Answers to problems are at the end of this document.

MD1. In class, we derived the "forward" velocity transformation, in which we find the velocity u' of an object in a new frame S',

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}.$$

In this problem, you will derive the "reverse" velocity transformation, in which the velocity u of an object in the frame S is expressed in terms of u' and v. You will do this in three different ways. (a) Use algebraic manipulation to solve the equation above for u. (b) Start with the "reverse" Lorentz transformations $x = \gamma(x' + vt')$ and $t = \gamma[t' + (v/c^2)x']$, and follow similar logic to what we did in class. (c) Use the trick you described in problem MC15.

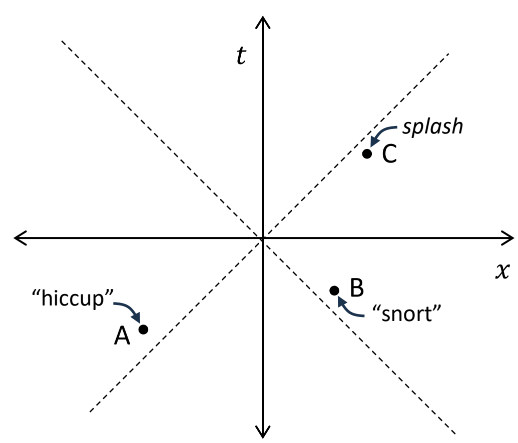
MD2. Event A occurs at coordinates $(x, ct) = (4\text{ m}, 6\text{ m})$. Event B occurs at coordinates $(7\text{ m}, 11\text{ m})$. (a) What is the numerical value of the spacetime interval $(\Delta s)^2$ between your two points? (b) Is the spacetime interval between these events *timelike*, *spacelike,* or *lightlike*, and how do you know?

MD3. Event A occurs at coordinates $(x, ct) = (2\text{ m}, -3\text{ m})$. (a) Find any point B such that the spacetime interval between points A and B is lightlike. (b) What is the numerical value of the spacetime interval $(\Delta s)^2$ between your two points?

MD4. Anna, Bob, and their roommate Carlos are quarantined together in a small apartment, and they are starting to get on each other's nerves. The space-time (Minkowski) diagram on the right shows three events that happen one afternoon, in the reference frame of their feckless cat, who is sleeping on the sofa. The dotted lines on the graph represent the speed of light, as usual.

- At point A, Anna hiccups.
- At point B, Bob emits a sudden snorting noise.
- At point C, Carlos spills his drink on his lap.
- Carlos says, "Darn it, Bob, you made me spill my drink!"
- Bob says, "Sorry Carlos, but I only snorted because Anna startled me when she hiccupped."

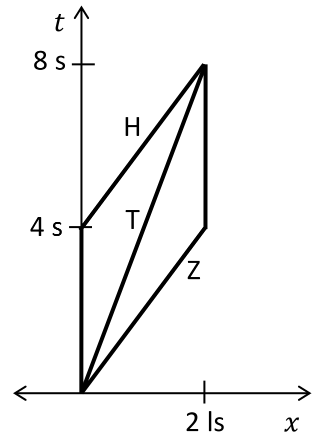
(a) Could Bob snorting have caused Carlos to spill his drink? Explain.
(b) Could Anna hiccupping have caused Bob to snort? Explain.



Anna says, "You are so full of it, Bob! In case you didn't notice, I was traveling in the positive x-direction at a high velocity, and in my reference frame, you snorted before I hiccupped. In fact, your snorting made me hiccup, so it's all your fault!"

(c) Is it possible that Bob's snort happened before Anna's hiccup in Anna's reference frame? Explain.
(d) Is it possible that Bob's snort caused Anna to hiccup? Explain.

MD5. A tortoise (top speed $0.25c$) and a hare (top speed $0.5c$) are having a race. They have agreed on a straight-line course that is 2 light-seconds long (its proper length). A zebra agrees to officiate. The Minkowski diagram on the right shows what happens when a starting gun sounds, as observed in the Earth reference frame:



- The tortoise, hare, zebra, and all spectators synchronize their watches to $t = 0$.
- The tortoise accelerates nearly instantaneously to his top speed of $0.25c$, which he maintains through the end of the race.
- The zebra accelerates nearly instantaneously to $0.5c$, and stops nearly instantaneously right at the finish line to observe the end of the race.
- The hare hot-dogs around at the starting line, mugging to cameras and waving to fans for a full four seconds. Only then does he accelerate nearly instantaneously to his top speed of $0.5c$, which he maintains through the end of the race.

(a) Draw a qualitative Minkowski diagram of the race in the reference frame of the tortoise.
(b) How much time passes on the tortoise's wrist watch while he is running the race?
(c) In the tortoise's reference frame, at what time does the zebra stop running, and at what time does the hare start running?
(d) According to the hare's wrist watch, how long does it take the hare to complete the course? (The total time, including the time he was just standing around showboating.)
(e) In the reference frame of the tortoise, what is the speed of the hare, once he starts running?

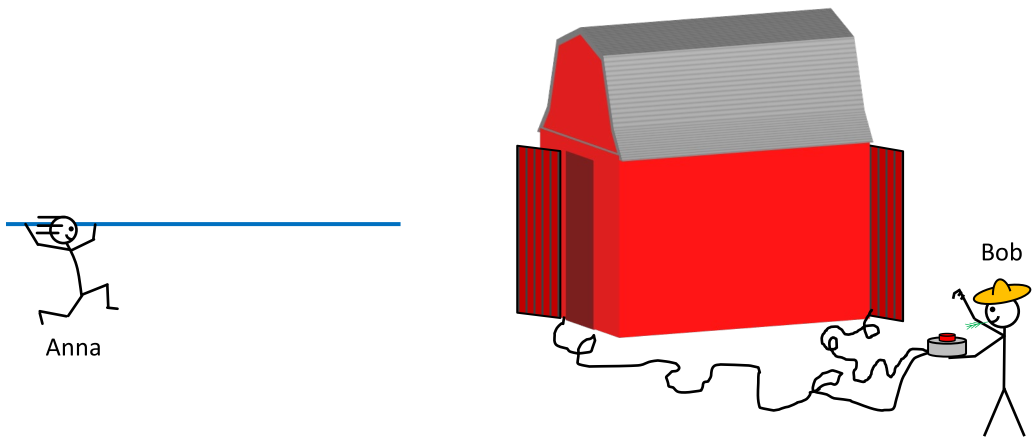
At the end of the race, the zebra declares the race to be a tie. The tortoise and hare each stare at their watches in amazement. The other animals all boo and throw trash onto the field.

(f) Do the tortoise, the hare, and the spectators agree or disagree about who crossed the finish line first? Explain.
(g) What do the tortoise, the hare, and the spectators measure for the time it took each competitor to complete the course? (That's six times to compare.) Do they agree or disagree about which competitor had the shortest time? Again, explain.

MD6. Two events occur at (x, ct) coordinates (4 m, 1 m) and (7 m, 6 m). (a) Is the spacetime interval between these events *timelike*, *spacelike,* or *lightlike*? (b) What is the proper time between these events? (c) What is the proper distance between these two events?

MD7. Two events occur at (x, ct) coordinates $(-3\text{ m}, -2\text{ m})$ and $(7\text{ m}, 4\text{ m})$. (a) Is the spacetime interval between these events *timelike*, *spacelike,* or *lightlike*? (b) What is the proper time between these events? (c) What is the proper distance between these two events?

MD8. Anna has recently taken up pole vaulting, and owns a 10 m long pole (its proper length). Bob has taken up farming, and owns an 8 m long barn (also its proper length) with a door on each end. Anna runs very fast towards the open barn; in fact, she runs so fast that in the reference frame of Bob and the barn, the pole's length is Lorentz contracted to just 6 meters. When Bob sees that Anna is exactly centered inside the barn, he



pushes a button that closes both doors very quickly.

- Bob says, "Hah! I've closed Anna and her pole inside my barn!"
- But in Anna's reference frame, it is the barn that is Lorentz contracted. Anna says, "My pole is 10 meters long, and your barn is clearly *less than* 8 meters long. Since your barn is shorter than my pole, you couldn't possibly close me and my pole inside of it!"

Who is correct? Anna, or Bob? (*Remarks: this problem is actually quite tricky, and we'll address it in our next class. Take five or ten minutes to think about it and write down your thoughts, but don't spend more time on it than that.*)

MD9. Consider the case of Anna's pole and Bob's barn, as represented in the Mathematica file pole_and_barn.nb.

(a) In Bob's reference frame, at what time does the front end of Anna's pole first enter the barn?
(b) In Anna's reference frame, at what time does the back end of her pole exit the barn?

The graphs in the Mathematica file all show what would happen if Anna and her pole crashed through the barn door and kept going. But what if the right barn door was so strong that it stopped Anna and her pole instantly? Would the two observers *agree* or would they *disagree*, about whether the left end of Anna's pole stops inside or outside the barn?

(c) In Bob's reference frame, at what time does the front of Anna's pole crash into the door? If the back end of Anna's pole stops instantly at that time, is the back end of the pole inside the barn or outside the barn?

(d) In Anna's reference frame, at what time does the front of Anna's pole crash into the door? If the back end of Anna's pole stops instantly at that time, is the back end of the pole inside the barn or outside the barn?

(e) Do your answers to parts (c) and (d) seem to present an additional paradox?

To resolve the paradox, we recall that there is a universal speed limit.

(f) What is the fastest that any object or information can travel, in any reference frame?

(g) In the reference frame of Anna, draw a line with a slope of -1 showing "information" about the collision traveling at the speed of light from the instant the right end of her pole hits the right barn door. (For this, you can either modify the Mathematica file, or you can create a careful freehand drawing using a pencil and a straight edge.)

(h) In Anna's reference frame, at about what time does information about the front end of Anna's pole hitting the door reach the back end of Anna's pole?

(i) At the time the information reaches the back end of the pole, is the back end of the pole *inside* the barn or *outside* the barn?

(j) Your friend is still stuck on the paradox you defined in part (e): "But if the pole stops instantly when the front end hits the far door, it seems like Bob would say the pole was entirely inside the barn, and Anna would say the back end was still outside. Isn't that a paradox?" Explain in a couple of sentences how you resolve this apparent paradox, focusing on your friend's use of the word "instantly."

MD10. Bob and Anna are the same age. While Bob stays on Earth, Anna sets off at a constant speed $v = 0.8c$ ($\gamma = 1.67$) to visit another planet. In the Earth reference frame, the planet is 24 ly away. Once there, she turns around and comes back, also at $v = 0.8c$. (a) According to Bob, how long is Anna gone? (b) According to Anna, how long is she gone? (c) Suppose that instead of staying on Earth, Bob is floating in a spaceship (engines off) just outside our solar system during Anna's journey. Does this affect your answers to parts (a) and (b)? (d) Suppose that instead of traveling to a specific planet, Anna simply travels away in a random direction to a distance 24 ly away from Bob (according to Bob). Does that change your answers to parts (a) and (b)? (e) Just before they reunite, Anna radios Bob, saying "Hey, from my spaceship, it looks like you had a velocity of $0.8c$ away from me, and then you turned around and had a velocity of $0.8c$ towards me. I expect that when I see you again, you will not have aged as much, and will look much younger than me." Bob expects Anna to be younger. But Anna also expects Bob to be younger. Can they both be right? Explain how to resolve this apparent paradox.

MD11. Suppose you drive for 2 hours at 70 mph up to Washington DC, and then return back to Richmond at the same speed. Due to time dilation, what is the difference between how much you aged, and how much your friend aged who stayed in Richmond? In answering this question, please do the calculation using a binomial series expansion for gamma, using the general fact that

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots.$$

(Presumably, you will only need to keep only the first order term in the above expression.) For very small numbers like these, the binomial series expansion is very handy!

Answers:

MD2. (a) $-16\ \mathrm{m}^2$ (b) timelike. (Why?)
MD3. (b) 0
MD4. (a) yes (b) no (c) yes (d) no
MD5. (b) 7.75 s (c) 3.61 s; 4.13 s (d) 7.48 s (e) $0.286c$ (f) They agree. (g) As a check for your results, the *sum* of all six times is 46.46 seconds. In fact, though their numbers differ, the tortoise, the hare, and all the other animals all agree on the ultimate outcome of the race, by any reasonable measure. (The other animals only booed and threw trash on the field because they got caught up in the moment, and it seemed like the right thing to do.)
MD6. (b) 4 m/c (c) n/a
MD7. (a) spacelike (b) n/a (c) 8 m
MD9. (a) $ct = -7.5$ m (b) $ct = +6$ m (c) $ct = 2.5$ m; back end is inside (d) $ct = -6.5$ m; back end is outside. (e) Yes, that's a humdinger of a paradox! (f) c (h) $ct \approx 3$ m (i) Inside the barn. Both Anna and Bob would agree on this point.
MD10. (a) 60 years (b) 36 years
MD11. The difference is 78 picoseconds.