

# Solutions for [Book Name]

Your Name

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# 1 Introduction

## 1.1 Background

### Exercise 1.1

Exercise description.

**Solution** Write your solution here.

### Exercise 1.2

Exercise description.

**Solution** Another solution here.

## 1.2 Advanced Topics

### Exercise 2.1

Exercise description.

**Solution** Write your solution here.

## 2 Introduction

### 2.1 Background

#### Exercise 1.1

Exercise description.

**Solution** Write your solution here.

#### Exercise 1.2

Exercise description.

**Solution** Another solution here.

### 2.2 Advanced Topics

#### Exercise 2.1

Exercise description.

**Solution** Write your solution here.

### 3 Transforms

#### 3.1 a

#### 3.2 b

#### 3.3 The Moment Generating Function

##### Exercise 3.1

##### Exercise 3.2

##### Exercise 3.3

##### Exercise 3.4

##### Exercise 3.5

- (a) Show that if  $X \sim N(\mu, \sigma^2)$ , then  $\mathbb{E}[X] = \mu$  and  $\text{Var}(X) = \sigma^2$ .
- (b) Let  $X_1 \sim N(\mu_1, \sigma_1^2)$  and  $X_2 \sim N(\mu_2, \sigma_2^2)$  be independent random variables. Show that  $X_1 + X_2$  is normally distributed, and find the mean and variance of  $X_1 + X_2$ .
- (c) Let  $X \sim N(0, \sigma^2)$ . Show that for  $n = 0, 1, 2, \dots$ ,

$$\mathbb{E}[X^{2n+1}] = 0,$$

and

$$\mathbb{E}[X^{2n}] = (2n-1)!! \cdot \sigma^{2n} = 1 \cdot 3 \cdot 5 \dots (2n-1) \cdot \sigma^{2n}.$$

Here,  $(2n-1)!!$  denotes the double factorial of  $2n-1$ .

##### Solution

- (a) Given a normal random variable  $X \sim N(\mu, \sigma^2)$ , its characteristic function  $\psi_X(t)$  is expressed as:

$$\psi_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}.$$

The expected value  $\mathbb{E}[X]$  is the coefficient of  $t$  in the Taylor expansion of  $\psi_X(t)$  around  $t = 0$ , which yields:

$$\mathbb{E}[X] = \left. \frac{d}{dt} \psi_X(t) \right|_{t=0} = \mu.$$

To find the variance  $\text{Var}(X)$ , we compute the second derivative of  $\psi_X(t)$  at  $t = 0$ :

$$\text{Var}(X) = \left. \frac{d^2}{dt^2} \psi_X(t) \right|_{t=0} - (\mu)^2 = \sigma^2.$$

- (b) Let  $X_1 \sim N(\mu_1, \sigma_1^2)$  and  $X_2 \sim N(\mu_2, \sigma_2^2)$  be independent random variables. To show that the sum  $X_1 + X_2$  is also normally distributed and to find its parameters, consider their moment generating functions:

$$\Psi_{X_1}(t) = e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2}, \quad \Psi_{X_2}(t) = e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2}.$$

Since  $X_1$  and  $X_2$  are independent, the MGF of their sum is the product of their MGFs:

$$\Psi_{X_1+X_2}(t) = \Psi_{X_1}(t) \cdot \Psi_{X_2}(t) = e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2} \cdot e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2}.$$

Simplify by combining the exponents:

$$\Psi_{X_1+X_2}(t) = e^{(\mu_1 + \mu_2)t + \frac{1}{2}(\sigma_1^2 + \sigma_2^2)t^2}.$$

This is the MGF of a normal distribution with mean  $\mu_1 + \mu_2$  and variance  $\sigma_1^2 + \sigma_2^2$ . Therefore,  $X_1 + X_2$  follows a normal distribution  $N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ .

- (c) Let  $X \sim N(0, \sigma^2)$ . The characteristic function  $\psi_X(t)$ , which also serves as the moment generating function in this context, is given by:

$$\psi_X(t) = e^{\frac{1}{2}\sigma^2 t^2}.$$

Expanding  $\psi_X(t)$  using a Taylor series around  $t = 0$  results in:

$$\psi_X(t) = \sum_{n=0}^{\infty} \frac{\frac{1}{2}\sigma^2 t^2}{n!} t^{2n} = 1 + \frac{1}{2}\sigma^2 t^2 + \frac{(\frac{1}{2}\sigma^2 t^2)^2}{2!} + \frac{(\frac{1}{2}\sigma^2 t^2)^3}{3!} + \dots = 1 + \frac{\sigma^2 t^2}{2} + \frac{\sigma^4 t^4}{2^2 \cdot 2!} + \frac{\sigma^6 t^6}{2^3 \cdot 3!} + \dots$$

This series only contains even powers of  $t$ , confirming that all coefficients of odd powers of  $t$  are zero, thus:

$$\mathbb{E}[X^{2n+1}] = 0$$

for all odd powers  $2n + 1$ . This occurs because the derivatives of  $\psi_X(t)$  at  $t = 0$  for odd orders are zero, as each term in the expansion of  $\psi_X(t)$  contains even powers.

For even powers, consider the coefficient of  $t^{2n}$  in the Taylor expansion:

$$\mathbb{E}[X^{2n}] = \left. \frac{d^{2n}}{dt^{2n}} \psi_X(t) \right|_{t=0} = \left. \frac{d^{2n}}{dt^{2n}} \left( \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{1}{2}\sigma^2 t^2 \right)^k \right) \right|_{t=0}$$

To see why  $\mathbb{E}[X^{2n}]$  equals  $(2n - 1)!!\sigma^{2n}$ , take the  $2n$ -th derivative:

$$\mathbb{E}[X^{2n}] = \frac{1}{n!} \left( \frac{1}{2}\sigma^2 \right)^n \cdot 2^n \cdot (2n)! = \sigma^{2n} \cdot (2n - 1)!!$$

This computation correctly reflects the product of the double factorial  $(2n - 1)!!$  which is the product of all odd numbers up to  $(2n - 1)$ , resulting in:

$$(2n - 1)!! = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n - 1) \cdot (\sigma^{2n}).$$

### Exercise 3.6

- (a) Show that if  $X \sim N(0, 1)$  then  $X^2 \sim \chi^2(1)$  by computing the moment generating function (MGF) of  $X^2$ , that is, by showing that

$$\psi_{X^2}(t) = \mathbb{E}[\exp(tX^2)] = \frac{1}{\sqrt{1 - 2t}} \quad \text{for } t < \frac{1}{2}.$$

- (b) Show that if  $X_1 \sim N(0, 1)$  and  $X_2 \sim N(0, 1)$  are independent, then  $X_1^2 + X_2^2$  is distributed as  $\chi^2(2)$  (which is equivalent to an exponential distribution with mean 2).

### Solution

- (a) Begin by recognizing the integral for the MGF:

$$\psi_{X^2}(t) = \int_{-\infty}^{\infty} e^{tx^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{x^2(t - \frac{1}{2})} dx.$$

This integral converges for  $t < \frac{1}{2}$ . Transform  $x$  to eliminate the variable change explicitly:

$$\frac{d(x\sqrt{1 - 2t})}{dx} = \sqrt{1 - 2t}, \quad dx = \frac{d(x\sqrt{1 - 2t})}{\sqrt{1 - 2t}}$$

Substitute directly:

$$\psi_{X^2}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x\sqrt{1 - 2t})^2}{2}} \frac{d(x\sqrt{1 - 2t})}{\sqrt{1 - 2t}} = \frac{1}{\sqrt{1 - 2t}}.$$

The integral of the standard normal density over the transformed variable is 1, leading to the final MGF expression for  $X^2$ .

- (b) Given that  $X_1 \sim N(0, 1)$  and  $X_2 \sim N(0, 1)$  are independent, to show that  $X_1^2 + X_2^2$  is distributed as  $\chi^2(2)$ , consider the moment generating functions (MGFs) of  $X_1^2$  and  $X_2^2$ , which are:

$$\psi_{X_1^2}(t) = \psi_{X_2^2}(t) = \frac{1}{\sqrt{1-2t}} \quad \text{for } t < \frac{1}{2}.$$

Since  $X_1^2$  and  $X_2^2$  are independent, the MGF of their sum,  $X_1^2 + X_2^2$ , is the product of their MGFs:

$$\psi_{X_1^2 + X_2^2}(t) = \psi_{X_1^2}(t) \cdot \psi_{X_2^2}(t) = \left( \frac{1}{\sqrt{1-2t}} \right)^2 = \frac{1}{1-2t}.$$

This MGF,  $\frac{1}{1-2t}$ , is the MGF of a  $\chi^2$  distribution with 2 degrees of freedom. The  $\chi^2(2)$  distribution is also known to be equivalent to an exponential distribution with mean 2, confirming the distribution of  $X_1^2 + X_2^2$ .

### 3.4 The Characteristic Function

#### Exercise 4.1

- (a) For a Bernoulli random variable  $X \sim \text{Be}(p)$ :

$$\varphi_{\text{Be}(p)}(t) = q + pe^{it}, \quad \text{where } q = 1 - p.$$

- (b) For a Binomial random variable  $Y \sim \text{Bin}(n, p)$ :

$$\varphi_{\text{Bin}(n,p)}(t) = (q + pe^{it})^n.$$

- (c) For a compound Poisson random variable  $Z$  with rate  $\lambda$  and jump size distribution  $C$ :

$$\varphi_C(t) = \frac{p}{1 - qe^{ist}},$$

assuming a specific relationship between the parameters  $p$  and  $q$ , and  $s$ .

- (d) For a compound Poisson random variable  $W$  with intensity  $m$  and jump size distribution  $P$ :

$$\varphi_{P * \theta(m)}(t) = \exp [m(e^{it} - 1)].$$

#### Solution

- (a) **Bernoulli Distribution**  $X \sim \text{Be}(p)$ :

$$\varphi_{\text{Be}(p)}(t) = \mathbb{E}[e^{itX}] = \sum_{x=0}^1 e^{itx} \Pr(X = x) = e^{it \cdot 0} \Pr(X = 0) + e^{it \cdot 1} \Pr(X = 1) = (1 - p) + pe^{it}.$$

This is exactly the expression given:  $q + pe^{it}$ , where  $q = 1 - p$ .

- (b) **Binomial Distribution**  $Y \sim \text{Bin}(n, p)$ : The characteristic function of a sum of independent identically distributed random variables (by the property often called the *factorization property*) is:

$$\varphi_{\text{Bin}(n,p)}(t) = [\varphi_{\text{Be}(p)}(t)]^n = (q + pe^{it})^n.$$

This uses the property that the characteristic function of the sum of independent random variables is the product of their characteristic functions.

- (c) **Geometric Distribution**:

$$\varphi_X(t) = \mathbb{E}[e^{itX}] = \sum_{x=0}^{\infty} e^{itx} \Pr(X = x) = \sum_{x=0}^{\infty} e^{itx} \frac{pq^x}{1-q} = \frac{p}{1 - qe^{it}},$$

where we used the formula for the sum of a geometric series  $\sum_{x=0}^{\infty} ar^x = \frac{a}{1-r}$  applied to  $e^{it}$  as  $r$ .

(d) **Compound Poisson Distribution** ( $W$ ) with intensity  $m$  and jump size distribution  $P$ :

The compound Poisson variable  $W$  can be expressed as  $W = \sum_{k=1}^N X_k$ , where  $N \sim \text{Poisson}(m)$  and  $X_k$  are iid random variables from the distribution  $P$ . The characteristic function  $\varphi_W(t)$  is given by the expectation:

$$\varphi_W(t) = \mathbb{E}[e^{itW}].$$

Given  $W$  conditioned on  $N$  being equal to  $n$ , the sum  $W = X_1 + X_2 + \dots + X_n$  and the  $X_k$ 's are independent. So, we write:

$$\mathbb{E}[e^{itW} \mid N = n] = \mathbb{E}[e^{it(X_1 + X_2 + \dots + X_n)}] = \prod_{k=1}^n \mathbb{E}[e^{itX_k}] = (\varphi_P(t))^n,$$

where  $\varphi_P(t)$  is the characteristic function of the distribution  $P$ .

The unconditional expectation is:

$$\varphi_W(t) = \sum_{n=0}^{\infty} \mathbb{E}[e^{itW} \mid N = n] \Pr(N = n) = \sum_{n=0}^{\infty} (\varphi_P(t))^n \frac{e^{-m} m^n}{n!}.$$

Using the Taylor series expansion for the exponential function, we have:

$$\varphi_W(t) = e^{-m} \sum_{n=0}^{\infty} \frac{[m\varphi_P(t)]^n}{n!} = e^{-m} e^{m\varphi_P(t)} = \exp[m(\varphi_P(t) - 1)].$$

This directly ties into the idea you suggested, where each  $e^{itx}$  term is weighted by its Poisson probability, which then sums to form the exponential series representation of  $\varphi_W(t)$ .

## Exercise 4.2

## Exercise 4.3

- Calculate the mean and variance of the Binomial distribution using its characteristic function.
- Calculate the mean and variance of the Poisson distribution using its characteristic function.
- Calculate the mean and variance of the Uniform distribution using its characteristic function.
- Calculate the mean and variance of the Exponential distribution using its characteristic function.

## Solution

(a) **Binomial Distribution:**

$$\text{Characteristic Function: } \varphi_X(t) = (1 - p + pe^{it})^n$$

Expansion of  $e^{it}$ :

$$e^{it} \approx 1 + it - \frac{t^2}{2}$$

Substitute and apply multinomial theorem:

$$\varphi_X(t) = (1 - p + p(1 + it - \frac{pt^2}{2}))^n$$

Expand using multinomial coefficients:

$$\varphi_X(t) \approx \sum_{x,y,z} \binom{n}{x,y,z} (1-p)^x (pit)^y \left(-\frac{pt^2}{2}\right)^z$$

Relevant terms up to  $t^2$ :

$$\varphi_X(t) \approx \binom{n}{n,0,0} (1-p)^n + \binom{n}{n-1,1,0} (1-p)^{n-1} (pit) + \binom{n}{n-2,0,2} (1-p)^{n-2} \left(-\frac{pt^2}{2}\right)$$

Mean  $E[X]$ :

$$E[X] = np$$

Variance  $\text{Var}(X)$ :

$$\text{Var}(X) = np(1 - p)$$

(b) **Poisson Distribution:**

$$\text{Characteristic Function: } \varphi_X(t) = e^{\lambda(e^{it} - 1)}$$

Expansion of  $e^{it}$ :

$$e^{it} \approx 1 + it - \frac{t^2}{2}$$

Substitute and expand:

$$\varphi_X(t) = e^{\lambda\left(1 + it - \frac{t^2}{2} - 1\right)} = e^{\lambda\left(it - \frac{t^2}{2}\right)}$$

Applying Taylor expansion to  $e^{\lambda\left(it - \frac{t^2}{2}\right)}$ :

$$\varphi_X(t) \approx 1 + \lambda\left(it - \frac{t^2}{2}\right) + \frac{\lambda^2}{2}\left(it - \frac{t^2}{2}\right)^2 + \dots$$

Relevant terms up to  $t^2$ :

$$\varphi_X(t) \approx 1 + (\lambda^2 + \lambda)it - \frac{\lambda t^2}{2}$$

Mean  $E[X]$ :

$$E[X] = \lambda$$

Second moment  $E[X^2]$ :

$$E[X^2] = \lambda^2 + \lambda$$

Variance  $\text{Var}(X)$ :

$$\text{Var}(X) = \lambda$$

(c) **Uniform Distribution:**

$$\text{Characteristic Function: } \varphi_X(t) = \frac{e^{itb} - e^{ita}}{it(b - a)}$$

Expansions of  $e^{itb}$  and  $e^{ita}$  (remember, we will divide by  $it(b - a)$  so we need terms up to  $t^3$ ):

$$e^{itb} \approx 1 + itb - \frac{t^2 b^2}{2} - i \frac{t^3 b^3}{6}, \quad e^{ita} \approx 1 + ita - \frac{t^2 a^2}{2} - i \frac{t^3 a^3}{6}$$

Substitute and simplify:

$$\varphi_X(t) = \frac{(1 + itb - \frac{t^2 b^2}{2} - i \frac{t^3 b^3}{6}) - (1 + ita - \frac{t^2 a^2}{2} - i \frac{t^3 a^3}{6})}{it(b - a)}$$

$$\varphi_X(t) \approx \frac{1}{it(b - a)} \left[ (1 - 1) + it(b - a) - \frac{t^2}{2}(b^2 - a^2) - i \frac{t^3}{6}(b^3 - a^3) \right] = \left[ 0 + 1 + it \frac{b + a}{2} + \frac{t^2}{6}(b^2 + a^2 - ab) \right]$$

Relevant terms up to  $t^2$ :

$$\varphi_X(t) \approx 1 + it \frac{b + a}{2} - \frac{t^2(b^2 + a^2 - ab)^2}{6}$$

Mean  $E[X]$ :

$$E[X] = \frac{b + a}{2}$$

Second moment  $E[X^2]$ :

$$E[X] = \frac{b^2 + a^2 - ab}{3}$$

Variance  $\text{Var}(X)$ :

$$\text{Var}(X) = E[X^2] - E[X]^2 = \frac{4(b^2 + a^2 - ab)}{4 \cdot 3} - \frac{3 \cdot (b + a)^2}{3 \cdot 2^2} = \frac{(b - a)^2}{12}$$



(d) **Exponential Distribution:**

$$\text{Characteristic Function: } \varphi_X(t) = \frac{1}{1 - it/\lambda}$$

Expand using Geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 \dots \therefore \varphi_X(t) \approx 1 + it/\lambda + (it/\lambda)^2 + o(t^2)$$

Relevant terms up to  $t^2$ :

$$\begin{aligned} \varphi_X(t) &\approx 1 + it \frac{1}{\lambda} - \frac{t^2}{2} \frac{2}{\lambda^2} \\ E[X] &= \frac{1}{\lambda} \quad E[X^2] = \frac{2}{\lambda^2} \quad \text{Var}(X) = \frac{1}{\lambda^2} \end{aligned}$$

(e) **Standard Normal Distribution:**

$$\text{Characteristic Function: } \varphi_X(t) = e^{-\frac{t^2}{2}}$$

Apply Taylor expansion to  $e^{-\frac{t^2}{2}}$ :

$$\varphi_X(t) \approx 1 - \frac{t^2}{2} + \frac{t^4}{8} - \frac{t^6}{48} + \dots$$

Relevant terms up to  $t^2$ :

$$\varphi_X(t) \approx 1 - \frac{t^2}{2}$$

Which yields:

$$E[X] = 0 \quad \text{Var}(X) = 1$$

**Exercise 4.4**

**Exercise 4.5**

**Exercise 4.6**

Use Theorem 4.9 to show that  $\varphi_{C(m,a)}(t) = e^{itm} \varphi_X(at) = e^{itm-a|t|}$

**Solution** Theorem 4.9 states that

$$\phi_{aX+b}(t) = e^{itb} \cdot \phi_X(at)$$

Physics teaches us that a Cauchy distribution is the dist of a  $x$ -intercept of a random ray going through the point  $C(m, a)$

Changing  $m$  is the same as moving the intercept by  $m$ , and changing  $a$  is the same as multiplying the intercept point, taking account the scaling already done by  $m$ .

It is therefore obvious that

$$\phi_{C(m,a)}(t) = e^{itm} \cdot \phi_X(at) = e^{itm} \cdot e^{-\|at\|} = \exp(itm - \|at\|)$$

**Exercise 4.7**

Show that if  $X, Y$  are iid, then  $X - Y$  has a symmetric distribution:

**Solution** yet again prove something obvious but with characteristic functions. If  $X \stackrel{d}{=} Y$  then

$$\phi_{X-Y}(t) = (\text{independent}) = \phi_X(t) \cdot \phi_Y(-t) = (\text{equidistributed}) = \phi_X(t) \cdot \phi_X(-t) = \phi_X(t) \cdot \overline{\phi_X(-t)} = \text{real}$$

**Exercise 4.8**

Show that one cannot find i.i.d R.V  $X$  and  $Y$  such that  $X - Y \in U(-1, 1)$

**Solution** We know that

$$\phi_{X-Y}(t) = (\text{independent}) = \phi_X(t) \cdot \phi_Y(-t) = (\text{equidistributed}) = \phi_X(t) \cdot \phi_X(-t) = \phi_X(t) \cdot \overline{\phi_X(-t)} = \|\phi_X(t)\|^2$$

Which is strictly positive, however this does not hold true for

$$\phi_{U(-1,1)} = \frac{\sin(t)}{t}$$