Computing and Minimalizing Schreyer Resolution

Senior Project

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May 10, 2022

Abstract

The main target of this project is to introduce an algorithm to compute a Schreyer resolution of a module over a polynomial ring, and the strategy we used is compute a Schreyer frame of the given model first and then fill in the information to get a Schreyer resolution. The definition and properties of the Gröbner basis and syzygy are investigated in the paper. They play an important role in computing the Schreyer resolutions since they help compute the kernels of the homomorphisms and the bases of the free modules. At the end of the paper, an algorithm minimalizing the free resolutions was discussed. Examples of computing and minimalizing a free module are included after giving the algorithms.

1 Introduction

Schreyer resolution of a module is an exact sequence of free modules over a polynomial ring, and computing the Schreyer resolution of a free module is an essential way to understand the invariants of the module, and It tell us how far an arbitrary R-module is from being a free R-module. This paper investigates an algorithm for computing and minimalizing a Schreyer resolution of a free module found by La Scala and Stillman [LS98]. The algorithm gives a minimal resolution in the graded case.

2 Preliminaries

Before discussing the Schreyer resolution, the definition of Gröbner bases and syzygies should first be introduced, since they are the primary building blocks of the resolution. The Gröbner bases and syzygies are of importance to build the differentials connecting the free modules.

2.1 Gröbner Basis

Since we seek for a basis for the polynomial ring with multiple variables, we want to define terms such as leading terms and leading monomials, so we need to understand the ordering of the monomials in $k[x_1, \cdots, x_n]$ for some field k.

Definition 2.1 (**Degree**)**.** Let $\gamma = (\gamma_1, \cdots, \gamma_n)$ be a n-tuple in $\mathbb{Z}^n_{\geq 0}$, then $x^\gamma \in k[x_1, \cdots, x_n]$ represents $x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_n^{\gamma_n}$. Then, we define the **degree** of a polynomial to be the sum of all components of the variables. For example, the degree of x^γ is $|(\gamma_1, \cdots, \gamma_n)| = \gamma_1 + \cdots + \gamma_n$.

Definition 2.2. For the polynomial ring $k[x_1, \cdots, x_n]$ with some field k, the order one the variables is defined as

$$x_1 > x_2 > \cdots > x_n \tag{1}$$

By convention, we always write the variables in a monomial and the monomials in a polynomial in descending order. Next, we introduce three monomial term orders in the following.

Definition 2.3 (**Lexicographic (Lex) Order**)**.** Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ be in $\mathbb{Z}^n_{\geq 0}$. We say $\alpha >_{\text{lex}} \beta$ if the leftmost nonzero entry of the vector difference $\alpha - \beta \in \mathbb{Z}^n$ is positive. We will write $x^\alpha >_{\text{lex}} x^\beta$ if $\alpha >_{\text{lex}} \beta$.

Definition 2.4 (**Graded Lex (GrLex) Order**). Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$. We say $\alpha >_{\text{grlex}} \beta$ if

$$|\alpha| = \sum_{i=1}^{n} \alpha_i > |\beta| = \sum_{i=1}^{n} \beta_i, \text{ or } |\alpha| = |\beta| \text{ and } \alpha >_{\text{lex}} \beta \tag{2}$$

Example 2.1. $x_1^2 x_2^5 x_3 >_{\text{grlex}} x_1 x_3^3$ *since for the coefficient tuples* $|(2,5,1)| = 8 > |(1,0,3)| = 4$*, then* $(2,5,1) >_{\text{grlex}} (1,0,3)$.

$x_1^5 x_2 x_3^2 >_{\text{grlex}} x_1 x_2^3 x_3^4$ *since for the coefficient tuple* $|(5,1,2)| = |(1,3,4)| = 8$*, and* $(5,1,2) - (1,3,4) = (4,-2,-2)$*, then* $(5,1,2) >_{\text{grlex}} (1,3,4)$.

Definition 2.5 (**Graded Reverse Lex (DegRevLex) Order**). Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$. We say $\alpha >_{\text{grevlex}} \beta$ if

$$|\alpha| = \sum_{i=1}^{n} \alpha_i > |\beta| = \sum_{i=1}^{n} \beta_i, \text{ or } |\alpha| = |\beta| \text{ and the rightmost nonzero entry of } \alpha - \beta \in \mathbb{Z}^n \text{ is negative.} \tag{3}$$

Example 2.2. $x_1^2 x_2^5 x_3 >_{\text{grevlex}} x_1 x_3^3$ *since for the coefficient tuples* $|(2,5,1)| = 8 > |(1,0,3)| = 4$*, then* $(2,5,1) >_{\text{grevlex}} (1,0,3)$.

$x_1^5 x_2 x_3^2 >_{\text{grevlex}} x_1 x_2^3 x_3^4$ *since for the coefficient tuple* $|(5,1,2)| = |(1,3,4)| = 8$*, and* $(5,1,2) - (1,3,4) = (4,-2,-2)$*, then* $(5,1,2) >_{\text{grevlex}} (1,3,4)$.

We choose to use the `DegRevLex` order for anywhere need the monomial term order in a polynomial ring in this paper because of the nice properties of the `DegRevLex` order for computer to do the computation. With the well-defined monomial ordering, we could define the basis with a nice property in describe the polynomial in an ideal: the Gröbner basis.

Definition 2.6 (**Gröbner Basis**). [CLO15] Fix a monomial order on the polynomial ring $k[x_1, \cdots, x_n]$. A finite subset $G = \{g_1, \cdots, g_t\}$ of an ideal $I \subseteq k[x_1, \cdots, x_n]$ different from $\{0\}$ is said to be **Gröbner basis** (or **standard basis**) of I if

$$\langle \text{LT}(g_1), \cdots, \text{LT}(g_t) \rangle = \langle \text{LT}(I) \rangle$$

Then, since $\langle LT(I) \rangle$ clearly contains $\langle \text{LT}(g_1), \cdots, \text{LT}(g_s) \rangle$, this set G is a Gröbner basis of I if $\langle \text{LT}(I) \rangle \subseteq \langle \text{LT}(g_1), \cdots, \text{LT}(g_s) \rangle$, that is, the leading term of all elements of the ideal can be written as a $k[x_1, \cdots, x_n]$-linear combination of elements in $\{\text{LT}(g_1), \cdots, \text{LT}(g_s)\}$. Furthermore, there are some properties of Gröbner basis as the followings.

- Dividing f by a Gröbner basis ($G = \{g_1, g_2, \cdots, g_t\}$), the remainder would be unique, but the quotient $q = \{q_1, \cdots, q_t\}$ would be different if the elements of G are listed in a different order.
- $f \in I$ if and only if 0 is a remainder of f on division by G
- Let $\overline{f}^F$ is the remainder on division of f by the ordered s-tuple $F = (f_1, f_2, \cdots, f_s)$, then $\overline{f}^G = 0$ if and only if $f \in I$

The uniqueness of the remainder can easily be checked by the division algorithm. Consider an example like $f = xy \in I \in \mathbb{Q}[x, y, z]$, and let $G_1 = \{x + z, y - z\}$ and $G_2 = \{y - z, x + z\}$ be a Gröbner basis written using two different orderings; then by the division algorithm one has

$$\text{For } G_1\text{: } f = y \cdot (x + z) - z \cdot (y - z) - z^2 \tag{4}$$

$$\text{For } G_2\text{: } f = x \cdot (y - z) + z \cdot (x + z) - z^2 \tag{5}$$

One could see that even though the remainders are the same, the quotients are different with two different bases. The second and third properties could be proved following the definition of Gröbner basis, since $\langle \text{LT}(I) \rangle \subseteq \langle \text{LT}(g_1), \cdots, \text{LT}(g_s) \rangle$.

Remark. An obstruction for a basis being a Gröbner basis is that the cancellation occurs in $f = ax^\alpha g_i - bx^\beta g_j$, where $g_i, g_j \in I$, and $\text{LT}(f) \in \langle \text{LT}(I) \rangle$, but $f \notin \langle \text{LT}(g_i), \text{LT}(g_j) \rangle$.

For the convenience and efficiency of the computation, we always want our Gröbner basis to be as small as possible while it still has the same properties introduced above. Thus, we could make the following two definitions of minimal and reduced Gröbner basis.

Definition 2.7 (**Minimal Gröbner Basis**). [CLO15] The **minimal** Gröbner basis is a Gröbner basis G for I such that:

- $\mathrm{LC}(p) = 1$ for all $p \in G$
- For all $p \in G$, none of $\mathrm{LT}(p)$ lies in $\langle \mathrm{LT}(G \backslash \{p\}) \rangle$

Definition 2.8 (**Reduced Gröbner Basis**). [CLO15] The **reduced** Gröbner basis is a Gröbner basis G for I such that:

- $\mathrm{LC}(p) = 1$ for all $p \in G$
- For all $p \in G$, none of the monomials in p lies in $\langle \mathrm{LT}(G \backslash \{p\}) \rangle$

Example 2.3. *Let $\{xy + yz, yz + xyz\}$ be a Gröbner basis of $I \in \mathbb{Q}[x, y, z]$ in the* `DegRevLex` *order, $\{xy + yz, yz + xyz\}$ is a minimal Gröbner basis, but it is not a reduced Gröbner basis. If we further reduced it, we could obtain $\{xy, yz\}$, which is a reduced Gröbner basis.*

2.2 Syzygy

Because Gröbner basis has such a pleasant property of describing polynomials with linear combination of basis elements and a unique remainder, it would be nice for one to know how to compute the Gröbner basis in an efficient way, so we introduce Buchberger's algorithm. However, we first need some more definitions to understand the algorithm. First, we need to define the S-polynomial and syzygy.

Definition 2.9 (**Least Common Multiple**). Let $f, g \in k[x_1, \cdots, x_n]$ be nonzero polynomials such that $\mathrm{multideg}(f) = \alpha$ and $\mathrm{multideg}(g) = \beta$, then let $\gamma = (\gamma_1, \cdots, \gamma_n)$, where $\gamma_i = \max(\alpha_i, \beta_i)$. Then x^γ is the least common multiple of $\mathrm{LM}(f)$ and $\mathrm{LM}(g)$, that is, $x^\gamma = lcm(\mathrm{LM}(f), \mathrm{LM}(g))$.

Definition 2.10 (**S-polynomial**). [CLO15] Let $f, g \in k[x_1, \cdots, x_n]$ be nonzero polynomials, and let $x^\gamma = \mathrm{lcm}(\mathrm{LM}(f), \mathrm{LM}(g))$.Then, the S-polynomial of f and g is defined as a polynomial that cancels leading terms of both the polynomials. To be specific,

$$S(f, g) = \tfrac{x^\gamma}{\mathrm{LT}(f)} \cdot f - \tfrac{x^\gamma}{\mathrm{LT}(g)} \cdot g \tag{6}$$

We can easily check the cancellation of the leading terms simply by the following calculation:
Let $f' = f - LT(f)$ and $g' = g - LT(g)$. Then:

$$\begin{aligned} \tfrac{x^\gamma}{\mathrm{LT}(f)} \cdot f - \tfrac{x^\gamma}{\mathrm{LT}(g)} \cdot g &= \tfrac{x^\gamma}{(\mathrm{LT}(f)+f')} - \tfrac{x^\gamma}{(\mathrm{LT}(g)+g')} && (7) \\ &= x^\gamma + \tfrac{x^\gamma}{\mathrm{LT}(f)} \cdot f' - x^\gamma - \tfrac{x^\gamma}{\mathrm{LT}(g)} \cdot g' && (8) \\ &= \tfrac{x^\gamma}{\mathrm{LT}(f)} \cdot f' - \tfrac{x^\gamma}{\mathrm{LT}(g)} \cdot g' && (9) \end{aligned}$$

As what $\frac{x^\gamma}{\mathrm{LT}(f)}$ and $\frac{x^\gamma}{\mathrm{LT}(g)}$ do in the previous calculation, they help to vanish the leading terms of the polynomials with performing the "dot product". To generalize this process, we define such monomials for any finite set of polynomials as the syzygy of this set of polynomials.

Definition 2.11 (**Syzygy**). [CLO15] Let $F = (f_1, \cdots, f_l) \in (k[x_1, \cdots, x_n])^l$. A **syzygy** on the leading terms $\mathrm{LT}(f_1), \cdots, \mathrm{LT}(f_l)$ of F is a l-tuple of polynomials $S = (s_1, , \cdots, s_l) \in (k[x_1, \cdots, x_n])^l$ such that

$$\sum_{i=1}^{l} s_i \cdot \mathrm{LT}(f_i) = 0 \tag{10}$$

Example 2.4. *Let $F = (3xy^2 + y, xyz + x^2z, 5y^3 + 7xyz)$, then $S = (yz, 2y^2, -xz)$ is a syzygy of F, because $3xy^2 \cdot yz + xyz \cdot 2y^2 + 5y^3 \cdot (-xz) = 3xy^3z + 2xy^3z - 5xy^3z = 0$.*

2.3 Buchberger's Algorithm

Now, we could outline the details of the Buchberger algorithm [CLO15].

Theorem 2.1 (**Buchberger's Algorithm**)**.** *Let $I = \langle f_1, \cdots, f_s \rangle \neq \{0\}$ be a polynomial ideal. Then a Gröbner basis for I can be constructed in a finite number of steps by the following algorithm:*

Algorithm 1 Buchberger's Algorithm

Input: F = $(f_1, \cdots, f_s)$
Output: a Gröbner basis $G = (g_1, \cdots, g_t)$ for I, with $F \subseteq G$

$G := F$
REPEAT
$G' := G$
for each pair$\{p, q\}, p \neq q$ in G' **do**
    $r := \overline{S(p,q)}^{G'}$
    if $r \neq 0$ **then**
        $G \ = G \cup \{r\}$
    end if
end for
UNTIL $G = G'$
return G

Proof. To prove the algorithm, we first need to understand the connection between S-polynomials and Gröbner bases by the theorem below.

Theorem 2.2 (Buchberger's Criterion)**.** *Let I be a polynomial ideal. Then a basis $G = \{g_1, \cdots, g_t\}$ of I is a Gröbner basis of I if and only if $\forall \{i, j\}$ such that $i \neq j$, the remainder on the division of $S(g_i, g_j)$ by G is zero.*

The proof of Theorem 2.2 could be referred to the proof of Theorem 6 of Chapter 2 §6 in Cox, Little, and O'Shea's book [CLO15].

Because for any pair of p, q, $S(p,q) = \sum_{i=1}^{t} h_i g_i + r$ for $g_i \in G'$ with $i = 1, \cdots, t$. Then we could show that $S(p,q) \in I$ since $p, q \in G' \subseteq I$ and the absorption property of I guarantee that the two polynomials summed when computing $S(p,q)$ are in I. Also by the absorption property, since $g_i \in G' \subseteq I$, each $h_i g_i \in I$. Therefore, $\sum_{i=1}^{t} h_i g_i \in I$ and $r = S(p,q) - \sum_{i=1}^{t} h_i g_i \in I$. Therefore, for each iteration, $G \ = G \cup \{r\} \subseteq I$, so G is a basis for I. Since the algorithm above ends when the $G = G'$, that is, when the remainder $r = 0$, $\forall p \neq q \in G'$, applying Theorem 2.2, G is a Gröbner basis of I at the end of the algorithm.

It is sufficient to show that the algorithm ends in finite steps. To prove that the algorithm terminates, we need the Ascending Chain Condition as follows.

Theorem 2.3 (The Ascending Chain Condition (ACC))**.** *Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be an ascending chain of ideals in $k[x_1, \cdots, x_n]$. Then there exists an $N \geq 1$ such that $I_N = I_{N+1} = I_{N+2} = \cdots$.*

The proof of Theorem 2.2 could be referred to the proof of Theorem 7 of Chapter 2 §5 of Cox, Little, and O'Shea's book [CLO15].

Since for each iteration, we have $G' \subseteq G$, then $\langle \mathrm{LT}(G') \rangle \subseteq \langle \mathrm{LT}(G) \rangle$. Let $G_0 = F$, and if we keep track of G from each iteration and label them as $G_1, G_2, \cdots$, then we have an ascending chain of ideals $\langle \mathrm{LT}(G_0) \rangle \subseteq \langle \mathrm{LT}(G_1) \rangle \subseteq \langle \mathrm{LT}(G_2) \rangle \subseteq \cdots$. Therefore, applying ACC, we know that $\exists N \geq 1$ such that $\langle \mathrm{LT}(G_N) \rangle = \langle \mathrm{LT}(G_{N+1}) \rangle = \langle \mathrm{LT}(G_{N+2}) \rangle = \cdots$. This shows that the algorithm ends at some point with a finite number of steps. □

Remark. The Buchberger's algorithm only produces a Gröbner Basis but it might not be the unique one, that is, it might include redundancy. The Gröbner basis produced by the algorithm is not necessarily a reduced Gröbner basis, since we only check for the leading terms of the polynomials using the S-polynomials instead of checking for every monomial in the polynomials. Moreover, it even does not promise a minimal Gröbner basis. For example, when we input Algorithm 1 with F, which is already a Gröbner basis with redundancy, which means that there is at least one polynomial $p \in F$

such that $\mathrm{LT}(p) \in \langle \mathrm{LT}(F \backslash \{p\}) \rangle$, the algorithm would output $G = F$ directly by Theorem 2.2, which is not minimal by definition.

2.4 Vector Space Spanned by the leading monomials

In the next definition, we define a way to build vector spaces from an ideal.

Definition 2.12. Let $S = k[x_1, \cdots, x_n]$ be a polynomial ring with a term order $>$ over a field k. For any set $G \subseteq S$, we let $\mathrm{in}(G)$ denote the **k-vector space spanned by the monomials** $\{\mathrm{LM}(f) : f \in G\}$.

Because $R = k[x_1, \cdots, x_n]$, the scalar multiples of $in(G)$ are well defined. Since the monomials of G are in R and R is a polynomial ring, $in(G)$ indeed defines a k-vector space, and this is indeed a monomial ideal generated by the leading terms of elements in G.

Proposition 2.1. *Let $S = k[x_1, \cdots, x_n]$ be a polynomial ring with a term order $>$ over a field k, $J \subset S$ and ideal and $R = S/J$ be a factor ring of S. Let N denote the k-vector space spanned by the set of standard monomials of R, that is, the set of monomials of S not in $in(J)$. Any element $f \in R$ can be written uniquely as the image of an element $g \in N$, and we set $lc(f) = lc(g)$ and $lm(f) = lm(g) \in N$. Thus, $N = \mathrm{in}(R)$ and $in(S) = N \bigoplus \mathrm{in}(J)$, as k-vector spaces.*

Proof. Let $m \in \mathrm{in}(S)$, then let $m = \sum_{i=1}^{l} q_i \cdot \mathrm{LM}(s_i)$ for $q_i \in k$ and $s_i \in S$. Suppose that in $\{\mathrm{LM}(s_1), \cdots, \mathrm{LM}(s_l)\}$, there are p of them in $\mathrm{in}(J)$. Then, by relabeling $s_i's$, we have $m = \sum_{i=1}^{p} q_i \cdot \mathrm{LM}(s_i) + \sum_{j=p+1}^{l} q_j \cdot \mathrm{LM}(s_j)$ for $\mathrm{LM}(s_i) \in \mathrm{in}(J)$ and $\mathrm{LM}(s_j) \in S \backslash J$. Let $u = \sum_{i=1}^{p} q_i \cdot \mathrm{LM}(s_i)$ and $v = \sum_{j=p+1}^{l} q_j \cdot \mathrm{LM}(s_j)$. Then, by definition, $u \in \mathrm{in}(J)$, $v \in N$, and $m = u + v$. Since we choose m arbitrarily and u, v to be unique by construction, we know that $in(S) = N \bigoplus \mathrm{in}(J)$. $\square$

3 Schreyer Frame and the Corresponding Schreyer Resolution

Before introducing the Schreyer resolutions and their frames, we need to first define the following terms in rings, and modules.

3.1 Free Module over A Factor Ring

Definition 3.1 (Power Product). Let $S = k[x_1, \cdots, x_n]$ be a polynomial ring with a term order $>$ over a field k. A **power product** t (or monomial of S) is $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in S$, where $\alpha_1, \cdots, \alpha_n$ are non-negative integers.

With this definition, we could uniquely write every nonzero element $f \in S$ as

$$f = c_1 \cdot m_1 + \cdots + c_l \cdot m_l \tag{11}$$

where for $i = 1, \cdots, l$, $c_i \in k$ and are nonzero, and m_i are monomials such that $m_1 > m_2 > \cdots > m_l$. In addition, we define the leading coefficient of $f \in S$ to be $lc(f) = c_1$, and the leading monomial of f, $\mathrm{LM}(f) = m_1$.

With such a factor ring, we could build a module from the ring.

Definition 3.2 (R-Module). Suppose that R is a ring, and 1 is its multiplicative identity. An R-module M consists of an Abelian group $(M, +)$ and an operation $\cdot : R \times M \to M$ such that for all r, s in R and x, y in M, we have

- $r \cdot (x + y) = r \cdot x + r \cdot y$
- $(r + s) \cdot x = r \cdot x + s \cdot x$
- $(rs) \cdot x = r \cdot (s \cdot x)$
- $1 \cdot x = x$

We define the elements and the term orders on the elements of a R-module as follows.

Definition 3.3 (**Monomials of a Module**)**.** Let F be a free module over R, and let $\hat{F}$ be the free S-module with the same rank as F and corresponding basis. A **monomial** of F is, by definition, any element $m = t \cdot e$, where $t \in N$ is a standard monomial and e is any element of the canonical basis of $\hat{F}$.

Definition 3.4 (**Term order on a Free Module**)**.** The term order on a free module F over k (a total order on the monomials of F) such that:

- if $m <_F n$, then $t \cdot m <_F t \cdot n$;
- if $s <_S t$, then $s \cdot e <_F t \cdot e$;

for all m, n monomials of F, s, t power products in S, and e any basis element of F.

Similar to a ring homomorphism, we have the following definition of R-module homomorpism that describe the mappings between R-modules.

Definition 3.5 (**R-module Homomorphism**)**.** A R-module homomorphism between M and N is a function $f : M \to N$ such that for any $x, y \in M$ and $r \in R$,

- $f(x + y) = f(x) + f(y)$,
- $f(rx) = rf(x)$

As a basis building block of the free resolution, the free R-module is defined as follows.

Definition 3.6 (**Free R-module**)**.** For a R-module M, the set $\mathcal{B} \subseteq M$ is a basis for M if:

- $\mathcal{B}$ is a spanning set for M;
- $\mathcal{B}$ is linearly independent.

A free R-module is a module with a basis.

Using the k-vector space spanned by leading monomials of elements of a subset of and ideal, we could define the Gröbner basis of a free module in a different way.

Definition 3.7 (**Gröbner Basis of a R-module**)**.** Fix a monomial order on the polynomial ring R and fix a term order on a R-module M. A finite set $G = \{g_1, \cdots, g_t\}$ of a submodule M of R^m different from $\{0\}$ is said to be **Gröbner basis** of M if

$$\langle \mathrm{LM}(g_1), \cdots, \mathrm{LM}(g_t) \rangle = \mathrm{in}(M)$$

where $\langle X \rangle$ is the submodule generated by elements of X for X be a subset of M.

We would soon see the Gröbner basis of a R-module is an essential tool for us to construct a sequence of R-homomorphisms of free R-modules.

3.2 Free Resolution

As the main structure of the resolutions and frames, complex and exact sequences must first be defined.

Definition 3.8 (**Complex and Exactness**)**.** Let $\Phi = \{\varphi_n : A_n \to A_{n-1}\}$ be a sequence of homomorphisms of R-modules for $n \in \mathbb{Z}$:

$$\Phi : \cdots \to A_{n+1} \xrightarrow{\varphi_{n+1}} A_n \xrightarrow{\varphi_n} A_{n-1} \to \cdots \tag{12}$$

Then, Φ is a **complex** if $\forall n \in \mathbb{Z}$, $\operatorname{im} \varphi_{n+1} \subseteq \ker \alpha_n$. If $\operatorname{im} \varphi_{i+1} = \ker \varphi_i$, Φ is **exact** in (homological) degree i. If Φ is exact in every homological degree, we say that Φ is exact. If Φ is exact at every homological degree except 0, then we say Φ is **acyclic**.

Then, we could introduce the definition of the free resolution.

Definition 3.9 (**Free Resolution**). Let $R = S/J$ be the factor ring as previously defined, and let F_i be free R-modules with a given (canonical) basis $\mathcal{E}_i$ of $\hat{F}_i$ and $M = F_0/I$. A **free resolution** of M is an acyclic complex of free R-modules:

$$\Phi : \cdots \to F_l \xrightarrow{\varphi_l} F_{l-1} \xrightarrow{\varphi_{l-1}} \cdots \xrightarrow{\varphi_1} F_0 \tag{13}$$

Define $\mathcal{C}_i = \varphi_i(\mathcal{E}_i)$ as the image of the given basis and define the level of an element f in C_i as $lev(f) = i$.

Proposition 3.1. *To (theoretically) construct a free resolution, we first choose a (finite) generating set of M, which gives a surjection $F_0 \twoheadrightarrow M$ where F_0 is a finite rank free R-module. The kernel of this map K_1 is again a finitely generated R-module because R is Noetherian. So we get a short exact sequence*

$$0 \hookrightarrow K_1 \hookrightarrow F_0 \twoheadrightarrow M \to 0 \tag{14}$$

where K_1 is a finitely generated R-module. We may now repeat this process: choose a finite generating set of K_1, which gives an onto map $\varphi'_1 : F_1 \to K_1$, where F_1 is again a finitely rank free R-module, which in turn gives rise to a short exact sequence

$$0 \hookrightarrow K_2 \hookrightarrow F_1 \twoheadrightarrow K_1 \to 0. \tag{15}$$

where K_2 is the kernel of the map Also, since the composition $\varphi_1 : F_1 \to F_0$ is a free R-module homomorphism between finite rank free R-modules, it may be represented by a matrix of finite size. Continuing this process, we fit all these short exact sequences into an acyclic complex of modules as in the following diagram, and this is a free resolution of M.

$$\begin{tikzcd}
 & & & & & & K_2 \arrow[dr, hook, "\iota_2"] & & & & & \\
\cdots \arrow[r] & F_l \arrow[rr, "\varphi_l"] \arrow[dr, "\varphi'_l"'] & & F_{l-1} \arrow[r, "\varphi_{l-1}"] & \cdots \arrow[r, "\varphi_3"] & F_2 \arrow[rr, "\varphi_2"] \arrow[ur, "\varphi'_2"] & & F_1 \arrow[rr, "\varphi_1"] \arrow[dr, "\varphi'_1"'] & & F_0 \arrow[r, two heads] & M \arrow[r] & 0 \\
 & & K_l \arrow[ur, hook, "\iota_l"'] & & & & & & K_1 \arrow[ur, hook, "\iota_1"'] & & &
\end{tikzcd} \tag{16}$$

Proof. To prove that this construction gives us a free resolution, we need to show that this sequence is exact at every homological degree except 0. Since by construction, K_1 is the kernel of the map $F_0 \twoheadrightarrow M$, then $K_1 \subseteq F_0$ and $\iota_1 : K_1 \hookrightarrow F_0$ is an inclusion. Therefore, the composition $\varphi_1 = \iota_1 \circ \varphi'_1 = \varphi'_1$ satisfies $\operatorname{im}\varphi_1 = \operatorname{im}\varphi'_1 = K_1$ since φ'_1 is an onto map sending to K_1. Thus, the sequence is exact at degree 1. Suppose the sequence is exact at degree t, then $F_t \to K_t$ must defines an onto map, and by construction, we have the short exact sequence,

$$0 \hookrightarrow K_{t+1} \hookrightarrow F_t \twoheadrightarrow K_t \to 0 \tag{17}$$

where K_{t+1}, F_t, and K_t are all finitely generated free R-modules, and K_{t+1} is the kernel of the map $F_t \twoheadrightarrow K_t$. Next, we could construct a finitely generated free R-module F_{t+1} such that there is an onto map $\varphi_{k+1} : F_{t+1} \twoheadrightarrow K_{t+1}$. Then the function composition gives us $\varphi_{t+1} = \iota_{t+1} \circ \varphi'_{t+1} = \varphi'_{t+1}$, and φ_{t+1} sends F_{t+1} to F_t. Hence, $\operatorname{im}\varphi_{t+1} = \operatorname{im}\varphi'_{t+1} = K_{t+1} = \ker\varphi_t$. Therefore, the sequence is exact at level $t+1$. Thus, by induction, we know that whole sequence is an acyclic complex, so it is a free resolution. □

When we are trying to practically compute the free resolution following the construction above, one might encounter a problem of computing the kernels of matrix over a polynomial ring, which, in this case, a polynomial ring R. This is why the technique of Gröbner basis are useful here. To be more specific, we could first find the Gröbner basis of each K_i and then compute a ordered set of polynomials that when multiplied to the bases of the Gröbner basis respectively, have the sum 0 like what a syzygy does, which is the kernel we want.

Remark. The construction above only defines ordering on each free R-modules F_i, but there is not a well-defined ordering on the whole sequence of R-module homomorphisms. Thus, we define a term ordering on the free resolution.

Definition 3.10 (**Term Ordering on Free Resolution**). Let $\tau = \{\tau_i\}$ be a sequence of term ordering on the modules F_i. We call τ a **term ordering on** Φ if it satisfies the following compatibility relationship:

$$s \cdot e_1 <_{\tau_i} t \cdot e_2 \text{ whenever } s \cdot \mathrm{LM}(\varphi_i(e_1)) <_{\tau_{i-1}} t \cdot \mathrm{LM}(\varphi_i(e_2)) \tag{18}$$

where e_1 and e_2 are elements of $\mathcal{E}_i$.

To describe the resolution Φ, we define the initial terms of Φ, Ξ as follows.

Definition 3.11 (**Initial Term of A Resolution**). Given Φ as previously described and a term ordering τ on Φ, define the **initial terms** of Φ, $\mathrm{in}(\Phi)$, as the sequence of (graded) R-homomorphisms:

$$\Xi = \mathrm{in}(\Phi) : \cdots \to F_l \xrightarrow{\xi_l} F_{l-1} \xrightarrow{\xi_{l-1}} \cdots \xrightarrow{\xi_2} F_1 \xrightarrow{\xi_1} F_0 \tag{19}$$

where $\xi_i(e) = \mathrm{LM}(\varphi_i(e))$, for all e in $\mathcal{E}_i$. That is, the differentials in $\mathrm{in}(\Phi)$ are obtained by taking the leading monomials of each entry in the differentials in Φ.

Remark. Since the homomorphisms defined in Ξ are $\xi_i(e) = \mathrm{LM}(\varphi_i(e))$, the compatibility relationship is satisfied.

$$\begin{aligned}\mathrm{LM}(s) \cdot e_1 <_{\tau_i} \mathrm{LM}(t) \cdot e_2 \text{ whenever } & \mathrm{LM}(s) \cdot \mathrm{LM}(\xi_i(e_1)) <_{\tau_{i-1}} \mathrm{LM}(t) \cdot \mathrm{LM}(\xi_i(e_2)) \\ & \Leftrightarrow \mathrm{LM}(s) \cdot \mathrm{LM}(\mathrm{LM}(\varphi_i(e_1))) <_{\tau_{i-1}} \mathrm{LM}(t) \cdot \mathrm{LM}(\mathrm{LM}(\varphi_i(e_2))) \\ & \Leftrightarrow \mathrm{LM}(s) \cdot \mathrm{LM}(\varphi_i(e_1)) <_{\tau_{i-1}} \mathrm{LM}(t) \cdot \mathrm{LM}(\varphi_i(e_2))\end{aligned} \tag{20}$$

where e_1 and e_2 are elements of $\mathcal{E}_i$. Thus, the term ordering of Ξ is also τ.

In order to practically compute a free resolution for a free module over a polynomial ring, we need to use the Gröbner basis, and if we use the Gröbner basis to compute the kernels connecting the whole resolution, we are actually creating a Schreyer resolution. We define the Schreyer resolution as the following. A Schreyer resolution is a free resolution with some restrictions.

Definition 3.12 (**Schreyer Resolution**). [LS98] A **Schreyer resolution** of an R-module $M = F_0/I$ is an exact sequence:

$$\Phi : \cdots \to F_l \xrightarrow{\varphi_l} F_{l-1} \xrightarrow{\varphi_{l-1}} \cdots \xrightarrow{\varphi_1} F_0, \tag{21}$$

such that:

- $\mathrm{coker}(\varphi_1) = M$
- There is a term ordering τ on Φ
- $\varphi_i(\mathcal{E}_i)$ forms a reduced Gröbner basis of $\mathrm{im}\,\varphi_i = \ker \varphi_{i-1}$ (for all i where $F_i \neq 0$)

Because a Schreyer resolution requires $\varphi_i(\mathcal{E}_i)$ to be a Gröbner basis, we want to focus on the initial term of the Schreyer resolution more. Therefore, we introduce the following definition of Schreyer frame, which is a similar term as the initial terms of a Schreyer resolution.

Definition 3.13 (**Schreyer Frame**). [LS98] A **Schreyer frame** of $M = F_0/I$ is an exact sequence of R-homomophisms:

$$\Xi : \cdots \to F_1 \xrightarrow{\xi_l} F_{l-1} \xrightarrow{\xi_{l-1}} \cdots \xrightarrow{\xi_2} F_1 \xrightarrow{\xi_1} F_0 \tag{22}$$

where each column is a monomial, and a term ordering on Ξ such that:

- $\xi_1(\mathcal{E}_1)$ is a minimal set of generators for $\mathrm{in}(I)$;
- $\xi_i(\mathcal{E}_i)$ is a minimal set of generators for $\mathrm{in}(\ker \xi_{i-1})$ for $i \geq 2$.

In this definition, we only trace how the leading monomials are changed in the resolution, where the homomorphisms $\xi_i(e_i) = \mathrm{LM}\,\varphi_i(e_i)$ for $e \in \mathcal{E}_1$. The restriction on the sets $\xi_i(\mathcal{E}_i)$ that it needs to be the minimal set such that $\xi_i(\mathcal{E}_i) = \mathrm{in}(\ker \xi_{i-1})$ makes sure that $\varphi_i(\mathcal{E}_i)$ forms a minimal Gröbner basis of $\ker \varphi_{i-1}$. From this construction, we could show that for any Schreyer resolution Φ, $\mathrm{in}(\Phi)$ is its corresponding Shreyer frame, and for any Shreyer frame Ξ, we could extend it to build a Shreyer resolution by including more information using the idea of syzygy.

With the definition of the Schreyer frame, we define some notations that would be useful when we prove the propositions later.

Let $e \in \mathcal{E}_{i-1}$, then

$$\mathcal{B}_i = \xi_i(\mathcal{E}_i) \tag{23}$$
$$\mathcal{E}_i(e) = \{\epsilon \in \mathcal{E}_i | \xi_i(\epsilon) = s \cdot e, \text{ for some power product } s\} \tag{24}$$

3.3 Computing Schreyer Frame

For that we want to use the Schreyer frame to get a Schreyer resolution for a free module, the first question we should answer is about how to find a Schreyer frame. We would start with the following definition.

Definition 3.14 (Colon Ideal). Let I, J be two ideals of commutative rings R, then the **colon ideal** $(I : J)$ is defined as

$$(I : J) = \{r \in R \mid rJ \subseteq I\} \tag{25}$$

If p is an polynomial in R, then

$$(I : p) = \{r \in R \mid rp \in I\} \tag{26}$$

Then, the next Lemma is crucial for us to build up a Schreyer frame, since we need to minimally generate $\mathrm{in}(\ker(\xi_{i-1}))$ to find $\mathcal{B}_i$.

Lemma 3.1. *[LS98]* $\mathrm{in}(ker\xi_i)$ *is minimally generated by*

$$\bigcup_{e \in \mathcal{E}_{i-1}} \bigcup_{j=2}^{r} \mathrm{mingens}((\mathrm{in}(J), t_1, \cdots, t_{j-1}) : t_j) \cdot \epsilon_j \tag{27}$$

where for each e in the outer union, if $\mathcal{E}_i(e) = \{\epsilon_1, \cdots, \epsilon_r\}$, then $\xi_i(\epsilon_j) = t_j \cdot e$, and mingens *defines the subset of the minimal genreators of the considered monomial ideal which do not lie in* $\mathrm{in}(J)$.

Proof. Suppose $\sum_j g_j \cdot m_j = 0$ where m_j are minimal generators of $\mathrm{in}(\mathrm{im}\,\xi_i)$ and g_j are standard polynomials of S. Let $s_k \cdot \epsilon$ be the monomial of the corresponding syzygy with $s_k = \mathrm{LM}(g_k)$. There are only two ways to cancelling $s_k \cdot m_k$:

$$s_k \cdot m_k = s_h \cdot m_h \text{ for some } h, \tag{28}$$
$$s_k \cdot t_k \text{ belongs to } \mathrm{in}(J), \tag{29}$$

where $m_k = t_k \cdot e$. Thus, if we compute the colon ideal with the previously computed power products and $\mathrm{in}(J)$, we could find minimal generated set of $\mathrm{in}(\ker \xi_i)$. ☐

Proposition 3.2. *Let $\Xi : \cdots \to F_l \xrightarrow{\xi_l} F_{l-1} \xrightarrow{\xi_{l-1}} \cdot \xrightarrow{2} F_1 \xrightarrow{\xi_1} F_0$ be a sequence of homomorphisms where each column is a monomial. To give a term ordering on Ξ, it is enough to define a term ordering on F_0 and place total orders on the set $\mathcal{E}_i(e) \neq \emptyset$, for every $i \geq 1$ and $e \in \mathcal{E}_{i-1}$.*

Proof. By induction on the level i, the term ordering on the free module F_i can be defined as follows. For any s, t power products, $\epsilon_1, \epsilon_2 \in \mathcal{E}_i$, $m = \xi_i(\epsilon_1)$, and $n = \xi_i(\epsilon_2)$, we put:

$$s \cdot \epsilon_1 < t \cdot \epsilon_2 \text{ iff } s \cdot m < t \cdots n, \text{ or} \tag{30}$$
$$s \cdot m = t \cdot n \text{ and } \epsilon_1 < \epsilon_2 \text{ w.r.t. the order on } \mathcal{E}_i(e) \tag{31}$$
$$\text{with } m, n \text{ multiples of } e \in \mathcal{E}_{i-1} \tag{32}$$

We could see that with this definition, there is a well-defined term ordering on Ξ, since $s \cdot \epsilon_1 < t \cdot \epsilon_2$ whenever $s \cdot m < t \cdots n$ follows the compatibility relationship defined in the Definition 3.10. ☐

Now, we could look at a specific example of computing a Schreyer frame.

Example 3.1. *Let K be a field of any characteristic and let the polynomial ring $S = K[x_1, \cdots, x_6]$ with the term order of* `DegRevLex`*. Consider the module $M = S/I$, for*

$$I = \langle x_2x_3, x_2x_4, x_5x_6, x_1x_2, x_4x_6, x_2x_5, x_1x_3x_5, x_1x_3x_6, x_3x_4x_5\rangle \tag{33}$$

Using the algorithm `Resolution` *for char$(K) \neq 2$ to the module M, consturct a Schreyer frame from the module M.*

Solution: We first want to compute the Schreyer frame Ξ for M. With this plan, the basis of the first level of the Schreyer frame is I, because it generates the kernel of the map $F_0 \to M$. Hence, we have

$$\mathcal{B}_1 = \{x_2x_3, x_2x_4, x_5x_6, x_1x_2, x_4x_6, x_2x_5, x_1x_3x_5, x_1x_3x_6, x_3x_4x_5\} \tag{34}$$

After ordering it using the `DegRevLex`, the first level basis is:

$$\mathcal{B}_1 = \{x_5x_6, x_4x_6, x_2x_5, x_1x_4, x_2x_3, x_1x_2, x_1x_3x_6, x_3x_4x_5, x_1x_3x_5\} \tag{35}$$

Then, we used the formula in the Lemma 3.1, to construct the basis of the next level by finding the monomials that knock a basis into the ideal once at a time.

$$j = 2 : (x_5x_6 : x_4x_6) = \{x_5\} \to \{x_5e_{1,2}\} \tag{36}$$
$$j = 3 : ((e_{1,1}, e_{1,2}) : x_2x_5) = \{x_6, x_4x_6\} = \{x_6\} \to \{x_6e_{1,3}\} \tag{37}$$
$$j = 4 : ((e_{1,1}, \cdots, e_{1,3}) : x_1x_4) = \{x_5x_6, x_6, x_2x_5\} = \{x_6, x_2x_5\} \to \{x_6e_{1,4}, x_2x_5e_{1,4}\} \tag{38}$$
$$j = 5 : ((e_{1,1}, \cdots, e_{1,4}) : x_2x_3) = \{x_5x_6, x_4x_6, x_5, x_1x_4\} = \{x_5, x_4x_6, x_1x_4\} \tag{39}$$
$$\to \{x_5e_{1,5}, x_4x_6e_{1,5}, x_1x_4e_{1,5}\} \tag{40}$$
$$j = 6 : ((e_{1,1}, \cdots, e_{1,5}) : x_1x_2) = \{x_5x_6, x_4x_6, x_5, x_4, x_3\} = \{x_5, x_4, x_3\} \tag{41}$$
$$\to \{x_5e_{1,6}, x_4e_{1,6}, x_3e_{1,6}\} \tag{42}$$
$$j = 7 : ((e_{1,1}, \cdots, e_{1,6}) : x_1x_3x_6) = \{x_5, x_4, x_2x_5, x_4, x_2, x_2\} = \{x_5, x_4, x_2\} \tag{43}$$
$$\to \{x_5e_{1,7}, x_4e_{1,7}, x_2e_{1,7}\} \tag{44}$$
$$j = 8 : ((e_{1,1}, \cdots, e_{1,7}) : x_3x_4x_5) = \{x_6, x_6, x_2, x_1, x_2, x_1x_2, x_1x_6\} = \{x_6, x_2, x_1\} \tag{45}$$
$$\to \{x_6e_{1,8}, x_2e_{1,8}, x_1e_{1,8}\} \tag{46}$$
$$j = 9 : ((e_{1,1}, \cdots, e_{1,8}) : x_1x_3x_5) = \{x_6, x_4x_6, x_2, x_4, x_2, x_2, x_6, x_4\} = \{x_6, x_4, x_2\} \tag{47}$$
$$\to \{x_6e_{1,9}, x_4e_{1,9}, x_2e_{1,9}\} \tag{48}$$

Therefore ,we have our second level basis of Ξ:

$$\begin{aligned}\mathcal{B}_2 = \{&x_5e_{1,2}, x_6e_{1,3}, x_6e_{1,4}, x_2x_5e_{1,4}, x_5e_{1,5}, x_4x_6e_{1,5}, x_1x_4e_{1,5}, x_5e_{1,6}, x_4e_{1,6}, x_3e_{1,6}, x_5e_{1,7},\\ &x_4e_{1,7}, x_2e_{1,7}, x_6e_{1,8}, x_2e_{1,8}, x_1e_{1,8}, x_6e_{1,9}, x_4e_{1,9}, x_2e_{1,9}\}\end{aligned} \tag{49}$$

After ordering it using the `DegRevLex`, the second level basis is:

$$\begin{aligned}\mathcal{B}_2 = \{&x_5e_{1,2}, x_6e_{1,3}, x_6e_{1,4}, x_5e_{1,5}, x_5e_{1,6}, x_4e_{1,6}, x_3e_{1,6}, x_6e_{1,8}, x_5e_{1,7}, x_6e_{1,9}, x_4x_6e_{1,5}, x_4e_{1,7},\\ &x_2e_{1,7}, x_2e_{1,8}, x_1e_{1,8}, x_4e_{1,9}, x_2x_5e_{1,4}, x_2e_{1,9}, x_1x_4e_{1,5}\}\end{aligned} \tag{50}$$

Renaming each basis with second-level subscripts and applying the Lemma 3.1, we have

$$
\begin{aligned}
e_{1,4} &: j=2 : (x_6, x_2x_5) = \{x_6\} \to \{x_6e_{2,17}\} && (51)\\
e_{1,5} &: j=2 : (x_5, x_4x_6) = \{x_5\} \to \{x_5e_{2,11}\} && (52)\\
e_{1,5} &: j=3 : ((x_5, x_4x_6) : x_1x_4) = \{x_5, x_6\} \to \{x_6e_{2,19}, x_5e_{2,19}\} && (53)\\
e_{1,6} &: j=2 : (x_5 : x_4) = \{x_5\} \to \{x_5e_{2,6}\} && (54)\\
e_{1,6} &: j=3 : ((x_5, x_4) : x_3) = \{x_5, x_4\} \to \{x_5e_{2,7}, x_4e_{2,7}\} && (55)\\
e_{1,7} &: j=2 : (x_5, x_4) = \{x_5\} \to \{x_5e_{2,12}\} && (56)\\
e_{1,7} &: j=3 : ((x_5, x_4) : x_2) = \{x_5, x_4\} \to \{x_5e_{2,13}, x_4e_{2,13}\} && (57)\\
e_{1,8} &: j=2 : (x_6, x_2) = \{x_6\} \to \{x_6e_{2,14}\} && (58)\\
e_{1,8} &: j=3 : ((x_6, x_2) : x_1) = \{x_6, x_2\} \to \{x_6e_{2,15}, x_2e_{2,15}\} && (59)\\
e_{1,9} &: j=2 : (x_6, x_4) = \{x_6\} \to \{x_6e_{2,16}\} && (60)\\
e_{1,9} &: j=3 : ((x_6, x_4) : x_2) = \{x_6, x_4\} \to \{x_6e_{2,18}, x_4e_{2,18}\} && (61)
\end{aligned}
$$

Then, we have the third level basis $\mathcal{B}_3$ of Ξ:

$$
\begin{aligned}
\mathcal{B}_3 = \{&x_6e_{2,17}, x_5e_{2,11}, x_6e_{2,19}, x_5e_{2,19}, x_5e_{2,6}, x_5e_{2,7}, x_4e_{2,7}, x_5e_{2,12}, x_5e_{2,13}, x_4e_{2,13}, x_6e_{2,14},\\
&x_6e_{2,15}, x_2e_{2,15}, x_6e_{2,16}, x_6e_{2,18}, x_4e_{2,18}\}
\end{aligned} \tag{62}
$$

After ordering it using the `DegRevLex`, the third level basis is:

$$
\begin{aligned}
\mathcal{B}_3 = \{&x_5e_{2,6}, x_5e_{2,7}, x_4e_{2,7}, x_5e_{2,11}, x_6e_{2,14}, x_5e_{2,12}, x_6e_{2,15}, x_6e_{2,16}, x_6e_{2,17}, x_5e_{2,13}, x_6e_{2,18},\\
&x_6e_{2,19}, x_4e_{2,13}, x_5e_{2,19}, x_2e_{2,15}, x_4e_{2,18}\}
\end{aligned} \tag{63}
$$

Applying the Lemma 3.1 once again, we find the following bases:

$$
\begin{aligned}
e_{2,19} &: j=2 : (x_6, x_5) = \{x_6\} \to \{x_6e_{3,14}\} && (64)\\
e_{2,7} &: j=2 : (x_5, x_4) = \{x_5\} \to \{x_5e_{3,3}\} && (65)\\
e_{2,13} &: j=2 : (x_5, x_4) = \{x_5\} \to \{x_5e_{3,13}\} && (66)\\
e_{2,15} &: j=2 : (x_6, x_2) = \{x_6\} \to \{x_6e_{3,15}\} && (67)\\
e_{2,18} &: j=2 : (x_6, x_4) = \{x_6\} \to \{x_6e_{3,16}\} && (68)\\
& && (69)
\end{aligned}
$$

The fourth-level basis $\mathcal{B}_4$ of Ξ is

$$\mathcal{B}_4 = \{x_6e_{3,14}, x_5e_{3,3}, x_5e_{3,13}, x_6e_{3,15}, x_5e_{3,16}\} \tag{70}$$

After ordering it using the `DegRevLex`, the last level basis is:

$$\mathcal{B}_4 = \{x_5e_{3,3}, x_6e_{3,14}, x_5e_{3,13}, x_6e_{3,15}, x_5e_{3,16}\} \tag{71}$$

Now, applying the Lemma 3.1 could not give us any basis from $\mathcal{B}_4$, then we know that we had found the complete Schreyer frame Ξ.

$$
\begin{aligned}
\mathcal{B}_1 &= \{x_5x_6, x_4x_6, x_2x_5, x_1x_4, x_2x_3, x_1x_2, x_1x_3x_6, x_3x_4x_5, x_1x_3x_5\} && (72)\\
\mathcal{B}_2 &= \{x_5e_{1,2}, x_6e_{1,3}, x_6e_{1,4}, x_5e_{1,5}, x_5e_{1,6}, x_4e_{1,6}, x_3e_{1,6}, x_6e_{1,8}, x_5e_{1,7}, x_6e_{1,9}, x_4x_6e_{1,5}, x_4e_{1,7}, && (73)\\
&\quad x_2e_{1,7}, x_2e_{1,8}, x_1e_{1,8}, x_4e_{1,9}, x_2x_5e_{1,4}, x_2e_{1,9}, x_1x_4e_{1,5}\} && (74)\\
\mathcal{B}_3 &= \{x_5e_{2,6}, x_5e_{2,7}, x_4e_{2,7}, x_5e_{2,11}, x_6e_{2,14}, x_5e_{2,12}, x_6e_{2,15}, x_6e_{2,16}, x_6e_{2,17}, x_5e_{2,13}, x_6e_{2,18}, && (75)\\
&\quad x_6e_{2,19}, x_4e_{2,13}, x_5e_{2,19}, x_2e_{2,15}, x_4e_{2,18}\} && (76)\\
\mathcal{B}_4 &= \{x_5e_{3,3}, x_6e_{3,14}, x_5e_{3,13}, x_6e_{3,15}, x_5e_{3,16}\}, && (77)
\end{aligned}
$$

and we know

$$\Xi : S^5 \xrightarrow{\xi_3} S^{16} \xrightarrow{\xi_2} S^{15} \xrightarrow{\xi_1} S^9 \to 0 \tag{78}$$

3.4 Computing Schreyer Resolution

Following our plan of getting a Schreyer resolution from a Schreyer frame, we prove the following proposition.

Proposition 3.3. *If Ξ is a Schreyer frame for free R-module M, then there exists a Schreyer resolution Φ such that $\Xi = \text{in}(\Phi)$.*

Proof. Since there F_0 is a free module with an ordering on it, then let $\mathcal{C}_1$ be a minimal Gröbner basis of the free module I, and we have $\text{in}(\mathcal{C}_1) = \text{in}(I)$ by the definition of the Gröbner basis. Hence, $\text{in}(\mathcal{C}_1) = \text{in}(\text{im}\,\varphi_1)$, where $I = K_1 = \text{im}\,\varphi_1$ referring to diagram 16. Then, $\text{in}(\mathcal{C}_1) = \text{in}(I) = \langle \xi_1(\mathcal{E}_1)\rangle = \langle \mathcal{B}_1\rangle$ since $\mathcal{B}_1 = \xi_i(\mathcal{E}_i)$ is a minimal set of generators for $\text{in}(I)$ by the meaning of Ξ being a Schreyer frame. Now, we know that for all element $m = t\cdot\epsilon \in \mathcal{B}_2$, we have $g = \varphi_1(\epsilon) \in \mathcal{C}_1$. In other words, we have $\varphi_1(m) = \varphi_1(t\cdot\epsilon) = t\cdot\varphi_1(\epsilon) = t\cdot g$.

Then, using our knowledge of syzygy, we could found sets of elements $m = t\cdot\epsilon$ in $\mathcal{B}_2$ such that the sum of all of those $t\cdot g$ equals to 0 ($\varphi_i(\epsilon) = g$). That is, these sets of elements are the syzygies of $\mathcal{E}_1$. The set of the sums of set of elements in $\mathcal{B}_2$ is the set $\mathcal{C}_2 = \varphi_2(\mathcal{E}_2)$.

Since there is a term ordering on Ξ, the ordering on F_1 follows this ordering on the frame, then $\text{in}(\ker\xi_1) = \text{in}(\ker\varphi_1)$. Thus, $\mathcal{C}_2$ is a minimal Gröbner basis of $\ker(\varphi_1)$ and $\text{in}(\mathcal{C}_2) = \langle\mathcal{B}_2\rangle$. By induction, we could construct $\mathcal{C}_i = \varphi_i(\mathcal{E}_i)$, and it is a minimal Gröbner basis of the $\ker(\varphi_{i-1})$, and $\text{in}(\varphi_i(\mathcal{E}_i)) = \text{in}(\mathcal{C}_1) = \langle\xi_i(\mathcal{E}_i)\rangle = \langle\mathcal{B}_i\rangle$. Thus, this gives a Schreyer resolution Φ, and $\Xi = \text{in}(\Phi)$. □

To practically construct the Schreyer resolution Φ, such that $\text{in}(\Phi) = \Xi$, we introduce a detailed algorithm discovered by La Scala and Stillman [LS98].

Algorithm 2 Resolution[$\bar{\mathcal{C}_1}$]

Input: a reduced Gröbner basis $\bar{\mathcal{C}_1}$ of I, and an ordered union $\mathcal{B}$ of bases of all levels
Output: the set of Gröbner bases $\mathcal{C}_1,\cdots,\mathcal{C}_l$, and the set of corresponding syzygies $\mathcal{H}_1,\cdots,\mathcal{H}_l$

$\mathcal{C}_i, \mathcal{H}_i := \emptyset (1 \le i \le l)$
while $\mathcal{B} \neq \emptyset$ **do**
  $m := min\mathcal{B}$
  $\mathcal{B} := \mathcal{B}\backslash\{m\}$
  $i := lev(m)$
  if $i = 1$ **then**
    $g :=$ the element of $\bar{\mathcal{C}_i}$ s.t. $lm(g) = m$
    $\mathcal{C}_1 := \mathcal{C}_1 \cup \{g\}$
    $\mathcal{H}_1 := \mathcal{H}_i \cup \{g\}$
    else
    $(f, g) :=$ Reduce[$m, \mathcal{C}_{i-1}$]
    $\mathcal{C}_i := \mathcal{C}_i \cup \{g\}$
    if $f \neq 0$ **then**
      $\mathcal{C}_{i-1} := \mathcal{C}_{i-1} \cup \{f\}$
      $\mathcal{B} := \mathcal{B}\backslash\{lm(f)\}$
      else
      $\mathcal{H}_i := \mathcal{H}_i \cup \{g\}$
    end if
  end if
end while
return $\mathcal{C}_i, \mathcal{H}_i (1 \le i \le l)$

When we implement the above algorithm, the term ordering on the free modules F_i are as defined in the Definition 3.4. we are required to show there is a total order (**strategy** of `Resolution`) on the set $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_{\updownarrow}$. We define a strategy s/t if one the following conditions holds:

- $\deg(m) - \text{lev}(m) \le \deg(n) - \text{lev}(n)$ and $\text{lev}(m) < \text{lev}(n)$;
- $\deg(m) < \deg(n)$ and $\text{lev}(m) = \text{lev}(n)$;

- $\deg(m) = \deg(n)$ and $\mathrm{lev}(m) > \mathrm{lev}(n)$;

then $m < n$, for all $m, n \in \mathcal{B}$.

The Algorithm `Reduce` in `Resolution` is elaborated in the Algorithm 3.

Algorithm 3 Reduce$[t \cdot \epsilon, \mathcal{C}_{i-1}]$

$f := t \cdot k$, where $\varphi_{i-1}(\epsilon) = k$
$g := t \cdot \epsilon$
while $f \neq 0$ and $lm(f) \in in\langle \mathcal{C}_{i-1} \rangle$ **do**
  choose $h \in \mathcal{C}_{i-1}$ s.t. $lm(h)|lm(f)$ **(**$h \neq f$ **at the first iteration)**
  $f := f - \frac{lc(f)lpp(f)}{lc(h)lpp(h)} h$
  $g := g - \frac{lc(f)lpp(f)}{lc(h)lpp(h)} e$, where $\varphi_{i-1}(e) = h$
end while
if $f \neq 0$ **then**
  $g := g - e$, where $\varphi_{i-1}(e) = f$
end if
return f, g

Proposition 3.4. *The resolution Φ produced by the Algorithm* `Resolution` *is a Schreyer resolution.*

Proof. By induction on i, we suppose that in Φ, the set $\mathcal{C}_i$ is a reduced Gröbner basis of the module $\ker \varphi_{i-1}$. From the procedures in the Algorithm above we have $\mathcal{B}_{i+1} \subset \mathrm{in}(\mathcal{C}_{i+1})$, which means $\mathcal{C}_{i+1}$ is a Gröbner basis of $\ker \varphi_i$. We have to prove now that $\mathcal{C}_{i+1}$ is a reduced Gröbner basis, that is

$$\mathcal{B}_{i+1} = \mathrm{in}(\mathcal{C}_{i+1}). \tag{79}$$

Let $f \in \mathcal{C}_{i+1}$ be an element produced by the algorithm from an basis element $m \in \mathcal{B}$. Then, it suffices to show $\mathrm{LM}(f) \in \mathcal{B}_{i+1}$. Indeed, we will show this by cases, when $m \in \mathcal{B}_{i+1}$ or when $m \notin \mathcal{B}_{i+1}$.

If $m \in \mathcal{B}_{i+1}$, then we know that the command `Reduce`$[m, \mathcal{C}_i]$ returns $(0, f)$ and $\mathrm{LM}(f) = m$. If $m \notin \mathcal{B}_{i+1}$, then if we tracing back to the algorithm, we know $m \in \mathcal{B}_{i+2}$, and now the command `Reduce`$[m, \mathcal{C}_{i+1}]$ gives (f, g), where f is the reductum of dividing m by elements of $\mathcal{C}_{i+1}$, and g is the syzygy tracing the elements that reduces $m - f$ to zero. Then, by the mean of reductum, we know that for all $p \in \mathcal{C}_{i+1}$ before the computation on m, $\mathrm{LM}(p) \nmid \mathrm{LM}(f)$. However, by the algorithm, f would be add to $\mathcal{C}_{i+1}$. Again by induction, $\mathrm{LM}(p) \in \mathcal{B}_{i+1}$, so that $\mathrm{LM}(f) \in \mathcal{B}_{i+1}$ because $\mathcal{B}_{i+1}$ is the minimal generating set for $\mathrm{in}(\ker \xi_i) = \mathrm{in}(\mathcal{C}_{i+1})$. □

Example 3.2. *We could look at an example given by La Scala and Stillman in 1998 [LS98] to see how the algorithm* `Resolution` *should be used.*

Let K be a field of any characteristic and let the polynomial ring $S = K[x_1, \cdots, x_6]$ with the term order of `DegRevLex`*. Consider the graded module $M = S/I$, for*

$$I = \langle x_2x_4x_5, x_0x_4x_5, x_2x_3x_5, x_1x_3x_5, x_0x_1x_5, x_1x_3x_4, x_0x_3x_4, x_1x_2x_4, x_0x_2x_3, x_0x_1x_2 \rangle \tag{80}$$

, and give the Schreyer frame Ξ:

$$\mathcal{B}_1 = \{x_2x_4x_5, x_0x_4x_5, x_2x_3x_5, x_1x_3x_5, x_0x_1x_5, x_1x_3x_4, x_0x_3x_4, x_1x_2x_4, x_0x_2x_3, x_0x_1x_2\} \tag{81}$$

$$\mathcal{B}_2 = \{x_4e_{1,3}, x_5e_{1,6}, x_5e_{1,7}, x_5e_{1,8}, x_2e_{1,2}, x_4e_{1,5}, x_2e_{1,4}, x_5e_{1,9}, x_3e_{1,5}, x_5e_{1,10}, x_3e_{1,8}, x_4e_{1,9}, \tag{82}$$

$$x_1e_{1,7}, x_4e_{1,10}, x_3e_{1,10}, x_0x_4e_{1,4}\} \tag{83}$$

$$\mathcal{B}_3 = \{x_5e_{2,11}, x_5e_{2,12}, x_4e_{2,9}, x_5e_{2,13}, x_5e_{2,14}, x_5e_{2,15}, x_4e_{2,15}, x_2e_{2,16}\} \tag{84}$$

$$\mathcal{B}_4 = \{x_5e_{3,7}\}, \tag{85}$$

Applying the algorithm `Resolution` *for $char(K) \neq 2$ to the module M, find the Schreyer resolution Φ such that* $\mathrm{in}(\Phi) = \Xi$.

Solution: Given the frame Ξ, we could implement the Algorithm 2 to construct a Schreyer resolution Φ corresponding to the frame. To start the algorithm, we should first identify the input by checking whether the first level of our frame gives a reduced Gröbner basis. Since the minimal generators of I are all monomials, for every $m \in \langle \mathrm{LT}(I) \rangle$, m is a linear combination of $\mathrm{LT}(\mathcal{C}_1)$. Therefore, $\hat{\mathcal{C}_1} = \mathcal{C}_1$. Then we want to order the union of all bases that we obtained in the frame Ξ using the term order in the resolution as defined before.

$$\begin{aligned}\mathcal{B} = \{&x_2x_4x_5, x_0x_4x_5, x_2x_3x_5, x_1x_3x_5, x_0x_1x_5, x_1x_3x_4, x_0x_3x_4, x_1x_2x_4, x_0x_2x_3, x_0x_1x_2, x_4e_{1,3},\\ &x_5e_{1,6}, x_5e_{1,7}, x_5e_{1,8}, x_2e_{1,2}, x_4e_{1,5}, x_2e_{1,4}, x_5e_{1,9}, x_3e_{1,5}, x_5e_{1,10}, x_3e_{1,8}, x_4e_{1,9}, x_1e_{1,7},\\ &x_4e_{1,10}, x_3e_{1,10}, x_5e_{2,11}, x_5e_{2,12}, x_4e_{2,9}, x_5e_{2,13}, x_0x_4e_{1,4}, x_5e_{2,14}, x_5e_{2,15}, x_4e_{2,15}, x_5e_{3,7},\\ &x_2e_{2,16}\}\end{aligned} \tag{86}$$

Given $\bar{\mathcal{C}_1}$ and $\mathcal{B}$, we could apply the algorithm `Resolution`. Since in Algorithm 2, we just add the first-level bases in $\mathcal{B}$ to $\mathcal{C}_1$ and $\mathcal{H}_1$, then we have:

$$\mathcal{C}_1 = \{x_2x_4x_5, x_0x_4x_5, x_2x_3x_5, x_1x_3x_5, x_0x_1x_5, x_1x_3x_4, x_0x_3x_4, x_1x_2x_4, x_0x_2x_3, x_0x_1x_2\} \tag{87}$$
$$\mathcal{H}_1 = \{x_2x_4x_5, x_0x_4x_5, x_2x_3x_5, x_1x_3x_5, x_0x_1x_5, x_1x_3x_4, x_0x_3x_4, x_1x_2x_4, x_0x_2x_3, x_0x_1x_2\} \tag{88}$$

Notice that, since there are some bases in the third level that come before the second level bases in the ordered list $\mathcal{B}$, we actually need to first apply the algorithm to them before applying the algorithm to the larger second level bases. Here, we need to be especially careful with the order in the list, since they use different $\mathcal{C}_2$ when they are computed.For example, when we process the basis $x_5e_{2,11}$, the $\mathcal{C}_2$ we use is

$$\begin{aligned}\mathcal{C}_2 = \{&x_4e_{1,3} - x_3e_{1,1}, x_5e_{1,6} - x_4e_{1,4}, x_5e_{1,7} - x_3e_{1,2}, x_5e_{1,8} - x_1e_{1,1}, x_2e_{1,2} - x_0e_{1,1},\\ &x_4e_{1,5} - x_1e_{1,2}, x_2e_{1,4} - x_1e_{1,3}, x_5e_{1,9} - x_0e_{1,3}, x_3e_{1,5} - x_0e_{1,4}, x_5e_{1,10} - x_2e_{1,5},\\ &x_3e_{1,8} - x_2e_{1,6}, x_4e_{1,9} - x_2e_{1,7}, x_1e_{1,7} - x_0e_{1,6}, x_4e_{1,10} - x_0e_{1,8}, x_3e_{1,10} - x_1e_{1,9}\}\end{aligned} \tag{89}$$

Applying the Algorithm 3 `Reduce` to $x_5e_{2,11}$ and the corresponding $\mathcal{C}_2$, we have f, g to start with $f = x_3x_5e_{1,8} - x_2x_5e_{1,6}$ and $g = x_5e_{2,11}$, since $\varphi(e_{2,1}) = x_3e_{1,8} - x_2e_{1,6}$. Based on the term order on R-module, $lm(f) = x_3x_5e_{1,8}$. We have $lm(f) \in in\langle(C_2)$, since $x_3x_5e_{1,8} = x_3 \cdot x_5e_{1,8}$. Therefore, in the first iteration of the while loop, we would get $h = x_5e_{1,8} - x_1e_{1,1}$, $f = -x_2x_5e_{1,6} + x_1x_3e_{1,1}$, and $g = x_5e_{2,11} - x_3e_{2,4}$.

Then again $lm(f) = x_2x_5e_{1,6} \in (\mathcal{C}_2)$ since $x_2x_5e_{1,6} = x_2 \cdot x_5e_{1,6}$. Therefore, in the second iteration of the while loop, we would get $h = x_5e_{1,6} - x_4e_{1,4}$, $f = -x_2x_4e_{1,4} + x_1x_3e_{1,1}$, and $g = x_5e_{2,11} - x_3e_{2,4} + x_2e_{1,2}$.

$lm(f) = x_2x_4e_{1,4} \in (\mathcal{C}_2)$ since $x_2x_4e_{1,4} = x_4 \cdot x_2e_{1,4}$. Therefore, in the third iteration of the while loop, we would get $h = x_2e_{1,4} - x_1e_{1,3}$, $f = -x_1x_4e_{1,3} + x_1x_3e_{1,1}$, and $g = x_5e_{2,11} - x_3e_{2,4} + x_2e_{1,2} + x_4e_{1,7}$.

Finally, $lm(f) = x_1x_4e_{1,3} \in (\mathcal{C}_2)$ since $x_1x_4e_{1,3} = x_1 \cdot x_4e_{1,3}$. Therefore, in the fourth iteration of the while loop, we would get $h = x_4e_{1,3} - x_3e_{1,1}$, $f = 0$, and $g = x_5e_{2,11} - x_3e_{2,4} + x_2e_{1,2} + x_4e_{1,7} + x_1e_{1,1}$.

Due to $f = 0$, this pair (f, g) would be entered in the next step of Algorithm 2 `Resolution`. Because $f = 0$, we could update C_3 and H_3 as follows.

$$\begin{aligned}\mathcal{C}_3 &= \{x_5e_{2,11} - x_3e_{2,4} + x_2e_{1,2} + x_4e_{1,7} + x_1e_{1,1}\}\\ \mathcal{H}_3 &= \{x_5e_{2,11} - x_3e_{2,4} + x_2e_{1,2} + x_4e_{1,7} + x_1e_{1,1}\}\end{aligned} \tag{90}$$

We will do a similar procedure to the remaining three third level bases that were ordered before the second level bases $x_0x_4e_{1,4}$. The three bases are

$$x_5e_{2,12},\ x_4e_{2,9},\ x_5e_{2,13} \tag{91}$$

For $x_5e_{2,12}$, it produces the element $x_5e_{2,12} - x_4e_{2,8} + x_2e_{2,3} - x_0e_{2,1} + x_3e_{2,5}$ into $\mathcal{C}_3$
Hence, we have

$$\begin{aligned}\mathcal{C}_3 &= \{x_5e_{2,11} - x_3e_{2,4} + x_2e_{1,2} + x_4e_{1,7} + x_1e_{1,1}, x_5e_{2,12} - x_4e_{2,8} + x_2e_{2,3} - x_0e_{2,1} + x_3e_{2,5}\} \\ \mathcal{H}_3 &= \{x_5e_{2,11} - x_3e_{2,4} + x_2e_{1,2} + x_4e_{1,7} + x_1e_{1,1}, x_5e_{2,12} - x_4e_{2,8} + x_2e_{2,3} - x_0e_{2,1} + x_3e_{2,5}\}\end{aligned} \tag{92}$$

However, when we process the basis $x_4e_{2,9}$, we need to consider more to compute the generator of the kernel. After the first iteration in the algorithm `Reduce`, we have $f = -x_0x_4e_{1,4} + x_1x_3e_{1,2}$, $g = x_4e_{2,9} - x_3e_{2,6}$. However, $f \notin \text{in}(\mathcal{C}_2)$. According to the Algorithm `Resolution`, we should now add the reductum $-x_0x_4e_{1,4} + x_1x_3e_{1,2}$ to $\mathcal{C}_2$ and add $x_4e_{2,9} - x_3e_{2,6}$ $\mathcal{H}_3$. Now, we could further reduce $f = x_4e_{2,9} - x_3e_{2,6}$ to zero using the current $\mathcal{C}_2$. Then, we would have $x_4e_{2,9} - x_3e_{2,6} - e_{2,16}$.

Thus, we have the following.

$$\begin{aligned}\mathcal{C}_2 = \{&x_4e_{1,3} - x_3e_{1,1}, x_5e_{1,6} - x_4e_{1,4}, x_5e_{1,7} - x_3e_{1,2}, x_5e_{1,8} - x_1e_{1,1}, x_2e_{1,2} - x_0e_{1,1}, \\ &x_4e_{1,5} - x_1e_{1,2}, x_2e_{1,4} - x_1e_{1,3}, x_5e_{1,9} - x_0e_{1,3}, x_3e_{1,5} - x_0e_{1,4}, x_5e_{1,10} - x_2e_{1,5}, \\ &x_3e_{1,8} - x_2e_{1,6}, x_4e_{1,9} - x_2e_{1,7}, x_1e_{1,7} - x_0e_{1,6}, x_4e_{1,10} - x_0e_{1,8}, x_3e_{1,10} - x_1e_{1,9}, \\ &- x_0x_4e_{1,4} + x_1x_3e_{1,2}\}\end{aligned} \tag{93}$$

$$\begin{aligned}\mathcal{C}_3 = \{&x_5e_{2,11} - x_3e_{2,4} + x_2e_{1,2} + x_4e_{1,7} + x_1e_{1,1}, x_5e_{2,12} - x_4e_{2,8} + x_2e_{2,3} - x_0e_{2,1} + x_3e_{2,5}, \\ &x_4e_{2,9} - x_3e_{2,6} - e_{2,16}\} \\ \mathcal{H}_3 = \{&x_5e_{2,11} - x_3e_{2,4} + x_2e_{1,2} + x_4e_{1,7} + x_1e_{1,1}, x_5e_{2,12} - x_4e_{2,8} + x_2e_{2,3} - x_0e_{2,1} + x_3e_{2,5}, \\ &x_4e_{2,9} - x_3e_{2,6}\}\end{aligned} \tag{94}$$

Notice that the newly added basis $-x_0x_4e_{1,4} + x_1x_3e_{1,2}$ has a leading monomial of $x_0x_4e_{1,4}$ which is the same as the remaining second level basis in $\mathcal{B}$. That means we could remove it from the list of bases $\mathcal{B}$ that we need to compute for free.

Next, we could complete C_3 except for the last third level basis $x_2e_{2,16}$, since it is greater than the fourth level basis $x_5e_{3,7}$.

$$\begin{aligned}\mathcal{C}_3 = \{&x_5e_{2,11} - x_3e_{2,4} + x_2e_{1,2} + x_4e_{1,7} + x_1e_{1,1}, x_5e_{2,12} - x_4e_{2,8} + x_2e_{2,3} - x_0e_{2,1} + x_3e_{2,5}, \\ &x_4e_{2,9} - x_3e_{2,6} - e_{2,16}, x_5e_{2,13} - x_1e_{2,3} + x_0e_{2,2} - e_{2,16}, \\ &x_5e_{2,14} - x_4e_{2,10} + x_0e_{2,4} - x_2e_{2,6} - x_1e_{2,5}, x_5e_{2,15} - x_3e_{2,10} + x_1e_{2,8} - x_2e_{2,9} - x_0e_{2,7}, \\ &x_4e_{2,15} - x_3e_{2,14} + x_1e_{2,12} - x_0e_{2,11} + x_2e_{2,13}\} \\ \mathcal{H}_3 = \{&x_5e_{2,11} - x_3e_{2,4} + x_2e_{1,2} + x_4e_{1,7} + x_1e_{1,1}, x_5e_{2,12} - x_4e_{2,8} + x_2e_{2,3} - x_0e_{2,1} + x_3e_{2,5}, \\ &x_4e_{2,9} - x_3e_{2,6}, x_5e_{2,13} - x_1e_{2,3} + x_0e_{2,2} - e_{2,16} \\ &x_5e_{2,14} - x_4e_{2,10} + x_0e_{2,4} - x_2e_{2,6} - x_1e_{2,5}, x_5e_{2,15} - x_3e_{2,10} + x_1e_{2,8} - x_2e_{2,9} - x_0e_{2,7}, \\ &x_4e_{2,15} - x_3e_{2,14} + x_1e_{2,12} - x_0e_{2,11} + x_2e_{2,13}\}\end{aligned} \tag{95}$$

Now, we should process the only level four basis $x_5e_{3,7}$. We would obtain $f = 2\cdot(x_2e_{2,16} + x_0x_4e_{2,7} + x_0x_1e_{2,1} - x_1x_3e_{3,5})$, $g = x_5e_{3,7} - x_4e_{3,6} + x_3e_{3,5} - x_1e_{3,2} + x_0e_{3,1} - x_2e_{3,4} - x_2e_{3,3}$. We could notice that $f \notin \text{in}(\mathcal{C}_3)$. Therefore, we should again add f to the previous set of generators, which is $\mathcal{C}_3$, and add g to $\mathcal{H}_4$.

$$\begin{aligned}\mathcal{C}_3 = \{&x_5e_{2,11} - x_3e_{2,4} + x_2e_{1,2} + x_4e_{1,7} + x_1e_{1,1}, x_5e_{2,12} - x_4e_{2,8} + x_2e_{2,3} - x_0e_{2,1} + x_3e_{2,5}, \\ &x_4e_{2,9} - x_3e_{2,6} - e_{2,16}, x_5e_{2,13} - x_1e_{2,3} + x_0e_{2,2} + e_{2,16}, \\ &x_5e_{2,14} - x_4e_{2,10} + x_0e_{2,4} - x_2e_{2,6} - x_1e_{2,5}, x_5e_{2,15} - x_3e_{2,10} + x_1e_{2,8} - x_2e_{2,9} - x_0e_{2,7}, \\ &x_4e_{2,15} - x_3e_{2,14} + x_1e_{2,12} - x_0e_{2,11} + x_2e_{2,13}, \\ &2\cdot(x_2e_{2,16} + x_0x_4e_{2,7} + x_0x_1e_{2,1} - x_1x_3e_{3,5})\}\end{aligned} \tag{96}$$

$$\mathcal{H}_4 = \{x_5e_{3,7} - x_4e_{3,6} + x_3e_{3,5} - x_1e_{3,2} + x_0e_{3,1} - x_2e_{3,4} - x_2e_{3,3}\} \tag{97}$$

In addition, with this updated $\mathcal{C}_3$, we could process the basis $x_5e_{3,7}$ to zero. Therefore, we have $\mathcal{C}_4$ computed.

$$\mathcal{C}_4 = \{x_5e_{3,7} - x_4e_{3,6} + x_3e_{3,5} - x_1e_{3,2} + x_0e_{3,1} - x_2e_{3,4} - x_2e_{3,3} - e_{3,8}\} \tag{98}$$

Similarly to the previous level, we do not need to process $x_2e_{2,16}$ since it was previously added for free.

Therefore, we now compute the entire Schreyer resolution with the Gröner bases of the kernels.

$$\mathcal{C}_1 = \{x_2x_4x_5, x_0x_4x_5, x_2x_3x_5, x_1x_3x_5, x_0x_1x_5, x_1x_3x_4, x_0x_3x_4, x_1x_2x_4, x_0x_2x_3, x_0x_1x_2\} \tag{99}$$

$$\mathcal{C}_2 = \{x_4e_{1,3} - x_3e_{1,1}, x_5e_{1,6} - x_4e_{1,4}, x_5e_{1,7} - x_3e_{1,2}, x_5e_{1,8} - x_1e_{1,1}, x_2e_{1,2} - x_0e_{1,1}, \tag{100}$$

$$x_4e_{1,5} - x_1e_{1,2}, x_2e_{1,4} - x_1e_{1,3}, x_5e_{1,9} - x_0e_{1,3}, x_3e_{1,5} - x_0e_{1,4}, x_5e_{1,10} - x_2e_{1,5}, \tag{101}$$

$$x_3e_{1,8} - x_2e_{1,6}, x_4e_{1,9} - x_2e_{1,7}, x_1e_{1,7} - x_0e_{1,6}, x_4e_{1,10} - x_0e_{1,8}, x_3e_{1,10} - x_1e_{1,9}, \tag{102}$$

$$- x_0x_4e_{1,4} + x_1x_3e_{1,2}\} \tag{103}$$

$$\mathcal{C}_3 = \{x_5e_{2,11} - x_3e_{2,4} + x_2e_{1,2} + x_4e_{1,7} + x_1e_{1,1}, x_5e_{2,12} - x_4e_{2,8} + x_2e_{2,3} - x_0e_{2,1} + x_3e_{2,5}, \tag{104}$$

$$x_4e_{2,9} - x_3e_{2,6} - e_{2,16}, x_5e_{2,13} - x_1e_{2,3} + x_0e_{2,2} + e_{2,16}, \tag{105}$$

$$x_5e_{2,14} - x_4e_{2,10} + x_0e_{2,4} - x_2e_{2,6} - x_1e_{2,5}, x_5e_{2,15} - x_3e_{2,10} + x_1e_{2,8} - x_2e_{2,9} - x_0e_{2,7}, \tag{106}$$

$$x_4e_{2,15} - x_3e_{2,14} + x_1e_{2,12} - x_0e_{2,11} + x_2e_{2,13}, \tag{107}$$

$$2 \cdot (x_2e_{2,16} + x_0x_4e_{2,7} + x_0x_1e_{2,1} - x_1x_3e_{3,5})\} \tag{108}$$

$$\mathcal{C}_4 = \{x_5e_{3,7} - x_4e_{3,6} + x_3e_{3,5} - x_1e_{3,2} + x_0e_{3,1} - x_2e_{3,4} - x_2e_{3,3} - e_{3,8}\} \tag{109}$$

$$\Phi : S^1 \xrightarrow{\varphi_3} S^8 \xrightarrow{\varphi_2} S^{16} \xrightarrow{\varphi_1} S^{10} \to 0 \tag{110}$$

4 Minimalize Schreyer Resolution

If we assume that the previous construction of module M is graded, which means that the modules are homogeneous, then we introduce the following way to minimalize the Schreyer resolutions [LS98].

Let $\hat{\mathcal{K}}_i$ be the subsets of $\mathcal{B}_i$ of the monomials corresponding to S-polynomials that reduce to nonzero elements in `Resolution`.

Algorithm 4 Minimalize[$\mathcal{H}_i$]

for $m \in \hat{\mathcal{K}}_i$ **do**
  (f, g) := (the reductum, the syzygy) obtained by m
  for $h \in \mathcal{H}_i$ **do**
    h_e := the component of h corresponding to $e \in \mathcal{E}_{i-1},\ \varphi_{i-1}(e) = f$
    $h := h + h_e \cdot g$
    $h :=$ Strip[h]
  end for
end for
return $\mathcal{H}_i$

and the algorithm `Strip` used above is described as follows.

Algorithm 5 Strip[h]

for $m \in \hat{K}_{i-1}$ **do**
  g := the syzygy of $\mathcal{C}_{i-1}$ obtained by m
  h_e := the component of h corresponding to $e \in \mathcal{E}_{i-1},\ \varphi_{i-1}(e) = g$
  $h := h - h_e \cdot e$
end for
return h

The proof of the Algorithm 4 could be referred to the Proposition 4.5 in the La Scala and Stillman's paper [LS98].

Example 4.1. *Minimalize the Schreyer resolution calculated in Example 3.2.*

To minimalize the Schreyer resolution, we could first find the differentials representing the homomorphisms φ_i.

$$\varphi_1 = \begin{bmatrix} -x_3 & 0 & 0 & -x_1 & -x_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_3 & 0 & x_2 & -x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1x_3 \\ x_4 & 0 & 0 & 0 & 0 & 0 & -x_1 & -x_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_4 & 0 & 0 & 0 & 0 & x_2 & 0 & -x_0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_0x_4 \\ 0 & 0 & 0 & 0 & 0 & x_4 & 0 & 0 & x_3 & -x_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_2 & 0 & -x_0 & 0 & 0 & 0 \\ 0 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_2 & x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & 0 & 0 & -x_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_5 & 0 & 0 & 0 & x_4 & 0 & 0 & -x_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_5 & 0 & 0 & 0 & x_4 & x_3 & 0 \end{bmatrix} \tag{111}$$

$$\varphi_2 = \begin{bmatrix} x_1 & -x_0 & 0 & 0 & 0 & 0 & 0 & 2x_0x_1 \\ x_2 & 0 & 0 & x_0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & -x_1 & 0 & 0 & 0 & 0 \\ -x_3 & 0 & 0 & 0 & x_0 & 0 & 0 & 0 \\ 0 & x_3 & 0 & 0 & -x_1 & 0 & 0 & -2x_1x_3 \\ 0 & 0 & -x_3 & 0 & -x_2 & 0 & 0 & 0 \\ x_4 & 0 & 0 & 0 & 0 & -x_0 & 0 & 2x_0x_4 \\ 0 & -x_4 & 0 & 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & x_4 & 0 & 0 & -x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_4 & -x_3 & 0 & 0 \\ x_5 & 0 & 0 & 0 & 0 & 0 & -x_0 & 0 \\ 0 & x_5 & 0 & 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & x_5 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & x_5 & 0 & -x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_5 & x_4 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 2x_2 \end{bmatrix} \tag{112}$$

$$\varphi_3 = \begin{bmatrix} x_0 \\ -x_1 \\ -x_2 \\ 0 \\ x_3 \\ -x_4 \\ x_5 \\ -1 \end{bmatrix} \tag{113}$$

We need to eliminate the constant terms in the matrices, which are -1 at the $(16, 3)$ entry of the matrix representing φ_2, 1 at the $(16, 4)$ entry of the matrix representing φ_2, and -1 at the $(8, 1)$ entry of the matrix representing φ_3. According to the algorithm, we could perform the column operation in φ_2 to add the third column to the fourth column to cancel the 1 in the fourth column.

Hence, we have

$$\varphi_2 = \begin{bmatrix} x_1 & -x_0 & 0 & 0 & 0 & 0 & 0 & 2x_0x_1 \\ x_2 & 0 & 0 & x_0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & -x_1 & 0 & 0 & 0 & 0 \\ -x_3 & 0 & 0 & 0 & x_0 & 0 & 0 & 0 \\ 0 & x_3 & 0 & 0 & -x_1 & 0 & 0 & -2x_1x_3 \\ 0 & 0 & -x_3 & -x_3 & -x_2 & 0 & 0 & 0 \\ x_4 & 0 & 0 & 0 & 0 & -x_0 & 0 & 2x_0x_4 \\ 0 & -x_4 & 0 & 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & x_4 & x_4 & 0 & -x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_4 & -x_3 & 0 & 0 \\ x_5 & 0 & 0 & 0 & 0 & 0 & -x_0 & 0 \\ 0 & x_5 & 0 & 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & x_5 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & x_5 & 0 & -x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_5 & x_4 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2x_2 \end{bmatrix} \tag{114}$$

Next, following the procedure in the algorithm `Minimalize`, we should remove the columns with constant entry and the column in the previous differential they use for constant entry.

In this case, we should remove the third column of φ_2 and the first column of φ_3, and the corresponding columns in the previous differnetial that are the 16^{th} column of φ_1 and the 8^{th} column of φ_2.

Then, we have

$$\varphi_1 = \begin{bmatrix} -x_3 & 0 & 0 & -x_1 & -x_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_3 & 0 & x_2 & -x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_4 & 0 & 0 & 0 & 0 & 0 & -x_1 & -x_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_4 & 0 & 0 & 0 & 0 & x_2 & 0 & -x_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_4 & 0 & 0 & x_3 & -x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_2 & 0 & -x_0 & 0 & 0 \\ 0 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_2 & x_1 & 0 & 0 \\ 0 & 0 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & 0 & 0 & -x_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_5 & 0 & 0 & 0 & x_4 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_5 & 0 & 0 & 0 & x_4 & x_3 \end{bmatrix} \tag{115}$$

$$\varphi_2 = \begin{bmatrix} x_1 & -x_0 & 0 & 0 & 0 & 0 \\ x_2 & 0 & x_0 & 0 & 0 & 0 \\ 0 & x_2 & -x_1 & 0 & 0 & 0 \\ -x_3 & 0 & 0 & x_0 & 0 & 0 \\ 0 & x_3 & 0 & -x_1 & 0 & 0 \\ 0 & 0 & -x_3 & -x_2 & 0 & 0 \\ x_4 & 0 & 0 & 0 & -x_0 & 0 \\ 0 & -x_4 & 0 & 0 & x_1 & 0 \\ 0 & 0 & x_4 & 0 & -x_2 & 0 \\ 0 & 0 & 0 & -x_4 & -x_3 & 0 \\ x_5 & 0 & 0 & 0 & 0 & -x_0 \\ 0 & x_5 & 0 & 0 & 0 & x_1 \\ 0 & 0 & x_5 & 0 & 0 & x_2 \\ 0 & 0 & 0 & x_5 & 0 & -x_3 \\ 0 & 0 & 0 & 0 & x_5 & x_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{116}$$

Correspondingly, we would have the generators for the kernels in the minimal case.

$$\mathcal{C}_1 = \{x_2x_4x_5, x_0x_4x_5, x_2x_3x_5, x_1x_3x_5, x_0x_1x_5, x_1x_3x_4, x_0x_3x_4, x_1x_2x_4, x_0x_2x_3, x_0x_1x_2\} \tag{117}$$

$$\mathcal{C}_2 = \{x_4e_{1,3} - x_3e_{1,1}, x_5e_{1,6} - x_4e_{1,4}, x_5e_{1,7} - x_3e_{1,2}, x_5e_{1,8} - x_1e_{1,1}, x_2e_{1,2} - x_0e_{1,1}, \tag{118}$$

$$x_4e_{1,5} - x_1e_{1,2}, x_2e_{1,4} - x_1e_{1,3}, x_5e_{1,9} - x_0e_{1,3}, x_3e_{1,5} - x_0e_{1,4}, x_5e_{1,10} - x_2e_{1,5}, \tag{119}$$

$$x_3e_{1,8} - x_2e_{1,6}, x_4e_{1,9} - x_2e_{1,7}, x_1e_{1,7} - x_0e_{1,6}, x_4e_{1,10} - x_0e_{1,8}, x_3e_{1,10} - x_1e_{1,9}\} \tag{120}$$

$$\mathcal{C}_3 = \{x_5e_{2,11} - x_3e_{2,4} + x_2e_{1,2} + x_4e_{1,7} + x_1e_{1,1}, x_5e_{2,12} - x_4e_{2,8} + x_2e_{2,3} - x_0e_{2,1} + x_3e_{2,5}, \tag{121}$$

$$x_5e_{2,13} - x_1e_{2,3} + x_0e_{2,2} + x_4e_{2,9} - x_3e_{2,6}, x_5e_{2,14} - x_4e_{2,10} + x_0e_{2,4} - x_2e_{2,6} - x_1e_{2,5}, \tag{122}$$

$$x_5e_{2,15} - x_3e_{2,10} + x_1e_{2,8} - x_2e_{2,9} - x_0e_{2,7}, \tag{123}$$

$$x_4e_{2,15} - x_3e_{2,14} + x_1e_{2,12} - x_0e_{2,11} + x_2e_{2,13}\} \tag{124}$$

Also, the minimal resolution is as follows:

$$\Phi : \; S^6 \xrightarrow{\varphi_2} S^{15} \xrightarrow{\varphi_1} S^{10} \to 0 \tag{125}$$

where φ_1 and φ_2 are the R-module homomorphisms represented using the matrices described above.

Using this minimal resolution, we could measure how far M is from being a free R-module.

5 Conclusion

Using the Gröbner bases and syzygies, we are able to compute a Schreyer resolution from constructing a Schreyer frame. We give the proof to the algorithm that we used to find the Schreyer resolution, and an example of computing the frame and using the frame to find the Schreyer resolution is discussed. In addition, the algorithm to minimalize a graded Schreyer resolution is introduced in the paper and an example is included to show how to use the algorithm. The process of computing the frame and the resolution by hand would be time-consuming, especially when we start with a module with a huge basis. However, the algorithms could be implemented using a computer algebra system like `Macaulay2` as tried in the paper by La Scala and Stillman [LS98].

References

[LS98] R. La Scala and M. Stillman. “Strategies for Computing Minimal Free Resolutions”. en. In: *Journal of Symbolic Computation* 26.4 (Oct. 1998), pp. 409–431. ISSN: 07477171. DOI: 10.1006/jsco.1998.0221. URL: https://linkinghub.elsevier.com/retrieve/pii/S0747717198902218 (visited on 04/17/2022).

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