefficients inferred from temperature measurements, as discussed next.

3 Fourier analysis

In Sec. 2 it was convenient to introduce the partial waves in terms of complex exponentials, c.f. Eq. (3), but it will be more convenient for the following Fourier analysis to switch to an equivalent real basis.

Let us first recall some basics of Fourier analysis (for more details see [4]). Fourier showed that all sufficiently smooth functions $f(t)$ with period[2] T can be represented as

$$f(t) = \tfrac{1}{2}a_0 + \sum_{m=1}^{\infty}\Big(a_m\cos(\omega_m t) + b_m\sin(\omega_m t)\Big), \tag{8}$$

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[2]Using the same symbol as for the temperature should not lead to ambiguities.

where $\omega_m = m \cdot 2\pi/T$, and

$$a_m = \frac{2}{T} \int_0^T dt\, f(t) \cos(\omega_m t)\,, \qquad b_m = \frac{2}{T} \int_0^T dt\, f(t) \sin(\omega_m t)\,. \tag{9}$$

Another useful real representation is

$$f(t) = \tfrac{1}{2} a_0 + \sum_{m=1}^{\infty} d_m \sin(\omega_m t + \varphi_m)\,,$$

with amplitudes and phases related to the coefficients (9) by

$$d_m = \sqrt{a_m^2 + b_m^2}\,, \qquad \tan \varphi_m = a_m/b_m\,. \tag{10}$$

This given, the general solution of Eq. (2) can be re-written as[3]

$$T(t,x) = \tfrac{1}{2} T_0\, e^{-\kappa_0 x} + \sum_{m=1}^{\infty} T_m\, e^{-\kappa_m x} \sin(\omega_m t - k_m x + \phi_m)\,, \tag{11}$$

with wavenumbers k_m and attenuation coefficients κ_m depending on ω_m, D and μ as discussed in Sec. 2, and which can be determined experimentally as follows. Note first that the time-dependent temperature at a particular point X with coordinate x can (as a T-periodic function) be expressed as a Fourier series,

$$T^X(t) = \tfrac{1}{2} T_0^X + \sum_{m=1}^{\infty} T_m^X \sin(\omega_m t + \varphi_m^X)\,.$$

Comparison with $T(t,x)$ in (11) shows that $T_m^X = T_m\, e^{-\kappa_m x}$ and $\varphi_m^X = \phi_m - k_m x$. Combining now the information from the temperature time series measured at two points P and Q on the bar, we find

$$\begin{aligned} k_m &= \frac{\varphi_m^P - \varphi_m^Q}{L}\,, \\ \kappa_m &= \frac{1}{L} \ln\left(\frac{T_m^P}{T_m^Q}\right)\,, \end{aligned} \tag{12}$$

where L denotes the distance between P and Q. Then relation (7) yields the diffusivity

$$D_m = \frac{\omega_m}{2 k_m \kappa_m}\,. \tag{13}$$

from analysing Fourier mode m. The diffusivity of the bar material should be the same for all m, but in practice there will only be a limited number of modes that allow to determine wavenumber and attenuation (12) with sufficient accuracy. This is a crucial circumstance to be also taken into consideration when planning the experiment.

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[3]To lighten the notation, I will drop the tilde in $\tilde{T}$ ($= T - T_e$) in Eq. (2), i.e. $T(t,x)$ in (11) denotes the difference to the environment temperature T_e.

To determine the diffusivity as described, you need the Fourier amplitudes T_n^X and phases φ_n^X for points $X = P, Q$. There are dedicated Fourier analysis programs and libraries, but you should develop your own package. To that end you need (besides the conversion (10) of Fourier coefficients) to compute the integrals (9), notably with integrands known only at certain 'grid points'. In our setting we will have temperature readings in regular time intervals, which we may choose as an integer fraction of the period of the heat input, $\Delta t = T/N$. A straight forward and robust approximation of the integral over the period of a function $g(t)$ in terms of its samples $g_i = g(i\Delta t)$, where $i = 0 \ldots N$, follows from the trapezoidal rule (see Fig. 2),[4]

$$\int_0^T dt\, g(t) \;\simeq\; \Delta t\, \frac{g_0 + g_1}{2} + \Delta t\, \frac{g_1 + g_2}{2} + \ldots + \Delta t\, \frac{g_{N-1} + g_N}{2}$$
$$= \; \Delta t\, [g_0 + g_1 + g_2 + \ldots + g_{N-1}]\,, \tag{14}$$

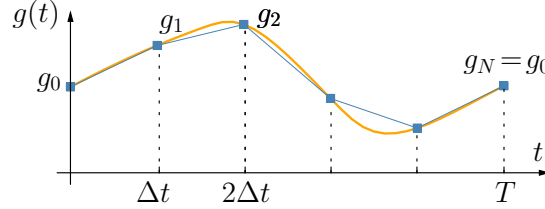where we used $g_N \equiv g_0$. Clearly, for smooth functions $g(t)$ the approximation will

Figure 2: Approximating an integral of a periodic function.

improve with increasing number N of samples. On the other hand, time-series from actual measurements usually have certain stochastic fluctuations (see Fig. 1), which are aspects to be taken into consideration when planning the experiment.

4 Apparatus and data acquisition

The apparatus for this experiment includes a brass bar[5] with several holes to place two thermocouples[6] for temperature measurments, and a heating/cooling unit which is controlled by an interface to a PC which also records the temperatures.

The heating unit can be switched on for a time T_{ON}, followed by a phase with duration T_{OFF} of active cooling. Both times can be set in a wide range. The number of

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[4]We note that when integrating over periodic functions it is not untypical that more sophisticated approaches, e. g. based on Simpson's rule, give larger errors than the simple formula (14).

[5]Brass comes in grades with different physical properties; the alloy used in this experiment has

$$\begin{aligned} \text{mass density} \quad \rho &= 8.45 \times 10^3 \,\mathrm{kg \cdot m^{-3}}, \text{ and} \\ \text{specific heat capacity} \quad c &= 385 \,\mathrm{J \cdot kg^{-1} \cdot K^{-1}}. \end{aligned}$$

[6]Thermocouples make use of the fact that junctions between dissimilar metals generate a temperature dependent e.m.f., which can be easily measured.

heating/cooling cycles is limited by the maximal time of 2 hours for one run of the experiment. The temperature readings of the thermocouples are recorded in time steps $\Delta t = 1\,\text{s}$, or multiple integers thereof. These settings are to be specified in the program *'Angstrom bar experiment COM4'*, see Fig. 3, which can be launched on the PC via its
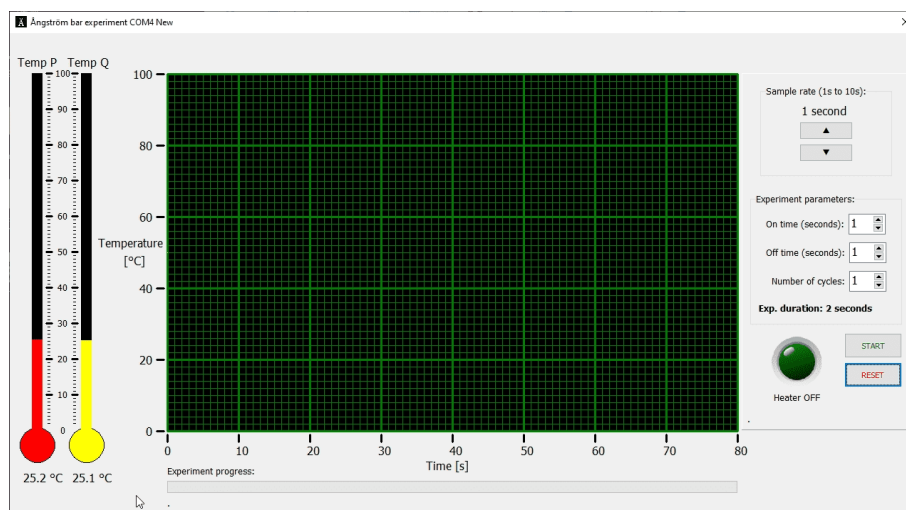

Figure 3: Setting parameters for the heating/cooling unit and data logging.

icon Å on the Windows desktop **after switching on, on the interface unit, the power for the heater and the cooler pump**. The captured data will be written to a CSV file with some header and 4 columns: time stamp (in s), heater status ($1 \equiv \text{ON}$, $0 \equiv \text{OFF}$) and the temperatures (in C°) at positions P and Q.

5 Objective of the practical

The ultimate objective is to determine the diffusivity D and the thermal conductivity α of the given brass alloy, including an estimate of the uncertainties. Note that the experiment, while straightforward to run, involves essential experimental physics skills: first finding 'good' parameter settings and then extracting accurate and reliable results from 'noisy' data – which requires a thorough understanding of the underlying physics and math tools.

References

[1] H. C. Bryant, *Heat waves and Ångström's method.* Am. J. Phys. **31** (1963) 325.

[2] https://en.wikipedia.org/wiki/Heat_equation

[3] https://www.youtube.com/watch?v=ToIXSwZ1pJU

[4] UCT PHYLAB2 manual, *Introduction to Fourier analysis.*