Example: Laplace PDE in Polar Coordinates

$\nabla^2 u(r,\theta) = 0$ with B.C $u|_{r=b} \begin{cases} 0 & \frac{\pi}{2} < \theta < \frac{3\pi}{2} \\ 200 & 0 < \theta < \frac{\pi}{2} \ \& \ \frac{3\pi}{2} < \theta < 2\pi \end{cases}$

With $0 < \theta < 2\pi$
$0 < r < b$

From Coordinate System

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

- Let $u = R(r)\Theta(\theta) = R\Theta$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$ $\Rightarrow$ Multiply by $\left(r^2\left(\frac{1}{R\Theta}\right)\right)$ Why? r^2?

2 reasons, we don't want r^2 with function Θ'' & solving equations with $\frac{1}{r^2}$ are hard.

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = K^2$$ Why positive? Let's see ODEs ☆

$$\begin{cases} r^2R'' + rR' - K^2R = 0 \\ \Theta'' + K^2\Theta = 0 \end{cases}$$

Let's do $\Theta(\theta)$ for $K=0$ $\quad \Theta''=0 \Rightarrow \boxed{\Theta(\theta) = A\theta + B}$

~~For $K\neq 0$~~, we recognize that $\Theta(\theta)$ is a periodic function.

- For periodic functions
  - Variable θ must be inside a periodic function such as sin/cos. Thus, $A=0$
  - Any constant K must be in integer form. Thus $K=n \Rightarrow$ integer

For $K\neq 0$ $\quad \Theta'' + K^2\Theta = 0 \Rightarrow K=n \quad \Theta'' + n^2\Theta = 0$

Solution: $\boxed{\Theta(\theta) = E\cos(n\theta) + F\sin(n\theta)}$ for $n \neq 0$

Here we see the importance of $K=n=$ integer

Also we see that if our original sign assumption ☆ would of been $-K^2$, Then ODEs would of been exponentials. Which is useless in periodic functions

Now we have

$$\begin{cases} r^2R'' + rR' - n^2R = 0 \\ \Theta'' + n^2\Theta = 0 \end{cases}$$

For function $R(r)$ with $n=0$

$r^2R'' + rR' = 0$ ← this can be solved multiple ways

$\div r$ ↳ $rR'' + R' = 0 \Rightarrow$ Product Rule definitions

$$\int \frac{d}{\cancel{dr}}[rR']\,\cancel{dr} = \int 0\,dr$$

$$rR' = C \Rightarrow \int \frac{d}{\cancel{dr}} R\,\cancel{dr} = \int \frac{C}{r}\,dr \Rightarrow \boxed{R = C\ln|r| + D}$$

For $n \neq 0$ $\quad r^2R'' + rR' - n^2R = 0$

From handout, this is Cauchy Euler: $x^2y'' + xy' + by = 0$ with $b = -n^2$

$$\boxed{R = Gr^n + Hr^{-n}}$$

$$u(r,\theta) = \underbrace{B(C\ln|r| + D)}_{u = R\Theta \text{ when } n=0} + (E\cos(n\theta) + F\sin(n\theta))(Gr^n + Hr^{-n})$$

$u = R\Theta$ when $n \neq 0$

Boundedness Conditions:

When a variable exists in the spatial/temporal range, The variable must be avoiding divergent terms.

In our equation $u(r,\theta)$

$r=0$ exists but when $r=0$, $u \to \infty$, therefore: term is eliminated

$r=0$ exists, $C\ln|r| \Rightarrow C\ln|0| \Rightarrow C(\infty)$ not possible $\therefore \Rightarrow C=0$

$r=0$ exists $Hr^{-n} \Rightarrow \frac{H}{0^n} \Rightarrow H(\infty)$ not possible for all n $\therefore \Rightarrow H=0$

$$u(r,\theta) = C_1 + \sum_{n=1}^{\infty} (C_2\cos(n\theta) + C_3\sin(n\theta))r^n$$

Recall: $u(b,\theta) = \begin{cases} 200 & 0 < \theta < \pi/2 \\ 0 & \pi/2 < \theta < 3\pi/2 \\ 200 & 3\pi/2 < \theta < 2\pi \end{cases}$

Remember all Constants in summation become an infinite # of Constants.

$$U(r,\theta) = C_1 + \sum_{n=1}^{\infty} r^n \left(P_n \cos(n\theta) + Q_n \sin(n\theta) \right)$$

At this point if we define our C_1, P_n, Q_n we are done

Apply dot product with an orthogonal function of your choice. This equation from PDEs is analogous to the Fourier Series Equation.

Lets do dot product with $\cos(m\theta)$

$$\int_a^b U(r,\theta)\cos(m\theta)\,d\theta = \int_a^b \cancel{C_1 \cos(m\theta)\,d\theta}^{\,0} + \int_a^b \sum_{n=1}^{\infty} r^n P_n \cos(n\theta)\cos(m\theta)\,d\theta$$

$$+ \int_a^b \sum_{n=1}^{\infty} r^n \cancel{Q_n \sin(n\theta)}^{\,0} \cos(m\theta)\,d\theta$$

Recall Fourier Series Handout

Now our orthogonality condition looks like

$$\int_a^b U(r,\theta)\cos(n\theta)\,d\theta = \int_a^b r^n P_n \cos(n\theta)\cos(n\theta)\,d\theta$$

Now all infinite terms are gone & P_n is constant

Lets apply our B.C's

$$\int_0^{2\pi} U(b,\theta)\cos(n\theta)\,d\theta = \int_0^{2\pi} b^n P \cos^2(n\theta)\,d\theta$$

Lets break this B.C integral into its components

$$\int_0^{\frac{\pi}{2}} 200\cos(n\theta)\,d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cancel{0\cos(n\theta)\,d\theta}^{\,0} + \int_{\frac{3\pi}{2}}^{2\pi} 200\cos(n\theta)\,d\theta$$

$$= P\int_0^{2\pi} b^n \cos^2(n\theta)\,d\theta$$

Now

$$P = \frac{\int_0^{\frac{\pi}{2}} 200\cos(n\theta)\,d\theta + \int_{\frac{3\pi}{2}}^{2\pi} 200\cos(n\theta)\,d\theta}{\int_0^{2\pi} b^n \cos^2(n\theta)\,d\theta} = \frac{\langle U(b,\theta), \cos(n\theta)\rangle}{\langle b^n \cos(n\theta), \cos(n\theta)\rangle}$$

Ex: Klein Gordon Equation (Hyperbollic)

$$u_{tt} = c^2 u_{xx} - hu \quad \text{with} \quad \begin{array}{l} u(0,t)=0 \\ u(L,t)=0 \end{array}$$

$u(x,t) = X(x)\,T(t)$

$$XT'' - c^2 X'' T + hXT = 0 \quad \cdot \frac{1}{TXc^2}$$

$$\frac{T''}{c^2 T} - \frac{X''}{X} + \frac{h}{c^2} = 0 \Rightarrow \frac{T''}{c^2T} + \frac{h}{c^2} = \frac{X''}{X} = -K^2$$

$$X'' + K^2 X = 0$$

$$T'' + (K^2c^2+h)T = 0$$

$$X(x) \begin{cases} A\cos(Kx) + B\sin(Kx) & K \neq 0 \\ Cx + D & K = 0 \end{cases}$$

$$T(t) \begin{cases} E\cos(\sqrt{K^2c^2+h}\;t) + F\sin(\sqrt{K^2c^2+h}\;t) & K \neq 0 \\ G\cos(\sqrt{h}\;t) + H\sin(\sqrt{h}\;t) & K=0 \end{cases}$$

For $u(x,t) = \underbrace{(A\cos(Kx) + B\sin(Kx))T_1(t)}_{K\neq 0} + \underbrace{(Cx+D)T_2(t)}_{K=0}$

$$u(0,t) = (A\cancelto{1}{\cos(0)} + B\cancelto{0}{\sin(0)})T_1(t) + (\cancelto{0}{C(0)} + D)T_2(t) = 0$$

$$AT_1(t) + DT_2(t) = 0 \Rightarrow A = D = 0$$

$$u(L,t) = (\cancelto{0}{A}\cos(KL) + B\sin(KL))T_1(t) + (CL + \cancelto{0}{D})T_2(t) = 0$$

L.I. $T_1(t)$: $B\sin(KL) = 0 \Rightarrow \boxed{K_n = \frac{n\pi}{L}}$

$T_2(t)$: $CL = 0 \Rightarrow C = 0$

$$u(x,t) = \sum_{n=1}^{\infty} \sin K_n x \left(P_n \cos\left(\sqrt{K_n^2c^2+h}\;t\right) + Q_n \sin\left(\sqrt{K_n^2c^2+h}\;t\right)\right)$$

This is the solution without I.C.s & because we have a summation with functions, we can apply the dot product.

Let I.C.s $\Rightarrow$ $u(x,0) = g_1(x)$, $u(x,4) = g_2(x)$

$$P_n = \frac{\langle g_1(x), \sin K_n x\rangle}{\langle \sin K_n x, \sin K_n x\rangle}, \quad \cancel{Q_n = \langle g_2(x) - P_n\cos\sqrt{K_n^2c^2+h}\,4, \sin K_n x\rangle}$$

$$Q_n = \frac{\langle g_2(x) - P_n\cos\sqrt{K_n^2c^2+h}\,(4), \sin K_n x\rangle}{\langle \sin\sqrt{K_n^2c^2+h}\,(4)\sin K_n x, \sin K_n x\rangle}$$

Where now everything is defined.

Ex: Non-Homogeneous Heat Equation

$$U_t = \alpha^2 U_{xx} + f(x,t)$$
$$U(0,t) = 0$$
$$U(1,t) = 0$$
$$U(x,0) = \phi(x)$$

Mindset: $f(x,t)$ can be broken into a Fourier Series Expansion

If we decompose $f(x,t)$, we will eventually break the solution of $U(x,t)$ as well. Ex: $U(x,t) = \sum_{n=1}^{\infty} U_n(x,t)$

Lets solve the homogeneous first.

Let $U_n = T_n(t) X_n(x)$

$$\frac{X''}{X} = \frac{T'}{\alpha^2 T} = -K^2 \Rightarrow \quad X'' + K^2 X = 0$$
$$X(x) = A\sin(Kx) + B\cos(Kx)$$

Apply B.C.s

$U(0,t)=0$ or $X(0)=0$: $\quad X(0) = A(0) + B(1) \Rightarrow B = 0 \Rightarrow X(x) = A\sin(Kx)$

$U(1,t)=0$ or $X(1)=0$: $\quad X(1) = A\sin(Kx) = 0 \Rightarrow K_n = n\pi$ ← eigenvalues

Formally/Mathematically Note: Eigenvalues permit the existance of non-trivial solution.

Now we know eigenfunction $= \sin(K_n x) = \sin(n\pi x)$

Eigenfunctions are basis for our problem. Thus lets decompose the non-homo term with our basis function.

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin n\pi x$$

coefficients that we can find with Dot Product

$$f_n = \frac{\langle f(x,t), \sin n\pi x \rangle}{\langle \sin n\pi x, \sin n\pi x \rangle}$$

Now: $f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin n\pi x$ ← Where $f_n(t)$ is defined!

Plug $X(x)$ into $u_n = T_n(t)X_n(x)$

$$u_n = \sum_{n=1}^{\infty} T_n (\sin n\pi x)$$

Note how the solution u_n is decomposed into its basis functions.

Now let's plug this new u_n into original PDE along with decomp. of the non-homo term.

$$u_t = \alpha^2 u_{xx} + f(x,t)$$

$$\sum_{n=1}^{\infty} T_n'(t) \sin n\pi x = -\alpha^2 \sum_{n=1}^{\infty} (n\pi)^2 T_n(t) \sin n\pi x + \sum_{n=1}^{\infty} f_n(t) \sin n\pi x$$

B.Cs & I.Cs change accordingly too

$u(0,t) = 0$ $\quad \sum_{n=1}^{\infty} T_n(t) \sin(0) = 0$

$u(1,t) = 0$ $\quad \sum_{n=1}^{\infty} T_n(t) \sin(n\pi) = 0$

These 2 we already used them & now give trivial solutions

$u(x,0) = \phi$ $\quad \sum_{n=1}^{\infty} T_n(0) \sin(n\pi x) = \phi(x)$

Now let's factor out our eigenbasis (eigenfunction) & show the Eigenfunction Decomposition

$$\sum_{n=1}^{\infty} \left[T_n'(t) + n\pi\alpha^2 T_n(t) - f_n(t) \right] \sin n\pi x = 0$$

~~Let's call the I.C $\Rightarrow a_n$~~

Now the solution is found if

$$\boxed{T_n'(t) + n\pi\alpha^2 T_n(t) - f_n(t) = 0} \quad \square$$

With I.C. $\quad \sum_{n=1}^{\infty} T_n(0) \sin n\pi x = \phi(x)$

Now, lets convert this I.C. into something we can use (i.e. no summation)

Let's call it

$$T_n(0) = \frac{\langle \phi(x), \sin n\pi x \rangle}{\langle \sin n\pi, \sin n\pi \rangle} = a_n$$

Note how equation $\square$ is a simple non-homo ODE

Now we are solving the following ODE

$$T_n' + n\pi\alpha^2 T_n = f_n$$

with $T_n(0) = a_n$

Where the homogereous part is: $\alpha + n\pi\alpha^2 = 0 \Rightarrow \alpha = -n\pi\alpha^2$

$T_{h,}$ not n $\rightarrow$ $\boxed{T_h = C_1 e^{-n\pi\alpha^2 t}}$

& the non-homo (particular) solution:

- Given that the non-homogeneous term is a general function, Thus, only a convolution solution might be used.

Recall we had

$$U_n = \sum_{n=1}^{\infty} T_n (\sin n\pi x)$$

$$U_n = \sum_{n=1}^{\infty} (T_h + T_p) \sin n\pi x$$

$$\boxed{U_n = \sum_{n=1}^{\infty} C_n e^{-n\pi\alpha^2 t} (\sin \pi n x) + T_{Particular} (\sin \pi n x)}$$

This equation still needs I.C.s?

Note: Recall that convolution gives $\Rightarrow$

$$y_p = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$ where $f(t), g(t)$ are interchangeable.

$$\Downarrow$$

$$T_{Part} = \int_0^t f_n(\tau)\, e^{-(n\pi\alpha^2)(t-\tau)}\, d\tau$$