Treaps and Skip Lists

COP3503 COMPUTER SCIENCE II

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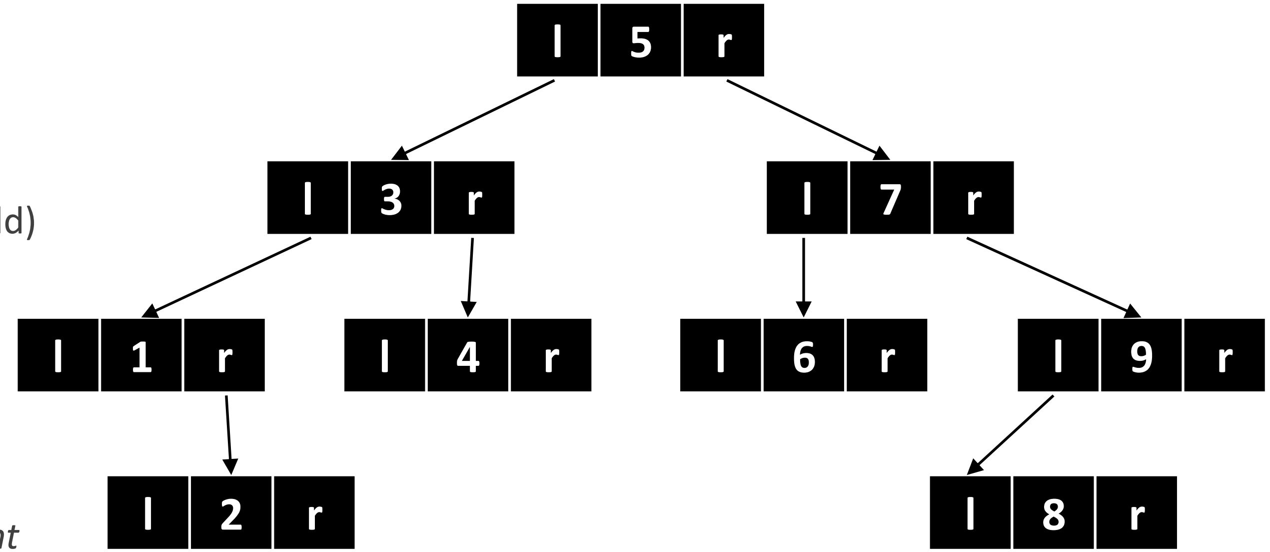
PORTIONS FROM SEAN SZUMLANSKI, MICHAEL MCALPIN, AND WIKIPEDIA

Review: Binary Search Trees

A *binary search tree* is a tree in which:

- Each node has at most two children
- Each node has at least the following data elements:
  - A **parent** pointer (NULL if the node is the root)
  - A **left** child pointer (NULL if the node has no left child)
  - A **right** child pointer (NULL if the node has no right child)
  - An indexing **key**
- The insertion and deletion behaviors guarantee that:
  - If node y is the left child of node x, or *any descendant* of the left child of x, then $y.key \leq x.key$
  - If node y is the right child of node x, or *any descendant* of the right child of x, then $y.key \geq x.key$



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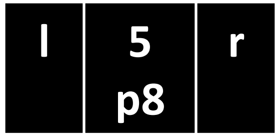
- Each node has an additional value, called *priority*



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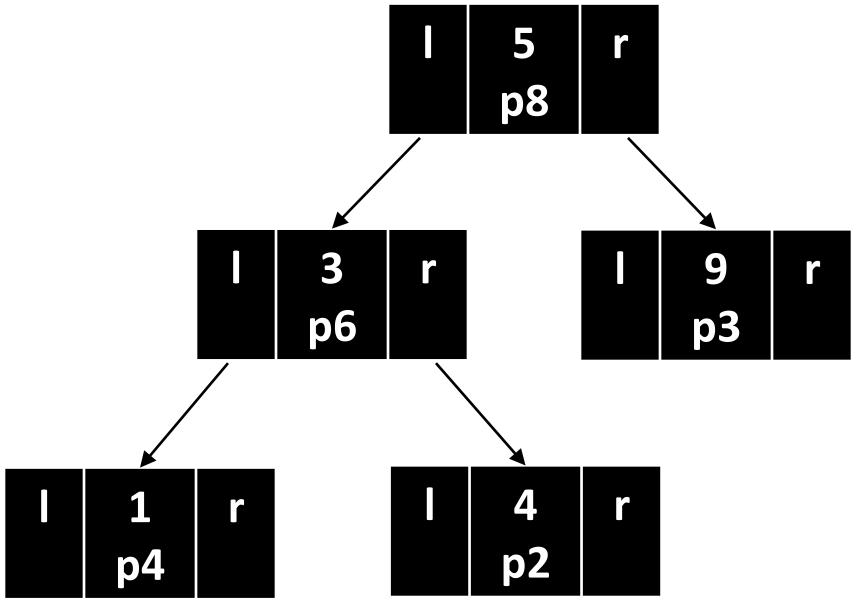

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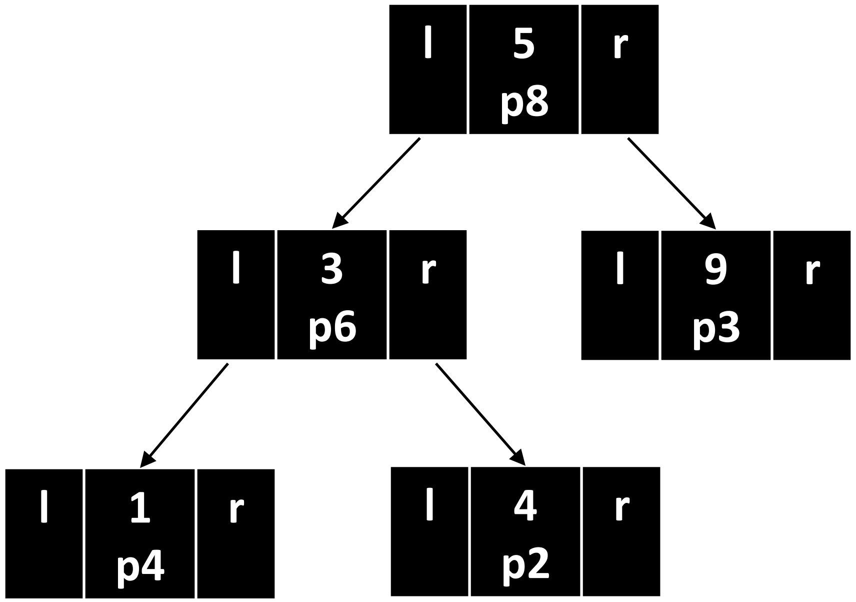

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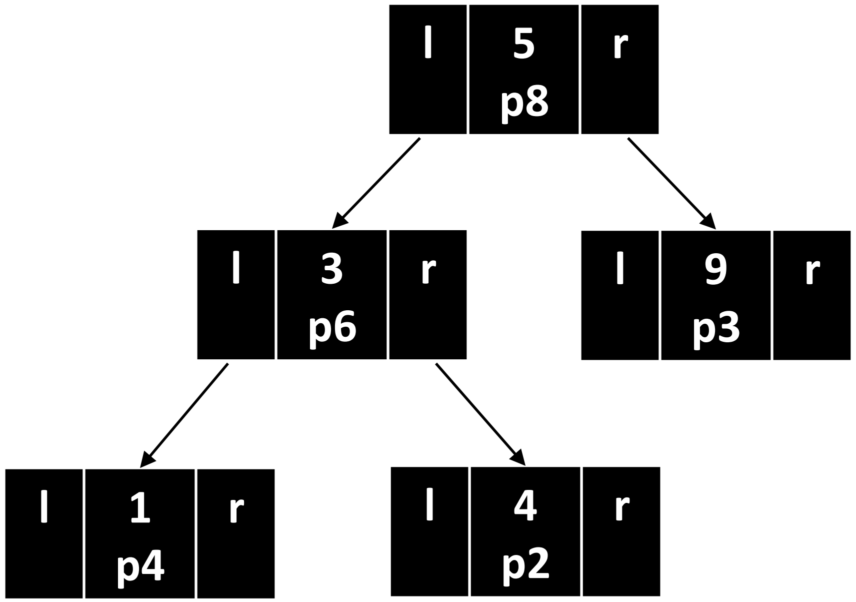

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- Which is $\mathcal{O}(\log n)$

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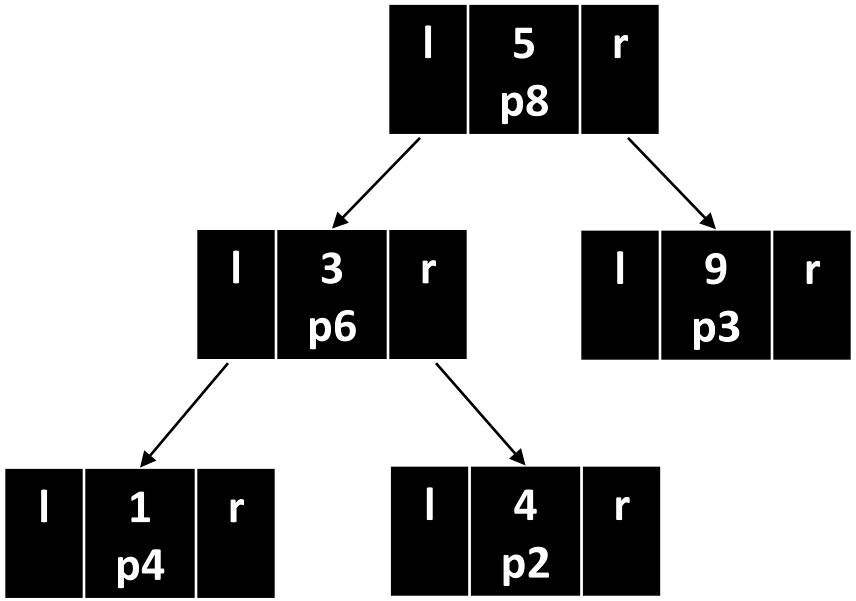

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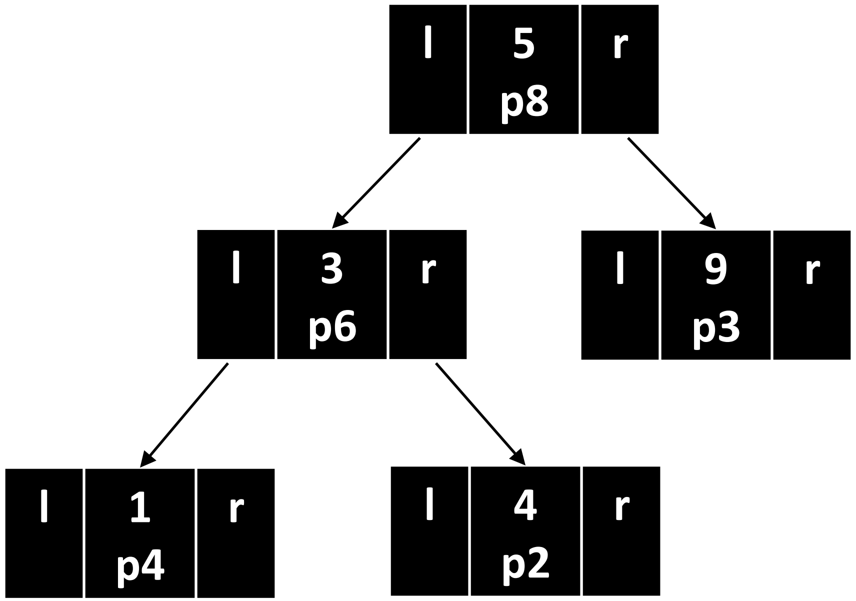
- So far, so good – but how do we maintain the heap property?



Insertion Into Treaps

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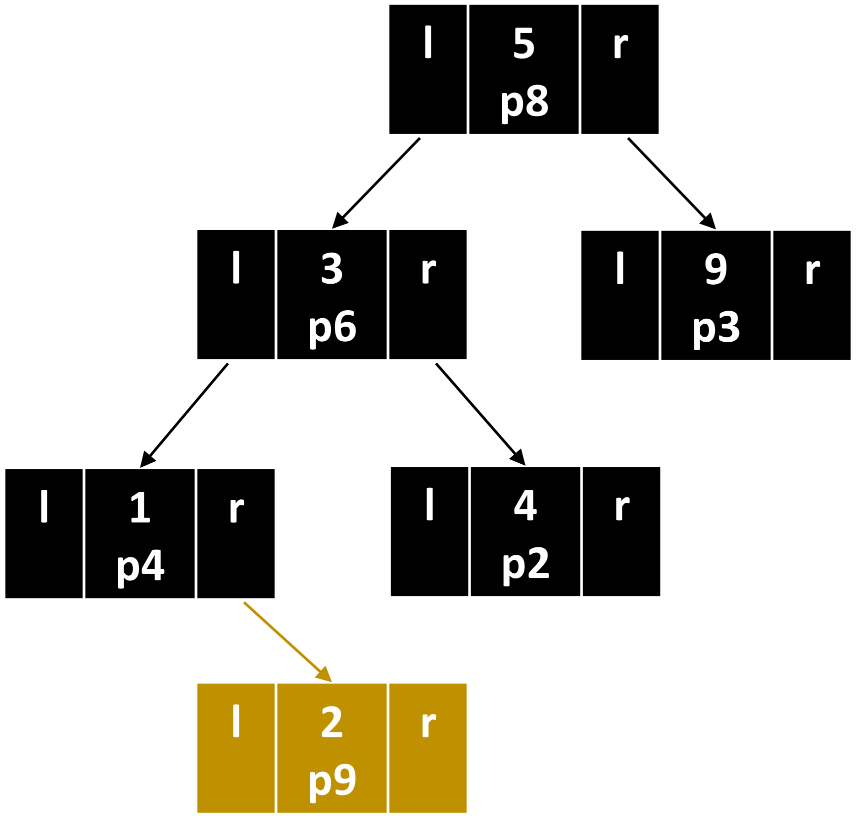

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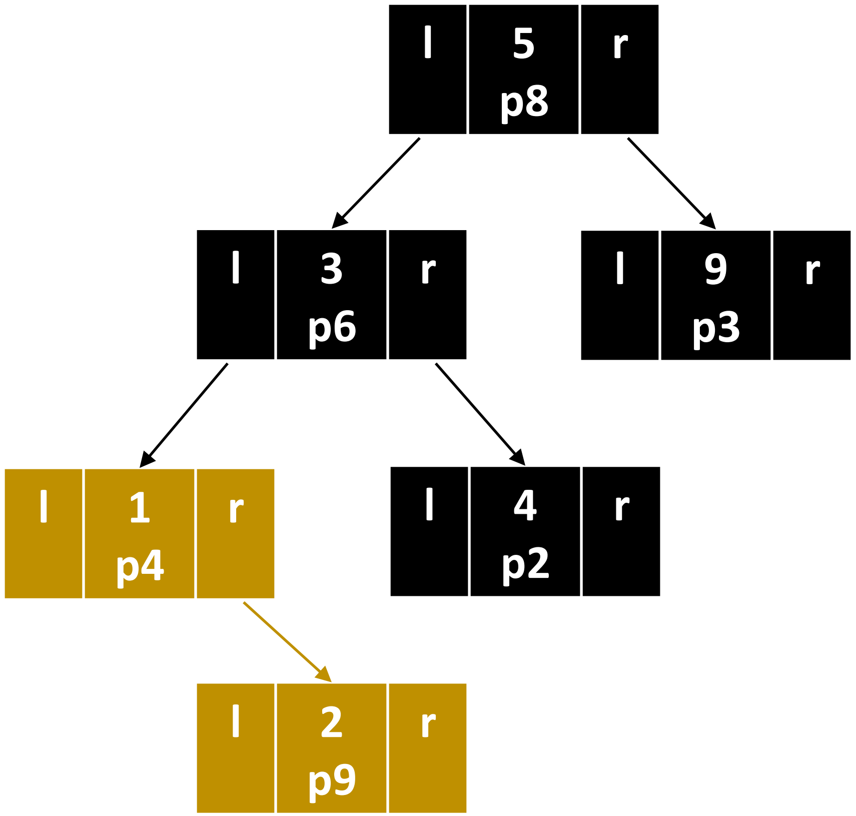

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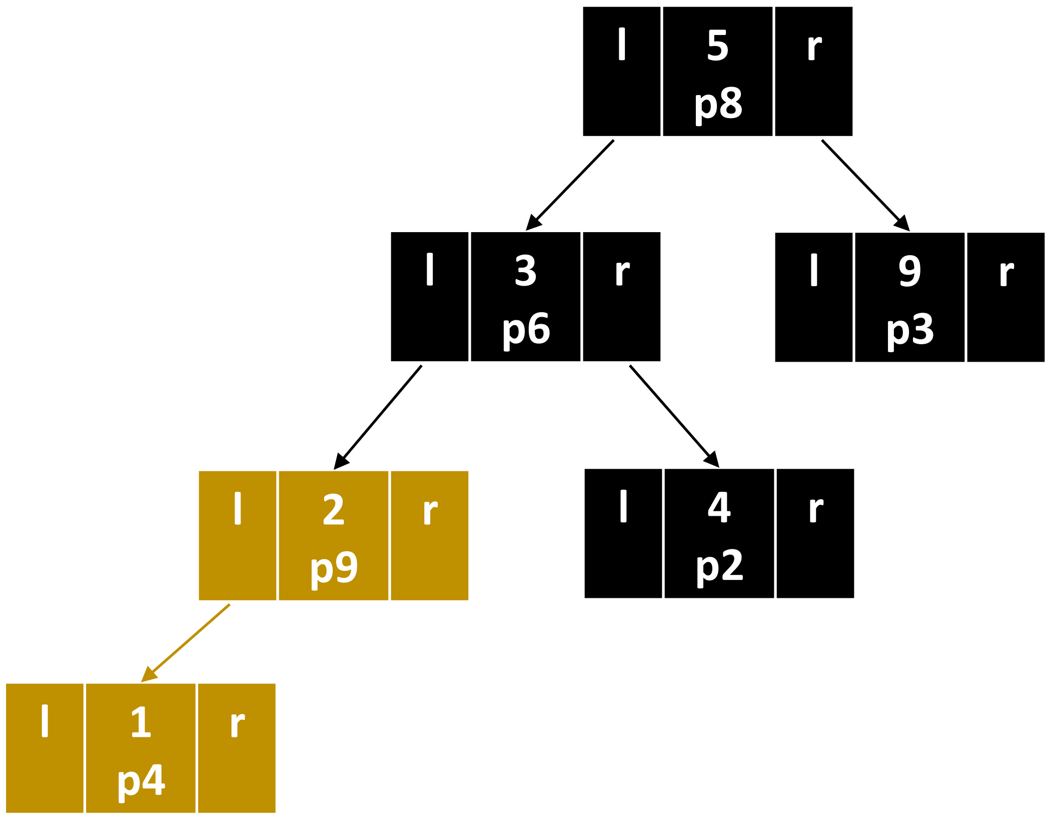

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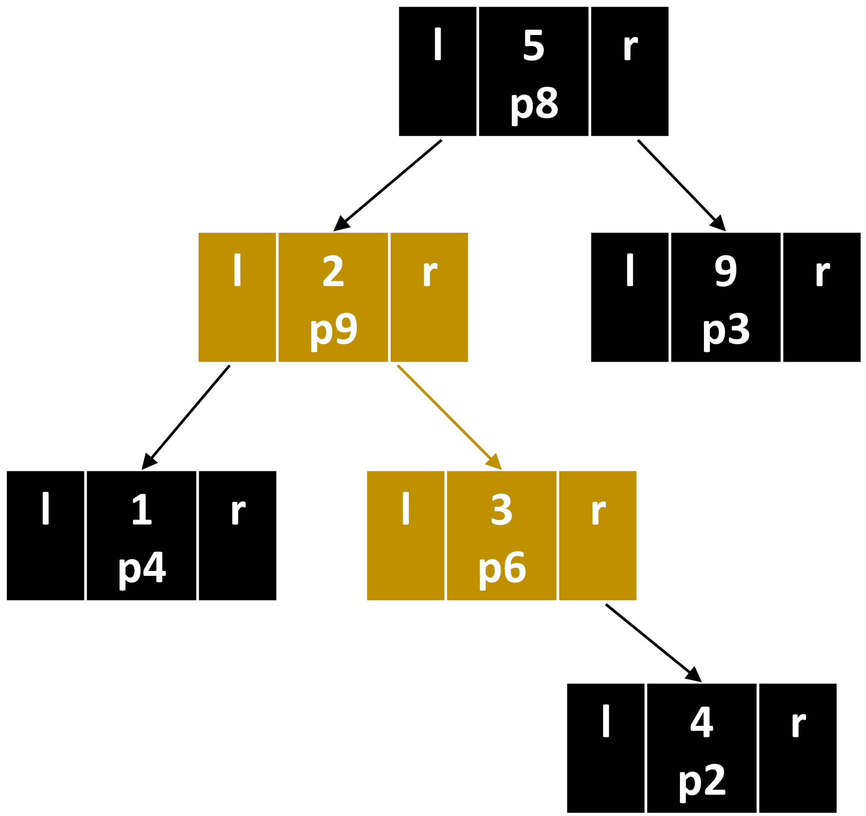

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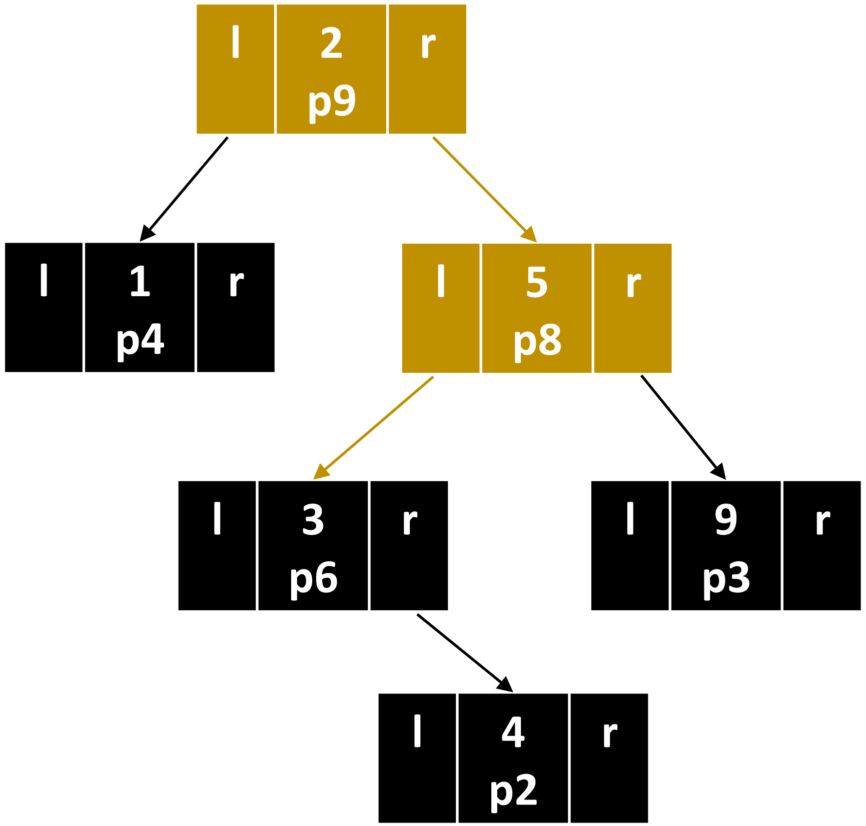

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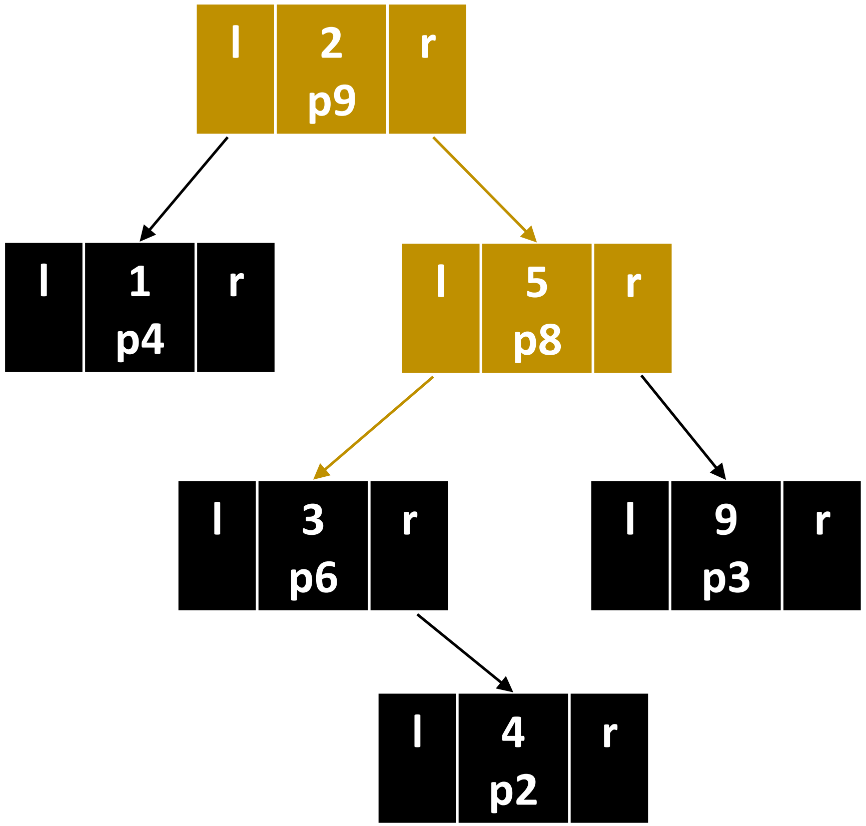

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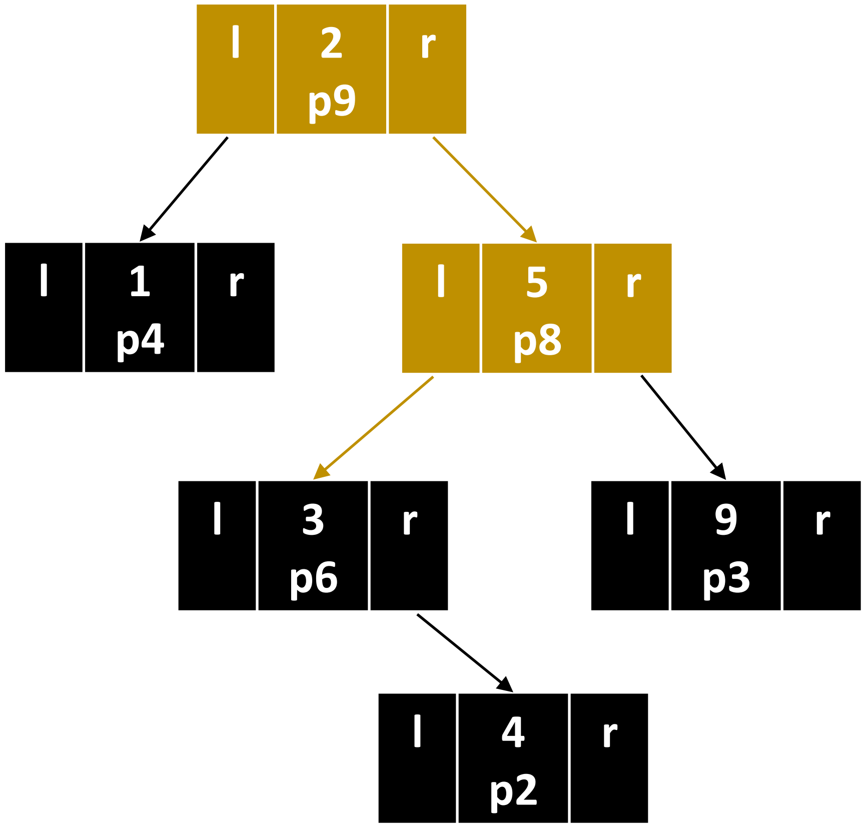
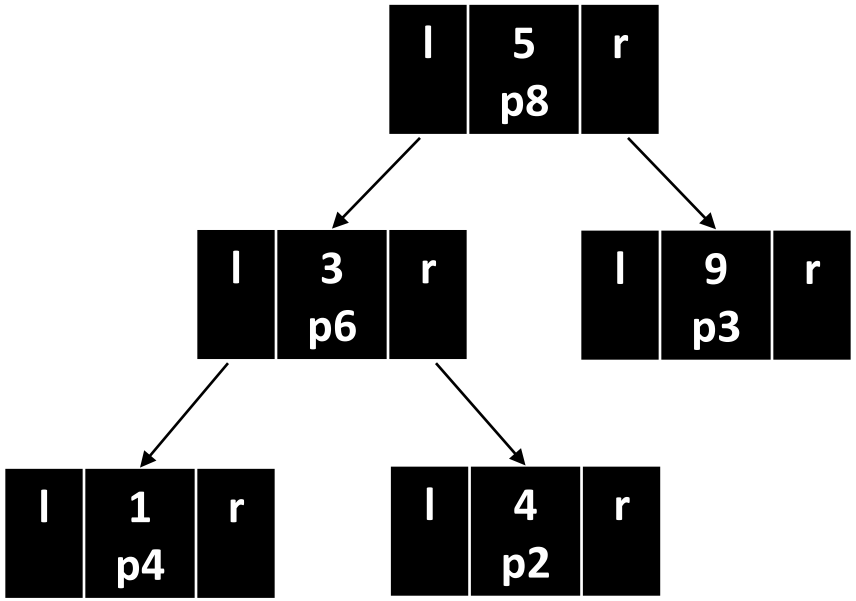
- The rotation is maximum $\mathcal{O}(h)$, which—again, *on average*—is $\mathcal{O}(\log n)$



Deletion from Treaps

- As always, leaves and nodes with one child are easy
- If a node has two children, replace it with its adjacent successor or predecessor…



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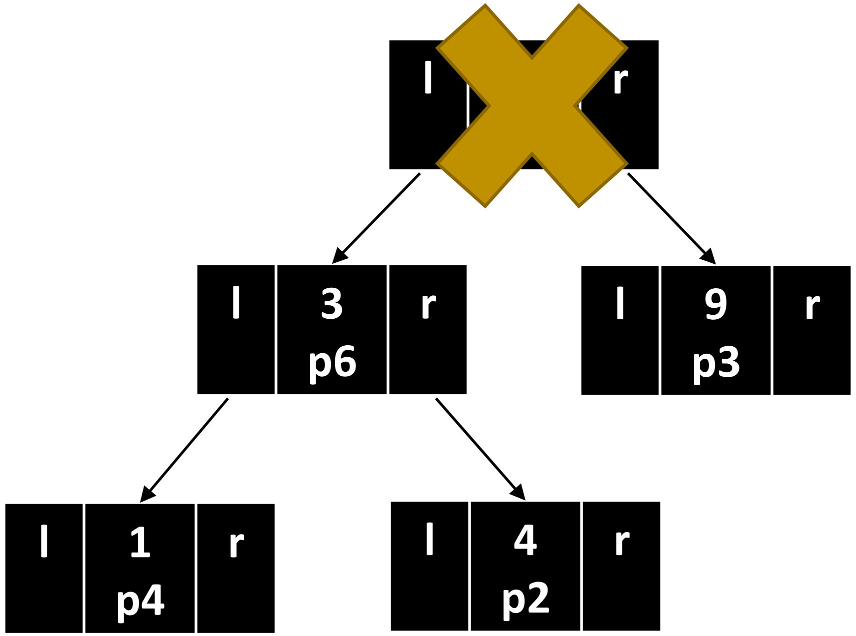

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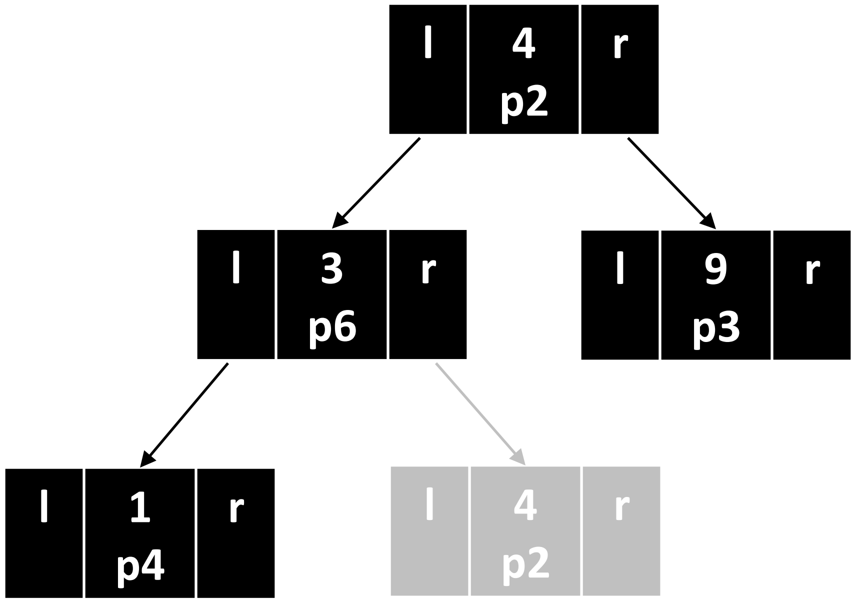

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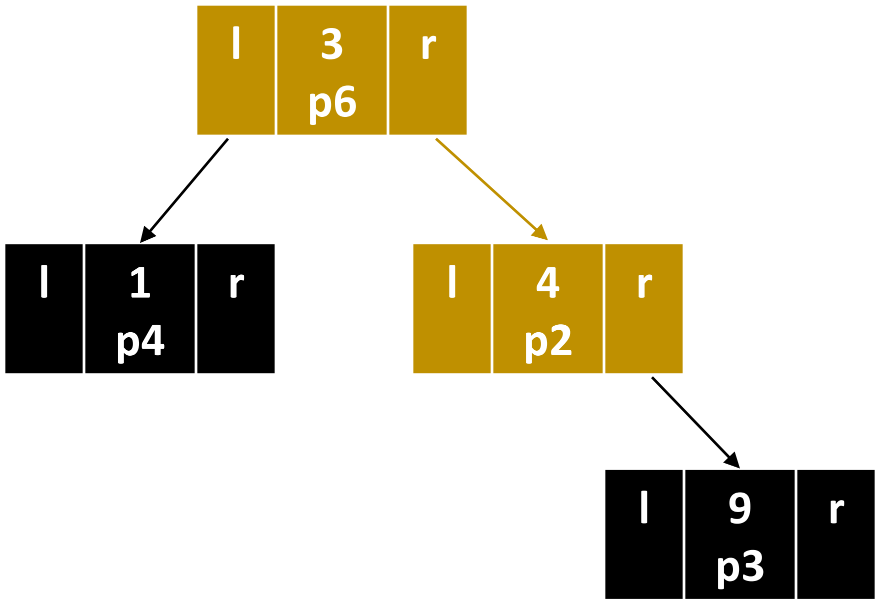

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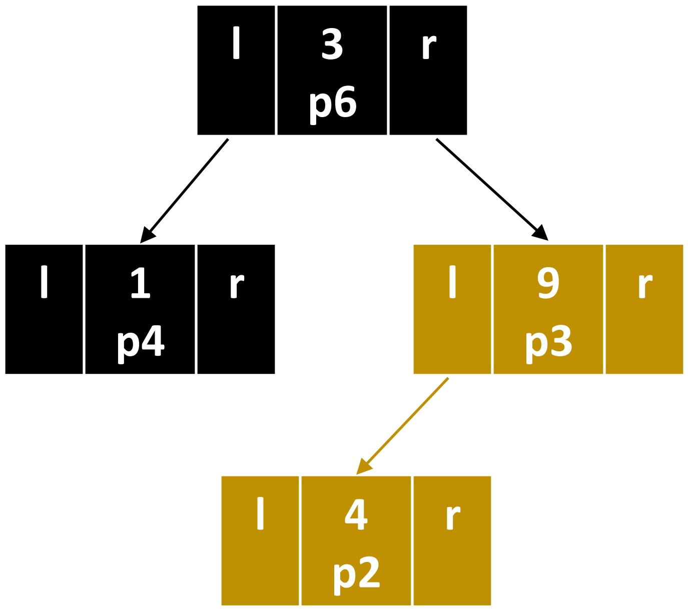

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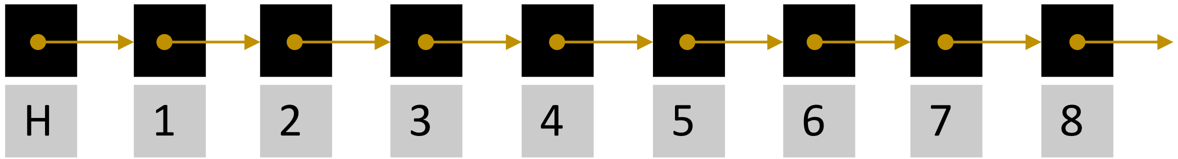

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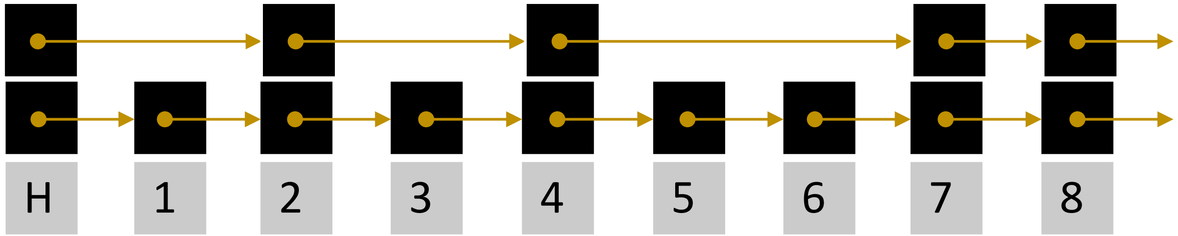

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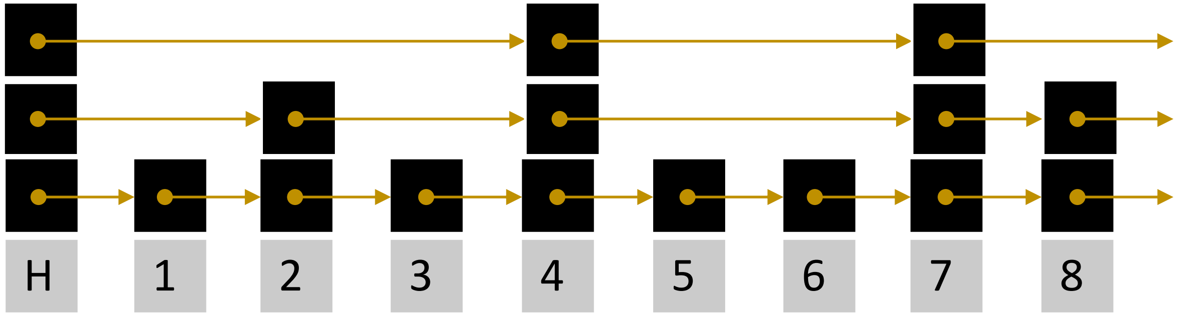

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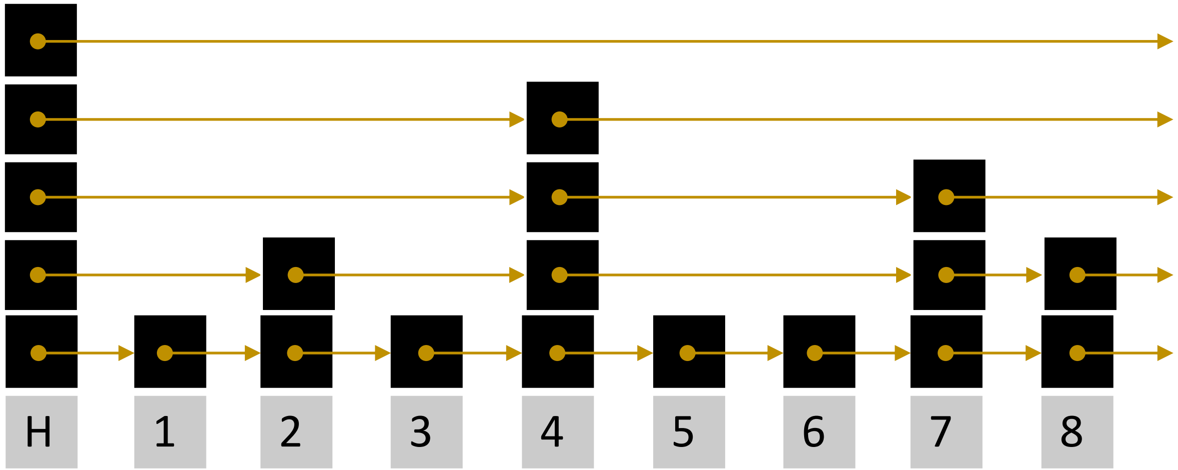
- But adds additional layers of pointers *on top* of it
- Each layer omits a certain number of the pointers from the previous later (usually about half)…
- …until all that's left is the head element
- Only needs about twice as many pointers as we had in the first place



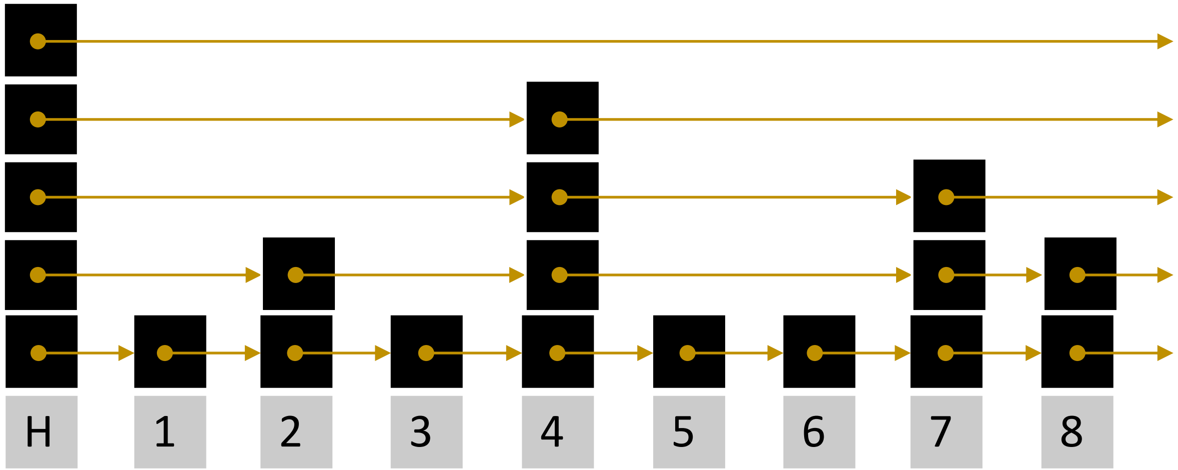
Searching Skip Lists

Searching for value v is easy:

- Let n be the node at the left of the top level
- Repeat
  - If $n.v = v$, succeed
  - If $n.v < v$, go right
  - If $n.v > v$ or $!n$, go left then down
  - If we've fallen off the bottom, fail

Obviously $\mathcal{O}(\log n)$ on average…

- …as long as we keep our cut-by-half property

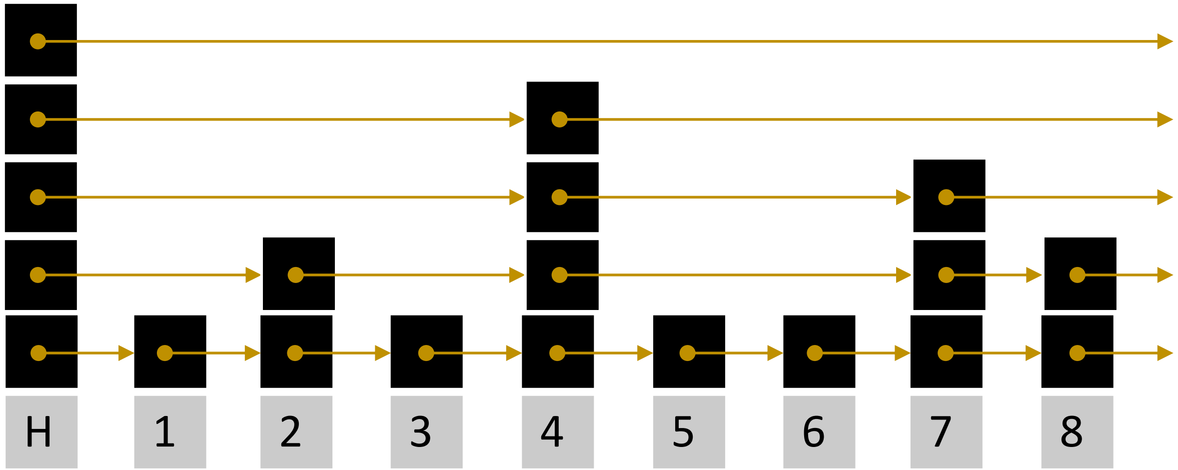


Insertion and Deletion

Insertion and deletion are *also* easy

Insertion:
- Find the spot the node should go
- Randomly choose the height of the node
- Insert it into the linked lists up to that level, as you normally would
- One search, plus $\log n$ insertions - $\mathcal{O}(\log n)$

Deletion:
- Find the node
- Delete it from all the linked lists it's in, as you normally would
- Dispose of the value
- One search, plus $\log n$ deletions - $\mathcal{O}(\log n)$



Re-Optimizing a Skip List

- We *could* theoretically end up deleting too many tall or short nodes
- We can fix this by just going through the list and re-balancing the height of each node
- Can use straight logarithmic boundaries if it's an internal structure…
- Or randomize it if we're worried about adversarial values
- $\mathcal{O}(n)$, but a fast $\mathcal{O}(n)$, since we're just doing *real* simple pointer operations
- Can attach this to list traversals that happen naturally (load/save, print, etc.)