Solutions for [Book Name]

Your Name

April 29, 2024

1 Chapter 1

Exercise 1.1

Exercise description.

Solution

Write your solution here.

Exercise 1.2

Exercise description.

Solution

Another solution here.

2 Chapter 2

Exercise 2.1

Exercise description.

Solution

Write your solution here.

Exercise 2.2

Exercise description.

Solution

Another solution here.

3 Chapter 3

Exercise 3.1

Exercise description.

Solution

Write your solution here.

Exercise 3.2

Exercise description.

(a) Show that if $X \sim N(\mu, \sigma^2)$, then $\mathbb{E}[X] = \mu$ and $\text{Var}(X) = \sigma^2$.

(b) Let $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ be independent random variables. Show that $X_1 + X_2$ is normally distributed, and find the mean and variance of $X_1 + X_2$.

(c) Let $X \sim N(0, \sigma^2)$. Show that for $n = 0, 1, 2, \ldots$,

$$\mathbb{E}[X^{2n+1}] = 0,$$

and

$$\mathbb{E}[X^{2n}] = (2n-1)!! \cdot \sigma^{2n} = 1 \cdot 3 \cdot 5 \ldots \cdot (2n-1) \cdot \sigma^{2n}.$$

Here, $(2n-1)!!$ denotes the double factorial of $2n-1$.

Solution

(a) Given a normal random variable $X \sim N(\mu, \sigma^2)$, its characteristic function $\Psi_X(t)$ is expressed as:

$$\Psi_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}.$$

The expected value $\mathbb{E}[X]$ is the coefficient of t in the Taylor expansion of $\Psi_X(t)$ around $t = 0$, which yields:

$$\mathbb{E}[X] = \frac{d}{dt}\Psi_X(t)\bigg|_{t=0} = \mu.$$

To find the variance $\text{Var}(X)$, we compute the second derivative of $\Psi_X(t)$ at $t = 0$:

$$\text{Var}(X) = \frac{d^2}{dt^2}\Psi_X(t)\bigg|_{t=0} - (\mu)^2 = \sigma^2.$$

(b) Let $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ be independent random variables. To show that the sum $X_1 + X_2$ is also normally distributed and to find its parameters, consider their moment generating functions:

$$\Psi_{X_1}(t) = e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2}, \quad \Psi_{X_2}(t) = e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2}.$$

Since X_1 and X_2 are independent, the MGF of their sum is the product of their MGFs:

$$\Psi_{X_1+X_2}(t) = \Psi_{X_1}(t) \cdot \Psi_{X_2}(t) = e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2} \cdot e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2}.$$

Simplify by combining the exponents:

$$\Psi_{X_1+X_2}(t) = e^{(\mu_1+\mu_2)t + \frac{1}{2}(\sigma_1^2+\sigma_2^2)t^2}.$$

This is the MGF of a normal distribution with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. Therefore, $X_1 + X_2$ follows a normal distribution $N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

(c) Let $X \sim N(0, \sigma^2)$. The characteristic function $\Psi_X(t)$, which also serves as the moment generating function in this context, is given by:

$$\Psi_X(t) = e^{\frac{1}{2}\sigma^2 t^2}.$$

Expanding $\Psi_X(t)$ using a Taylor series around $t = 0$ results in:

$$\Psi_X(t) = \sum_{n=0}^{\infty} \frac{\frac{1}{2}\sigma^2 t^2}{n!} t^{2n} = 1 + \frac{1}{2}\sigma^2 t^2 + \frac{(\frac{1}{2}\sigma^2 t^2)^2}{2!} + \frac{(\frac{1}{2}\sigma^2 t^2)^3}{3!} + \ldots = 1 + \frac{\sigma^2 t^2}{2} + \frac{\sigma^4 t^4}{2^2 \cdot 2!} + \frac{\sigma^6 t^6}{2^3 \cdot 3!} + \ldots$$

This series only contains even powers of t, confirming that all coefficients of odd powers of t are zero, thus:

$$\mathbb{E}[X^{2n+1}] = 0$$

for all odd powers $2n + 1$. This occurs because the derivatives of $\Psi_X(t)$ at $t = 0$ for odd orders are zero, as each term in the expansion of $\Psi_X(t)$ contains even powers.

For even powers, consider the coefficient of t^{2n} in the Taylor expansion:

$$\mathbb{E}[X^{2n}] = \frac{d^{2n}}{dt^{2n}} \Psi_X(t) \bigg|_{t=0} = \frac{d^{2n}}{dt^{2n}} \left(\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{2} \sigma^2 t^2 \right)^k \right) \bigg|_{t=0}$$

To see why $\mathbb{E}[X^{2n}]$ equals $(2n-1)!!\sigma^{2n}$, take the $2n$-th derivative:

$$\mathbb{E}[X^{2n}] = \frac{1}{n!} \left(\frac{1}{2} \sigma^2 \right)^n \cdot 2^n \cdot (2n)! = \sigma^{2n} \cdot (2n-1)!!$$

This computation correctly reflects the product of the double factorial $(2n-1)!!$ which is the product of all odd numbers up to $(2n-1)$, resulting in:

$$(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1).$$

4 Chapter 4

Exercise 4.1

Exercise description.

Solution

Write your solution here.

Exercise 4.2

Exercise description.

Solution

Another solution here.