# Time Series Analysis

## Lag operator


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# Introduction

## Operator

The operator is summarized by describing the value of a typical element of $\{y_t\}_{t=-\infty}^{\infty}$ in terms of the corresponding elements of $\{x_t\}_{t=-\infty}^{\infty}$

## Lag Operator

The lag operation is represented by symbol $L$:

$$Ly_t = y_{t-1}$$

For any integer $k$,

$$L^k y_t = y_{t-k}$$

# Introduction

For example, the process defined by

$$y_t = (a + bL)Ly_t$$

is exactly the same as:

$$y_t = (aL + bL^2)y_t = ay_{t-1} + by_{t-2}$$

An expression such as $(aL + bL^2)y_t$ is referred to as *polynomial in the lag operator*

# First-Order Difference Equations

Consider a first-order difference equation:

$$y_t = \phi y_{t-1} + u_t \tag{1}$$

We can solve this equation by recursive substitution approach or lag operator approach.

For recursive approach, we can calculate the value of $y$ at time $t-1$ in equation (1):

$$\begin{aligned} y_t &= \phi(y_{t-2} + u_{t-1}) + u_t && (2) \\ &= \phi^2 y_{t-2} + \phi u_{t-1} + u_t && (3) \end{aligned}$$

# First-Order Difference Equations

Now we have the relationship between $y_t$ and $y_{t-2}$. We can calculate the value of $y$ at time $t-2$ in equation (3):

$$\begin{aligned} y_t &= \phi(y_{t-3} + u_{t-2}) + \phi u_{t-1} + u_t \\ &= \phi^3 y_{t-3} + \phi^2 u_{t-2} + \phi u_{t-1} + u_t \end{aligned}$$

Continuing recursively in this fashion, we can have the relationship between $y_t$ and $y_0$:

$$y_t = \phi^t y_0 + \phi^{t-1} u_1 + \phi^{t-2} u_2 + \cdots + \phi u_{t-1} + u_t \tag{4}$$

This procedure is known as solving the difference equation (4) by *recursive substitution*

# First-Order Difference Equations

Now we rewrite equation (1) by using lag operator approach as:

$$y_t - \phi L y_t = u_t$$

or

$$(1 - \phi L) y_t = u_t \tag{5}$$

Next consider multiplying both sides of equation by the following operator:

$$(1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^{t-1} L^{t-1})$$

The result would be:

$$\begin{aligned} &(1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^{t-1} L^{t-1})(1 - \phi L) y_t \\ &= (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^{t-1} L^{t-1}) u_{t-1} \end{aligned} \tag{6}$$

# First-Order Difference Equations

$$(1-\phi^t L^t)y_t = (1+\phi L+\phi^2 L^2+\phi^3 L^3+\cdots+\phi^{t-1}L^{t-1})u_{t-1}$$

The result would be:

$$y_t = \phi^t y_0 + \phi^{t-1}u_1 + \phi^{t-2}u_2 + \cdots + \phi u_{t-1} + u_t \tag{7}$$

Which is the same as the recursive substitution approach

If $|\phi| < 1$, the term $\phi^t y_0$ would be negligible and the same as operator $(1 - \phi L)^{-1}$:

$$(1 - \phi L)^{-1} = \lim_{j \to \infty} (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^j L^j) \tag{8}$$

Thus equation (6) can be represented by:

$$(1 - \phi L)^{-1}(1 - \phi L) y_t = (1 - \phi L)^{-1} u_t$$

which can get the same result as equation 7

# Second-Order Difference Equations

Consider a second-order difference equation:

$$y_t = \phi y_{t-1} + \phi^2 y_{t-2} + u_t \tag{9}$$

Rewrite as lag operator form:

$$(1 - \phi_1 L - \phi_2 L^2) y_t = u_t \tag{10}$$

If we factor this polynomial as:

$$(1 - \phi_1 L - \phi_2 L^2) = (1 - \lambda_1 L)(1 - \lambda_2 L)$$

# Second-Order Difference Equations

we can have the following proposition:

## Proposition

*Factoring the polynomial* $(1 - \phi_1 L - \phi_2 L^2)$ *as:*

$$(1 - \phi_1 L - \phi_2 L^2) = (1 - \lambda_1 L)(1 - \lambda_2 L) \tag{11}$$

*is the same calculation as finding the eigenvalues of the matrix* $\mathbf{F}$*:*

$$\mathbf{F} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \tag{12}$$

*The eigenvalues* $\lambda_1$ *and* $\lambda_2$ *of* $\mathbf{F}$ *are the same as the parameters* $\lambda_1$ *and* $\lambda_2$ *in equation (11)*

# Pth-Order Difference Equations

The proposition of pth-order difference equations as follows:

## Proposition

*Factoring a pth-order polynomial in the lag operator as:*

$$(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p) = (1 - \lambda_1 L)(1 - \lambda_2 L) \ldots (1 - \lambda_p L) \quad (13)$$

*is the same calculation as finding the eigenvalues of the matrix* $\mathbf{F}$*:*

$$\mathbf{F} = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \ldots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \quad (14)$$

*The eigenvalues* $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ *of* $\mathbf{F}$ *are the same as the parameters* $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ *in equation (13)*

# Initial Conditons and Unbounded Sequence

Consider a *perfect-foresight model*:

$$r = (P_{t+1} - P_t/P_t + D_t/P_t) \qquad r > 0 \tag{15}$$

Where $P_t$ is the price of a stock, $D_t$ is dividend payment, total return is $r$. Multiply by $P_t$ and rewrite the equation:

$$P_{t+1} = (1 + r)P_t - D_t \tag{16}$$

Equation (16) is the same as the first-order difference equation and it implies that:

$$P_{t+1} = (1+r)^{t+1}P_0 - (1+r)^t D_0 - (1+r)^{t-1} D_1 - \cdots - (1+r)D_{t-1} - D_t \tag{17}$$

If the sequence $\{D_0, D_1, \ldots, D_t\}$ and the initial value of $P_0$ were given, then we could determine $\{P_0, P_1, \ldots, P_t\}$.

# Initial Conditons and Unbounded Sequence

If only the sequence are given, we can not determine $\{P_0, P_1, \ldots, P_t\}$

Suppose that dividends are constant over time, equation (17) becomes

$$P_{t+1} = (1+r)^{t+1}[P_0 - (D/r)] + (D/r) \tag{18}$$

If $P_0 = D/r$, equation (18) implies that:

$$P_t = D/r$$

If $P_0 > D/r$, equation (18) is consistent with APT and consists bubble in stock price

However, the assumption for constant dividend can be relaxed

# Initial Conditons and Unbounded Sequence

If we rewrite the equation (16) as:

$$P_t = \frac{1}{1+r}[P_{t+1} + D_t] \tag{19}$$

Continuing recursive substitution for $T$ periods:

$$\begin{aligned} P_t = [\frac{1}{1+r}]^T P_{t+T} + [\frac{1}{1+r}]^T D_{t+T-1} \\ + \cdots + [\frac{1}{1+r}]^2 D_{t+1} + [\frac{1}{1+r}] D_t \end{aligned} \tag{20}$$

# Initial Conditons and Unbounded Sequence

If finite world resources put an upper limit on stock price, then:

$$\lim_{T\to\infty}[\frac{1}{1+r}]^T P_t = 0$$

If $\{D_t\}_{t=-\infty}^{\infty}$ is also bounded, equation (20) as:

$$P_t = \sum_{j=0}^{\infty}[\frac{1}{1+r}]^{j+1} D_{t+j} \tag{21}$$

which is the solution for the present value of future dividends with time-varying discount rate