Review: ODEs Homogeneous

$$\boxed{ay'' + by' + cy = 0}$$

For a, b, c = constant coefficients $\Rightarrow$ $a\lambda^2 + b\lambda + c = 0$

Characteristic Equation

Real λ's $\Rightarrow$ $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

Complex $\lambda \pm \beta i$ $\Rightarrow$ $y = e^{\lambda x}(C_1 \cos(\beta(x)) + C_2 \sin(\beta(x)))$

Imag $\pm \beta i$ $\Rightarrow$ $y = C_1 \cos(\beta x) + C_2 \sin(\beta x)$

For non-constant coefficients

$$a(x)y'' + b(x)y' + c(x)y = 0$$

$a(x), b(x), c(x)$ functions of x

Cauchy - Euler Equation

$$\boxed{a x^2 y'' + b x^1 y' + c x^0 y = 0}$$

Here $a(x) = x^2$ $a = 1$

$b(x) = x^1$ $b = 1$

$c(x) = x^0$ $c = n$

$$\boxed{\text{Descending \# of exponents \& derivatives}}$$

Ex: For $a = b = 1$ & c = any constant

$$\boxed{y(x) = C_1 x^c + C_2 x^{-c}}$$

Bessel Equation

$$\boxed{x^2 y'' + x y' + (x^2 - \nu^2) y = 0}$$

where $\nu \Rightarrow$ defines order type

Ex: $\nu = 0$ Zeroth Order

$$x^2 y'' + x^1 y' + x^2 y = 0$$

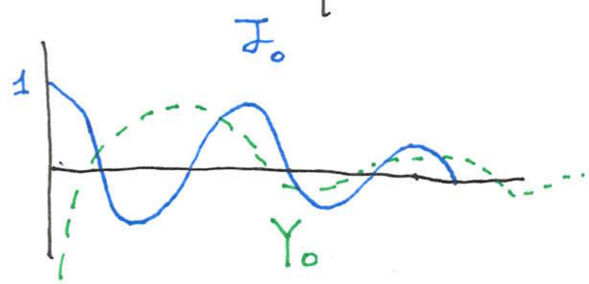
Solution Obtained through Power Series

$$\boxed{y = C_1 J_0(x) + C_2 Y_0(x)}$$

$\nu = 0$ Zeroth Order

J_ν = Bessel of First Kind $\qquad$ Y_ν = Bessel of Second Kind

Recall Shapes



Note:

$J_0(0) = 1$, $J_1 - J_4 = 0$ @ $= 0$

$Y_0(0) = \infty$, $Y_1 - Y_4 = \infty$ @ $= 0$

Review

ODEs Non-homogeneous

$$ay'' + by' + cy = F(x)$$

* Obtain homogeneous solution first

Then, based on problem we can apply

$y_{Particular}$

Undetermined Coefficients

Guess a solution of similar form to $F(x)$

Ex: $y_p = Ae^{ax}$

Variation of Parameters

Construct a solution

$$y = v_1 y_{h_1} + v_2 y_{h_2}$$

Annihilator Method

DtD

Derive till Death

* Power Series Approach, Laplace Methods, Eigenvalues/vectors usually not used in basic PDEs

Review

PDEs: Deriving a function in terms of 2 or more variables

Main PDEs
$$\begin{cases} U_t = c^2 \nabla^2 U & \text{Heat Equation} \\ U_{tt} = c^2 \nabla^2 U & \text{Wave Equation} \\ \nabla^2 U = 0 & \text{Laplace Equation} \end{cases}$$

All 3 types can be described with ∇^2, which defines both # of dimensions & coordinate type.

1D $\Rightarrow$ $\nabla^2 U = \frac{\partial^2 U}{\partial x^2}$ only (or $\frac{\partial^2 U}{\partial y^2}$, or $\frac{\partial^2 U}{\partial z^2}$)

$\nabla^2 U = U_{xx}$ Cartesian Only!

2D $\Rightarrow$
$$\left.\begin{aligned} \nabla^2 U &= \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \\ \nabla^2 U &= U_{xx} + U_{yy} \end{aligned}\right\} \text{Cartesian 2D}$$

$$\left.\begin{aligned} \nabla^2 U &= \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} \\ \nabla^2 U &= U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} \end{aligned}\right\} \text{Polar 2D}$$

3D
$$\left.\begin{aligned} \nabla^2 U &= \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \\ \nabla^2 U &= U_{xx} + U_{yy} + U_{zz} \end{aligned}\right\} \text{Cartesian 3D}$$

$$\left.\begin{aligned} \nabla^2 U &= \left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2}\right) \\ \nabla^2 U &= U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} + U_{zz} \end{aligned}\right\} \text{Polar 3D aka as Cylindrical}$$

$$\left.\begin{aligned} \nabla^2 U &= \frac{1}{\rho^2}\left[\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial U}{\partial \rho}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial U}{\partial \phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 U}{\partial \theta^2}\right] \\ \nabla^2 U &= \frac{1}{\rho^2}\left[\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial U}{\partial \rho}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\, U_\phi\right) + \frac{1}{\sin^2\phi}U_{\theta\theta}\right] \end{aligned}\right\} \text{Spherical}$$

or $\frac{\partial}{\partial \rho}(\rho^2 U_\rho)$

Spherical & Cylindrical can be simplified with symmetries.

Ex: Sphere or Circle where radius doesn't grow like a ball. $\Rightarrow$ implies $\frac{\partial^2 U}{\partial r^2} = 0$

Review

Orthogonality / Dot Product.

Start with 2 vectors $\underline{A}$, $\underline{B}$

Define dot product for vectors

as $\underline{A} \cdot \underline{B} = \|A\| \, \|B\| \cos\theta$

If $\underline{A} \cdot \underline{B} = 0$ Then ~~A~~ $\underline{A}$ & ~~B~~ $\underline{B}$ are called orthogonal vectors.

Start with 2 functions $Q(x)$ $P(x)$

Define dot product for functions

as $\langle P, Q \rangle = \int_a^b P(x)\, Q(x)\, w(x)\, dx$

where $w(x)$ depends on coordinate type.

Cartesian	$w(x) = 1$
Polar	$w(x) = r$
Spherical	$w(x) = r^2$

If $\langle P, Q \rangle = 0$, Then $P(x)$ & $Q(x)$ are called orthogonal.

In PDEs "orthogonality" means that you take the dot product of functions to get rid of terms, because orthogonal terms will be zero.

① Separation of Variables based on function

$$\alpha^2 U_{xx} = U_t$$

B.C. $\frac{\partial u}{\partial x}\Big|_{x=0} = 0 \quad u\Big|_{x=\ell} = 1$ I.C. $u\Big|_{t=0} = 0$

$$U = X(x)\,T(t) \Rightarrow \alpha^2 X'' T = X\dot{T}$$

$$\frac{X''}{X} = \frac{\dot{T}}{\alpha^2 T} = -K^2$$

Constant is squared to simplify solution of ODEs, <u>sign</u> choice arises from obtaining a <u>valid</u> (convergent) solution.

First ODE

$$\frac{X''}{X} = -K^2 \Rightarrow X'' + K^2 X = 0 \quad (I)$$

Second ODE

$$\frac{\dot{T}}{\alpha^2 T} = -K^2 \Rightarrow \dot{T} + K^2\alpha^2 T = 0 \quad (II)$$

Since $K=0$ gives a different ODE than $K \neq 0$ we find both solutions.

$$\text{For (I)} \begin{cases} X(x) = A\cos Kx + B\sin Kx & K \neq 0 \\ X(x) = D + Ex & K = 0 \end{cases}$$

Pause: What about if $\frac{X''}{X} = +K^2$

$\lambda^2 - K^2 = 0 \Rightarrow \lambda = \pm K \quad X(x) = Ae^{Kx} + Be^{-Kx}$

However, Ae^{Kx} is <u>divergent</u> therefore bad.

$$\text{For (II)} \begin{cases} T(t) = Fe^{-K^2\alpha^2 t} & K \neq 0 \\ T(t) = G & K = 0 \end{cases}$$

If $\frac{\dot{T}}{\alpha^2 T} = K^2$ Then, $T(t) = Fe^{K^2\alpha^2 t}$

Again, solution grows exponentially! <u>Bad</u>

Now $u = X(t)\,T(t) = \boxed{\begin{matrix} X(t)\,T(t) \\ K=0 \end{matrix}} + \boxed{\begin{matrix} X(t)\,T(t) \\ K\neq 0 \end{matrix}}$

Superposed Solution

$\cancel{u = (A\cos Kx + B}$

$$u = G(D + Ex) + Fe^{-K^2\alpha^2 t}(A\cos Kx + B\sin Kx)$$

Combining Constants

$$u(x,t) = C_1 + C_2 x + e^{-k^2\alpha^2 t}\left(C_3 \cos kx + C_4 \sin kx\right)$$

② Apply Boundary Conditions First!!!

Tip: Start w/ zero valued B.C.s.

For $\boxed{\frac{\partial u}{\partial x}\Big|_{x=0}}$

$$\frac{\partial u}{\partial x} = C_2 + e^{-k^2\alpha^2 t}\left(-C_3 k \sin(kx) + C_4 k \cos(kx)\right)$$

$$\frac{\partial u}{\partial x}\Big|_{x=0} = C_2 + e^{-k^2\alpha t}\left(\cancel{-C_3 k \sin(0)}^{\,0} + C_4 k \cancel{\cos(0)}^{\,1}\right)$$

$$\Rightarrow \quad C_2 = 0$$

L.I. on t $\quad e^{-k^2\alpha t} C_4 k = 0 \Rightarrow C_4 = 0$

New Solution

$$\boxed{u(x,t) = C_1 + e^{-k^2\alpha^2 t} C_3 \cos(kx)}$$

For $\cancel{\frac{\partial u}{\partial x}}$ $\boxed{u\big|_{x=\ell} = 1}$ $\quad u(\ell,t) = C_1 + e^{-k^2\alpha^2 t}\left(C_3 \cos(k\ell)\right) = 1$

L.I. on t

$$C_1 = 1$$

$$e^{-k^2\alpha^2 t} C_3 \cos(k\ell) = 0$$

$C_3 \neq 0 \rightarrow$ For 2 reasons. If $C_3 = 0$ Then $u(x,t) = 0$ which is trivial (useless).

Also, $\cos(k\ell)$ can also be 0, Thus, $0(0)$ is also useless. $\therefore$ Just worry about $\cos(k\ell)$

$$\cos(k\ell) = 0$$

$$k\ell = \frac{(2n-1)\pi}{2} \Rightarrow k_n = \frac{(2n-1)\pi}{2\ell}$$

There are multiple values that make $\cos(k\ell) = 0$, Thus we need all of them labeled as "k_n". These are eigenvalues, & they ~~d~~ come from the eigenfunction $\cos(k\ell)$.

In other words: Eigenfunctions are functions that give eigenvalues. These arise from spatial (x, y, z, r) problems <u>only</u>. Eigenfunctions do <u>not</u> need to be in the general solution where eigenvalues are placed.

* Eigenfunctions can be a "condition" that comes from B.C.s & give eigenvalues.

Eigenvalues <u>do</u> appear in general solution.

Now: $u(x,t) = 1 + \sum_{1}^{\infty} H_n e^{-K_n^2 \alpha^2 t} \cos(K_n x)$

We plugged K_n to our equation. Therefore we need a sum for all "n" terms.

Before we had C_3, & because we have "n" terms, each term has a constant "H_n" to them.

(3) Apply Initial Conditions!!

For $\boxed{u\big|_{t=0} = 0}$

$$u(x,0) = 1 + \sum_{1}^{\infty} H_n \cancel{e^{0}}^{\,1} \cos K_n x = 0$$

$$\Rightarrow \quad 1 + \sum_{1}^{\infty} H_n \cos K_n x = 0$$

Refer to Orthogonality Handout.

We will cancel all orthogonal terms by applying dot product in between them.

Note: Dot Product is applied to all equation

We know $\boxed{\cos K_n x \text{ is orthogonal to } \cos K_m x}$

"Orthogonality" Condition

$$\int_a^b \left(1+\sum_1^{\infty} H_n \cos K_n x\right)\cos K_m\, dx = \int_a^b 0 \xrightarrow{} \cancel{\cos K_m\, dx}^{\,0}$$

$$\int_a^b \cos K_m\, dx + \sum_1^{\infty} H_n \int_a^b \cos K_n x \cos K_m x\, dx = 0$$

All constants can be taken out

Limits a, b of integral, come from spatial problem in this case $0 \le x \le L$

Now we see the dot product $\int_a^b P(x)\,Q(x)\,dx$ which will cancel out all terms until just $m=n$ is left.

$$\int_0^L \cos K_m\, dx + H_m \int_0^L \cos(K_m x)^2\, dx = 0$$

Solving for $H_m \Rightarrow$

$$H_m = \frac{-\int_0^L \cos K_m dx}{\int_0^L (\cos K_m)^2\, dx}$$

Now Plug to final solution

$$U(x,t) = 1 + \sum_1^{\infty} H_m e^{-K_m^2 \alpha^2 t} \cos(K_m t)$$

Note: doesn't matter if its m or n at the end, they're equal.

H_m is now defined & can be plugged in, but usually is not put in equation to look nicer. Also the integrals in H_m are usually too hard to solve by hand.