第十一讲：有限与无限

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2017 年 12 月 13 日

请独立完成作业，不得抄袭。
若参考了其它资料，请给出引用。
鼓励讨论，但需独立书写解题过程。

第一部分　作业

题目 (UD:20.4)

(a) Show that the positive rationals $\mathbb{Q}^+$ and the negative rationals$\mathbb{Q}^-$ are equivalent.

(b) Show that the even and odd integers are equivalent.

解答：

(a) Define f: $\mathbb{Q}^+ \to \mathbb{Q}^-$ by f(x)=-x

Firstly, we can easily note that it is well-defined.

Then, for every y$\in$ $\mathbb{Q}^-$, there is one and only one x $\in$ $\mathbb{Q}^+$ that satisfies f(x)=-x

Therefore f is bijective.

$\therefore$ the positive rationals $\mathbb{Q}^+$ and the negative rationals$\mathbb{Q}^-$ are equivalent.

(b) Define f: {2k|k$\in$Z} $\to$ {2k-1|k$\in$Z} by f(x)=x+1

Firstly, we can easily note that it is well-defined.

Then, for every y$\in$ $\mathbb{Q}^-$, there is one and only one x $\in$ $\mathbb{Q}^+$ that satisfies f(x)=-x

Therefore f is bijective.

$\therefore$ the even and odd integers are equivalent.

题目 (UD:20.8)

Prove Theorem 20.6 working with the outline given in the text.

解答：

Firstly, for every x$\in$ A$\cup$B, x$\in$ A or $\in$B (but not both), therefore there exists only one y that satisfies H(x)=y for every x$\in$ A$\cup$B

$\therefore$ H(x) is well-defined.

Then, for every y$\in$ C$\cup$D, y$\in$C or $\in$D (but not both). We just illusrate the case that y$\in$C, for y$\in$D, it is similar

$\because$ f is bijective and A∩B=∅

$\therefore$ for every y∈C, there is one and only one x∈A that satisfies f(x)=y and there is no x∈B that satisfies f(x)=y.

$\therefore$ H(x) is bijective.

$\therefore$ H(x) is well-defined and bijective.

$\therefore$ A∪B ≈ C∪D.

题目 (UD:20.9)

(a) Suppose that A and B are nonempty finite sets and A∩B=∅. Show that there exist integers n and m such that $A \approx \{1,2,\ldots,n\}$ and $B \approx \{n+1,\ldots,n+m\}$.

(b) Prove Corollary 20.8.

解答:

(a) Since A is a nonempty finite set, we can define f: A→{1,2,3,...,n} for some ninN

$\therefore$ A≈{1,2,3,..., n} for some n∈N

Similarly, we can define g: B→ {1,2,3,...,m} for some m∈N

Then we define a bijective function h: {1,2,3,..., m} → {n+1,n+2,...,n+m} by f(x)=x+n

By Theorem 15.6, the comosition h∘g: B→ {n+1,n+2,...,n+m} for some n and m ∈N is bijective.

$\therefore B \approx \{n+1,\ldots,n+m\}$.

(b) let C={1,2,3,...,n}, D={n+1,n+2,...,n+m} Therefore C∩D=∅

Define

$$H(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

By Theorem 20.6, we can have A∪B is finite.

题目 (UD:20.10)

Prove Theorem 20.14 below. We suggest that you start by working Problem 15.14 if you have not already done so.

***Theorem*20.14.**

Let A,B,C and D be nonempty sets with A≈C and B≈D. Then A×B ≈ C×D.

解答:

$\because$ A≈B and C≈D

$\therefore$ there exist a bijective function f: A→B and a bijective function g: C→D

Define H: A×C → B×D by H(a,c)=(f(a),g(c)).

According to Problem 15.14, we can conclude that H is bijective.

$\therefore$ A×B ≈ C×D

题目 (UD:21.7)

Show that $\mathbb{Q}$ is infinite.

解答:

We know that N⊊Z, adn Theorem 21.3 tells us that N is infinite. Since Corollary 20.11 says that every subset of a finite set is finite, or set Q must be infinite.

题目 (UD:21.9)

Let A be a set, and suppose that B is an infinite subset of A. Show that A must be infinite.

解答:

If A is finite, by Corollary 20.11, we have B is finite. It contradicts with that B is infinite. Therefore A must be infinite.

题目 (UD:21.10)

Suppose that A is an infinite set, B is a finite set and f: A→B is a function. Show that there exists b ∈B such that $f^{-1}(\{b\})$ is infinite.

解答:

Let we suppose for all b∈B , we have $f^{-1}(\{b\})$ is finite.
By Theorem 20.12, we have $\bigcup_{b\in B} f^{-1}(\{b\})$ is finite.
And because $\bigcup_{b\in B} f^{-1}(\{b\})$=A
Therefore A is finite, but it contradcits with that A is infinite.
Therefore what we suppose is wrong.
Therefore there exists b ∈B such that $f^{-1}(\{b\})$ is infinite.

题目 (UD:21.11)

Let X be an infinite set, and A and B be finite subsets of X. Answer each of the following, giving reasons for your answers:
(a) Is $A \cap B$ finite or infinite?
(b) Is A\B finite or infinite?
(c) Is X\A finite or infinite?
(d) Is A∪B finite or infinite?
(e) If f: A→X is a one-to-one function, is $f(A)$ finite or infinite?

解答:

(a) A∩B is finite.
∵ A and B are finite
∴ A∵ A∪B is finite.

$\because$ A∩B $\subseteq$ A∪B

$\therefore$ A∩B is finite.

(b) A\B is finite.

$\because$ A is finite

$\because$ A\B $\subseteq$ A

$\therefore$ A\B is finite.

(c) X\A is infinite

Let we suppose X\A is finite

$\because$ A is finite

$\therefore$ X=A∪ (X\A) is finite, but it contradicts with that X is infinite.

$\therefore$ what we suppose is wrong.

$\therefore$ X\A is infinite

(d) A∪B is finite.

By Theorem 20.12, we can conclude that.

(e) f(A) is finite.

$\because$ A is one-to-one.

$\therefore$ A→ f(A) is bijective

By Theorem 21.6, there is a unique positive integer n such that A≈{1,. . . ,n}.

$\therefore$ f(A) $\approx \{1,\ldots,n\}$

$\therefore$ f(A) is finite.

题目 (UD:21.16)

(a) Suppose that A is a finite set and B⊆A. We showed that B is finite. Show that |B|≤|A|.

(b) Suppose that A is a finite set and B⊆A. Show that if B≠A, then |B|<|A|.

(c) Show that if two finite sets A and B satisfy B⊆A and |A|≤|B|, then A=B.

解答:

(a) A=B∪(B\A)

(i) if (B\A)≠∅ According to Problem 20.9, we can have B≈{1,2,. . . ,n} , (B\A)={n+1,. . . ,n+m} and A={1,. . . ,n+m}

$\therefore$ |A|=|B|+|(B\A)|

(ii) if (B\A)=∅, then A=B and |(B\A)|=0

$\therefore$ |A|=|B|+|(B\A)|

$\because$ |(B\A)| $\geq$ 0

$\therefore$ |B|≤|A|.

(b)A=B∪(B\A)

$\because$ B≠A

$\therefore$ (B\A)≠∅

$\therefore$ According to Problem 20.9, we can have B≈{1,2,. . . ,n} , (B\A)={n+1,. . . ,n+m} and

A={1,. . . ,n+m}

$\therefore$ |A|=|B|+|(B\A)|

$\because$ $|(B\backslash A)| > 0$

$\therefore$ |B|<|A|.

(c)

$\because$ B satisfy B⊆A

$\therefore$ |B|≤|A| and A=B∪(B\A)

$\because$ |A|≤|B|

$\therefore$ |A|=|B|

$\therefore$ $|(B\backslash A)| = 0$

$\therefore$ (B\A)=∅

$\therefore$ A=B

题目 (UD:21.17)

Suppose that A and B are finite sets and f: A→B is one-to-one. Show that |A|≤|B|.

解答:

$\because$ A and B are finite sets

$\therefore$ $\{1,\ldots,n\}$≈A for some n∈N and B≈$\{1,\ldots,m\}$ for some m∈N

$\therefore$ Define g: $\{1,\ldots,n\}$→A is bijective and h: B→$\{1,\ldots,m\}$ is bijective

and $\because$ f is one-to-one,

$\therefore$ h∘f∘g: $\{1,\ldots,n\}\to\{1,\ldots,m\}$ is one-to-one.

$\therefore$ we can easily conclude that $\{1,\ldots,n\} \subseteq \{1,\ldots,m\}$

$\therefore$ n≤m

$\therefore$ |A|≤|B|.

题目 (UD:21.18)

Let A and B be sets with A finite. Let f: A→B. Prove that $|\text{ran}(f)| \leq |A|$.

解答:

Let we suppose $|\text{ran}(f)| > |A|$

$\therefore$ there exists an x which relates to at least two different y's

$\therefore$ it contradicts with f is a well-defined function

$\therefore$ what we suppose is not right.

$\therefore$ $|\text{ran}(f)| \leq |A|$.

题目 (UD:21.19)

Let A be a finite set. Show that a function f: A→A is one-to-one if and only if it is onto. Is this still true if A is infinite?

解答:

(a) f: A→A is one-to-one.

Let we suppose f is not onto.

∴ ran(f)⊊A

∴ |ran(f)|<|A|

∴ we can note that g: A→ran(f) is also one-to-one

∴ |A|≤|ran(f)|

∴ |A|≤|ran(f)|<|A|, but it contradicts

∴ what we suppose is not right

∴ f is onto

(b) f: A→A is onto.

Let we suppose f is not one-to-one

∴ there exist one y∈A that there are at least two x's ∈ A that satisfy f(x)=y, we define them x_1 and x_2

∴ we define F: A\{x_1} is still onto.

∴ |A| ≤ |A\{x_1}| =|A| -1,

∴ it contradicts

∴ what we suppose is wrong

∴ f is one-to-one

∴ f: A→A is one-to-one if and only if it is onto.

(c) it is not true if A is infinite.

题目 (UD:22.1)

Give an example, if possible, of each of the following:

(a) a countably infinite collection of pairwise disjoint finite sets whose union is countably infinite;(See Problem 8.11 for the definition of pairwise disjoint.)

(b) a countably infinite collection of nonempty sets whose union is finite;

(c) a countably infinite collection of pairwise disjoint nonempty sets whose union is finite.

解答:

(a) $\bigcup_{I\in N}$\{I\}

(b) not exist

(c) not exist

题目 (UD:22.2)

Which of the following sets are finite? countably infinite? uncountable? (Be careful-don't apply theorems for finite sets to infinite sets!) Give reasons for your answers for each of the following:

(a) $\{1/n\text{: } n\in \mathbb{Z}\backslash\{0\}\}$;

(b) $\mathbb{R} \setminus \mathbb{N}$;

(c) $\{x \in \mathbb{Z}\text{:}|x\text{-}7|<|x|\}$;

(d) $2\mathbb{Z} \times 3\mathbb{Z}$;

(e) the set of all lines with rational slopes;

(f) $\mathbb{Q} \setminus \{0\}$;

(g) $\mathbb{N} \setminus \{1,3\}$.

解答:

(a) countably infinite.

As $\mathbb{Q}$ is countably infinite and the asked set is a subset of it.

(b) uncountable, the proof is similar to Theorem 22.12.

(c) countably infinite. The set is $\{4,5,6,\ldots\}$ and we can easily prove that $\{4,5,6,\ldots\}\approx$ N.

(d) countably infinite. By Corollary 22.10

(e) uncountable, let y=ax+b, b can be irrational number.

(f) countably infinite.

As $\mathbb{Q}$ is countably infinite and the asked set is a subset of it.

(g) countably infinite.

As $\mathbb{N}$ is countably infinite and the asked set is a subset of it.

题目 (UD:22.3)

Is the set of all infinite sequences of 0's and 1's finite, countably infinite, or uncountable? Guess and then prove, please.

解答:

We can easily note that this set is not finite.

Let us suppose this set is countably infinite. And we have shown that $\mathbb{Z}^+$ is countably infinite, there exists a bijective function f: $\mathbb{Z}^+ \rightarrow$ (0,111111...). We list the values of f using the sequence of 1 or 0

$f(1) = a_{11}a_{12}a_{13}\ldots$

$f(2) = a_{21}a_{22}a_{23}\ldots$

$f(3) = a_{31}a_{32}a_{33}\ldots$

$\ldots$

Since f is onto, each sequence of 0's and 1's appears in this list.

The odd thing is that we can construct a number $b=f(1) = b_1b_2b_3\ldots$

that is not in the list.

if a_{11}=0, then b_1=1 else b_1=1, and similarly we relates b_n with a_{nn} by assigning a different value from a_{nn} to b_n

Therefore we construct a b that is not in the list, it contradicts with the assertion that f is onto.

$\therefore$ what we suppose is not right

$\therefore$ this set is not countably infinite and isn't finite.

$\therefore$ this set is uncoountable.

题目 (UD:22.6)

Prove Corollary 22.4.

解答:

let A be the countable set. There are two cases.

(i) A is finite

According to Corollay 20.11, we can conclude that every subset of A is finite.

Therefore every subset of A is countable.

(ii) A is infinite.

Therefore A is countable infinite.

Therefore A$\approx$N

Therefore we can construct a bijective function f: N$\rightarrow$A

Select an arbitrary subset of A, name it B

Therefore B $\subseteq$A

if B is finite, we can easily note that B is countable.

if B is infinite, we can construct a bijective function g: N$\rightarrow$B

$\therefore$ B is countable

$\therefore$ Every subset of a countable set is countable.

题目 (UD:22.9)

There is another way to show that $\mathbb{Q}$ is countable. Turn the outline below into a proof by describing the counting process. (Don't try to find a formula for the function.)

解答:

We can count the elements of Q^+ in the way illustrated in the grapg below.

And we have proved that $Q^- \approx Q^+$. Then $Q=Q^+ \cup Q^- \cup \{0\}$ is finite and countable, so Q$\approx$N

题目 (UD:23.2)

(a) In $\mathbb{R}$, find the distance of the number 1 to the number 3 in the usual metric and in the discrete metric.

(b) In $\mathbb{R}^2$, find the distance of the point (1,3) to the point (2,5) in the usual metric, the taxicab metric, the max metric, and the discrete metric.

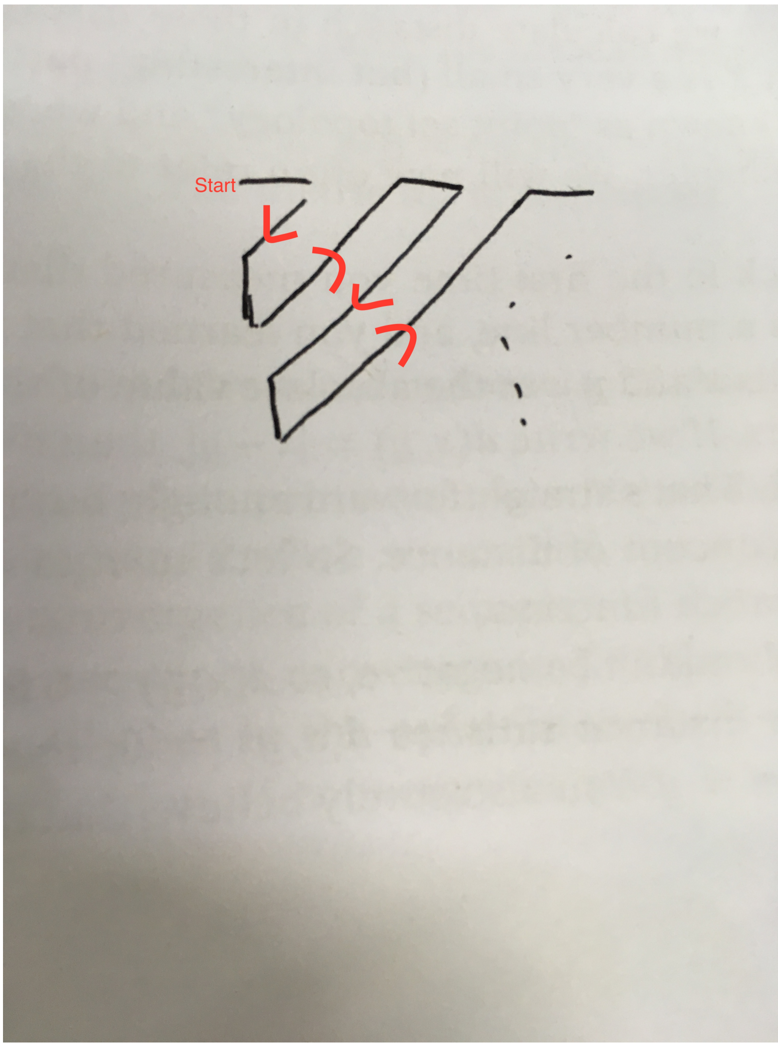

图 1: counting solution

解答:

(a) $d_u(1,3)=2$, $d_d(1.3)=1$

(b) $d_u((1,3),(2,5))=\sqrt{5}$

$d_{tc}((1,3),(2,5))=3$

$d_m((1,3),(2,5))=2$

$d_d((1,3),(2,5))=1$

题目 (UD:23.3)

(a) Sketch the set $\{(x, y) \in \mathbb{R}^2\colon\ d_u((x, y), (0, 0) < 1)\}$, where d_u is the usual metric.

(b) Sketch the set $\{(x, y) \in \mathbb{R}^2\colon\ d_{tc}((x, y), (0, 0) < 1)\}$, where d_{tc} is the taxicab metric.

(c) Sketch the set $\{(x, y) \in \mathbb{R}^2\colon\ d_m((x, y), (0, 0) < 1)\}$, where d_m is the max metric.

(d) Sketch the set $\{(x, y) \in \mathbb{R}^2\colon\ d_d((x, y), (0, 0) < 1)\}$, where d_d is the discrete metric.

(e) Sketch the set $\{(x, y, z) \in \mathbb{R}^3\colon\ d_u((x, y, z), (0, 0, 0) < 1)\}$, where d_u is the usual metric. (See Example 23.2 for the definition if you need it.)

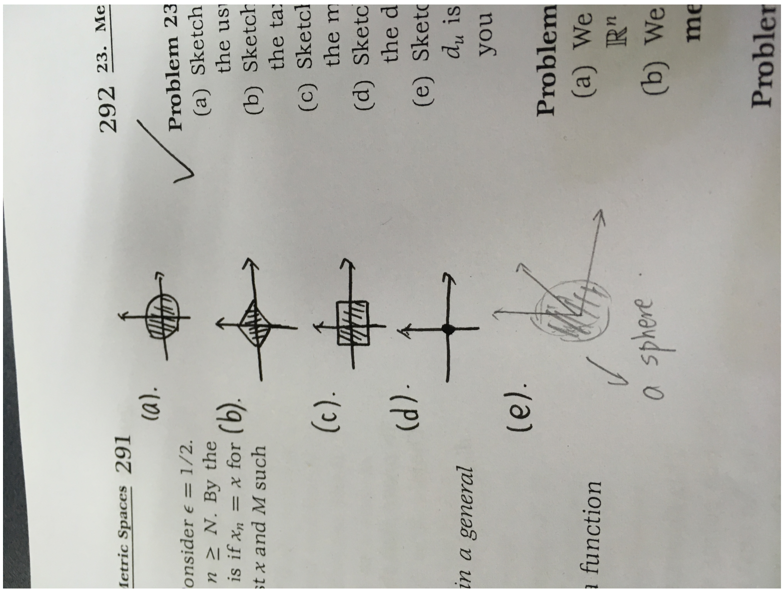

图 2: sketching solution

解答:

题目 (UD:23.10)

Let X be the space of polynomials with real coefficients. Define a function d from X×X $\to \mathbb{R}$ by $d(p, q) = |p(0), q(0)|$. Is d a metric? If so, prove it. If not,why not?

解答:

No, it doesn't satisfy definiteness: let p=x, q=x^2, then |p(0)=q(0)|, but p≠q.

第二部分 订正

题目 (UD: 10.5)

Let X be a nonempty set with an equivalence relation $\sim$ on it. Prove that for all elements x and y in X, the equality E_x=E_y holds if and only if x$\sim$y.

错因分析： 简述错误原因（可选）。

订正：

(1)

$\because$ [x]=[y]

$\therefore$ x$\in$[x]=[y]

$\therefore$ x$\sim$y

(2)

for a$\in$[x]

$\therefore$a$\sim$x$\sim$y

$\therefore$a$\sim$y

$\therefore$a$\in$[y]

$\therefore$ [x]$\subseteq$[y]

Similarly, we can have [y]$\subseteq$[x]

$\therefore$ [x]=[y]

第三部分 反馈

你可以写：

- 对课程及教师的建议与意见
- 教材中不理解的内容
- 希望深入了解的内容
- 等