Computing Schreyer Resolutions

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Goal

- Find a Schreyer resolution of a given module over a polynomial ring (by first computing a Schreyer frame)

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$$\Phi : \cdots \to F_l \xrightarrow{\varphi_l} F_{l-1} \xrightarrow{\varphi_{l-1}} \cdots \xrightarrow{\varphi_1} F_0 \tag{1}$$

R-module

Let $R = k[x_1, \cdots, x_n]$ be the polynomial ring over a field k.

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Let $R = k[x_1, \cdots, x_n]$ be the polynomial ring over a field k.

Definition (R-Module)

Suppose that R is a ring, and 1 is its multiplicative identity. An R-module M consists of an Abelian group $(M, +)$ and an operation $\cdot : R \times M \to M$ such that for all r, s in R and x, y in M, we have

- $r \cdot (x + y) = r \cdot x + r \cdot y$
- $(r + s) \cdot x = r \cdot x + s \cdot x$
- $(rs) \cdot x = r \cdot (s \cdot x)$
- $1 \cdot x = x$

Like a "vector space" over the ring R.
i.e. the scalar multiplications are actions of multiplying by a ring element in a module.

R-module homomorphism

Definition (R-module Homomorphism)

A R-module homomorphism between M and N is a function $f : M \to N$ such that for any $x, y \in M$ and $r \in R$,

- $f(x + y) = f(x) + f(y)$,
- $f(rx) = rf(x)$

A group homomorphism under addition that commutes with "scalar multiplication".

Free R-module

Definition (Free R-module)

For a R-module M, the set $\mathcal{B} \subseteq M$ is a basis for M if:

- $\mathcal{B}$ is a spanning set for M;
- $\mathcal{B}$ is linearly independent.

A free R-module is a module with a basis.
Example: R^n has basis $e_1, ..., e_n$ where e_i is the i^{th} standard basis vector.

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- Let K be a field, then all K-modules are free modules.
- Not every module is a free R-module.

Motivating Problem

- Find a spanning set $\mathcal{B}_0$ of M, and use it to define a homomorphism from a free R-module R^{m_0} onto M

$$R^{m_0} \twoheadrightarrow M$$

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- Find the the non-trivial kernel of the map $R^{m_0} \twoheadrightarrow M$

$$\begin{tikzcd} & R^{m_0} \arrow[r, two heads] & M \\ K_1 \arrow[ur, hook, "\iota_1"'] & & \end{tikzcd}$$

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R^{m_1} \arrow[dr, two heads, "\varphi_1'"'] & & R^{m_0} \arrow[r, two heads] & M \\
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- Find the kernel of φ_1'

$$\begin{tikzcd}
K_2 \arrow[dr, hook, "\iota_2"] & & & & \\
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- Using function composition, we define φ_1, a free R-module homomorphism

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- Using function composition, we define φ_1, a free R-module homomorphism
- Iterating this process

$$
\begin{tikzcd}
 & & & & & & K_2 \arrow[dr, hook, "\iota_2"] & & & & \\
\cdots \arrow[r] & R^{m_l} \arrow[rr, "\varphi_l"] \arrow[dr, two heads, "\varphi_l'"'] & & R^{m_{l-1}} \arrow[r, "\varphi_{l-1}"] & \cdots \arrow[r, "\varphi_3"] & R^{m_2} \arrow[rr, "\varphi_2"] \arrow[ur, two heads, "\varphi_2'"] & & R^{m_1} \arrow[rr, "\varphi_1"] \arrow[dr, two heads, "\varphi_1'"'] & & R^{m_0} \arrow[r, two heads] & M \\
 & & K_l \arrow[ur, hook, "\iota_l"'] & & & & & & K_1 \arrow[ur, hook, "\iota_1"'] & &
\end{tikzcd}
$$

Finding the Kernel?

Example

Let $R = \mathbb{R}[x, y, z]$ be a ring, $I = \langle x^2, xy, xz \rangle$ be an ideal in R, and $M = R/I$ be a R-module. How far is M from being a free R-module?

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$$
\varphi_1 = \varphi_i' = \begin{bmatrix} x^2 & xy & xz \end{bmatrix}
$$

$$
\left\{ \begin{bmatrix} y \\ -x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ z \\ -y \end{bmatrix}, \begin{bmatrix} z \\ 0 \\ -x \end{bmatrix} \right\} \tag{2}
$$

But how should we compute K_2?

Term Orders

Definition (Graded Lex (GrLex) Order)

Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$. We say $\alpha >_{\text{grlex}} \beta$ if

$$|\alpha| = \sum_{i=1}^{n} \alpha_i > |\beta| = \sum_{i=1}^{n} \beta_i, \text{ or } |\alpha| = |\beta| \text{ and } \alpha >_{\text{lex}} \beta \tag{3}$$

Example

Compare $x_1^5 x_2 x_3^2$ and $x_1 x_2^3 x_3^4$ using the GrLex Order.

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Example

Compare $x_1^5 x_2 x_3^2$ and $x_1 x_2^3 x_3^4$ using the GrLex Order.

$$x_1^5 x_2 x_3^2 >_{\text{grlex}} x_1 x_2^3 x_3^4$$

Reason: the coefficient tuple $|(5,1,2)| = |(1,3,4)| = 8$, and $(5,1,2) > (1,3,4)$

Term Orders

Definition (Term order on a Free Module)

The term order on a free module F over k (a total order on the monomials of F) such that:

- if $m <_F n$, then $t \cdot m <_F t \cdot n$;
- if $s <_R t$, then $s \cdot e <_F t \cdot e$;

for all m, n monomials of F, s, t power products in R, and e any basis element of F.

Gröbner Basis

Definition (**Gröbner Basis**)

Fix a monomial order on the polynomial ring R and fix a term order on a R-module M. A finite set $G = \{g_1, \cdots, g_t\}$ of a submodule M of R^m different from $\{0\}$ is said to be **Gröbner basis** of M if

$$\langle \mathrm{LT}(g_1), \cdots, \mathrm{LT}(g_t) \rangle = \langle \mathrm{LT}(M) \rangle$$

where $\langle X \rangle$ is the submodule generated by elements of X where X be a subset of M.

Definition (**Reduced Gröbner Basis**)

The **reduced** Gröbner basis is a Gröbner basis G for M such that:

- $\mathrm{LC}(p) = 1$ for all $p \in G$
- For all $p \in G$, none of the monomials in p lies in $\langle \mathrm{LT}(G \backslash \{p\}) \rangle$

Schreyer Resolution and Schreyer Frame

Definition (Free resolution)

Let R be the polynomial ring as previously defined, and let F_i be free R-modules with a given (canonical) basis $\mathcal{E}_i$ of F_i and $M = F_0/I$. A **free resolution** of M is a sequence of homomorphisms of free R-modules:

$$\Phi : \cdots \to F_l \xrightarrow{\varphi_l} F_{l-1} \xrightarrow{\varphi_{l-1}} \cdots \xrightarrow{\varphi_1} F_0 \to 0 \tag{4}$$

such that $\operatorname{im} \varphi_i = \ker \varphi_{i-1}$ for $i \geq 1$ and $\operatorname{coker} \varphi_1 = M$.

Define $\mathcal{C}_i = \varphi_i(\mathcal{E}_i)$ as the image of the given basis and define the level of an element f in $\mathcal{C}_i$ as $lev(f) = i$.

Schreyer resolution and Schreyer Frame

Definition (Term Ordering on a Resolution)

Let $\tau = \{\tau_i\}$ be a sequence of term ordering on the modules F_i. We call τ a **term ordering on** Φ if it satisfies the following compatibility relationship:

$$s \cdot e_1 <_{\tau_i} t \cdot e_2 \text{ whenever } s \cdot \mathrm{LM}(\varphi_i(e_1)) <_{\tau_{i-1}} t \cdot \mathrm{LM}(\varphi_i(e_2)) \tag{5}$$

where e_1 and e_2 are elements of $\mathcal{E}_i$.

Schreyer resolution and Schreyer Frame

Definition (**Schreyer Resolution**)

A **Schreyer resolution** of an R-module $M = F_0/I$ is free resolution:

$$\Phi : \cdots \to F_l \xrightarrow{\varphi_l} F_{l-1} \xrightarrow{\varphi_{l-1}} \cdots \xrightarrow{\varphi_1} F_0, \tag{6}$$

such that:

- There is a term ordering $\tau = \{\tau_i\}$ on Φ
- $\varphi_i(\mathcal{E}_i)$ forms a reduced Gröbner basis of $\operatorname{im} \varphi_i = \ker \varphi_{i-1}$ (for all i where $F_i \neq 0$) with respect to the term order τ_{i-1}.

Schreyer Resolution and Schreyer Frame

Definition (Initial Term of A Resolution)

Given Φ as previously described and a term ordering τ on Φ, define the **initial terms** of Φ, $\mathrm{in}(\Phi)$, as the sequence of (graded) R-homomorphisms:

$$\Xi = \mathrm{in}(\Phi) : \cdots \to F_l \xrightarrow{\xi_l} F_{l-1} \xrightarrow{\xi_{l-1}} \cdots \xrightarrow{\xi_2} F_1 \xrightarrow{\xi_1} F_0 \tag{7}$$

where $\xi_i(e) = \mathrm{LM}(\varphi_i(e))$, for all e in $\mathcal{E}_i$. That is, the differentials in $\mathrm{in}(\Phi)$ are obtained by taking the leading monomials of each entry in the differentials in Φ.

Schreyer Resolution and Schreyer Frame

Definition (Schreyer Frame)

A **Schreyer frame** of $M = F_0/I$ is a sequence of R-homomophisms:

$$\Xi : \cdots \to F_l \xrightarrow{\xi_l} F_{l-1} \xrightarrow{\xi_{l-1}} \cdots \xrightarrow{\xi_2} F_1 \xrightarrow{\xi_1} F_0 \tag{8}$$

where each column is a monomial, and a term ordering on Ξ such that:

- $\xi_1(\mathcal{E}_1)$ is a minimal set of generators for $\mathrm{in}(I)$;
- $\xi_i(\mathcal{E}_i)$ is a minimal set of generators for $\mathrm{in}(\ker \xi_{i-1})$ for $i \geq 2$.

Computing the Schreyer Frame

Definition (Colon Ideal)

Let I be an ideal of a commutative ring R, let p be an polynomial in R, then the **colon ideal** is defined as:

$$(I : p) = \{r \in R \mid rp \in I\} \tag{9}$$

Computing the Schreyer Frame

Let $e \in \mathcal{E}_{i-1}$, then

$$\mathcal{B}_i = \xi_i(\mathcal{E}_i) \tag{10}$$

$$\mathcal{E}_i(e) = \{\epsilon \in \mathcal{E}_i | \xi_i(\epsilon) = s \cdot e, \text{ for some power product } s\} \tag{11}$$

Lemma

In a Schreyer frame Ξ, $\mathrm{in}(\ker \xi_i)$ is minimally generated by

$$\bigcup_{e \in \mathcal{E}_{i-1}} \bigcup_{j=2}^{r} \mathrm{mingens}((t_1, \cdots, t_{j-1}) : t_j) \cdot \epsilon_j \tag{12}$$

where for each e in the outer union, if $\mathcal{E}_i(e) = \{\epsilon_1, \cdots, \epsilon_r\}$, then $\xi_i(\epsilon_j) = t_j \cdot e$, and $\mathrm{mingens}$ defines the minimal generators of the considered monomial ideal.

Example

$$\bigcup_{e \in \mathcal{E}_{i-1}} \bigcup_{j=2}^{r} \text{mingens}((t_1, \cdots, t_{j-1}) : t_j) \cdot \epsilon_j \tag{13}$$

Example

Let $R = \mathbb{R}[x, y, z]$ be a ring, $I = \langle x^2, xy, xz \rangle$ be an ideal in R, and $M = R/I$ be a R-module. Computing the Schreyer frame of M.

- $\mathcal{B}_1 = \{x^2 e_{0,1}, xye_{0,1}, xze_{0,1}\}$

- $\mathcal{E}_0 = \{e_{0,1}\}$
- $\mathcal{E}_1 = \{e_{1,1}, e_{1,2}, e_{1,3}\}$

Example

$$\bigcup_{e \in \mathcal{E}_{i-1}} \bigcup_{j=2}^{r} \operatorname{mingens}((t_1, \cdots, t_{j-1}) : t_j) \cdot \epsilon_j \tag{13}$$

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- $\mathcal{B}_1 = \{x^2 e_{0,1}, xye_{0,1}, xze_{0,1}\}$
- $\mathcal{B}_2 = \{xe_{1,2}, xe_{1,3}, ye_{1,3}\}$

- $\mathcal{E}_0 = \{e_{0,1}\}$
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Example

$$\bigcup_{e\in\mathcal{E}_{i-1}}\bigcup_{j=2}^{r}\operatorname{mingens}((t_1,\cdots,t_{j-1}):t_j)\cdot\epsilon_j \tag{13}$$

Example

Let $R=\mathbb{R}[x,y,z]$ be a ring, $I=\langle x^2,xy,xz\rangle$ be an ideal in R, and $M=R/I$ be a R-module. Computing the Schreyer frame of M.

- $\mathcal{B}_1=\{x^2e_{0,1},xye_{0,1},xze_{0,1}\}$
- $\mathcal{B}_2=\{xe_{1,2},xe_{1,3},ye_{1,3}\}$

- $\mathcal{E}_0=\{e_{0,1}\}$
- $\mathcal{E}_1=\{e_{1,1},e_{1,2},e_{1,3}\}$
- $\mathcal{E}_2=\{e_{2,1},e_{2,2},e_{2,3}\}$

Example

$$\bigcup_{e \in \mathcal{E}_{i-1}} \bigcup_{j=2}^{r} \mathrm{mingens}((t_1, \cdots, t_{j-1}) : t_j) \cdot \epsilon_j \tag{13}$$

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- $\mathcal{B}_1 = \{x^2 e_{0,1}, xye_{0,1}, xze_{0,1}\}$
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Example

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Example

Let $R = \mathbb{R}[x, y, z]$ be a ring, $I = \langle x^2, xy, xz \rangle$ be an ideal in R, and $M = R/I$ be a R-module. Computing the Schreyer frame of M.

- $\mathcal{B}_1 = \{x^2 e_{0,1}, xy e_{0,1}, xz e_{0,1}\}$
- $\mathcal{B}_2 = \{x e_{1,2}, x e_{1,3}, y e_{1,3}\}$
- $\mathcal{B}_3 = \{x e_{2,3}\}$

- $\mathcal{E}_0 = \{e_{0,1}\}$
- $\mathcal{E}_1 = \{e_{1,1}, e_{1,2}, e_{1,3}\}$
- $\mathcal{E}_2 = \{e_{2,1}, e_{2,2}, e_{2,3}\}$
- $\mathcal{E}_3 = \{e_{3,1}\}$

Compute the Schreyer Resolution

Theorem

If Ξ is a Schreyer frame for a R-module M, then there exists a Schreyer resolution Φ such that $\Xi = \mathrm{in}(\Phi)$.

Computing the Schreyer Resolution

Algorithm 2 Resolution[$\bar{\mathcal{C}}_1$]

Input: a reduced Gröbner basis $\bar{\mathcal{C}}_1$ of I, and an ordered union $\mathcal{B}$ of bases of all levels
Output: the set of Gröbner bases $\mathcal{C}_1, \cdots, \mathcal{C}_l$, and the set of corresponding syzygies $\mathcal{H}_1, \cdots, \mathcal{H}_l$

$\mathcal{C}_i, \mathcal{H}_i := \emptyset (1 \leq i \leq l)$
while $\mathcal{B} \neq \emptyset$ **do**
  $m := min \mathcal{B}$
  $\mathcal{B} := \mathcal{B} \backslash \{m\}$
  $i := lev(m)$
  if $i = 1$ **then**
    $g :=$ the element of $\bar{\mathcal{C}}_i$ s.t. $lm(g) = m$
    $\mathcal{C}_1 := \mathcal{C}_1 \cup \{g\}$
    $\mathcal{H}_1 := \mathcal{H}_i \cup \{g\}$
    else
    $(f, g) :=$ Reduce[$m, \mathcal{C}_{i-1}$]
    $\mathcal{C}_i := \mathcal{C}_i \cup \{g\}$
    if $f \neq 0$ **then**
      $\mathcal{C}_{i-1} := \mathcal{C}_{i-1} \cup \{f\}$
      $\mathcal{B} := \mathcal{B} \backslash \{lm(f)\}$
      else
      $\mathcal{H}_i := \mathcal{H}_i \cup \{g\}$
    end if
  end if
end while
return $\mathcal{C}_i, \mathcal{H}_i (1 \leq i \leq l)$

Example

Example

Given the Schreyer frame in the previous example, create a Schreyer resolution corresponding to it.

- $\mathcal{B}_1 = \{x^2 e_{0,1}, xye_{0,1}, xze_{0,1}\}$
- $\mathcal{B}_2 = \{xe_{1,2}, xe_{1,3}, ye_{1,3}\}$
- $\mathcal{B}_3 = \{xe_{2,3}\}$

- $\mathcal{C}_1 = \{x^2 e_{0,1}, xye_{0,1}, xze_{0,1}\}$

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- $\mathcal{B}_2 = \{xe_{1,2}, xe_{1,3}, ye_{1,3}\}$
- $\mathcal{B}_3 = \{xe_{2,3}\}$

- $\mathcal{C}_1 = \{x^2 e_{0,1}, xye_{0,1}, xze_{0,1}\}$
- $\mathcal{C}_2 = \{xe_{1,2} - ye_{1,1}, xe_{1,3} - ze_{1,1}, ye_{1,3} - ze_{1,2}\}$

Example

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Given the Schreyer frame in the previous example, create a Schreyer resolution corresponding to it.

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- $\mathcal{B}_2 = \{xe_{1,2}, xe_{1,3}, ye_{1,3}\}$
- $\mathcal{B}_3 = \{xe_{2,3}\}$

- $\mathcal{C}_1 = \{x^2 e_{0,1}, xye_{0,1}, xze_{0,1}\}$
- $\mathcal{C}_2 = \{xe_{1,2} - ye_{1,1}, xe_{1,3} - ze_{1,1}, ye_{1,3} - ze_{1,2}\}$
- $\mathcal{C}_3 = \{xe_{2,3} - ye_{2,2} + ze_{2,1}\}$

The End

Thank you for your attention!