STA 365: Assignment 1

Professor Boris Babic

Due Date: Sunday, March 6, 2022 (11:59pm via Quercus)

Instructions.

This is the first assignment. It is to be treated like a take-home exam. Accordingly, unlike the homework, this assignment should be completed independently (on your own) without help or consultation from your colleagues, or from anyone else (except the course instructor and TAs). The TAs will be instructed to flag answers that look sufficiently similar.

You are allowed to use the course lecture notes, the recommended textbooks, and external sources. However, you must cite the sources you rely on, and you should be using these sources to inform/help you in developing your own answer, not copying them.

You are encouraged to type out your answers in LaTeX or a word processor. If you need to handwrite your responses, make sure that they are clear and legible, and that you scan a high quality image. What cannot be read will be marked as incomplete.

Where a problem requires the use of R, you must produce your associated R code. While you may use other software, solutions will be provided only in R.

Problem 1. (25 points)

Suppose that $y \sim \text{Exp}(\theta)$ with density

$$\pi(y) = \theta \exp(-\theta y).$$

Suppose that the prior for θ is Gamma(α, β) with density

$$\pi(\theta) \propto \theta^{\alpha-1} \exp(-\beta\theta).$$

(a) We observe only that $y \geq 10$ without observing the exact value of y.

(i) What is the posterior distribution $\pi(\theta|y \geq 10)$, as a function of α and β?

(ii) What is the posterior mean of θ?

(iii) What is the posterior variance of θ?

(10 points)

(b) We are now told that y is exactly 15.

(i) What is the posterior distribution $\pi(\theta|y = 15)$?

(ii) What is the posterior mean of θ?

(iii) What is the posterior variance of θ?

(10 points)

(c) We assign weakly informative priors to θ in both parts (a) and (b) by setting $\alpha = 0.001$ and $\beta = 0.001$. Which posterior variance of θ is higher, the one from part (a) or the one from part (b)?

(5 points)

Problem 2. (25 points)

Suppose $y \sim \text{Galenshore}(a, \theta)$, given by

$$\pi(y) = \frac{2}{\Gamma(a)} \theta^{2a} y^{2a-1} \exp(-\theta^2 y^2)$$

for $y > 0, \theta > 0, a > 0$. Assume that a is known. It may be helpful to know that for this density,

$$\text{E}[y] = \frac{\Gamma(a + 1/2)}{\theta\Gamma(a)}, \text{ and } \text{E}[y^2] = \frac{a}{\theta^2}.$$

You can also assume that $y_1, ..., y_n, y_{n+1} \sim \text{Galenshore}(a, \theta)$ for any $n > 1$.

a) Identify the natural conjugate prior for θ.

b) Find $\pi(\theta|y_1, ...y_n)$.

c) Find $\text{E}[\theta|y_1, ...y_n]$.

d) Find $\pi(\widetilde{y}|y_1, ...y_n)$.

e) Identify the Jeffreys' prior for θ.

(5 points each)

Problem 3. (25 points)

(a) Suppose that we are interested in flight times between Toronto and Washington DC. We will assume that the times are normally distributed and wish to make inferences on μ. We know from previous observations that the average flight time is about 1.5 hours and so we set μ_0 to 1.5. Suppose we set τ_0 to 1. Hence, the prior for μ is $\text{Normal}(\mu_0, \tau_0^2)$.

Now suppose we observe $y_1...y_n$ which can be summarized by $\overline{y} = 1.6$ and $s^2 = 0.01$. Find $\pi(\mu|y_1, ...y_n, \sigma^2 = s^2)$. (5 points)

(b) It is a common problem for measurements to be observed in rounded form. For a simple example, suppose we weigh an object five times and measure weights, rounded to the nearest pound, of 10,10,12,11,9. Assume that unrounded measurements are normally distributed with a noninformative prior distribution on the mean μ and variance σ^2, given by

$$\pi(\mu, \sigma^2) \propto (\sigma^2)^{-1}.$$

(i) Give the posterior distribution for (μ, σ^2) obtained by pretending that the observations are exact, unrounded measurements. (10 points)

(ii) Give the posterior distribution for (μ, σ^2) treating the measurements as rounded. (10 points)

Problem 4. (25 points)

(a) Find Jeffreys' priors for the unknown parameters in the following models:

(i) $Y \sim \mathrm{N}(\mu, \sigma^2)$ with μ known and σ^{-2} (precision) as the parameter that we need to specify the prior for. (5 points)

(ii) $Y \sim \mathrm{N}(\mu, \sigma^2)$ and we need a prior for (μ, σ^{-2}). (10 points)

(b) Let

$$L(\theta, a) = \begin{cases} k_0(\theta - a), & \text{if } \theta \geq a, \\ k_1(a - \theta), & \text{if } \theta < a. \end{cases}$$

Find the Bayes estimator under this loss function. Recall that the Bayes estimator a is the quantity that minimizes the posterior expected loss given by

$$\mathrm{E}[L(\theta, a)|y] = \int_{\Omega} L(\theta, a)\pi(\theta, y)d\theta.$$

(10 points)