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Conservative System Analysis for Glider Dynamics

A conservative system involves a system of differential equations where quantities (such as energy and momentum) are conserved over time. This means that our given quantities are not created nor destroyed in a system, but rather converted in a way where we do not lose the total amount of quantities we started with. Analyzing these types of systems helps us understand how much of a quantity is distributed into other forms of a quantity or quantities. When energy is conserved, the total amount of energy remains constant. When energy is not conserved, the total amount of energy changes over time due to energy being lost or even gained.

It is quite interesting how these systems are used in everyday phenomena – especially the ones we do not expect. With this concept at our disposal, we can generate models in many fields like Physics, Engineering, Economics, and Chemistry. It is important to note that the mathematics involved depends on the system being analyzed. Generally, we would expect to utilize concepts from Algebra, Calculus, Linear Algebra, Differential Equations, and even more subsets of those fields such as bifurcation theory and nondimensionalization to help with the derivation of these systems. That is when our math modeling knowledge comes into play because we need to find and determine the stability of equilibria. Doing so will give us an idea of where these quantities are gravitating or repelling to throughout time, which will give us a better understanding and interpretation of these systems. Now, one huge caveat in these systems is that there's always some sort of external force that may remove energy from the system. With that said, it is important to know that a conservative system is never truly conservative, but nevertheless, it is still useful in providing approximations to systems with extremely low frictional forces. On top of that, conservative systems can still give us an idea of what we are working with.

The two Differential Equations from our glider example (6.5.14) are derived from the conservation of energy and conservation of momentum principles. The total energy of the glider must be conserved, meaning that the sum of the kinetic energy and potential energy is constant. The total momentum of the glider is also conserved, so the sum of its linear momentum and angular momentum remains constant as well. These two ideas are the bread and butter to find exactly what we need. Without going too in-depth with the specifics of deriving both differential equations for this model, we will present the general thought process since the work requires some clever algebraic geometric interpretations. To start, we can manipulate the basic kinematic equations from physics to form the equation of motion:

$$m\dot{v} = -mgsin(\theta) - Dv^2$$

Let us break down exactly what is happening here. This above equation is the equation of motion for an object with mass m moving through space (or a fluid) along with a drag force that is proportional to the square of the velocity. Specifically, the first term on the right-hand side is the gravitational force acting upon our glider at an angle θ to the horizonal – since gravity is pulling down on the glider, we have a negative sign in front. The second term is the drag force upon the glider – since drag is in the opposite direction, the sign is also negative. As for the left-hand side, this comes from Newton's Second Law of Motion where $\sum F = ma$. Since acceleration is the derivative of velocity, we have $\sum F = ma = m\dot{v}$. So, the net force upon the glider is a gravitational force, $-mgsin(\theta)$, and a drag force, $-Dv^2$. This should make intuitive sense since those are the only forces acting upon the glider in the air. Here, we will not be considering any other factors such as turbulence. Through some nondimensionalization and some trigonometric properties, we can obtain the first differential equation:

$$\dot{v} = -\sin(\theta) - Dv^2$$

We can also look at the second differential equation we are given for this problem and analyze it:

$$v\dot{\theta} = -\cos(\theta) + v^2$$

This differential equation represents the angular rate of change of the glider's velocity with respect to the horizonal. It is possible for us to isolate the $\dot{\theta}$ variable but keeping the differential equation in this form will be helpful for analysis later. As stated earlier, the trigonometric terms represent the effects of gravity. Since we are now looking at the velocity with respect to the horizonal, we should be considering the left and right directions – this is why we have a $-\cos(\theta)$ appear in this equation. Also, for both differential equations, the v^2 terms amplify the effects of drag and lift. In this case, the v^2 term is positive since the lift force is generated by the shape of the glider's wings. Without getting too much into the specifics of this motion, the shape of the wings develop a difference in pressure above and below the wings, which ultimately results in an upward force and opposes gravity. In simpler terms, the lift force acts perpendicular to the direction of motion. One final thought to mention is that lift is proportional to the square of velocity, which is why we specifically have the v^2 term in both equations.

With the derivation now complete, we will begin analyzing our glider problem through the lens of these two differential equations. We are going to specifically focus on the drag force and how it affects this system. As a reminder, and to bring everything together, we have the following differential equations:

$$\begin{cases} \dot{v} = -\sin(\theta) - Dv^2 \\ v\dot{\theta} = -\cos(\theta) + v^2 \end{cases}$$

First, we will suppose there is no drag $(D = 0)$. With this information, we will show that $E(v, \theta) = v^3 - 3vcos(\theta)$ is a conserved quantity. Before doing this analysis, it is important to discuss why we should even care about this quantity – what is $E(v, \theta)$? From our introduction, we discussed how energy is always conserved, implying that the total energy can be written as a constant. In this problem, we have a glider in the air with potential and kinetic energy. From physics, we should note that kinetic energy is given by $E_K = \frac{1}{2}mv^2$ and potential energy is given by $E_P = mgh$, where m is the mass of the glider, g is the acceleration due to gravity, and h represents the altitude of the glider. Adding these two together and rewriting them using trigonometric identities will give us the specific energy of the glider: $v^3 - 3vcos(\theta)$. At this point, we need to show $E(v, \theta) = v^3 - 3vcos(\theta)$ is a conserved quantity. Again, a quantity is conserved when its value remains constant. If this is true, the derivative of our function needs to be 0:

$$E = v^3 - 3vcos(\theta)$$

$$\dot{E} = 3v^2\dot{v} - 3\dot{v}cos(\theta) + 3v\dot{\theta}\sin(\theta)$$

Note that $D = 0$, so $\dot{v} = -\sin(\theta)$. By our given differential equations, we can substitute for $\dot{v}$ and $v\dot{\theta}$ to get the following:

$$\dot{E} = 3v^2(-\sin(\theta)) - 3(-\sin(\theta))\cos(\theta) + 3(-\cos(\theta) + v^2)\sin(\theta)$$

$$\dot{E} = -3v^2\sin(\theta) + 3\sin(\theta)\cos(\theta) - 3\cos(\theta)\sin(\theta) + 3v^2\sin(\theta)$$

$$\boxed{\dot{E} = 0}$$

Now we know that $E(v, \theta)$ is some constant value. Remember that this only works for the case where $D = 0$. If we have $D > 0$, then the quantity $E(v, \theta)$ is no longer conserved – in fact, for $D > 0$, air resistance will cause a dissipation of energy and the specific energy of the glider will decrease. This makes intuitive sense since there is air resistance converting energy into other forms such as heat.

It is time to visualize what is going on by creating a phase portrait. We can determine the equilibrium points through the nullclines and classify these points via the Jacobian matrix. Since there are infinite periodic solutions for θ, setting the domain to $\theta \in [0,2\pi)$ should be done here to examine one cycle – we will assume that this behavior repeats periodically. When determining our equilibria, we need to set both $\dot{v} = \dot{\theta} = 0$ and find our values for v and θ in the form (v, θ). After doing some algebra and precalculus, we end up with equilibrium points $(1,0)$ and $(-1,0)$. Using the Jacobian, we can determine the stability and behavior of these equilibrium points. Consider the Jacobian of this system:

$$J = \begin{bmatrix} 0 & -\cos{(\theta)} \\ 1 + \dfrac{\cos(\theta)}{v^2} & \dfrac{\sin(\theta)}{v} \end{bmatrix}$$

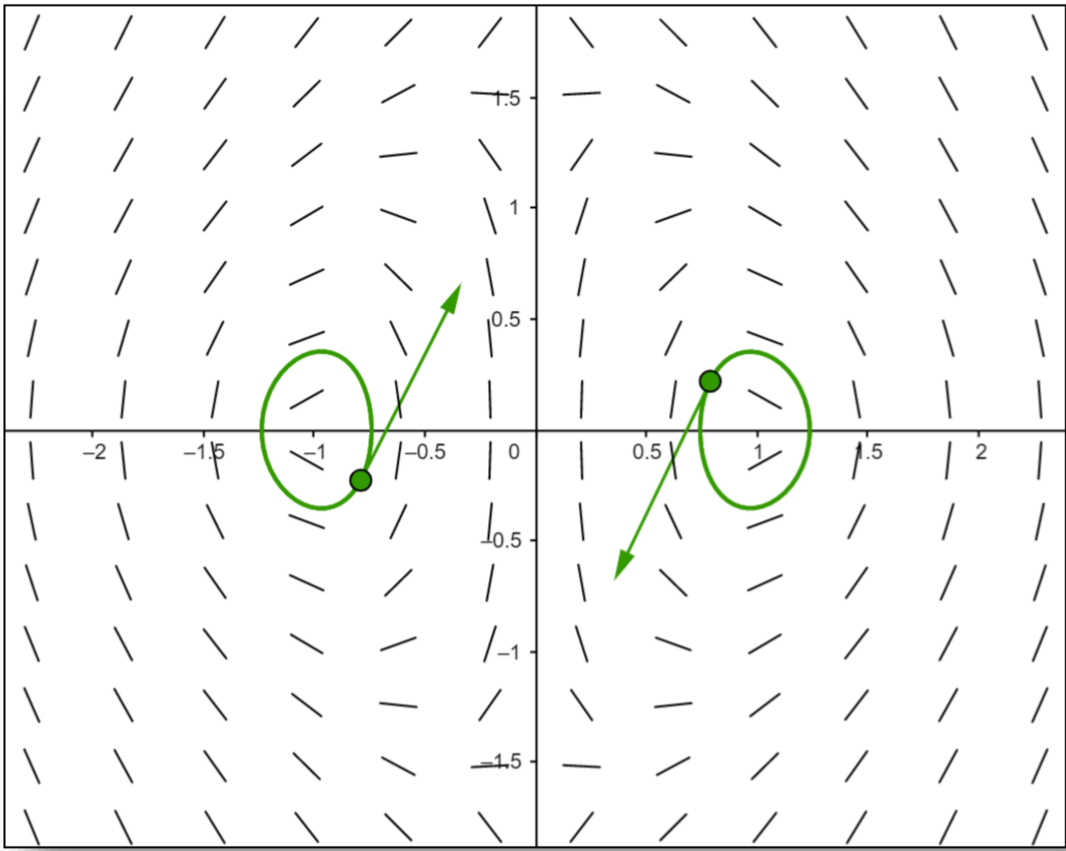
We can now evaluate the Jacobian at each equilibrium point to determine the stability of these equilibria.

$$J_{(1,0)} = \begin{bmatrix} 0 & -\cos{(0)} \\ 1 + \dfrac{\cos(0)}{1^2} & \dfrac{\sin(0)}{1} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$$

$$J_{(-1,0)} = \begin{bmatrix} 0 & -\cos{(0)} \\ 1 + \dfrac{\cos(0)}{(-1)^2} & \dfrac{\sin(0)}{-1} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$$

For this system, the Jacobian resulted in the same matrix, giving us two center equilibria at (1,0) and (-1,0) since both eigenvalues for each matrix are purely imaginary. It is time to plot our phase portrait and interpret our results now that we have an idea of what is going on in our system.

geogebra.org was used for the following model:



In terms of the glider flying in the air, we see that the motion is periodic and stable as the glider stays near these center equilibria. If the glider moves far enough from these points, the stable circular motion will shift to a non-circular unstable motion, resulting in the glider moving away from our equilibria indefinitely. Considering that we are in the bounds of a conservative system, other external forces like drag and turbulence will not disrupt these oscillations. Therefore, it is safe to say that the periodic motion of the glider will be maintained indefinitely when there is no drag. This makes sense because if we had a glider that was gliding in the air without any sort of air resistance, the glider would keep on gliding forever since it is experiencing zero net force.

This is not practical in the real world, but it gives us an idea of how outside forces such as air resistance affect an object. Of course, a more practical use would be studying the non-conservative model where $D > 0$, but that would not be considered a conservative system anymore.

Overall, conservative systems are extremely useful in seeing the behavior of a multitude of phenomena in our world, but we should be considerate of the fact that no system is truly conservative in a practical sense. As stated earlier, there is always a caveat in these systems where there exists some sort of external force directly impacting the system. We should keep in mind their uses, but not get too ahead of ourselves when analyzing them by understanding their limitations.