STA 365: Applied Bayesian Statistics

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Week 7A: Hypothesis Testing

Bayesian Approach

- Priors:

$$\pi(H_0), \pi(H_1) = 1 - \pi(H_0)$$

- Posteriors:

$$\pi(H_0|y) = \frac{\pi(H_0)f(y|H_0)}{\pi(H_0)f(y|H_0) + \pi(H_1)f(y|H_1)}$$

$$\pi(H_1|y) = \frac{\pi(H_1)f(y|H_1)}{\pi(H_1)f(y|H_1) + \pi(H_0)f(y|H_0)}$$

- Prior odds:

$$\Omega' = \frac{\pi(H_0)}{\pi(H_1)}$$

- Posterior odds:

$$\Omega'' = \frac{\pi(H_0|y)}{\pi(H_1|y)} = \frac{\pi(H_0)f(y|H_0)}{\pi(H_1)f(y|H_1)} \rightarrow \Omega'' = \Omega' \times LR$$

More on Losses

Bayesian Decision Theory

	H_0 True	H_1 True
Decide H_0	0	L_{II}
Decide H_1	L_I	0

$$\mathrm{E}\left[\text{Decide } H_0\right] = 0 \times \pi(H_0|y) + L_{II} \times \pi(H_1|y)$$

$$\mathrm{E}\left[\text{Decide } H_1\right] = L_I \times \pi(H_0|y) + 0 \times \pi(H_1|y)$$

$$\text{Decide } H_1 \text{ if} L_I \pi(H_0|y) < L_{II} \pi(H_1|y)$$

Odds and Likelihood Ratios

Decide H_1 if

$$\Omega'' < \frac{L_{II}}{L_I}$$

$$\Omega' \times LR < \frac{L_{II}}{L_I}$$

$$LR < \frac{L_{II}/L_I}{\Omega'}$$

The last line suggests that only relative severity of each type of error is what matters.

Example

Example

Let $X \sim \mathrm{N}(\mu, 1)$, $n = 1$, $H_0 : \mu = 0, H_1 : \mu = 1.645$
If $L_{II}/L_I = 1/2$ and $\Omega' = 1$, express the decision rule.

If $X = 1$, what will you do?

Solution

Decide H_1 if $LR < \frac{1}{2}$

$$\begin{aligned} LR &= \frac{f(x|H_0)}{f(x|H_1)} \\ &= \frac{\exp\left[-\frac{x^2}{2}\right]}{\exp\left[-\frac{(x-1.645)^2}{2}\right]} \\ &= \exp\left[-\frac{x^2}{2} + \frac{(x-1.645)^2}{2}\right] \\ &= \exp\left[\frac{(x-1.645)^2}{2} - \frac{x^2}{2}\right] \\ &= \exp\left[\frac{(x-1.645)^2 - x^2}{2}\right] \\ &= \exp\left[\frac{x^2 + 1.645^2 - 2x1.645 - x^2}{2}\right] \\ &= \exp\left[\frac{1.645(1.645-2x)}{2}\right] \end{aligned}$$

If $x = 1$, $LR \approx 0.7 \rightarrow$ decide H_0.

Standard One-Tailed Test

$H_0 : \theta \leq \theta_0$
$H_1 : \theta > \theta_0$
Priors:

$$\pi(H_0) = \int_{-\infty}^{\theta_0} f(\theta) d\theta$$

$$\pi(H_1) = \int_{\theta_0}^{\infty} f(\theta) d\theta$$

Posteriors:

$$\pi(H_0|y) = \int_{-\infty}^{\theta_0} f(\theta|y) d\theta$$

$$\pi(H_1|y) = \int_{\theta_0}^{\infty} f(\theta|y) d\theta$$

Bayes Factor

$$\Omega'' = \Omega' \times LR \rightarrow LR = \frac{\Omega''}{\Omega'} = BF$$

Example

Example

- Let $X \sim \mathrm{N}(\mu, 1)$, $n = 1$, $L_{II}/L_I = 1/2$.
- Let $\mu \sim \mathrm{N}(\mu_0 = 0, \tau_0 = 3)$.
- Consider $H_0 : \mu \leq 0$ vs $H_1 : \mu > 0$.
- Suppose $x = 1$.
- What should you decide?

Example

Solution

Note that $\sigma_0^2 = 1/3$ and $\tau = 1$. And $\mu|x \sim \mathrm{N}(\mu_1, \sigma^2/\tau_1)$, where

$$\mu_1 = \frac{n\tau + \tau_0\mu_0}{n + \tau_0} = \frac{1+0}{4} = 1/4$$

$$\tau_1 = 4$$

$$\Omega' = \frac{\pi(H_0)}{\pi(H_1)} = \frac{\pi(\mu \leq 0)}{\pi(\mu > 0)} = \frac{\Pr\left(Z \leq \frac{0-0}{1/\sqrt{1/3}}\right)}{\Pr\left(Z > \frac{0-0}{1/\sqrt{1/3}}\right)} = 1$$

$$\Omega'' = \frac{\pi(H_0|x)}{\pi(H_1|x)} = \frac{\pi(\mu \leq 0|x)}{\pi(\mu > 0|x)} = \frac{\Pr\left(Z \leq \frac{0-0.25}{0.5}\right)}{\Pr\left(Z > \frac{0-0.25}{0.5}\right)} = \frac{0.31}{0.69} = 0.45$$

Since $L_{II}/L_I = 1/2$ and $\Omega'' \leq 1/2$, decide H_1.

Two Tailed Test

Example

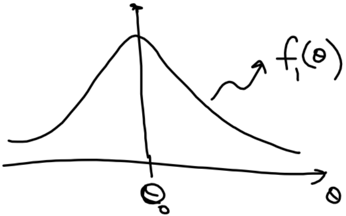
Suppose $X_1, ... X_n \overset{\text{iid}}{\sim} f(x|\theta)$. And consider

$$H_0 : \theta = \theta_0 \text{ vs. } H_1 : \theta \neq \theta_0.$$

Priors

- Priors:

$$\pi(H_0) = \pi_0 \text{ and } \pi(H_1) = \pi_1 \sim f_1(\theta)$$



Two Tailed Test

Remaining Ingredients

- Likelihoods:

$$f(x|H_0) = f(x|\theta_0)$$

$$f(x|H_1) = \int_{-\infty}^{\infty} f(x|\theta) f_1(\theta) d\theta = f_1(x) \rightarrow \text{PPD under } H_1$$

- Full predictive:

$$f(x) = \pi_0 f(x|\theta_0) + \pi_1 f_1(x)$$

- Posteriors:

$$\begin{aligned}
\pi(H_0|x) &= \frac{\pi(H_0) f(x|H_0)}{\pi(H_0) f(x|H_0) + [1 - \pi(H_0)] f(x|H_1)} \\
&= \frac{\pi_0 f(x|\theta_0)}{\pi_0 f(x|\theta_0) + \pi_1 f_1(x)} \\
&= \frac{\pi_0 f(x|\theta_0)}{f(x)} \\
\pi(H_1|x) &= \frac{\pi_1 f_1(x)}{f(x)}
\end{aligned}$$