Übung 4 Lösungen

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1

Gegeben:

Die Dichte einer hypergeometrischen Verteilung:

$$f(x; M, K, n) = \frac{\binom{K}{x}\binom{M-K}{n-x}}{\binom{M}{n}} I_{\{0,1,\ldots,n\}}(x)$$

Mit $M \in \mathbb{N},\ K = 0, 1, \ldots, M$ und $n = 1, 2, \ldots, M$

Gesucht:

Erwartungswert $E(X)$ und Varianz $Var(X)$

Lösung:

Da X diskret ist:

$$\begin{aligned}
E(X) &= \sum_{x \in R(X)} x f(x) \\
&= \sum_{x=0}^{n} x \frac{\binom{K}{x}\binom{M-K}{n-x}}{\binom{M}{n}} \\
&= \\
Var(X) &= (\sum_{x\in}^{R} x^2 f(x)) - E(X)^2 \\
&= (\sum_{x=0}^{n} x^2 \frac{\binom{K}{x}\binom{M-K}{n-x}}{\binom{M}{n}}) - (\frac{nK}{M})^2 \\
&= (\sum_{x=1}^{n} x^2 \frac{\binom{K}{x}\binom{M-K}{n-x}}{\binom{M}{n}}) - (\frac{nK}{M})^2 \\
&= (\sum_{x=1}^{n} x^2 \frac{\frac{K}{x}\binom{K-1}{x-1}\binom{M-K}{n-x}}{\binom{M}{n}}) - (\frac{nK}{M})^2 \\
&= (K \sum_{x=1}^{n} x \frac{\binom{K-1}{x-1}\binom{M-K}{n-x}}{\binom{M}{n}}) - (\frac{nK}{M})^2 \\
\text{Setze } y = x - 1 \qquad &= (K \sum_{y=0}^{n-1} (y+1) \frac{\binom{K-1}{y}\binom{M-K}{n-(y+1)}}{\binom{M}{n}}) - (\frac{nK}{M})^2
\end{aligned}$$

$$= \left[K\left(\sum_{y=0}^{n-1} y \frac{\binom{K-1}{y}\binom{M-K}{n-(y+1)}}{\binom{M}{n}} + \frac{\binom{K-1}{y}\binom{M-K}{n-(y+1)}}{\binom{M}{n}}\right)\right] - \left(\frac{nK}{M}\right)^2$$

$$= \left[K\left(\sum_{y=1}^{n-1} y \frac{\frac{K-1}{y}\binom{K-2}{y-1}\binom{M-K}{n-(y+1)}}{\binom{M}{n}} + \sum_{y=0}^{n-1} \frac{\binom{K-1}{y}\binom{M-K}{n-(y+1)}}{\binom{M}{n}}\right)\right] - \left(\frac{nK}{M}\right)^2$$

$$= \left[K\left((K-1)\sum_{y=1}^{n-1} \frac{\binom{K-2}{y-1}\binom{M-K}{n-(y+1)}}{\binom{M}{n}} + \sum_{y=0}^{n-1} \frac{\binom{K-1}{y}\binom{M-K}{n-(y+1)}}{\binom{M}{n}}\right)\right] - \left(\frac{nK}{M}\right)^2$$

Setze $z = y - 1$
$$= \left[K\left((K-1)\sum_{z=0}^{n-2} \frac{\binom{K-2}{z}\binom{M-2-K+2}{n-2-(z+2)+2}}{\frac{M}{n}\frac{M-1}{n-1}\binom{M-2}{n-2}} + \sum_{y=0}^{n-1} \frac{\binom{K-1}{y}\binom{M-1-K+1}{n-1-(y+1)+1}}{\frac{M}{n}\binom{M-1}{n-1}}\right)\right] - \left(\frac{nK}{M}\right)^2$$

$$= \left[K\left((K-1)\frac{n}{M}\frac{n-1}{M-1} \underbrace{\sum_{z=0}^{n-2} \frac{\binom{K-2}{z}\binom{(M-2)-(K-2)}{(n-2)-z}}{\binom{M-2}{n-2}}}_{=1,\text{ da } Z\sim Hypergeom(M-2,K-2,n-2)} + \frac{n}{M} \underbrace{\sum_{y=0}^{n-1} \frac{\binom{K-1}{y}\binom{(M-1)-(K-1)}{(n-1)-y+}}{\binom{M-1}{n-1}}}_{=1,\text{ da } Y\sim Hypergeom(M-1,K-1,n-1)}\right)\right]$$
$$- \left(\frac{nK}{M}\right)^2$$

$$= K\left((K-1)\frac{n}{M}\frac{n-1}{M-1} + \frac{n}{M}\right) - \left(\frac{nK}{M}\right)^2$$

$$= (K-1)\frac{nK}{M}\frac{n-1}{M-1} + \frac{nK}{M} - \left(\frac{nK}{M}\right)^2$$

$$= \frac{nK}{M}\left((K-1)\frac{n-1}{M-1} + 1 - \frac{nK}{M}\right)$$

$$= \frac{nK}{M}\left(\frac{(K-1)(n-1)M}{M(M-1)} + \frac{M(M-1)}{M(M-1)} - \frac{nK(M-1)}{M(M-1)}\right)$$

$$= \frac{nK}{M}\left(\frac{KnM - nM - KM + M + M^2 - M - nKM + nK}{M(M-1)}\right)$$

$$= \frac{nK}{M}\left(\frac{M^2 - nM - KM + nk}{M(M-1)}\right)$$

$$= \frac{nK}{M}\left(\frac{(M-n)(M-k)}{M(M-1)}\right)$$

2

Gegeben:

Dichte einer Betaverteilung:

$$f(x;\alpha,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} I_{(0,1)}(x), \qquad \alpha,\beta > 0$$

Wobei B die Betafunktion ist:

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \Gamma(z) = \int_0^\infty t^{z-1}e^t \; dt \qquad \text{(Gammafunktion)}$$

Gesucht:

Erwartungswert $E(X)$ und Varianz $Var(X)$

Lösung:

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x;\alpha,\beta)dx \\
&= \int_0^1 x \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx \\
&= \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha}(1-x)^{\beta-1}dx \\
&= \frac{1}{B(\alpha,\beta)} \int_0^1 \frac{B(\alpha+1,\beta)}{B(\alpha+1,\beta)} x^{(\alpha+1)-1}(1-x)^{\beta-1}dx \\
&= \frac{1}{B(\alpha,\beta)} B(\alpha+1,\beta) \underbrace{\int_0^1 \frac{1}{B(\alpha+1,\beta)} x^{(\alpha+1)-1}(1-x)^{\beta-1}dx}_{=1 \text{ für } X\sim Beta(\alpha+1,\beta)} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+1+\beta)} \\
&= \frac{\Gamma(\alpha+\beta)\alpha\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta)(\alpha+\beta)\Gamma(\alpha+\beta)} \\
&= \frac{\alpha}{\alpha+\beta} \\
Var(X) &= \left(\int_{-\infty}^{\infty} x^2 f(x;\alpha,\beta)dx\right) - E(X)^2 \\
&= \left(\int_0^1 x^2 \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx\right) - \left(\frac{\alpha}{\alpha+\beta}\right)^2 \\
&= \left(\frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha+1}(1-x)^{\beta-1}dx\right) - \left(\frac{\alpha}{\alpha+\beta}\right)^2 \\
&= \left(\frac{1}{B(\alpha,\beta)} B(\alpha+2,\beta) \underbrace{\int_0^1 \frac{1}{B(\alpha+2,\beta)} x^{(\alpha+2)-1}(1-x)^{\beta-1}dx}_{=1 \text{ für } X\sim Beta(\alpha+2,\beta)}\right) - \left(\frac{\alpha}{\alpha+\beta}\right)^2 \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+2+\beta)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2 \\
&= \frac{\Gamma(\alpha+\beta)(\alpha+1)\alpha\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta)(\alpha+\beta+1)(\alpha+\beta)\Gamma(\alpha+\beta)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2 \\
&= \frac{\alpha^2+\alpha}{(\alpha+\beta)(\alpha+\beta+1)} \\
&= \frac{\alpha}{\alpha+\beta}\left(\frac{\alpha+1}{\alpha+\beta+1} - \frac{\alpha}{\alpha+\beta}\right) \\
&= \frac{\alpha}{\alpha+\beta}\left(\frac{(\alpha+1)(\alpha+\beta)-\alpha(\alpha+\beta+1)}{(\alpha+\beta+1)(\alpha+\beta)}\right) \\
&= \frac{\alpha}{\alpha+\beta}\left(\frac{\alpha^2+\alpha\beta+\alpha+\beta-\alpha^2-\alpha\beta-\alpha}{(\alpha+\beta+1)(\alpha+\beta)}\right) \\
&= \frac{\alpha}{\alpha+\beta} \frac{\beta}{(\alpha+\beta+1)(\alpha+\beta)} \\
&= \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}
\end{aligned}
$$

3

Gegeben:

Die Dichte einer Poissonverteilung:

$$f(x;\Theta) = \frac{e^{-\Theta}\Theta^x}{x!} I\{0,1,...\}(x), \qquad \Theta > 0$$

Wobei Θ einer Gammaverteilung mit der Dichte:

$$g(\Theta;r,\lambda) = \frac{\lambda^r}{\Gamma(r)} \Theta^{r-1} e^{-\lambda\Theta} I_{0,\infty}(\Theta), \qquad r,\lambda > 0$$

folgt.

Zu Zeigen:

Die Mischung von Poissonverteilungen

$$\int_0^\infty f(x;\Theta) \cdot g(\Theta;r,\lambda) d\Theta$$

folgt einer negativ Binomialverteilung mit den Parametern r und $p = \frac{\lambda}{\lambda+1}$

Lösung:

$$\begin{aligned}
\int_0^\infty f(x;\Theta) \cdot g(\Theta;r,\lambda) d\Theta &= \int_0^\infty \frac{e^{-\Theta}\Theta^x}{x!} I_{\{0,1,...\}}(x) \cdot \frac{\lambda^r}{\Gamma(r)} \Theta^{r-1} e^{-\lambda\Theta} d\Theta \\
&= \frac{\lambda^r}{\Gamma(r)x!} I_{\{0,1,...\}}(x) \int_0^\infty e^{-\Theta}\Theta^x \Theta^{r-1} e^{-\lambda\Theta} d\Theta \\
&= \frac{\lambda^r}{\Gamma(r)x!} I_{\{0,1,...\}}(x) \int_0^\infty \Theta^{(x+r)-1} e^{-(1+\lambda)\Theta} d\Theta \\
&= \frac{\lambda^r}{\Gamma(r)x!} I_{\{0,1,...\}}(x) \int_0^\infty \frac{\Gamma(r+x)}{(\lambda+1)^{r+x}} \frac{(\lambda+1)^{r+x}}{\Gamma(r+x)} \Theta^{(x+r)-1} e^{-(1+\lambda)\Theta} d\Theta \\
&= \frac{\lambda^r}{\Gamma(r)x!} I_{\{0,1,...\}}(x) \frac{\Gamma(r+x)}{(\lambda+1)^{r+x}} \underbrace{\int_0^\infty \underbrace{\frac{(\lambda+1)^{r+x}}{\Gamma(r+x)} \Theta^{(x+r)-1} e^{-(1+\lambda)\Theta}}_{\Theta\sim\Gamma(r+x,\lambda+1)} d\Theta}_{=1} \\
&= \frac{\lambda^r\Gamma(r+x)}{\Gamma(r)x!(\lambda+1)^{r+x}} I_{\{0,1,...\}}(x) \\
&= \frac{\lambda^r\Gamma(r+x)}{\Gamma(r)x!(\lambda+1)^r(\lambda+1)^x} I_{\{0,1,...\}}(x) \\
&= \frac{\Gamma(r+x)}{\Gamma(r)x!(\lambda+1)^x} \left(\frac{\lambda}{\lambda+1}\right)^r I_{\{0,1,...\}}(x) \\
&= \frac{\Gamma(r+x)}{\Gamma(r)x!(\lambda+1)^x} \left(\frac{\lambda}{\lambda+1}\right)^r I_{\{0,1,...\}}(x)
\end{aligned}$$

4

a)

Gegeben:

Die Dichte der Normalverteilung ist:

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$

Zu Zeigen:

$$A = \int_{-\infty}^{\infty} f(x;\mu,\sigma)dx = 1$$

Lösung:

Wir zeigen, dass $A^2 = 1$:

$$A^2 = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)} dx \int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)} dx$$

d)

Zu Zeigen:

Die momenterzeugende Funktion einer standardnormalverteilten ZV ist

$$M(t) = e^{\frac{t^2}{2}}$$

Lösung:

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f(x) dx \\ &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\cdot 1} e^{\left(-\frac{1}{2}\right)\left(\frac{x-0}{1}\right)^2} dx \qquad = \end{aligned}$$