Faculty of Electrical Eng, Mathematics and Computer Science

Lab Assignment 7
WI4243FEM
Finite-element analysis for Applied Physics
Fred Vermolen

1 Problem Statement

We consider a square reservoir (a porous medium) with several wells where water is extracted. This is an important application in countries like Bangladesh where fresh water is extracted from the subsurface. Far away from the reservoir, the water pressure is equal to the hydrostatic pressure. Since we are not able to consider an infinite domain, we use a mixed boundary condition which models the transfer of water over the boundary to locations far away. To this extent, we consider a square domain with length 2 in meter, that is $\Omega = (-1, 1) \times (-1, 1)$ with its boundary $\partial\Omega$. In this assignment, we consider a steady-state equilibrium determined by Darcy's Law for fluid velocity, given by

$$\mathbf{v} = -\frac{k}{\mu}\nabla p, \tag{1}$$

where p, k, μ and $\mathbf{v}$, respectively denote the fluid pressure, permeability of the porous medium, viscosity of water and the fluid flow velocity. Since we only consider a plane section of the reservoir in this assignment, the effect of gravity is not important. Next to Darcy's Law, we consider incompressibility, where the extraction wells are treated as point sinks (this assumption can be justified by the fact that the well diameter is much smaller than the dimensions of the porous medium), that extract at the same rate in each direction, hence

$$\nabla \cdot \mathbf{v} = -\sum_{p=1}^{n_{well}} Q_p \delta(\mathbf{x} - \mathbf{x}_p) = 0, \quad (x, y) \in \Omega, \tag{2}$$

where Q_p denotes the water extraction rate by well k, which is located at $\mathbf{x}_p$. We use the convention $\mathbf{x} = (x, y)$ to represent the spatial coordinates. Further, $\delta(.)$ represents the Dirac Delta Distribution, which is characterised by

$$\begin{cases} \delta(\mathbf{x}) = 0, & \text{if } \mathbf{x} \neq \mathbf{0}, \\ \int_\Omega \delta(\mathbf{x}) d\Omega = 1, & \text{where } \Omega \text{ contains the origin.} \end{cases} \tag{3}$$

Next to the above partial differential equation, we consider the boundary condition

$$\mathbf{v} \cdot \mathbf{n} = K(p - p^H), \quad (x, y) \in \partial\Omega. \tag{4}$$

Here K denotes the transfer rate coefficient of the hormon between the boundary of the domain and its surroundings, and p^H represents the hydrostatic pressure. For the computations, we use the following values:

Table 1: Values of input parameters

Symbol	Value	Unit
Q_p	50	m^2/s
k	10^{-7}	m^2
μ	$1.002 \cdot 10^{-3}$	$Pa \cdot s$
K	10	m/s
p^H	10^6	Pa

We consider six wells, located at

$$\begin{cases} x_p = 0.6\cos(\dfrac{2\pi(p-1)}{5}), \\ \\ x_p = 0.6\sin(\dfrac{2\pi(p-1)}{5}), \end{cases}$$

$p \in \{1, \dots, 5\}$ and for $p = 6$, we have $x_6 = 0$ and $y_6 = 0$. In order to solve this problem, one needs to consider the following questions:

1. Give the partial differential equation and boundary condition in terms of the pressure p.

2. Give the weak formulation of the problem (partial differential equation + boundary condition). *Hint:* $\int_\Omega \delta(\mathbf{x}) f(\mathbf{x}) d\Omega = f(\mathbf{0})$.

3. Give the Galerkin equations (the system of linear equations).

4. Give the element matrix and element vector for the internal elements. Distinguish between cases where the point sink lies inside or outside the considered element.

5. Give the element matrix and element vector for the boundary elements.

6. In order to solve the problem, you need to determine whether each internal element (triangle) contains a cell. We will determine whether cell with index p, with position $\mathbf{x}_p$, is in the element e_k with vertices $\mathbf{x}_{k1}$, $\mathbf{x}_{k2}$ and $\mathbf{x}_{k3}$. We do so by testing the following criterion:

$$|\Delta(\mathbf{x}_p, \mathbf{x}_{k2}, \mathbf{x}_{k3})| + |\Delta(\mathbf{x}_{k1}, \mathbf{x}_p, \mathbf{x}_{k3})| + |\Delta(\mathbf{x}_{k1}, \mathbf{x}_{k2}, \mathbf{x}_p)| : \begin{cases} = |e_k|, & \mathbf{x}_p \in \overline{e}_k \\ \\ > |e_k|, & \mathbf{x}_p \notin \overline{e}_k. \end{cases} \tag{5}$$

   Here $\Delta(\mathbf{x}_p, \mathbf{x}_q, \mathbf{x}_r)$ denotes the triangle with vertices $\mathbf{x}_p$, $\mathbf{x}_q$ and $\mathbf{x}_r$, and $|\Delta(\mathbf{x}_{k1}, \mathbf{x}_{k2}, \mathbf{x}_{k3})|$ denotes its area. Further, $e_k = \Delta(\mathbf{x}_{k1}, \mathbf{x}_{k2}, \mathbf{x}_{k3})$ represents the triangular element k with vertices $\mathbf{x}_{k1}$, $\mathbf{x}_{k2}$ and $\mathbf{x}_{k3}$ and $\overline{e}_k$ includes the boundaries of element e_k. Express the area of the triangles in terms of the nodal points (*Hint: Use the determinant as in Chapter 6 of the book.*). To implement the above criterion whether a cell is within an element, use a tolerance of eps in matlab because of possible rounding errors.

   Remark: As an alternative to the above procedure, you may also consider the barycentric coordinates, which are the linear basis functions ϕ_{k1}, ϕ_{k2} and ϕ_{k3}, and see whether their values are in the interval $[0, 1]$ for $\mathbf{x}_p$, then, $\mathbf{x}_p$ is within the triangular element e_k.

7. Program the finite-element code (GenerateElementMatrix, GenerateElementVector, GenerateBoundaryElementMatrix, GenerateBoundaryElementVector), and evaluate the solution. Use mesh refinement to evaluate the quality of the solution. Plot your solution in terms of contour plot and a three-dimensional surface plot.

8. Use Darcy's Law, equation (1), to compute the velocities in both directions, by writing both components of equation (1) in a weak form, and by subsequent derivation of the Galerkin equations. Implement this where you solve the resulting systems of linear equations:

$$M\underline{v}_x = C_x\underline{p}, \qquad M\underline{v}_y = C_y\underline{p}, \tag{6}$$

to get $\underline{v}_x$ and $\underline{v}_y$. Note that the files Give a plot of the components by adjusting the files GenerateElementMatrix, GenerateElementVector, GenerateBoundaryElementMatrix, GenerateBoundaryElementVector, and now also BuildMatricesandVectors. In the file WI4243Post, you can add

figure(4); quiver(x,y,vx',vy'); axis([-1 1 -1 1]);

9. Perform various simulations where you let the transfer coefficient K range between 0.00001 and 10000. Show the contour plots, and give the values of the minimal pressure (which is important from an engineering point of view). Explain your results.

10. What happens if $K = 0$? Explain the results.