PDE Final Overview

- Understand Dimension & coordinates (1D, 2D, 3D, spherical, etc)
- Separation of Variables

  Based on equation (only care about variables in derivatives)

Ex: $U_{xx} + U_{yy} + U_{zz} = 0$

$$\Rightarrow \quad U = X(x)\,Y(y)\,Z(z)$$

$$-\frac{\hbar^2}{2m}\Psi_{xx} + V\Psi = E\Psi_t$$

$$\Rightarrow \quad \Psi = X(x)\,T(t)$$

- You obtain as many ODE's as variables in derivatives

$U_{xx} + U_{yy} + U_{zz} = 0$ 3 variables, 3 ODEs

$$X''YZ + XY''Z + XYZ'' = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

- ODE's obtained come from equation type & coordinates

Usually: Cylindrical / Polar $\Rightarrow$ Cauchy-Euler ODE

Spherical $\Rightarrow$ Bessel ODE

- Boundary conditions give: Eigenvalues & Eigenfunctions
- Initial conditions give: Orthogonality Condition

  In general, after I. C.

$$F(x) = \sum_{n=1}^{\infty} H_n P_n$$

Function that will be subject to orthogonality

all things that don't have summation

Constant to be found

- Take dot product with P_m

$$\int_a^b F(x)\,P_m\,dx = \int_a^b \sum_{n=1}^{\infty} H_n P_n P_m\, w(x)\,dx$$

$w(x)$ depends on coordinates

- Solving for $H_n = H_m$

$$H_m = \frac{\langle F(x), P_m\rangle}{\langle P_m, P_m\rangle}$$

P_m sometimes are eigenfunctions

or

$$H_m = \frac{\int_a^b w(x) F(x)\, P_m(x)\, dx}{\int_a^b w(x) (P_m)^2\, dx}$$

In general, linear 2^{nd} order PDEs are:

$$A u_{xx} + 2B u_{xy} + C u_{yy} + D u_x + E u_y + F u = f$$

Classification		Example		$B^2 - AC$
parabollic:	$B^2 - AC = 0$	$\alpha^2 u_{xx} = u_t$	$\leftarrow$ with $y=t$	$(0)-(\alpha^2)(0)=0$
hyperbollic:	$B^2 - AC > 0$	$c^2 u_{xx} = u_{tt}$	$\leftarrow$ with $y=t$	$(0)-c^2(-1)>0$
elliptic:	$B^2 - AC < 0$	$u_{xx} + u_{yy} = 0$		$(0)-(1)(1)<0$

In Cartesian Coordinates, we encounter the following:

Heat/Diffusion PDE	Wave PDE	Laplace PDE
$\alpha^2 u_{xx} = u_t$	$c^2 u_{xx} = u_{tt}$	$u_{xx} + u_{yy} = 0$
$X'' + K^2 X = 0$	$X'' + K^2 X = 0$	$X'' + K^2 X = 0$
$\dot{T} + \alpha^2 K^2 T = 0$	$\ddot{T} + K^2 T = 0$	$Y'' - K^2 Y = 0$
$\sin(Kx)$	$\sin(Kx)$	$\sin(Kx)$
$\cos(Kx)$	$\cos(Kx)$	$\cos(Kx)$
$e^{-\lambda t}$	$\sin(Kt)$	$\sinh(Ky)$
	$\cos(Kt)$	$\cosh(Ky)$

Recall that # of variables in derivatives = # of ODEs.
Also # of functions in solution = # of highest derivatives added.
Exponentials are only desired in the diffusion equation (usually, but for this basic 3 PDEs, yes).
So, to manage that, real roots from ODEs can be manipulated:

From Laplace PDE $Y'' - K^2 Y = 0 \Rightarrow \lambda^2 - K^2 = 0$

$\lambda = \pm K$ real!

Solution is: $C_1 e^{-\lambda y} + C_2 e^{\lambda y}$, let's mess with constants

$$\left(\frac{A+B}{2}\right) e^{-\lambda y} + \left(\frac{A-B}{2}\right) e^{\lambda y}$$

$$\frac{A}{2} e^{-\lambda x} + \frac{B}{2} e^{-\lambda y} + \frac{A}{2} e^{\lambda y} - \frac{B}{2} e^{\lambda y} \Rightarrow$$

$$\frac{A}{2}\left(e^{-\lambda y} + e^{\lambda y}\right) + \frac{B}{2}\left(e^{\lambda y} - e^{\lambda y}\right) = A\left(\frac{e^{-\lambda y} + e^{\lambda y}}{2}\right) + B\left(\frac{e^{\lambda y} - e^{-\lambda y}}{2}\right)$$

$$Y(y) = A \cosh(\lambda y) + B \sinh(\lambda y)$$

What are eigenfunctions anyway?

We need to understand eigenvectors

For any system (matrix) of equations A, there is a unique decomposition on a unique eigenvalue(s) with corresponding eigenvectors.

Such decomp. looks like:

$$A v = \lambda v \qquad \text{Ex: } A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

From EVP we know: $\lambda_1 = -1 \quad \lambda_2 = 4$

The eigenvectors have an infinite number of options

Let's take $\underline{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \underline{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Thus:

$$A v_1 = \lambda_1 v_1 \qquad A v_2 = \lambda_2 v_2$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = -1\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{Bmatrix} 2 \\ 3 \end{Bmatrix} = 4\begin{Bmatrix} 2 \\ 3 \end{Bmatrix}$$

Now, we can "decompose" a matrix into eigenvectors/values

$$A\underline{v}_1 + A\underline{v}_2 = v_1\lambda_1 + \underline{v}_2\lambda_2$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{Bmatrix} 2 \\ 3 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}-1 + \begin{Bmatrix} 2 \\ 3 \end{Bmatrix}4$$

$$\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} + \begin{Bmatrix} 8 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} + \begin{Bmatrix} 8 \\ 12 \end{Bmatrix}$$

These terms are linearly independent, and thus form part of "basis" vectors.

Recap: basis vectors

- Vectors that are LI & span (cover) all the space defined.

Ex: $i = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ Both i & j are LI & cover all 2D space.

(You can create any other vector in terms of these basis vectors)

Just like i, j are basis. Eigenvectors can ← not always!
form a basis (called eigenbasis) of a space defined by its unique eigenvalues.

Ex: $\underline{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad$ $\underline{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Is a basis on a space defined by 1D $\lambda_1 = -1$

Is a basis on a space defined by 1D $\lambda_2 = 4$

· Euclidean 3D Space has eigenvectors/values

$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad$ $\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad$ $\underline{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_1 = 1$ $\quad$ $\lambda_2 = 1$ $\quad$ $\lambda_3 = 1$

The number of repeated eigenvalues define dimension.

This same mentality applies to functions!
A decomposition is applied to any function & break it into its corresponding eigenvalues & eigenfunctions. These eigenfunctions can be a "basis" function that lets you write any function in terms of its basis function.
In fourier series, the basis functions are $\sin(n\pi x)$ & $\cos(n\pi x)$.

Note: All these exist in a "function" space

EVP for Matrices.

Easy one: $|A - \lambda \underline{I}| = 0$ $\quad A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

Which gives: $\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) - 6 = 0$

$\hookrightarrow \lambda_1 = -1$
$\lambda_2 = 4$

In PDEs, after applying our Boundary Conditions we can arrive to an EVP for functions instead of matrices. It's just not so obvious.

Recall from handout of Separation of Variables:

Ex: $\boxed{\alpha^2 U_{xx} = U_t}$

Where our solution before B.C.s was:

$$U(x,t) = C_1 + C_2 x + e^{-K^2\alpha^2 t}(C_3 \cos Kx + C_4 \sin Kx)$$

Lets apply both B.C's without solving anything.

① $\left.\frac{\partial U}{\partial x}\right|_{x=0} = C_2 + e^{-K^2\alpha t}(-C_3 K \overset{0}{\cancel{\sin(0)}} + C_4 K \overset{1}{\cancel{\cos(0)}}) = 0$

② $U\big|_{x=\ell} = C_2\ell + C_1 + e^{-K^2\alpha^2 t}(C_3 \cos(K\ell) + C_4 \sin(K\ell)) = 1$

Let's take L.I. terms on time for both Equations

① $t^0: C_2 = 0$ $\qquad e^{-K^2\alpha t}: (-C_3K(0) + C_4K) = 0$

② $t^0: C_2\ell + C_1 = 1$ $\qquad e^{-K^2\alpha t}: (C_3\cos(K\ell) + C_4 \sin(K\ell)) = 0$

from here we get

$C_2 = 0$
$C_1 = 1$

here we are leaving the 0 term in, to make it into matrix form

$$\begin{bmatrix} 0 & K \\ \cos(K\ell) & \sin(K\ell) \end{bmatrix} \begin{Bmatrix} C_3 \\ C_4 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad *$$

Now this is the famous EVP! $\Rightarrow$ for functions

To find eigenfunctions/values simply take the determinant & find K, which $K = \lambda$ = eigenvalues.

Orthogonality & Fourier Series

From previous handout we know 2 functions are orthogonal if $\underbrace{\int_a^b P(x)\, Q(x)\, w(x)\, dx = 0}_{\text{Called Dot Product of Functions}}$

Similarly, the following identities can be found:

$$\int_{-\ell}^{\ell} \cos\frac{m\pi x}{\ell} \cos\frac{n\pi x}{\ell}\, dx = \begin{cases} 0 & m \neq n \\ \ell & m = n \neq 0 \\ 2\ell & m = n = 0 \end{cases}$$

$$\int_{-\ell}^{\ell} \sin\frac{m\pi x}{\ell} \sin\frac{n\pi x}{\ell}\, dx = \begin{cases} 0 & m \neq n \\ \ell & m = n \neq 0 \end{cases}$$

$$\int_{-\ell}^{\ell} \cos\frac{m\pi x}{\ell} \sin\frac{n\pi x}{\ell}\, dx = 0 \quad \text{for all } m, n$$

$$\int_{-\ell}^{\ell} \sin\frac{n\pi x}{\ell}\, b_0\, dx = 0 \quad \text{for all } n$$

$$\int_{-\ell}^{\ell} \cos\frac{n\pi x}{\ell}\, b_0\, dx = 0 \quad \text{for all } n$$

With this, let us review Fourier Series.

Any continuous & periodic function can be represented as a sum of cosines & sines

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\right)$$

Usually, a_0, a_n, b_n are given in a book. With the dot product we can find them easily:

Let's find a_n, which has a $\cos\left(\frac{n\pi x}{\ell}\right)$, thus, lets multiply by $\cos\left(\frac{m\pi x}{\ell}\right)$ & integrate from $[-\cancel{\ell}^{\,a}, \cancel{\ell}^{\,b}]$

$$\int_a^b f(x) \cos\left(\frac{m\pi x}{\ell}\right) dx = \int_a^b a_0 \cos\left(\frac{m\pi x}{\ell}\right) dx + \int_a^b \sum_{n=1}^{\infty} a_n \underbrace{\cos\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{m\pi x}{\ell}\right)}_{0 \text{ except when } m=n} dx$$

$$+ \int_a^b \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{m\pi x}{\ell}\right) dx$$

From Table Above

$$\Rightarrow \int_a^b f(x) \cos\left(\frac{m\pi x}{\ell}\right) dx = a_n \int_a^b \cos\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{m\pi x}{\ell}\right) dx$$

at this point all summation terms are cancelled & a_n is a constant

$$\Rightarrow a_n = \frac{\int_a^b f(x) \cos\left(\frac{m\pi x}{\ell}\right) dx}{\int_a^b \cos\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{m\pi x}{\ell}\right) dx} = \boxed{\frac{1}{\ell} \int_a^b f(x) \cos\left(\frac{n\pi x}{\ell}\right) dx = a_n}$$

If $\ell = 2\pi$, $a = -\pi$, $b = \pi$, this is a Fourier Series Coefficient.