2 Bode Plots

Bode plots play an important role in frequency domain analysis and design.

Bode plots of a transfer function $G(s)$ **include two graphs:**

➢**Magnitude plot:** $20\log_{10}|G(j\omega)|$, **(dB-decibels) in linear scale versus the frequency in log scale.**

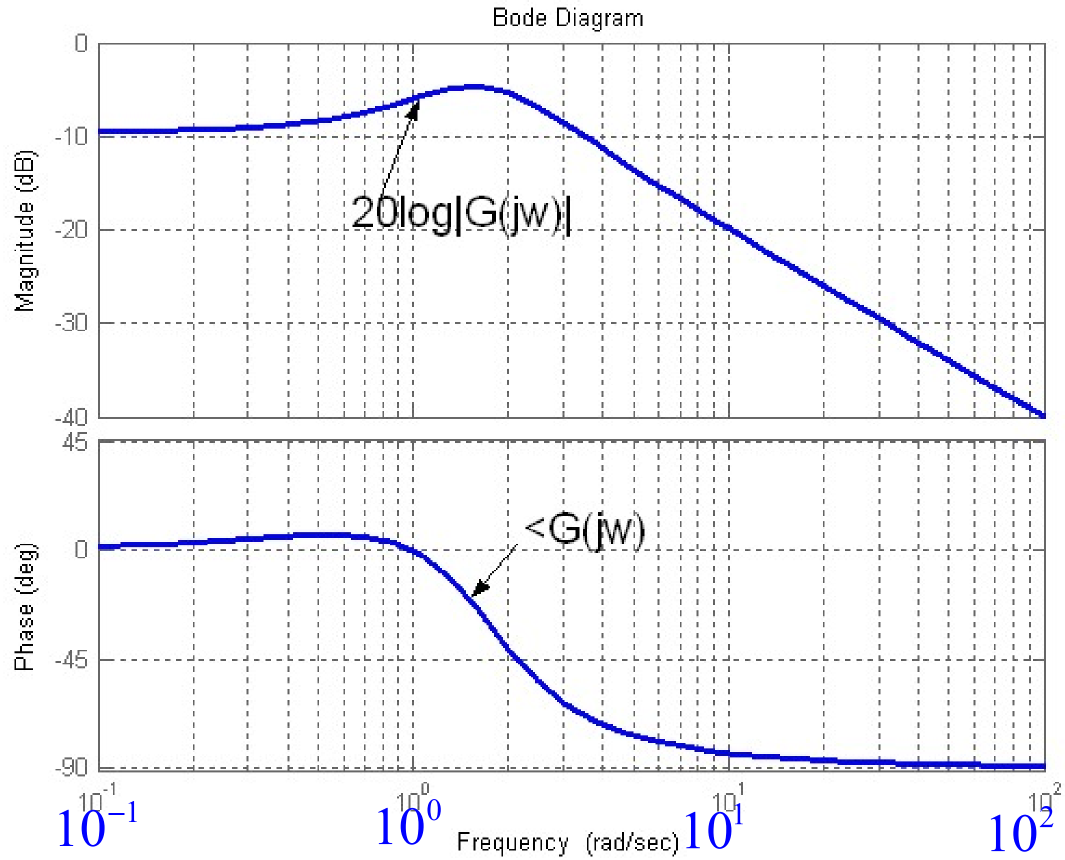
➢**Phase plot:** $\phi(\omega) = \angle G(j\omega)(\text{degrees})$ **in linear scale versus** ω **in log scale.**

Note: Bode plots are plotted on semilog papers with the horizontal axis in log scale and the vertical axis in linear scale.

Example 2.1

Consider the transfer function $G(s) = \dfrac{s+1}{s^2+2s+3}$

Bode plots of $G(s)$ **are**



Matlab commands:
num=[1 1];
den=[1 2 3];
bode(num,den)

ω	10^{-1}	10^{0}	10^{1}	10^{2}
$\log\omega$	-1	0	1	2

Advantages of Bode plots:

- **Multiplying operation in frequency domain is converted to addition operation. Specifically, given a transfer function**

$$G(s) = G_1(s)G_2(s)$$

The frequency response is $G(j\omega) = G_1(j\omega)G_2(j\omega)$. **Then,**

$$20\log|G(j\omega)| = 20\log|G_1(j\omega)| + 20\log|G_2(j\omega)|$$

$$\angle G(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega)$$

- **Because the frequency is depicted using log scale, a much wider range of system behavior can be displayed in a single plot.**

- **Bode plots can be determined experimentally.**

- **Compensators can be easily designed with Bode plots through loop shaping.**

Understanding Bode Plots

To understand the magnitude and phase responses of a transfer function, we first study the Bode plots of some basic factors that a transfer function generally contains:

- K
- $\dfrac{K}{s^N}$ or Ks^N
- $\dfrac{1}{\tau s+1}$ or $(\tau s+1)$
- $\left[\left(\dfrac{s}{\omega_n}\right)^2+2\zeta\left(\dfrac{s}{\omega_n}\right)+1\right]^{\pm 1}$

For example,

$$G(s)=\frac{10(s+1)}{s(s^2+0.4s+4)}=\frac{2.5}{s}\frac{s+1}{(s^2/4+0.2s/2+1)}$$

Bode Plots of Basic Factors

1) $G(j\omega)=K$

The magnitude is simply $20\log|K|$. For $K>0,\ \angle K=0^0$.

For $K<0, \angle K=-180^0$.

2) $G(s)=\dfrac{K}{s^N}$ or $G(j\omega)=\dfrac{K}{(j\omega)^N}\quad (K>0)$

Magnitude in dB:

linear in $\log\omega$

$$20\log\left|\frac{K}{(j\omega)^N}\right|=20\log K-20N\log\omega$$

- **For a decade increment in ω (horizontal axis), the gain change is $-20N$ dB (vertical axis).**
- **When $\omega = 1$, the gain is $20\log K$.**
- **Hence, the magnitude plot is a straight line with slope of $-20N$ dB/dec, passing through $20\log K$ at $\omega = 1$.**

Phase in degrees:

$$\phi(\omega) = \angle K - \angle (j\omega)^N = -90^0 N$$

which is constant for all frequencies.

$$\angle j^N = 90 \times N$$

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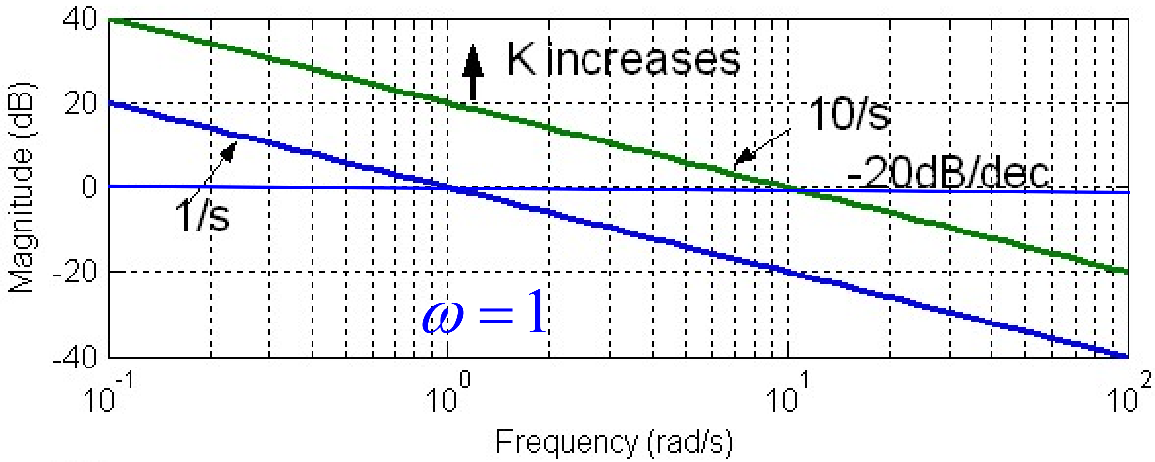
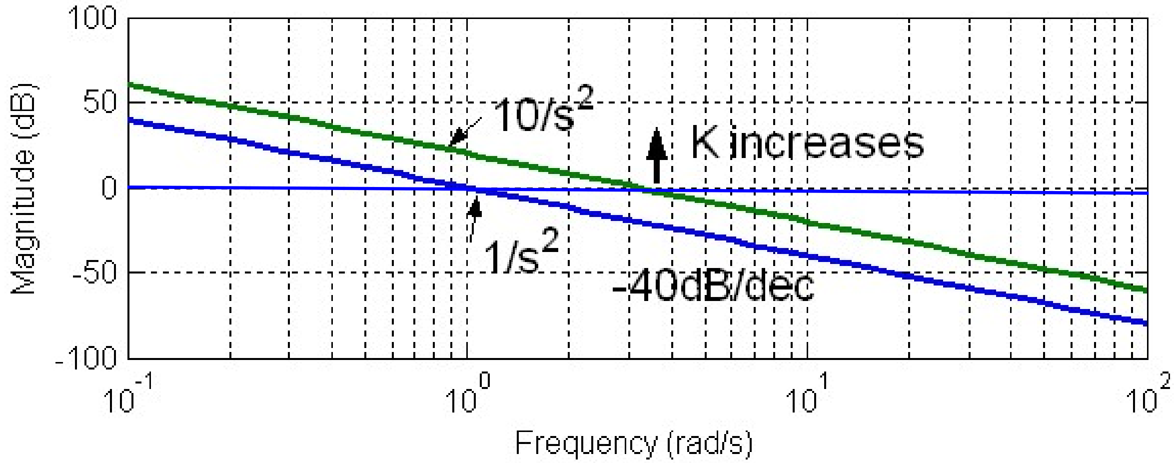
Example 2.2

$$\frac{K}{s}$$



$$\frac{K}{s^2}$$



When $K = 1, \quad \left|\frac{1}{j\omega}\right| = \frac{1}{\omega}$

When $K \neq 1,$

$$20\log|K\frac{1}{\omega}|$$

$$= 20\log K + 20\log\frac{1}{\omega}$$

$$\left|\frac{1}{(j\omega)^2}\right| = \frac{1}{\omega^2}$$

On the other hand, for $G(s) = Ks^N$**, the magnitude and phase are respectively** $20\log K + 20N\log\omega$ **(a straight line of slope 20*N* dB/dec) and** $90^0 N$ **.**

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3) $G(s)=\dfrac{1}{\tau s+1}$ or $G(j\omega)=\dfrac{1}{j\omega\tau+1}$

- **Magnitude in dB:**

$20\log 1-20\log|1+j\omega\tau|$

$$20\log\left|\frac{1}{j\tau\omega+1}\right|=-20\log\sqrt{1+(\omega\tau)^2}$$

- **It has two asymptotes:**

✓ **For** $\omega<<1/\tau$, $20\log\left|\dfrac{1}{j\omega\tau+1}\right|\approx 0$

✓ **For** $\omega>>1/\tau$,

$$20\log\left|\frac{1}{j\omega\tau+1}\right|\approx -20\log\omega\tau=-20\log\omega-20\log\tau$$

(**a straight line of slope of -20 dB/dec**).

- **The intersect of the high frequency asymptote with the low frequency asymptote is named as *corner frequency*, i.e., at the corner frequency**

$$-20\log\omega\tau = 0 \rightarrow \omega = \frac{1}{\tau}$$

- **The actual magnitude at the corner frequency is**

$$20\log\left|\frac{1}{j\omega\tau+1}\right| = -20\log\sqrt{2} = -3 \text{ dB}$$

Therefore, the magnitude plot of $(j\omega\tau+1)^{-1}$ **approximately looks like:**

✓ **A horizontal 0 dB line and –20 dB/dec line intersect at** $\omega = \frac{1}{\tau}$.

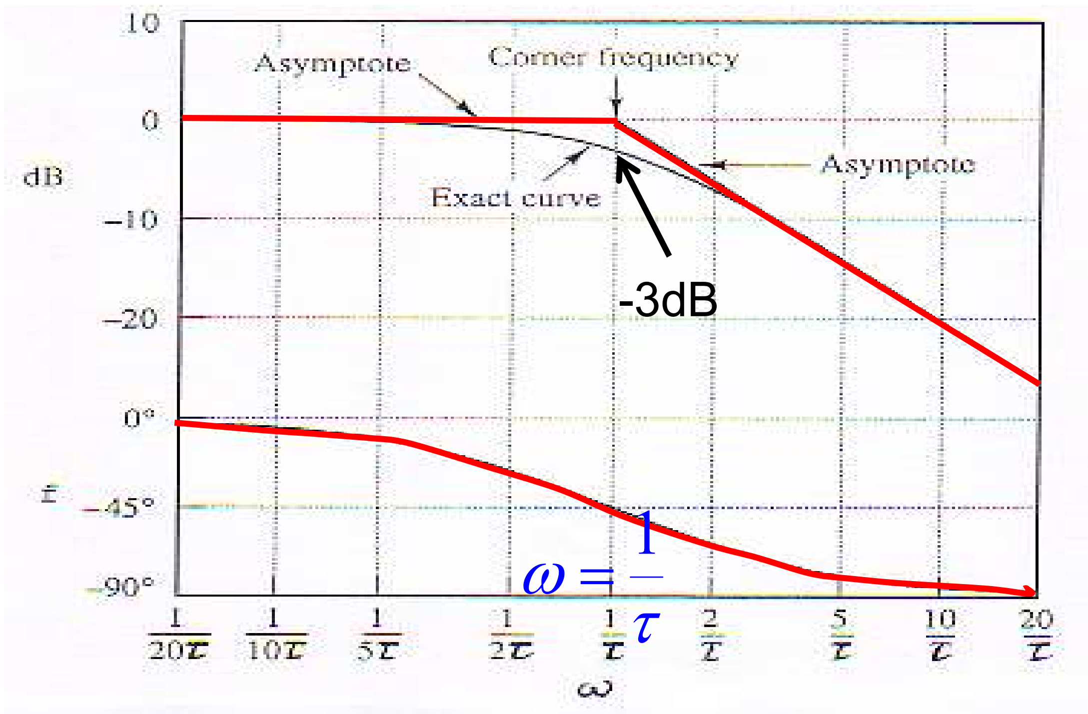
Phase in degrees: $\phi(\omega) = \angle G(j\omega) = -\tan^{-1}(\omega\tau)$

- **For** $\omega << 1/\tau, \quad \phi(\omega) \approx 0^0$
- **For** $\omega >> 1/\tau, \quad \phi(\omega) = \angle - j\tau\omega \approx -90^0$
- **At** $\omega = 1/\tau, \phi(\omega) = \angle(j+1)^{-1} = -45^0$

The magnitude and phase plots of $(j\omega\tau+1)^{-1}$ **are shown in the figure below.**

- $s \to 0, \dfrac{1}{1+\tau s} \to 1$

  $s \to \infty, \dfrac{1}{1+\tau s} \to \dfrac{1}{\tau s}$



Bode plots of $(j\omega\tau + 1)$:

- The Bode plots of $(j\omega\tau + 1)$ are the mirror images of those of $(j\omega\tau + 1)^{-1}$ because

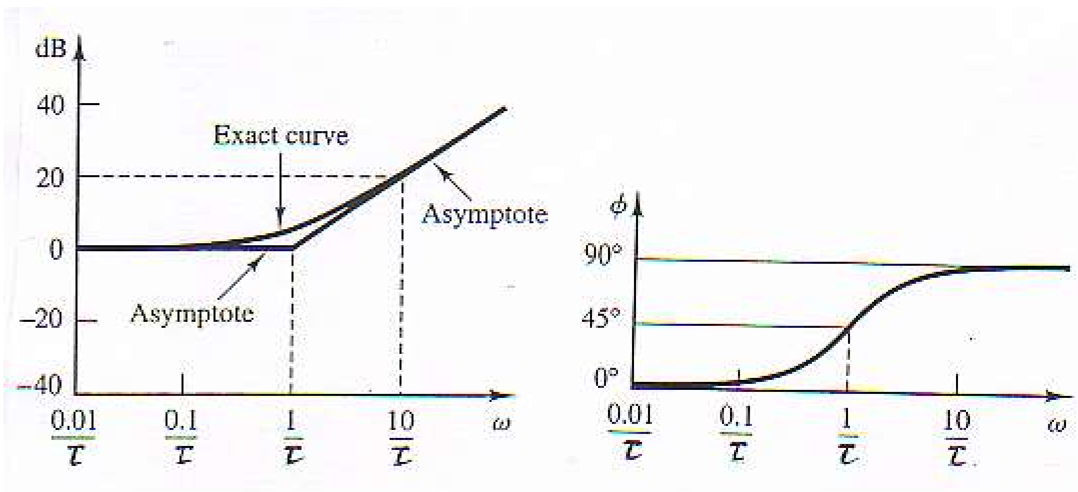
$$20\log|j\omega\tau + 1| = -20\log\left|\frac{1}{j\omega\tau + 1}\right|$$

Magnitude of 1st order pole

$$\angle(j\omega\tau + 1) = -\angle\frac{1}{j\omega\tau + 1}$$

Phase of 1st order pole

- Hence, the magnitude and phase plots are shown below.



4). $$G(j\omega)=\left[\left(\frac{j\omega}{\omega_n}\right)^2+2\zeta\left(\frac{j\omega}{\omega_n}\right)+1\right]^{-1}\qquad G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Recall that ω_n is referred to as the *natural frequency* and ζ the *damping ratio*.

The magnitude and phase of $G(j\omega)$ are respectively

$$20\log|G(j\omega)|=-20\log\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}$$

$$\phi(\omega)=-\tan^{-1}\frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}$$

Magnitude plot:

- **It has two asymptotes:**

✓ **For** $\omega << \omega_n$ **,** $20\log|G(j\omega)| \approx 0$ **dB/dec (0 dB line)** $\Longrightarrow$

✓ **For** $\omega >> \omega_n$,

$$\approx -20\log\left(\frac{\omega}{\omega_n}\right)^2$$

$$20\log|G(j\omega)| \approx -40\log\left(\frac{\omega}{\omega_n}\right) = -40\log\omega + 40\log\omega_n$$

$\omega = \omega_n$ **is the corner frequency. At the corner frequency,**

$$20\log|G(j\omega)| = -20\log(2\zeta) = 20\log\frac{1}{2\zeta}$$

The magnitude plot can be approximated by

✓ **A horizontal line of magnitude of 0 dB before** $\omega = \omega_n$.

✓ **A straight line with slope -40 dB/dec after** $\omega = \omega_n$.

✓ **There will be a peak around the corner frequency and**

$$20\log|G(j\omega_n)| = 20\log\frac{1}{2\zeta}$$

Note that the smaller the ζ the larger the peak around the corner frequency.

Phase plot:

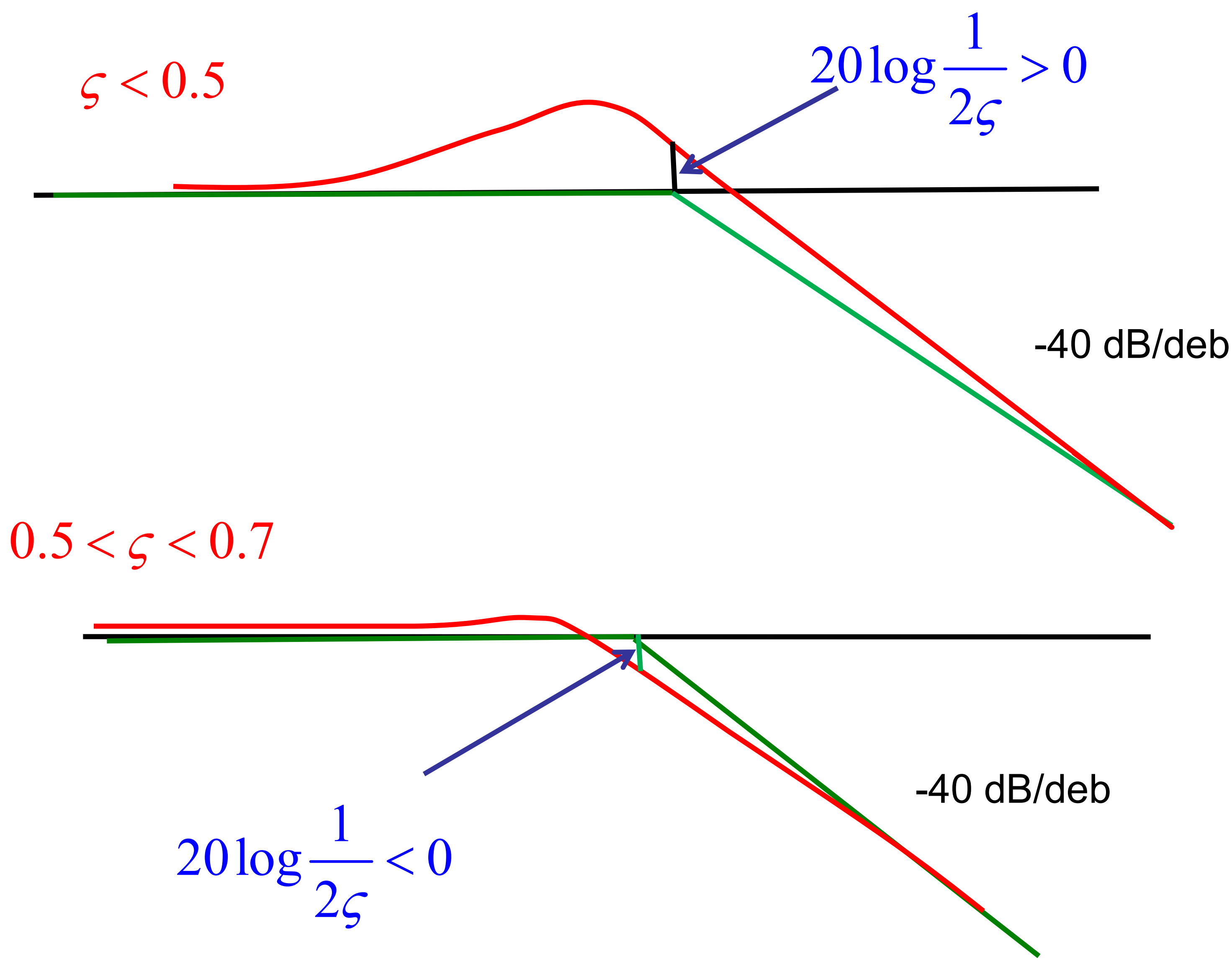
$$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \to 1 \ (s \to 0)$$

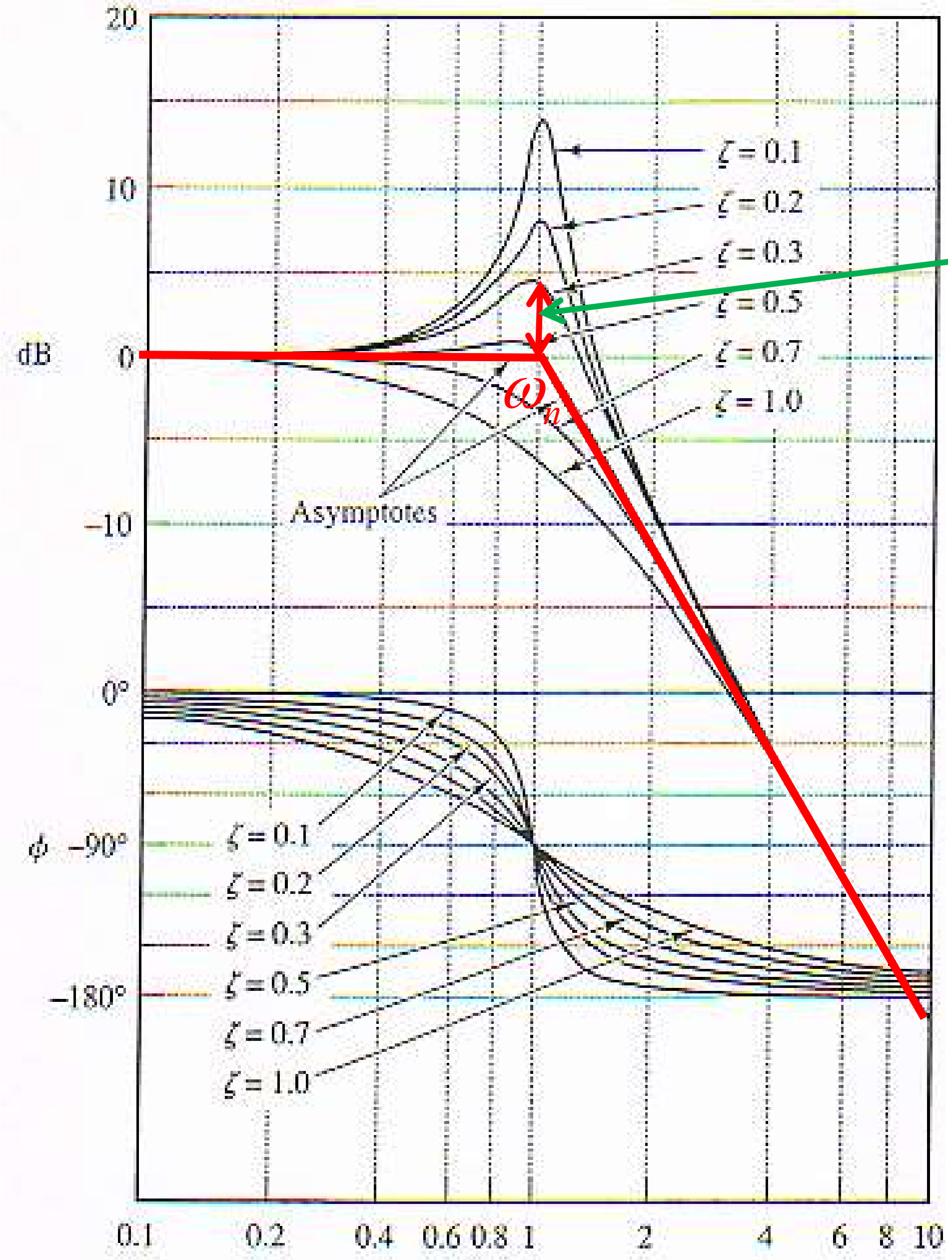
- ✓ **When** $\omega \to 0, \phi(\omega) \to 0^0$

$$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \to \frac{\omega_n^2}{s^2}, s \to \infty$$

- ✓ **When** $\omega \to \infty, \phi(\omega) \to -180^0$
- ✓ **At the corner frequency** $\omega = \omega_n$, $\phi(\omega) = -90^0$ **regardless of** ζ

The phase plot is skew symmetric about the inflection point $\phi(\omega_n) = -90^0$. There are no simple ways to sketch the phase plot in general.


$\varsigma < 0.5$
$20\log\frac{1}{2\varsigma} > 0$
-40 dB/deb
$0.5 < \varsigma < 0.7$
-40 dB/deb
$20\log\frac{1}{2\varsigma} < 0$



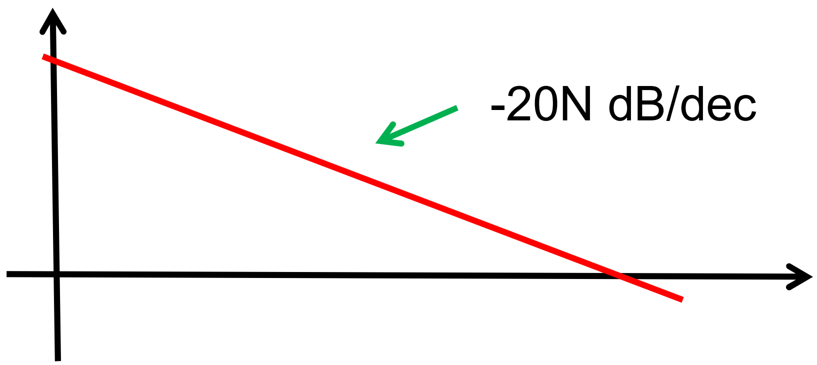
$$20\log\frac{1}{2\varsigma}$$

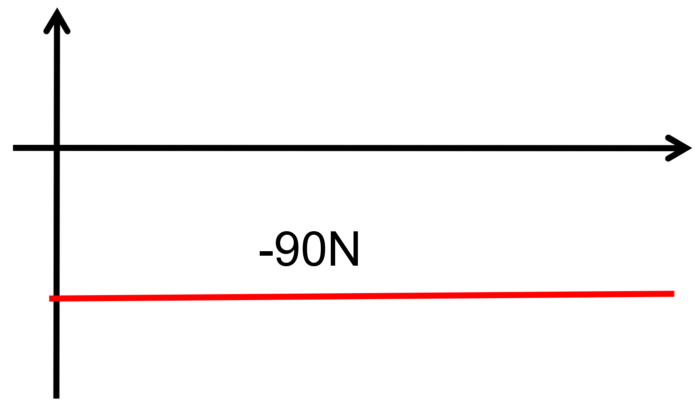
Q: What are the Bode plots of

$$G(j\omega)=\left(\frac{j\omega}{\omega_n}\right)^2+2\zeta\left(\frac{j\omega}{\omega_n}\right)+1\,(\zeta<1)?$$

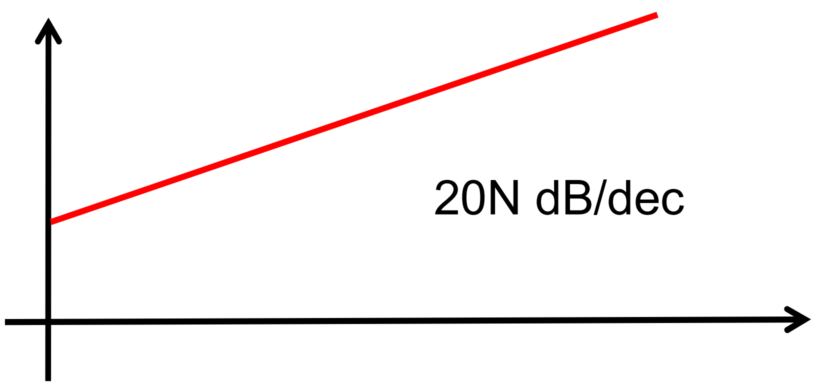
Summary

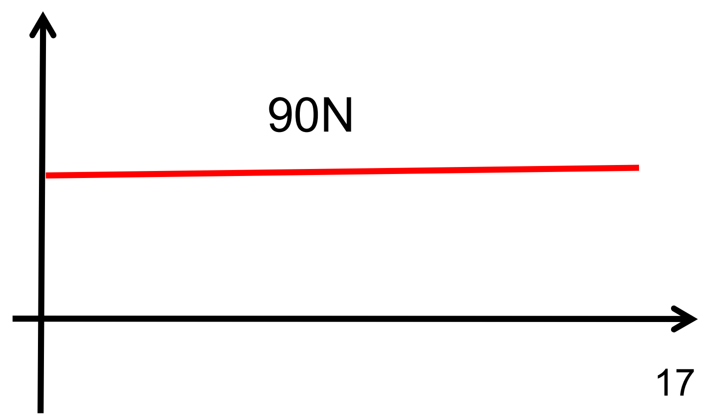
- K (K>0): Magnitude: 20log K; Phase: 0
- Integrators: $\dfrac{K}{s^N}$



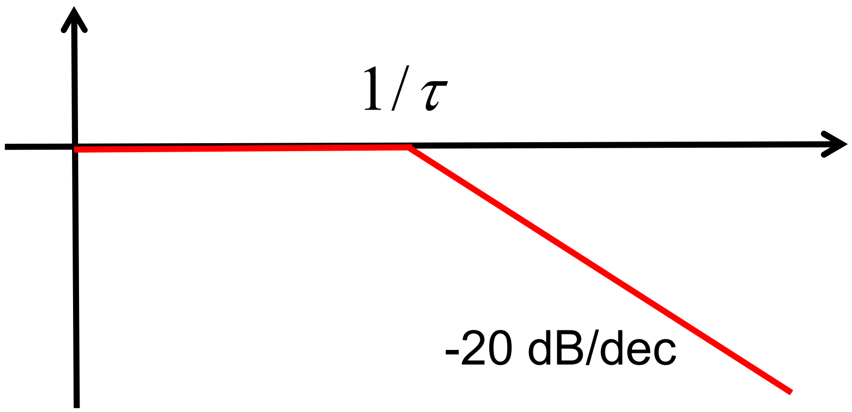


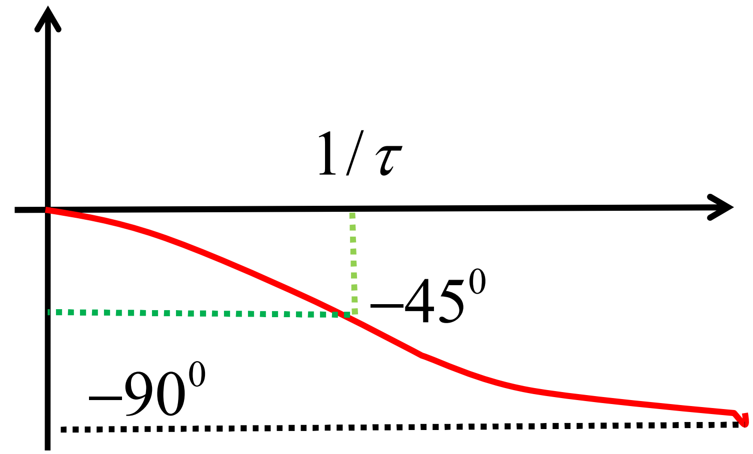
- Ks^N



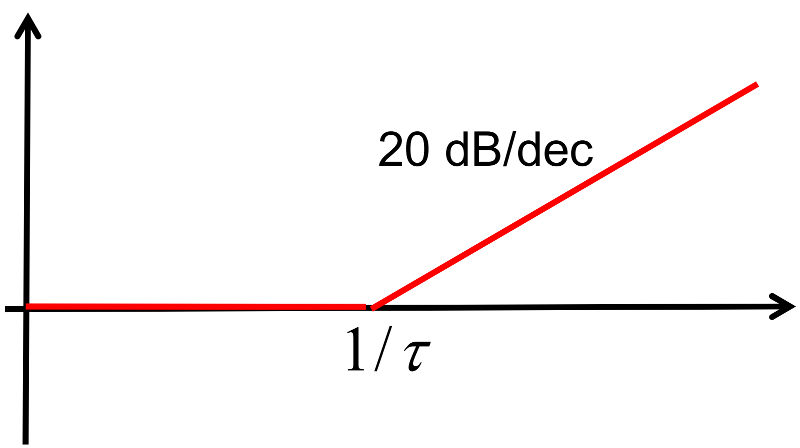


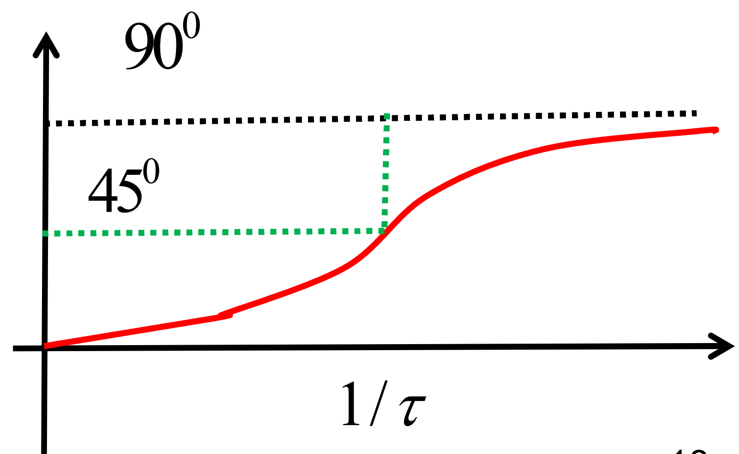
- $\dfrac{1}{1+\tau s}$



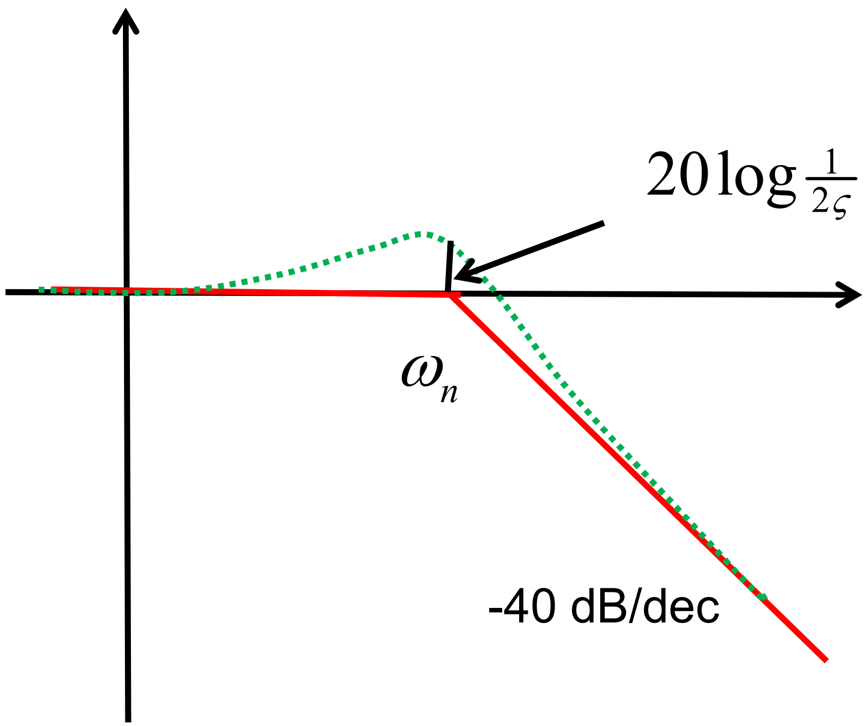


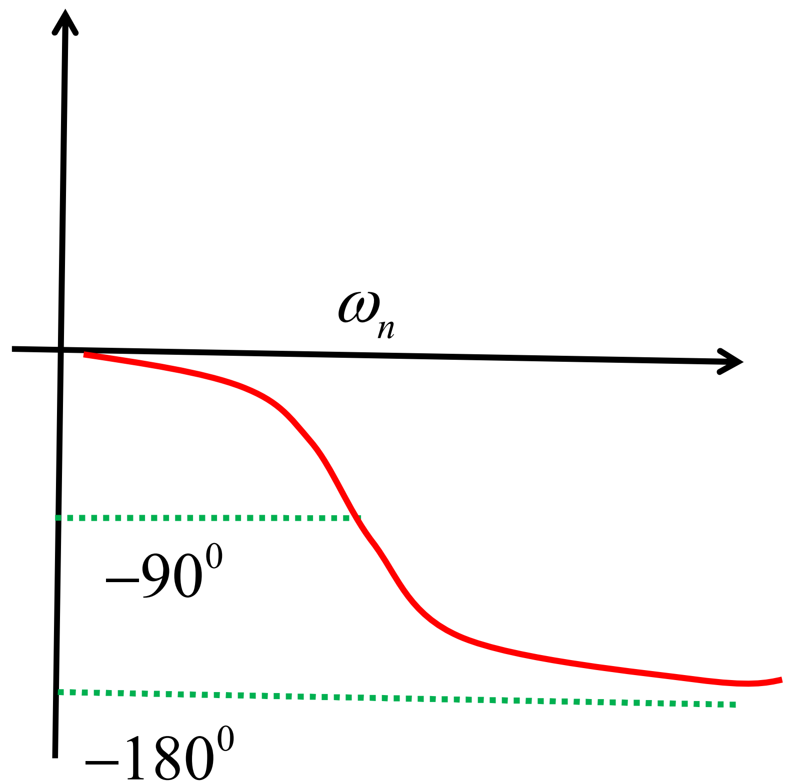
- $1+\tau s$





- $$\frac{1}{\left(\frac{s}{\omega_n}\right)^2 + 2\varsigma\left(\frac{s}{\omega_n}\right) + 1}$$





3. Bode Plots of Transfer Functions

Bode plots of a transfer function (system) can be obtained using Matlab (experiment). However, to interpret the Bode plots, we shall take a look at how basic factors of the transfer function affect the Bode plots.

Example 3.1

$$G(s) = \frac{2000(s+0.5)}{s(s+10)(s+50)}$$

- **Normalization**

$$G(s) = \frac{2(s/0.5+1)}{s(s/10+1)(s/50+1)}$$

Basic factors and their associated corner frequencies:

- ✓ $\frac{2}{s}$
- ✓ $(s/0.5+1)$ **(corner freq. at 0.5 rad/s)**
- ✓ $\frac{1}{s/10+1}$ **(corner freq. at 10 rad/s)**
- ✓ $\frac{1}{s/50+1}$ **(corner freq. at 50 rad/s)**

Ascending order of corner frequencies

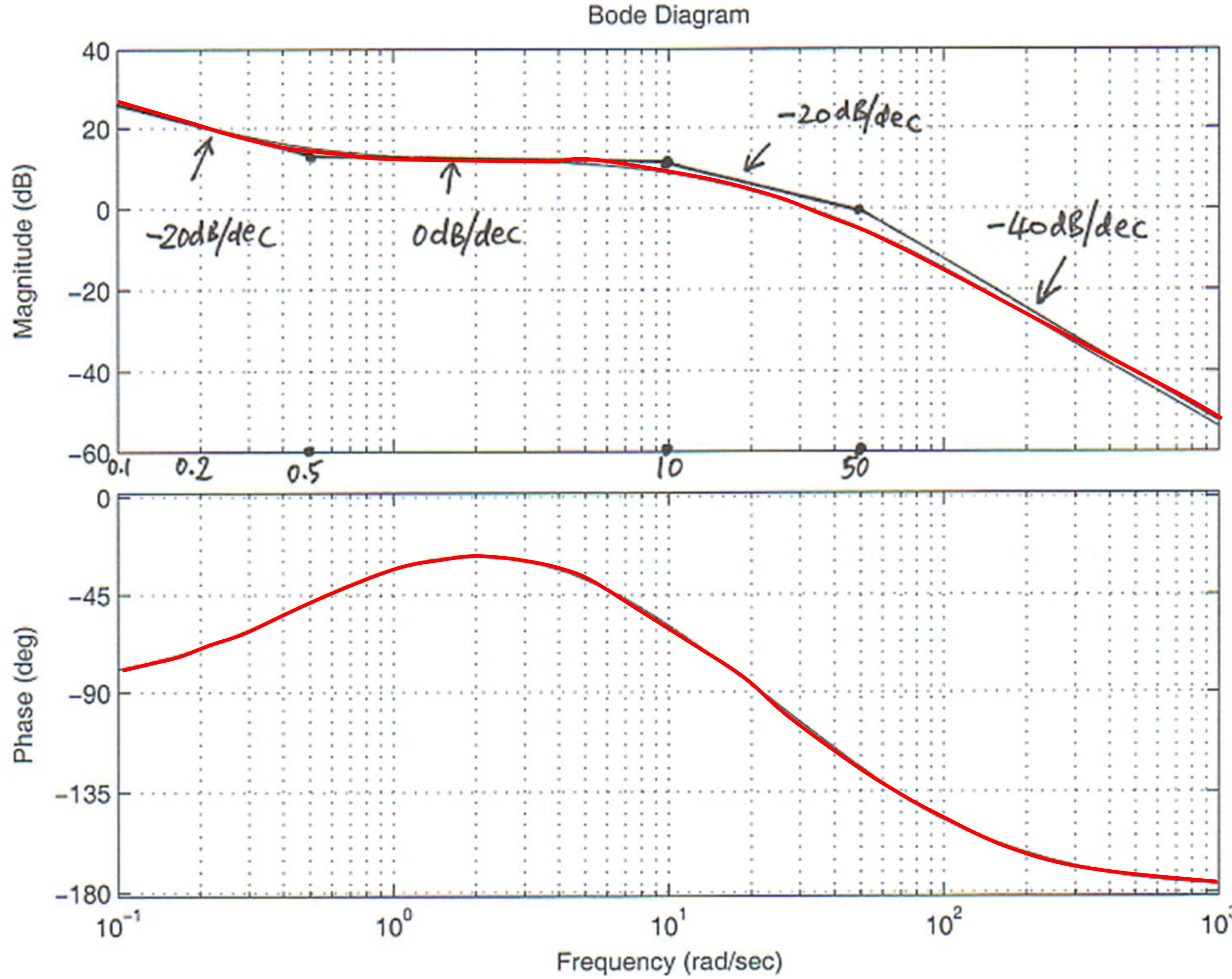
- **Bode plots**

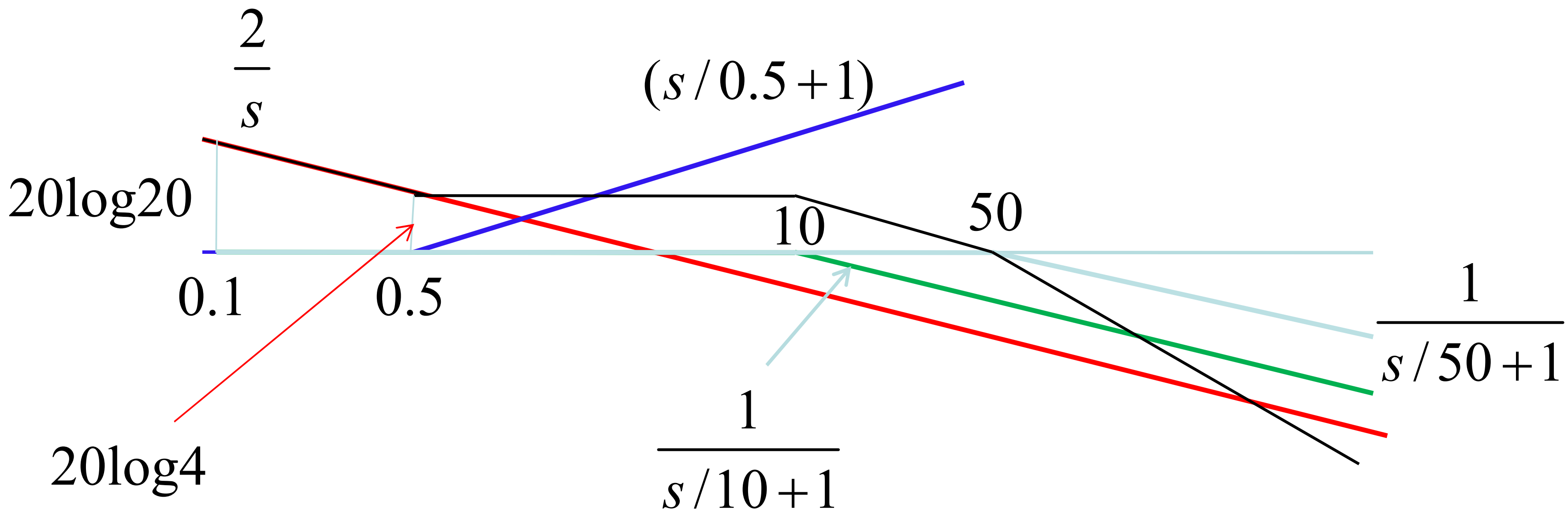
  Magnitude plot is the summation of those of basic factors. Phase is given by

$$\phi(\omega) = -90^0 + \tan^{-1}(\omega/0.5) - \tan^{-1}(\omega/10) - \tan^{-1}(\omega/50)$$

ω (rad/s)	0.1	0.5	1	10	20	50	100	200
$\phi(\omega)(^0)$	-79.4	-48.4	-33.4	-59	-86.7	-124.3	-148	-163

Using Matlab





$\frac{2}{s}$
$(s/0.5+1)$
20log20
10
50
0.1
0.5
$\frac{1}{s/50+1}$
$\frac{1}{s/10+1}$
20log4

Observations:

The asymptotic magnitude plot is a piecewise straight line with gradient change only at each corner frequency:

- ✓ **The –20 dB/dec line at low frequency is due to the integrator.**
- ✓ **If the corner frequency associates with a first order zero, the gradient will be increased by 20 dB/dec.**
- ✓ **If the corner frequency associates with a first order pole, the gradient will be reduced by 20 dB/dec.**
- ✓ **The phase starts from -90^0 due to the integrator. It increases because of the (minimum phase) zero and decreases afterwards due to the poles.**

Example 3.2 (Complex poles)

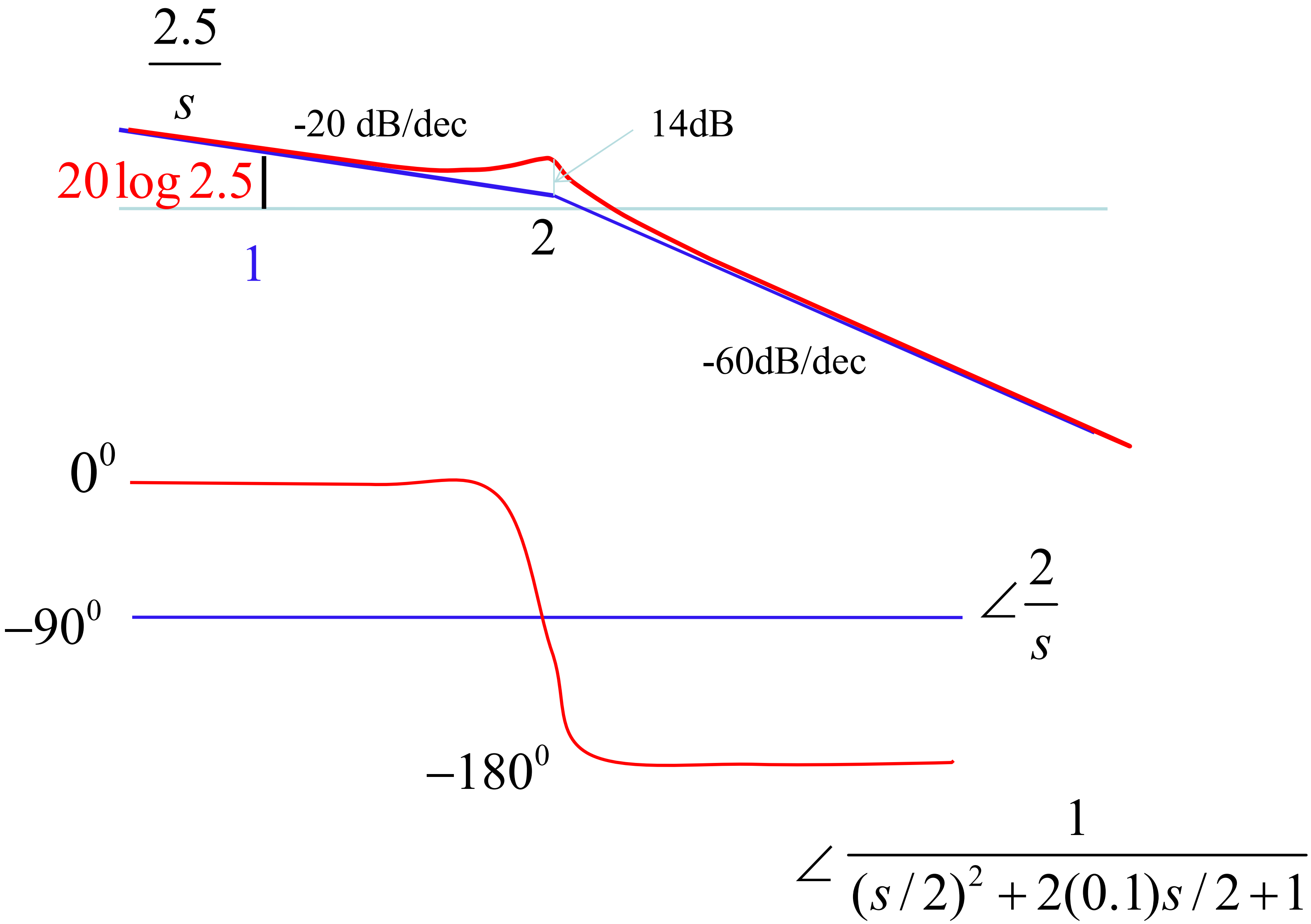
$$G(s) = \frac{10}{s(s^2 + 0.4s + 4)}$$

Normalization:

$$G(s) = \frac{2.5}{s} \frac{1}{s^2/4 + 2(0.1)s/2 + 1}$$

Basic factors:

- ✓ $2.5/s$
- ✓ $\dfrac{1}{(s/2)^2 + 2(0.1)s/2 + 1}$ **(corner freq.=2)**


$\frac{2.5}{s}$
-20 dB/dec
14dB
$20\log 2.5$
2
1
-60dB/dec
0^0
-90^0
$\angle \frac{2}{s}$
-180^0
$\angle \frac{1}{(s/2)^2 + 2(0.1)s/2 + 1}$

Magnitude plot:

✓ Below the corner frequency 2 rad/s draw 2.5/s which is a –20dB/dec line pass through $20\log 2.5$ at $\omega = 1$.

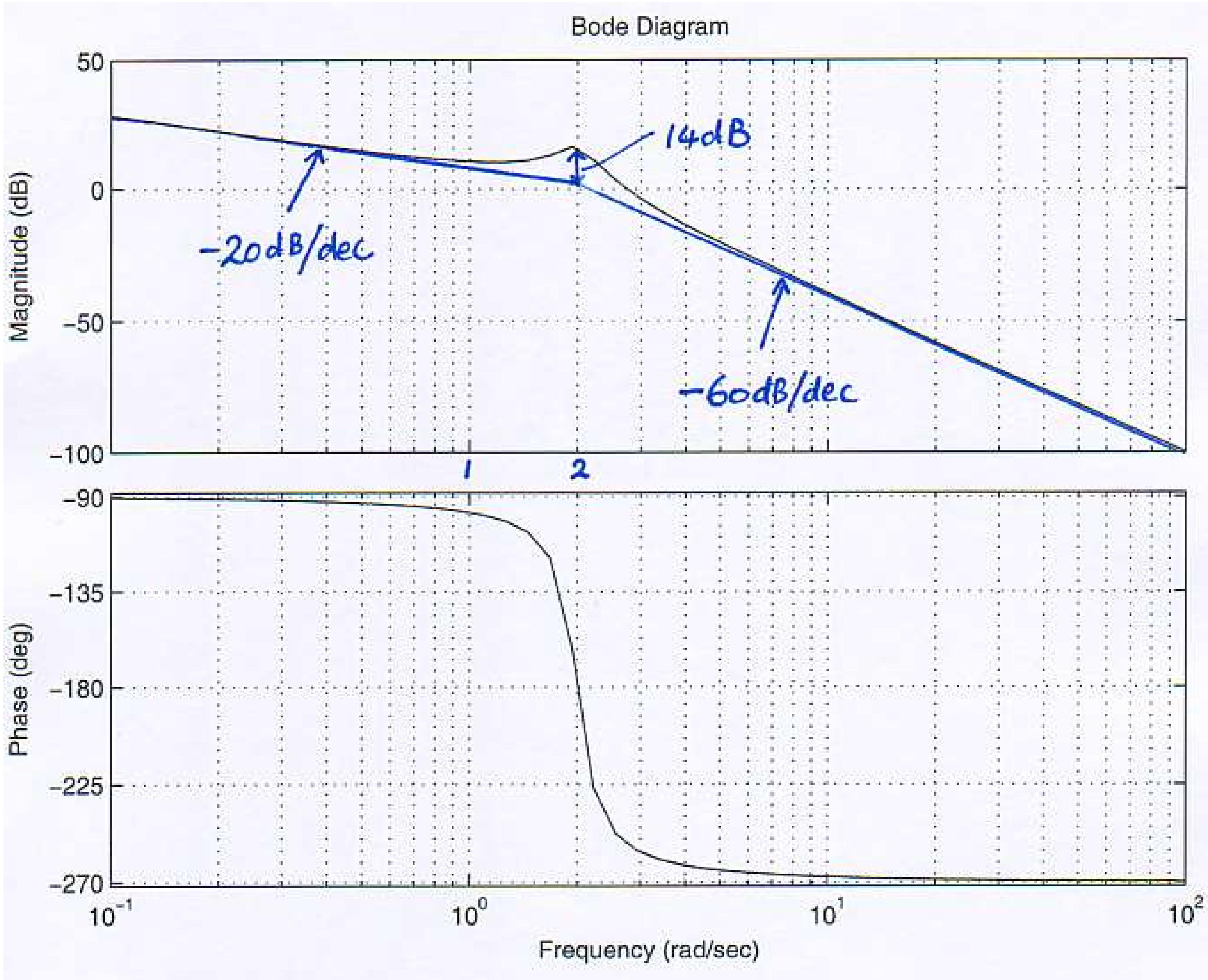
✓ After the corner frequency 2 rad/s, the pair of complex poles contributes another –40 dB/dec, thus results in a line of –60dB/dec.

✓ At $\omega = \omega_n = 2$, make correction based on damping ratio ζ :

$$20\log\frac{1}{2\zeta} = 20\log\frac{1}{0.2} = 14 \text{ dB}$$

Phase: $\phi(\omega) = -90^0 - \tan^{-1}\dfrac{0.2\omega/2}{1-(\omega/2)^2}$

ω	0.2	1	1.8	2.0	2.2	3.0	10
$\phi(\omega)(^0)$	-91	-98	-134	-180	-224	-257	-268


Bode Diagram
50
0
-50
-100
Magnitude (dB)
-20dB/dec
14dB
-60dB/dec
1
2
-90
-135
-180
-225
-270
Phase (deg)
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

Example 3.3:

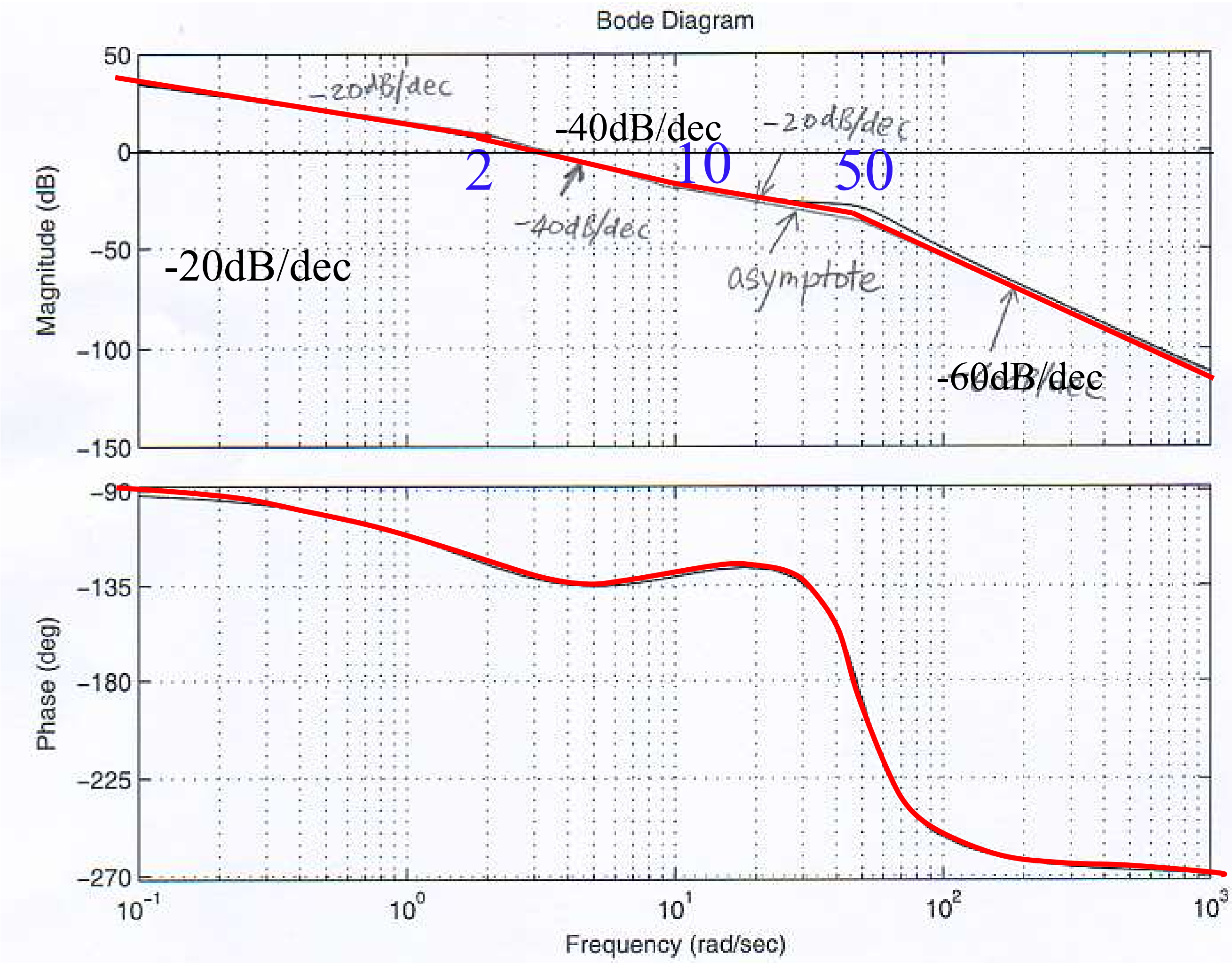
$$G(s)=\frac{2500(s+10)}{s(s+2)(s^2+30s+2500)}$$

$$=\frac{5(0.1s+1)}{s(0.5s+1)[(s/50)^2+0.6(s/50)+1]}$$

Basic factors:

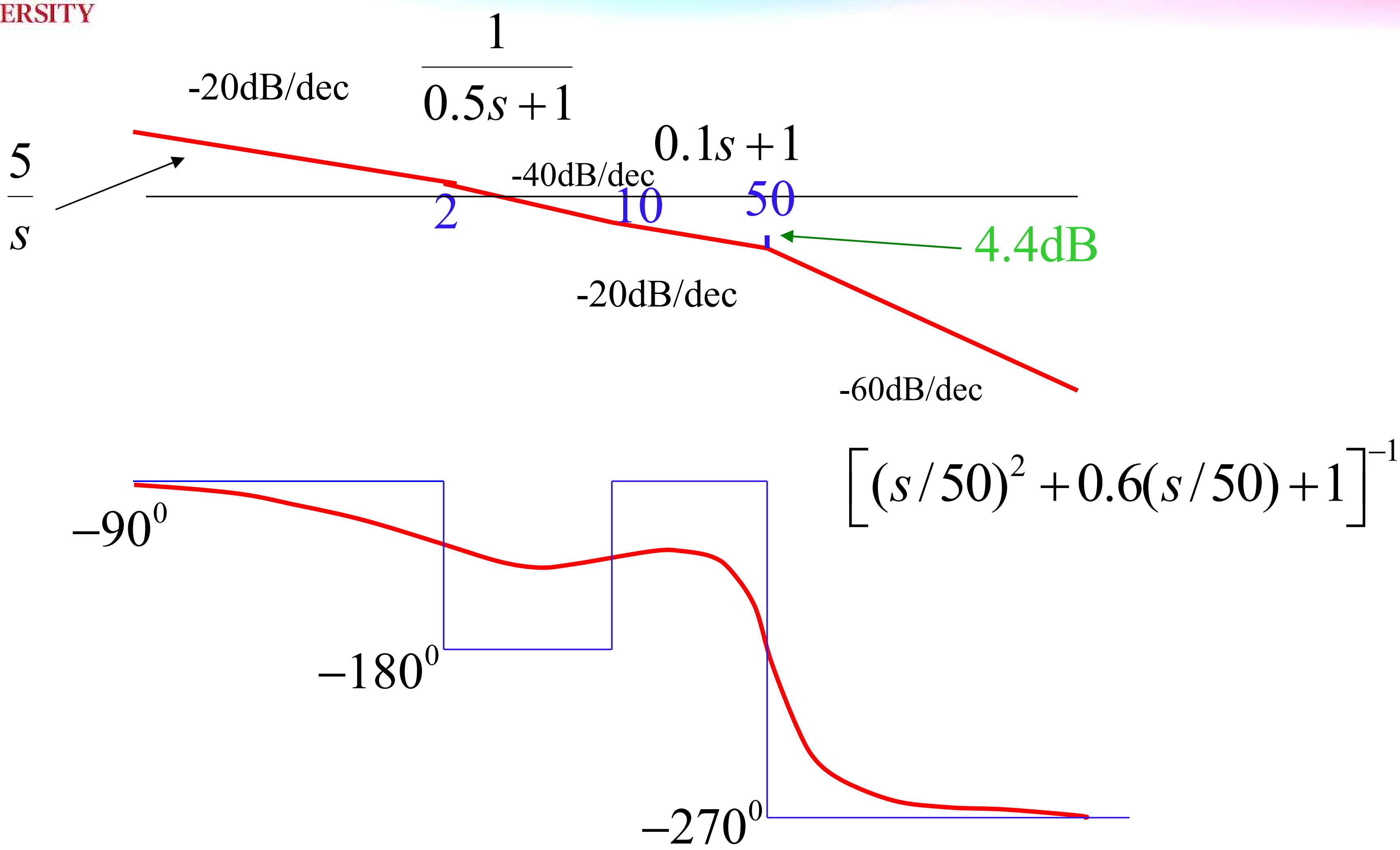
- ✓ $\frac{5}{s}$
- ✓ $\frac{1}{0.5s+1}$ **(corner frequency at 2 rad/s)**
- ✓ $0.1s+1$ **(corner frequency at 10 rad/s)**
- ✓ $\left[(s/50)^2+0.6(s/50)+1\right]^{-1}$ **(corner frequency at 50 rad/s and $\zeta=0.3$).**

Correction at 50 rad/s: $20\log\frac{1}{2\zeta}=4.4dB$



You may try to use the Java applet on the website
http://csd.newcastle.edu.au/control//index.html

to try Bode plots if you don't have Matlab at home.



-20dB/dec
$\frac{1}{0.5s+1}$
$\frac{5}{s}$
$0.1s+1$
-40dB/dec
2
10
50
4.4dB
-20dB/dec
-60dB/dec
$\left[(s/50)^2+0.6(s/50)+1\right]^{-1}$
-90^0
-180^0
-270^0

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Summary

- Bode plots of a transfer function can be derived by summing bode plots of each basic factor
- Change of gradient of asymptote happens at corner frequency:

+20 dB/dec $\Rightarrow$ first order zero $\tau s \pm 1$

-20 dB/dec $\Rightarrow$ first order pole $\dfrac{1}{\tau s + 1}$

+ 40 dB/dec $\Rightarrow$ second order zeros

-40 dB/dec $\Rightarrow$ second order ploes

$$(\tau s + 1)^{\pm 2}$$

or

$$\left[\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1\right]^{\pm 1}$$

Exercises:

3.1 Sketch the Bode plots of the following transfer functions:

$$(a)\ G(s)=\frac{s+a}{s+b}\ \ (a>b>0)$$

$$(b)\ G(s)=\frac{1}{s^2(s+1)}$$

$$(c)\ G(s)=\frac{s}{(s+1)(s+10)}$$