Partial Differential Equations with Applications

Examples to supplement Chapter 2 on Second Order PDEs

Example 1 (The Linear Wave Equation, $u_{tt} - c^2 u_{xx} = 0$.) The *Linear Wave Equation* in **laboratory coordinates** is:

$$u_{tt} - c^2 u_{xx} = 0.$$

We define characteristic co-ordinates ξ, η:

$$\xi = x + ct \quad \text{and} \quad \eta = x - ct,$$

which gives the following **differential operator** relations (*chain rule*):

$$\frac{\partial}{\partial t} + c\frac{\partial}{\partial x} = 2c\frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial t} - c\frac{\partial}{\partial x} = -2c\frac{\partial}{\partial \eta}.$$

Thus, the second order linear wave equation, in laboratory co-ordinates, can be written as:

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) u = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = -4c^2\frac{\partial^2 u}{\partial \xi \partial \eta}.$$

This shows that, **in terms of the characteristic co-ordinates**, the linear wave equation can be written

$$u_{\xi\eta} = 0.$$

This can easily be integrated to give **d'Alembert's solution**

$$u(\xi, \eta) = f(\xi) + g(\eta), \quad \text{or} \quad u(x,t) = f(x+ct) + g(x-ct),$$

where f and g are arbitrary functions.

Example 2 (Laplace's Equation in Polar Coordinates.) Consider Laplace's equation in rectangular Cartesian coordinates

$$\Delta\, u \equiv u_{xx} + u_{yy} = 0.$$

In *vector calculus* we learn that

$$\Delta u = \nabla \cdot \nabla u = \text{div grad}\, u,$$

and that we can write ∇ and the scalar product in *any* set of *orthogonal coordinates*. By doing this we can derive the formula for Δ in any of these coordinate systems.

Here we use a *direct change of coordinates* to derive the same formula. The calculation is not as elegant, but doesn't rely on orthogonality, so is a more general approach.

We have

$$\left.\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} r &= \sqrt{x^2+y^2} \\ \tan\theta &= \frac{y}{x} \end{aligned}\right.$$

We first calculate the derivatives of r and θ:

$$r_x = \frac{x}{r} = \cos\theta, \quad r_y = \frac{y}{r} = \sin\theta, \quad \theta_x = -\frac{y}{r^2} = -\frac{\sin\theta}{r}, \quad \theta_y = \frac{x}{r^2} = \frac{\cos\theta}{r}, \tag{1}$$

and

$$\begin{aligned} r_{xx} &= \frac{y^2}{r^3} = \frac{\sin^2\theta}{r}, \quad r_{xy} = -\frac{xy}{r^3} = -\frac{\sin 2\theta}{2r}, \quad r_{yy} = \frac{x^2}{r^3} = \frac{\cos^2\theta}{r}, \\ \theta_{xx} &= \frac{2xy}{r^4} = \frac{\sin 2\theta}{r^2}, \quad \theta_{xy} = \frac{y^2 - x^2}{r^4} = -\frac{\cos 2\theta}{r^2}, \quad \theta_{yy} = -\frac{2xy}{r^4} = -\frac{\sin 2\theta}{r^2}. \end{aligned} \tag{2}$$

In the case of polar coordinates **we know in advance** *both the transformation and its inverse, so it is possible to explicitly calculate all of these expressions in terms of the new coordinates.*

In the calculations of this chapter, part of the problem is to actually **find** *these coordinates, so we can't make these changes until* **after** *these are found. However, ultimately, this must be done.*

Now consider a function $f(r(x,y), \theta(x,y))$ and use the chain rule to obtain

$$f_x = r_x\, f_r + \theta_x\, f_\theta \quad \text{and} \quad f_y = r_y\, f_r + \theta_y\, f_\theta.$$

Notice that the first of these formulae is "homogeneous" in x, having one x "downstairs" in each term. The second equation is obtained from the first by just replacing x by y.

Taking f to be u we differentiate again to get an expression for u_{xx}:

$$u_{xx} = r_{xx} u_r + \theta_{xx} u_\theta + r_x \frac{\partial}{\partial x}(u_r) + \theta_x \frac{\partial}{\partial x}(u_\theta), \tag{3}$$

with

$$\frac{\partial}{\partial x}(u_r) = r_x\, u_{rr} + \theta_x\, u_{r\theta}, \quad \frac{\partial}{\partial x}(u_\theta) = r_x\, u_{\theta r} + \theta_x\, u_{\theta\theta},$$

where we have just used the above formulae with $f = u_r$ and $f = u_\theta$.

Substituting these expressions into (3), we get

$$u_{xx} = r_{xx}\, u_r + \theta_{xx}\, u_\theta + r_x^2\, u_{rr} + 2 r_x \theta_x\, u_{r\theta} + \theta_x^2\, u_{\theta\theta}.$$

It is obvious that we can just change the "label" x to y in this formula and obtain

$$u_{yy} = r_{yy}\, u_r + \theta_{yy}\, u_\theta + r_y^2\, u_{rr} + 2 r_y \theta_y\, u_{r\theta} + \theta_y^2\, u_{\theta\theta}.$$

These expressions are again "homogeneous" in x and (respectively) y.

The expressions on the right contain **second order** *terms like u_{rr}, but also two first order terms u_r and u_θ. Such first derivatives will always arise if the transformation is* **nonlinear**.

Piecing these formulae together, we obtain

$$u_{xx} + u_{yy} = (r_x^2 + r_y^2) u_{rr} + 2(r_x\theta_x + r_y\theta_y) u_{r\theta} + (\theta_x^2 + \theta_y^2) u_{\theta\theta} + (r_{xx} + r_{yy}) u_r + (\theta_{xx} + \theta_{yy}) u_\theta.$$

After substituting from (1) and (2), we obtain

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r^2}\, u_{\theta\theta} + \frac{1}{r}\, u_r.$$

Example 3 (The Linear Wave Equation Revisited) The *Linear Wave Equation* in *laboratory coordinates* is:

$$u_{yy} - \gamma^2 u_{xx} = 0,$$

having

$$a = -\gamma^2, \quad b = 0, \quad c = 1, \quad \Delta = b^2 - ac = \gamma^2 > 0,$$

so is *hyperbolic.*

The characteristic curves $\xi =$ const. and $\eta =$ const. are respectively defined as solutions of:

$$\frac{dy}{dx} = -\frac{1}{\gamma^2}(0 + \gamma) = -\frac{1}{\gamma} \quad \Rightarrow \quad dx + \gamma dy = 0,$$

$$\frac{dy}{dx} = -\frac{1}{\gamma^2}(0 - \gamma) = \frac{1}{\gamma} \quad \Rightarrow \quad dx - \gamma dy = 0,$$

Solutions are given by:

$$\xi = x + \gamma y \quad \text{and} \quad \eta = x - \gamma y,$$

as before.

Example 4 ($yu_{xx} + (x+y)u_{xy} + xu_{yy} = 0$) For this equation

$$a = y\,, \quad b = \frac{1}{2}(x+y)\,, \quad c = x\,, \quad \Delta = b^2 - ac = \frac{1}{4}(x-y)^2 > 0,$$

so the equation is hyperbolic.

The characteristics are defined by:

$$\frac{dy}{dx} = \frac{x}{y} \quad \text{and} \quad \frac{dy}{dx} = 1,$$

giving:

$$d\xi = 2(xdx - ydy) \quad \Rightarrow \quad \xi = x^2 - y^2,$$

$$d\eta = dx - dy \quad \Rightarrow \quad \eta = x - y.$$

To calculate B we need:

$$\xi_x = 2x\,, \quad \xi_y = -2y\,, \quad \eta_x = 1\,, \quad \eta_y = -1.$$

We then have:

$$B = 2xy - (x+y)^2 + 2xy = -(x-y)^2 = -\eta^2.$$

We also have:

$$L\xi = y\xi_{xx} + (x+y)\xi_{xy} + x\xi_{yy} = -2(x-y) = -2\eta \quad \textbf{and} \quad L\eta = 0.$$

The equation now takes the form

$$\hat{L}u = -2\eta^2 u_{\xi\eta} - 2\eta u_\xi = 0.$$

<u>General Solution.</u>

In this form it is possible to calculate the general solution (as we did for the linear wave equation in Example 1)

$$0 = \eta u_{\xi\eta} + u_\xi = \frac{\partial}{\partial\xi}(\eta u_\eta + u) = \frac{\partial^2}{\partial\xi\partial\eta}(\eta u).$$

The solution is:

$$\eta u = f(\xi) + \hat{g}(\eta),$$

so

$$u(x,y) = \frac{f(x^2-y^2)}{x-y} + g(x-y)$$

for arbitrary functions f and g, **determined by initial conditions**.

<u>A Specific Initial Value Problem.</u>

When $u(x,0) = x^3$ **and** $u_y(x,0) = -x^2$

$$\frac{f(x^2)}{x} + g(x) = x^3, \quad \frac{f(x^2)}{x^2} - g'(x) = -x^2,$$

giving $xg'(x) + g(x) = 2x^3$ **and then**

$$g(x) = \frac{1}{2}x^3 + \frac{c}{x} \quad \Rightarrow \quad f(x^2) = \frac{1}{2}x^4 - c.$$

The final result is

$$u(x,y) = \frac{1}{2}\,\frac{(x^2-y^2)^2}{x-y} + \frac{1}{2}\,(x-y)^3 = (x-y)(x^2+y^2).$$

The terms with c *cancelled out.*

Example 5 $(u_{xx} - y^2 u_{yy} + u_x = 0)$ For this equation

$$a = 1\,, \quad b = 0\,, \quad c = -y^2\,, \quad \Delta = b^2 - ac = y^2 > 0,$$

so the equation is hyperbolic.
The characteristics are defined by:

$$\frac{dy}{dx} = \frac{1}{a}(b \pm \sqrt{\Delta}) = \pm y$$

giving:

$$\xi = x - \ln y, \quad \eta = x + \ln y.$$

B is given by

$$B = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y = 1 + 0 + (-y^2)\left(-\frac{1}{y^2}\right) = 2,$$

and

$$L\xi = \xi_{xx} - y^2\xi_{yy} = -1, \quad L\eta = 1, \quad u_x = u_\xi + u_\eta.$$

Piecing this together, we get

$$4u_{\xi\eta} - u_\xi + u_\eta + u_\xi + u_\eta = 0 \quad \Rightarrow \quad 2u_{\xi\eta} + u_\eta = 0.$$

<u>General Solution.</u>

In this form, the equation can be integrated

$$(2u_\xi + u)_\eta = 0 \quad \Rightarrow \quad 2u_\xi + u = \alpha(\xi),$$

where $\alpha(\xi)$ is an arbitrary function. This can then be solved by using an integrating factor

$$\left(e^{\frac{1}{2}\xi}u\right)_\xi = \frac{1}{2}e^{\frac{1}{2}\xi}\alpha(\xi) \quad \Rightarrow \quad u = (f(\xi)+g(\eta))\,e^{-\frac{1}{2}\xi} = (f(x-\ln y)+g(x+\ln y)\sqrt{y}\,e^{-\frac{1}{2}x}$$

Alternative Choice for ξ and η.

We could have chosen

$$\xi = y\,e^{-x} \quad \text{and} \quad \eta = y\,e^{x},$$

which would have lead to

$$B = -2\xi\eta, \quad L\xi = \xi, \quad L\eta = \eta, \quad u_x = \eta\,u_\eta - \xi\,u_\xi \quad \Rightarrow \quad -4\xi\eta\,u_{\xi\eta} + 2\eta\,u_\eta = 0.$$

Again, the general solution can be found

$$(2\xi\,u_\xi - u)_\eta = 0 \quad \Rightarrow \quad \left(\frac{u}{\sqrt{\xi}}\right)_\xi = \alpha(\xi) \quad \Rightarrow \quad u = \sqrt{\xi}(f(\xi)+g(\eta)) = \sqrt{y}e^{-\frac{1}{2}x}(f(ye^{-x})+g(ye^{x})),$$

which is the same solution as before.

Example 6 $(u_{xx} + 4u_{xy} + 4u_{yy} = 0)$ For this equation

$$a = 1, \quad b = 2, \quad c = 4, \quad \Delta = b^2 - ac = 0,$$

so the equation is parabolic.

We define ξ by:

$$\frac{dy}{dx} = \frac{b}{a} = 2 \quad \Rightarrow \quad \xi = 2x - y.$$

We choose $\eta = y$, so

$$\mathcal{J} = \xi_x\eta_y - \xi_y\eta_x = 2 \neq 0.$$

Thus,

$$A = B = 0 \quad \text{and} \quad C = \eta_x^2 + 4\eta_x\eta_y + 4\eta_y^2 = 4.$$

Therefore:

$$\hat{L}u = 4u_{\eta\eta} = 0 \quad \Rightarrow \quad u = f(\xi) + g(\xi)\eta,$$

where f and g are arbitrary functions.

In $x - y$ co-ordinates:

$$u(x,y) = f(2x-y) + yg(2x-y).$$

Remark 1 *Since L has constant coefficients, we can choose ξ and η to be* **linear** *in x and y, so no extra first order terms arise in L.*

Example 7 $(y^2u_{xx} + 2yu_{xy} + u_{yy} = u_x)$ For this equation

$$a = y^2, \quad b = y, \quad c = 1, \quad \Delta = b^2 - ac = 0,$$

so the equation is parabolic.

We define ξ by:

$$\frac{dy}{dx} = \frac{b}{a} = \frac{1}{y} \quad \Rightarrow \quad d(2x - y^2) = 0 \quad \Rightarrow \quad \xi = 2x - y^2.$$

so

$$A = B = 0 \quad \text{and} \quad C = y^2\eta_x^2 + 2y\eta_x\eta_y + \eta_y^2.$$

We choose $\eta = y$, so

$$\mathcal{J} = \xi_x\eta_y - \xi_y\eta_x = 2 \neq 0 \quad \text{and} \quad C = 1.$$

We have:

$$L = \hat{L} + (L\xi)\partial_\xi + (L\eta)\partial_\eta, \quad \text{and} \quad L\xi = -2, \quad L\eta = 0,$$

so

$$L = \hat{L} - 2\partial_\xi = \partial_\eta^2 - 2\partial_\xi.$$

We also have, by the chain rule, $\partial_x = 2\partial_\xi$, so the equation is written

$$(\partial_\eta^2 - 2\partial_\xi)u = 2\partial_\xi u,$$

giving the **heat equation**:

$$u_{\eta\eta} = 4u_\xi.$$

Example 8 ($u_{xx} + u_{xy} + u_{yy} = 0$) For this equation

$$a = c = 1, \quad b = \frac{1}{2}, \quad \Delta = b^2 - ac = -\frac{3}{4} < 0,$$

so the equation is elliptic.

The **Beltrami equations** are:

$$\xi_x = \frac{2}{\sqrt{3}}\left(\frac{1}{2}\eta_x + \eta_y\right) \quad \text{and} \quad \xi_y = -\frac{2}{\sqrt{3}}\left(\eta_x + \frac{1}{2}\eta_y\right),$$

with integrability conditions

$$\partial_y\left(\frac{1}{2}\eta_x + \eta_y\right) + \partial_x\left(\eta_x + \frac{1}{2}\eta_y\right) = 0.$$

Any pair of functions ξ and η will suffice. Choosing $\eta_x = 0$, we have $\eta_{yy} = 0$. A **simple solution** is $\eta = y$, giving

$$\xi_x = \frac{2}{\sqrt{3}} \quad \text{and} \quad \xi_y = -\frac{1}{\sqrt{3}} \quad \Rightarrow \quad \xi = \frac{2}{\sqrt{3}}\left(x - \frac{1}{2}y\right).$$

With this choice, we have $B = 0$ and:

$$A = C = \eta_x^2 + \eta_x\eta_y + \eta_y^2 = 1,$$

so the equation reduces to **Laplace's equation**:

$$u_{\xi\xi} + u_{\eta\eta} = 0.$$

Example 9 $(y^2u_{xx} + u_{yy} = 0)$ For this equation

$$a = y^2, \quad b = 0, \quad c = 1, \quad \Delta = b^2 - ac = -y^2 < 0,$$

so the equation is elliptic.

The **Beltrami equations** are:

$$\xi_x = \frac{\eta_y}{y} \quad \text{and} \quad \xi_y = -y\eta_x \quad \Rightarrow \quad \left(\frac{\eta_y}{y}\right)_y + (y\eta_x)_x = 0.$$

We are free to choose any solution.

Suppose $\eta_x = 0$. Then

$$\left(\frac{\eta_y}{y}\right)_y = 0 \quad \Rightarrow \quad \eta_y = 2y \quad \Rightarrow \quad \eta = y^2,$$

where we have **chosen** specific integration constants.

The **Beltrami equations** then give ξ:

$$\xi_x = \frac{\eta_y}{y} = 2 \quad \text{and} \quad \xi_y = -y\eta_x = 0,$$

with solution $\xi = 2x$.

With this choice of ξ and η, $B = 0$ and

$$A = C = y^2\eta_x^2 + \eta_y^2 = 4y^2 = 4\eta, \qquad L\xi = 0, \qquad L\eta = 2,$$

giving

$$L = \hat{L} + (L\xi)\partial_\xi + (L\eta)\partial_\eta = 4\eta\left(\partial_\xi^2 + \partial_\eta^2\right) + 2\partial_\eta.$$

The equation takes the form:

$$2\eta\left(u_{\xi\xi} + u_{\eta\eta}\right) + u_\eta = 0.$$

Example 10 (Tricomi's Equation: $u_{xx} + xu_{yy} = 0$**)** For this equation

$$a = 1, \quad b = 0, \quad c = x, \quad \Delta = b^2 - ac = -x,$$

so the equation is **hyperbolic** when $x < 0$ and **elliptic** when $x > 0$.

The Hyperbolic Region.

When $x < 0$, we have $\Delta > 0$ and

$$\frac{dy}{dx} = \pm\sqrt{-x} \quad \Rightarrow \quad \begin{aligned} \xi &= y + \tfrac{2}{3}(-x)^{3/2}, \\ \eta &= y - \tfrac{2}{3}(-x)^{3/2}. \end{aligned}$$

Then:

$$B = \xi_x\eta_x + x\xi_y\eta_y = -(-x)^{1/2}(-x)^{1/2} + x = 2x,$$

and

$$L\xi = \frac{1}{2}(-x)^{-1/2}, \quad L\eta = -\frac{1}{2}(-x)^{-1/2}.$$

This gives

$$Lu = 4xu_{\xi\eta} + \frac{1}{2}(-x)^{-1/2}\left(u_\xi - u_\eta\right) = 4x\left(u_{\xi\eta} - \frac{1}{8}(-x)^{-3/2}\left(u_\xi - u_\eta\right)\right).$$

Since $\xi - \eta = \frac{4}{3}(-x)^{3/2}$,

$$Lu = 0 \quad \Rightarrow \quad u_{\xi\eta} - \frac{1}{6}\frac{1}{\xi - \eta}\left(u_\xi - u_\eta\right) = 0,$$

which is the **canonical hyperbolic form** of Tricomi's equation.

The Elliptic Region.

When $x > 0$, the equation is **elliptic**.

With $\delta = \sqrt{x}$ **Beltrami's equations** are:

$$\xi_x = \sqrt{x}\eta_y \quad \text{and} \quad \xi_y = -\frac{\eta_x}{\sqrt{x}},$$

with integrability conditions:

$$\partial_y\left(\sqrt{x}\eta_y\right) + \partial_x\left(\frac{\eta_x}{\sqrt{x}}\right) = 0.$$

Again, **we choose** $\eta_x = 0$, so $(\sqrt{x}\eta_y)_y = 0$ or $\eta_{yy} = 0$.

We choose $\eta = y$, so

$$\xi_x = \sqrt{x} \quad \text{and} \quad \xi_y = 0 \quad \Rightarrow \quad \xi = \frac{2}{3}x^{3/2}.$$

With this choice of ξ and η, we have $B = 0$ and:

$$A = C = \eta_x^2 + x\eta_y^2 = x, \qquad L\xi = \xi_{xx} = \frac{1}{2}x^{-1/2}, \qquad L\eta = 0.$$

We find

$$x\left(u_{\xi\xi} + u_{\eta\eta}\right) + \frac{1}{2}x^{-1/2}u_\xi = 0.$$

The **canonical elliptic form** of **Tricomi's equation** is:

$$u_{\xi\xi} + u_{\eta\eta} + \frac{1}{3\xi}u_\xi = 0.$$