Time Series Analysis

Stationary ARMA Process

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Limits of Deterministic Sequences

Let $\{c_T\}_{T=1}^{\infty}$ denote a sequence of deterministic numbers. The sequence is said to *converge* to c if for any $\epsilon > 0$, there exists an N such that $|c_T - c| < \epsilon$ whenever T:

$$\lim_{T \to \infty} c_T = c$$

Or,

$$c_T \to c$$

Convergence in Probability

Consider a sequence of random variables $\{X_T\}_{T=1}^{\infty}$. The sequence is said to *converge in probability* to c if for any $\epsilon > 0$ and any $\delta > 0$, there exists an N whenever T:

$$P\{|X_T - c| > \delta\} < \epsilon$$

When the above is satisfied, the number c is called the plim, of the sequence $\{X_T\}_{T=1}^{\infty}$:

$$plim X_T = c$$

Or,

$$X_T \xrightarrow{p} c$$

Convergence in Mean Square

Consider a sequence of random variables $\{X_T\}_{T=1}^{\infty}$. The sequence is said to *mean square convergence* to c if for any $\epsilon > 0$, there exists an N whenever T:

$$E(X_T - c)^2 < \epsilon$$

Which can be indicated as:

$$X_T \xrightarrow{m.s.} c$$

Chebyshev's Inequality

Let X be a random variable with $E(|X|^r)$ finite for some $r > 0$. Then, for any $\delta > 0$ and any value of c,

$$P\{|X - c| > \delta\} \leq \frac{E|X - c|^r}{\delta^r}$$

An implication of Chebyshev's inequality is that,

if $X_T \xrightarrow{m.s.} c$, then $X_T \xrightarrow{p} c$.

Law of Large Numbers for i.i.d.

Consider $\{Y_t\}$ is i.i.d. with mean μ and variance σ^2.
If the sample mean $\bar{Y}_T = (1/T)\sum_{t=1}^{T} Y_t$ has expectation μ, then the variance:

$$Var(\bar{Y}_T) = E(\bar{Y}_T - \mu)^2 = (1/T^2)\sum_{t=1}^{T} Var(Y_t) = \sigma^2/T$$

Since $\sigma^2/T \to 0$ as $T \to \infty$, this means that $\bar{Y}_T \xrightarrow{m.s.} \mu$, implying $\bar{Y}_T \xrightarrow{p} \mu$

Convergence in Distribution

Consider a sequence of random variables $\{X_T\}_{T=1}^{\infty}$, and let $F_{X_T}(X)$ denote the cumulative distribution function. Suppose that there exists a cumulative distribution function:

$$\lim_{T\to\infty} F_{X_T}(X) = F_X(X)$$

When $F_X(X)$ is of a common form, such as the cumulative distribution function for a $N(\mu, \sigma^2)$ variable, we will equivalently write:

$$X_T \xrightarrow{L} N(\mu, \sigma^2)$$

Central Limit Theorem

We have seen that the sample mean $\bar{Y}_T$ for an i.i.d. sequence has a degenerate probability density as $T \to \infty$, collapsing toward a point mass at μ, as the sample size grows.

The central limit theorem is the result that as T increases, $\sqrt{T}(\bar{Y}_T - \mu)$ converges in distribution to a Gaussian random variable:

$$\sqrt{T}(\bar{Y}_T - \mu) \xrightarrow{L} N(0, \sigma^2)$$

Limit Theorems for Serially Dependent Observations

The previous section stated the law of large numbers and central limit theorem for independent and identically distributed random variables with finite second moments. This section develops analogous results for heterogeneously distributed variables th various forms of serial dependence. We first develop a law of large numbers for a general weakly stationary conditions:

$$E(Y_t) = \mu \qquad \text{for all t}$$

$$E[Y_t - \mu]^2 = \sigma^2 \qquad \text{for all t}$$

$$E[Y_t - \mu][Y_{t-j} - \mu] = \gamma_j \qquad \text{for all t and j}$$

Limit Theorems for Serially Dependent Observations

If the properties of the sample mean gives that:

$$\bar{Y}_T = (1/T)\sum_{t=1}^{T} Y_t$$

Taking expectations of the equation reveals that the sample mean provides an unbiased estimate of the population mean:

$$E(\bar{Y}_T) = \mu$$

The sample mean must satisfy:

$$(a) \qquad \bar{Y}_T \xrightarrow{m.s.} \mu$$

Limit Theorems for Serially Dependent Observations

For equation (a):

$$
\begin{aligned}
E(\bar{Y}_T - \mu)^2 &= E[(1/T)\sum_{t=1}^{T}(Y_t - \mu)]^2 \\
&= (1/T^2)E\{[(Y_1 - \mu) + (Y_2 - \mu) + \dots (Y_t - \mu)] \\
&\quad \cdot [(Y_1 - \mu) + (Y_2 - \mu) + \dots (Y_t - \mu)]\} \\
&= (1/T^2)E\{(Y_1 - \mu)[(Y_1 - \mu) + \cdots + (Y_t - \mu)] \\
&\quad + (Y_2 - \mu)[(Y_1 - \mu) + \cdots + (Y_t - \mu)] \\
&\quad + \dots (Y_t - \mu)[(Y_1 - \mu) + \cdots + (Y_t - \mu)]\} \\
&= (1/T^2)E\{[\gamma_0 + \gamma_1 + \gamma_2 + \cdots + \gamma_{t-1}] \\
&\quad + [\gamma_1 + \gamma_0 + \gamma_1 + \cdots + \gamma_{t-2}] \\
&\quad + \cdots + [\gamma_{t-1} + \gamma_{t-2} + \cdots + \gamma_0]\}
\end{aligned}
$$

Thus,

$$E(\bar{Y}_T - \mu)^2 = (1/T^2)\{T\gamma_0 + 2(T-1)\gamma_1 \\ + 2(T-2)\gamma_2 + 2(T-3)\gamma_3 + \cdots + 2\gamma_{T-1}\}$$

Or,

$$E(\bar{Y}_T - \mu)^2 = (1/T)\{\gamma_0 + [(T-1)/T](2\gamma_1) \\ + [(T-2)/T](2\gamma_2) + [(T-3)/T](2\gamma_3) \\ + \cdots + [1/T](2\gamma_{T-1})\}$$

Limit Theorems for Serially Dependent Observations

It is easy to see that this expression goes to zero as the sample size grows:

$$
\begin{aligned}
T \cdot E(\bar{Y}_T - \mu)^2 &= |\gamma_0 + [(T-1)/T]2\gamma_1 \\
&\quad + [(T-2)/T]2\gamma_2 + [(T-3)/T]2\gamma_3 \\
&\quad + \cdots + [1/T]2\gamma_{T-1}| \\
&\leq \{|\gamma_0| + [(T-1)/T]2|\gamma_1| \\
&\quad + [(T-2)/T]2|\gamma_2| + \cdots + [(1/T]2|\gamma_{T-1}|\} \\
&\leq \{|\gamma_0| + 2|\gamma_1| + 2|\gamma_2| + \cdots + 2|\gamma_{T-1}|\}
\end{aligned}
$$

Hence, $T \cdot E(\bar{Y}_T - \mu)^2 < \infty$, and so $E(\bar{Y}_T - \mu)^2 \rightarrow 0$ as claimed

Ergodicity

A stationary process is said to be *ergodic for the mean* if:

$$\bar{y} = (1/T)\sum_{i=1}^{T} y_i \tag{1}$$

converges in probability to $E(Y_t)$ as $T \to \infty$, which provided that the autocovariance γ_j goes to zero sufficiently quickly as j becomes large

From equation (6) we can know that ergodicity is about time average rather than ensemble average