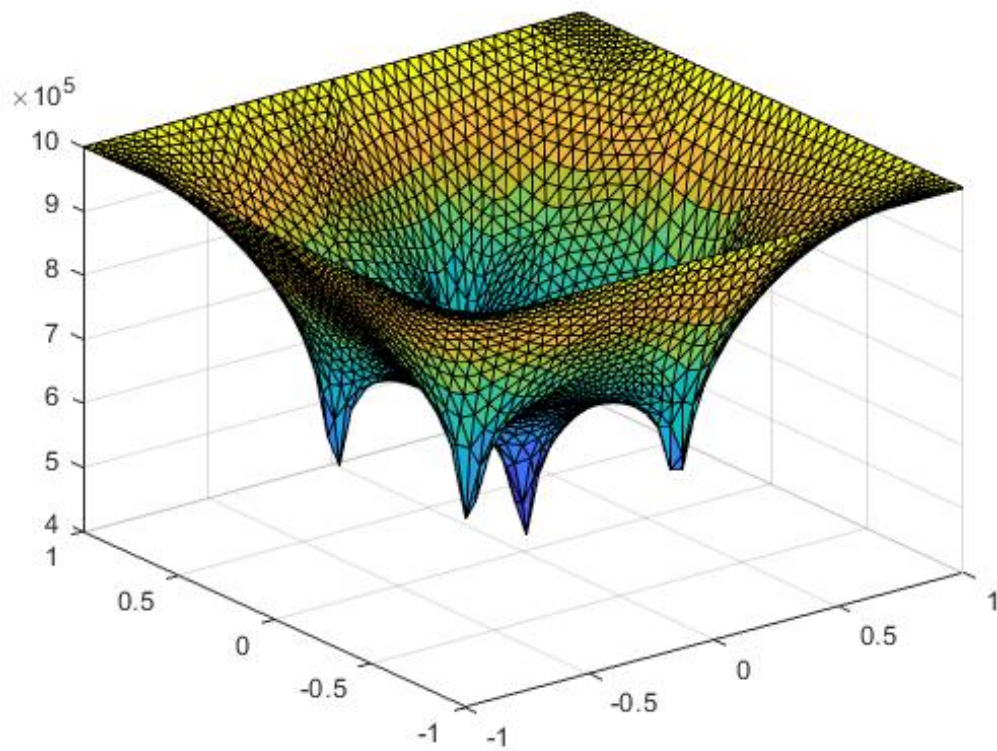


Finite Elements

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February 27, 2018



Preface

This report was written in order to better understand and demonstrate what one can do with the theory of finite elements in combination with MATLAB. Solving boundary value problem through finite elements will help engineers understand difficult dynamics of systems. To show this first a general 1D problem with boundary conditions is presented, solved and computed. Secondly a real life problem is presented, where the flow velocity and pressure within a square reservoir for water filtration is calculated. This is done by solving the boundary value problem of a square reservoir of domain $\Omega = (-1; 1) \times (-1; 1)$ and its boundary conditions, then adapting the 1D MATLAB code to this 2D case and finally computing and plotting velocity flow charts and a 3D surface plot.

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Chapter 1

1D-case

On the 1D interval of $x = [0, 1]$, we consider a steady-state convection-diffusion-reaction equation, with homogeneous Neumann boundary conditions. The following equations apply to this domain:

$$\begin{cases} -D\Delta u + \lambda u = f(x), \\ -D\frac{du}{dx}(0) = 0, \\ -D\frac{du}{dx}(1) = 0 \end{cases} \quad (1.1)$$

In this report Δ denotes the laplacian operator. The function $f(x)$ is a given function, where D and λ are positive real constants. In order to solve this boundary value problem (BVP), first the interval is divided in $n-1$ elements ($n =$ positive integer). This results in the domain being divided in elements: $e_i = [x_i, x_{i+1}]$ where $i = 1, 2, \dots, n$.

In order to solve this BVP, the solutions for the given equations will first be calculated and then computed using MATLAB codes.

1.1 Boundary value problem 1D

In order to find the Weakform of the given equations(1.1), both sides are multiplied by a test function $\phi(x)$ and then integrate both sides over the domain Ω . In the equations $\phi(x)$ is written as ϕ

$$\int_{\Omega} \phi(-D\Delta u + \lambda u) d\Omega = \int_{\Omega} \phi f(x) d\Omega \quad (1.2)$$

Now by rewriting and then using partial integration the following equation can be found:

$$\int_{\Omega} (\nabla \cdot (-D\phi \cdot \nabla u) + D\nabla\phi \cdot \nabla u + \phi\lambda u) d\Omega = \int_{\Omega} \phi f(x) d\Omega \quad (1.3)$$

Applying Gauss on the first term on the left side of equation(1.3):

$$\int_{\Omega} \vec{n} \cdot (-D\phi \nabla u) d\tau + \int_{\Omega} (D\nabla\phi \cdot \nabla u + \phi\lambda u) d\Omega = \int_{\Omega} \phi f(x) d\Omega \quad (1.4)$$

Using the boundary conditions from equations(1.1) the boundary integral equals to 0 and then the following weak formulation(WF) is found:

(WF):

$$\begin{cases} \text{find } u \in \Sigma = \{u \text{ smooth}\} \text{ Such that:} \\ \int_{\Omega} (D\nabla\phi \cdot \nabla u + \phi\lambda u) d\Omega = \int_{\Omega} \phi f(x) d\Omega \\ \forall \phi \in \Sigma \end{cases} \quad (1.5)$$

The next step is to substitute the Galerkin equations into the found differential equation, where u is replaced by $\sum_{j=1}^n c_j \phi_j$ and $\phi(x) = \phi_i(x)$ with $i = [1, \dots, n]$. Filling this in equation (1.5) the following equation is found:

$$\sum_{j=1}^n c_j \int_0^1 (D \nabla \phi_i \cdot \nabla \phi_j + \lambda \phi_i \phi_j) d\Omega = \int_0^1 \phi_i f(x) d\Omega \quad (1.6)$$

Which is of the form of $S\vec{c} = \vec{f}$

1.2 Element matrix

Now the found Galerkin equations can be used to compute S_{ij} the element matrix, over a generic line element e_i .

$$S\vec{c} = \sum_{j=1}^n c_j \int_0^1 (D \nabla \phi_i \cdot \nabla \phi_j + \lambda \phi_i \phi_j) d\Omega \quad (1.7)$$

$$S_{ij} = \sum_{l=1}^{n-1} S_{ij}^{e_k} \quad (1.8)$$

Now to solve S we solve the following equation, over the internal line element.

$$S_{ij}^{e_k} = -D \int_{e_k} \nabla \phi_i \cdot \nabla \phi_j d\Omega + \lambda \int_{e_k} \phi_i \phi_j dx \quad (1.9)$$

1.3 Element vector

Again the found Galerkin Equations(1.6) are used in order to compute the element vector f_i over a generic line-element.

$$f_i^{e_k} = \int_{e_k} \phi_i f dx \quad (1.10)$$

$$f_i^{e_k} = \frac{|x_k - x_{k-1}|}{(1+1+0)!} f(\vec{x}) = \frac{|x_k - x_{k-1}|}{2} \begin{bmatrix} f_{k-1}^{e_n} \\ f_k^{e_n} \end{bmatrix} \quad (1.11)$$

1.4 Boundary value problem 1D MATLAB routine

1.4.1 mesh and elmat code

The first step in order to solve the BVP is to write a MATLAB routine that generates an equidistant distribution of points over the given interval of $[0, 1]$ (generate a mesh with $n-1$ elements).

```
1 function [ x ] = GenerateMesh(int, N_elem)
2 %GenerateMesh Creates a mesh for 1D problems
3
4 x = linspace(int(1,1),int(1,2),N_elem);
5 end
```

Using the codes to generate a mesh and the elmat, it is easier to use this 1D problem and adapt to a higher dimensional problem. The next step is to write a code that generates a two dimensional array, called the elmat.

```

1  function [ elmat ] = GenerateTopology( N_elem )
2  %GenerateTopology Creates the topology for a 1D problem given mesh 'x'.
3
4  elmat = zeros(N_elem,2);
5  elmat(i,1) = i;
6  elmat(i,2) = i + 1;
7  end

```

1.4.2 Element matrix

Now that the base MATLAB codes are made the element matrix and element vector codes can be written. The first step in this process is, is to compute the element matrix S_{elem} .

```

1  function [ Selem ] = GenerateElementMatrix( k, elmat, D, lambda, mesh)
2  %GenerateElementMatrix Creates element matrix S_ek
3
4  Selem = zeros(2,2);
5
6  i = elmat(k,1);
7  j = elmat(k,2);
8
9  x1 = mesh(i);
10 x2 = mesh(j);
11
12 element_length = abs(x1-x2);
13
14 slope = 1/element_length;
15
16 for m = 1:2
17     for n = 1:2
18         if m == n
19             Selem(m,n) = element_length*((-1)^(abs(m-n))*D*slope^2
20                 + (2)*lambda/6);
21         else
22             Selem(m,n) = element_length*((-1)^(abs(m-n))*D*slope^2
23                 + (1)*lambda/6);
24         end
25     end
26 end
27 end

```

1.4.3 Assemble matrix S

To generate a n-by-n matrix S, the sum over the connections of the vertices in each element matrix, over all elements has to be calculated. The following code computes this matrix:

```

1  function [ S ] = AssembleMatrix( N_elem, int, lambda, D)
2  % global N_elem
3
4  elmat = GenerateTopology(N_elem);
5
6  S = zeros(N_elem,N_elem);
7
8  for i = 1:N_elem-1
9      Selem = GenerateElementMatrix(i, elmat, int, N_elem, D, lambda);
10     for j = 1:2
11         for k = 1:2
12             S(elmat(i,j), elmat(i,k)) =
13                 S(elmat(i,j), elmat(i,k)) + Selem(j,k);
14         end
15     end
16 end
17 end

```


All the previous code will generate a large matrix S , from the element matrices S_{elem} over each element.

1.4.4 Element vector MATLAB routine

The next step In order to solve the equation $S\vec{c} = \vec{f}$ is to create a code to generate the element vector. This element vector provides information about node i and node $i+1$, which are the vertices of element e_i .

```

1  function [ felem ] = GenerateElementVector( i, elmat, mesh )
2  %GenerateElementVector Creates element vector f_ek
3
4
5  felem = [0;0];
6
7  k1 = elmat(i,1);
8  k2 = elmat(i,2);
9
10 x1 = mesh(k1);
11 x2 = mesh(k2);
12
13 element_length = abs(x1-x2);
14
15 felem = (element_length/2*arrayfun(@functionBVP,[x1,x2]))';
16
17 end

```

Where the function $f(x)$ from the BVP is defined in the following function. The different definitions of $f(x)$ will be used in different assignments.

```

1  function [f] = functionBVP(x)
2  f = 1;
3  %f = sin(20*x);
4  %f = x;
5  end

```

To generate the vector f , the sum over the connections of the vertices in each element matrix, over all elements $i \in \{1, \dots, n-1\}$ has to be calculated.

```

1  function [ f ] = AssembleVector( N_elem, int, lambda, D )
2
3  f = zeros(N_elem,1);
4  elmat = GenerateTopology(N_elem);
5
6  for i = 1:N_elem-1
7      felem = GenerateElementVector(i, elmat, int, N_elem);
8      for j = 1:2
9          f(elmat(i,j)) = f(elmat(i,j)) + felem(j);
10     end
11 end

```

1.4.5 Computing S and f

Now if the previous MATLAB codes are run the following happens. First, a mesh and 1D topology are build. These are needed for the S matrix and f vector. The second step is to calculate the S matrix and f vector themselves through the found equations of section 1.2 and 1.3. The final step is to use the found matrix and vector to solve the equation $Su = \vec{f}$.

1.5 Main program

The main program is simply written by assembling the previous created MATLAB code AssembleMatrix and AssembleVector and deviding the vector f by the matrix S.

```
1 function [ u ] = SolveBVP( N_elem, int, lambda, D )
2
3 S = AssembleMatrix( N_elem, int, lambda, D );
4 f = AssembleVector( N_elem, int, lambda, D );
5
6 %% Calculate u
7 x = linspace(int(1),int(2),N_elem);
8
9 u = S\f;
10 plot(x,u);
```

The result of running this MATLAB code is shown in figure(1.1).

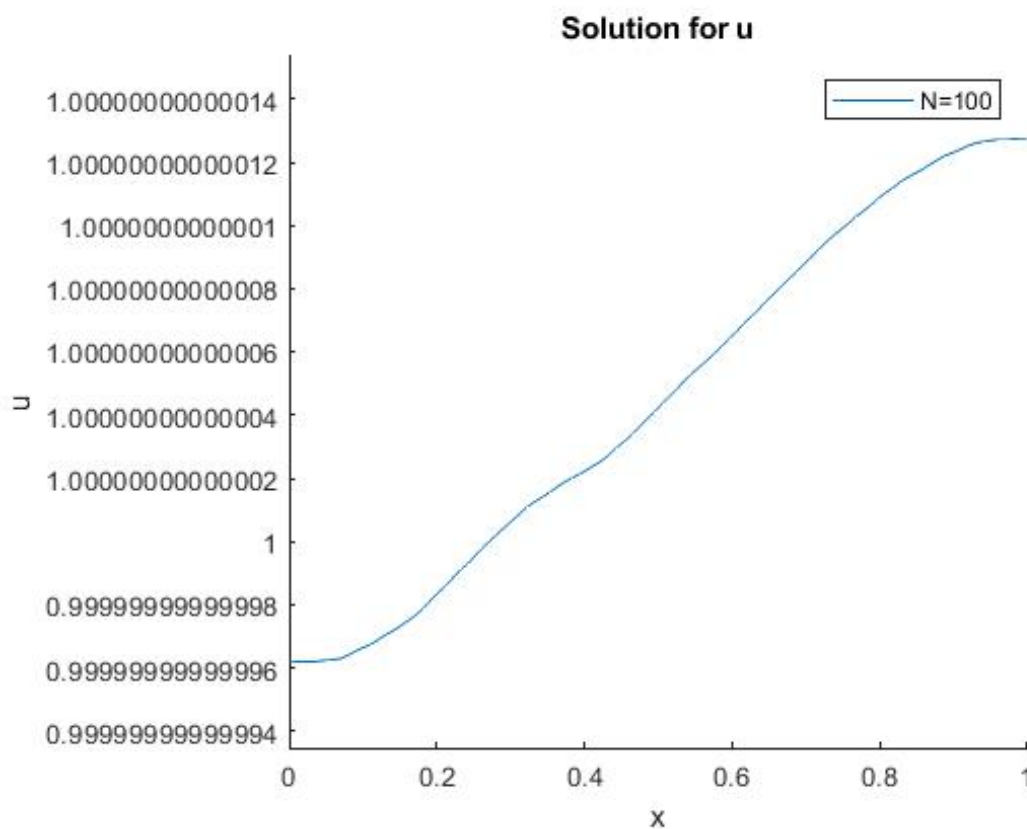


Figure 1.1: showing the calculated u versus x, with $N = 100, f(x)=1$

Figure 1.1 shows the solution for $u = \vec{f}/S$ for $N = 100$ and $f(x) = 1$. The domain is divided by equal spaced elements (follows from $f(x) = 1$). Even though it is not a very stable or smooth curve, the solution does meet the requirement for the boundary condition where $du/dx = 0$ at $x = 0$ and $x = 1$.

1.6 Solution for u

The final step is to combine all the codes in a main code to solve $Su = \vec{f}$. This code can be found in Appendix A. Previously the S matrix and f vector were computed for $n = 100$. Now u will be calculated for $f(x) = 1$, $D = 1$, $\lambda = 1$ and $N = 100$. The result of this is plotted in figure(1.2).

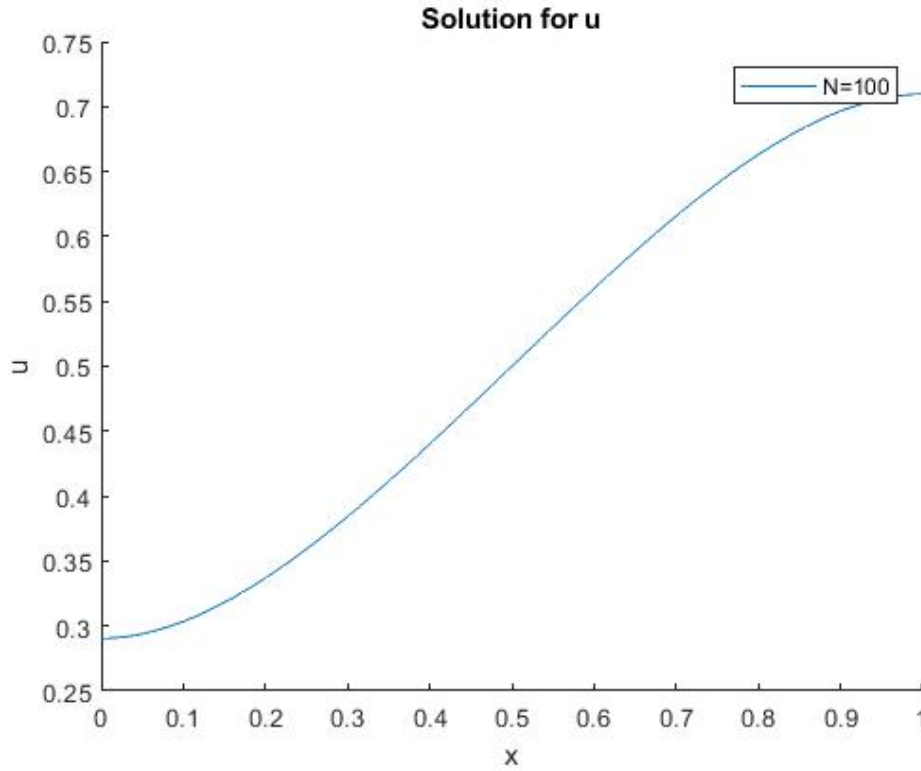


Figure 1.2: showing the calculated u versus x , with $N = 100$, $f(x) = x$

Figure(1.2) shows the solution of $u = \vec{f}/S$ for $N = 100$ elements. In this case $f(x)=x$ is used as function to divide the elements. Compared to the case where $f(x)=1$ it can be seen that now a smoother curve is found. The solution found meets the requirement for the boundary condition where $du/dx = 0$ at $x = 0$ and $x = 1$.

1.7 Experiment

The next step is to see what happens when changing $f(x)$ to $f(x) = \sin(20x)$ and to see the difference for several values for N ($n = 10, 20, 30, 40, 80, 160$).

```
1     function [f] = functionBVP(x)
2         f = sin(20*x);
3         %f = x;
4         %f = 1;
5     end
6
7     figure
8     hold on
9
10    for N_elem = [10 20 40 80 100 160]
11        mesh = GenerateMesh(int,N_elem);
12        elmat = GenerateTopology(N_elem);
13        S = AssembleMatrix( N_elem, lambda, D, mesh, elmat);
14        f = AssembleVector( N_elem, mesh, elmat);
15
16        x = linspace(int(1),int(2),N_elem);
17
18        u = S\f;
19        plot(x,u);
20
21        legend('N=100')
22        title('Solution for u')
23        xlabel('x')
24        ylabel('u')
25        ax.box='on'
26    end
27    hold off
```

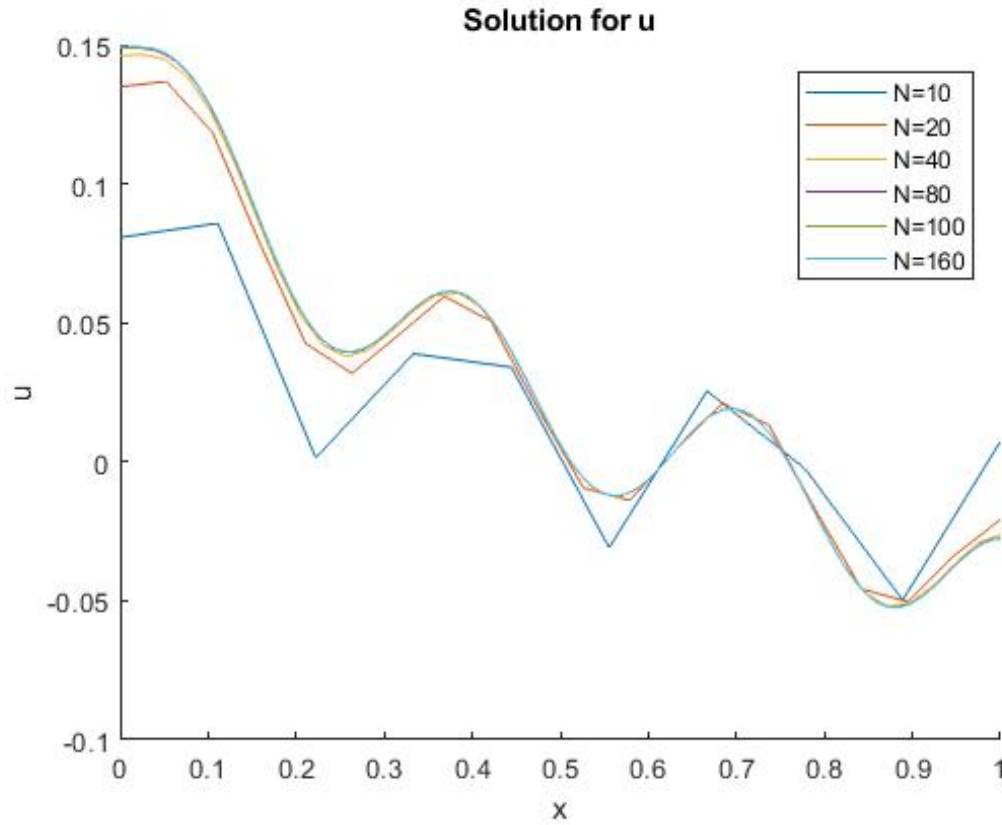


Figure 1.3: showing the calculated u versus x , with $N = [10, 20, 40, 80, 100, 160]$, $f(x) = \sin(x)$.

Looking at figure(1.3) one shows the solution for u when picking $f(x) = \sin(x)$. As the number of elements N is increased, it can be seen that an increasingly smooth line is drawn. The created MATLAB code divides the domain $x = [0, 1]$ in N elements and then tries to solve the found equation while being consistent with the boundary condition. The boundary condition states that $du/dx = 0$ at $x = 0$ and $x = 1$. For low numbers of N this is not achieved, but the higher number of elements shows that above $N = 40$ the solution meets the boundary condition. Another thing one can observe is how the curve is similar to a $\sin(x)$, where a higher N value shows a better fit to a sinus.

Chapter 2

2D-case

The obvious next step after solving a 1 dimensional BVP is to adept the 1D solutions into code to solve a 2 dimensional BVP. To do this, a real life problem is going to be solved. In 3rd world countries one of the big issues is the supply of fresh water. One way is to do this is to take square reservoirs, which is a porous medium, with several wells where water is extracted from the subsurface. The water pressure is equal to the hydrostatic pressure. As this is not on an infinite domain, mixed boundary conditions are used. These boundary conditions represent a model for the transfer of the water over the boundary to locations far away. To this extent, a square domain is considered with length 2 in meter: $\Omega = (-1; 1) \times (-1; 1)$ with boundary $\partial\Omega$. Darcy's law for fluid determines the steady state equilibrium of this BVP, given by equation (2.1):

$$\vec{v} = -\frac{k}{\mu}\nabla p \quad (2.1)$$

Where p, k, μ and \vec{v} , respectively denote the fluid pressure, permeability of the porous medium, viscosity of water and the fluid flow velocity. In this BVP the effect of gravity will not play a part as the problem is looked at in 2D. An accompanying assumption is incompressibility, so the extraction wells are treated as point sinks. This assumption can be made as the well its diameter is much smaller than the dimensions of the square reservoir. The extraction wells extract at the same rate in each direction, leading to the following boundary conditions (2.2).

$$\nabla \cdot \vec{v} = -\sum_{p=1}^{n_{well}} Q_p \delta(\vec{x} - \vec{x}_p) = 0, \quad (x, y) \in \Omega \quad (2.2)$$

Where Q_p denotes the water extraction rate by well k, which is located at x_p . Here x equals $(x; y)$, the spatial coordinates. The convention $\vec{x} = (x; y)$ to represent the spatial coordinates is used. The dirac Delta Distribution is characterized by equation(2.3).

$$\begin{cases} \delta(\vec{x}) = 0, & \vec{x} \neq 0 \\ \int_{\Omega} \delta(\vec{x}) d\Omega = 1, & \text{where } \Omega \text{ contains the origin.} \end{cases} \quad (2.3)$$

For this BVP the following boundary condition is considered:

$$\vec{v} \cdot \vec{n} = K(p - p^H), \quad (x, y) \in \partial\Omega \quad (2.4)$$

Where K denotes the transfer rate coefficient of the water between the boundary of the domain and its surroundings. The constant p^H represents the hydrostatic pressure.

The boundary $\partial\Omega$ is divided into four parts described by the sides of the square domain Ω . $\partial\Omega_1$ is the part with $x = -1$, $\partial\Omega_2$ is the part with $y = 1$, $\partial\Omega_3$ is the part with $x = 1$, $\partial\Omega_4$ is the part with $y = -1$.

In order to solve this BVP the values needed for all the constants are given in table (2.1).

Table 2.1: Values of input parameters

| Symbol | Value | Unit |
|--------|-----------------------|--------------|
| Q_p | 50 | m^2/s |
| k | 10^{-7} | m^2 |
| μ | $1.002 \cdot 10^{-3}$ | $Pa \cdot s$ |
| K | 10 | m/s |
| p^H | 10^6 | Pa |

In this BVP six wells are considered, located at:

$$\begin{cases} x_p = 0.6 \cos(\frac{2\pi(p-1)}{5}) \\ x_p = 0.6 \sin(\frac{2\pi(p-1)}{5}) \end{cases} \quad (2.5)$$

For $p \in \{1, \dots, 5\}$ and for $p = 6$ we have $x_6 = 0$ and $y_6 = 0$.

2.1 Boundary value problem 2D

The first step to solving these equations using finite elements is to find the boundary value problem to solve. This is done by filling in equation(2.1) in both equation 2.2 and the boundary condition(2.4) in order to find the BVP in terms of p:

$$\text{BVP} \begin{cases} -\frac{k}{\mu} \Delta \vec{p} = -\sum_{p=1}^{n_{well}} Q_p \delta(\vec{x} - \vec{x}_p) = 0, & (x, y) \in \Omega \\ -\frac{k}{\mu} \nabla \vec{p} \cdot \vec{n} = -\frac{k}{\mu} \frac{\partial p}{\partial n} = K(p - p^H), & (x, y) \in \partial\Omega \end{cases} \quad (2.6)$$

The next step is to compute the weak formulation using the previous found BVP(2.6). By multiplying both sides by test function $\phi(x) = \alpha_i + \beta_i x + \gamma_i y$ and integrating both sides over the domain Ω the weak formulation can be found.

$$\int_{\Omega} \phi(\vec{x}) \nabla \cdot (-\frac{k}{\mu} \nabla \vec{p}) d\Omega = \int_{\Omega} -\sum_{p=1}^{n_{well}} \phi(\vec{x}) Q_p \delta(\vec{x} - \vec{x}_p) d\Omega \quad (2.7)$$

Using integrating by parts on the left side of equation(2.7) results in:

$$\int_{\Omega} \nabla \cdot [\phi(\vec{x}) (-\frac{k}{\mu} \nabla \vec{p})] + \frac{k}{\mu} \nabla \phi(\vec{x}) \cdot \nabla p d\Omega = - \int_{\Omega} \sum_{p=1}^{n_{well}} \phi(\vec{x}) Q_p \delta(\vec{x} - \vec{x}_p) d\Omega \quad (2.8)$$

Next is to apply Gauss's Theorem on the first term of the left side.

$$\int_{\partial\Omega} \vec{n} \cdot [\phi(\vec{x}) (-\frac{k}{\mu} \nabla \vec{p})] d\tau + \int_{\Omega} \frac{k}{\mu} \nabla \phi(\vec{x}) \cdot \nabla p d\Omega = - \int_{\Omega} \sum_{p=1}^{n_{well}} \phi(\vec{x}) Q_p \delta(\vec{x} - \vec{x}_p) d\Omega \quad (2.9)$$

Switching the integral and summation on the right side of equation(2.9) and simplifying terms:

$$\int_{\partial\Omega} (\phi(\vec{x}) (-\frac{k}{\mu} \frac{\partial p}{\partial n})) d\tau + \int_{\Omega} \frac{k}{\mu} \nabla \phi(\vec{x}) \cdot \nabla p d\Omega = - \sum_{p=1}^{n_{well}} \int_{\Omega} \phi(\vec{x}) Q_p \delta(\vec{x} - \vec{x}_p) d\Omega \quad (2.10)$$

Equation(2.10) can be simplified, by using the boundary conditions (equation(2.6)) and the following property, into equation(2.12):

$$\int_{\Omega} \delta(\vec{x}) f(\vec{x}) d\Omega = f(0) \quad (2.11)$$

$$\int_{\partial\Omega} \phi(\vec{x}) K(p - p^H) d\tau + \int_{\Omega} \frac{k}{\mu} \nabla \phi(\vec{x}) \cdot \nabla p d\Omega = - \sum_{p=1}^{n_{well}} \phi(\vec{x}_p) Q_p \quad (2.12)$$

Rearranging equation(2.12) so that the unknowns are on the left and the constant parts on the right leads to the following WF:

$$(WF) \begin{cases} \text{Find } p \in \Sigma = \{p \text{ smooth}\} \text{ Such that:} \\ \int_{\partial\Omega} \phi(\vec{x}) K p d\tau + \int_{\Omega} \frac{k}{\mu} \nabla \phi(\vec{x}) \cdot \nabla p d\Omega = - \sum_{p=1}^{n_{well}} \phi(\vec{x}_p) Q_p + \int_{\partial\Omega} \phi(\vec{x}) K p^H d\tau \\ \forall \phi \in \Sigma \end{cases} \quad (2.13)$$

To solve the WF the Galerkin equations are applied, where p is replaced by $\sum_{j=1}^n c_j \phi_j$ and $\phi(x) = \phi_i(x)$.

$$\sum_{j=1}^n c_j \int_{\partial\Omega} \phi_i K \phi_j d\tau + \int_{\Omega} \frac{k}{\mu} \nabla \phi_i(x) \cdot \nabla \phi_j d\Omega = - \sum_{p=1}^{n_{well}} \phi_i(x_p) Q_p + \int_{\partial\Omega} \phi_i K p^H d\tau \quad (2.14)$$

Equation(2.14) now is of the form $S\vec{c} = \vec{f}$ and, like with the 1D problem, \vec{c} can be computed. First the element and boundary elements are determined from the Galerkin equations.

2.2 Element matrix and element vector

First the galerkin equation is seperated in its element and boundary components. The element matrix $S_{ij}^{e_k}$ and the element vector $f_i^{e_k}$ are given in equations (2.15) and 2.16 respectively. For the element matrix, boundary element matrix and boundary element vector Newton-Côtes theorem and Holand-Bell theorem are applied to simplify the equations into a set of equations that can be used for computing with MATLAB code.

$$S_{ij}^{e_k} = \int_{e_k} \frac{k}{\mu} \nabla \phi_i \cdot \nabla \phi_j d\Omega = (\beta_i \beta_j + \gamma_i \gamma_j) \frac{k}{\mu} \frac{|\Delta e_k|}{2} \quad (2.15)$$

$$f_i^{e_k} = - \sum_{p=1}^{n_{well}} \phi_i(\vec{x}_p) Q_p \quad (2.16)$$

2.3 Boundary matrix and boundary vector

The boundary matrix $S_{ij}^{be_l}$ and boundary vector $f_i^{be_l}$ can be found in the following equations:

$$S_{ij}^{be_l} = \int_{be_l} K \phi_i \phi_j dx = K \frac{|be_l|}{6} (1 + \delta_{ij}) \quad (2.17)$$

$$f_i^{be_l} = K p^H \int_{be_l} \phi_i dx = K p^H \frac{|be_l|}{2} \quad (2.18)$$

Where δ_{ij} is the Kronicker delta. It is assumed there are no wells on the boundary.

2.4 Wells within an internal element

To solve the BVP in 2D, one of the aspects that need to be determined is whether each internal element contains a well. So it must be determined whether the well with index p and position x_p is contained within element e_k with vertices x_{k1} , x_{k2} and x_{k3} .

This can be done according the following criterion:

$$|\Delta(\vec{x}_p, \vec{x}_{k2}, \vec{x}_{k3})| + |\Delta(\vec{x}_{k1}, \vec{x}_p, \vec{x}_{k3})| + |\Delta(\vec{x}_{k1}, \vec{x}_{k2}, \vec{x}_p)| : \begin{cases} = |e_k|, \vec{x}_p \in \vec{e}_k \\ > |e_k|, \vec{x}_p \notin \vec{e}_k \end{cases} \quad (2.19)$$

In the criterion $\Delta(\vec{x}_p, \vec{x}_q, \vec{x}_r)$ denotes the triangle with vertices \vec{x}_p , \vec{x}_q and \vec{x}_r , where $|\Delta(\vec{x}_{k1}, \vec{x}_{k2}, \vec{x}_{k3})|$ denote its area. The triangular element k is given by $e_k = \Delta(\vec{x}_{k1}, \vec{x}_{k2}, \vec{x}_{k3})$ with vertices \vec{x}_{k1} , \vec{x}_{k2} and \vec{x}_{k3} and \vec{e}_k includes the boundary of element e_k . To solve this BVP a certain tolerance has to be accounted for in the MATLAB code. However, while using this method it proved difficult to determine a correct tolerance to ensure every well was contained in a single element.

Therefore, an alternative method was used. A check was done whether the linear basis functions ϕ_{k1} , ϕ_{k2} and ϕ_{k3} all have values in the interval $[0, 1]$ at position \vec{x}_p . If this is true, then the well at \vec{x}_p is within the triangular element e_k .

If it is found that an element does contain a well, we see in equation (2.16) that $\phi_i(\vec{x}_p)$ for $i = \{k1, k2, k3\}$ must be found. The following code does this by determining α_i , β_i , and γ_i and subsequently filling \vec{x}_p into the found $\phi_i(\vec{x})$.

```

1  for index1=1:topology
2      xc(index1) = x(elmat(i,index1));
3      yc(index1) = y(elmat(i,index1));
4  end;
5
6  Delta = det([1 xc(1) yc(1);1 xc(2) yc(2);1 xc(3) yc(3)]);
7
8  B_mat = [1 xc(1) yc(1);1 xc(2) yc(2);1 xc(3) yc(3)] \ eye(3);
9
10 alpha = B_mat(1,1:3);
11 beta  = B_mat(2,1:3);
12 gamma = B_mat(3,1:3);
13
14 felem = zeros(1,topology);
15
16 if ~exist('u','var') % Only if u is already know can the calculation of the velocity ...
17     begin.
18     for N = 1:N_wells
19         for index3 = 1:topology
20             phi_p(index3) = alpha(index3)*xp(N) + beta(index3)*yp(N);
21         end
22         if (phi_p(1) ≤ 1) && (phi_p(1) ≥ 0) && (phi_p(2) ≤ 1) && (phi_p(2) ≥ 0) && ...
23             (phi_p(3) ≤ 1) && (phi_p(3) ≥ 0);
24             for index1 = 1:topology
25                 felem(index1) = felem(index1) + -Qp*phi_p(index1);
26             end
27         end
28     end
29 end

```

2.5 Generating MATLAB code

Similar as with the 1D BVP, MATLAB code is written in order to generate a mesh, element matrix, element vector, boundary element matrix and boundary element vector. These codes can be found in appendix B.1 through B.6. The scripts have been written such that they can be used to find both the pressure field and the velocity field (the derivation for the velocities is found in the next section).

2.6 Velocities

In order to find the velocities in x,y-direction, Darcy's law is used to compute the speed in both directions.

In order to find v_x and v_y the first step is to rewrite equation(2.20).

$$\vec{v} = -\frac{k}{\mu} \nabla p \quad (2.20)$$

$$v_x = -\frac{k}{\mu} \frac{\partial p}{\partial x} \quad (2.21)$$

$$v_y = -\frac{k}{\mu} \frac{\partial p}{\partial y} \quad (2.22)$$

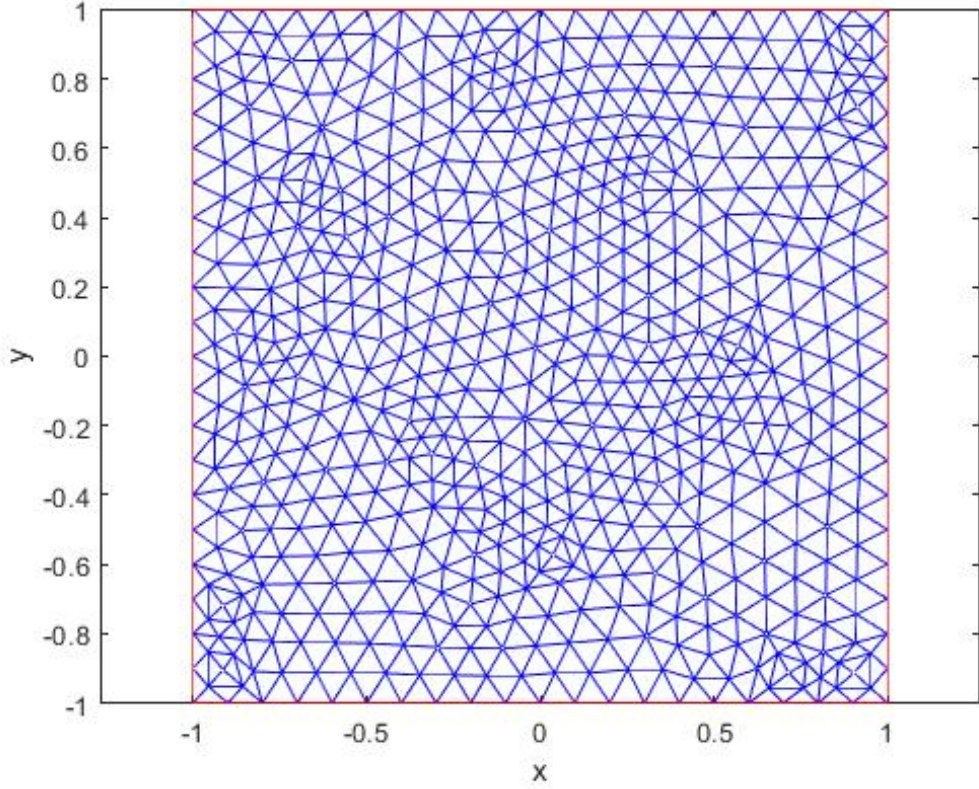


Figure 2.1: Triangle element mesh over the square reservoir domain $\Omega = (-1; 1) \times (-1; 1)$.

Equation(2.24) shows the relation between \vec{v} and pressure p .

$$\vec{v} \cdot \vec{n} = k(p - p^H) \text{ on } \partial\Omega \quad (2.23)$$

This relation gives the boundary conditions for v_x and v_y :

$$v_x(x = -1) = -k(p - p^H) \quad (2.24)$$

$$v_x(x = 1) = k(p - p^H) \quad (2.25)$$

$$v_y(y = -1) = -k(p - p^H) \quad (2.26)$$

$$v_y(y = 1) = k(p - p^H) \quad (2.27)$$

In order to find the weak form, again, the test function ϕ and integration over the domain Ω is used. Here follows the derivation for finding v_x , the steps for deriving v_y are similar.

$$\int_{\Omega} \phi v_x d\Omega = -\frac{k}{\mu} \int_{\Omega} \phi \frac{\partial p}{\partial x} d\Omega \quad (2.28)$$

Partial integration is applied on the right side term.

$$\int_{\Omega} \phi v_x d\Omega = -\frac{k}{\mu} \left\{ \int_{\Omega} \frac{\partial}{\partial x} (\phi p) - p \frac{\partial \phi}{\partial x} d\Omega \right\} \quad (2.29)$$

Rewriting the integral:

$$\int_{\Omega} -\frac{k}{\mu} \frac{\partial \phi p}{\partial x} dx dy = \int_{-1}^1 -\frac{k}{\mu} [\phi p]_{-1}^1 dy \quad (2.30)$$

The surface integral turns into a set of line integrals along parts of the boundary $\partial\Omega$.

$$\int_{\Omega} -\frac{k}{\mu} \frac{\partial \phi p}{\partial x} dx dy = \int_{-1}^1 -\frac{k}{\mu} (\phi(x=1, y)p(x=1, y)) + \frac{k}{\mu} (\phi(x=-1)p(x=-1, y)) dy \quad (2.31)$$

Simplifying the previous equations.

$$\int_{\Omega} -\frac{k}{\mu} \frac{\partial \phi p}{\partial x} dx dy = \int_{\partial\Omega_3} -\frac{k}{\mu} \phi p d\tau + \int_{\partial\Omega_1} \frac{k}{\mu} \phi p d\tau \quad (2.32)$$

Inserting this into equation(2.29) the following equation is found.

$$\int_{\Omega} \phi v_x = \frac{k}{\mu} \left\{ \int_{\partial\Omega_3} -\phi p d\tau + \int_{\partial\Omega_1} \phi p d\tau + \int_{\Omega} p \frac{\partial \phi}{\partial x} d\Omega \right\} \quad (2.33)$$

The boundary conditions is used to rewrite p in the two boundary integrals.

$$\text{on } \begin{cases} \partial\Omega_3 : -v_x = K(p - p^H) \rightarrow p = -\frac{v_x}{K} + p^H \\ \partial\Omega_1 : v_x = K(p - p^H) \rightarrow p = \frac{v_x}{K} + p^H \end{cases} \quad (2.34)$$

The following weakform is derived:

$$\text{WF} \begin{cases} \text{Find } v_x \in \Sigma = \{v_x \text{ smooth}\}, \text{ such that} \\ \int_{\Omega} \phi v_x d\Omega + \int_{\partial\Omega_3} -\frac{k}{\mu} \frac{1}{K} \phi v_x d\tau + \int_{\partial\Omega_1} -\frac{k}{\mu} \frac{1}{K} \phi v_x d\tau = \int_{\partial\Omega_3} -\frac{k}{\mu} \phi p^H d\tau + \int_{\partial\Omega_1} \frac{k}{\mu} \phi p^H d\tau + \int_{\Omega} \frac{k}{\mu} p \frac{\partial \phi}{\partial x} d\Omega \\ \forall \phi \in \Sigma \end{cases} \quad (2.35)$$

To find the system of equations the following equations are filled in equation(2.35) $\phi(\vec{x}) = \phi_i(\vec{x}) = \alpha_i + \beta_i x + \gamma_i y$ and $v_x \approx \sum_{j=1}^n c_j \phi_j(\vec{x})$.

$$\sum_{j=1}^n c_j \left\{ \int_{\Omega} \phi_i \phi_j d\Omega + \int_{\partial\Omega_3} -\frac{k}{\mu} \frac{1}{K} \phi_i \phi_j d\tau + \int_{\partial\Omega_1} -\frac{k}{\mu} \frac{1}{K} \phi_i \phi_j d\tau \right\} = \int_{\partial\Omega_3} -\frac{k}{\mu} \phi_i p^H d\tau + \int_{\partial\Omega_1} \frac{k}{\mu} \phi_i p^H d\tau + \int_{\Omega} \frac{k}{\mu} p \beta_i d\Omega \quad (2.36)$$

From this system of equations the element matrix and element vector are extracted.

$$S_{ij} = \int_{\Omega} \phi_i \phi_j d\Omega + \int_{\partial\Omega_3} -\frac{k}{\mu} \frac{1}{K} \phi_i \phi_j d\tau + \int_{\partial\Omega_1} -\frac{k}{\mu} \frac{1}{K} \phi_i \phi_j d\tau \quad (2.37)$$

$$f_i = \int_{\partial\Omega_3} -\frac{k}{\mu} \phi_i p^H d\tau + \int_{\partial\Omega_1} \frac{k}{\mu} \phi_i p^H d\tau + \int_{\Omega} \frac{k}{\mu} p \beta_i d\Omega \quad (2.38)$$

Now separating contributions to both S_{ij} and f_i from boundary and internal elements into $S_{ij}^{be_l}$, $S_{ij}^{e_k}$, $f_i^{be_l}$ and $f_i^{e_k}$ such that:

$$S_{ij} = \sum_{l=1}^{n_{be}} S_{ij}^{be_l} + \sum_{k=1}^{n_e} S_{ij}^{e_k} \quad (2.39)$$

$$f_i = \sum_{l=1}^{n_{be}} f_i^{be_l} + \sum_{k=1}^{n_e} f_i^{e_k} \quad (2.40)$$

Applying Newton-Côtes theorem and Holand-Bell theorem results in the following new expressions for the (boundary) element-matrix and -vector are found.

$$S_{ij}^{e_k} = \int_{e_k} \phi_i \phi_j d\Omega = \frac{|\Delta e_k|}{24} \quad (2.41)$$

$$S_{ij}^{be_l} = \int_{be_l} -\frac{k}{\mu} \phi_i \phi_j d\tau = \frac{k}{\mu} \frac{1}{k} \frac{|be_l|}{6} (1 + \delta_{ij}) \quad (2.42)$$

$$f_i^{e_k} = \int_{e_k} \frac{k}{\mu} p \beta_i d\Omega = \frac{k}{\mu} \beta_i \sum_{m \in \{k_1, k_2, k_3\}} p(\vec{x}_m) \frac{|\Delta e_k|}{6} \quad (2.43)$$

$$f_i^{be_l} = \int_{be_l} \pm \frac{k}{\mu} \phi_i p^H d\tau = \pm \frac{k}{\mu} p^H \frac{|be_l|}{2} \quad (2.44)$$

With '+' if be_l is on $\partial\Omega_1$ and '-' if it is on $\partial\Omega_3$

Since the pressure field p was previously calculated, all the necessary information to compute v_x (and similarly v_y) is now derived. In the following plots the velocities for $K = 10 \text{ m/s}$ are shown using a vector plot, contour plot and a 3D surface plot.

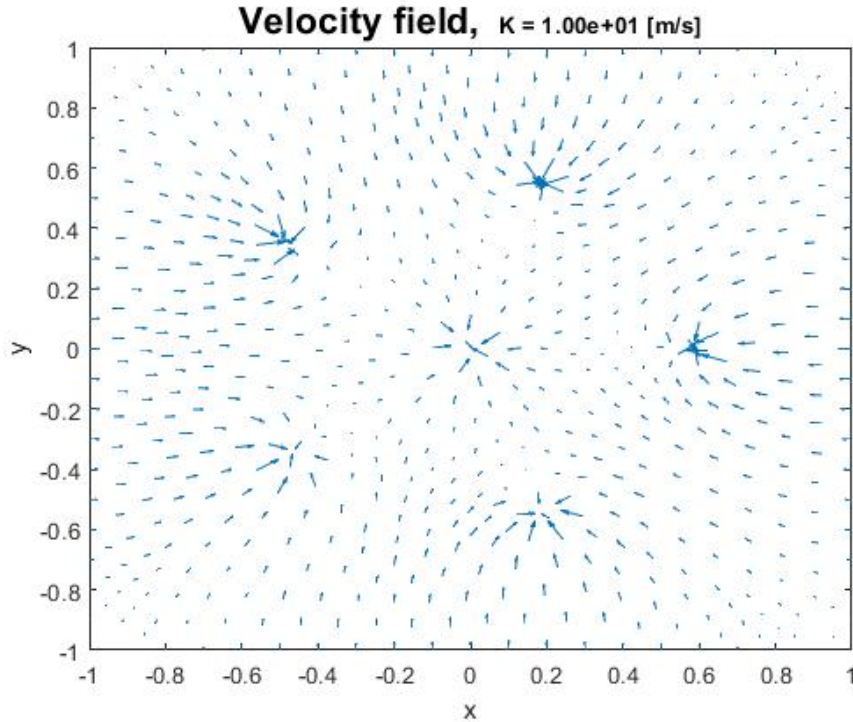


Figure 2.2: Arrow velocity plot, indicating the direction and velocity of the water(longer arrows indicate higher velocity).

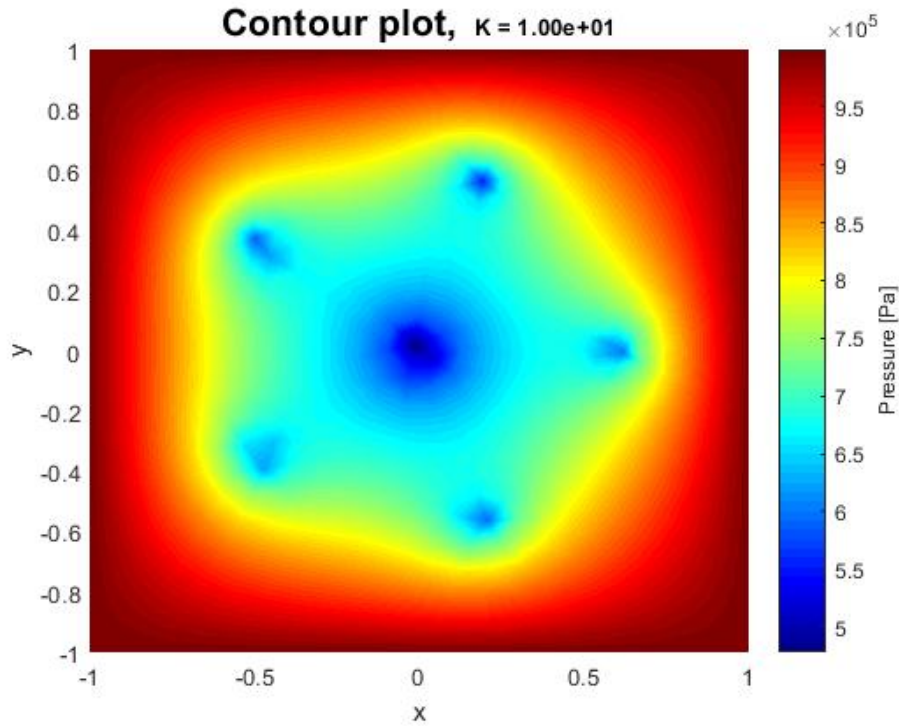


Figure 2.3: Contour plot of the velocity in the square reservoir on domain $\Omega = (-1, 1) \times (-1, 1)$, showing six areas where a drop of about two times the pressure can be observed, compared to the boundary pressure.

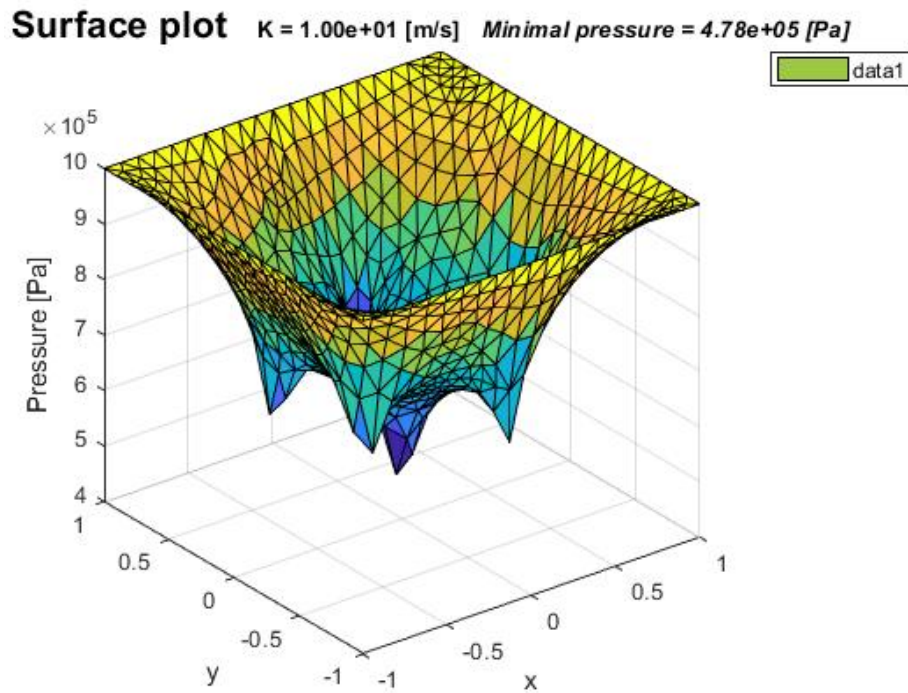


Figure 2.4: 3D surface plot

2.7 Varying constant K

Now that the velocities have been calculated the last thing to vary is, is the factor K . A few different plots will now follow in which the transfer coefficient K has different values between 0.00001 and 10000.

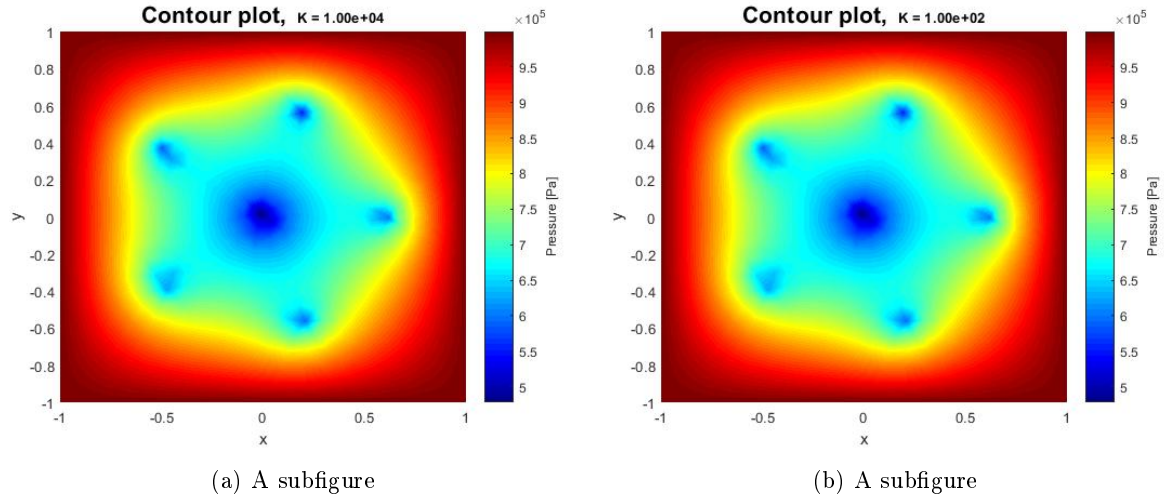


Figure 2.5: A figure with two subfigures

Show the contour plots, and give the values of the minimal pressure (which is important from an engineering point of view). Explain your results.

The final step is to determine what happens when $K = 0$ and why? When looking at our previous plots, when varying K from 0.00001 to 10000 it can be derived that When $K = 0$, the transfer coefficient of the water between the boundary of the square reservoir and its surroundings is zero. In the following plot the K factor is set to 0:

Appendix A

1D-case full script

```
1 clear all
2 close all
3
4 %%Finite Element 1D
5 %% Parameters
6
7 N_elem = 100; %Number of elements
8 int = [0,1]; %Interval
9 lambda = 1;
10 D = .1;
11
12 %% Mesh & Topology
13
14 mesh = GenerateMesh(int,N_elem);
15 elmat = GenerateTopology(N_elem); %1D topology!!
16
17 %% Assemble Matrix & Vector
18
19 S = AssembleMatrix( N_elem, lambda, D, mesh, elmat);
20 f = AssembleVector( N_elem, mesh, elmat);
21
22 %% Calculate u
23 x = linspace(int(1),int(2),N_elem);
24
25 u = S\f;
26
27 hold on
28 plot(x,u);
29 legend('N=100')
30 title('Solution for u')
31 xlabel('x')
32 ylabel('u')
33 ax.box='on'
34 hold off
35
36
37 % For this part change the function in functionBVP.m to 'f = sin(20*x)'
38
39 figure
40 hold on
41
42 for N_elem = [10 20 40 80 100 160]
43     mesh = GenerateMesh(int,N_elem);
44     elmat = GenerateTopology(N_elem);
45     S = AssembleMatrix( N_elem, lambda, D, mesh, elmat);
46     f = AssembleVector( N_elem, mesh, elmat);
47
48     x = linspace(int(1),int(2),N_elem);
49
50     u = S\f;
51     plot(x,u);
52
53
```

```
54 end
55
56 legend('N=10','N=20','N=40','N= 80','N=100','N=160')
57 title('Solution for u')
58 xlabel('x')
59 ylabel('u')
60 ax.box='on'
61 hold off
```


Appendix B

2D-case

B.1 Generate mesh

```
1 clear all
2
3 Geometry = 'squareg';
4
5 DiffCoeff = 1;
6 h_transfer = 1;
7 u_inf = 1;
8
9
10 % Geometry = 'squareg'; % gives square [-1,1] x [-1,1]
11 % Geometry = 'circleg'; % gives unit circle centered at origin
12 % Geometry = 'lshapeg'; % gives L-shape
13
14 [p,e,t] = initmesh(Geometry);
15 [p,e,t] = refinemesh(Geometry,p,e,t); % gives gridrefinement
16 [p,e,t] = refinemesh(Geometry,p,e,t); % gives second gridrefinement
17 % [p,e,t] = refinemesh(Geometry,p,e,t); % gives third gridrefinement
18 pdemesh(p,e,t); % plots the geometry and mesh
19
20 x = p(1,:); y = p(2,:);
21 n = length(p(1,:));
22
23 elmat = t(1:3,:);
24 elmat = elmat';
25 elmatbnd = e(1:2,:);
26 elmatbnd = elmatbnd';
27 % h
28 topology = 3; topologybnd = 2;
```

B.2 Generate element matrix

```
1 clear xc
2 clear yc
3 clear Selem
4
5 for index1 = 1:topology
6     xc(index1) = x(elmat(i,index1));
7     yc(index1) = y(elmat(i,index1));
8 end;
9
10 Delta = det([1 xc(1) yc(1); 1 xc(2) yc(2); 1 xc(3) yc(3)]);
11 B_mat = [1 xc(1) yc(1); 1 xc(2) yc(2); 1 xc(3) yc(3)] \ eye(3);
12
13 alpha = B_mat(1,1:3);
14 beta = B_mat(2,1:3);
15 gamma = B_mat(3,1:3);
```

```

16
17 for index1 = 1:topology
18     for index2 = 1:topology
19         if ~exist('u','var')
20             Selem(index1,index2) =
21                 abs(Delta)/2*(k/mu)*(beta(index1)*beta(index2)+gamma(index1)*gamma(index2));
22         else
23             Selem(index1,index2) = abs(Delta)/24;
24         end
25     end;
26 end;

```

B.3 Generate element vector

```

1
2 % Module for element mass matrix for reactive term
3 %
4 % Output: felem ===== vector of two components
5 %
6 % felem(1), felem(2) to be computed in this routine.
7
8 clear xc
9 clear yc
10 clear felem
11
12 for index1=1:topology
13     xc(index1) = x(elmat(i,index1));
14     yc(index1) = y(elmat(i,index1));
15 end;
16
17 Delta = det([1 xc(1) yc(1);1 xc(2) yc(2);1 xc(3) yc(3)]);
18
19 B_mat = [1 xc(1) yc(1);1 xc(2) yc(2);1 xc(3) yc(3)] \ eye(3);
20
21 alpha = B_mat(1,1:3);
22 beta = B_mat(2,1:3);
23 gamma = B_mat(3,1:3);
24
25 felem = zeros(1,topology);
26
27 if ~exist('u','var') % Only if u is already know can the calculation of the velocity ...
28     begin.
29     for N = 1:N_wells
30         for index3 = 1:topology
31             phi_p(index3) = alpha(index3) + beta(index3)*xp(N) + gamma(index3)*yp(N);
32         end
33         if (phi_p(1) ≤ 1) && (phi_p(1) ≥ 0) && (phi_p(2) ≤ 1) && (phi_p(2) ≥ 0) && ...
34             (phi_p(3) ≤ 1) && (phi_p(3) ≥ 0);
35         for index1 = 1:topology
36             felem(index1) = felem(index1) + -Qp*phi_p(index1);
37         end
38     %         N_Test = N_Test + 1;
39
40 % Components of f are zero except for those elements with a well! So no
41 % other contributions!
42 %     else
43 %         for index1 = 1:topology
44 %             global_index = elmat(N,index1);
45 %             end
46 %         end
47 %     end
48 else
49     switch direction
50     case 1 % x direction
51         for index1 = 1:topology
52             felem(index1) = felem(index1) + ...
53                 (k/mu)*(abs(Delta)/6)*beta(index1)*(u(elmat(i,1))+u(elmat(i,2))+u(elmat(i,3)));

```

```

53         end
54     case 2 % y direction
55         for index1 = 1:topology
56             felem(index1) = felem(index1) + ...
                    (k/mu)*(abs(Delta)/6)*gamma(index1)*(u(elmat(i,1))+u(elmat(i,2))+u(elmat(i,3)));
57         end
58     end
59 end

```

B.4 Generate Boundary element matrix

```

1  clear xc
2  clear yc
3  clear BMelem
4
5  for index1=1:topologybnd
6      xc(index1) = x(elmatbnd(i,index1));
7      yc(index1) = y(elmatbnd(i,index1));
8  end;
9
10 lek = sqrt((xc(2)-xc(1))^2 + (yc(2)-yc(1))^2);
11
12 for index1=1:topologybnd
13     if ~exist('u','var')
14         BMelem(index1,index1) = K*lek/2; % NC used! not HB!!
15     else
16
17         BMelem(index1,index1) = -(k/(mu*K))*lek/6;
18     end
19 end;

```

B.5 Generate boundary element vector:

```

1  clear xc
2  clear yc
3  clear bfelem
4
5  for index1 = 1:topologybnd
6      xc(index1) = x(elmatbnd(i,index1));
7      yc(index1) = y(elmatbnd(i,index1));
8  end;
9
10 lek = sqrt((xc(2)-xc(1))^2+(yc(2)-yc(1))^2);
11
12 if ~exist('u','var')
13     for index1 = 1:topologybnd
14         bfelem(index1) = K*pH*lek/2*u_inf; %what is u_inf?
15     end;
16 else
17     for index1 = 1:topologybnd
18         bfelem(index1) = ((k*pH)/mu)*lek/2*u_inf; %what is u_inf?
19     % bfelem(index1) = -(k/mu)*lek/6*u(elmat(i,ind1));
20 end
21 end

```

B.6 Buildmatrices and vectors

```

1 % This routine constructs the large matrices and vector.
2 % The element matrices and vectors are also dealt with.
3 % First the internal element contributions

```

```

4 % First Initialisation of large discretisation matrix, right-hand side vector
5
6 % Treatment of the internal (triangular) elements
7
8 if ~exist('u', 'var')
9
10     S      = sparse(n,n); % stiffness matrix
11     f      = zeros(n,1); % right-hand side vector
12
13     for i = 1:length(elmat(:,1)) % for all internal elements
14         GenerateElementMatrix; % Selem
15         for ind1 = 1:topology
16             for ind2 = 1:topology
17                 S(elmat(i,ind1),elmat(i,ind2)) = S(elmat(i,ind1),elmat(i,ind2)) + ...
18                     Selem(ind1,ind2);
19             end;
20         end;
21
22         GenerateElementVector; % felem
23         for ind1 = 1:topology
24             f(elmat(i,ind1)) = f(elmat(i,ind1)) + felem(ind1);
25         end;
26     end;
27 % Next the boundary contributions
28
29     for i = 1:length(elmatbnd(:,1)); % for all boundary elements extension of mass ...
30         matrix M and element vector f
31         GenerateBoundaryElementMatrix; % BMelem
32         for ind1 = 1:topologybnd
33             for ind2 = 1:topologybnd
34                 S(elmatbnd(i,ind1),elmatbnd(i,ind2)) = ...
35                     S(elmatbnd(i,ind1),elmatbnd(i,ind2)) + BMelem(ind1,ind2);
36             end;
37         end;
38         GenerateBoundaryElementVector; % bfelem
39         for ind1 = 1:topologybnd
40             f(elmatbnd(i,ind1)) = f(elmatbnd(i,ind1)) + bfelem(ind1);
41         end;
42     end;
43
44     Sx      = sparse(n,n); % stiffness matrix
45
46     fx      = zeros(n,1); % right-hand side vector
47
48     left_nodes = find(p(1,:) == -1);
49     top_nodes = find(p(2,:) == 1);
50     right_nodes = find(p(1,:) == 1);
51     bottom_nodes = find(p(2,:) == -1);
52
53     bnd1_nodes = ismember(elmatbnd,left_nodes);
54     bnd1 = find(bnd1_nodes(:,1) == 1 & bnd1_nodes(:,2) == 1);
55
56     bnd2_nodes = ismember(elmatbnd,top_nodes);
57     bnd2 = find(bnd2_nodes(:,1) == 1 & bnd2_nodes(:,2) == 1);
58
59     bnd3_nodes = ismember(elmatbnd,right_nodes);
60     bnd3 = find(bnd3_nodes(:,1) == 1 & bnd3_nodes(:,2) == 1);
61
62     bnd4_nodes = ismember(elmatbnd,bottom_nodes);
63     bnd4 = find(bnd4_nodes(:,1) == 1 & bnd4_nodes(:,2) == 1);
64
65     direction = 1;
66
67     for i = 1:length(elmat(:,1)) % for all internal elements
68         GenerateElementMatrix; % Selem
69         for ind1 = 1:topology
70             for ind2 = 1:topology
71                 if elmat(i,ind1) == elmat(i,ind2)

```

```

73         Sx(elmat(i,ind1),elmat(i,ind2)) = Sx(elmat(i,ind1),elmat(i,ind2)) ...
           + 2*Selem(ind1,ind2);
74     else
75         Sx(elmat(i,ind1),elmat(i,ind2)) = Sx(elmat(i,ind1),elmat(i,ind2)) ...
           + Selem(ind1,ind2);
76     end
77     end;
78 end;
79 GenerateElementVector; % felem
80 for ind1 = 1:topology
81     fx(elmat(i,ind1)) = fx(elmat(i,ind1)) + felem(ind1);
82 end;
83 end;
84
85 % Next the boundary contributions
86
87
88
89 for j = 1:length(bnd1); % left boundary
90     i = bnd1(j);
91     GenerateBoundaryElementMatrix; % BMelem
92     for ind1 = 1:topologybnd
93         for ind2 = 1:topologybnd
94             if elmatbnd(i,ind1) == elmatbnd(i,ind2)
95                 Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) = ...
                   Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) + 2*BMelem(ind1,ind2);
96             else
97                 Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) = ...
                   Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) + BMelem(ind1,ind2);
98             end;
99         end
100     end;
101     GenerateBoundaryElementVector; % bfelem
102     for ind1 = 1:topologybnd
103         fx(elmatbnd(i,ind1)) = fx(elmatbnd(i,ind1)) + bfelem(ind1);
104     end;
105 end;
106
107 for j = 1:length(bnd3); % right boundary
108     i = bnd3(j);
109     GenerateBoundaryElementMatrix; % BMelem
110     for ind1 = 1:topologybnd
111         for ind2 = 1:topologybnd
112             if elmatbnd(i,ind1) == elmatbnd(i,ind2)
113                 Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) = ...
                   Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) + 2*BMelem(ind1,ind2);
114             else
115                 Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) = ...
                   Sx(elmatbnd(i,ind1),elmatbnd(i,ind2)) + BMelem(ind1,ind2);
116             end;
117         end
118     end;
119     GenerateBoundaryElementVector; % bfelem
120     for ind1 = 1:topologybnd
121         fx(elmatbnd(i,ind1)) = fx(elmatbnd(i,ind1)) - bfelem(ind1);
122     end;
123 end;
124
125 direction = 2;
126
127 Sy      = sparse(n,n); % stiffness matrix
128
129 fy      = zeros(n,1); % right-hand side vector
130
131 for i = 1:length(elmat(:,1)) % for all internal elements
132     GenerateElementMatrix; % Selem
133     for ind1 = 1:topology
134         for ind2 = 1:topology
135             if elmat(i,ind1) == elmat(i,ind2)
136                 Sy(elmat(i,ind1),elmat(i,ind2)) = Sy(elmat(i,ind1),elmat(i,ind2)) ...
                   + 2*Selem(ind1,ind2);
137             else

```

```

138         Sy(elmat(i,ind1),elmat(i,ind2)) = Sy(elmat(i,ind1),elmat(i,ind2)) ...
           + Selem(ind1,ind2);
139     end
140 end;
141 end;
142 GenerateElementVector; % felem
143 for ind1 = 1:topology
144     fy(elmat(i,ind1)) = fy(elmat(i,ind1)) + felem(ind1);
145 end;
146 end;
147
148 % Next the boundary contributions
149
150
151
152 for j = 1:length(bnd2); % left boundary
153     i = bnd2(j);
154     GenerateBoundaryElementMatrix; % BMelem
155     for ind1 = 1:topologybnd
156         for ind2 = 1:topologybnd
157             if elmatbnd(i,ind1) == elmatbnd(i,ind2)
158                 Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) = ...
                   Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) + 2*BMelem(ind1,ind2);
159             else
160                 Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) = ...
                   Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) + BMelem(ind1,ind2);
161             end;
162         end
163     end;
164     GenerateBoundaryElementVector; % bfelem
165     for ind1 = 1:topologybnd
166         fy(elmatbnd(i,ind1)) = fy(elmatbnd(i,ind1)) - bfelem(ind1);
167     end;
168 end;
169
170 for j = 1:length(bnd4); % right boundary
171     i = bnd4(j);
172     GenerateBoundaryElementMatrix; % BMelem
173     for ind1 = 1:topologybnd
174         for ind2 = 1:topologybnd
175             if elmatbnd(i,ind1) == elmatbnd(i,ind2)
176                 Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) = ...
                   Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) + 2*BMelem(ind1,ind2);
177             else
178                 Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) = ...
                   Sy(elmatbnd(i,ind1),elmatbnd(i,ind2)) + BMelem(ind1,ind2);
179             end;
180         end
181     end;
182     GenerateBoundaryElementVector; % bfelem
183     for ind1 = 1:topologybnd
184         fy(elmatbnd(i,ind1)) = fy(elmatbnd(i,ind1)) + bfelem(ind1);
185     end;
186 end;
187 end

```

B.7 Compute u and v_x/v_y

```

1 % Construction of linear problem
2
3 BuildMatricesandVectors;
4
5 % Solution of linear problem
6
7 u = S \ f;
8
9 BuildMatricesandVectors;
10
11 vx = Sx \ fx;

```

```
12 vy = Sy \ fy;
```

B.8 Full script

```
1 close all
2 clear all
3
4 %% 2D Assignment
5 % Lab Assignment 7
6
7 %% Create Mesh
8 WI4243Mesh
9
10 %% Parameters
11
12 Qp = 50;           % [m^2/s]
13 k = 10^-7;         % [m^2]
14 mu = 1.002*10^-3;  % [Pa*s]
15 K = 10000;         % [m/s]
16 pH = 10^6;         % [Pa]
17 N_wells = 6;       % number of wells
18
19 epsilon1 = 0.03;
20 N_Test = 0;
21 %% Coordinates of wells
22
23 for i = 1:N_wells-1;
24 xp(i) = 0.6*cos((2*pi)*(i-1)/(N_wells-1));
25 yp(i) = 0.6*sin((2*pi)*(i-1)/(N_wells-1));
26 end
27
28 xp(N_wells) = 0;
29 yp(N_wells) = 0;
30 clear i;
31
32
33 %% Compute Problem
34 WI4243Comp
35
36 %% Post
37 hold on
38
39 figure(2);
40 ax.BoxStyle = 'full';
41 hold off
42 trisurf(elmat,x,y,u)
43 xlabel('x'); ylabel('y'); zlabel('Pressure [Pa]');
44 title(['\bf\fontsize{16}3D Surface plot \fontsize{10} K = ' num2str(K,'%10.2e\n') ' ...
45        [m/s] \it Minimal pressure = ' num2str(min(u), '%10.2e\n') ' [Pa]']);
46
47 lgd = legend();
48 % title(lgd,['3D Surface plot, K = ' num2str(K) '\it Minimal pressure = ' ...
49 %        num2str(Pressure_minimum)]);
50
51 % title( {'Title';'subtitle'} )
52
53 figure(3);
54 trisurf(elmat,x,y,u);
55 xlabel('x'); ylabel('y');
56 title(['\bf\fontsize{16}Contour plot, \fontsize{10} K = ',num2str(K,'%10.2e\n')]);
57 view(2); shading interp; colormap jet; colorbar; set(gcf,'renderer','zbuffer')
58 h = colorbar; ylabel(h, 'Pressure [Pa]');
59
60
61 figure(4); quiver(x,y,vx',vy'); axis([-1 1 -1 1]);
62 xlabel('x'); ylabel('y');
63 title(['\bf\fontsize{16}Velocity field, \fontsize{10} K = ' num2str(K,'%10.2e\n') ' ...
64        [m/s]']);
65 %% Velocity part
```

