Time Series Analysis

Lag operator

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Introduction

Operator

The operator is summarized by describing the value of a typical element of $\{y_t\}_{t=-\infty}^{\infty}$ in terms of the corresponding elements of $\{x_t\}_{t=-\infty}^{\infty}$

Lag Operator

The lag operation is represented by symbol L:

$$Ly_t = y_{t-1}$$

For any integer k,

$$L^k y_t = y_{t-k}$$

Introduction

For example, the process defined by

$$y_t = (a + bL)Ly_t$$

is exactly the same as:

$$y_t = (aL + bL^2)y_t = ay_{t-1} + by_{t-2}$$

An expression such as $(aL + bL^2)y_t$ is referred to as *polynomial in the lag operator*

First-Order Difference Equations

Consider a first-order difference equation:

$$y_t = \phi y_{t-1} + u_t \tag{1}$$

We can solve this equation by recursive substitution approach or lag operator approach.

For recursive approach, we can calculate the value of y at time $t-1$ in equation (1):

$$\begin{aligned} y_t &= \phi(y_{t-2} + u_{t-1}) + u_t && (2) \\ &= \phi^2 y_{t-2} + \phi u_{t-1} + u_t && (3) \end{aligned}$$

Now we have the relationship between y_t and y_{t-2}. We can calculate the value of y at time $t-2$ in equation (3):

$$\begin{aligned} y_t &= \phi(y_{t-3} + u_{t-2}) + \phi u_{t-1} + u_t \\ &= \phi^3 y_{t-3} + \phi^2 u_{t-2} + \phi u_{t-1} + u_t \end{aligned}$$

Continuing recursively in this fashion, we can have the relationship between y_t and y_0:

$$y_t = \phi^t y_0 + \phi^{t-1} u_1 + \phi^{t-2} u_2 + \cdots + \phi u_{t-1} + u_t \tag{4}$$

This procedure is known as solving the difference equation (4) by *recursive substitution*

First-Order Difference Equations

Now we rewrite equation (1) by using lag operator approach as:

$$y_t - \phi L y_t = u_t$$

or

$$(1 - \phi L) y_t = u_t \tag{5}$$

Next consider multiplying both sides of equation by the following operator:

$$(1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^{t-1} L^{t-1})$$

The result would be:

$$\begin{aligned}(1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^{t-1} L^{t-1})(1 - \phi L) y_t \\ = (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^{t-1} L^{t-1}) u_{t-1}\end{aligned} \tag{6}$$

First-Order Difference Equations

$$(1-\phi^t L^t)y_t = (1+\phi L+\phi^2 L^2+\phi^3 L^3+\cdots+\phi^{t-1}L^{t-1})u_{t-1}$$

The result would be:

$$y_t = \phi^t y_0 + \phi^{t-1}u_1 + \phi^{t-2}u_2 + \cdots + \phi u_{t-1} + u_t \tag{7}$$

Which is the same as the recursive substitution approach

If $|\phi| < 1$, the term $\phi^t y_0$ would be negligible and the same as operator $(1-\phi L)^{-1}$:

$$(1-\phi L)^{-1} = \lim_{j\to\infty}(1+\phi L+\phi^2 L^2+\phi^3 L^3+\cdots+\phi^j L^j) \tag{8}$$

Thus equation (6) can be represented by:

$$(1-\phi L)^{-1}(1-\phi L)y_t = (1-\phi L)^{-1}u_t$$

which can get the same result as equation 7

Second-Order Difference Equations

Consider a second-order difference equation:

$$y_t = \phi y_{t-1} + \phi^2 y_{t-2} + u_t \tag{9}$$

Rewrite as lag operator form:

$$(1 - \phi_1 L - \phi_2 L^2) y_t = u_t \tag{10}$$

If we factor this polynomial as:

$$(1 - \phi_1 L - \phi_2 L^2) = (1 - \lambda_1 L)(1 - \lambda_2 L)$$

Second-Order Difference Equations

we can have the following proposition:

Proposition

Factoring the polynomial $(1 - \phi_1 L - \phi_2 L^2)$ *as:*

$$(1 - \phi_1 L - \phi_2 L^2) = (1 - \lambda_1 L)(1 - \lambda_2 L) \tag{11}$$

is the same calculation as finding the eigenvalues of the matrix $\mathbf{F}$*:*

$$\mathbf{F} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \tag{12}$$

The eigenvalues λ_1 *and* λ_2 *of* $\mathbf{F}$ *are the same as the parameters* λ_1 *and* λ_2 *in equation (11)*

Pth-Order Difference Equations

The proposition of pth-order difference equations as follows:

Proposition

Factoring a pth-order polynomial in the lag operator as:

$$(1-\phi_1 L-\phi_2 L^2-\cdots-\phi_p L^p)=(1-\lambda_1 L)(1-\lambda_2 L)\ldots(1-\lambda_p L) \quad (13)$$

is the same calculation as finding the eigenvalues of the matrix $\mathbf{F}$*:*

$$\mathbf{F}=\begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \ldots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \quad (14)$$

The eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ *of* $\mathbf{F}$ *are the same as the parameters* $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ *in equation (13)*

Initial Conditons and Unbounded Sequence

Consider a *perfect-foresight model*:

$$r = (P_{t+1} - P_t/P_t + D_t/P_t) \qquad r > 0 \tag{15}$$

Where P_t is the price of a stock, D_t is dividend payment, total return is r. Multiply by P_t and rewrite the equation:

$$P_{t+1} = (1 + r)P_t - D_t \tag{16}$$

Equation (16) is the same as the first-order difference equation and it implies that:

$$\mathrm{P}_{t+1} = (1 + r)^{t+1}P_0 - (1 + r)^t D_0 - (1 + r)^{t-1}D_1 - \cdots - (1 + r)D_{t-1} - D_t \tag{17}$$

If the sequence $\{D_0, D_1, \ldots, D_t\}$ and the initial value of P_0 were given, then we could determine $\{P_0, P_1, \ldots, P_t\}$

Initial Conditons and Unbounded Sequence

If only the sequence are given, we can not determine $\{P_0, P_1, \ldots, P_t\}$

Suppose that dividends are constant over time, equation (17) becomes

$$P_{t+1} = (1+r)^{t+1}[P_0 - (D/r)] + (D/r) \tag{18}$$

If $P_0 = D/r$, equation (18) implies that:

$$P_t = D/r$$

If $P_0 > D/r$, equation (18) is consistent with APT and consists bubble in stock price

However, the assumption for constant dividend can be relaxed

Initial Conditons and Unbounded Sequence

If we rewrite the equation (16) as:

$$P_t = \frac{1}{1+r}[P_{t+1} + D_t] \tag{19}$$

Continuing recursive substitution for T periods:

$$\begin{aligned} P_t = [\frac{1}{1+r}]^T P_{t+T} + [\frac{1}{1+r}]^T D_{t+T-1} \\ + \cdots + [\frac{1}{1+r}]^2 D_{t+1} + [\frac{1}{1+r}] D_t \end{aligned} \tag{20}$$

Initial Conditons and Unbounded Sequence

If finite world resources put an upper limit on stock price, then:

$$\lim_{T\to\infty}[\frac{1}{1+r}]^T P_t = 0$$

If $\{D_t\}_{t=-\infty}^{\infty}$ is also bounded, equation (20) as:

$$P_t = \sum_{j=0}^{\infty}[\frac{1}{1+r}]^{j+1} D_{t+j} \tag{21}$$

which is the solution for the present value of future dividends with time-varying discount rate