Solutions for [Book Name]

Your Name

April 30, 2024

1 Chapter 1

Exercise 1.1

Exercise description.

Solution

Write your solution here.

Exercise 1.2

Exercise description.

Solution

Another solution here.

2 Chapter 2

Exercise 2.1

Exercise description.

Solution

Write your solution here.

Exercise 2.2

Exercise description.

Solution

Another solution here.

3 Chapter 3

Exercise 3.1

Solution

Exercise 3.2

Solution

Exercise 3.3

Solution

Exercise 3.4

Solution

Exercise 3.5

(a) Show that if $X \sim N(\mu, \sigma^2)$, then $\mathbb{E}[X] = \mu$ and $\mathrm{Var}(X) = \sigma^2$.

(b) Let $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ be independent random variables. Show that $X_1 + X_2$ is normally distributed, and find the mean and variance of $X_1 + X_2$.

(c) Let $X \sim N(0, \sigma^2)$. Show that for $n = 0, 1, 2, \ldots$,

$$\mathbb{E}[X^{2n+1}] = 0,$$

and

$$\mathbb{E}[X^{2n}] = (2n-1)!! \cdot \sigma^{2n} = 1 \cdot 3 \cdot 5 \ldots \cdot (2n-1) \cdot \sigma^{2n}.$$

Here, $(2n-1)!!$ denotes the double factorial of $2n-1$.

Solution

(a) Given a normal random variable $X \sim N(\mu, \sigma^2)$, its characteristic function $\psi_X(t)$ is expressed as:

$$\psi_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}.$$

The expected value $\mathbb{E}[X]$ is the coefficient of t in the Taylor expansion of $\psi_X(t)$ around $t = 0$, which yields:

$$\mathbb{E}[X] = \frac{d}{dt}\psi_X(t)\bigg|_{t=0} = \mu.$$

To find the variance $\mathrm{Var}(X)$, we compute the second derivative of $\psi_X(t)$ at $t = 0$:

$$\mathrm{Var}(X) = \frac{d^2}{dt^2}\psi_X(t)\bigg|_{t=0} - (\mu)^2 = \sigma^2.$$

(b) Let $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ be independent random variables. To show that the sum $X_1 + X_2$ is also normally distributed and to find its parameters, consider their moment generating functions:

$$\Psi_{X_1}(t) = e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2}, \quad \Psi_{X_2}(t) = e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2}.$$

Since X_1 and X_2 are independent, the MGF of their sum is the product of their MGFs:

$$\Psi_{X_1+X_2}(t) = \Psi_{X_1}(t) \cdot \Psi_{X_2}(t) = e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2} \cdot e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2}.$$

Simplify by combining the exponents:

$$\Psi_{X_1+X_2}(t) = e^{(\mu_1+\mu_2)t + \frac{1}{2}(\sigma_1^2+\sigma_2^2)t^2}.$$

This is the MGF of a normal distribution with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. Therefore, $X_1 + X_2$ follows a normal distribution $N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

(c) Let $X \sim N(0, \sigma^2)$. The characteristic function $\psi_X(t)$, which also serves as the moment generating function in this context, is given by:

$$\psi_X(t) = e^{\frac{1}{2}\sigma^2 t^2}.$$

Expanding $\psi_X(t)$ using a Taylor series around $t = 0$ results in:

$$\psi_X(t) = \sum_{n=0}^{\infty} \frac{\frac{1}{2}\sigma^2 t^2}{n!} t^{2n} = 1 + \frac{1}{2}\sigma^2 t^2 + \frac{(\frac{1}{2}\sigma^2 t^2)^2}{2!} + \frac{(\frac{1}{2}\sigma^2 t^2)^3}{3!} + \ldots = 1 + \frac{\sigma^2 t^2}{2} + \frac{\sigma^4 t^4}{2^2 \cdot 2!} + \frac{\sigma^6 t^6}{2^3 \cdot 3!} + \ldots$$

This series only contains even powers of t, confirming that all coefficients of odd powers of t are zero, thus:

$$\mathbb{E}[X^{2n+1}] = 0$$

for all odd powers $2n+1$. This occurs because the derivatives of $\psi_X(t)$ at $t = 0$ for odd orders are zero, as each term in the expansion of $\psi_X(t)$ contains even powers.

For even powers, consider the coefficient of t^{2n} in the Taylor expansion:

$$\mathbb{E}[X^{2n}] = \frac{d^{2n}}{dt^{2n}} \psi_X(t) \bigg|_{t=0} = \frac{d^{2n}}{dt^{2n}} \left(\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{2}\sigma^2 t^2 \right)^k \right) \bigg|_{t=0}$$

To see why $\mathbb{E}[X^{2n}]$ equals $(2n-1)!!\sigma^{2n}$, take the $2n$-th derivative:

$$\mathbb{E}[X^{2n}] = \frac{1}{n!} \left(\frac{1}{2}\sigma^2 \right)^n \cdot 2^n \cdot (2n)! = \sigma^{2n} \cdot (2n-1)!!$$

This computation correctly reflects the product of the double factorial $(2n-1)!!$ which is the product of all odd numbers up to $(2n-1)$, resulting in:

$$(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1) \cdot (\sigma^{2n}).$$

Exercise 3.6

(a) Show that if $X \sim N(0, 1)$ then $X^2 \sim \chi^2(1)$ by computing the moment generating function (MGF) of X^2, that is, by showing that

$$\psi_{X^2}(t) = \mathbb{E}[\exp(tX^2)] = \frac{1}{\sqrt{1-2t}} \quad \text{for} \quad t < \frac{1}{2}.$$

(b) Show that if $X_1 \sim N(0, 1)$ and $X_2 \sim N(0, 1)$ are independent, then $X_1^2 + X_2^2$ is distributed as $\chi^2(2)$ (which is equivalent to an exponential distribution with mean 2).

Solution

(a) Begin by recognizing the integral for the MGF:

$$\psi_{X^2}(t) = \int_{-\infty}^{\infty} e^{tx^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{x^2(t-\frac{1}{2})} \, dx.$$

This integral converges for $t < \frac{1}{2}$. Transform x to eliminate the variable change explicitly:

$$\frac{d(x\sqrt{1-2t})}{dx} = \sqrt{1-2t}, \quad dx = \frac{d(x\sqrt{1-2t})}{\sqrt{1-2t}}$$

Substitute directly:

$$\psi_{X^2}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x\sqrt{1-2t})^2}{2}} \frac{d(x\sqrt{1-2t})}{\sqrt{1-2t}} = \frac{1}{\sqrt{1-2t}}.$$

The integral of the standard normal density over the transformed variable is 1, leading to the final MGF expression for X^2.

(b) Given that $X_1 \sim N(0,1)$ and $X_2 \sim N(0,1)$ are independent, to show that $X_1^2 + X_2^2$ is distributed as $\chi^2(2)$, consider the moment generating functions (MGFs) of X_1^2 and X_2^2, which are:

$$\psi_{X_1^2}(t) = \psi_{X_2^2}(t) = \frac{1}{\sqrt{1-2t}} \quad \text{for} \quad t < \frac{1}{2}.$$

Since X_1^2 and X_2^2 are independent, the MGF of their sum, $X_1^2 + X_2^2$, is the product of their MGFs:

$$\psi_{X_1^2+X_2^2}(t) = \psi_{X_1^2}(t) \cdot \psi_{X_2^2}(t) = \left(\frac{1}{\sqrt{1-2t}}\right)^2 = \frac{1}{1-2t}.$$

This MGF, $\frac{1}{1-2t}$, is the MGF of a χ^2 distribution with 2 degrees of freedom. The $\chi^2(2)$ distribution is also known to be equivalent to an exponential distribution with mean 2, confirming the distribution of $X_1^2 + X_2^2$.

4 Chapter 4

Exercise 4.1

(a) For a Bernoulli random variable $X \sim \text{Be}(p)$:

$$\varphi_{\text{Be}(p)}(t) = q + pe^{it}, \quad \text{where } q = 1-p.$$

(b) For a Binomial random variable $Y \sim \text{Bin}(n,p)$:

$$\varphi_{\text{Bin}(n,p)}(t) = (q + pe^{it})^n.$$

(c) For a compound Poisson random variable Z with rate λ and jump size distribution C:

$$\varphi_C(t) = \frac{p}{1 - qe^{ist}},$$

assuming a specific relationship between the parameters p and q, and s.

(d) For a compound Poisson random variable W with intensity m and jump size distribution P:

$$\varphi_{P*\theta(m)}(t) = \exp\left[m(e^{it} - 1)\right].$$

Solution

(a) **Bernoulli Distribution** $X \sim \text{Be}(p)$:

$$\varphi_{\text{Be}(p)}(t) = \mathbb{E}[e^{itX}] = \sum_{x=0}^{1} e^{itx} \Pr(X = x) = e^{it\cdot 0} \Pr(X = 0) + e^{it\cdot 1} \Pr(X = 1) = (1-p) + pe^{it}.$$

This is exactly the expression given: $q + pe^{it}$, where $q = 1 - p$.

(b) **Binomial Distribution** $Y \sim \text{Bin}(n,p)$: The characteristic function of a sum of independent identically distributed random variables (by the property often called the *factorization property*) is:

$$\varphi_{\text{Bin}(n,p)}(t) = [\varphi_{\text{Be}(p)}(t)]^n = (q + pe^{it})^n.$$

This uses the property that the characteristic function of the sum of independent random variables is the product of their characteristic functions.

(c) **Compound Poisson Distribution** (Specific Case):

$$\varphi_C(t) = \mathbb{E}[e^{itC}] = \sum_{x=0}^{\infty} e^{itx} \Pr(C = x) = \sum_{x=0}^{\infty} e^{itx} \frac{pq^x}{1-q} = \frac{p}{1 - qe^{it}},$$

where we used the formula for the sum of a geometric series $\sum_{x=0}^{\infty} ar^x = \frac{a}{1-r}$ applied to e^{it} as r.

(d) **Poisson Distribution**: Assuming a Poisson process with parameter λ and counting the number of events N with intensity m:

$$\varphi_{P*\theta(m)}(t) = \mathbb{E}\left[e^{it\sum_{k=1}^{N} X_k}\right] = \mathbb{E}\left[\prod_{k=1}^{N} e^{itX_k}\right] = \exp\left[m(e^{it}-1)\right].$$

This follows from the definition of the characteristic function of the Poisson distribution and using the independence of increments X_k. Each X_k contributes e^{it} to the product, and the expectation over the Poisson-distributed count N produces the exponential function.

Exercise 4.2

Exercise description.

Solution

Another solution here.