Finite Elements

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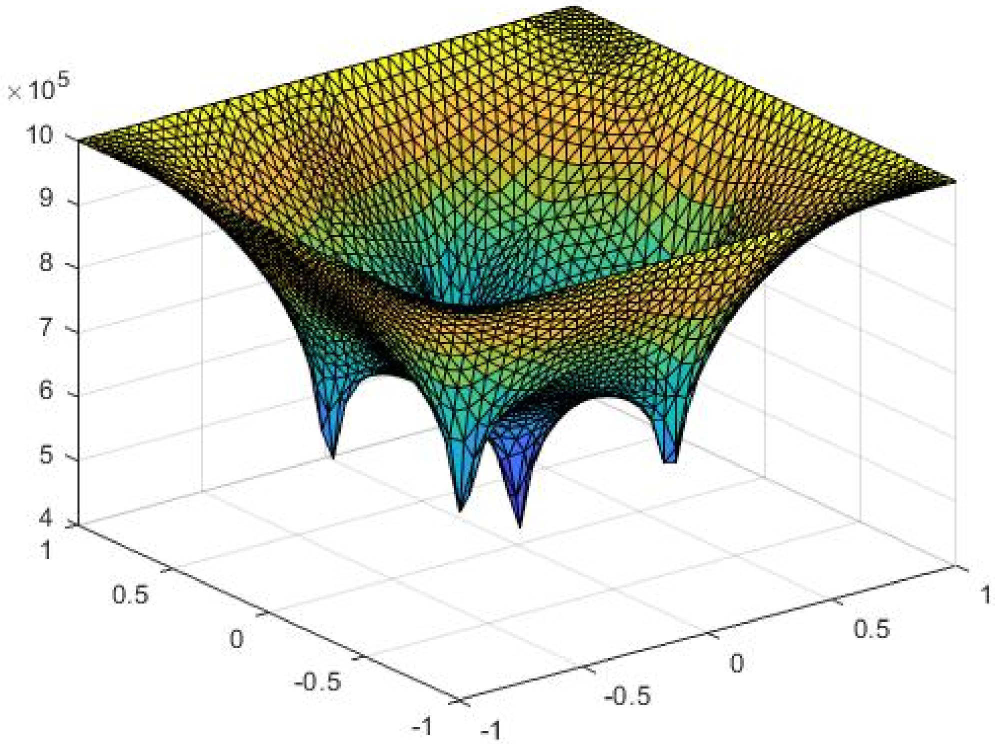

Contents

Chapter 1

1D-case

On the 1D interval of $x = [0, 1]$, we consider a steady-state convection-diffusion-reaction equation, with homogeneous Neumann boundary conditions. The following equations apply to this domain:

$$\begin{cases} -D\triangle u + \lambda u = f(x), \\ -D\frac{du}{dx}(0) = 0, \\ -D\frac{du}{dx}(1) = 0 \end{cases} \tag{1.1}$$

In this report $\triangle$ denotes the laplacian operator. The function f(x) is a given funtion, where D and ∇ are positive real constants. In order to solve this boundary value problem (BVP), first the interval is divided in n-1 elements(n = positive integer). This results in the domain being divided in elements: $e_i = [x_i, x_{i-1}]$ where $i = 1, 2, .., n$.

In order to solve this BVP, the solutions for the given equations will first be calculated and then computed using MATLAB codes.

1.1 Boundary value problem 1D

In order to find the Weakform of the given equations(1.1), both sides are multiplied by a test function $\phi(x)$ and then integrate both sides over the domain Ω. In the equations $\phi(x)$ is written as ϕ

$$\int_\Omega \phi(-D\triangle u + \lambda u)d\Omega = \int_\Omega \phi f(x)d\Omega \tag{1.2}$$

Now by rewriting and then using partial integration the following equation can be found:

$$\int_\Omega (\nabla \cdot (-D\phi \cdot \nabla u) + D\nabla\phi\nabla u + \phi\lambda u)d\Omega = \int_\Omega \phi f(x)d\Omega \tag{1.3}$$

Applying Gauss on the first term on the left side of equation(1.3):

$$\int_\Omega \vec{n} \cdot (-D\phi\nabla u)d\tau + \int_\Omega (D\nabla\phi \cdot \nabla u + \phi\lambda u)d\Omega = \int_\Omega \phi f(x)d\Omega \tag{1.4}$$

Using the boundary conditions from equations(1.1) the boundary integral equals to 0 and then the following weak formulation(WF) is found:

(WF):

$$\begin{cases} \text{find u } \epsilon \sum = \{u \ smooth\} \text{ Such that:} \\ \int_\Omega (D\nabla\phi \cdot \nabla u + \phi\lambda u)d\Omega = \int_\Omega \phi f(x)d\Omega \\ \forall\phi \in \sum \end{cases} \tag{1.5}$$

The next step is to substitute the Galerkin equations into the found diferential equation, where u is replaced by $\sum_{j=1}^{n} c_i\phi_j$ and $\phi(x) = \phi_i(x)$ with $i = [1, .., n]$. Filling this in equation (1.5) the following equation is found:

$$\sum_{j=1}^{n} c_i \int_0^1 (D\nabla\phi_i \cdot \nabla\phi_j + \lambda\phi_i\phi_j)d\Omega = \int_0^1 \phi_i f(x)d\Omega \tag{1.6}$$

Which is of the form of $S\vec{c} = \vec{f}$

1.2 Element matrix

Now the found Galerkin equations can be used to compute S_{ij} the element matrix, over a generic line element e_i.

$$S\vec{c} = \sum_{j=1}^{n} c_i \int_0^1 (D\nabla\phi_i \cdot \nabla\phi_j + \lambda\phi_i\phi_j)d\Omega \tag{1.7}$$

$$S_{ij} = \sum_{l=1}^{n-1} S_{ij}^{e_k} \tag{1.8}$$

Now to solve S we solve the following equation, over the internal line element.

$$S_{ij}^{e_k} = -D \int_{e_k} \nabla\phi_i \cdot \nabla\phi_j d\Omega + \lambda \int_{e_k} \phi_i\phi_j dx \tag{1.9}$$

1.3 Element vector

Again the found Galerkin Equations(1.6) are used in order to compute the element vector f_i over a generic line-element.

$$f_i^{e_k} = \int_{e_k} \phi_i f dx \tag{1.10}$$

$$f_i^{e_k} = \frac{|x_k - x_{k-1}|}{(1+1+0)!} f(\vec{x}) = \frac{|x_k - x_{k-1}|}{2} \begin{bmatrix} f_{k-1}^{e_n} \\ f_k^{e_n} \end{bmatrix} \tag{1.11}$$

1.4 Boundary value problem 1D MATLAB routine

1.4.1 mesh and elmat code

The first step in order to solve the BVP is to write a MATLAB routine that generates an equidistant distrubition of points over the given interval of $[0, 1]$(generate a mesh with n-1 elements).

```
function [ x ] = GenerateMesh(int, N_elem)
%GenerateMesh Creates a mesh for 1D problems

x = linspace(int(1,1),int(1,2),N_elem);
end
```

Using the codes to generate a mesh and the elmat, it is easier to use this 1D problem and adapt to a higher dimensional problem. The next step is to write a code that generates a two dimensional array, called the elmat.

```
function [ elmat ] = GenerateTopology( N_elem )
%GenerateTopology Creates the topology for a 1D problem given mesh 'x'.

elmat = zeros(N_elem,2);
elmat(i,1) = i;
elmat(i,2) = i + 1;
end
```

1.4.2 Element matrix

Now that the base MATLAB codes are made the element matrix and element vector codes can be written. The first step in this process is, is to compute the element matrix S_{elem}.

```
function [ Selem ] = GenerateElementMatrix( k, elmat, D, lambda, mesh)
%GenerateElementMatrix Creates element matrix S_ek

Selem = zeros(2,2);

i = elmat(k,1);
j = elmat(k,2);

x1 = mesh(i);
x2 = mesh(j);

element_length = abs(x1-x2);

slope = 1/element_length;

for m = 1:2
    for n = 1:2
        if m == n
            Selem(m,n) = element_length*((-1)^(abs(m-n))*D*slope^2
            + (2)*lambda/6);
        else
            Selem(m,n) = element_length*((-1)^(abs(m-n))*D*slope^2
            + (1)*lambda/6);
        end
    end
end
end
```

1.4.3 Assemble matrix S

To generate a n-by-n matrix S, the sum over the connections of the vertices in each element matrix, over all elements has to be calculated. The following code computes this matrix:

```
function [ S ] = AssembleMatrix( N_elem, int, lambda, D)
% global N_elem

elmat = GenerateTopology(N_elem);

S = zeros(N_elem,N_elem);

for i = 1:N_elem-1
    Selem = GenerateElementMatrix(i, elmat, int, N_elem, D, lambda);
    for j = 1:2
        for k = 1:2
            S(elmat(i,j), elmat(i,k)) =
            S(elmat(i,j), elmat(i,k)) + Selem(j,k);
        end
    end
end
end
```

All the previous code will generate a large matrix S, from the element matrices S_{elem} over each element.

1.4.4 Element vector MATLAB routine

The next step In order to solve the equation $S\vec{c} = f$ is to create a code to generate the element vector. This element vector provides information about node i and node $i+1$, which are the vertices of element e_i.

```
function [ felem ] = GenerateElementVector( i, elmat, mesh )
%GenerateElementVector Creates element vector f_ek


felem = [0;0];

k1 = elmat(i,1);
k2 = elmat(i,2);

x1 = mesh(k1);
x2 = mesh(k2);

element_length = abs(x1-x2);

felem = (element_length/2*arrayfun(@functionBVP,[x1,x2]))';

end
```

Where the function $f(x)$ from the BVP is defined in the following function. The different definitions of $f(x)$ will be used in different assignments.

```
function [f] = functionBVP(x)
f = 1;
%f = sin(20*x);
%f = x;
end
```

To generate the vector f, the sum over the connections of the vertices in each element matrix, over all elements $i \epsilon \{1, ..., n-1\}$ has to be calculated.

```
function [ f ] = AssembleVector( N_elem, int, lambda, D )

f = zeros(N_elem,1);
elmat = GenerateTopology(N_elem);

for i = 1:N_elem-1
    felem = GenerateElementVector(i, elmat, int, N_elem);
    for j = 1:2
        f(elmat(i,j)) = f(elmat(i,j)) + felem(j);
    end
end
```

1.4.5 Computing S and f

Now if the previous matlab codes are run the following happens. First, a mesh and 1D topology are build. These are needed for the S matrix and f vector. The second step is to calculate the S matrix and f vector themselves through the found equations of section 1.2 and 1.3. The final step is to use the found matrix and vector to solve the equation $Su = \vec{f}$.

1.5 Main program

The main program is simply written by assembling the previous created MATLAB code AssembleMatrix and AssembleVector and deviding the vector f by the matrix S.

```
function [ u ] = SolveBVP(  N_elem, int, lambda, D )

S = AssembleMatrix( N_elem, int, lambda, D);
f = AssembleVector( N_elem, int, lambda, D);

%% Calculate u
x = linspace(int(1),int(2),N_elem);

u = S\f;
plot(x,u);
```

The result of the plot is shown in figure(1.1).

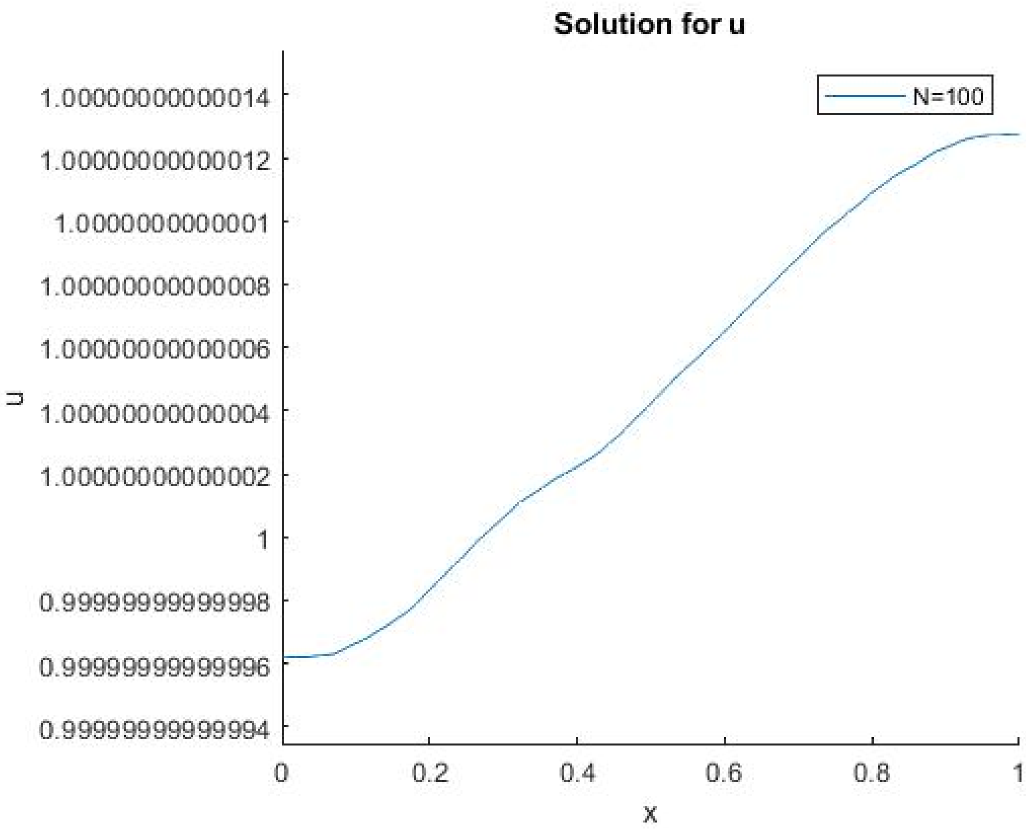

Figure 1.1: showing the calculated u versus x, with N = 100,f(x)=1

1.6 Solution for u

The final step is to combine all the codes in a main code to solve $Su = \vec{f}$. This code can be found in Appendix A. Previously the S matrix and f vector were computed for $n = 100$. Now u will be calculated for $f(x) = 1$, $D = 1$, $\lambda = 1$ and $N = 100$. The result of this is plotted in figure(1.2).

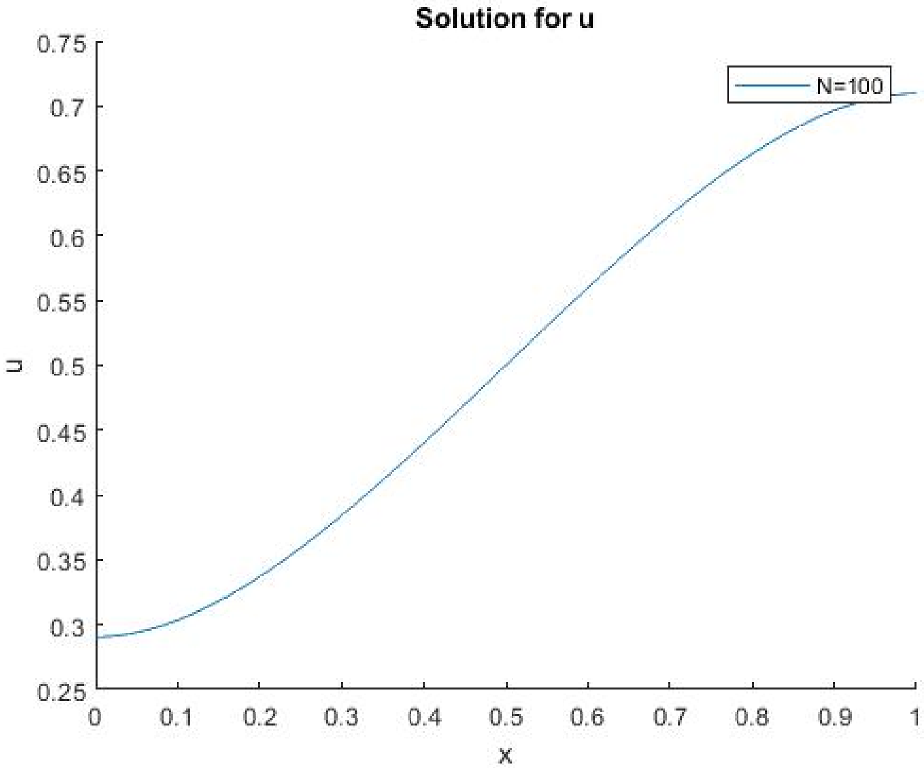

Figure 1.2: showing the calculated u versus x, with N = 100,f(x)=x

1.7 Experiment

The next step is to see what happens when changing f(x) to $f(x) = sin(20x)$ and to see the difference for several values for n $(n = 10, 20, 30, 40, 80, 160)$

```
function [f] = functionBVP(x)
    f = sin(20*x);
    %f = x;
    %f = 1;
end

figure
hold on

for N_elem = [10 20 40 80 100 160]
mesh = GenerateMesh(int,N_elem);
elmat = GenerateTopology(N_elem);
S = AssembleMatrix( N_elem, lambda, D, mesh, elmat);
f = AssembleVector( N_elem, mesh, elmat);

x = linspace(int(1),int(2),N_elem);

u = S\f;
plot(x,u);

legend('N=100')
title('Solution for u')
xlabel('x')
ylabel('u')
ax.box='on'
end
hold off
```

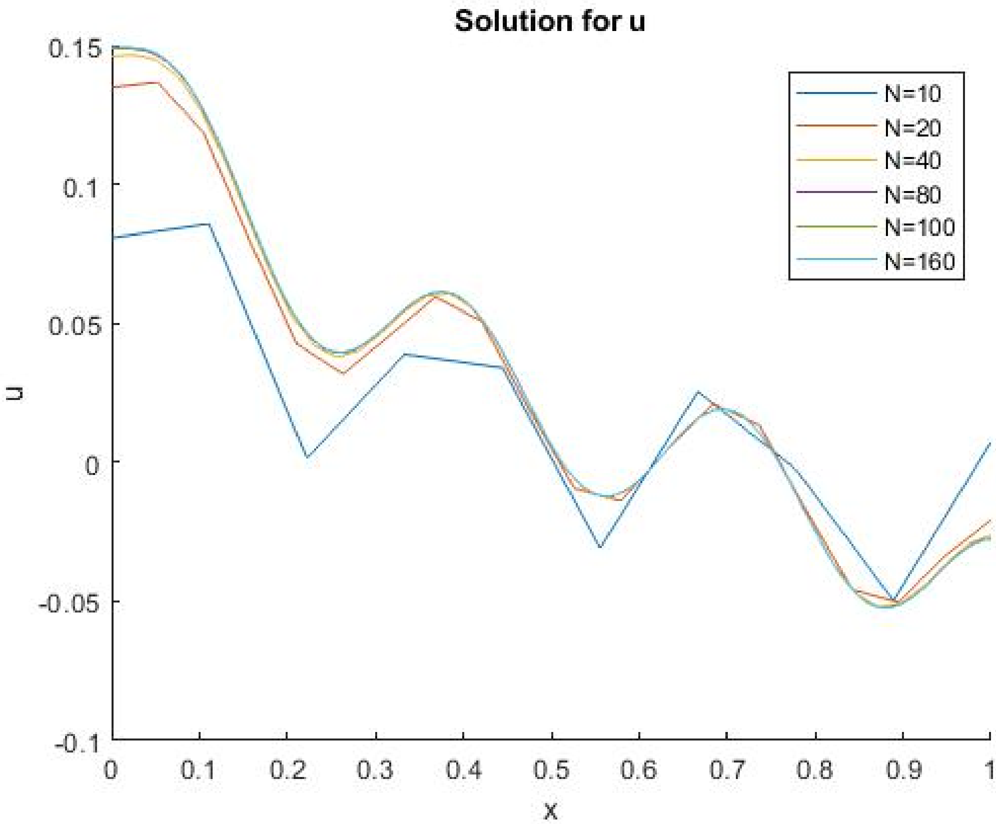

Figure 1.3: showing the calculated u versus x, with $N = [10, 20, 40, 80, 100, 160]$, $f(x) = sin(x)$

Chapter 2

2D-case

The obvious next step after solving a 1 dimensional BVP is to adept the 1D solutions into code to solve a 2 dimensional BVP. To do this, a real life problem is going to be solved. In 3rd world countries one of the big issues is the supply of fresh water. One way is to do this is to take square reservoirs, which is a porous medium, with several wells where water is extracted from the subsurface. The water pressure is equal to the hydrostatic pressure. As this is not on an infinite domain, mixed boundary conditions are used. These boundary conditions represent a model for the transfer of the water over the boundary to locations far away. To this extent, a square domain is considered with length 2 in meter: $\Omega = (-1;1) \times (-1;1)$ with boundary $\partial\Omega$. Darcy's law for fluid determines the steady state equilibrium of this BVP, given by equation (2.1):

$$\vec{v} = -\frac{k}{\mu}\nabla p \tag{2.1}$$

Where p, k, μ and v, respectively denote the fluid pressure, permeability of the porous medium, viscosity of water and the fluid flow velocity. In this BVP the effect of gravity will not play a part as the problem is looked at in 2D. An accompanying assumption is incompressibility, so the extraction wells are treated as point sinks. This assumption can be made as the well its diameter is much smaller than the dimensions of the square reservoir. The extraction wells extract at the same rate in each direction, leading to the following boundary conditions (2.2).

$$\nabla \cdot \vec{v} = -\sum_{p=1}^{n_{well}} Q_p \delta(\phi(\vec{x} - \phi(\vec{x}_p) = 0, \quad (x, y) \in \Omega \tag{2.2}$$

Where Q_p denotes the water extraction rate by well k, which is located at x_p. Here x equals $(x; y)$, the spatial coordinates. The convention $\vec{x} = (x; y)$ to represent the spatial coordinates is used. The dirac Delta Distribution is characterized by equation(2.3).

$$\begin{cases} \delta(\vec{x}) = 0, \quad \vec{x} \neq 0 \\ \int_\Omega \delta(\vec{x}) d\Omega = 1, \quad \text{where } \Omega \text{ contains the origin.} \end{cases} \tag{2.3}$$

For this BVP the following boundary condition is considered:

$$\vec{v} \cdot \vec{n} = K(p - p^H),\ (x, y)\ \in\ \partial\Omega \tag{2.4}$$

Where K denotes the transfer rate coefficient of the water between the boundary of the domain and its surroundings. The constant p^H represents the hydrostatic pressure.

The boundary $\partial\Omega$ is divided into four parts described by the side of the square domain Ω. $\partial\Omega_1$ is the part with $x = -1$, $\partial\Omega_2$ is the part with $y = 1$, $\partial\Omega_3$ is the part with $x = 1$, $\partial\Omega_4$ is the part with $y = -1$.

In order to solve this BVP the values needed for all the constants are given in table (2.1).

Table 2.1: Values of input parameters

Symbol	Value	Unit
Q_p	50	m^2/s
k	10^{-7}	m^2
μ	$1.002 \cdot 10^{-3}$	$Pa \cdot s$
K	10	m/s
p^H	10^6	Pa

In this BVP six wells are considered, which are located at:

$$\begin{cases} x_p = 0.6\cos(\frac{2\pi(p-1)}{5}) \\ x_p = 0.6\sin(\frac{2\pi(p-1)}{5}) \end{cases} \tag{2.5}$$

$p \in \{1, \dots, 5\}$ and for $p = 6$ we have $x_6 = 0$ and $y_6 = 0$.

2.1 Boundary value problem 2D

The first step to solving these equations using finite elements is to find to find the boundary value problem to solve. This is done by filling in equation(2.1) in both equation 2.2 and the boundary condition(2.4) in order to find the BVP in terms of p:

$$\text{BVP} \begin{cases} -\frac{k}{\mu}\triangle\vec{p} = -\sum\limits_{p=1}^{n_{\text{well}}} Q_p\delta(\vec{x}-\vec{x}_p) = 0, & (x,y) \in \Omega \\ -\frac{k}{\mu}\nabla\vec{p}\cdot\vec{n} = -\frac{k}{\mu}\frac{dp}{dn} = K(p-p^H), & (x,y) \in \partial\Omega \end{cases} \tag{2.6}$$

The next step is to compute the weak formulation using the previous found BVP(2.6). By multiplying both sides by test function $\phi(x) = \alpha_i + \beta_i x + \gamma_i y$ and integrating both sides over the domain Ω the weak formulation can be found.

$$\int_\Omega \phi(\vec{x})\nabla\cdot(-\frac{k}{\mu}\nabla\vec{p})d\Omega = \int_\Omega -\sum_{p=1}^{n_{well}} \phi(\vec{x})Q_p\delta(\vec{x}-\vec{x}_p) \tag{2.7}$$

Using integrating by parts on the left side of equation(2.7) results in:

$$\int_\Omega \nabla\cdot\left[\phi(\vec{x})(-\frac{k}{\mu}\nabla\vec{p})\right] + \frac{k}{\mu}\nabla\phi(\vec{x})\cdot\nabla p d\Omega = -\int_\Omega \sum_{p=1}^{n_{well}} \phi(\vec{x})Q_p\delta(\vec{x}-\vec{x}_p)d\Omega \tag{2.8}$$

Next is to apply Gauss on the first term of the left side.

$$\int_{d\Omega} \vec{n}\cdot(\phi(\vec{x})(-\frac{k}{\mu}\nabla\vec{p}))d\tau + \int_\Omega \frac{k}{\mu}\nabla\phi(\vec{x})\cdot\nabla p d\Omega = -\int_\Omega \sum_{p=1}^{n_{well}} \phi(\vec{x})Q_p\delta(\vec{x}-\vec{x}_p)d\Omega \tag{2.9}$$

Switching the integral and summation on the right side of equation(2.9) and simplifying terms:

$$\int_{d\Omega} (\phi(\vec{x})(-\frac{k}{\mu}\frac{d\vec{p}}{dn}))d\tau + \int_\Omega \frac{k}{\mu}\nabla\phi(\vec{x})\cdot\nabla p d\Omega = -\sum_{p=1}^{n_{well}} \int_\Omega \phi(\vec{x})Q_p\delta(\vec{x}-\vec{x}_p)d\Omega \tag{2.10}$$

The right side of equation(2.10) can be simplied using the boundary conditions (equation(2.6)) and the following property into equation(2.12):

$$\int_\Omega \delta(\vec{x})f(\vec{x})d\Omega = f(0) \tag{2.11}$$

$$\int_{\delta\Omega} \phi(\vec{x})K(p - p^H)d\tau + \int_{\Omega} \frac{k}{\mu}\nabla\phi(\vec{x}) \cdot \nabla p d\Omega = -\sum_{p=1}^{n_{well}} \phi(\vec{x}_p)Q_p \tag{2.12}$$

Rearranging equation(2.12) so that the variable parts are on the left and the constant parts on the right leads to the following WF:

(WF):

$$\begin{cases} \text{find p } \epsilon \sum = \{p \ smooth\} \text{ Such that:} \\ \int_{\delta\Omega} \phi(\vec{x})Kpd\tau + \int_{\Omega} \frac{k}{\mu}\nabla\phi(\vec{x}) \cdot \nabla p d\Omega = -\sum_{p=1}^{n_{well}} \phi(\vec{x}_p)Q_p + \int_{\delta\Omega} \phi(\vec{x})Kp^H d\tau \\ \forall\phi \ \epsilon \sum \end{cases} \tag{2.13}$$

To solve the WF the Galerkin equations are applied, where p is replaced by $\sum_{j=1}^{n} c_j\phi_j$ and $\phi(x) = \phi_i(x)$.

$$\sum_{j=1}^{n} c_i \int_{\partial\Omega} \phi_i K\phi_j d\tau + \int_{\Omega} \frac{k}{\mu}\nabla\phi(x) \cdot \nabla\phi_j d\Omega = -\sum_{p=1}^{n_{well}} \phi(x_p)Q_p + \int_{\partial\Omega} \phi_i K p^H d\tau \tag{2.14}$$

Equation(2.14) now is of the form $S\vec{c} = \vec{f}$ and, like with the 1D problem, $\vec{c}$ can be computed. First the element and boundary elements are determined from the Galerkin equations.

2.2 Element matrix and element vector

First the galerkin equation is seperated in its element and boundary components. The element matrix $S_{ij}^{e_k}$ and the element vector $f_i^{e_k}$ are given in equations (2.15) and 2.16 respectively. For the element matrix, boundary element matrix and boundary element vector Newton-Côtes theorem and Holand-Bell theorem are applied to simplify the equations into a set of equations that can be used for computing with MATLAB code.

$$S_{ij}^{e_k} = \int_{e_k} \frac{k}{\mu}\nabla\phi_i \cdot \nabla\phi_j d\Omega = (\beta_i\beta_j + \gamma_i\gamma_j)\frac{k}{\mu}\frac{|\triangle e_k|}{2} \tag{2.15}$$

$$f_i^{e_k} = -\sum_{p=1}^{n_{well}} \phi_i(\vec{x}_p)Q_p \tag{2.16}$$

2.3 Boundary matrix and boundary vector

The boundary matrix $S_{ij}^{be_l}$ and boundary vector $f_i^{be_l}$ can be found in the following equations:

$$S_{ij}^{be_l} = \int_{be_l} K\phi_i\phi_j dx = K\frac{|be_l|}{6}(1 + \delta_{ij}) \tag{2.17}$$

$$f_i^{be_l} = Kp^H \int_{be_l} \phi_i dx = Kp^H \frac{|be_l|}{2} \tag{2.18}$$

Where δ_{ij} is the Kronicker delta.

2.4 Wells within an internal element

To solve the BVP in 2D, one of the aspects that need to be determined is whether each internal element contains a well. This is done by determining whether well with index p and position x_p is contained within element e_k with vertices x_{k1}, x_{k2} and x_{k3}. This is done according the following criterion:

$$|\Delta(\vec{x}_p, \vec{x}_{k2}, \vec{x}_{k3})| + |\Delta(\vec{x}_{k1}, \vec{x}_p, \vec{x}_{k3})| + |\Delta(\vec{x}_{k1}, \vec{x}_{x2}, \vec{x}_p) : \begin{cases} = |e_k|, \ \vec{x}_p \ \in, \vec{e}_k \\ > |e_k|, \ \vec{x}_p \ \notin \ \vec{e}_k \end{cases} \tag{2.19}$$

In the criterion $\Delta(\vec{x}_p, \vec{x}_q, \vec{x}_r)$ denotes the triangle with vertices $\vec{x}_p$, $\vec{x}_q$ and $\vec{x}_r$, where $|\Delta(\vec{x}_{k1}, \vec{x}_{k2}, \vec{x}_{k3})|$ denote its area. The triangular element k is given by $e_k = \Delta(\vec{x}_{k1}, \vec{x}_{k2}, \vec{x}_{k3})$ with vertices $\vec{x}_{k1}$, $\vec{x}_{k2}$ and $\vec{x}_{k3}$ and $\vec{e_k}$ includes the boundary of element e_k. To solve this BVP a certain tolerance has to be accounted for in the Matlab code. However, while using this method it proved difficult to determine a correct tolerance to ensure every well was contained in a single element.

Therefore, an alternative method was used instead. A check was done whether the linear basis functions ϕ_{k1}, ϕ_{k2} and ϕ_{k3} all have values in the interval $[0, 1]$ at position $\vec{x}_p$. If this is true, then the well at $\vec{x}_p$ is within the triangular element e_k.

If it is found that an element does contain a well, we see in equation (2.16) that $\phi_i(\vec{x_p})$ for $i = \{k1, k2, k3\}$ is found. The following code does this by determining α_i, β_i, and γ_i and subsequently filling $\vec{x}_p$ into the found $\phi_i(\vec{x})$.

```
for index1=1:topology
    xc(index1) = x(elmat(i,index1));
    yc(index1) = y(elmat(i,index1));
end;

Delta = det([1 xc(1) yc(1);1 xc(2) yc(2);1 xc(3) yc(3)]);

B_mat = [1 xc(1) yc(1);1 xc(2) yc(2);1 xc(3) yc(3)] \  eye(3);

alpha = B_mat(1,1:3);
beta  = B_mat(2,1:3);
gamma = B_mat(3,1:3);

felem = zeros(1,topology);

if ¬exist('u','var')  % Only if u is already know can the calculation of the velocity ...
    begin.
    for N = 1:N_wells
        for index3 = 1:topology
            phi_p(index3) = alpha(index3) + beta(index3)*xp(N) + gamma(index3)*yp(N);
        end
        if (phi_p(1) ≤ 1) && (phi_p(1) ≥ 0) && (phi_p(2) ≤ 1) && (phi_p(2) ≥ 0) && ...
             (phi_p(3) ≤ 1) && (phi_p(3) ≥ 0);
            for index1 = 1:topology
                felem(index1) = felem(index1) + -Qp*phi_p(index1);
            end

        end
end
```

2.5 Generating MATLAB code

Similar as with the 1D BVP, MATLAB code is written in order to generate a mesh, element matrix, element vector, boundary element matrix and boundary element vector. These codes can be found in appendix B.1 through B.6. The scripts have been written such that they can be used to find both the pressure field and the velocity field(the derivation fort the velocities is found in the next section).

Using the MATLAB codes a nice mesh can be found.

2.6 Velocities

In order to find the velocities in x,y-direction, Darcy's law is used to compute the speed in both directions.

In order to find v_x and v_y the first step is to rewrite equation(2.20).

$$\vec{v} = -\frac{k}{\mu}\nabla p \tag{2.20}$$

$$v_x = -\frac{k}{\mu}\frac{dp}{dx} \tag{2.21}$$

$$v_y = -\frac{k}{\mu}\frac{dp}{dy} \tag{2.22}$$

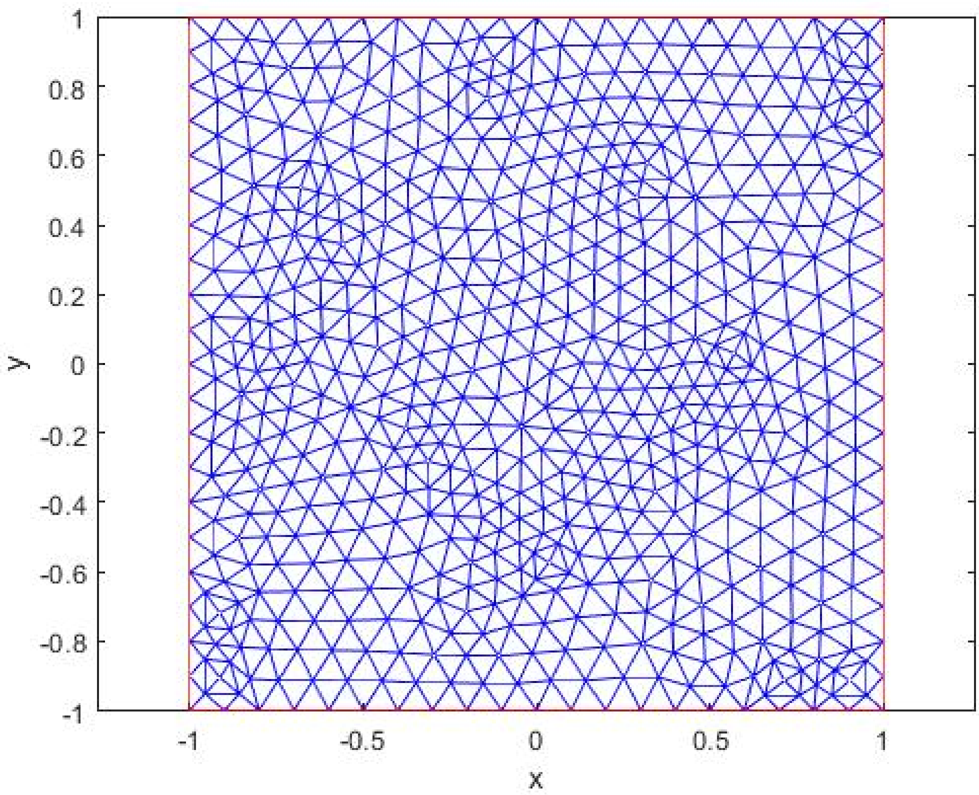

Figure 2.1: Triangle element mesh over the square reservoir domain $\Omega = (-1;1) \times (-1;1)$.

Equation(2.24) shows the relation between $\vec{v}$ and pressure p.

$$\vec{v} \cdot \vec{n} = k(p - p^H) \text{ on } \partial\Omega \tag{2.23}$$

This relation gives the boundary conditions for v_x and v_y:

$$v_x(x = -1) = -k(p - p^H) \tag{2.24}$$

$$v_x(x = 1) = k(p - p^H) \tag{2.25}$$

$$v_y(y = -1) = -k(p - p^H) \tag{2.26}$$

$$v_y(y = 1) = k(p - p^H) \tag{2.27}$$

In order to find the weak form, again, the test function ϕ is used and integrated over the domain Ω. Here follows the derivation for finding v_x, the steps for deriving v_y are similar.

$$\int\limits_\Omega \phi v_x d\Omega = -\frac{k}{\mu} \int\limits_\Omega \phi \frac{dp}{dx} d\Omega \tag{2.28}$$

Partial integration is applied on the right side term.

$$\int\limits_\Omega \phi v_x d\Omega = -\frac{k}{\mu} \{ \int\limits_\Omega \frac{d}{dx}(\phi p) - p\frac{d\phi}{x} d\Omega \} \tag{2.29}$$

Rewriting the integral:

$$\int_{\Omega} -\frac{k}{\mu}\frac{d\phi p}{dx}dxdy = \int_{-1}^{1} -\frac{k}{\mu}[\phi p]_{-1}^{1}dy \tag{2.30}$$

The surface integral turns into a set of line integrals along parts of the boundary $\partial\Omega$.

$$\int_{\Omega} -\frac{k}{\mu}\frac{d\phi p}{dx}dxdy = \int_{-1}^{1} -\frac{k}{\mu}(\phi(x=1,y)p(x=1,y)) + \frac{k}{\mu}(\phi(x=-1)p(x=-1,y))dy \tag{2.31}$$

Simplifying the previous equations.

$$\int_{\Omega} -\frac{k}{\mu}\frac{d\phi p}{dx}dxdy = \int_{\partial\Omega_3} -\frac{k}{\mu}\phi p d\tau + \int_{\partial\Omega_1} \frac{k}{\mu}\phi p d\tau \tag{2.32}$$

Inserting this into equation(2.29) the following equation is found.

$$\int_{\Omega} \phi v_x = \frac{k}{\mu}\{\int_{\partial\Omega_3} -\phi p d\tau + \int_{\partial\Omega_1} \phi p d\tau + \int_{\Omega} p\frac{\phi_i}{dx}d\Omega\} \tag{2.33}$$

The boundary conditions is used to rewrite p in the two boundary integrals.

$$\text{on} \begin{cases} \partial\Omega_3 : \ -v_x = k(p-p^H) \ \rightarrow \ p = -\frac{v_x}{k} + p^H \\ \partial\Omega_1 : \quad v_x = k(p-p^H) \ \rightarrow \ p = \frac{v_x}{k} + p^H \end{cases} \tag{2.34}$$

The following weakform is derived:

Find $v_x \in \Sigma = \{v_x \text{ smooth}\}$, such that

$$\int_{\Omega} \phi v_x d\Omega + \int_{d\Omega_3} -\frac{k}{\mu}\frac{1}{k}\phi v_x dy + \int_{d\Omega_1} -\frac{k}{\mu}\frac{1}{k}\phi v_x dy = \int_{d\Omega_3} -\frac{k}{\mu}\phi p^H dy + \int_{d\Omega_1} \frac{k}{\mu}\phi p^H dy + \int_{\Omega} \frac{k}{\mu}p\frac{d\phi}{dx}d\Omega \tag{2.35}$$

for all $\phi \in \Sigma$.

To find the system of equations the following equations are filled in equation(2.35) $\phi(\vec{x}) = \phi_i(\vec{x}) = \alpha_i + \beta_i x + \gamma_i y$ and $v_x \approx \sum_{j=1}^{n} c_j\phi_j(\vec{x})$.

$$S_{ij} = \sum_{j=1}^{n} c_j\{\int_{\Omega} \phi_i\phi_j d\Omega + \int_{\partial\Omega_3} -\frac{k}{\mu}\frac{1}{k}\phi_i\phi_j d\tau + \int_{\partial\Omega_1} -\frac{k}{\mu}\frac{1}{k}\phi_i\phi_j d\tau\} \tag{2.36}$$

$$f_i = \int_{\partial\Omega_3} -\frac{k}{\mu}\phi_i p^H d\tau + \int_{\partial\Omega_1} \frac{k}{\mu}\phi_i p^H d\tau + \int_{\Omega} \frac{k}{\mu}p\frac{d\phi_i}{dx}d\Omega \tag{2.37}$$

Now separating contributions to both S_{ij} and f_i from boundary and internal elements into $S_{ij}^{be_l}$, $S_{ij}^{e_k}$, $f_i^{be_l}$ and $f_i^{e_k}$ such that:

$$S_{ij} = \sum_{l=1}^{n_{be}} S_{ij}^{be_l} + \sum_{k=1}^{n_e} S_{ij}^{e_k} \tag{2.38}$$

$$f_i = \sum_{l=1}^{n_{be}} f_i^{be_l} + \sum_{k=1}^{n_e} f_i^{e_k} \tag{2.39}$$

Applying Newton-Côtes theorem and Holand-Bell theorem results in the following new expressions for the (boundary) element-matrix and -vector are found.

$$S_{ij}^{e_k} = \int_{e_k} \phi_i\phi_j = \frac{|\triangle_{ek}|}{24} \tag{2.40}$$

$$S_{ij}^{be_l} = \int_{be_l} -\frac{k}{\mu}\phi_i\phi_j dy = \frac{k}{\mu}\frac{1}{k}\frac{|be_l|}{6}(1+\delta_{ij}) \tag{2.41}$$

$$f_i^{e_n} = \int_{e_n} \frac{k}{\mu} p \beta_i d\Omega = \frac{k}{\mu} \beta_i \sum_{m = \in \{k_1, k_2, k_3\}} p(\vec{x}_m) \frac{|\triangle e_n|}{6} \tag{2.42}$$

$$f_i^{be_l} = \int_{be_l} \pm \frac{k}{\mu} \phi_i p^H dy = \pm \frac{k}{\mu} p^H \frac{|be_l|}{2} \tag{2.43}$$

With '+' if be_l is on $\partial\Omega_1$ and '−' if it is on $\partial\Omega_3$

Since the pressure field p was previously calculated, all the necessary information to calculate and compute v_x (and similarly v_y) is now derived. In the following plots the velocities for $K = 10m/s$ are shown using a vector plot, contour plot and a 3D surface plot.

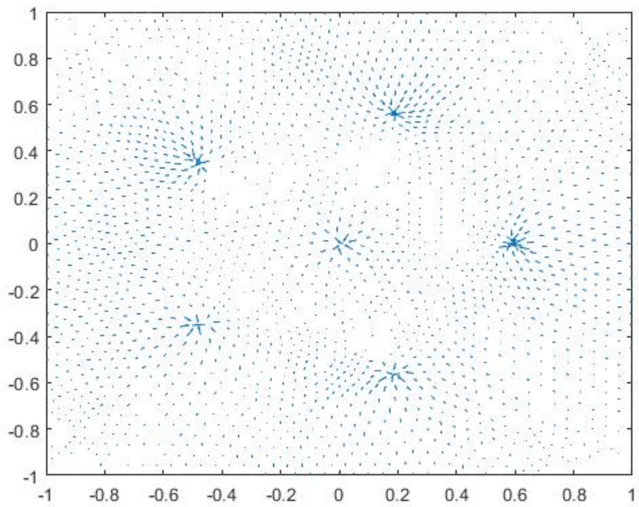

Figure 2.2: Arrow velocity plot, indicating the direction and velocity of the water(longer arrows indicate higher velocity).

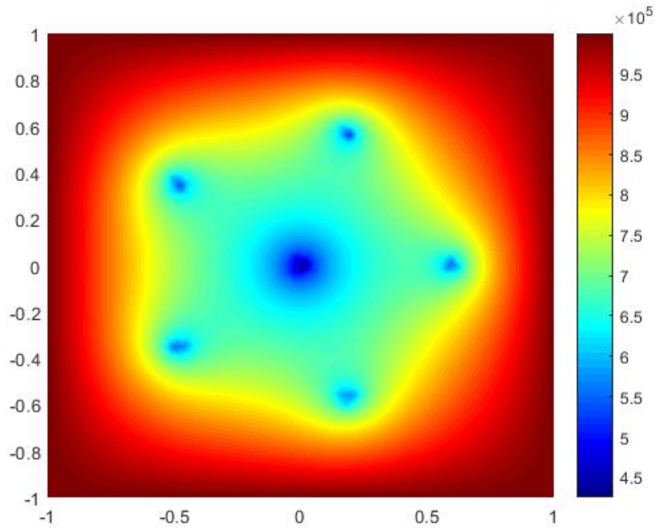

Figure 2.3: Contour plot of the velocity in the square reservoir on domain $\Omega = (-1, 1) \times (-1, 1)$, showing six areas where a drop of about two times the pressure can be observed, compared to the boundary pressure.

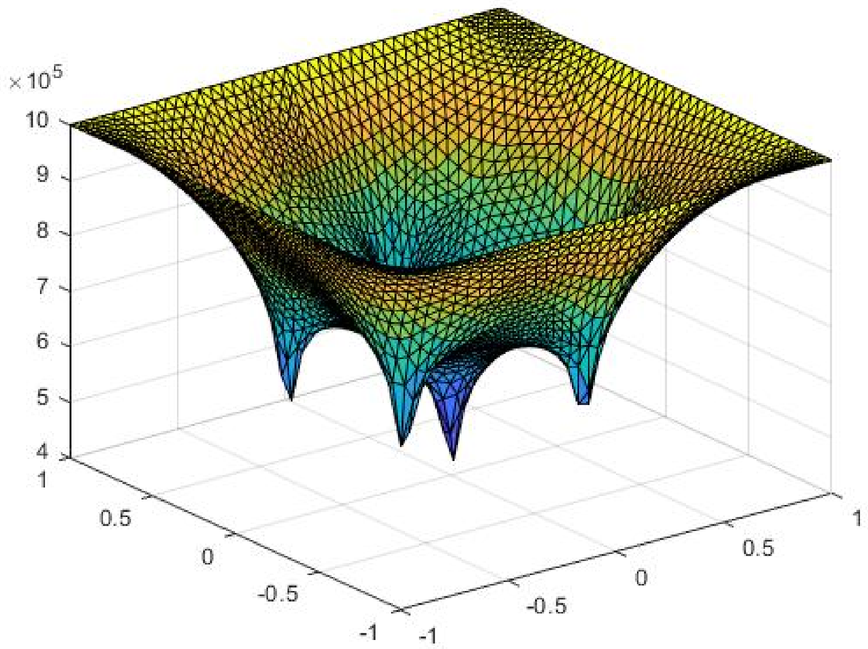

Figure 2.4: 3D surface plot

2.7 Varrying constant K

Now that the velocities have been calculated the last thing to vary is, is the transfer K factor. A few different plots will now follow in which the transfer coefficient K has different values between 0.00001 and 10000.

Show the contour plots, and give the values of the minimal pressure (which is important from an engineering point of view). Explain your results.

The final step is to determine what happens when $K = 0$ and why? When looking at our previous plots, when varying K from 0.00001 to 10000 it can be derived that When $K = 0$, the transfer coefficient of the water between the boundary of the square reservour and its surroundings is zero. In the following plot the K factor is set to 0:

Appendix A

1D-case full script

```
clear all
close all

%%Finite Element 1D
%% Parameters

N_elem = 100; %Number of elements
int = [0,1]; %Interval
lambda = 1;
D = .1;

%% Mesh & Topology

mesh = GenerateMesh(int,N_elem);
elmat = GenerateTopology(N_elem); %1D topology!!

%% Assemble Matrix & Vector

S = AssembleMatrix( N_elem, lambda, D, mesh, elmat);
f = AssembleVector( N_elem, mesh, elmat);

%% Calculate u
x = linspace(int(1),int(2),N_elem);

u = S\f;

hold on
plot(x,u);
legend('N=100')
title('Solution for u')
xlabel('x')
ylabel('u')
ax.box='on'
hold off


% For this part change the function in functionBVP.m to 'f = sin(20*x)'

figure
hold on

for N_elem = [10 20 40 80 100 160]
    mesh = GenerateMesh(int,N_elem);
    elmat = GenerateTopology(N_elem);
    S = AssembleMatrix( N_elem, lambda, D, mesh, elmat);
    f = AssembleVector( N_elem, mesh, elmat);

    x = linspace(int(1),int(2),N_elem);

    u = S\f;
    plot(x,u);


```

```
end

legend('N=10','N=20','N=40','N= 80','N=100','N=160')
title('Solution for u')
xlabel('x')
ylabel('u')
ax.box='on'
hold off
```

Appendix B

2D-case

B.1 Generate mesh

```
clear all

Geometry = 'squareg';

DiffCoeff = 1;
h_transfer = 1;
u_inf = 1;


% Geometry = 'squareg'; % gives square [-1,1] x [-1,1]
% Geometry = 'circleg'; % gives unit circle centered at origin
% Geometry = 'lshapeg'; % gives L-shape

[p,e,t] = initmesh(Geometry);
[p,e,t] = refinemesh(Geometry,p,e,t); % gives gridrefinement
[p,e,t] = refinemesh(Geometry,p,e,t); % gives second gridrefinement
%[p,e,t] = refinemesh(Geometry,p,e,t); % gives third gridrefinement
pdemesh(p,e,t); % plots the geometry and mesh

x = p(1,:); y = p(2,:);
n = length(p(1,:));

elmat = t(1:3,:);
elmat = elmat';
elmatbnd = e(1:2,:);
elmatbnd = elmatbnd';
% h
topology = 3; topologybnd = 2;
```

B.2 Generate element matrix

```
clear xc
clear yc
clear Selem

for index1 = 1:topology
    xc(index1) = x(elmat(i,index1));
    yc(index1) = y(elmat(i,index1));
end;

Delta = det([1 xc(1) yc(1);1 xc(2) yc(2);1 xc(3) yc(3)]);
B_mat = [1 xc(1) yc(1);1 xc(2) yc(2);1 xc(3) yc(3)] \ eye(3);

alpha = B_mat(1,1:3);
beta  = B_mat(2,1:3);
gamma = B_mat(3,1:3);
```

```

for index1 = 1:topology
    for index2 = 1:topology
        if ¬exist('u','var')
            Selem(index1,index2) =
            abs(Delta)/2*(k/mu)*(beta(index1)*beta(index2)+gamma(index1)*gamma(index2));
        else
            Selem(index1,index2) = abs(Delta)/24;
        end
    end;
end;
```

B.3 Generate element vector

```

% Module for element mass matrix for reactive term
%
% Output: felem  ====== vector of two components
%
% felem(1), felem(2) to be computed in this routine.

clear xc
clear yc
clear felem

for index1=1:topology
    xc(index1) = x(elmat(i,index1));
    yc(index1) = y(elmat(i,index1));
end;

Delta = det([1 xc(1) yc(1);1 xc(2) yc(2);1 xc(3) yc(3)]);

B_mat = [1 xc(1) yc(1);1 xc(2) yc(2);1 xc(3) yc(3)] \  eye(3);

alpha = B_mat(1,1:3);
beta  = B_mat(2,1:3);
gamma = B_mat(3,1:3);

felem = zeros(1,topology);

if ¬exist('u','var')  % Only if u is already know can the calculation of the velocity ...
    begin.
    for N = 1:N_wells
        for index3 = 1:topology
            phi_p(index3) = alpha(index3) + beta(index3)*xp(N) + gamma(index3)*yp(N);
        end

        if (phi_p(1) ≤ 1) && (phi_p(1) ≥ 0) && (phi_p(2) ≤ 1) && (phi_p(2) ≥ 0) && ...
             (phi_p(3) ≤ 1) && (phi_p(3) ≥ 0);
            for index1 = 1:topology
                felem(index1) = felem(index1) + -Qp*phi_p(index1);
            end

%          N_Test = N_Test + 1;

% Components of f are zero except for those elements with a well! So no
% other contributions!
%      else
%          for index1 = 1:topology
%          global_index = elmat(N,index1);
%          end
        end
    end
else
    switch direction
        case 1 % x direction
            for index1 = 1:topology
                felem(index1) = felem(index1) + ...
                     (k/mu)*(abs(Delta)/6)*beta(index1)*(u(elmat(i,1))+u(elmat(i,2))+u(elmat(i,3)));
```

```
            end
        case 2 % y direction
            for index1 = 1:topology
                felem(index1) = felem(index1) + ...
                    (k/mu)*(abs(Delta)/6)*gamma(index1)*(u(elmat(i,1))+u(elmat(i,2))+u(elmat(i,3)));
            end
    end
end
```

B.4 Generate Boundary element matrix

```
clear xc
clear yc
clear BMelem

for index1=1:topologybnd
    xc(index1) = x(elmatbnd(i,index1));
    yc(index1) = y(elmatbnd(i,index1));
end;

lek = sqrt((xc(2)-xc(1))^2 + (yc(2)-yc(1))^2);

for index1=1:topologybnd
    if ¬exist('u', 'var')
        BMelem(index1,index1) = K*lek/2;  % NC used! not HB!!
    else

        BMelem(index1,index1) = -(k/(mu*K))*lek/6;
    end
end;
```

B.5 Generate boundary element vector:

```
clear xc
clear yc
clear bfelem

for index1 = 1:topologybnd
    xc(index1) = x(elmatbnd(i,index1));
    yc(index1) = y(elmatbnd(i,index1));
end;

lek = sqrt((xc(2)-xc(1))^2+(yc(2)-yc(1))^2);

if ¬exist('u','var')
    for index1 = 1:topologybnd
        bfelem(index1) = K*pH*lek/2*u_inf;   %what is u_inf?
    end;
else
    for index1 = 1:topologybnd
        bfelem(index1) = ((k*pH)/mu)*lek/2*u_inf;   %what is u_inf?
%         bfelem(index1) = -(k/mu)*lek/6*u(elmat(i,ind1));
    end
end
```

B.6 Buildmatrices and vectors

```
% This routine constructs the large matrices and vector.
% The element matrices and vectors are also dealt with.
% First the internal element contributions
```

```
% First Initialisation of large discretisation matrix, right-hand side vector

% Treatment of the internal (triangular) elements

if ¬exist('u', 'var')

    S       = sparse(n,n); % stiffness matrix
    f       = zeros(n,1); % right-hand side vector

    for i = 1:length(elmat(:,1)) % for all internal elements
        GenerateElementMatrix; % Selem
        for ind1 = 1:topology
            for ind2 = 1:topology
                S(elmat(i,ind1),elmat(i,ind2))  = S(elmat(i,ind1),elmat(i,ind2)) + ...
                    Selem(ind1,ind2);
            end;
        end;

        GenerateElementVector; % felem
        for ind1 = 1:topology
            f(elmat(i,ind1)) = f(elmat(i,ind1)) + felem(ind1);
        end;
    end;

% Next the boundary contributions

    for i = 1:length(elmatbnd(:,1)); % for all boundary elements extension of mass ...
        matrix M and element vector f
    GenerateBoundaryElementMatrix; % BMelem
        for ind1 = 1:topologybnd
            for ind2 = 1:topologybnd
                S(elmatbnd(i,ind1),elmatbnd(i,ind2)) = ...
                    S(elmatbnd(i,ind1),elmatbnd(i,ind2)) + BMelem(ind1,ind2);
            end;
        end;
        GenerateBoundaryElementVector; % bfelem
        for ind1 = 1:topologybnd
            f(elmatbnd(i,ind1)) = f(elmatbnd(i,ind1)) + bfelem(ind1);
        end;
    end;

else

    Sx      = sparse(n,n); % stiffness matrix

    fx      = zeros(n,1); % right-hand side vector

    left_nodes = find(p(1,:) == -1);
    top_nodes = find(p(2,:) == 1);
    right_nodes = find(p(1,:) == 1);
    bottom_nodes = find(p(2,:) == -1);

    bnd1_nodes = ismember(elmatbnd,left_nodes);
    bnd1 = find(bnd1_nodes(:,1) == 1 & bnd1_nodes(:,2) == 1);

    bnd2_nodes = ismember(elmatbnd,top_nodes);
    bnd2 = find(bnd2_nodes(:,1) == 1 & bnd2_nodes(:,2) == 1);

    bnd3_nodes = ismember(elmatbnd,right_nodes);
    bnd3 = find(bnd3_nodes(:,1) == 1 & bnd3_nodes(:,2) == 1);

    bnd4_nodes = ismember(elmatbnd,bottom_nodes);
    bnd4 = find(bnd4_nodes(:,1) == 1 & bnd4_nodes(:,2) == 1);


    direction = 1;

    for i = 1:length(elmat(:,1)) % for all internal elements
        GenerateElementMatrix; % Selem
        for ind1 = 1:topology
            for ind2 = 1:topology
                if elmat(i,ind1) == elmat(i,ind2)
```

```
                    Sx(elmat(i,ind1),elmat(i,ind2)) = Sx(elmat(i,ind1),elmat(i,ind2)) ...
                        + 2*Selem(ind1,ind2);
                else
                    Sx(elmat(i,ind1),elmat(i,ind2)) = Sx(elmat(i,ind1),elmat(i,ind2)) ...
                        + Selem(ind1,ind2);
                end
            end;
        end;
        GenerateElementVector; % felem
        for ind1 = 1:topology
            fx(elmat(i,ind1)) = fx(elmat(i,ind1)) + felem(ind1);
        end;
    end;

% Next the boundary contributions



    for j = 1:length(bnd1); % left boundary
        i = bnd1(j);
        GenerateBoundaryElementMatrix; % BMelem
        for ind1 = 1:topologybnd
            for ind2 = 1:topologybnd
                if elmatbnd(i,ind1) == elmatbnd(i,ind2)
                    Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) = ...
                        Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) + 2*BMelem(ind1,ind2);
                else
                    Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) = ...
                        Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) + BMelem(ind1,ind2);
                end;
            end
        end;
        GenerateBoundaryElementVector; % bfelem
        for ind1 = 1:topologybnd
            fx(elmatbnd(i,ind1)) = fx(elmatbnd(i,ind1)) + bfelem(ind1);
        end;
    end;

    for j = 1:length(bnd3); % right boundary
        i = bnd3(j);
        GenerateBoundaryElementMatrix; % BMelem
        for ind1 = 1:topologybnd
            for ind2 = 1:topologybnd
                if elmatbnd(i,ind1) == elmatbnd(i,ind2)
                    Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) = ...
                        Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) + 2*BMelem(ind1,ind2);
                else
                    Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) = ...
                        Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) + BMelem(ind1,ind2);
                end;
            end
        end;
        GenerateBoundaryElementVector; % bfelem
        for ind1 = 1:topologybnd
            fx(elmatbnd(i,ind1)) = fx(elmatbnd(i,ind1)) - bfelem(ind1);
        end;
    end;

    direction = 2;

    Sy      = sparse(n,n); % stiffness matrix

    fy      = zeros(n,1); % right-hand side vector

    for i = 1:length(elmat(:,1)) % for all internal elements
        GenerateElementMatrix; % Selem
        for ind1 = 1:topology
            for ind2 = 1:topology
                if elmat(i,ind1) == elmat(i,ind2)
                    Sy(elmat(i,ind1),elmat(i,ind2)) = Sy(elmat(i,ind1),elmat(i,ind2)) ...
                        + 2*Selem(ind1,ind2);
                else
```

```
                    Sy(elmat(i,ind1),elmat(i,ind2)) = Sy(elmat(i,ind1),elmat(i,ind2)) ...
                        + Selem(ind1,ind2);
                end
            end;
        end;
        GenerateElementVector; % felem
        for ind1 = 1:topology
            fy(elmat(i,ind1)) = fy(elmat(i,ind1)) + felem(ind1);
        end;
    end;

% Next the boundary contributions



    for j = 1:length(bnd2); % left boundary
        i = bnd2(j);
        GenerateBoundaryElementMatrix; % BMelem
        for ind1 = 1:topologybnd
            for ind2 = 1:topologybnd
                if elmatbnd(i,ind1) == elmatbnd(i,ind2)
                    Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) = ...
                        Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) + 2*BMelem(ind1,ind2);
                else
                    Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) = ...
                        Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) + BMelem(ind1,ind2);
                end;
            end
        end;
        GenerateBoundaryElementVector; % bfelem
        for ind1 = 1:topologybnd
            fy(elmatbnd(i,ind1)) = fy(elmatbnd(i,ind1)) - bfelem(ind1);
        end;
    end;

    for j = 1:length(bnd4); % right boundary
        i = bnd4(j);
        GenerateBoundaryElementMatrix; % BMelem
        for ind1 = 1:topologybnd
            for ind2 = 1:topologybnd
                if elmatbnd(i,ind1) == elmatbnd(i,ind2)
                    Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) = ...
                        Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) + 2*BMelem(ind1,ind2);
                else
                    Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) = ...
                        Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) + BMelem(ind1,ind2);
                end;
            end
        end;
        GenerateBoundaryElementVector; % bfelem
        for ind1 = 1:topologybnd
            fy(elmatbnd(i,ind1)) = fy(elmatbnd(i,ind1)) + bfelem(ind1);
        end;
    end;
end
```

B.7 Compute u and v_x/v_y

```
% Construction of linear problem

BuildMatricesandVectors;

% Solution of linear problem

u = S \ f;

BuildMatricesandVectors;

vx = Sx \ fx;
```

```
vy = Sy \ fy;
```

B.8 Full script

```
close all
clear all

%% 2D Assignment
% Lab Assignment 7

%% Create Mesh
WI4243Mesh

%% Parameters

Qp = 50;               % [m^2/s]
k = 10^-7;             % [m^2]
mu = 1.002*10^-3;      % [Pa*s]
K = 10000;                % [m/s]
pH = 10^6;             % [Pa]
N_wells = 6;           % number of wells

epsilon1 = 0.03;
N_Test = 0;
%% Coordinates of wells

for i = 1:N_wells-1;
xp(i) = 0.6*cos((2*pi)*(i-1)/(N_wells-1));
yp(i) = 0.6*sin((2*pi)*(i-1)/(N_wells-1));
end

xp(N_wells) = 0;
yp(N_wells) = 0;
clear i;


%% Compute Problem
WI4243Comp

%% Post
hold on

figure(2);
ax.BoxStyle = 'full';
hold off
trisurf(elmat,x,y,u)
xlabel('x'); ylabel('y'); zlabel('Pressure [Pa]');
title(['\bf\fontsize{16}3D Surface plot  \fontsize{10} K = ' num2str(K,'%10.2e\n') ' ...
    [m/s]  \it Minimal pressure = ' num2str(min(u), '%10.2e\n') ' [Pa]']);

lgd = legend();
% title(lgd,['3D Surface plot, K = ' num2str(K) '\it Minimal pressure = ' ...
    num2str(Pressure_minimum)]);

% title( {'Title';'subtitle'} )

figure(3);
trisurf(elmat,x,y,u);
xlabel('x'); ylabel('y');
title(['\bf\fontsize{16}Contour plot, \fontsize{10} K = ',num2str(K,'%10.2e\n')]);
view(2); shading interp; colormap jet; colorbar; set(gcf,'renderer','zbuffer')
h = colorbar; ylabel(h, 'Pressure [Pa]');


figure(4); quiver(x,y,vx',vy'); axis([-1 1 -1 1]);
xlabel('x'); ylabel('y');
title(['\bf\fontsize{16}Velocity field, \fontsize{10} K = ' num2str(K,'%10.2e\n') ' ...
    [m/s]']);
%% Velocity part
```

```
hold off
```