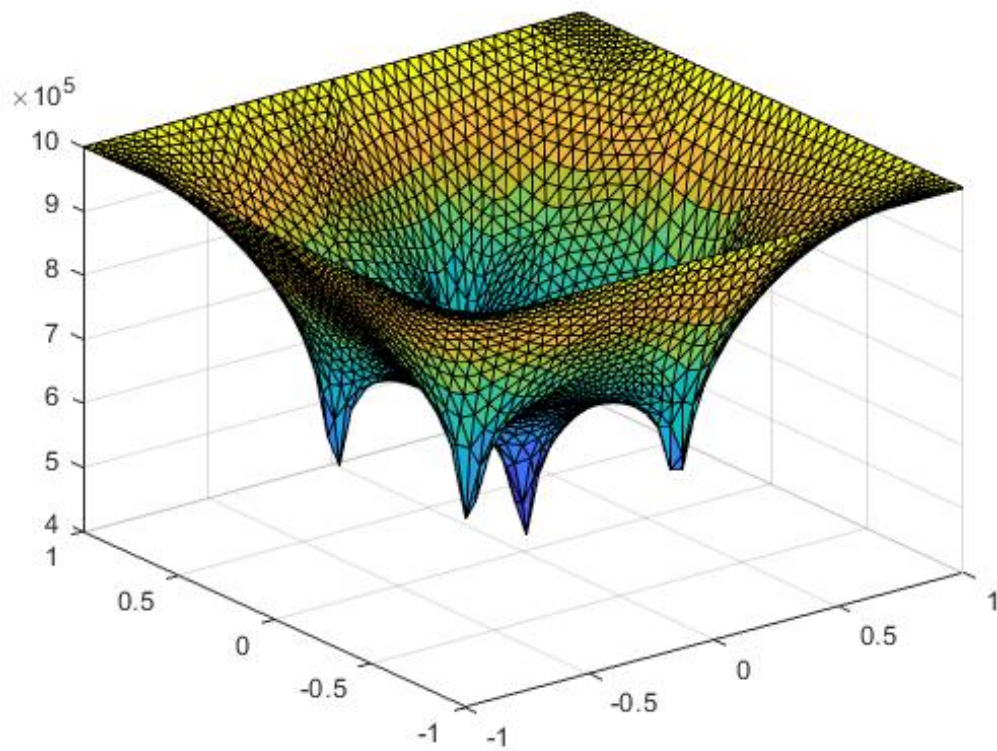


Finite Elements

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Chapter 1

1D-case

On the 1D interval of $x = [0, 1]$, we consider a steady-state convection-diffusion-reaction equation, with homogeneous Neumann boundary conditions. The following equations apply to this domain:

$$\begin{cases} -D\Delta u + \lambda u = f(x), \\ -D\frac{du}{dx}(0) = 0, \\ -D\frac{du}{dx}(1) = 0 \end{cases} \quad (1.1)$$

In this report Δ denotes the laplacian operator. The function $f(x)$ is a given function, where D and λ are positive real constants. In order to solve this boundary value problem (BVP), first the interval is divided in $n-1$ elements ($n =$ positive integer). This results in the domain being divided in elements: $e_i = [x_i, x_{i+1}]$ where $i = 1, 2, \dots, n$.

In order to solve this BVP, the solutions for the given equations will first be calculated and then computed using MATLAB codes.

1.1 Boundary value problem 1D

In order to find the Weakform of the given equations(1.1), both sides are multiplied by a test function $\phi(x)$ and then integrate both sides over the domain Ω . In the equations $\phi(x)$ is written as ϕ

$$\int_{\Omega} \phi(-D\Delta u + \lambda u) d\Omega = \int_{\Omega} \phi f(x) d\Omega \quad (1.2)$$

Now by rewriting and then using partial integration the following equation can be found:

$$\int_{\Omega} (\nabla \cdot (-D\phi \cdot \nabla u) + D\nabla\phi \cdot \nabla u + \phi\lambda u) d\Omega = \int_{\Omega} \phi f(x) d\Omega \quad (1.3)$$

Applying Gauss on the first term on the left side of equation(1.3):

$$\int_{\Omega} \vec{n} \cdot (-D\phi \nabla u) d\tau + \int_{\Omega} (D\nabla\phi \cdot \nabla u + \phi\lambda u) d\Omega = \int_{\Omega} \phi f(x) d\Omega \quad (1.4)$$

Using the boundary conditions from equations(1.1) the boundary integral equals to 0 and then the following weak formulation(WF) is found:

(WF):

$$\begin{cases} \text{find } u \in \Sigma = \{u \text{ smooth}\} \text{ Such that:} \\ \int_{\Omega} (D\nabla\phi \cdot \nabla u + \phi\lambda u) d\Omega = \int_{\Omega} \phi f(x) d\Omega \\ \forall \phi \in \Sigma \end{cases} \quad (1.5)$$

The next step is to substitute the Galerkin equations into the found differential equation, where u is replaced by $\sum_{j=1}^n c_i \phi_j$ and $\phi(x) = \phi_i(x)$ with $i = [1, \dots, n]$. Filling this in equation (1.5) the following equation is found:

$$\sum_{j=1}^n c_i \int_0^1 (D \nabla \phi_i \cdot \nabla \phi_j + \lambda \phi_i \phi_j) d\Omega = \int_0^1 \phi_i f(x) d\Omega \quad (1.6)$$

Which is of the form of $S\vec{c} = \vec{f}$

1.2 Element matrix

Now the found Galerkin equations can be used to compute S_{ij} the element matrix, over a generic line element e_i .

$$S\vec{c} = \sum_{j=1}^n c_i \int_0^1 (D \nabla \phi_i \cdot \nabla \phi_j + \lambda \phi_i \phi_j) d\Omega \quad (1.7)$$

$$S_{ij} = \sum_{l=1}^{n-1} S_{ij}^{e_k} \quad (1.8)$$

Now to solve S we solve the following equation, over the internal line element.

$$S_{ij}^{e_k} = -D \int_{e_k} \nabla \phi_i \cdot \nabla \phi_j d\Omega + \lambda \int_{e_k} \phi_i \phi_j dx \quad (1.9)$$

1.3 Element vector

Again the found Galerkin Equations(1.6) are used in order to compute the element vector f_i over a generic line-element.

$$f_i^{e_k} = \int_{e_k} \phi_i f dx \quad (1.10)$$

$$f_i^{e_k} = \frac{|x_k - x_{k-1}|}{(1 + 1 + 0)!} f(\vec{x}) = \frac{|x_k - x_{k-1}|}{2} \begin{bmatrix} f_{k-1}^{e_n} \\ f_k^{e_n} \end{bmatrix} \quad (1.11)$$

1.4 Boundary value problem 1D MATLAB routine

1.4.1 mesh and elmat code

The first step in order to solve the BVP is to write a MATLAB routine that generates an equidistant distribution of points over the given interval of $[0, 1]$ (generate a mesh with $n-1$ elements).

```
1 function [ x ] = GenerateMesh(int, N_elem)
2 %GenerateMesh Creates a mesh for 1D problems
3
4
5 % int = [0,1];
6 % N_elem = 100;
7
8 x = linspace(int(1,1),int(1,2),N_elem);
```

Using the codes to generate a mesh and the elmat, it is easier to use this 1D problem and adapt to a higher dimensional problem. The next step is to write a code that generates a two dimensional array, called the elmat.

```

1 function [ elmat ] = GenerateTopology( N_elem )
2 %GenerateTopology Creates the topology for a 1D problem given mesh 'x'.
3
4 % global N_elem
5 elmat = zeros(N_elem,2);
6 elmat(i,1) = i;
7 elmat(i,2) = i + 1;

```

1.4.2 Element matrix

Now that the base MATLAB codes are made the element matrix and element vector codes can be written. The first step in this process is, is to compute the element matrix S_{elem} .

```

1 function [ Selem ] = GenerateElementMatrix( k, elmat, D, lambda, mesh)
2 %GenerateElementMatrix Creates element matrix S_ek
3
4
5 Selem = zeros(2,2);
6
7 i = elmat(k,1);
8 j = elmat(k,2);
9
10 x1 = mesh(i);
11 x2 = mesh(j);
12
13 element_length = abs(x1-x2);
14
15 slope = 1/element_length;
16
17 for m = 1:2
18     for n = 1:2
19         if m == n
20             Selem(m,n) = element_length*((-1)^(abs(m-n))*D*slope^2
21                 + (2)*lambda/6);
22         else
23             Selem(m,n) = element_length*((-1)^(abs(m-n))*D*slope^2
24                 + (1)*lambda/6);
25         end
26     end
27 end
28
29
30 end

```

1.4.3 Assemble matrix S

To generate a n-by-n matrix S, the sum over the connections of the vertices in each element matrix, over all elements has to be calculated. The following code computes this matrix:

```

1 function [ S ] = AssembleMatrix( N_elem, int, lambda, D)
2 % global N_elem
3
4 elmat = GenerateTopology(N_elem);
5
6 S = zeros(N_elem,N_elem);
7
8 for i = 1:N_elem-1
9     Selem = GenerateElementMatrix(i, elmat, int, N_elem, D, lambda);
10    for j = 1:2
11        for k = 1:2
12            S(elmat(i,j), elmat(i,k)) =
13                S(elmat(i,j), elmat(i,k)) + Selem(j,k);
14        end
15    end
16 end

```

All the previous code will generate a large matrix S , from the element matrices S_{elem} over each element.

1.4.4 Element vector MATLAB routine

The next step In order to solve the equation $S\vec{c} = F$ is to create a code to generate the element vector. This element vector provides information about node i and node $i+1$, which are the vertices of element e_i .

```

1  function [ felem ] = GenerateElementVector( i, elmat, mesh )
2  %GenerateElementVector Creates element vector f_ek
3
4
5  felem = [0;0];
6
7  k1 = elmat(i,1);
8  k2 = elmat(i,2);
9
10 x1 = mesh(k1);
11 x2 = mesh(k2);
12
13 element_length = abs(x1-x2);
14
15 felem = (element_length/2*arrayfun(@functionBVP,[x1,x2]))';
16
17 end

```

To generate the vector f , the sum over the connections of the vertices in each element matrix, over all elements $i \in \{1, \dots, n-1\}$ has to be calculated.

```

1  function [ f ] = AssembleVector( N_elem, int, lambda, D )
2
3  f = zeros(N_elem,1);
4  elmat = GenerateTopology(N_elem);
5
6  for i = 1:N_elem-1
7      felem = GenerateElementVector(i, elmat, int, N_elem);
8      for j = 1:2
9          f(elmat(i,j)) = f(elmat(i,j)) + felem(j);
10     end
11 end

```

1.4.5 Computing S and f

Now if the previous matlab codes are run the following happens. Firstly a mesh and 1D topology is build, which is needed for the S matrix and f vector. The second step is to calculate the S matrix and f vector through the found equations of section 1.2 and 1.3. The final step is to use the found matrix and vector to solve the equation $Su = \vec{f}$.

1.5 Main program

The main program is simple written by assembling the previous created matlab code `AssembleMatrix` and `AssembleVector` and deviding the vector f by the matrix S .

```

1  function [ u ] = SolveBVP( N_elem, int, lambda, D )
2
3  S = AssembleMatrix( N_elem, int, lambda, D);
4  f = AssembleVector( N_elem, int, lambda, D);
5
6  %% Calculate u
7  x = linspace(int(1),int(2),100);

```

```

8
9     u = S\f;
10    plot(x,u);

```

The result of the plot is shown in figure(1.1).

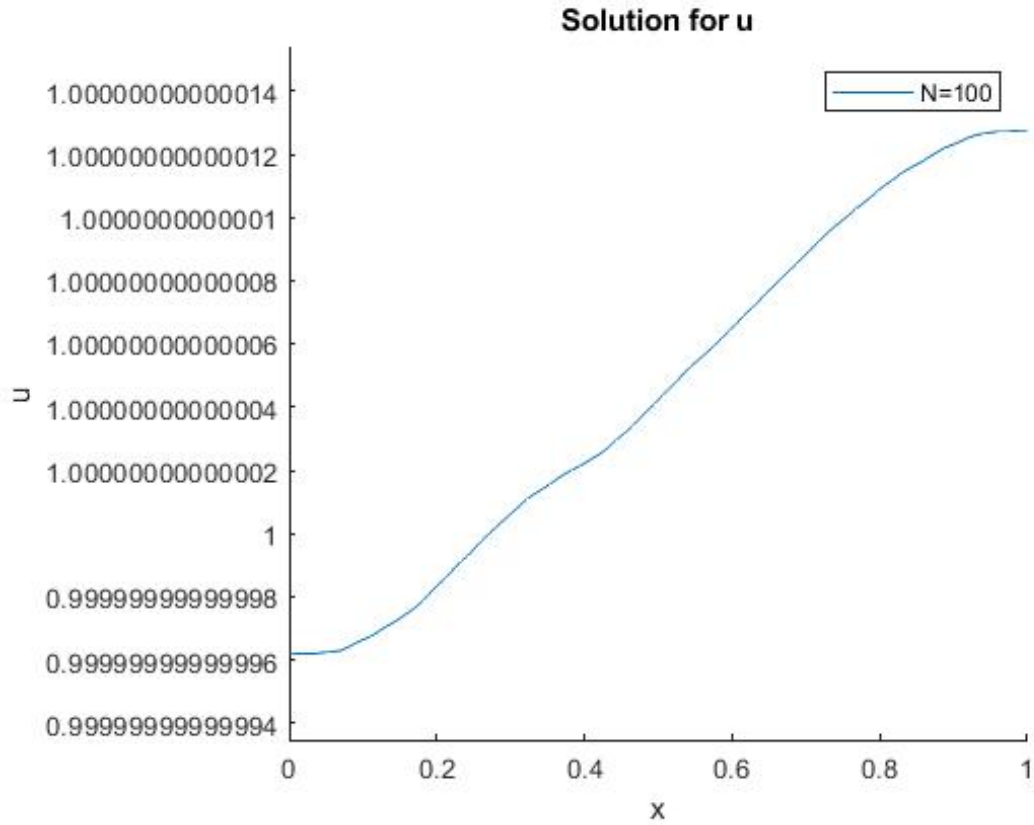


Figure 1.1: showing the calculated u versus x , with $N = 100, f(x)=1$

1.6 Solution for u

The final step is to combine all the codes in a main code to solve $Su = \vec{f}$. This code can be found in Appendix A. Previously the S matrix and f vector were computed for $n = 100$. Now u will be calculated for $f(x) = 1$, $D = 1$, $\nabla = 1$ and $N = 100$. The result of this is plotted in figure(1.2).

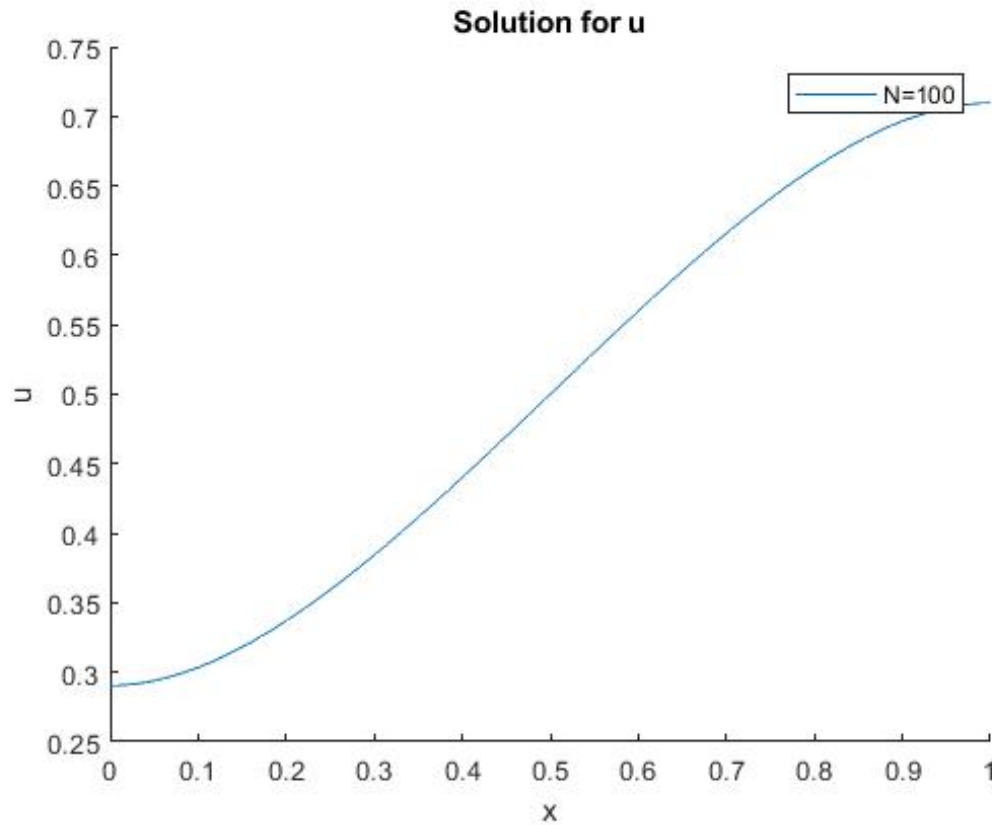


Figure 1.2: showing the calculated u versus x , with $N = 100, f(x)=x$

1.7 Experiment

The next step is to see what happens when changing $f(x)$ to $f(x) = \sin(20x)$ and to see the difference for several values for n ($n = 10, 20, 30, 40, 80, 160$)

```

1  function [f] = functionBVP(x)
2      f = sin(20*x);
3
4      %f = x;
5      %f = 1;
6
7  end
8
9
10 figure
11 hold on
12
13 for N_elem = [10 20 40 80 100 160]
14     mesh = GenerateMesh(int,N_elem);
15     elmat = GenerateTopology(N_elem);
16     S = AssembleMatrix( N_elem, lambda, D, mesh, elmat);
17     f = AssembleVector( N_elem, mesh, elmat);
18
19     x = linspace(int(1),int(2),N_elem);
20
21     u = S\f;
22     plot(x,u);
23
24     legend('N=100')
25     title('Solution for u')
26     xlabel('x')
27     ylabel('u')

```

```
28     ax.box='on'  
29     end  
30     hold off
```

Figure 1.3

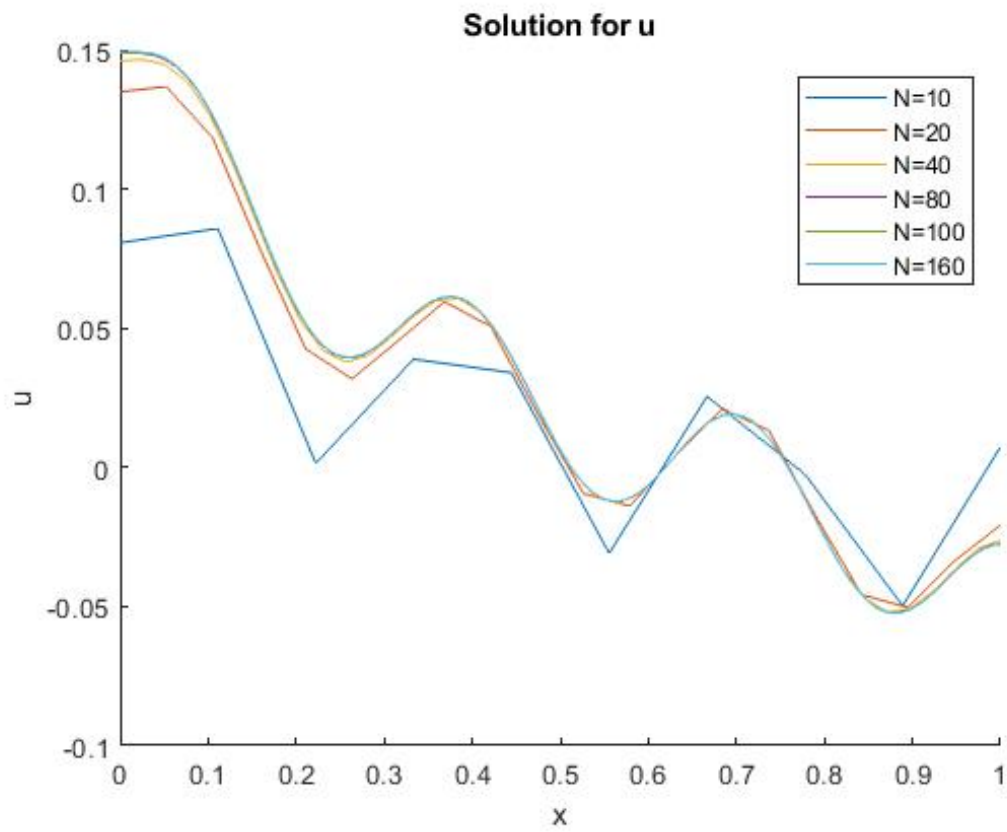


Figure 1.3: showing the calculated u versus x , with $N = [10 \ 20 \ 40 \ 80 \ 100 \ 160]$, $f(x) = \sin(x)$

Chapter 2

2D-case

The obvious next step after solving a 1 dimensional boundary value problem BVP is to addept the 1D solutions into code to solve a 2 dimensional boundary value problem. To do this a real life problem is going to be solved. In 3rd world countries one of the big issues is the supply of fresh water. One way is to do this is to take square reservoirs, which is a porous medium, with several wells where water is extracted from the subsurface. The water pressure is equal to the hydrostatic pressure. As this is not on an infinite domain, mixed boundary conditions are used. These boundary conditions represent a model for the transfer of the water over the boundary to locations far away. To this extent, a square domain is considered with length 2 in meter: $\Omega = (-1; 1) \times (-1; 1)$ with boundary $d\Omega$. Darcy's law for fluid determins the steady state equilibrium of this BVP, given by equation(2.1) :

$$\vec{v} = -\frac{k}{\mu} \nabla p \quad (2.1)$$

where p , k , μ and v , respectively denote the fluid pressure, permeability of the porous medium, viscosity of water and the fluid flow velocity. In this BVP the effect of gravity will not play a part as the problem is looked at in 2D. An accompanying assumption is incompressibility, so the extraction wells are treated as point sinks. This assumption can be made as the well its diameter is much smaller than the dimensions of the square reservoir. The extraction wells extract at the same rate in each direction, leading to the following boundary conditions(2.2).

$$\nabla \cdot \vec{v} = - \sum_{p=1}^{n_{well}} Q_p \delta(\phi(\vec{x}) - \phi(\vec{x}_p)) = 0, (x, y) \in \Omega \quad (2.2)$$

where Q_p denotes the water extraction rate by well k , which is located at x_p . Here x equals $(x; y)$, the spatial coordinates. We use the convention $x = (x; y)$ to represent the spatial coordinates. The dirac Delta Distribution is characterized by equation(2.3).

$$\begin{cases} \delta(\vec{x}) = 0, \vec{x} \neq 0 \\ \int_{\Omega} \delta(\vec{x}) d\Omega = 1, \text{ where } \Omega \text{ contains the origin.} \end{cases} \quad (2.3)$$

For this BVP the following boundary condition is considered:

$$\vec{v} \cdot \vec{n} = K(p - p^H), (x, y) \in \partial\Omega \quad (2.4)$$

Here k denotes the transfer rate coefficient of the horizon between the boundary of the domain and its surroundings. The constant p^H represents the hydrostatic pressure. In order to solve this BVP the values needed for all the constants are given in table (2.1).

In this BVP six wells are considered, which are located at:

$$\begin{cases} x_p = 0.6 \cos(\frac{2\pi(p-1)}{5}) \\ y_p = 0.6 \sin(\frac{2\pi(p-1)}{5}) \end{cases} \quad (2.5)$$

Table 2.1: Values of input parameters

Symbol	Value	Unit
Q_p	50	m^2/s
k	10^{-7}	m^2
μ	$1.002 \cdot 10^{-3}$	$Pa \cdot s$
K	10	m/s
p^H	10^6	Pa

2.1 Boundary value problem 2D

The first step to solving these equations using finite elements is to find the boundary value problem to solve. This is done by filling in equation(2.1) in both equation 2.2 and the boundary condition(2.4) in order to find the BVP in terms of p:

BVP:

$$\begin{cases} -\frac{k}{\mu} \Delta \vec{p} = -\sum_{p=1}^{n_{well}} Q_p \delta(\vec{x} - \vec{x}_p) = 0, & (x, y) \in \Omega \\ -\frac{k}{\mu} \nabla \vec{p} \cdot \vec{n} = -\frac{k}{\mu} \frac{dp}{dn} = K(p - p^H), & (x, y) \in \partial\Omega \end{cases} \quad (2.6)$$

The next step is to compute the weak formulation using the previous found BVP(2.6). By multiplying both sides by $\phi(\vec{x})$ and integrating both sides over the domain Ω the weak formulation can be found.

$$\int_{\Omega} \phi(\vec{x}) \nabla \cdot \left(-\frac{k}{\mu} \nabla \vec{p} \right) d\Omega = \int_{\Omega} -\sum_{p=1}^{n_{well}} \phi(\vec{x}) Q_p \delta(\vec{x} - \vec{x}_p) d\Omega \quad (2.7)$$

Using integrating by parts on the left side of equation(2.7) results in:

$$\int_{\Omega} \nabla \cdot \phi(\vec{x}) \left(-\frac{k}{\mu} \nabla \vec{p} \right) + \frac{k}{\mu} \nabla \phi(\vec{x}) \cdot \nabla p d\Omega = - \int_{\Omega} \sum_{p=1}^{n_{well}} \phi(\vec{x}) Q_p \delta(\vec{x} - \vec{x}_p) d\Omega \quad (2.8)$$

Next is to apply Gauss on the first term of the left side.

$$\int_{\Omega} \vec{n} \cdot \left(\phi(\vec{x}) \left(-\frac{k}{\mu} \nabla \vec{p} \right) \right) d\tau + \int_{\Omega} \frac{k}{\mu} \nabla \phi(\vec{x}) \cdot \nabla p d\Omega = - \int_{\Omega} \sum_{p=1}^{n_{well}} \phi(\vec{x}) Q_p \delta(\vec{x} - \vec{x}_p) d\Omega \quad (2.9)$$

Switching the integral and summation on the right side of equation(2.9) and simplifying terms:

$$\int_{\Omega} \left(\phi(\vec{x}) \left(-\frac{k}{\mu} \frac{dp}{dn} \right) \right) d\tau + \int_{\Omega} \frac{k}{\mu} \nabla \phi(\vec{x}) \cdot \nabla p d\Omega = - \sum_{p=1}^{n_{well}} \int_{\Omega} \phi(\vec{x}) Q_p \delta(\vec{x} - \vec{x}_p) d\Omega \quad (2.10)$$

The right side of equation(2.10) can be simplified using the boundary conditions (equation(2.6)) and the following property into equation(2.12):

$$\int_{\Omega} \delta(\vec{x}) f(\vec{x}) d\Omega = f(0) \quad (2.11)$$

$$\int_{d\Omega} \phi(\vec{x}) K(p - p^H) \delta\Gamma + \int_{\Omega} \frac{k}{\mu} \nabla \phi(\vec{x}) \cdot \nabla p d\Omega = - \sum_{p=1}^{n_{well}} \phi(\vec{x}_p) Q_p \quad (2.12)$$

Rearranging equation(2.12) so that the variable parts are on the left and the constant parts on the right leads to the following WF:

(WF):

$$\begin{cases} \text{find } p \in \Sigma = \{p \text{ smooth}\} \text{ Such that:} \\ \int_{d\Omega} \phi(\vec{x}) K p \delta\Gamma + \int_{\Omega} \frac{k}{\mu} \nabla \phi(\vec{x}) \cdot \nabla p d\Omega = - \sum_{p=1}^{n_{well}} \phi(\vec{x}_p) Q_p + \int_{d\Omega} \phi(\vec{x}) K p^H \delta\Gamma \\ \forall \phi \in \Sigma \end{cases} \quad (2.13)$$

To solve the WF the Galerkin equations are applied, where p is replaced by $\sum_{j=1}^n c_j \phi_j$ and $\phi(x) = \phi_i(x)$ with $i = [1, \dots, n]$.

$$\sum_{j=1}^n c_j \int_{d\Omega} \phi_i K \phi_j d\Gamma + \int_{\Omega} \frac{k}{\mu} \nabla \phi(x) \cdot \nabla \phi_j d\Omega = - \sum_{p=1}^{n_{well}} \phi(x_p) Q_p + \int_{d\Omega} \phi_i K p^H \delta\tau \quad (2.14)$$

Equation(2.14) now is of the form $S\vec{c} = \vec{f}$ and like with the 1D problem can be computed. First the element and boundary elements are determined from the Galerkin equations.

2.2 Element matrix and element vector

First the galerkin equation is seperated in its element and boundary components. The element matrix $S_{ij}^{e_k}$ and the element vector $f_i^{e_k}$ are given in equations (2.15) and 2.16 respectively.

$$S_{ij}^{e_k} = \int_{e_k} \frac{k}{\mu} \nabla \phi_i \cdot \nabla \phi_j d\Omega = (\beta_i \beta_j + \gamma_i \gamma_j) \frac{k}{\mu} \frac{|\Delta e_k|}{2} \quad (2.15)$$

$$f_i^{e_k} = - \sum_{p=1}^{n_{well}} \phi(x_p) Q_p \quad (2.16)$$

2.3 Boundary matrix and boundary vector

The boundary matrix $S_{ij}^{b_l}$ and boundary vector $f_i^{b_l}$ can be found in the following equations:

$$S_{ij}^{b_{e_l}} = \int_{b_l} K \phi_i \phi_j dx = K \frac{|b_{e_l}|}{6} (1 + \delta_{ij}) \quad (2.17)$$

$$f_i^{b_{e_l}} = K p^H \int_{b_l} \phi_i dx = K p^H \frac{|b_{e_l}|}{2} \quad (2.18)$$

2.4 Cells within an internal element

To solve the BVP in 2D one of the aspects that need to be determined are whether each internal element contains a cell. This is done by determining whether cell with index p and position x_p is contained within element e_k with vertices x_{k1} , x_{k2} and x_{k3} . This is done according the following criterion:

$$|\delta(x_p, x_{k2}, x_{k3})| + |\delta(x_{k1}, x_p, x_{k3})| + |\delta(x_{k1}, x_{k2}, x_p)| : \begin{cases} = |e_k|, & x_p \in \vec{e_k} \\ > |e_k|, & x_p \notin \vec{e_k} \end{cases} \quad (2.19)$$

In the criterion $\delta(x_p, x_q, x_r)$ denotes the triangle with vertices x_p , x_q and x_r , where $|\delta(x_{k1}, x_{k2}, x_{k3})|$ denote its area. The triangular element k is given by $e_k = \delta(x_{k1}, x_{k2}, x_{k3})$ with vertices x_1 , x_2 and x_3 and $\vec{e_k}$ includes the boundary of element e_k . To solve this BVP a certain tolerance has to be accounted for in the Matlab code.

2.5 Generating MATLAB code

Similar as with the 1D BVP, code is written in order to generate a mesh, element matrix, element vector, boundary element matrix and boundary element vector. These codes can be found in appendix B.1 through B.6.

2.6 Assignment 8

Now Darcy's law is used to compute the velocity in both directions, by implementing the found WF(equation (2.13)) and the Galerkin equations in the resulting system of linear equations(2.20).

$$M \vec{v_x} = C_x \vec{p}, \quad M \vec{v_y} = C_y \vec{p} \quad (2.20)$$

In order to find \vec{v}_x and \vec{v}_y equations(2.20) need to be solved. First writing out the x and y parts of the velocity gives:

$$\vec{v} = -\frac{k}{\mu} \nabla p \quad (2.21)$$

$$\vec{v}_x = -\frac{k}{\mu} \frac{dp}{dx} \quad (2.22)$$

$$\vec{v}_y = -\frac{k}{\mu} \frac{dp}{dy} \quad (2.23)$$

Equation(2.24) shows the relation between \vec{v} and pressure p.

$$\vec{v} \cdot \vec{n} = k(p - p^H) \text{ on } d\Omega \quad (2.24)$$

This relation is used to rewrite equations(2.22 and 2.23):

$$v_x(x = -1) = -k(p - p^H) \quad (2.25)$$

$$v_x(x = 1) = k(p - p^H) \quad (2.26)$$

$$v_y(y = -1) = -k(p - p^H) \quad (2.27)$$

$$v_y(y = 1) = k(p - p^H) \quad (2.28)$$

The Galerkin approach is used again to solve the equations by approximating v_x .

$$v_x \approx v_x^{(n)} = \sum_{j=1}^n c_j \phi_j \quad (2.29)$$

Here $\phi(\vec{x}) = \phi(\vec{x})_i$. Filling equations(2.29) and $\phi(\vec{x})_i$ in the exuations for v_x

$$\int_{\Omega} \phi v_x d\Omega = -\frac{k}{\mu} \int_{\Omega} \phi \frac{dp}{dx} d\Omega \quad (2.30)$$

Partial integration is applied on the right side term.

$$\int_{\Omega} \phi v_x d\Omega = -\frac{k}{\mu} \left\{ \int_{\Omega} \frac{d}{dx} (\phi p) - p \frac{d\phi}{dx} d\Omega \right\} \quad (2.31)$$

Rewriting the integral:

$$\int_{\Omega} -\frac{k}{\mu} \frac{d\phi p}{dx} dx dy = \int_{-1}^1 = -\frac{k}{\mu} [\phi p] dy \quad (2.32)$$

Evaluating the right side of equation(2.31) by filling in the previous conditions for specific points (1,1) and (-1,-1):

$$= \int_{-1}^1 -\frac{k}{\mu} (\phi(x=1, y)p(x=1, y)) - \frac{k}{\mu} (\phi(x=-1)p(x=-1, y)) dy \quad (2.33)$$

Simplifying the equation:

$$= \int_{d\Omega_3} -\frac{k}{\mu} \phi p dy + \int_{d\Omega_1} \frac{k}{\mu} \phi p dy \quad (2.34)$$

Now the total combined equation found is:

$$\int_{\Omega} \phi v_x = -\frac{k}{\mu} \left\{ \int_{\Omega} \frac{d}{dx} (\phi p) - p \frac{d\phi}{dx} d\Omega \right\} \quad (2.35)$$

Find v_x stationary:

$$\int_{\Omega} \phi v_x = \frac{k}{\mu} \left\{ \int_{d\Omega_3} -\phi p dy + \int_{d\Omega_1} \phi p dy + \int_{\Omega} p \frac{d\phi}{dx} d\Omega \right\} \quad (2.36)$$

Again applying the Galerkin equations to (2.36) the following conditions are found (2.38).

$$\phi(\vec{x}) = \phi(\vec{x})_i \quad v_x \approx v_x^{(n)} = \sum_{j=1}^n c_j \phi(\vec{x})_j \quad (2.37)$$

$$\text{on } \begin{cases} d\Omega_3 : -v_x = k(p - p^H) \rightarrow p = -\frac{v_x}{k} + p^H \\ d\Omega_1 : v_x = k(p - p^H) \rightarrow p = \frac{v_x}{k} + p^H \end{cases} \quad (2.38)$$

Now writing the equation where v_x is filled in so that an equation is found that depends on p .

$$\int_{\Omega} \phi v_x d\Omega + \int_{d\Omega_3} -\frac{k}{\mu} \frac{1}{k} \phi v_x dy + \int_{d\Omega_1} -\frac{k}{\mu} \frac{1}{k} \phi v_x dy = \int_{d\Omega_3} -\frac{k}{\mu} \phi p^H dy + \int_{d\Omega_1} \frac{k}{\mu} \phi p^H dy + \int_{\Omega} \frac{k}{\mu} p \frac{d\phi}{dx} d\Omega \quad (2.39)$$

For the right side of equation (2.39) the galerkin equations are filled in to find a $\phi(\vec{x}) = \phi(\vec{x})_i$
 $v_x \approx \sum_{j=1}^n c_j \phi(\vec{x})_j \quad \phi(\vec{x}) = \alpha_i + \beta_i x + \gamma_i y$

$$\sum_{j=1}^n c_j \left\{ \int_{\Omega} \phi_i \phi_j d\Omega + \int_{d\Omega_3} -\frac{k}{\mu} \frac{1}{k} \phi_i \phi_j dy + \int_{d\Omega_1} -\frac{k}{\mu} \frac{1}{k} \phi_i \phi_j dy \right\} \quad (2.40)$$

$$\sum_{j=1}^n c_j \left\{ \int_{d\Omega_3} -\frac{k}{\mu} \phi_i p^H dy + \int_{d\Omega_1} \frac{k}{\mu} \phi_i p^H dy + \int_{\Omega} \frac{k}{\mu} p \frac{d\phi_i}{dx} d\Omega \right\} \quad (2.41)$$

Applying Newton-Cotes theorem and , the element matrix (2.42), boundary element matrix (2.43), element vector (2.44) and boundary element vector (2.45) is found.

$$S_{ij}^{e_k} = \int_{e_k} \phi_i \phi_j = \frac{|\Delta_{ek}|}{24} \quad (2.42)$$

$$S_{ij}^{be_l} = \int_{be_l} -\frac{k}{\mu} \phi_i \phi_j dy = \frac{k}{\mu} \frac{1}{k} \frac{|be_l|}{6} (1 + \delta_{ij}) \quad (2.43)$$

$$f_i^{e_n} = \int_{e_n} \frac{k}{\mu} p \beta_i d\Omega = \frac{k}{\mu} \beta_i \sum_{m \in \{k_1, k_2, k_3\}} p(\vec{x}_m) \frac{|\Delta_{en}|}{6} \quad (2.44)$$

$$f_i^{be_l} = \int_{be_l} \pm \frac{k}{\mu} \phi_i p^H dy = \pm \frac{k}{\mu} p^H \frac{|be_l|}{2} \quad (2.45)$$

2.7 Assignment 9

2.8 Assignment 10

What happens if $K = 0$? Explain the results.

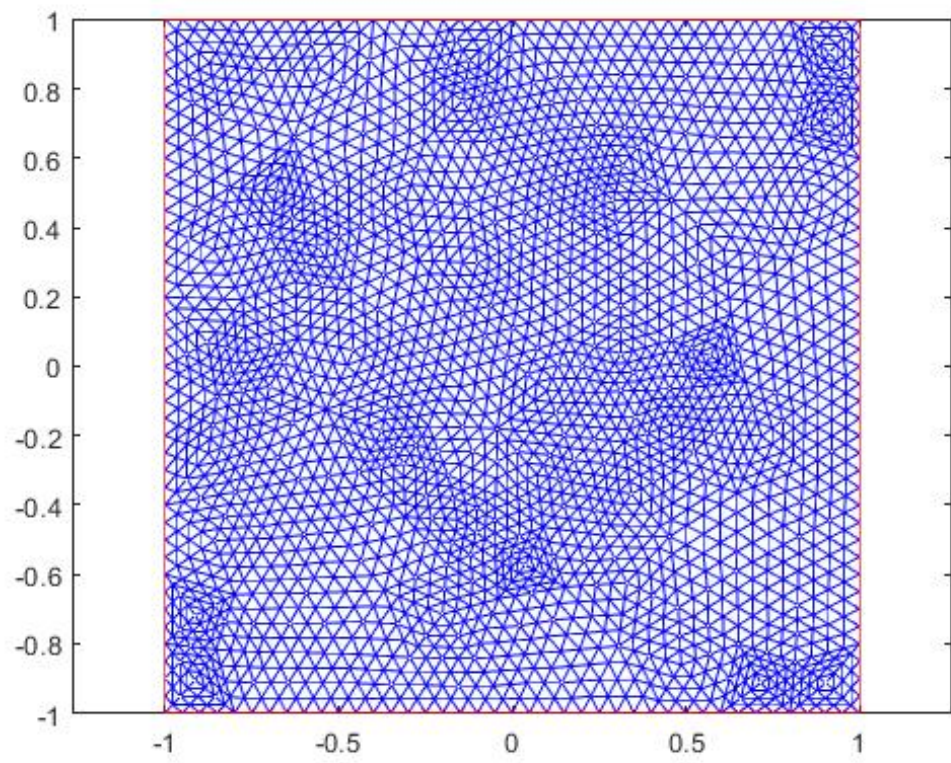


Figure 2.1: caption

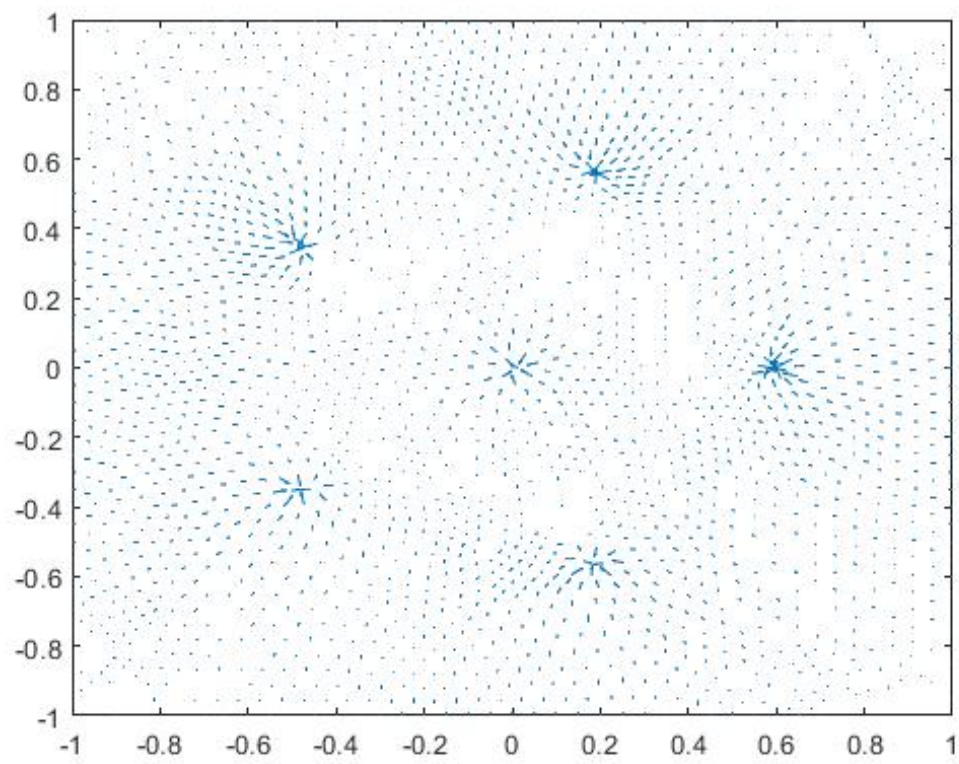


Figure 2.2: caption

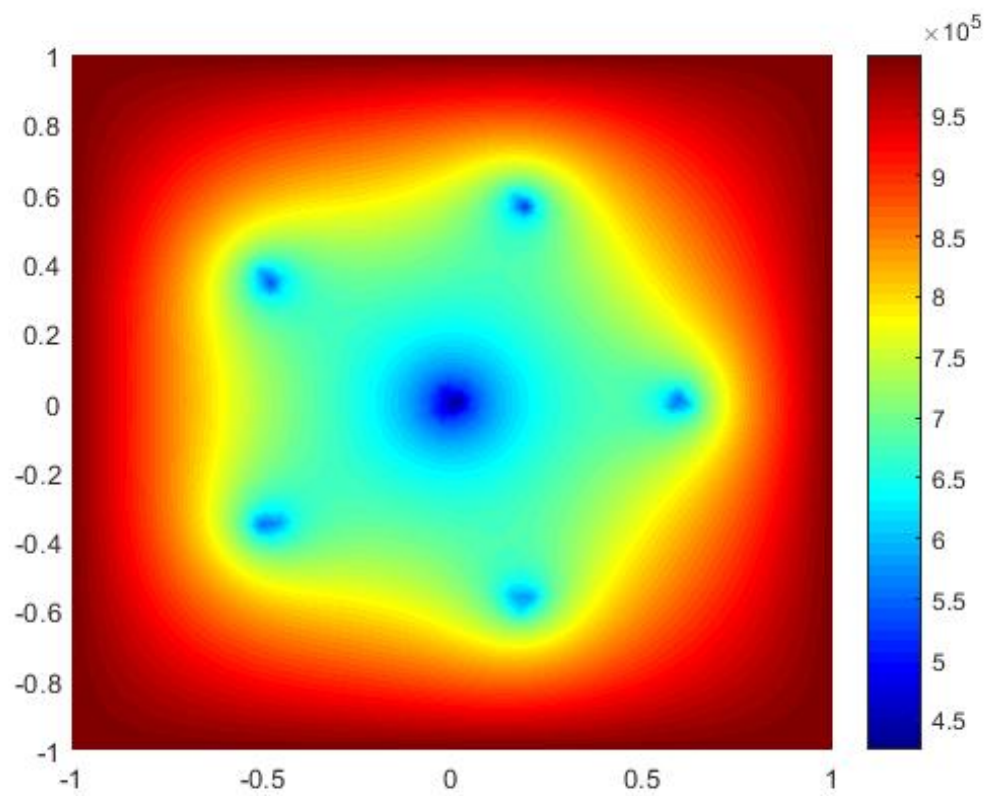


Figure 2.3: caption

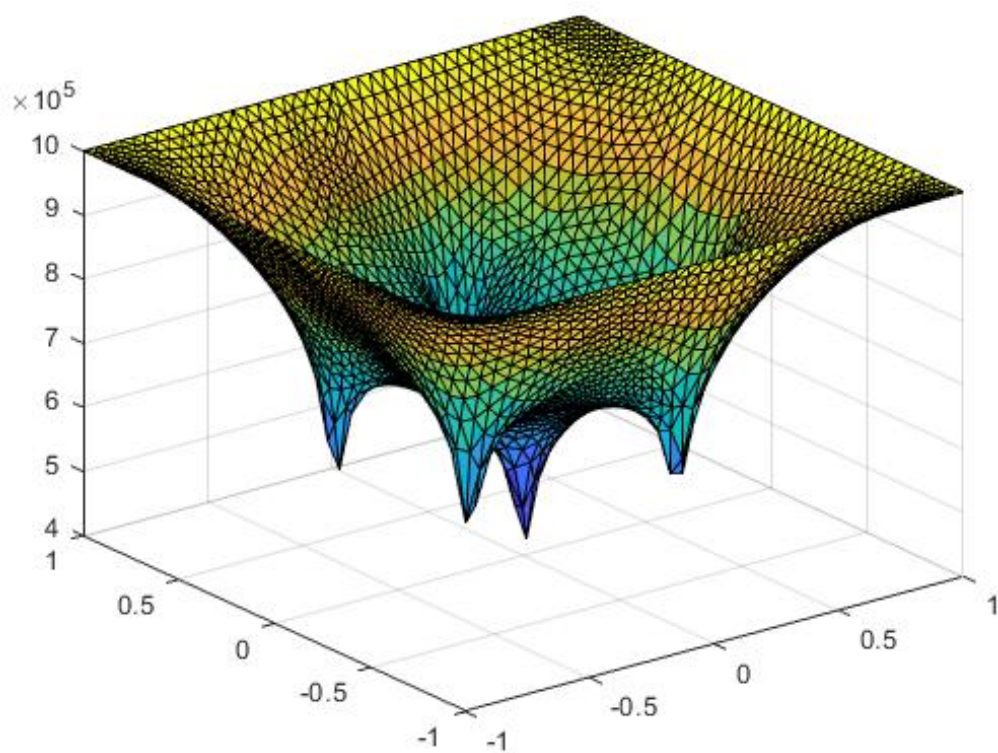


Figure 2.4: caption

Appendix A

1D-case full script

```
1 clear all
2 close all
3
4 %%Finite Element 1D
5 %% Parameters
6
7 N_elem = 100; %Number of elements
8 int = [0,1]; %Interval
9 lambda = 1;
10 D = .1;
11
12 %% Mesh & Topology
13
14 mesh = GenerateMesh(int,N_elem);
15 elmat = GenerateTopology(N_elem); %1D topology!!
16
17 %% Assemble Matrix & Vector
18
19 S = AssembleMatrix( N_elem, lambda, D, mesh, elmat);
20 f = AssembleVector( N_elem, mesh, elmat);
21
22 %% Calculate u
23 x = linspace(int(1),int(2),N_elem);
24
25 u = S\f;
26
27 hold on
28 plot(x,u);
29 legend('N=100')
30 title('Solution for u')
31 xlabel('x')
32 ylabel('u')
33 ax.box='on'
34 hold off
35
36
37 % For this part change the function in functionBVP.m to 'f = sin(20*x)'
38
39 figure
40 hold on
41
42 for N_elem = [10 20 40 80 100 160]
43     mesh = GenerateMesh(int,N_elem);
44     elmat = GenerateTopology(N_elem);
45     S = AssembleMatrix( N_elem, lambda, D, mesh, elmat);
46     f = AssembleVector( N_elem, mesh, elmat);
47
48     x = linspace(int(1),int(2),N_elem);
49
50     u = S\f;
51     plot(x,u);
52
53
```

```
54 end
55
56 legend('N=10','N=20','N=40','N= 80','N=100','N=160')
57 title('Solution for u')
58 xlabel('x')
59 ylabel('u')
60 ax.box='on'
61 hold off
62 \chapter{2D-case full script}
```

Appendix B

2D-case

B.1 Generate mesh

```
1 clear all
2
3 Geometry = 'squareg';
4
5 DiffCoeff = 1;
6 h_transfer = 1;
7 u_inf = 1;
8
9
10 % Geometry = 'squareg'; % gives square [-1,1] x [-1,1]
11 % Geometry = 'circleg'; % gives unit circle centered at origin
12 % Geometry = 'lshapeg'; % gives L-shape
13
14 [p,e,t] = initmesh(Geometry);
15 [p,e,t] = refinemesh(Geometry,p,e,t); % gives gridrefinement
16 [p,e,t] = refinemesh(Geometry,p,e,t); % gives second gridrefinement
17 % [p,e,t] = refinemesh(Geometry,p,e,t); % gives third gridrefinement
18 pdemesh(p,e,t); % plots the geometry and mesh
19
20 x = p(1,:); y = p(2,:);
21 n = length(p(1,:));
22
23 elmat = t(1:3,:);
24 elmat = elmat';
25 elmatbnd = e(1:2,:);
26 elmatbnd = elmatbnd';
27 % h
28 topology = 3; topologybnd = 2;
```

B.2 Generate element matrix

```
1 clear xc
2 clear yc
3 clear Selem
4
5 for index1 = 1:topology
6 xc(index1) = x(elmat(i,index1));
7 yc(index1) = y(elmat(i,index1));
8 end;
9
10 Delta = det([1 xc(1) yc(1); 1 xc(2) yc(2); 1 xc(3) yc(3)]);
11 B_mat = [1 xc(1) yc(1); 1 xc(2) yc(2); 1 xc(3) yc(3)] \ eye(3);
12
13 alpha = B_mat(1,1:3);
14 beta = B_mat(2,1:3);
15 gamma = B_mat(3,1:3);
```

```

16
17 for index1 = 1:topology
18 for index2 = 1:topology
19 if ~exist('u','var')
20 Selem(index1,index2) =
21 abs(Delta)/2*(k/mu)*(beta(index1)*beta(index2)+gamma(index1)*gamma(index2));
22 else
23 Selem(index1,index2) = abs(Delta)/24;
24 end
25 end;
26 end;

```

B.3 Generate element vector

```

1 clear xc
2 clear yc
3 clear felem
4
5 for index1=1:topology
6 xc(index1) = x(emat(i,index1));
7 yc(index1) = y(emat(i,index1));
8 end;
9
10 Delta = det([1 xc(1) yc(1);1 xc(2) yc(2);1 xc(3) yc(3)]);
11 B_mat = [1 xc(1) yc(1);1 xc(2) yc(2);1 xc(3) yc(3)] \ eye(3);
12
13 alpha = B_mat(1,1:3);
14 beta = B_mat(2,1:3);
15 gamma = B_mat(3,1:3);
16
17 felem = zeros(1,topology);
18
19 % for N = 1:N_wells
20 % Delta_23 = det([1 xp(N) yp(N);1 xc(2) yc(2);1 xc(3) yc(3)]);
21 % Delta_13 = det([1 xc(1) yc(1);1 xp(N) yp(N);1 xc(3) yc(3)]);
22 % Delta_12 = det([1 xc(1) yc(1);1 xc(2) yc(2);1 xp(N) yp(N)]);
23 %
24 % if Delta + epsilon1*Delta >= abs(Delta_23) + abs(Delta_13)
25 + abs(Delta_12) || Delta - epsilon1*Delta >= abs(Delta_23)
26 + abs(Delta_13) + abs(Delta_12) ;
27 % N_Test = N_Test + 1;
28 % for index1 = 1:topology
29 % felem(index1) = felem(index1) +
30 % -Qp*(1-abs(beta(index1))*abs(xc(index1)-
31 % xp(index1))-abs(gamma(index1))*abs(yc(index1)-yp(index1)));
32 %
33 % end
34 %
35 % i
36 % % Components of f are zero except for those elements with a well! So no
37 % % other contributions!
38 % % else
39 % % for index1 = 1:topology
40 % % global_index = emat(N,index1);
41 % %
42 % %
43 % % end
44 % end
45 % end
46 if ~exist('u','var')
47 for N = 1:N_wells
48 for index3 = 1:topology
49 phi_p(index3) = alpha(index3) + beta(index3)*xp(N)
50 + gamma(index3)*yp(N);
51 end
52
53 if (phi_p(1) <= 1) && (phi_p(1) >= 0) && (phi_p(2) <= 1) && (phi_p(2)
54 >= 0) && (phi_p(3) <= 1) && (phi_p(3) >= 0);
55 for index1 = 1:topology

```

```

56 phi_p
57 felem(index1) = felem(index1) +
58 -Qp*phi_p(index1); %*(1-abs(beta(index1))*abs(xc(index1)
59 -xp(index1))-abs(gamma(index1))*abs(yc(index1)-yp(index1))) ;
60 felem
61 end
62 i
63
64 N_Test = N_Test + 1;
65 % Components of f are zero except for those elements with a well! So no
66 % other contributions!
67 %     else
68 %         for index1 = 1:topology
69 %             global_index = elmat(N,index1);
70 %
71 %
72 %         end
73 end
74 end
75 else
76 switch direction
77 case 1 % x direction
78 for index1 = 1:topology
79 felem(index1) = felem(index1) +
80 (k/mu)*(abs(Delta)/6)*beta(index1)*(u(elmat(i,1))+u(elmat(i,2))+u(elmat(i,3)));
81 end
82
83 case 2 % y direction
84 for index1 = 1:topology
85 felem(index1) = felem(index1) +
86 (k/mu)*(abs(Delta)/6)*gamma(index1)*(u(elmat(i,1))+u(elmat(i,2))+u(elmat(i,3)));
87 end
88 end
89 end

```

B.4 Generate Boundary matrix

```

1 clear xc
2 clear yc
3 clear BMelem
4
5 for index1=1:topologybnd
6 xc(index1) = x(elmatbnd(i,index1));
7 yc(index1) = y(elmatbnd(i,index1));
8 end;
9
10 lek = sqrt((xc(2)-xc(1))^2 + (yc(2)-yc(1))^2);
11
12 for index1=1:topologybnd
13 if ~exist('u', 'var')
14 BMelem(index1,index1) = K*lek/2; % NC used! not HB!!
15 else
16
17 BMelem(index1,index1) = -(k/(mu*K))*lek/6;
18 end
19 end;

```

B.5 Generate boundary element vector:

```

1 clear xc
2 clear yc
3 clear bfelem
4
5 for index1 = 1:topologybnd

```



```

6 xc(index1) = x(elmatbnd(i,index1));
7 yc(index1) = y(elmatbnd(i,index1));
8 end;
9
10 lek = sqrt((xc(2)-xc(1))^2+(yc(2)-yc(1))^2);
11
12 if ~exist('u','var')
13 for index1 = 1:topologybnd
14 bfelem(index1) = K*pH*lek/2*u_inf; %what is u_inf?
15 end;
16 else
17 for index1 = 1:topologybnd
18 bfelem(index1) = ((k*pH)/mu)*lek/2*u_inf; %what is u_inf?
19 % bfelem(index1) = -(k/mu)*lek/6*u(elmat(i,ind1));
20 end
21 end

```

B.6 Buildmatrices and vectors

```

1 % This routine constructs the large matrices and vector.
2 % The element matrices and vectors are also dealt with.
3 % First the internal element contributions
4 % First Initialisation of large discretisation matrix, right-hand side vector
5
6 % Treatment of the internal (triangular) elements
7
8 if ~exist('u','var')
9
10 S = sparse(n,n); % stiffness matrix
11
12 f = zeros(n,1); % right-hand side vector
13
14 for i = 1:length(elmat(:,1)) % for all internal elements
15 GenerateElementMatrix; % Selem
16 for ind1 = 1:topology
17 for ind2 = 1:topology
18 S(elmat(i,ind1),elmat(i,ind2)) = S(elmat(i,ind1),elmat(i,ind2)) + Selem(ind1,ind2);
19 end;
20 end;
21
22 GenerateElementVector; % felem
23 for ind1 = 1:topology
24 f(elmat(i,ind1)) = f(elmat(i,ind1)) + felem(ind1);
25 end;
26 end;
27
28 % Next the boundary contributions
29
30 for i = 1:length(elmatbnd(:,1)); % for all boundary elements extension of mass matrix ...
31 M and element vector f
32 GenerateBoundaryElementMatrix; % BMelem
33 for ind1 = 1:topologybnd
34 for ind2 = 1:topologybnd
35 S(elmatbnd(i,ind1),elmatbnd(i,ind2)) = S(elmatbnd(i,ind1),elmatbnd(i,ind2)) + ...
36 BMelem(ind1,ind2);
37 end;
38 end;
39 GenerateBoundaryElementVector; % bfelem
40 for ind1 = 1:topologybnd
41 f(elmatbnd(i,ind1)) = f(elmatbnd(i,ind1)) + bfelem(ind1);
42 end;
43 end;
44
45 Sx = sparse(n,n); % stiffness matrix
46
47 fx = zeros(n,1); % right-hand side vector
48

```



```

49 left_nodes = find(p(1,:) == -1);
50 top_nodes = find(p(2,:) == 1);
51 right_nodes = find(p(1,:) == 1);
52 bottom_nodes = find(p(2,:) == -1);
53
54 bnd1_nodes = ismember(elmatbnd,left_nodes);
55 bnd1 = find(bnd1_nodes(:,1) == 1 & bnd1_nodes(:,2) == 1);
56
57 bnd2_nodes = ismember(elmatbnd,top_nodes);
58 bnd2 = find(bnd2_nodes(:,1) == 1 & bnd2_nodes(:,2) == 1);
59
60 bnd3_nodes = ismember(elmatbnd,right_nodes);
61 bnd3 = find(bnd3_nodes(:,1) == 1 & bnd3_nodes(:,2) == 1);
62
63 bnd4_nodes = ismember(elmatbnd,bottom_nodes);
64 bnd4 = find(bnd4_nodes(:,1) == 1 & bnd4_nodes(:,2) == 1);
65
66
67 direction = 1;
68
69 for i = 1:length(elmat(:,1)) % for all internal elements
70 GenerateElementMatrix; % Selem
71 for ind1 = 1:topology
72 for ind2 = 1:topology
73 if elmat(i,ind1) == elmat(i,ind2)
74 Sx(elmat(i,ind1),elmat(i,ind2)) = Sx(elmat(i,ind1),elmat(i,ind2)) + 2*Selem(ind1,ind2);
75 else
76 Sx(elmat(i,ind1),elmat(i,ind2)) = Sx(elmat(i,ind1),elmat(i,ind2)) + Selem(ind1,ind2);
77 end
78 end;
79 end;
80 GenerateElementVector; % felem
81 for ind1 = 1:topology
82 fx(elmat(i,ind1)) = fx(elmat(i,ind1)) + felem(ind1);
83 end;
84 end;
85
86 % Next the boundary contributions
87
88
89
90 for j = 1:length(bnd1); % left boundary
91 i = bnd1(j);
92 GenerateBoundaryElementMatrix; % BMelem
93 for ind1 = 1:topologybnd
94 for ind2 = 1:topologybnd
95 if elmatbnd(i,ind1) == elmatbnd(i,ind2)
96 Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) = Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) + ...
97     2*BMelem(ind1,ind2);
98 else
99 Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) = Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) + ...
100     BMelem(ind1,ind2);
101 end;
102 end
103 end;
104 GenerateBoundaryElementVector; % bfelem
105 for ind1 = 1:topologybnd
106 fx(elmatbnd(i,ind1)) = fx(elmatbnd(i,ind1)) + bfelem(ind1);
107 end;
108 end;
109
110 for j = 1:length(bnd3); % right boundary
111 i = bnd3(j);
112 GenerateBoundaryElementMatrix; % BMelem
113 for ind1 = 1:topologybnd
114 for ind2 = 1:topologybnd
115 if elmatbnd(i,ind1) == elmatbnd(i,ind2)
116 Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) = Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) + ...
117     2*BMelem(ind1,ind2);
118 else
119 Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) = Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) + ...
120     BMelem(ind1,ind2);
121 end;
122 end
123 end;
124 GenerateBoundaryElementVector; % bfelem
125 for ind1 = 1:topologybnd
126 fx(elmatbnd(i,ind1)) = fx(elmatbnd(i,ind1)) + bfelem(ind1);
127 end;
128 end;

```

```

118 end
119 end;
120 GenerateBoundaryElementVector; % bfelem
121 for ind1 = 1:topologybnd
122     fx(elmatbnd(i,ind1)) = fx(elmatbnd(i,ind1)) - bfelem(ind1);
123 end;
124 end;
125
126 direction = 2;
127
128 Sy = sparse(n,n); % stiffness matrix
129
130 fy = zeros(n,1); % right-hand side vector
131
132 for i = 1:length(elmat(:,1)) % for all internal elements
133     GenerateElementMatrix; % Selem
134     for ind1 = 1:topology
135         for ind2 = 1:topology
136             if elmat(i,ind1) == elmat(i,ind2)
137                 Sy(elmat(i,ind1),elmat(i,ind2)) = Sy(elmat(i,ind1),elmat(i,ind2)) + 2*Selem(ind1,ind2);
138             else
139                 Sy(elmat(i,ind1),elmat(i,ind2)) = Sy(elmat(i,ind1),elmat(i,ind2)) + Selem(ind1,ind2);
140             end
141         end;
142     end;
143     GenerateElementVector; % felem
144     for ind1 = 1:topology
145         fy(elmat(i,ind1)) = fy(elmat(i,ind1)) + felem(ind1);
146     end;
147 end;
148
149 % Next the boundary contributions
150
151
152
153 for j = 1:length(bnd2); % left boundary
154     i = bnd2(j);
155     GenerateBoundaryElementMatrix; % BMelem
156     for ind1 = 1:topologybnd
157         for ind2 = 1:topologybnd
158             if elmatbnd(i,ind1) == elmatbnd(i,ind2)
159                 Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) = Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) + ...
160                     2*BMelem(ind1,ind2);
161             else
162                 Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) = Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) + ...
163                     BMelem(ind1,ind2);
164             end;
165         end;
166     end;
167     GenerateBoundaryElementVector; % bfelem
168     for ind1 = 1:topologybnd
169         fy(elmatbnd(i,ind1)) = fy(elmatbnd(i,ind1)) - bfelem(ind1);
170     end;
171 end;
172
173 for j = 1:length(bnd4); % right boundary
174     i = bnd4(j);
175     GenerateBoundaryElementMatrix; % BMelem
176     for ind1 = 1:topologybnd
177         for ind2 = 1:topologybnd
178             if elmatbnd(i,ind1) == elmatbnd(i,ind2)
179                 Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) = Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) + ...
180                     2*BMelem(ind1,ind2);
181             else
182                 Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) = Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) + ...
183                     BMelem(ind1,ind2);
184             end;
185         end;
186     end;
187     GenerateBoundaryElementVector; % bfelem
188     for ind1 = 1:topologybnd
189         fy(elmatbnd(i,ind1)) = fy(elmatbnd(i,ind1)) + bfelem(ind1);
190     end;
191 end;

```

```

187 end;
188 end

```

B.7 Compute u and v_x/v_y

```

1 % Construction of linear problem
2
3 BuildMatricesandVectors;
4
5 % Solution of linear problem
6
7 u = S \ f;
8
9 BuildMatricesandVectors;
10
11 vx = Sx \ fx;
12 vy = Sy \ fy;

```

B.8 Full script

```

1 close all
2 clear all
3
4 %% 2D Assignment
5 % Lab Assignment 7
6
7 %% Create Mesh
8 WI4243Mesh
9
10 %% Parameters
11
12 Qp = 50; % [m^2/s]
13 k = 10^-7; % [m^2]
14 mu = 1.002*10^-3; % [Pa*s]
15 K = 10000; % [m/s]
16 pH = 10^6; % [Pa]
17 N_wells = 6; % number of wells
18
19 epsilon1 = 0.03;
20 N_Test = 0;
21 %% Coordinates of wells
22
23 for i = 1:N_wells-1;
24 xp(i) = 0.6*cos((2*pi)*(i-1)/(N_wells-1));
25 yp(i) = 0.6*sin((2*pi)*(i-1)/(N_wells-1));
26 end
27
28 xp(N_wells) = 0;
29 yp(N_wells) = 0;
30 clear i;
31
32
33 %% Compute Problem
34 WI4243Comp
35
36 %% Post
37 WI4243Post
38
39 %% Velocity part

```