CP4

Numerical Integration | Simple Pendulum

Nathan Ngqwebo

NGQLIN011

Department of Science, University of Cape Town

PHY2004W | Intermediate Physics

Dr Spencer Wheaton

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This computational activity is an exercise in numerical integration techniques used in studying non-linear systems without analytical solutions. All code was produced using Python (Van Rossum & Drake, 2009) with the packages: NumPy (Harris et al., 2020), and Matplotlib (Hunter, 2007) used for data wrangling and plotting.

Numerical Methods

Derivative Approximation

We are tasked with deriving a first order approximation of the first derivative (1) using the Taylor expansion of $f(t+\Delta t)$ while making note of the leading error term ε_L.

$$f(t+\Delta t) = f(t) + \boldsymbol{f'(t)}\Delta t + \frac{f''(t)\Delta t^2}{2!} + \frac{f'''(t)\Delta t^3}{3!} + \cdots$$

$$f(t+\Delta t) - f(t) = \Delta t\left(\boldsymbol{f'(t)} + \frac{f''(t)\Delta t}{2!} + \frac{f'''(t)\Delta t^2}{3!} + \cdots\right)$$

$$\frac{f(t+\Delta t) - f(t)}{\Delta t} = \boldsymbol{f'(t)} + \frac{f''(t)\Delta t}{2!} + \frac{f'''(t)\Delta t^2}{3!} + \cdots$$

$$\frac{f(t+\Delta t) - f(t)}{\Delta t} = \boldsymbol{f'(t)} + \varepsilon_L$$

$$\boldsymbol{f'(t)} \approx \frac{f(t+\Delta t) - f(t)}{\Delta t} \tag{1}$$

$$\text{Where,} \qquad \varepsilon_L = \frac{f''(t)\Delta t}{2!} + \frac{f'''(t)\Delta t^2}{3!} + \cdots$$

Note, as $\Delta t \to 0,\ \varepsilon_L \to 0$ thus making the truncated approximation (1) more accurate.

Second Order Approximation

We also derived a second order approximation (4) for the first derivative using the Taylor expansions of $f(t+\Delta t)$ and $f(t-\Delta t)$

$$f(t+\Delta t) = f(t) + \boldsymbol{f'(t)}\Delta t + \frac{f''(t)\Delta t^2}{2!} + \frac{f'''(t)\Delta t^3}{3!} + \cdots \tag{2}$$

$$f(t-\Delta t) = f(t) - \boldsymbol{f'(t)}\Delta t + \frac{f''(t)\Delta t^2}{2!} - \frac{f'''(t)\Delta t^3}{3!} + \cdots \tag{3}$$

$$(2) - (3) = 2(\boldsymbol{f'(t)}\Delta t) + 2\left(\frac{f'''(t)\Delta t^3}{3!}\right) + 2\left(\frac{f^{(5)}(t)\Delta t^5}{5!}\right) + \cdots$$

$$f(t+\Delta t) - f(t-\Delta t) = 2\Delta t\left(\boldsymbol{f'(t)} + \frac{f'''(t)\Delta t^2}{3!} + \frac{f^{(5)}(t)\Delta t^4}{5!} + \cdots\right)$$

$$\frac{f(t+\Delta t) - f(t-\Delta t)}{2\Delta t} = \boldsymbol{f'(t)} + \frac{f'''(t)\Delta t^2}{3!} + \frac{f^{(5)}(t)\Delta t^4}{5!} + \cdots$$

$$\frac{f(t+\Delta t) - f(t-\Delta t)}{2\Delta t} = \boldsymbol{f'(t)} + \varepsilon_L$$

$$\boldsymbol{f'(t)} \approx \frac{f(t+\Delta t) - f(t-\Delta t)}{2\Delta t} \tag{4}$$

Amplitude dependence on Period

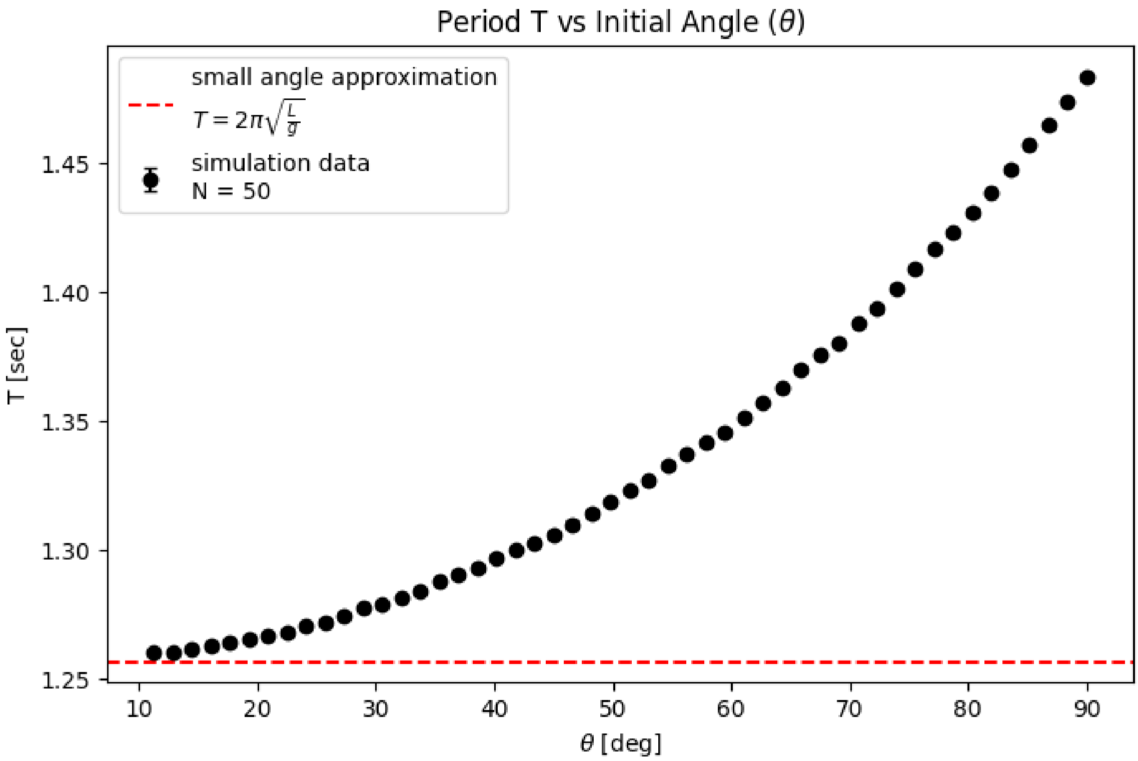

Figure 1

Scatter Plot of 50 Non-Linear Simple Pendulum Experiments Period vs Initial Angle Data with Horizontal Line Implying the Same Experiment under the Small Angle Assumption.

If we were to plot the negative initial angles as well, we would see that the period dependence on θ follows a parabolic relationship. As we expect, making the small angle approximation ensures that the system remains independent on the initial angle and is simply given by: $T = 2\pi\sqrt{\frac{g}{L}}$.

Euler Method Energy Non-Conservation

Here, we use the small angle approximation to show that energy is not conserved when using the Euler method for numerical integration:

$$\frac{d^2\theta}{dt^2} + \Omega_0^2 = 0 \tag{5}$$

$$\frac{d\omega}{dt} = -\Omega_0^2 \qquad\qquad \frac{d\theta}{dt} = \omega$$

$$\omega_{t+1} \approx \omega_t - \Omega_0^2 \Delta t \qquad\qquad \theta_{t+1} \approx \theta_t + \omega \Delta t$$

$$E_t = \frac{1}{2} mL^2 \omega_t^2 + mgL(1 - \cos\theta_t)$$

$$E_{t+1} = \frac{1}{2} mL^2 \omega_{t+1}^2 + mgL(1 - \cos\theta_{t+1})$$

$$\Rightarrow E_{t+1} - E_t = \frac{1}{2} mL^2 [\omega_{t+1}^2 - \omega_t^2] + mgL(\cos\theta_t - \cos\theta_{t+1})$$

$$\Delta E = \frac{mL^2}{2}[(\omega_t - \Omega_0^2 \Delta t)^2 - \omega_t^2] + mgL[\cos\theta_t - \cos(\theta_t + \omega_t \Delta t)]$$

$$= \frac{mL^2}{2}[\Omega_0^4 \Delta t^2 - 2\omega\Omega_0^2 \Delta t] + mgL[\cos\theta_t - \cos(\theta_t + \omega_t \Delta t)]$$

Note that $mgL[\cos\theta_t - \cos(\theta_t + \omega_t \Delta t)]$ always represents either a positive vertical displacement from equilibrium or zero displacement, exactly at equilibrium.

Also $\Omega_0^4 \Delta t^2 - 2\omega\Omega_0^2 \Delta t$ is always small compared to the potential energy whenever it is negative (near the peak of an oscillation). Therefore ΔE is always positive, and the total energy of the system will increase with each time step thus making the Euler method energy non-conservative.

References

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