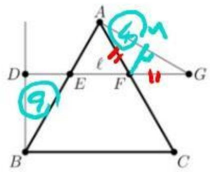
2024/01/25 19:00-21:00

等边三角形 ABC 的边长为 840. 点 D 与 A 位于线 BC 的同一侧，使得 $\overline{BD} \perp \overline{BC}$. 经过点 D 的线 ℓ 与线 BC 平行，且与边 $\overline{AB}$ 和 $\overline{AC}$ 分别相交于点 E 和 F 点。点 G 位于 ℓ 上，使得 F 在 E 和 G 点之间，$\triangle AFG$ 是等腰的，且 $\triangle AFG$ 的面积与 $\triangle BED$ 的面积的比值为 $8:9$. 求 AF.

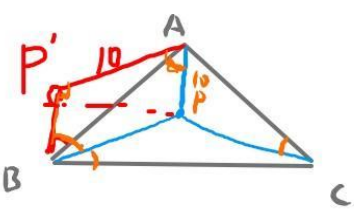


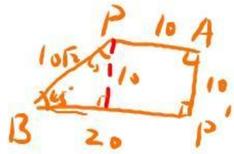
相似三角形面积之比 = 相似比的平方

$AF = AE = \frac{2}{5} \cdot 840 = 168 \times 2 = \boxed{336}$

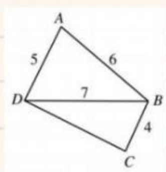
2. **(2023 AIME II Problem 3)** Let $\triangle ABC$ be an isosceles triangle with $\angle A = 90^\circ$. There exists a point P inside $\triangle ABC$ such that $\angle PAB = \angle PBC = \angle PCA$ and $AP = 10$. Find the area of $\triangle ABC$.

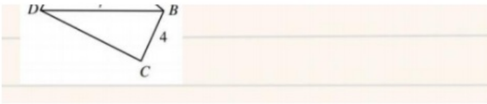
设 $\triangle ABC$ 为等腰三角形，$\angle A = 90^\circ$. $\triangle ABC$ 内存在一个点 P，使得 $\angle PAB = \angle PBC = \angle PCA$ 且 $AP = 10$. 求 $\triangle ABC$ 的面积。





4. 如下图所示，四边形 $ABCD$ 满足 $\angle A + \angle C = 180^\circ$. 求 CD 的长度。



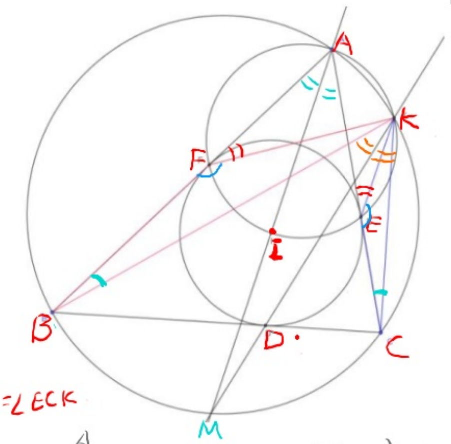


$\cos C = -\frac{1}{5}$.

3. Let ABC be a triangle with $AB > AC$. Its circumcircle is Γ and its incentre is I. Let D be the contact point of the incircle of ABC with BC.

Let K be the point on Γ such that $\angle AKI$ is a right angle.

Prove that AI and KD meet on Γ.



$\angle AKI = 90°$.

需证: $\frac{BK}{CK} = \frac{BD}{CD}$ ⇒ DK是∠BKC角平分线.

$BD = BF$　$CD = CE$

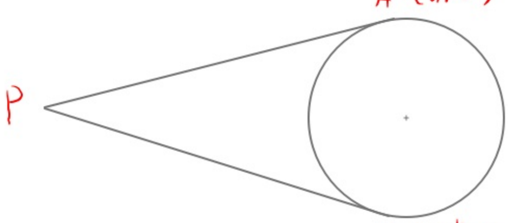
需证 $\frac{BK}{CK} = \frac{BF}{CE}$

需证 $\triangle BFK \sim \triangle CEK$ ← $\angle FBK = \angle ECK$，$\angle AFK = \angle AEK$.

[角平分线定理]

AD是角平分线⇔ $\frac{AB}{AC} = \frac{BD}{DC}$

[切线长定理]

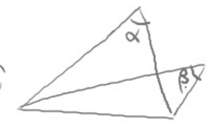


$PA = PB$

[四点共圆] ① 四边形对角和 = 180°

$\alpha + \beta = 180°$

②



$\alpha = \beta$

⇒ 可推出四点共圆.

[圆周角] 在同一个圆中，同弧或等弧所对应的圆周角相等.

圆周角为同弧的圆心角的一半 ⇒ 直径对应的圆周角 = 90°

以直径为斜边的直角三角形，其直角落在圆上

[三角形的内心]

⇒ AI角平分∠A

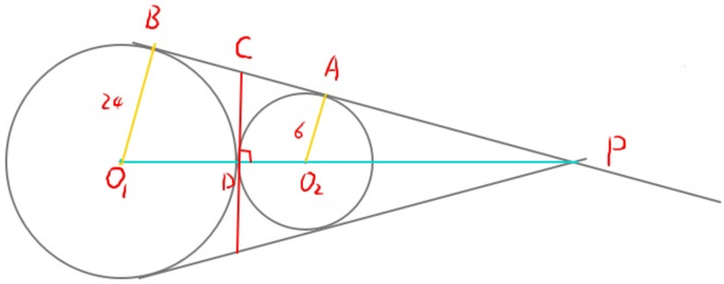
⇒ I是△ABC角平分线的交点

[三角形的内心]

⇒ I是△ABC角平分线的交点

7. **(2022 AIME II Problem 7)** A circle with radius 6 is externally tangent to a circle with radius 24. Find the area of the triangular region bounded by the three common tangent lines of these two circles.

半径为 6 的圆与半径为 24 的圆外切。求这两个圆的三条公切线所围成的三角形区域的面积。



$PO_2 = 10$ $\quad PA = 8$

$PD = PO_2 + O_2D = 10 + 6 = 16$

$CD = \frac{6}{8} \cdot 16 = 12$

$S = CD \cdot PD = 12 \cdot 16 = \boxed{192}$

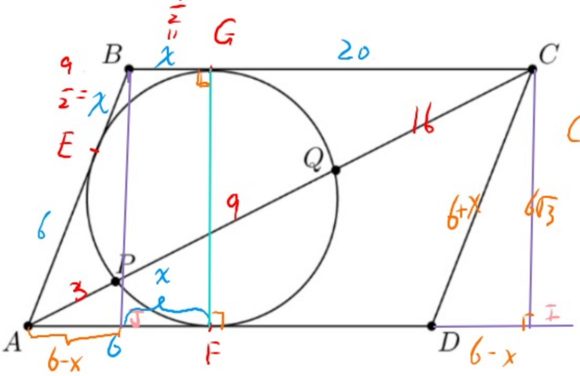
设 $ABCD$ 为平行四边形，且 $\angle BAD < 90°$. 与边 $\overline{DA}$, $\overline{AB}$, $\overline{BC}$ 相切的圆与对角线 $\overline{AC}$ 相交于点 P 和 Q, 且 $AP < AQ$（如图所示）。假设 $AP = 3$, $PQ = 9$ 且 $QC = 16$. 则 $ABCD$ 的面积可以用 $m\sqrt{n}$, 的形式表示，其中 m 和 n 是正整数，n 不能被任何素数的平方整除。求 $m+n$.

圆幂定理

$AF = AE = 6$

$CG = 20$

设 $BG = BE = x$



$CI = \sqrt{28^2 - 26^2}$

$= \sqrt{108}$

$\Rightarrow AD = BC = 20 + \frac{9}{2} = \frac{49}{2}$

$CI = 6\sqrt{3}$.

$S = AD \cdot CI = \frac{49}{2} \cdot 6\sqrt{3} = 147\sqrt{3} \Rightarrow 147 + 3 = \boxed{150}$

Rt△DIC 中用勾股定理.

$(6+x)^2 = (6-x)^2 + 108$.

$12x = -12x + 108$

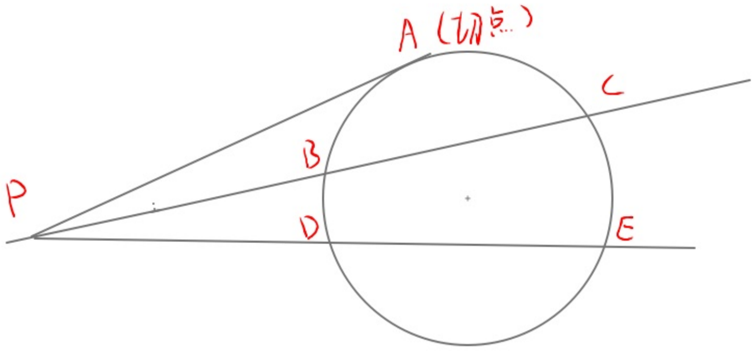
$24x = 108$

$x = \frac{9}{2}$

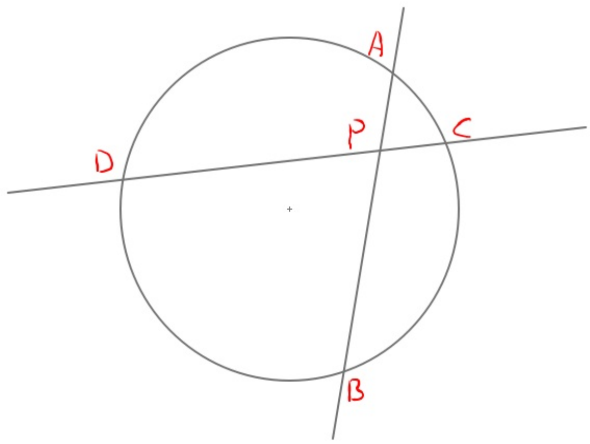
[圆幂定理 power of a point]

$PB \cdot PC = PD \cdot PE = PA^2$

A (切点)



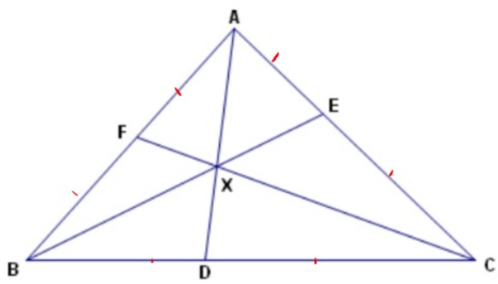
$PB \cdot PC = PD \cdot PE = PA^2$



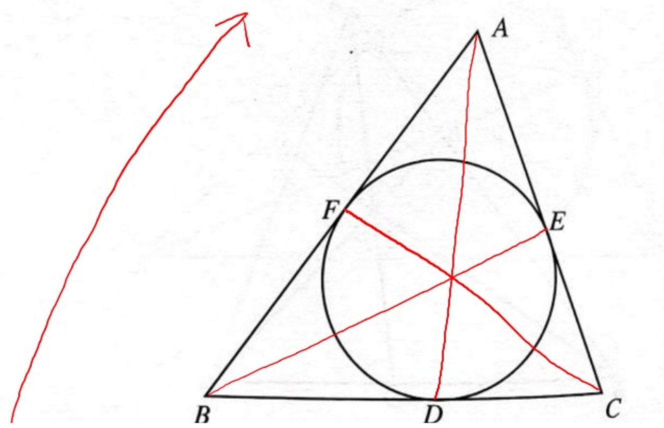
$PA \cdot PB = PC \cdot PD$

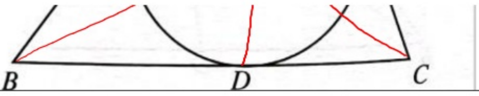
Ceva 定理： 设 ABC 为三角形，D, E, F 分别为边 BC, CA, AB 上的点。直线 AD, BE, CF 交于一点当且仅当

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$



$\triangle ABC$ 的内切圆分别切 BC，CA，AB 于 D，E，F.
求证：AD，BE，CF 共点.
这点称为 Gergonne 点.



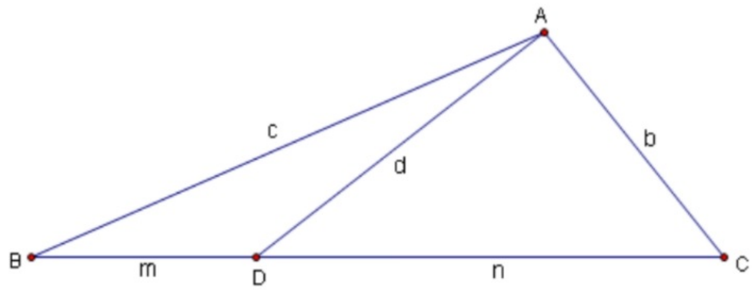


$\frac{\cancel{BD}}{\cancel{CD}}\cdot\frac{\cancel{CE}}{\cancel{EA}}\cdot\frac{\cancel{AF}}{\cancel{FB}}=1$

[切线长定理]　$BD=BF$　$CD=CE$　$AE=AF$

Stewart 定理：给定一个三角形 $\triangle ABC$，其边长为 a,b,c，边对应的顶点分别为 A，B，C. 在边 BC 上取一点 D，使得 $BD=m$，$DC=n$. 连接 AD, 我们有 $AD=d$. 则我们有 $b^2m+c^2n=amn+d^2a$.

$a=BC$.

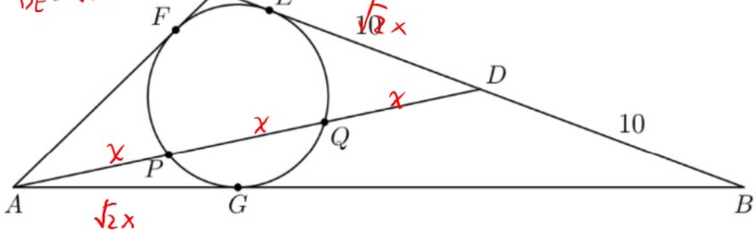


Problem

Triangle ABC has $BC=20$. The incircle of the triangle evenly trisects the median AD. If the area of the triangle is $m\sqrt{n}$ where m and n are integers and n is not divisible by the square of a prime, find $m+n$.

由圆幂定理：$AF=\sqrt2 x$　$AG=\sqrt2 x$.

$DE=\sqrt2 x$



由切线长定理. $CE=CF$　$BE=BG$

$AC=10$，$BC=20$，$AB=AG+BG=AG+BE=10+2\sqrt2 x$.

根据 Stewart 定理.　$10^2\cdot10+(10+2\sqrt2 x)^2\cdot10=2000+(3x)^2\cdot20$.

AC=10 , BC=20 , AB= ...

根据 Stewart 定理. $10^2 \cdot 10 + (10+2\sqrt{2}x)^2 \cdot 10 = 2000 + (3x)^2 \cdot 20$

$(\cancel{80x^2} + 40\sqrt{2}\cancel{x}) = \cancel{180x^2} \cdot 100x^2$

$x = 4\sqrt{2}$

$AB = 10 + 2\sqrt{2} \cdot 4\sqrt{2} = 26.$

$s = \frac{a+b+c}{2}$

[海伦公式] $S = \sqrt{s(s-a)(s-b)(s-c)}$

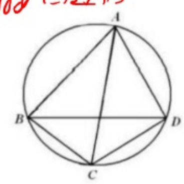
$= \sqrt{28 \cdot 18 \cdot 8 \cdot 2}$

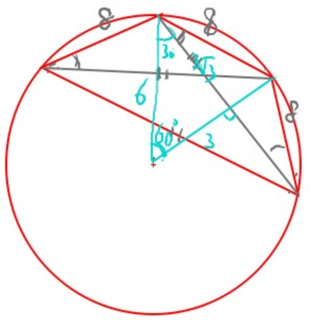
$= 4\sqrt{28 \cdot 18}$

$= 24\sqrt{7 \cdot 2} = 24\sqrt{14}.$

Ptolemy's Theorem: If quadrilateral *ABCD* is a cyclic quadrilateral, then *AC* × *BD* = *AB* × *CD* + *AD* × *BC*.

圆内接四边形





对角线 $6\sqrt{3}$.

Ptolemy·定理.

$(6\sqrt{3})^2 = 8x + 8^2$