Example

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Example 0.1. *Let K be a field of any characteristic and let the polynomial ring $S = K[x_1, \cdots, x_6]$ with the term order of* `DegRevLex`*. Consider the module $M = S/I$, for*

$$I = \langle x_2x_3, x_2x_4, x_5x_6, x_1x_2, x_4x_6, x_2x_5, x_1x_3x_5, x_1x_3x_6, x_3x_4x_5\rangle \tag{1}$$

Using the algorithm `Resolution` *for char$(K) \neq 2$ to the module M, consturct a Schreyer frame from the module M.*

Solution: We first want to compute the Schreyer frame Ξ for M. With this plan, the basis of the first level of the Schreyer frame is I, because it generates the kernel of the map $F_0 \to M$. Hence, we have

$$\mathcal{B}_1 = \{x_2x_3, x_2x_4, x_5x_6, x_1x_2, x_4x_6, x_2x_5, x_1x_3x_5, x_1x_3x_6, x_3x_4x_5\} \tag{2}$$

After ordering it using the `DegRevLex`, the first level basis is:

$$\mathcal{B}_1 = \{x_5x_6, x_4x_6, x_2x_5, x_1x_4, x_2x_3, x_1x_2, x_1x_3x_6, x_3x_4x_5, x_1x_3x_5\} \tag{3}$$

Then, we used the formula in the Lemma **??**, to construct the basis of the next level by finding the monomials that knock a basis into the ideal once at a time.

$$j = 2 : (x_5x_6 : x_4x_6) = \{x_5\} \to \{x_5e_{1,2}\} \tag{4}$$
$$j = 3 : ((e_{1,1}, e_{1,2}) : x_2x_5) = \{x_6, x_4x_6\} = \{x_6\} \to \{x_6e_{1,3}\} \tag{5}$$
$$j = 4 : ((e_{1,1}, \cdots, e_{1,3}) : x_1x_4) = \{x_5x_6, x_6, x_2x_5\} = \{x_6, x_2x_5\} \to \{x_6e_{1,4}, x_2x_5e_{1,4}\} \tag{6}$$
$$j = 5 : ((e_{1,1}, \cdots, e_{1,4}) : x_2x_3) = \{x_5x_6, x_4x_6, x_5, x_1x_4\} = \{x_5, x_4x_6, x_1x_4\} \tag{7}$$
$$\to \{x_5e_{1,5}, x_4x_6e_{1,5}, x_1x_4e_{1,5}\} \tag{8}$$
$$j = 6 : ((e_{1,1}, \cdots, e_{1,5}) : x_1x_2) = \{x_5x_6, x_4x_6, x_5, x_4, x_3\} = \{x_5, x_4, x_3\} \tag{9}$$
$$\to \{x_5e_{1,6}, x_4e_{1,6}, x_3e_{1,6}\} \tag{10}$$
$$j = 7 : ((e_{1,1}, \cdots, e_{1,6}) : x_1x_3x_6) = \{x_5, x_4, x_2x_5, x_4, x_2, x_2\} = \{x_5, x_4, x_2\} \tag{11}$$
$$\to \{x_5e_{1,7}, x_4e_{1,7}, x_2e_{1,7}\} \tag{12}$$
$$j = 8 : ((e_{1,1}, \cdots, e_{1,7}) : x_3x_4x_5) = \{x_6, x_6, x_2, x_1, x_2, x_1x_2, x_1x_6\} = \{x_6, x_2, x_1\} \tag{13}$$
$$\to \{x_6e_{1,8}, x_2e_{1,8}, x_1e_{1,8}\} \tag{14}$$
$$j = 9 : ((e_{1,1}, \cdots, e_{1,8}) : x_1x_3x_5) = \{x_6, x_4x_6, x_2, x_4, x_2, x_2, x_6, x_4\} = \{x_6, x_4, x_2\} \tag{15}$$
$$\to \{x_6e_{1,9}, x_4e_{1,9}, x_2e_{1,9}\} \tag{16}$$

Therefore ,we have our second level basis of Ξ:

$$\begin{aligned}\mathcal{B}_2 = \{&x_5e_{1,2}, x_6e_{1,3}, x_6e_{1,4}, x_2x_5e_{1,4}, x_5e_{1,5}, x_4x_6e_{1,5}, x_1x_4e_{1,5}, x_5e_{1,6}, x_4e_{1,6}, x_3e_{1,6}, x_5e_{1,7},\\ &x_4e_{1,7}, x_2e_{1,7}, x_6e_{1,8}, x_2e_{1,8}, x_1e_{1,8}, x_6e_{1,9}, x_4e_{1,9}, x_2e_{1,9}\}\end{aligned} \tag{17}$$

After ordering it using the `DegRevLex`, the second level basis is:

$$\begin{aligned}\mathcal{B}_2 = \{&x_5e_{1,2}, x_6e_{1,3}, x_6e_{1,4}, x_5e_{1,5}, x_5e_{1,6}, x_4e_{1,6}, x_3e_{1,6}, x_6e_{1,8}, x_5e_{1,7}, x_6e_{1,9}, x_4x_6e_{1,5}, x_4e_{1,7},\\ &x_2e_{1,7}, x_2e_{1,8}, x_1e_{1,8}, x_4e_{1,9}, x_2x_5e_{1,4}, x_2e_{1,9}, x_1x_4e_{1,5}\}\end{aligned} \tag{18}$$

Renaming each basis with second-level subscripts and applying the Lemma ??, we have

$$e_{1,4} : j = 2 : (x_6, x_2x_5) = \{x_6\} \to \{x_6e_{2,17}\} \tag{19}$$
$$e_{1,5} : j = 2 : (x_5, x_4x_6) = \{x_5\} \to \{x_5e_{2,11}\} \tag{20}$$
$$e_{1,5} : j = 3 : ((x_5, x_4x_6) : x_1x_4) = \{x_5, x_6\} \to \{x_6e_{2,19}, x_5e_{2,19}\} \tag{21}$$
$$e_{1,6} : j = 2 : (x_5 : x_4) = \{x_5\} \to \{x_5e_{2,6}\} \tag{22}$$
$$e_{1,6} : j = 3 : ((x_5, x_4) : x_3) = \{x_5, x_4\} \to \{x_5e_{2,7}, x_4e_{2,7}\} \tag{23}$$
$$e_{1,7} : j = 2 : (x_5, x_4) = \{x_5\} \to \{x_5e_{2,12}\} \tag{24}$$
$$e_{1,7} : j = 3 : ((x_5, x_4) : x_2) = \{x_5, x_4\} \to \{x_5e_{2,13}, x_4e_{2,13}\} \tag{25}$$
$$e_{1,8} : j = 2 : (x_6, x_2) = \{x_6\} \to \{x_6e_{2,14}\} \tag{26}$$
$$e_{1,8} : j = 3 : ((x_6, x_2) : x_1) = \{x_6, x_2\} \to \{x_6e_{2,15}, x_2e_{2,15}\} \tag{27}$$
$$e_{1,9} : j = 2 : (x_6, x_4) = \{x_6\} \to \{x_6e_{2,16}\} \tag{28}$$
$$e_{1,9} : j = 3 : ((x_6, x_4) : x_2) = \{x_6, x_4\} \to \{x_6e_{2,18}, x_4e_{2,18}\} \tag{29}$$

Then, we have the third level basis $\mathcal{B}_3$ of Ξ:

$$\begin{aligned}\mathcal{B}_3 = \{&x_6e_{2,17}, x_5e_{2,11}, x_6e_{2,19}, x_5e_{2,19}, x_5e_{2,6}, x_5e_{2,7}, x_4e_{2,7}, x_5e_{2,12}, x_5e_{2,13}, x_4e_{2,13}, x_6e_{2,14},\\ &x_6e_{2,15}, x_2e_{2,15}, x_6e_{2,16}, x_6e_{2,18}, x_4e_{2,18}\}\end{aligned} \tag{30}$$

After ordering it using the `DegRevLex`, the third level basis is:

$$\begin{aligned}\mathcal{B}_3 = \{&x_5e_{2,6}, x_5e_{2,7}, x_4e_{2,7}, x_5e_{2,11}, x_6e_{2,14}, x_5e_{2,12}, x_6e_{2,15}, x_6e_{2,16}, x_6e_{2,17}, x_5e_{2,13}, x_6e_{2,18},\\ &x_6e_{2,19}, x_4e_{2,13}, x_5e_{2,19}, x_2e_{2,15}, x_4e_{2,18}\}\end{aligned} \tag{31}$$

Applying the Lemma ?? once again, we find the following bases:

$$e_{2,19} : j = 2 : (x_6, x_5) = \{x_6\} \to \{x_6e_{3,14}\} \tag{32}$$
$$e_{2,7} : j = 2 : (x_5, x_4) = \{x_5\} \to \{x_5e_{3,3}\} \tag{33}$$
$$e_{2,13} : j = 2 : (x_5, x_4) = \{x_5\} \to \{x_5e_{3,13}\} \tag{34}$$
$$e_{2,15} : j = 2 : (x_6, x_2) = \{x_6\} \to \{x_6e_{3,15}\} \tag{35}$$
$$e_{2,18} : j = 2 : (x_6, x_4) = \{x_6\} \to \{x_6e_{3,16}\} \tag{36}$$
$$\tag{37}$$

The fourth-level basis $\mathcal{B}_4$ of Ξ is

$$\mathcal{B}_4 = \{x_6e_{3,14}, x_5e_{3,3}, x_5e_{3,13}, x_6e_{3,15}, x_5e_{3,16}\} \tag{38}$$

After ordering it using the `DegRevLex`, the last level basis is:

$$\mathcal{B}_4 = \{x_5e_{3,3}, x_6e_{3,14}, x_5e_{3,13}, x_6e_{3,15}, x_5e_{3,16}\} \tag{39}$$

Now, applying the Lemma ?? could not give us any basis from $\mathcal{B}_4$, then we know that we had found the complete Schreyer frame Ξ.

$$\mathcal{B}_1 = \{x_5x_6, x_4x_6, x_2x_5, x_1x_4, x_2x_3, x_1x_2, x_1x_3x_6, x_3x_4x_5, x_1x_3x_5\} \tag{40}$$
$$\mathcal{B}_2 = \{x_5e_{1,2}, x_6e_{1,3}, x_6e_{1,4}, x_5e_{1,5}, x_5e_{1,6}, x_4e_{1,6}, x_3e_{1,6}, x_6e_{1,8}, x_5e_{1,7}, x_6e_{1,9}, x_4x_6e_{1,5}, x_4e_{1,7}, \tag{41}$$
$$x_2e_{1,7}, x_2e_{1,8}, x_1e_{1,8}, x_4e_{1,9}, x_2x_5e_{1,4}, x_2e_{1,9}, x_1x_4e_{1,5}\} \tag{42}$$
$$\mathcal{B}_3 = \{x_5e_{2,6}, x_5e_{2,7}, x_4e_{2,7}, x_5e_{2,11}, x_6e_{2,14}, x_5e_{2,12}, x_6e_{2,15}, x_6e_{2,16}, x_6e_{2,17}, x_5e_{2,13}, x_6e_{2,18}, \tag{43}$$
$$x_6e_{2,19}, x_4e_{2,13}, x_5e_{2,19}, x_2e_{2,15}, x_4e_{2,18}\} \tag{44}$$
$$\mathcal{B}_4 = \{x_5e_{3,3}, x_6e_{3,14}, x_5e_{3,13}, x_6e_{3,15}, x_5e_{3,16}\}, \tag{45}$$

and we know

$$\Xi : S^5 \xrightarrow{\xi_3} S^{16} \xrightarrow{\xi_2} S^{15} \xrightarrow{\xi_1} S^9 \to 0 \tag{46}$$
$$\tag{47}$$

Example 0.2. *Let K be a field of any characteristic and let the polynomial ring $S = K[x_1, \cdots, x_6]$ with the term order of* `DegRevLex`. *Consider the graded module $M = S/I$, for*

$$I = \langle x_2x_3, x_2x_4, x_5x_6, x_1x_2, x_4x_6, x_2x_5, x_1x_3x_5, x_1x_3x_6, x_3x_4x_5\rangle \tag{48}$$

Using the algorithm `Resolution` *for $char(K) \neq 2$ to the module M, find the minimal Schreyer resolution.*

Solution: Given the frame Ξ, we could implement the Algorithm **??** to construct a Schreyer resolution Φ corresponding to the frame. To start the algorithm, we should first identify the input by checking whether the first level of our frame gives a reduced Gröbner basis. Since the minimal generators of I are all monomials, then for every $m \in \langle \mathrm{LT}(I)\rangle$, m is a linear combination of $\mathrm{LT}(\mathcal{C}_1)$. Therefore, $\hat{\mathcal{C}_1} = \mathcal{C}_1$. Then we want to order the union of all bases we obtained in the frame Ξ.

$$\begin{aligned}\mathcal{B} = \{&x_5x_6, x_4x_6, x_2x_5, x_1x_4, x_2x_3, x_1x_2, x_5e_{1,2}, x_6e_{1,3}, x_6e_{1,4}, x_1x_3x_6, x_3x_4x_5, x_5e_{1,5}, x_1x_3x_5,\\ &x_5e_{1,6}, x_4e_{1,6}, x_3e_{1,6}, x_6e_{1,8}, x_5e_{1,7}, x_6e_{1,9}, x_4x_6e_{1,5}, x_4e_{1,7}, x_2e_{1,7}, x_2e_{1,8}, x_1e_{1,8}, x_4e_{1,9},\\ &x_2x_5e_{1,4}, x_5e_{2,6}, x_2e_{1,9}, x_5e_{2,7}, x_1x_4e_{1,5}, x_4e_{2,7}, x_5e_{2,11}, x_6e_{2,14}, x_5e_{2,12}, x_6e_{2,15}, x_6e_{2,16},\\ &x_6e_{2,17}, x_5e_{2,13}, x_6e_{2,18}, x_6e_{2,19}, x_4e_{2,13}, x_5e_{2,19}, x_2e_{2,15}, x_4e_{2,18}, x_5e_{3,3}, x_6e_{3,14}, x_5e_{3,13},\\ &x_6e_{3,15}, x_5e_{3,16}\}\end{aligned} \tag{49}$$

Given $\bar{\mathcal{C}_1}$ and $\mathcal{B}$, we could apply the algorithm `Resolution`. Since in Algorithm **??**, we just add the first-level bases in $\mathcal{B}$ to $\mathcal{C}_1$ and $\mathcal{H}_1$, then we have:

$$\mathcal{C}_1 = \{x_5x_6, x_4x_6, x_2x_5, x_1x_4, x_2x_3, x_1x_2, x_1x_3x_6, x_3x_4x_5, x_1x_3x_5\} \tag{50}$$

$$\mathcal{H}_1 = \{x_5x_6, x_4x_6, x_2x_5, x_1x_4, x_2x_3, x_1x_2, x_1x_3x_6, x_3x_4x_5, x_1x_3x_5\} \tag{51}$$

Notice that since there are some bases in the second level that come before the first-level bases in the ordered list $\mathcal{B}$, we actually need to first apply the algorithm to them before applying the algorithm to the larger first-level bases. For example, when we process the basis $x_5e_{1,2}$, the $\mathcal{C}_1$ we are using is $\{x_5x_6, x_4x_6, x_2x_5, x_1x_4, x_2x_3, x_1x_2\}$. However, in this example, we do not have any bases in the second level that cannot be vanished using the smaller bases in the first level. Similarly, we also have some bases from the third level appearing before the second-level bases in the list. They are

$$x_5e_{2,6},\ x_5e_{2,7} \tag{52}$$

Here, we need to be especially careful with the order in the list, since they are using different $\mathcal{C}_2$ when they are computed. For $x_5e_{2,6}$, we have

$$\begin{aligned}\mathcal{C}_2 = \{&x_5e_{1,2} - x_4e_{1,1}, x_6e_{1,3} - x_2e_{1,1}, x_6e_{1,4} - x_1e_{1,2}, x_5e_{1,5} - x_3e_{1,3}, x_5e_{1,6} - x_1e_{1,3},\\ &x_4e_{1,6} - x_2e_{1,4}, x_3e_{1,6} - x_1e_{1,5}, x_6e_{1,8} - x_3x_4e_{1,1}, x_5e_{1,7} - x_1x_3e_{1,1},\\ &x_6e_{1,9} - x_1x_3e_{1,1}, x_4x_6e_{1,5} - x_2x_3e_{1,2}, x_4e_{1,7} - x_1x_3e_{1,2}, x_2e_{1,7} - x_1x_6e_{1,5},\\ &x_2e_{1,8} - x_4x_5e_{1,5}, x_1e_{1,8} - x_3x_5e_{1,4}, x_4e_{1,9} - x_3x_4e_{1,4}, -x_4x_5e_{1,6} + x_2x_5e_{1,4}\}\end{aligned} \tag{53}$$

Applying Algorithm **??** `Reduce` to $x_5e_{2,6}$ and the corresponding $\mathcal{C}_2$, we have f, g to start with $f = x_4x_5e_{1,6} - x_2x_5e_{1,4}$ and $g = x_5e_{2,6}$, since $\varphi(e_{2,6}) = -x_2e_{1,4} + x_4e_{1,6}$. Based on the term order on R-module, $lm(f) = x_4x_5e_{1,6}$. We have $lm(f) \in in\langle \mathcal{C}_2\rangle$, since $x_4x_5e_{1,6} = (-1) \cdot x_2x_5e_{1,4}$. Therefore, in the first iteration of the while loop, we would get $h = -x_4x_5e_{1,6} + x_2x_5e_{1,4}$, $f = 0$, and $g = x_5e_{2,6} + e_{2,17}$. Because $f = 0$, this pair (f, g), would be input in the next step of Algorithm **??** `Resolution`. Because $f = 0$, we could update C_3 and H_3 as follows.

$$\begin{aligned}\mathcal{C}_3 &= \{e_{2,17} + x_5e_{2,6}\}\\ \mathcal{H}_3 &= \{e_{2,17} + x_5e_{2,6}\}\end{aligned} \tag{54}$$

Then for $x_5e_{2,7}$, we have

$$
\begin{aligned}
\mathcal{C}_2 = \{&x_5e_{1,2} - x_4e_{1,1}, x_6e_{1,3} - x_2e_{1,1}, x_6e_{1,4} - x_1e_{1,2}, x_5e_{1,5} - x_3e_{1,3}, x_5e_{1,6} - x_1e_{1,3},\\
&x_4e_{1,6} - x_2e_{1,4}, x_3e_{1,6} - x_1e_{1,5}, x_6e_{1,8} - x_3x_4e_{1,1}, x_5e_{1,7} - x_1x_3e_{1,1},\\
&x_6e_{1,9} - x_1x_3e_{1,1}, x_4x_6e_{1,5} - x_2x_3e_{1,2}, x_4e_{1,7} - x_1x_3e_{1,2}, x_2e_{1,7} - x_1x_6e_{1,5},\\
&x_2e_{1,8} - x_4x_5e_{1,5}, x_1e_{1,8} - x_3x_5e_{1,4}, x_4e_{1,9} - x_3x_4e_{1,4}, -x_4x_5e_{1,6} + x_2x_5e_{1,4},\\
&x_2e_{1,9} - x_1x_3e_{1,3}\}
\end{aligned}
\tag{55}
$$

This time, we have $f = x_3x_5e_{1,6} - x_1x_5e_{1,5}$ and $g = x_5e_{2,7}$, since $\varphi(e_{2,7}) = x_3e_{1,6} - x_1e_{1,5}$. Based on the term order on R-module, $lm(f) = x_3x_5e_{1,6}$. We have $lm(f) \in in\langle \mathcal{C}_2 \rangle$, since $x_3x_5e_{1,6} = x_3 \cdot x_5e_{1,6}$. Therefore, in the first iteration of the while loop, we would get $h = x_5e_{1,6} - x_1e_{1,3}$, $f = -x_1x_5e_{1,5} + x_1x_3e_{1,3}$, and $g = x_5e_{2,7} - x_3e_{2,5}$. Then, again $lm(f) = x_1x_5e_{1,5} \in \langle \mathcal{C}_2 \rangle$ since $x_1x_5e_{1,5} = x_1 \cdot x_5e_{1,5}$. Therefore, in the second iteration of the while loop, we would get $h = x_5e_{1,5} + x_3e_{1,3}$, $f = 0$, and $g = x_5e_{2,7} - x_3e_{2,5} + x_1e_{2,4}$. Because $f = 0$, this pair (f, g), would be input in the next step of Algorithm **??** `Resolution`. Because $f = 0$, we could update C_3 and H_3 as follows.

$$
\begin{aligned}
\mathcal{C}_3 &= \{e_{2,17} + x_5e_{2,6}, x_5e_{2,7} - x_3e_{2,5} + x_1e_{2,4}\}\\
\mathcal{H}_3 &= \{e_{2,17} + x_5e_{2,6}, x_5e_{2,7} - x_3e_{2,5} + x_1e_{2,4}\}
\end{aligned}
\tag{56}
$$

Next, we could look at the remaining bases from level 2 to compute the complete $\mathcal{C}_2$.

$$
\begin{aligned}
\mathcal{C}_2 = \{&x_5e_{1,2} - x_4e_{1,1}, x_6e_{1,3} - x_2e_{1,1}, x_6e_{1,4} - x_1e_{1,2}, x_5e_{1,5} - x_3e_{1,3}, x_5e_{1,6} - x_1e_{1,3},\\
&x_4e_{1,6} - x_2e_{1,4}, x_3e_{1,6} - x_1e_{1,5}, x_6e_{1,8} - x_3x_4e_{1,1}, x_5e_{1,7} - x_1x_3e_{1,1},\\
&x_6e_{1,9} - x_1x_3e_{1,1}, x_4x_6e_{1,5} - x_2x_3e_{1,2}, x_4e_{1,7} - x_1x_3e_{1,2}, x_2e_{1,7} - x_1x_6e_{1,5},\\
&x_2e_{1,8} - x_4x_5e_{1,5}, x_1e_{1,8} - x_3x_5e_{1,4}, x_4e_{1,9} - x_3x_4e_{1,4}, -x_4x_5e_{1,6} + x_2x_5e_{1,4},\\
&x_2e_{1,9} - x_1x_3e_{1,3}, -x_3x_4e_{1,6} + x_1x_4e_{1,5}\},
\end{aligned}
\tag{57}
$$

and $\mathcal{H}_2 = \mathcal{C}_2$ since there is no reductum produced from f when divided by C_1

Since all third-level bases are smaller compare to the forth-level bases using the term order on the resolution, we could now compute the compute $\mathcal{C}_3$ and $\mathcal{H}_3$ as follows.

$$
\begin{aligned}
\mathcal{C}_3 = \{&e_{2,17} + x_5e_{2,6}, x_5e_{2,7} - x_3e_{2,5} + x_1e_{2,4}, e_{2,19} + x_4e_{2,7}, x_5e_{2,11} - x_4x_6e_{2,4} - x_3x_4e_{2,2} + e_{2,1},\\
&x_6e_{2,14} - x_2e_{2,8} + x_5e_{2,11} + e_{2,1}, x_5e_{2,12} - x_2e_{2,9} + x_1x_3e_{2,1}, x_6e_{2,15} - x_1e_{2,8} + x_3x_5e_{2,3} + x_1x_3e_{2,1}\}\\
\mathcal{H}_3 = \{&e_{2,17} + x_5e_{2,6}, x_5e_{2,7} - x_3e_{2,5} + x_1e_{2,4}, e_{2,19} + x_4e_{2,7}, x_5e_{2,11} - x_4x_6e_{2,4} - x_3x_4e_{2,2} + e_{2,1},\\
&x_6e_{2,14} - x_2e_{2,8} + x_5e_{2,11} + e_{2,1}, x_5e_{2,12} - x_2e_{2,9} + x_1x_3e_{2,1}, x_6e_{2,15} - x_1e_{2,8} + x_3x_5e_{2,3} + x_1x_3e_{2,1}\}
\end{aligned}
\tag{58}
$$