Lösungen zum Übungsblatt 3

1.

$$E(X) = \sum_{x=1}^{\infty} x\,\theta(1-\theta)^{x-1}$$

$$\begin{array}{ccccccc}
x=1 & x=2 & x=3 & & & \\
\hline
= \quad \theta & +\theta(1-\theta) & +\theta(1-\theta)^2 & +\ldots+ & = & \theta\sum_{x=0}^{\infty}(1-\theta)^x \\
 & +\theta(1-\theta) & +\theta(1-\theta)^2 & +\ldots+ & = & \theta(1-\theta)\sum_{x=0}^{\infty}(1-\theta)^x \\
 & & +\theta(1-\theta)^2 & +\ldots+ & = & \theta(1-\theta)^2\sum_{x=0}^{\infty}(1-\theta)^x
\end{array}$$

$$\begin{aligned}
\Longrightarrow \quad E(X) &= \theta\sum_{x=0}^{\infty}(1-\theta)^x + \theta(1-\theta)\sum_{x=0}^{\infty}(1-\theta)^x + \theta(1-\theta)^2\sum_{x=0}^{\infty}(1-\theta)^x \\
&= \frac{1}{1-(1-\theta)}\left(\theta\sum_{x=0}^{\infty}(1-\theta)^x\right) = \frac{1}{\theta}
\end{aligned}$$

2. Nichtzentrale Momente:

$$\mu'_r = E(X^r) = 1^r\cdot\theta + 0^r(1-\theta) = \theta$$

Zentrale Momente:

$$\mu_2 = E\big[(X-\theta)^2\big] = (-\theta)^2(1-\theta) + (1-\theta)^2\theta = \theta(1-\theta)$$

$$E\left[X^2 - 2E(X)X + E[(X)^2]\right] = \theta - 2\theta^2 + \theta^2 = \theta(1-\theta)$$

$$\begin{aligned}
E[(X-\theta)^3] &= E\left(X^3 - 3X^2E(X) + 3XE[(X)^2] - E[(X)^3]\right) \\
&= \theta - 3\theta^2 + 3\theta^3 - \theta^3 = \theta - 3\theta^2 + 2\theta^2 \\
&= \theta(1-\theta)(1-2\theta)
\end{aligned}$$

$$\begin{aligned}
E[(X-\theta)^4] &= E\left[X^4 - 4X^3E(X) + 6X^2E[(X)^2] - 4XE[(X)^3] + E[(X)^4]\right] \\
&= (-\theta)^4(1-\theta) + (1-\theta)^4\theta \\
&= \theta(1-\theta)(1-3\theta+3\theta^2)
\end{aligned}$$

3. Hier alternative Lösung durch Substitution:

$$\mathrm{E}(X^r) = \int_0^\infty x^r \lambda e^{-\lambda x} dx$$

Hinweis: Gammafunktion

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx \qquad a \in \mathbb{R}_0; \qquad \Gamma(n) = (n-1)! \qquad n \in \mathbb{N}$$

Substitution: $\quad \lambda x = y \qquad dx = dy\lambda^{-1}$

$$\begin{aligned} \Rightarrow \mathrm{E}(X^r) &= \int_0^\infty \left(\frac{y}{\lambda}\right)^r \lambda e^{-y} \lambda^{-1} dy = \lambda^{-r} \int_0^\infty y^r e^{-y} dy \\ &= \lambda^{-r}\Gamma(r+1) \;\Rightarrow\; \mathrm{E}(X^r) = \frac{r!}{\lambda^r} \end{aligned}$$

4. (a)

$$\mathrm{E}[(X-b)^2] = \int_{-\infty}^{\infty} \underbrace{(x-b)^2}_{x^2 - 2bx + b^2} f(x)dx \longrightarrow \min!$$

$$x^2 - 2bx + b^2 \longrightarrow \int x^2 f(x)dx - 2b \int x f(x)dx + b^2 \int f(x)dx$$

$$\begin{aligned} \frac{\partial \mathrm{E}[(X-b)^2]}{\partial b} &= -2\int x f(x)dx + 2b \int f(x)dx = -2\mathrm{E}(X) + 2b \overset{!}{=} 0 \\ &\Longrightarrow b = \mathrm{E}(X) \qquad \text{q.e.d.} \end{aligned}$$

(b)

Leibniz-Regel

$$\begin{aligned} I &= \int_{l(z)}^{h(z)} f(s,z)ds \\ \frac{\partial I}{\partial z} &= \int_{l(z)}^{h(z)} \frac{\partial f}{\partial z} ds + \frac{\partial h}{\partial z} f(h(z),z) - \frac{\partial l}{\partial z} f(l(z),z) \end{aligned}$$

$$\begin{aligned}
\mathrm{E}(|X-b|) &= \int\limits_{-\infty}^{\infty} |x-b| f(x)dx = \int\limits_{b}^{\infty} (x-b)f(x)dx - \int\limits_{-\infty}^{b} (x-b)f(x)dx \\
\frac{\partial \mathrm{E}(|X-b|)}{\partial b} &= -\int\limits_{b}^{\infty} f(x)dx + 0 - 1\cdot(b-b)f(b) - \\
& \left(-\int\limits_{-\infty}^{b} f(x)dx + 1\cdot(b-b)f(b) - 0\right) \stackrel{!}{=} 0
\end{aligned}$$

<u>Hinweis</u>: $z = b$

$$\left.\begin{array}{lll} h(z) & \Rightarrow & b, \infty \\ l(z) & \Rightarrow & -\infty, b \end{array}\right| \; (x-b)f(x) = f(s,z)$$

$$\begin{aligned}
&\Longrightarrow \quad -\int\limits_{b}^{\infty} f(x)dx + \int\limits_{-\infty}^{b} f(x)dx = \frac{\partial \mathrm{E}(|X-b|)}{\partial b} \\
&\Longrightarrow \quad 1 - \int\limits_{b}^{\infty} f(x)dx - \int\limits_{b}^{\infty} f(x)dx = 1 - 2\int\limits_{b}^{\infty} f(x)dx \stackrel{!}{=} 0 \\
&\Longrightarrow \quad \frac{1}{2} = \int\limits_{b}^{\infty} f(x)dx = \int\limits_{-\infty}^{b} f(x)dx \Rightarrow b = x_{\mathrm{MEDIAN}}
\end{aligned}$$

5. Eine symmetrische Dichte um $x = c$ impliziert:

$$f(c + x_0) = f(c - x_0) \quad \forall x_0 \in D(x)$$

Es gilt:

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{c} xf(x)dx + \int_{c}^{\infty} xf(x)dx$$

$$\begin{array}{llll} x & = c - x_0 & \Leftrightarrow \; dx = -dx_0 & \text{[1. Integral]} \\ x & = c + x_0 & \Leftrightarrow \; dx = dx_0 & \text{[2. Integral]} \end{array}$$

Substitution

$$\mathrm{E}(X) = -\int_{+\infty}^{0} (c-x_0)f(c-x_0)dx_0 + \int_0^{\infty} (c+x_0)f(c+x_0)dx_0$$

$$= \int_0^{\infty} (c-x_0)f(c-x_0)dx_0 + \int_0^{\infty} (c+x_0)f(c+x_0)dx_0$$

$$= \overbrace{\int_0^{\infty} cf(c-x_0)dx_0 - \int_0^{\infty} x_0 f(c-x_0)dx_0}^{} + \overbrace{\int_0^{\infty} cf(c+x_0)dx_0 + \int_0^{\infty} x_0 f(c+x_0)dx_0}^{}$$

$$= \underbrace{\int_0^{\infty} cf(c-x_0)dx_0 + \int_0^{\infty} cf(c+x_0)dx_0}_{} \underbrace{- \int_0^{\infty} x_0 f(c-x_0)dx_0 + \int_0^{\infty} x_0 f(c+x_0)dx_0}_{=0}$$

$$= 2c\int_0^{\infty} f(c+x_0)dx_0 \quad \leftarrow \text{(wg. Symmetrie)} \uparrow$$

Für $x = c + x_0 \;\Leftrightarrow\; dx = dx_0$

$$\mathrm{E}(X) = 2c\int_c^{\infty} f(x)dx = 2c(1-F(c)) = 2c\left(1-\frac{1}{2}\right) = c$$

$$\begin{aligned}
\mu_3 &\stackrel{!}{=} 0 \\
\mathrm{E}[(X-\mu)^3] &= \int_{-\infty}^{\infty} (X-\mu)^3 f(x)dx \\
&= \int_{-\infty}^{\mu} (X-\mu)^3 f(x)dx + \int_{\mu}^{\infty} (X-\mu)^3 f(x)dx \qquad \text{(wg. Symmetrie)}
\end{aligned}$$

$$\begin{aligned}
x = x_0 + \mu &\quad\Leftrightarrow\quad dx = dx_0 && \text{[1. Integral]} \\
x = -x_0 + \mu &\quad\Leftrightarrow\quad dx = -dx_0 && \text{[2. Integral]}
\end{aligned}$$

Substitution

$$\begin{aligned}
\mathrm{E}[(X-\mu)^3] &= \int_{-\infty}^{0} x_0^3 f(x_0+\mu)dx_0 + \int_0^{-\infty} (-x_0)^3 f(-x_0+\mu)(-dx_0) \\
&= \int_{-\infty}^{0} x_0^3 f(x_0+\mu)dx_0 - \int_{-\infty}^{0} (-x_0)^3 f(-x_0+\mu)(-dx_0) \\
\text{(Rücksubstitution)} &= \int_{-\infty}^{0} x^3 f(x)dx - \int_{-\infty}^{0} x^3 f(x)dx \\
&= 0
\end{aligned}$$

6. (a)

$$\mu^{2k+1} = \int_{-\infty}^{\infty} x^{2k+1} f(x)dx = \int_{-\infty}^{0} x^{2k+1} f(x)dx + \int_{0}^{\infty} x^{2k+1} f(x)dx$$

Substituiere im 1. Integral $x = -x_0$, $dx = -dx_0$ und im 2. $x = x_0$ und $dx = dx_0$. Dann ergibt sich (wegen Symmetrie der Dichtefunktion und weil $2k+1$ ungerade)

$$\begin{aligned} & -\int_{\infty}^{0} (-x_0)^{2k+1} f(x_0)dx_0 + \int_{0}^{\infty} x_0^{2k+1} f(x_0)dx_0 \\ = & \int_{0}^{\infty} (-x_0^{2k+1} + x_0^{2k+1}) f(x_0)dx_0 = 0 \ . \end{aligned}$$

(b) Per Induktion:

Induktionsanfang: Sei $k = 1$, dann gilt

$$\mu_2 \ = \ \operatorname{var}(X) = \prod_{i=1}^{1}(2 - 2i + 1) = 1 \ .$$

Induktionsvoraussetzung: für $\exists k \in \mathbb{N}$ gilt, dass

$$\mu_{2k} \ = \ \prod_{i=1}^{k}(2k - 2i + 1) \quad \text{für } k = 1, 2, \ldots$$

Induktionsbehauptung: dann gelte für $k+1$

$$\mu_{2(k+1)} \ = \ \prod_{i=1}^{k+1}(2(k+1) - 2i + 1) \quad \text{für } k = 1, 2, \ldots$$

Induktionsschritt: Folgern von k auf $k+1$.

Es gilt

$$\begin{aligned} \mu_{2k} &= \int_{-\infty}^{\infty} x^{2k} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \frac{1}{\sqrt{2\pi}} \left\{ \left[\frac{1}{\sqrt{2\pi}} \frac{x^{2k+1}}{2k+1} e^{-\frac{x^2}{2}} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{1}{2k+1} x^{2k+1} (-x) e^{-\frac{x^2}{2}} \right\} \\ &= 0 + \frac{1}{2k+1} \int_{-\infty}^{\infty} x^{2k+2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \frac{1}{2k+1} \mu_{2(k+1)} \ . \end{aligned}$$

Also ist

$$\begin{aligned}
\mu_{2k}\cdot(2k+1) &= \mu_{2(k+1)} = \prod_{i=1}^{k}(2k-2i+1)\cdot(2k+1)\\
&= [(2k-1)(2k-3)\ldots 1](2k+1)\\
&= \prod_{i=1}^{k+1}[2(k+1)-2i+1]\ .
\end{aligned}$$

7.

$$\begin{aligned}
\int\limits_{-\infty}^{\infty} xf(x)dx &= \int\limits_{-\infty}^{\infty} x\frac{1}{\pi(1+x^2)}dx\\
&= \frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{x}{1+x^2}dx = \frac{1}{\pi}\left[\frac{1}{2}\ln(1+x^2)\right]_{-\infty}^{\infty}\\
&= 0
\end{aligned}$$

Aber:

$$\begin{aligned}
\int\limits_{-\infty}^{\infty} |x|f(x)dx &= \int\limits_{-\infty}^{\infty} |x|\frac{1}{\pi(1+x^2)}dx\\
&= 2\int\limits_{0}^{\infty}\frac{x}{\pi(1+x^2)}dx = \frac{2}{\pi}\int\limits_{0}^{\infty}\frac{x}{1+x^2}dx\\
&= \frac{2}{\pi}\left[\frac{1}{2}\ln(1+x^2)\right]_{0}^{\infty} = \infty
\end{aligned}$$

$\Longrightarrow$ kein Erwartungswert; keine höheren Momente; Existenzbedingung nicht erfüllt

8. (a) i.

$$M_X(t) = \int_0^1 e^{xt}dx = \frac{1}{t}\left(e^t-1\right)$$

Für $t\to 0$ De L'Hôpitalsche Regel:

$$\frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)} \text{ sofern } \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0} \text{ oder } \frac{\infty}{\infty}$$

Hier:

$$\lim_{t\to 0}\frac{e^t-1}{t}=\lim_{t\to 0}\frac{e^t}{1}=1$$

$$M_X(t)=\begin{cases}\frac{1}{t}(e^t-1) & \text{für } t\neq 0\\ 1 & \text{für } t=0\end{cases}$$

ii.

$$\begin{aligned}M_X(t) &= \int_0^\infty e^{tx}\lambda e^{-\lambda x}dx=\int_0^\infty \lambda e^{(t-\lambda)x}dx\\ &= \left[\frac{\lambda}{t-\lambda}e^{(t-\lambda)x}\right]_0^\infty\\ &= \begin{cases}\frac{\lambda}{\lambda-t} & \text{für } t<\lambda,\\ 0 & \text{sonst.}\end{cases}\end{aligned}$$

iii.

$$\begin{aligned}M_X(t) &= \int_0^\infty e^{tx}xe^{-x}dx=\int_0^\infty xe^{(t-1)x}dx\\ &= \left[x\frac{1}{(t-1)}e^{(t-1)x}\right]_0^\infty-\int_0^\infty\frac{1}{(t-1)}e^{(t-1)x}dx\\ &= \left[\frac{x}{(t-1)}e^{(t-1)x}\right]_0^\infty-\left[\frac{1}{(t-1)^2}e^{(t-1)x}\right]_0^\infty\end{aligned}$$

Für $t<1$ ist $M_X(t)$ endlich, somit existiert die MGF $=\frac{1}{(t-1)^2}$.

[Partielle Integration: $x=f(x)$; $e^{(t-1)x}=g'(x)$]

iv. X hat einen beschränkten Wertebereich $\rightarrow$ alle Momente existieren und somit die MGF

$$\begin{aligned}M_X(t) &= \sum_{x=0}^{3}e^{tx}\frac{1}{8}\binom{3}{x}\\ &= \frac{1}{8}\sum_{x=0}^{3}\binom{3}{x}[e^t]^x\cdot 1^{3-x}\end{aligned}$$

$$\left[\text{Binomialtheorem: } (z+y)^n=\sum_{r=0}^{n}\binom{n}{r}z^ry^{n-r}\quad ; n=1,2,\ldots\right]$$

$$= \frac{1}{8}(e^t + 1)^3$$

(b) i.

$$M_X(t) = \begin{cases} \frac{1}{t}(e^t - 1) & \text{für } t \neq 0 \\ 1 & \text{für } t = 0 \end{cases}$$

$$\begin{aligned} M_X'(t) &= -\frac{1}{t^2}(e^t - 1) + \frac{1}{t}e^t = \frac{(t-1)e^t + 1}{t^2} \\ \mathrm{E}(X) &= M_X'(0) = \lim_{t\to 0} \frac{te^t - e^t + 1}{t^2} \underset{\substack{\uparrow \\ \text{De L'Hôpital}}}{=} \lim_{t\to 0} \frac{te^t}{2t} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} M_X''(t) &= \frac{2}{t^3}(e^t - 1) - \frac{e^t}{t^2} - \frac{1}{t^2}e^t + \frac{1}{t}e^t \\ &= \frac{2e^t}{t^3} - \frac{2}{t^3} - \frac{2e^t}{t^2} + \frac{e^t}{t} \\ \mathrm{E}(X^2) &= M_X''(0) = \lim_{t\to 0} \frac{2e^t - 2 - 2te^t + t^2e^t}{t^3} \underset{\substack{\uparrow \\ \text{De L'Hôpital}}}{=} \frac{1}{3} \end{aligned}$$

ii.

$$\mathrm{E}(X) = M_X'(t)|_{t=0} = \left.\frac{\lambda}{(t-\lambda)^2}\right|_{t=0} = \frac{1}{\lambda}$$

$$\mathrm{E}(X^2) = M_X''(t)|_{t=0} = \left.\frac{2\lambda}{(t-\lambda)^3}\right|_{t=0} = \frac{2}{\lambda^2}$$

iii.

$$\begin{aligned} M_X(t) &= \frac{1}{(t-1)^2} \\ M_X'(t) &= -\frac{2}{(t-1)^3} & \Rightarrow M_X'(0) = \mathrm{E}(X) = 2 \\ M_X''(t) &= \frac{6}{(t-1)^4} & \Rightarrow M_X''(0) = \mathrm{E}(X^2) = 6 \end{aligned}$$

iv.

$$\begin{aligned} M_X(t) &= \frac{1}{8}(e^t+1)^3 \\ M'_X(t) &= \frac{3}{8}(e^t+1)^2 e^t & \Rightarrow M'_X(0) = \mathrm{E}(X) = \frac{3}{2} \\ M''_X(t) &= \frac{3}{8}[e^t(e^t+1)^2 + e^{2t}2(e^t+1)] & \Rightarrow M''_X(0) = \mathrm{E}(X^2) = 3 \end{aligned}$$

9. (a)

$$M_{\underline{X}}(\underline{t}) = \frac{2}{(t_2-1)(t_1+t_2-2)}$$

(b)

$$\begin{aligned} \mathrm{E}(X_1) &= \frac{1}{2} & \mathrm{E}(X_1^2) = \frac{1}{2} \\ \mathrm{E}(X_2) &= \frac{3}{2} & \mathrm{E}(X_2^2) = \frac{7}{2} \end{aligned}$$

(c) Die Randdichte von X_1 ergibt sich über $M_{X_1}(t_1, t_2 = 0)$. Damit ist

$$f(x_1) = 2e^{-2x_1} I_{(0,\infty)}(x_1) \quad \text{mit } X_1 \sim \mathrm{Exp}(2)$$

10.

$$\mathrm{corr}(X_1, X_2) = \frac{\mathrm{cov}(X_1, X_2)}{\sqrt{\mathrm{var}(X_1)}\sqrt{\mathrm{var}(X_2)}}$$

$$\begin{aligned} \mathrm{E}(X_1) &= \frac{1}{e} & \mathrm{E}(X_1^2) = \frac{1}{e} \\ \mathrm{E}(X_2) &= \frac{1}{e^2} & \mathrm{E}(X_2^2) = \frac{1}{e^2} \\ \mathrm{E}(X_1 X_2) &= 0 \end{aligned}$$

$$\begin{aligned} \mathrm{corr}(X_1, X_2) &= \frac{0 - \frac{1}{e^3}}{\sqrt{\frac{1}{e} - \frac{1}{e^2}}\sqrt{\frac{1}{e^2} - \frac{1}{e^4}}} = \frac{-\sqrt{e^{-6}}}{\sqrt{e^{-1} - e^{-2}}\sqrt{e^{-2} - e^{-4}}} \\ &= \frac{-1}{\sqrt{e^6(e^{-1} - e^{-2})(e^{-2} - e^{-4})}} = \frac{-1}{(e-1)\sqrt{e+1}} \end{aligned}$$

11. (a)

$$Z_1 = \frac{1}{n}X_1 + \ldots + \frac{1}{n}X_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\begin{aligned}
M_{Z_1}(t) = \mathrm{E}(e^{tz}) &= \prod_{i=1}^{n} \mathrm{E}\left(e^{\frac{X_i}{n}t}\right) \\
&= \prod_{i=1}^{n} M_X\left(\frac{t}{n}\right) \\
&= \prod_{i=1}^{n} e^{\mu\frac{t}{n}+\frac{1}{2}\sigma^2\left(\frac{t}{n}\right)^2} \\
&= e^{\mu\frac{t}{n}+\frac{1}{2}\sigma^2\left(\frac{t}{n}\right)^2+\ldots+\mu\frac{t}{n}+\frac{1}{2}\sigma^2\left(\frac{t}{n}\right)^2} \\
&= e^{\mu t+\frac{\sigma^2 t}{2n}}
\end{aligned}$$

$$\Rightarrow Z_1 \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

(b)

$$\begin{aligned}
M_{Z_2}(t) &= M_{X_1}\left(\frac{10}{9+n}t\right)\prod_{i=2}^{n} M_{X_i}\left(\frac{t}{9+n}\right) \\
&= e^{\frac{\mu 10}{9+n}t+\frac{1}{2}\sigma^2\frac{100}{(9+n)^2}t^2+\frac{\mu}{9+n}t+\frac{1}{2}\sigma^2\frac{1}{(9+n)^2}t^2+\ldots} \\
&= e^{\mu\left(\frac{10}{9+n}+\frac{n-1}{9+n}\right)t+\frac{t^2\sigma^2}{2}\left(\frac{100}{(9+n)^2}+\frac{n-1}{(9+n)^2}\right)} \\
&= e^{\mu t+\frac{t^2\sigma^2}{2}\left(\frac{n+99}{(9-n)^2}\right)}
\end{aligned}$$

$$\Rightarrow Z_2 \sim \mathcal{N}\left(\mu, \frac{n+99}{(9+n)^2}\sigma^2\right)$$

(c)

$\underline{Y_1} = \underline{X}$

$$\begin{aligned}
M_{\underline{X}}(\underline{t}) &= \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} e^{t_1x_1+\ldots+t_nx_n} f(x_1,\ldots,x_n)dx_1\ldots dx_n \\
&= \prod_{i=1}^{n}\int_{-\infty}^{\infty} e^{t_ix_i}f(x_i)dx_i \\
&= \prod_{i=1}^{n} M_{X_i}(t_i) = e^{\mu\sum_{i=1}^{n}t_i+\frac{\sigma^2}{2}\sum_{i=1}^{n}t_i^2}
\end{aligned}$$

$\Rightarrow$ multivariate ZV $\quad \underline{Y_1} \sim \mathcal{N}\left(\underline{\mu}, \Sigma\right)$, wobei $\sigma_{ij} = 0 \quad \forall i \neq j$.

(d)

$$\underline{Y_2} = (10X_1, X_2, \ldots, X_n) \qquad \text{analog zu (c)}$$

$$\begin{aligned} M_{\underline{Y_2}}(\underline{t}) &= M_{X_1}(t_1 10) M_{X_2}(t_2) \ldots M_{X_n}(t_n) \\ &= e^{\mu\left(10t_1+\sum_{i=2}^{n} t_i\right)+\frac{\sigma^2}{2}\left(100t_1^2+\sum_{j=2}^{n} t_j^2\right)} \end{aligned}$$

$\Rightarrow$ multivariate ZV

12. (a)

$$\begin{aligned} \underline{X} &= (X_1, X_2), \quad \underline{\mu} = (\mu_1, \mu_2), \quad \text{und} \quad \underline{t} = (t_1, t_2) \\ \underline{X} &\sim \mathcal{N}_2\left(\underline{\mu}, \Sigma\right), \text{ mit } \Sigma = \operatorname{cov}(\underline{X}) = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \underline{Z} &= (Z_1, Z_2) \\ Z_1 &= aX_1 + bX_2 \\ Z_2 &= cX_1 + dX_2 \end{aligned}$$

Da $\underline{X}$ multivariat normalverteilt ist und $\underline{Z}$ eine Linearkombination von $\underline{X}$, nämlich $\underline{Z} = \mathbf{A}\underline{X}$

$$\mathbf{Z} = \mathbf{A} \times \mathbf{X} = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

gilt, dass

$$\underline{Z} \sim \mathcal{MN}\left(\mathbf{A}\underline{\mu}, \mathbf{A}\Sigma\mathbf{A}'\right), \text{ mit } \Sigma = \operatorname{cov}(\underline{X}) .$$

Da

$$\begin{aligned} \mathrm{E}(\underline{Z}) &= \mu_{\underline{Z}} = \mathbf{A}\underline{\mu} \\ \operatorname{cov}(\underline{Z}) &= \Sigma_{\underline{Z}} = \mathbf{A}\Sigma\mathbf{A}' \end{aligned}$$

und wg. Eingdeutigkeitstheorem (3.16), MGF $\Leftrightarrow$ PDF, ist die MGF von $\underline{Z}$

$$\begin{aligned} M_{\underline{Z}}(\underline{t}) &= \exp\left\{\underline{t}(\mathbf{A}\underline{\mu})' + \frac{1}{2}\underline{t}(\mathbf{A}\Sigma\mathbf{A}')\underline{t}'\right\} \\ &= \exp\left\{(\underline{t}\mathbf{A})\underline{\mu}' + \frac{1}{2}\underline{t}\mathbf{A}\Sigma(\underline{t}\mathbf{A}')\right\} \\ &\Rightarrow M_{\underline{X}}(\underline{t}\mathbf{A}) \end{aligned}$$

(b)

$$
\begin{aligned}
M_{aX_1+bX_2}(\underline{t}) &= \exp\left\{(a\mu_1 + b\mu_2)\underline{t} + \frac{1}{2}\sigma^2\underline{t}^2\right\} \\
\sigma^2 &= \operatorname{var}(aX_1 + bX_2) = a^2\sigma_{11} + b^2\sigma_{22} + 2ab\sigma_{12}
\end{aligned}
$$

13.

$$
\begin{aligned}
M_{X_i}(t) &= \mathrm{E}[e^{xt}] = e^{i(e^t-1)} \\
Y_i &= \frac{X_i - i}{\sqrt{i}} = \frac{X_i}{\sqrt{i}} - \frac{i}{\sqrt{i}} = \frac{X_i}{\sqrt{i}} - \sqrt{i} \\
\Rightarrow M_{Y_i}(t) &= \mathrm{E}\left[e^{t\cdot\left(\frac{X_i-i}{\sqrt{i}}\right)}\right] = \mathrm{E}\left[e^{t\cdot\left(\frac{X_i}{\sqrt{i}}-\sqrt{i}\right)}\right] \\
&= \mathrm{E}\left[e^{-t\sqrt{i}} \cdot e^{t\cdot\left(\frac{X_i}{\sqrt{i}}\right)}\right] \\
&= e^{-\sqrt{i}\cdot t} \cdot \underbrace{\mathrm{E}\left[e^{\frac{X_i t}{\sqrt{i}}}\right]}_{=M_{X_i}\left(\frac{t}{\sqrt{i}}\right)} \\
&= e^{-\sqrt{i}\cdot t} \cdot \exp\left\{i\left(e^{\frac{t}{\sqrt{i}}} - 1\right)\right\} \\
&= \exp\left\{i\left(e^{\frac{t}{\sqrt{i}}} - 1\right) - \sqrt{i}\cdot t\right\} \\
\text{[Taylorreihenentw.]} &= \exp\left\{i\left[\left(\frac{t}{i^{\frac{1}{2}}}\right)^0 + \left(\frac{t^1}{i^{\frac{1}{2}}\cdot 1!}\right) + \left(\frac{t^2}{i^{\frac{2}{2}}\cdot 2!}\right) + \left(\frac{t^3}{i^{\frac{3}{2}}\cdot 3!}\right) + \ldots - 1\right] - \sqrt{i}\cdot t\right\} \\
&= \exp\left\{i\left(1 + \frac{t}{\sqrt{i}} + \frac{t^2}{i\cdot 2!} + \frac{t^3}{i^{\frac{3}{2}}\cdot 3!} + \frac{t^4}{i^2\cdot 4!} + \ldots - 1\right) - \sqrt{i}\cdot t\right\} \\
&= \exp\left\{i\left(\frac{t}{\sqrt{i}} + \frac{t^2}{i\cdot 2!} + \frac{t^3}{i^{\frac{3}{2}}\cdot 3!} + \frac{t^4}{i^2\cdot 4!} + \ldots\right) - \sqrt{i}\cdot t\right\} \\
&= \exp\left\{\frac{it}{\sqrt{i}} + \frac{it^2}{i\cdot 2!} + \frac{it^3}{i^{\frac{3}{2}}\cdot 3!} + \frac{it^4}{i^2\cdot 4!} + \ldots - \sqrt{i}\cdot t\right\} \\
&= \exp\left\{\sqrt{i}\cdot t + \frac{it^2}{i\cdot 2!} + \frac{it^3}{i^{\frac{3}{2}}\cdot 3!} + \frac{it^4}{i^2\cdot 4!} + \ldots - \sqrt{i}\cdot t\right\} \\
&= \exp\left\{\frac{t^2}{2!} + \frac{t^3}{i^{\frac{1}{2}}\cdot 3!} + \frac{t^4}{i\cdot 4!} + \ldots\right\} \\
\lim_{i\to\infty} &= \exp\left\{\frac{t^2}{2} + 0 + 0 + \ldots\right\} = \exp\left\{\frac{t^2}{2}\right\} \\
\Rightarrow Y &\sim \mathcal{N}(0,1)
\end{aligned}
$$

14. (a) Der Träger ist endlich, somit bestehen Momente beliebiger Ordnung ($| x_{ij} | < \infty; \forall(i,j)$).

(b) Es gilt:

$$f(x_1) = \begin{cases} 0{,}2 & \text{für } x_1 = 2 \\ 0{,}5 & \text{für } x_1 = 4 \\ 0{,}3 & \text{für } x_1 = 8 \end{cases}$$

$$f(x_2) = \begin{cases} 0{,}2 & \text{für } x_2 = 1 \\ 0{,}5 & \text{für } x_2 = 2 \\ 0{,}3 & \text{für } x_2 = 5 \end{cases}$$

Gesucht:

$$\mathrm{var}(\underline{X}) = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} \qquad \text{Varianz-Kovarianz-Matrix}$$

$$\begin{aligned}
\mathrm{E}(X_1) &= \sum_{x_1 \in R(X_1)} x_1 \cdot f(x_1) = 2 \cdot 0{,}2 + 4 \cdot 0{,}5 + 8 \cdot 0{,}3 = 4{,}8 \\
\mathrm{E}(X_1^2) &= \sum_{x_1 \in R(X_1)} x_1^2 \cdot f(x_1) = 2^2 \cdot 0{,}2 + 4^2 \cdot 0{,}5 + 8^2 \cdot 0{,}3 = 28 \\
\mathrm{E}(X_2) &= \sum_{x_2 \in R(X_2)} x_2 \cdot f(x_2) = 1 \cdot 0{,}2 + 2 \cdot 0{,}5 + 5 \cdot 0{,}3 = 2{,}7 \\
\mathrm{E}(X_2^2) &= \sum_{x_2 \in R(X_2)} x_2^2 \cdot f(x_2) = 1^2 \cdot 0{,}2 + 2^2 \cdot 0{,}5 + 5^2 \cdot 0{,}3 = 9{,}7 \\
\mathrm{E}(X_1 X_2) &= \sum_{x_1 \in R(X_1)} \sum_{x_2 \in R(X_2)} x_1 \cdot x_2 \cdot f(x_1, x_2) \\
&= 2 \cdot 1 \cdot 0{,}1 + 4 \cdot 1 \cdot 0{,}1 + 2 \cdot 2 \cdot 0{,}1 + 4 \cdot 2 \cdot 0{,}3 \\
&\quad + 8 \cdot 2 \cdot 0{,}1 + 4 \cdot 5 \cdot 0{,}1 + 8 \cdot 5 \cdot 0{,}2 = 15
\end{aligned}$$

$$\begin{aligned}
\mathrm{var}(X_1) &= \sigma_1^2 = \mathrm{E}(X_1^2) - [\mathrm{E}(X_1)]^2 = 28 - (4{,}8)^2 = 4{,}96 \\
\mathrm{var}(X_2) &= \sigma_2^2 = \mathrm{E}(X_2^2) - [\mathrm{E}(X_2)]^2 = 9{,}7 - (2{,}7)^2 = 2{,}41 \\
\mathrm{cov}(X_1, X_2) &= \sigma_{12} = \mathrm{E}(X_1 X_2) - \mathrm{E}(X_1)\mathrm{E}(X_2) = 15 - 4,8 \cdot 2{,}7 = 2{,}04 \\
&\Downarrow \\
\mathrm{var}(\underline{X}) &= \begin{pmatrix} 4{,}96 & 2{,}04 \\ 2{,}04 & 2{,}41 \end{pmatrix}
\end{aligned}$$

(c)

$$\mathrm{E}(X_1 \mid X_2) = \begin{cases} 2\frac{0,1}{0,2} + 4\frac{0,1}{0,2} = 3 & \text{für } x_2 = 1 \\ 2\frac{0,1}{0,5} + 4\frac{0,3}{0,5} + 8\frac{0,1}{0,5} = 4,4 & \text{für } x_2 = 2 \\ 4\frac{0,1}{0,3} + 8\frac{0,2}{0,3} \approx 6\frac{2}{3} & \text{für } x_2 = 5 \end{cases}$$

$$\mathrm{E}(X_2 \mid X_1) = \begin{cases} 1\frac{0,1}{0,2} + 2\frac{0,1}{0,2} = 1,5 & \text{für } x_1 = 2 \\ 1\frac{0,1}{0,5} + 2\frac{0,3}{0,5} + 5\frac{0,1}{0,5} = 2,4 & \text{für } x_1 = 4 \\ 2\frac{0,1}{0,3} + 5\frac{0,2}{0,3} = 4 & \text{für } x_1 = 8 \end{cases}$$

$$f[\mathrm{E}(X_1 \mid X_2)] = \begin{cases} 0,2 & \text{für } \mathrm{E}(\cdot) = 3 \\ 0,5 & \text{für } \mathrm{E}(\cdot) = 4,4 \\ 0,3 & \text{für } \mathrm{E}(\cdot) = 6\frac{2}{3} \end{cases}$$

$$f[\mathrm{E}(X_2 \mid X_1)] = \begin{cases} 0,2 & \text{für } \mathrm{E}(\cdot) = 1,5 \\ 0,5 & \text{für } \mathrm{E}(\cdot) = 2,4 \\ 0,3 & \text{für } \mathrm{E}(\cdot) = 4 \end{cases}$$

$$\begin{aligned} \mathrm{var}(X_1 \mid X_2) &= \mathrm{E}[(X_1 - \mathrm{E}(X_1 \mid X_2))^2] \\ &= \begin{cases} (2-3)^2\frac{0,1}{0,2} + (4-3)^2\frac{0,1}{0,2} = 1 & \text{für } x_2 = 1 \\ (2-4,4)^2\frac{0,1}{0,5} + (4-4,4)^2\frac{0,3}{0,5} + (8-4,4)^2\frac{0,1}{0,5} = 3,84 & \text{für } x_2 = 2 \\ (4-6\frac{2}{3})^2\frac{0,1}{0,3} + (8-6\frac{2}{3})^2\frac{0,2}{0,3} \approx 3,5556 & \text{für } x_2 = 5 \end{cases} \end{aligned}$$

$$\mathrm{E}[\mathrm{E}(X_1 \mid X_2)] = 3 \cdot 0,2 + 4,4 \cdot 0,5 + 6\frac{2}{3} \cdot 0,3 = 4,8 = \mathrm{E}(X_1)$$

$$\begin{aligned} \mathrm{var}(X_1) &= \mathrm{E}[\mathrm{var}(X_1 \mid X_2)] + \mathrm{var}[\mathrm{E}(X_1 \mid X_2)] \\ \mathrm{E}[\mathrm{var}(X_1 \mid X_2)] &= 1 \cdot 0,2 + 3,84 \cdot 0,5 + 3,5556 \cdot 0,3 \approx 3,1867 \\ \mathrm{var}[\mathrm{E}(X_1 \mid X_2)] &= 3^2 \cdot 0,2 + 4,4^2 \cdot 0,5 + (6\frac{2}{3})^2 \cdot 0,3 - 4,8^2 \approx 1,7733 \\ \mathrm{var}(X_1) &= 3,1867 + 1,7733 = 4,96 \end{aligned}$$

Die Ergebnisse, insbesondere die Zwischenergebnisse, sind gerundet.

(d) Die Koordinaten der Regressionsfunktion sind mit $[\mathrm{E}(X_1 \mid X_2); X_2]$ gegeben $\Longrightarrow Q_1(3;1)$, $Q_2(4,4;2)$ und $Q_3(6\frac{2}{3};5)$.

15.

$$f(\underline{x}) = \begin{cases} \frac{36}{(1+x_1+x_2)^5(1+x_3)^4} & \text{für } x_1, x_2, x_3 > 0 \\ 0 & \text{sonst} \end{cases}$$

(a) Stochastische Unabhängigkeit: $f(\underline{x}) = f(x_1)f(x_2)f(x_3)$

$$\begin{aligned} f(x_1) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3)dx_2dx_3 = 3(1+x_1)^{-4}I_{(0,\infty)}(x_1) \\ f(x_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3)dx_1dx_3 = 3(1+x_2)^{-4}I_{(0,\infty)}(x_2) \\ f(x_3) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3)dx_1dx_2 = 3(1+x_3)^{-4}I_{(0,\infty)}(x_3) \end{aligned}$$

$$\Longrightarrow \qquad f(x_1, x_2, x_3) \neq f(x_1)f(x_2)f(x_3)$$

Die Zufallsvariablen X_1, X_2 und X_3 sind nicht gemeinsam stochastisch unabhängig.

(b)

$$\text{cov}(\underline{X}) = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & \frac{3}{4} \end{pmatrix}$$

Durch Berechnung der entsprechenden Integrale:

$$\begin{aligned} \text{E}(X_i) &= \int_{-\infty}^{\infty} x_i f(x_i)dx_i \\ \text{E}(X_1) &= \text{E}(X_2) = \text{E}(X_3) = \frac{1}{2} \\ \text{var}(X_i) &= \int_{-\infty}^{\infty} [x_i - \text{E}(X_i)]^2 f(x_i)dx_i \\ \text{var}(X_1) &= \text{var}(X_2) = \text{var}(X_3) = \frac{3}{4} \\ \text{E}(X_1X_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2 f(x_1, x_2)dx_1dx_2 = \frac{1}{2} \\ \text{cov}(X_1, X_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [x_1x_2 - \text{E}(X_1)\text{E}(X_2)] f(x_1, x_2)dx_1dx_2 = \frac{1}{4} \\ \text{E}(X_2X_3) &= \text{E}(X_2) \cdot \text{E}(X_3) = \frac{1}{4}, \text{ da } X_2 \perp X_3 \\ \text{cov}(X_2, X_3) &= 0 \end{aligned}$$

Gilt analog für $\mathrm{E}(X_1 X_3)$ und $\mathrm{cov}(X_1, X_3)$, da ebenfalls paarweise stochastisch unabhängig.

(c)

$$\begin{aligned}
\mathrm{E}(X_1 X_3) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 x_3) f(x_1, x_3) dx_1 dx_3 = \frac{1}{4} \\
\mathrm{E}(X_1|X_3) &= \mathrm{E}(X_1) \text{ und } \mathrm{var}(X_1|X_3) = \mathrm{var}(X_1), \text{ da } X_1 \perp X_3
\end{aligned}$$

(d)

$$\begin{aligned}
&\text{Für } X_1 : \\
\mathrm{E}(X_1|X_2, X_3) &= \int_{-\infty}^{\infty} x_1 \frac{f(x_1, x_2, x_3)}{f(x_2, x_3)} dx_1 = \frac{1}{3}(1 + x_2) I_{(0,\infty)}(x_2) \\
&\text{Für } X_3 : \\
\mathrm{E}(X_3|X_1, X_2) &= \int_{-\infty}^{\infty} x_3 \frac{f(x_1, x_2, x_3)}{f(x_1, x_2)} dx_3 = \frac{1}{2}
\end{aligned}$$

16.

$$\mathrm{E}(X) = \int_0^{\infty} \big[1 - F(x)\big] dx$$

Durch grafische Darstellung ergibt sich äquivalent:

$$\int_0^1 F^{-1}(z) dz \; .$$

Nach Substitution von $z = F(x)$ ergibt sich

$$\int_{F(0)=0}^{F(\infty)=1} F^{-1}(z) dz = \int_0^{\infty} F^{-1}[F(x)] f(x) dx = \int_0^{\infty} x f(x) dx = \mathrm{E}(X) \; .$$

17.

$$\begin{aligned}
\mathrm{cov}(X, Y) &= \mathrm{E}\left[(X - \mathrm{E}(X))(Y - \mathrm{E}(Y))\right] \\
&= \mathrm{E}\left[XY - \mathrm{E}(X)Y - X\mathrm{E}(Y) + \mathrm{E}(X)\mathrm{E}(Y)\right] \\
&= \mathrm{E}(XY) - \mathrm{E}\left[\mathrm{E}(X)Y\right] - \mathrm{E}\left[X\mathrm{E}(Y)\right] + \mathrm{E}\left[\mathrm{E}(X)\mathrm{E}(Y)\right] \\
&= \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y) - \mathrm{E}(X)\mathrm{E}(Y) + \mathrm{E}(X)\mathrm{E}(Y) \\
&= \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)
\end{aligned}$$