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1. Case 1 : $\Sigma_i = \sigma^2 I$

$g_i(x) = W_i^t x + W_{i0}$, where $W_i = \frac{\mu_i}{\sigma^2}$ and $W_{i0} = -\frac{1}{2\sigma^2}\mu_i^t \mu + \ln P(W_i)$

$= -\frac{1}{2\sigma^2}\mu_i^2 + \ln P(W_i)$

$W_j = \frac{\mu_j}{\sigma^2}$

$W_{j0} = -\frac{1}{2\sigma^2}\mu_j^t \mu + \ln P(W_j)$

$\Rightarrow g_i(x_0) = g_j(x_0)$, $i \neq j$

$W_i^t x_0 + W_{i0} = W_j^t x_0 + W_{j0}$

$\Rightarrow (W_i^t - W_j^t) x_0 = W_{j0} - W_{i0}$

$\Rightarrow \frac{1}{\sigma^2}(\mu_i - \mu_j) x_0 = -\frac{1}{2\sigma^2}\mu_j^2 + \ln P(W_j) + \frac{1}{2\sigma^2}\mu_i^2 - \ln P(W_i)$

$= \frac{1}{2\sigma^2}(\mu_i^2 - \mu_j^2) + \ln \frac{P(W_j)}{P(W_i)}$

$\Rightarrow (\mu_i - \mu_j) x_0 = \frac{1}{2}(\mu_i^2 - \mu_j^2) + \sigma^2 \ln \frac{P(\mu_j)}{P(\mu_i)}$

$\Rightarrow x_0 = \frac{1}{2} \cdot \frac{(\mu_i^2 - \mu_j^2)}{\mu_i - \mu_j} + \frac{\sigma^2}{\mu_i - \mu_j} \ln \frac{P(\mu_j)}{P(\mu_i)}$

$\ln \frac{x}{y} = \ln \frac{1}{\frac{y}{x}}$

$= -\ln \frac{y}{x}$

$= \frac{1}{2}(\mu_i + \mu_j) + \frac{\sigma^2(\mu_i - \mu_j)}{(\mu_i - \mu_j)^2} \ln \frac{P(\mu_j)}{P(\mu_i)}$

$= \frac{1}{2}(\mu_i + \mu_j) - \frac{\sigma^2(\mu_i - \mu_j)}{\|\mu_i - \mu_j\|^2} \ln \frac{P(\mu_i)}{P(\mu_j)}$

2. Case 2 : $\Sigma_i = \Sigma$

$$g_i(x) = (\Sigma^{-1}\mu_i)^t x + w_{i0}$$

where $w_{i0} = -\frac{1}{2}\mu_i^t \Sigma^{-1}\mu_i + \ln P(w_i)$

$$\Rightarrow g_i(x_0) = g_j(x_0)$$

$$\Rightarrow (\Sigma^{-1}\mu_i)^t x_0 + w_{i0} = (\Sigma^{-1}\mu_j)^t x_0 + w_{j0}$$

$$\Rightarrow [\Sigma^{-1}\mu_i - \Sigma^{-1}\mu_j]^t x_0 = w_{j0} - w_{i0}$$

$$\Rightarrow [\Sigma^{-1}(\mu_i - \mu_j)]^t x_0 = w_{j0} - w_{i0}$$

$$\Rightarrow (\mu_i - \mu_j)^t (\Sigma^{-1})^t x_0 = w_{j0} - w_{i0}$$

$$= -\frac{1}{2}\mu_j^t \Sigma^{-1}\mu_j + \ln P(w_j) + \frac{1}{2}\mu_i^t \Sigma^{-1}\mu_i - \ln P(w_i)$$

$$= \frac{1}{2}(\mu_i - \mu_j)^t \Sigma^{-1}(\mu_i + \mu_j) - \ln\frac{P(w_i)}{P(w_j)}$$

$$(AB)^t = A^t B^t$$

$$\Rightarrow (\Sigma^{-1})^t = \Sigma^{-1}$$

$$\mu_i^t \Sigma^{-1}\mu_y = \mu_y^t \Sigma^{-1}\mu_i$$

$$\Rightarrow x_0 = \frac{\frac{1}{2}(\mu_i - \mu_j)^t \Sigma^{-1}(\mu_i + \mu_j) - \ln\frac{P(w_i)}{P(w_j)}}{(\mu_i - \mu_j)^t (\Sigma^{-1})^t}$$

$$= \frac{1}{2}(\mu_i + \mu_j) - \frac{\ln\frac{P(w_i)}{P(w_j)}}{(\mu_i - \mu_j)^t \Sigma^{-1}}$$

$$= \frac{1}{2}(\mu_i + \mu_j) - \frac{\ln\frac{P(w_i)}{P(w_j)}}{(\mu_i - \mu_j)^t \Sigma^{-1}} \cdot \frac{\mu_i - \mu_j}{\mu_i - \mu_j}$$

$$= \frac{1}{2}(\mu_i + \mu_j) - \frac{(\mu_i - \mu_j)\ln\frac{P(w_i)}{P(w_j)}}{(\mu_i - \mu_j)\Sigma^{-1}(\mu_i - \mu_j)^t}$$

3. $\mu_1 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}; \Sigma_1 \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \Sigma_1^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

$\mu_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}; \Sigma_2 \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \Sigma_2^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

$g_i(x) = x^t W_i x + w_i^t x + w_{i0}$

where $W_i = -\frac{1}{2} \Sigma_i^{-1}$

$w_i = \Sigma_i^{-1} \mu_i$

$w_{i0} = -\frac{1}{2} \mu_i^t \Sigma_i^{-1} \mu_i - \frac{1}{2} \ln |\Sigma_i| + \ln P(\omega_i)$

$\Rightarrow g_i(x) = g_j(x)$

$\Rightarrow x^t W_i x + w_i^t x + w_{i0} = x^t W_j x + w_j^t x + w_{j0}$

$\Rightarrow x^t W_i x + w_i^t x - x^t W_j x - w_j^t x = w_{j0} - w_{i0}$

$\Rightarrow x^t W_i x + w_i^t x - x^t W_j x - w_j^t x = -\frac{1}{2} \mu_j^t \Sigma_j^{-1} \mu_j - \frac{1}{2} \ln |\Sigma_j| + \ln P(\omega_j)$

$+ \frac{1}{2} \mu_i^t \Sigma_i^{-1} \mu_i + \frac{1}{2} \ln |\Sigma_i| + \ln P(\omega_i)$

$= \frac{1}{2} \left[\mu_i^t \Sigma_i^{-1} \mu_i - \mu_j^t \Sigma_j^{-1} \mu_j - \ln \frac{\Sigma_j}{\Sigma_i} \right] - \ln \frac{P(\omega_i)}{P(\omega_j)}$

$= \frac{1}{2} \left[[3, 6] \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} - [3, -2] \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} - \ln \frac{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}}{\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}} \right] - \ln \frac{\frac{1}{2}}{\frac{1}{2}}$

$= \frac{1}{2} \left[[3, 6] \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} - [3, -2] \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} - \ln \frac{2 \times 2 - 0 \times 0}{\frac{1}{2} \times 2 - 0 \times 0} \right] - \ln 1$

$= \frac{1}{2} \left[[3, 6] \begin{bmatrix} 2 \times 3 + 0 \times 3 \\ 0 \times 3 + \frac{1}{2} \times 6 \end{bmatrix} - [3, -2] \begin{bmatrix} \frac{1}{2} \times 3 + 0 \times 3 \\ 0 \times (-2) + \frac{1}{2}(-2) \end{bmatrix} - \ln 4 \right] - 0$

$= \frac{1}{2} \left[[3, 6] \begin{bmatrix} 6 \\ 3 \end{bmatrix} - [3, -2] \begin{bmatrix} \frac{3}{2} \\ -1 \end{bmatrix} - \ln 4 \right]$

$= \frac{1}{2} \left[3 \times 6 + 6 \times 3 - \left(3 \times \frac{3}{2} + (-2)(-1) \right) - \ln 4 \right]$

$= \frac{1}{2} \left[36 - \frac{9}{2} - 2 - \ln 4 \right]$

$= \frac{1}{2} (29.5 - \ln 4)$

$\Rightarrow W_1 = -\frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}$

$W_2 = -\frac{1}{2} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}$

$\Rightarrow w_1 = \Sigma_1^{-1} \mu_1 = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

$w_2 = \Sigma_2^{-1} \mu_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -1 \end{bmatrix}$

$x^t W_i x + w_i^t x - x^t W_j x - w_j^t x$

$= [x_1, x_2] \begin{bmatrix} -1 & 0 \\ 0 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [6, 3] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - [x_1, x_2] \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - [\frac{3}{2}, -1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= [x_1, x_2] \begin{bmatrix} -x_1 + 0 \\ 0 - \frac{1}{4} x_2 \end{bmatrix} + (6x_1 + 3x_2) - [x_1, x_2] \begin{bmatrix} -\frac{1}{4} x + 0 \\ 0 + (-\frac{1}{4}) x_2 \end{bmatrix} - (\frac{3}{2} x_1 + (-x_2))$

$= [x_1, x_2] \begin{bmatrix} -x_1 \\ -\frac{1}{4} x_2 \end{bmatrix} + 6x_1 + 3x_2 - [x_1, x_2] \begin{bmatrix} -\frac{1}{4} x_1 \\ -\frac{1}{4} x_2 \end{bmatrix} - \frac{3}{2} x_1 + x_2$

$= -x_1^2 + x_2 (-\frac{1}{4} x_2) + 6x_1 + 3x_2 - [(-\frac{1}{4} x_1^2) + x_2 (-\frac{1}{4} x_2)] - \frac{3}{2} x_1 + x_2$

$= -x_1^2 - \frac{1}{4} x_2^2 + 6x_1 + 3x_2 + \frac{1}{4} x_1^2 + \frac{1}{4} x_2^2 - \frac{3}{2} x_1 + x_2$

$= -\frac{3}{4} x_1^2 + \frac{9}{2} x_1 + 4x_2$

$\Rightarrow -\frac{3}{4} x_1^2 + \frac{9}{2} x_1 + 4x_2 = \frac{1}{2} (29.5 - \ln 4)$

$= 14.75 - 0.69 = 14.06$

$\Rightarrow 4x_2 = 14.06 + \frac{3}{4} x_1^2 - \frac{9}{2} x_1$

$\Rightarrow x_2 = 3.514 + \frac{3}{16} x_1^2 - \frac{9}{8} x_1$

$\Rightarrow x_2 = 3.514 + 0.1875 x_1^2 - 1.125 x_1$