第九讲：关系及其基本性质

姓名：丁保荣　　**学号**：171860509

2017 年 11 月 27 日

请独立完成作业，不得抄袭。
若参考了其它资料，请给出引用。
鼓励讨论，但需独立书写解题过程。

第一部分　作业

题目 (UD:10.2)

Let X={1,2,3,4,5}.

(a) If possible, define a relation on X that is an equivalence relation.

(b) If possible, define a relation on X that is reflexive, but neither symmetric nor transitive.

(c) If possible, define a relation on X that is symmetric, but neither reflexive nor transitive.

(d) If possible, define a relation on X that is transitive, but neither reflexive nor symmetric.

解答：

(a) define x∼y if and only if x=y

(b) define x∼y if and only if 0≤x-y≤1

(c) define x∼y if and only if x=6-y

(d) define x∼y :1∼2 , 2∼3 and 1∼3

题目 (UD:10.4)

Define a relation ∼ on R^2 as follows: For $(x_1,x_2),(y_1,y_2) \in R^2$, we say that $(x_1,x_2) \sim (y_1,y_2)$ if and only if both x_1-y_1 and x_2-y_2 are even integers. Is this relation an equivalence relation? Why or why not?

解答：

Yes, this relation is an equivalence.

(i) for reflexive:

$\because$ for $(x_1,x_2) \sim (y_1,y_2)$, we have both x_1-y_1 and x_2-y_2 are even integers

$\therefore$ we can let x_1-y_1 and x_2-y_2 are 0

$\therefore$ we have x_1=y_1 and x_2=y_2

$\therefore$ we have $(x_1,x_2) \sim (x_1,x_2)$

$\therefore$ this relation is reflexive

(ii) for symmetric

$\because$ for $(x_1,x_2) \sim (y_1,y_2)$, we have both x_1-y_1 and x_2-y_2 are even integers

$\therefore$ we also have both y_1-x_1 and y_2-x_2 are even integers

$\therefore$ $(y_1,y_2) \sim (x_1,x_2)$

$\therefore$ this relation is symmetric

(iii) for transitive

$\because$ for $(x_1,x_2) \sim (y_1,y_2)$, we have both x_1-y_1 and x_2-y_2 are even integers

$\therefore$ we also have for $(y_1,y_2) \sim (z_1,z_2)$, we have both y_1-z_1 and y_2-z_2 are even integers

$\therefore$ we have both x_1-z_1 and x_2-z_2 are even integers

$\therefore$ $(x_1,x_2) \sim (z_1,z_2)$

$\therefore$ this relation is transitive

题目 (UD:10.5)

Let X be a nonempty set with an equivalence relation $\sim$ on it. Prove that for all elements x and y in X, the equality E_x=E_y holds if and only if x$\sim$y.

解答:

(1)

$\because$ we have E_x=E_y

$\therefore$ $\{y\in X:x\sim y\}=\{x\in X:y\sim x\}$

$\therefore$ x$\sim$y

(2)

$\because$ we have x$\sim$y and this relation is an equivalence

$\therefore$ y$\sim$x

$\therefore$ $\{y\in X:x\sim y\}=\{x\in X:y\sim x\}$

$\therefore$ E_x=E_y

Based on (1) and (2), we have for all elements x and y in X, the equality E_x=E_y holds if and only if x$\sim$y.

题目 (UD:10.8)

Recall that a ***polynomial*** p over R is an expression of the form $p(x)=a_nx^n + a_{n-1}x^{n-1} + ... + a_1x^1 + a_0$ where each $a_j \in$ R and n$\in$ N. The largest integer j such that $a_j \neq 0$ is the degree of p. We define the degree of the constant polynomial p=0 to be -∞. (A polynomial over R defines a function p:R $\rightarrow$ R.)

(a) Define a relation on the set of polynomials by p $\sim$ q if and only if p(0)=q(0). Is this

an equivalence relation? If so, what is the equivalence class of the polynomial given by p(x)=x?
(b) Define a relation on the set of polynomials by p $\sim$ q if and only the degree of p is the same as the degree of q. Is this an equivalence relation? If so, what is E_r if r(x)=3x+5?
(c) Define a relation on the set of polynomials by p $\sim$ q if and only the degree of p is less than or equal to the degree of q. Is this an equivalence relation? If so, what is E_r where r(x)=x^2?

解答:

(a) yes.
$E_{p(x)=x}$={q(x)=$a_n x^n$ + $a_{n-1}x^{n-1}$ + ... + $a_1 x^1$:$a_j \in$ R and n$\in$ N}
(b) yes.
$E_{r(x)=3x+5}$={q(x)=ax+b : a$\neq$0}
(c) No

题目 (UD:11.3)

(a) For each r$\in$R, let A_r={(x,y,z)$\in$ R^3 : x+y+z=r}. Is this a partition of R^3? If so, give a geometric description of the partitioning sets.
(b) For each r$\in$R, let A_r={(x,y,z)$\in$ R^3 : $x^2+y^2+z^2=r^2$}. Is this a partition of R^3? If so, give a geometric description of the partitioning sets.

解答:

(a) yes.
Each partitioning set is a plane.
(b) yes.
when r=0,the partitioning set is a dot.
for other r, the partitioning set is a sphere(only the surface).

题目 (UD:11.7)

Consider the set P of polynomials with real coefficients. Decide whether or not each of the following collection of sets determines a partition of P. If you decide that it does determine a partition, show it carefully. If you decide that it does not determine a partition, list the part(s) of the definition that is (are) not satisfied and justify your claim with an example. (See Problem 10.8 for more information about polynomials.)
(a) For m$\in$ N, let A_m denote the set of polynomials of degree m.
(b) For c$\in$ R, let A_c denote the set of polynomials such that p(0)=c.
(c) For a polynomial q, let A_q denote the set of all polynomials p such that q is a factor

of p; that is, there is a polynomial r such that p=qr.

(d) For c$\in$ R, let A_c denote the set of polynomials such that p(c)=0.

解答:

(a) this doesn't determine a partition of P.

$\because$ the degree of (p(x)=0) is -∞

$\therefore$the set of (A_m) doesn't include B={p(x)=0}

(b) this determines a partition of P.

(i) we can easily note that A_c is nonempty

(ii) first, we can easily conclude that $\bigcup_{c\in R}A_c \subseteq$ P

then, for every polynomial $\in$ P, we can denote it by p(x)=$a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x^1$ + c where each $a_j \in$ R and n$\in$ N.

$\therefore$ p(0)=0

$\therefore$ p(x) $\in A_c$ for some c $\in$ R

$\therefore$ P $\subseteq \bigcup_{c\in R}A_c$

$\therefore \bigcup_{c\in R}A_c$ =P

(iii) If $A_a \cap A_b \neq \emptyset$ (a,b$\in$R)

$\therefore$ there exist a p(x)$\in A_a \cap A_b$

$\therefore$ p(0)=a and p(0)=b

$\therefore$ a=b

$\therefore A_a = A_b$

Based on (i),(ii) and (iii), this determines a partition of P

(c) this doesn't determine a partition of P.

A_1=P while $A_x \subsetneqq$ P and $A_x \neq \emptyset$

$\therefore A_1 \cap A_x \neq \emptyset$ and $A_1 \neq A_x$

$\therefore$ this doesn't satisfy the third condition

$\therefore$ this doesn't determine a partition of P.

(d) this doesn't determine a partition of P.

for p(x)=x^2+1, we cannot find c$\in$R to let p(c)=0

$\therefore$ p(x)=$x^2+1 \notin A_c$ for any of c$\in$R

$\therefore \bigcup_{c\in R}A_c \neq$P

$\therefore$ this doesn't determine a partition of P.

题目 (UD:11.8)

For two nonempty disjoint sets, I and J, let $\{A_\alpha\colon \alpha \in I\}$ be a partition of R^+ and $\{A_\alpha\colon \alpha \in J\}$ be a partition of $R^- \cup \{0\}$. Prove that $\{A_\alpha : \alpha \in I \cup J\}$ is a partition of R.

解答:

(i) we can easily note that for every $\alpha \in I \cup J$, A$\alpha \neq \emptyset$

(ii)

$\because$ {A_α: $\alpha \in$ I} is a partition of R^+ and {A_α: $\alpha \in$ J} is a partition of $R^- \cup \{0\}$

$\therefore \bigcup_{\alpha \in I} A\alpha = R^+$ and $\bigcup_{\alpha \in J} A\alpha = R^- \cup \{0\}$

$\therefore \bigcup_{\alpha \in I} A\alpha \cup \bigcup_{\alpha \in J} A\alpha = R^+ \cup (R^- \cup \{0\})$

$\therefore \bigcup_{\alpha \in I \cup J} A\alpha = R$

(iii)

if both i and j $\in$ I(or J) and $A_i \cap A_j \neq \emptyset$, we conclude from the condition that A_i=A_j

if i$\in$ I and j$\in$ J we can conclude from the condition that $A_i \cap A_j = \emptyset$

$\therefore$ for i,j$\in$ I$\cup$J, if $A_i \cap A_j \neq \emptyset$, we can conclude that A_i=A_j

Based on (i), (ii) and (iii), {A_α : $\alpha \in$ I$\cup$J } is a partition of R.

题目 (UD:11.9)

Let X be a nonempty set and {A_α : $\alpha \in$ I} be a partition of X.

(a) Let B be a subset of X such that $A_\alpha \cap$ B $\neq \emptyset$ for every $\alpha \in$ I. Is {$A_\alpha \cap B$: $\alpha \in$ I} a partition of B? Prove it or give a counterexample.

(b) Suppose further that $A_\alpha \neq$ X for every $\alpha \in$ I. Is {X\ A_α: $\alpha \in$ I} a partition of X? Prove it or give a counterexample.

解答:

(a) yes, {$A_\alpha \cap B$: $\alpha \in$ I} is a partition of B

(i) we can easily see from the condition that $A_\alpha \cap$ B $\neq \emptyset$ for every $\alpha \in$ I

(ii)

first, for every $\alpha \in$ I, $A_\alpha \cap$ B $\subseteq$ B

$\therefore \bigcup_{\alpha \in I}$ ($A_\alpha \cap$ B) $\subseteq$ B

then, $\because$ B$\subseteq$ X and X=$\bigcup_{\alpha \in I} A_\alpha$

$\therefore$ B$\subseteq \bigcup_{\alpha \in I} A_\alpha$

$\therefore$ B$\subseteq \bigcup_{\alpha \in I}$ ($A_\alpha \cap$ B)

$\therefore \bigcup_{\alpha \in I}$ ($A_\alpha \cap$ B) = B

(iii)

if $A_i \cap B \cap A_j \cap B \neq \emptyset$ (i,j$\in$ I)

$\therefore A_i \cap A_j \neq \emptyset$ (i,j$\in$ I)

$\therefore$ acoodint to the condition , A_i=A_j

$\therefore A_i \cap B = A_j \cap B$

Based on (i), (ii) and (iii), {$A_\alpha \cap B$: $\alpha \in$ I} is a partition of B

(b) No

Let X={1,2,3} A_1={1} A_2={2} A_3={3}

$\therefore$ we have X\ A_1={2,3} ,X\ A_2={1,3} ,X\ A_3={1,2}

$\therefore A_1 \cap A_2$ ={3} $\neq \emptyset$ but $A_1 \neq A_2$

$\therefore$ it doesn't satisfy the third condition.

题目 (UD:12.10)

Let S and T be nonempty bounded subsets of R.

(a) Show that sup(S$\cup$T) $\geq$ sup S, and sup(S$\cup$T) $\geq$ sup T.

(b) Show that sup(S$\cup$T) = max{sup S, sup T}.

(c) Try to state the results of (a) and (b) in English, without using mathematical symbols.

解答:

(a)

$\therefore$ for all x $\in$ S,we have x$\leq$sup S and for all y $\in$ T,we have y$\leq$sup T

there are three cases: S>T,S<T,S=T

(i) S>T

$\therefore$ sup S$\geq$ x and sup S$\geq$ sup T $\geq$ y

$\therefore$ sup S is the upper bound of S$\cup$T

if there is M<sup S that M is also the upper bound of S$\cup$T, then M is also the upper bound of S

$\therefore$ it contradicts with that sup S is the least upper bound of S

$\therefore$ sup S is the supremum of S$\cup$T

(ii)S<T

it is similar to (i)

$\therefore$ sup T is the supremum of S$\cup$T

(iii)S=T

following the above, we can easily conclude that sup T or sup S is the supremum of S$\cup$T

$\therefore$ we can conclude that sup{S$\cup$T)=max(sup S, sup T)}

$\therefore$ sup(S$\cup$T) $\geq$ sup S, and sup(S$\cup$T) $\geq$ sup T.

(b)

I have done it in (a)

$\therefore$ we can conclude that sup{S$\cup$T)=max(sup S, sup T)}

(c)

for (a): the supremum of the union of sets S and T is greater than the supremum of set S or that of set T

for (b): the supremum of the union of sets S and T is the maximum of the supremum of set S and that of set T

题目 (UD:12.13b)

Let $\sim$ denote a relation on a set S. The relation $\sim$ is called a $\boldsymbol{partialorder}$ if the following three conditions are satisfied.

(i) (Reflexive property) For all x$\in$ S, we have x$\sim$x.

(ii) (Transitive property) For all x,y,z$\in$S, if x$\sim$y and y$\sim$z,then x$\sim$z.

(iii)(Antisymmetric property) For all x,y$\in$ S, if x$\sim$y and y$\sim$x, then x=y.

The relation $\sim$ is a ***totalorder*** on the set S if, in addition, (iv) below is satisfied.

(iv) For all x,y$\in$S, either x$\sim$y or y$\sim$x.

(b) Let A be a set containing at least two elements. We define an order on P(A) using the regular set inclusion $\Subset$. Show that (P(A),$\Subset$) is a partial order, but not a total order.

解答:

(b)

(i) for every x$\in \mathcal{P}$(A), we can easily conclude that x$\subseteq$x

$\therefore$ x$\sim$x

(ii) for every x,y,z$\in\mathcal{P}$(A) and x$\sim$y and y$\sim$z, we have x$\subseteq$y and y$\subseteq$z

$\therefore$ x$\subseteq$z

$\therefore$ x$\sim$z

(iii) for every x,y$\in\mathcal{P}$(A) and x$\sim$y and y$\sim$x, we have x$\subseteq$y and y$\subseteq$x

$\therefore$ x=y

(iv) for z,w$\in$A, let x={z}, y={w}, we have x,y$\in \mathcal{P}$(A), but x$\nsubseteq$y and y$\nsubseteq$x

Based on (i), (ii), (iii), and (iv) ,we can conclude that (P(A),$\Subset$) is a partial order, but not a total order.

题目 (UD:12.16)

You showed in Problem 12.13 that (P(Z),$\Subset$) is a partial order. For every nonempty subset $\mathcal{A}$ of $\mathcal{P}(\mathcal{Z})$ we say that U$\in \mathcal{P}(\mathcal{Z})$ is an upper set of $\mathcal{A}$, if X$\Subset$U for all X$\in \mathcal{A}$. A nonempty set $\mathcal{A} \Subset \mathcal{P}(\mathcal{Z})$ will be called an upper bounded set if there is an upper set of $\mathcal{A}$ in $\mathcal{P}(\mathcal{Z})$. We say $U_0 \in \mathcal{P}(\mathcal{Z})$ is a least upper set if (i) U_0 is an upper set of $\mathcal{A}$ and (ii) if U is another upper set of $\mathcal{A}$, then $U_0 \Subset$ U.

(a) Let $\mathcal{B}$={{ 1,2,5,7},{2,8,10},{2,5,8}}. Show that $\mathcal{B}$ is an upper bounded set and find a least upper set of $\mathcal{B}$, if there is one.

(b) Prove that every nonempty subset of $\mathcal{P}(\mathcal{Z})$ is upper bounded.

(c) Define "lower set," "lower bounded set," and "greatest lower set."

(d) Let $\mathcal{A}$ be a nonempty subset of $\mathcal{P}(\mathcal{Z})$. Using union and intersection, find an expression for least upper set of $\mathcal{A}$ and greatest lower set of $\mathcal{A}$.

(e) Prove that ($\mathcal{P}(\mathcal{Z})$,$\Subset$) has the "least upper set property" (in other words, show every upper bounded set has a least upper set).

解答:

(a) Let U_0={1,2,5,7,8,10}

$\therefore$ for every X$\in \mathcal{B}$, we X$\subseteq U_0$

$\therefore \mathcal{B}$ is an upper bounded set

And the least upper set of $\mathcal{B}$ is U_0={1,2,5,7,8,10}

(b) Let U=$\mathcal{Z}$

$\therefore$ for every $\mathcal{A} \subseteq \mathcal{P}(\mathcal{Z})$ and every X$\in \mathcal{A}$, we have X $\subseteq \mathcal{Z}$

$\therefore$ X $\subseteq$ U

$\therefore \mathcal{A}$ is an upper bounded set

$\therefore$ every nonempty subset of $\mathcal{P}(\mathcal{Z})$ is upper bounded.

(c)

For every nonempty subset $\mathcal{A}$ of $\mathcal{P}(\mathcal{Z})$ we say that U$\in \mathcal{P}(\mathcal{Z})$ is a lower set of $\mathcal{A}$, if U$\subseteq$X for all X$\in \mathcal{A}$. A nonempty set $\mathcal{A} \subseteq \mathcal{P}(\mathcal{Z})$ will be called an lower bounded set if there is a lower set of $\mathcal{A}$ in $\mathcal{P}(\mathcal{Z})$. We say $U_0 \in \mathcal{P}(\mathcal{Z})$ is a greatest upper set if (i) U_0 is a lower set of $\mathcal{A}$ and (ii) if U is another lower set of $\mathcal{A}$, then U $\subseteq U_0$.

(d)

the least upper set of $\mathcal{A}$=$\bigcup A_\alpha$ (for every A$\alpha\in\mathcal{A}$)

the greatest lower set of $\mathcal{A}$=$\bigcap A_\alpha$ (for every A$\alpha\in\mathcal{A}$)

(e)

for every upper bounded set $\mathcal{A}$, we can let U_0=$\bigcup A_\alpha$ (for every A$\alpha\in\mathcal{A}$)

$\therefore$ we can easily conclude that for every X$\in \mathcal{A}$, we have X$\subseteq U_0$

$\therefore U_0$ is an upper set of $\mathcal{A}$

Then ,we need to prove $U_0$is the least one

we set a set $U_1 \subsetneq U_0$

there is x$\notin U_1$ but $\in \bigcup A_\alpha$ (for every A$\alpha\in\mathcal{A}$)

then there is x$\notin U_1$ but $\in A_\alpha$ (A$\alpha\in\mathcal{A}$)

we can let X=Aα , then X$\nsubseteq U_1$

$\therefore U_1$ is not an upper set of $\mathcal{A}$

$\therefore U_0$is the least upper set of $\mathcal{A}$

$\therefore$ every upper bounded set has a least upper set

and we prove that every nonempty subset of $\mathcal{P}(\mathcal{Z})$ is upper bounded in (b)

$\therefore$ ($\mathcal{P}(\mathcal{Z})$,$\Subset$) has the "least upper set property"

题目 (UD:12.20)

Suppose we define ∞ to be an object that satisfies a$\leq \infty$ for all a$\in$ R. Prove that $\infty \neq$ R.

解答:

Let we suppose $\infty \in$ R

$\therefore \infty$+1 $\in$ R

$\therefore \infty$+1 $> \infty$

but it contradicts with a$\leq \infty$ for all a$\in$ R

$\therefore$ what we suppose is not right

$\therefore \infty \neq$ R.

题目 (UD:12.22)

Prove that if a is a rational number, then there is an irrational number b such that a<b.

解答:

let b=a+$\sqrt{2}$

$\because$ $\sqrt{2}$ is an irrational number and a is a rational number

$\therefore$ b=a+$\sqrt{2}$ is am irrational number and b>a

$\therefore$ if a is a rational number, then there is an irrational number b such that a<b.

题目 (UD:12.23)

Prove that for two arbitrary real numbers a and b with a<b, there is an irrational number c such that a<c<b.(Hint: Consider a/$\sqrt{2}$ and b/$\sqrt{2}$.)

解答:

we can find an x that a/$\sqrt{2}$<x<b/$\sqrt{2}$

$\therefore$ according to Theorem 12.11 we have find an x that is a rational number

let x=$\sqrt{2}$c

$\therefore$ c is an irrational number and a<c<b

$\therefore$ for two arbitrary real numbers a and b with a<b, there is an irrational number c such that a<c<b.

第二部分　订正

题目 (题号)

题目。

错因分析:　简述错误原因（可选）。

订正:

正确解答。

第三部分　反馈

你可以写:

- 对课程及教师的建议与意见

- 教材中不理解的内容
- 希望深入了解的内容
- 等