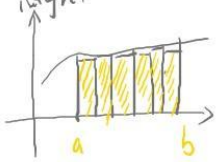
3. **(2015 AIME I Problem 12)** Consider all 1000-element subsets of the set $\{1, 2, 3, \ldots, 2015\}$. From each such subset choose the least element. The arithmetic mean of all of these least elements is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

考虑集合 $\{1, 2, 3, \ldots, 2015\}$ 的所有 1000 个元素子集。从每个这样的子集中选择最小的元素。所有这些最小元素的算术平均值为 $\frac{p}{q}$，其中 p 和 q 是互质正整数。求$p+q$。

$\binom{2015}{1000}$ 个 1000元素子集

Riemann integral



Lebesgue integral

Dirichlet function

$$\delta(x) = \begin{cases} 0 & x \in 无理数 \\ 1 & x \in \mathbb{Q} \end{cases}$$

$$\int \delta = 1 \cdot \overset{0}{\overset{\|}{\mathrm{Vol}(\mathbb{Q})}} + 0 \cdot \cancel{\mathrm{Vol}}(无理数)$$
$$= 0$$

$$\frac{m}{n} = \frac{1\binom{2014}{999} + 2\binom{2013}{999} + \cdots + 1016\binom{999}{999}}{\binom{2015}{1000}}$$

7. $\binom{m}{m} + \binom{m+1}{m} + \ldots + \binom{n}{m} = \binom{n+1}{m+1}$

8. $\binom{n}{0}\binom{m}{k} + \binom{n}{1}\binom{m}{k-1} + \ldots + \binom{n}{k}\binom{m}{0} = \binom{n+m}{k}$

$$\binom{2015}{1000} + \binom{2014}{1000} + \cdots + \binom{1000}{1000} = \binom{2016}{1001}.$$

$$\frac{m}{n} = \frac{\binom{2016}{1001}}{\binom{2015}{1000}} = \frac{\frac{2016!}{\cancel{1001!}\,\cancel{1015!}}}{\frac{2015!}{\cancel{1000!}\,\cancel{1015!}}} = \frac{2016}{1001} = \frac{288}{143} =$$

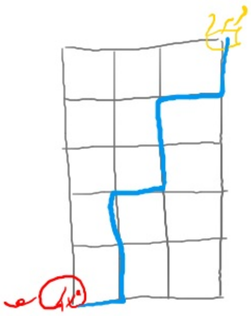
$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

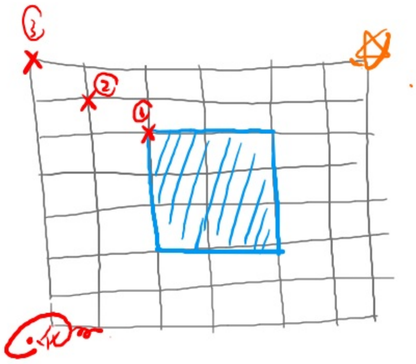
$$\binom{n}{\cdot} = \frac{n!}{\quad} \qquad \binom{n}{\cdot} = \binom{n}{\cdot}$$

$$\binom{n}{i}=\frac{n!}{i!(n-i)!}$$

$$\binom{n}{i}=\binom{n}{n-i}$$

$$\binom{8}{3}=\binom{8}{5}$$





① $\binom{7}{2}\cdot\binom{6}{2}$

② $\binom{7}{1}\cdot\binom{6}{1}$

③ 1

所有路径 $=\left[\binom{7}{2}\cdot\binom{6}{2}+\binom{7}{1}\cdot\binom{6}{1}+1\right]\times 2.$

7. **(2021 AIME I Problem 4)** Find the number of ways 66 identical coins can be separated into three nonempty piles so that there are fewer coins in the first pile than in the second pile and fewer coins in the second pile than in the third pile.

找出将 66 相同硬币分成三堆非空硬币的方法数，使得第一堆中的硬币比第二堆中的硬币少，第二堆中的硬币比第三堆中的硬币少。

(*). $x_1+x_2+x_3=66$ 且 $x_1,x_2,x_3\geq 1$ 的整数，问这个方程有多少解.

$$\binom{66-1}{3-1}=\binom{65}{2}=\frac{65\times 64}{2}$$

$x_1+x_2+x_3=66$ 且 $x_1\geq 2$, $x_2\geq 0$, $x_3\geq 1$ 的整数 问这个方程有多少解.

这个方程有多少解.

$x_1' = x_1 - 1 \geq 1$

$x_2' = x_2 + 1 \geq 1$

$x_3' = x_3 \geq 1$

$x_1' + x_2' + x_3' = x_1 - \cancel{1} + x_2 + \cancel{1} + x_3 = 66$ 且 $x_i' \geq 1$ 的整数.

原方程的解有 $\binom{65}{2}$ 个.

$x_1 + x_2 + x_3 = 66$ 且 x_1, x_2 是正偶数, $x_3 \geq 1$. 问这个方程有多少正整数解

$x_1 = 2 \cdot x_1'$, 其中 $x_1' \geq 1$

$x_2 = 2 \cdot x_2'$, 其中 $x_2' \geq 1$

$2x_1' + 2x_2' + x_3 = 66$ 设 $x_3 = 2 \cdot x_3'$, with $x_3' \geq 1$

$\Rightarrow 2x_1' + 2x_2' + 2x_3' = 66 \Rightarrow x_1' + x_2' + x_3' = 33$ with $x_1', x_2', x_3' \geq 1$

$\binom{33-1}{3-1} = \binom{32}{2} = \frac{32 \times 31}{2}$.

7. **(2021 AIME I Problem 4)** Find the number of ways 66 identical coins can be separated into three nonempty piles so that there are fewer coins in the first pile than in the second pile and fewer coins in the second pile than in the third pile.

找出将 66 相同硬币分成三堆非空硬币的方法数，使得第一堆中的硬币比第二堆中的硬币少，第二堆中的硬币比第三堆中的硬币少。

$x_1 + x_2 + x_3 = 66$ with $1 \leq x_1 < x_2 < x_3$ 设有 k.

$\binom{65}{2} = \underbrace{6 \cdot k}_{x_1, x_2, x_3 \text{ 两两不等}} + \underbrace{3 \cdot 31}_{x_1, x_2, x_3 \text{ 恰有两个相等}} + \underbrace{1}_{x_1 = x_2 = x_3}$

$x_1 = x_2$

x_1, x_2, x_3 两两不等

$x_1 = x_2 \neq x_3$

原解 $\Rightarrow$ $2x_1 + x_3 = 66 \rightarrow \binom{32}{1} = 32$

$32 - 1 = 31$

$x_1 = x_2 \neq x_3$

$k = 3\ 31$

	①	②	③
31			
+29 $\rightarrow$	1		
+28			
+26 $\rightarrow$	2		
+25 $\rightarrow$	3		
+23 $\rightarrow$	4		
+22 $\rightarrow$	5		
+20			
+19 $\rightarrow$	6		
+17			
16 $\rightarrow$	7		
14			
13			
11			
10			
8			
7			
5			
4			
2			
1			

9. **(2023 AIME I Problem 11)** Find the number of subsets of $\{1, 2, 3, \ldots, 10\}$ that contain exactly one pair of consecutive integers. Examples of such subsets are $\{\mathbf{1}, \mathbf{2}, 5\}$ and $\{1, 3, \mathbf{6}, \mathbf{7}, 10\}$.

求 $\{1, 2, 3, \ldots, 10\}$ 中恰好包含一对连续整数的子集的数量。此类子集的示例为 $\{\mathbf{1}, \mathbf{2}, 5\}$ 和 $\{1, 3, \mathbf{6}, \mathbf{7}, 10\}$.

$\{\cancel{1}, 2, 3, \cancel{4}, \cancel{5}, 6, 7, \cancel{8}, 9, 10\}$

令 a_n 表示 $\{1,2,3, \ldots n\}$ 中无连续整数的子集的个数.

$a_4 \cdot a_2$ = 恰含有一个连续整数 $\{6,7\}$ 的子集的个数

$a_3 \cdot a_3$ = $\{5,6\}$

$a_2 \cdot a_4$ = $\{4,5\}$

$a_1 \cdot a_5$ = $\{3,4\}$

a_6 = $\{2,3\}$

a_7 = $\{1,2\}$

$a_5 \cdot a_1$ = $\{7,8\}$

a_6 = $\{8,9\}$

a_7 = $\{9,10\}$

$2a_7 + 2a_6 + 2a_5 \cdot a_1 + 2a_4 \cdot a_2 + a_3 \cdot a_3$

$\{1,2,3, \ldots n\}$

$a_n = a_{n-2} + a_{n-1}$

包含 n　　不包含 n

$a_1 = 2$

$a_2 = 3$

11. **(2004 AIME I Problem 3)** A convex polyhedron P has 26 vertices, 60 edges, and 36 faces, 24 of which are triangular and 12 of which are quadrilaterals. A space diagonal is a line segment connecting two non-adjacent vertices that do not belong to the same face. How many space diagonals does P have?

补集法

凸多面体 P 有 26 个顶点、60 个边和 36 个面，其中 24 个是三角形，

diagonals does P have?

凸多面体 P 有 26 个顶点、60 个边和 36 个面，其中 24 个是三角形，12 个是四边形。空间对角线是连接不属于同一面的两个不相邻顶点的线段。 P 有多少条空间对角线？

棱　　面对角线　　空间对角线

$\binom{26}{2} = 60 + 2\cdot 12 + x.$

$\Rightarrow x = \checkmark$

12. **(2021 AIME II Problem 6)** For any finite set S, let $|S|$ denote the number of elements in S. Find the number of ordered pairs (A, B) such that A and B are (not necessarily distinct) subsets of $\{1, 2, 3, 4, 5\}$ that satisfy

$$|A| \cdot |B| = |A \cap B| \cdot |A \cup B|$$

对于任何有限集 S，令 $|S|$ 表示 S 中的元素数量。求有序对 (A, B) 的数量，使得 A 和 B 是$\{1, 2, 3, 4, 5\}$ 的（不一定不同）子集，且满足

$$|A| \cdot |B| = |A \cap B| \cdot |A \cup B|$$

$|A\cup B| = |A| + |B| - |A\cap B|.$

原式 $\Rightarrow |A|\cdot|B| = |A\cap B|\cdot\left(|A|+|B| - |A\cap B|\right)$

$\Rightarrow |A\cap B|^2 - |A\cap B|\cdot|A| - |A\cap B|\cdot|B| + |A|\cdot|B| = 0$

$\Rightarrow (|A\cap B| - |A|)(|A\cap B| - |B|) = 0.$

$\Rightarrow |A\cap B| = |A|$ 或 $|A\cap B| = |B|$

$\Rightarrow A\subseteq B$ 或 $B\subseteq A.$

1° $A\subseteq B$ 这样的有序数对 (A, B) 有多少个.

$|A| = 0 \Rightarrow A=\phi.$ B有 2^5 种

$|A| = 1 \Rightarrow$ A有5种， B有 2^4 种.

$|A| = 2 \Rightarrow$ A有 $\binom{5}{2}$ B有 2^3 种.

$|A| = 3 \Rightarrow$ A有 $\binom{5}{3}$ B有 2^2 种

$|A|=3 \Rightarrow A$有$\binom{5}{3}$ B有2^2种.

$|A|=4 \Rightarrow A$有$\binom{5}{4}$ B有2^1种.

$|A|=5 \Rightarrow A$有$\binom{5}{5}$ B有2^0种

$$\Rightarrow \quad 1\cdot 2^5+5\cdot 2^4+\binom{5}{2}\cdot 2^3+\binom{5}{3}\cdot 2^2+\binom{5}{4}\cdot 2^1+\binom{5}{5}2^0.$$

Zou 和 Chou 正在通过互相进行 6 场比赛来练习 100 米的冲刺。Zou 赢得了第一场比赛。如果他们赢得了上一场比赛，他们赢得下一场比赛的概率为 $\frac{2}{3}$，但如果他们输掉了上一场比赛，则只有 $\frac{1}{3}$ 的概率赢得下一场比赛。Zou 在 6 场比赛中恰好赢得 5 场的概率为 $\frac{m}{n}$，其中 m 和 n 是互素的正整数。求 $m+n$.

ZCZZZZ $\quad \frac{1}{3}\times\frac{1}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}$

ZZCZZZ $\quad \frac{2}{3}\times\frac{1}{3}\times\frac{1}{3}\times\frac{2}{3}\times\frac{2}{3}$

ZZZCZZ $\quad \frac{2}{3}\times\frac{2}{3}\times\frac{1}{3}\times\frac{1}{3}\times\frac{2}{3}$

ZZZZCZ $\quad \frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{1}{3}\times\frac{1}{3}.$

ZZZZZC $\quad \frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{1}{3}.$

$$4\times\left(\frac{1}{3}\right)^2\cdot\left(\frac{2}{3}\right)^3+\left(\frac{2}{3}\right)^4\cdot\frac{1}{3}.$$

$$=\frac{48}{3^5}=\frac{16}{3^4}$$

$\boxed{097}$

天气预报说，明天阴天概率是 $\frac{2}{3}$，晴天概率是 $\frac{1}{3}$ A 阴天投篮投中概率为 $\frac{1}{10}$ 晴天……投中概率为 $\frac{1}{5}$

明天投中多少个篮球的数学期望

条件概率 条件概率

数学期望：

$$E=\underset{\text{阴天}}{\frac{1}{10}\cdot\frac{2}{3}}+\underset{\text{晴天}}{\frac{1}{5}\cdot\frac{1}{3}}=\frac{2}{30}+\frac{2}{30}=\frac{4}{30}=\frac{2}{15}$$

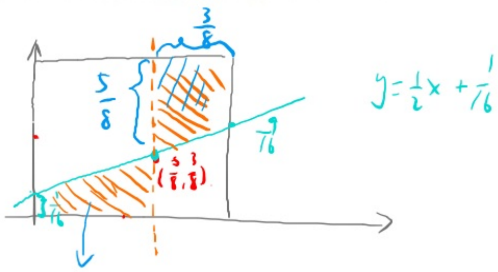
7. **(2020 AIME II Problem 2)** Let P be a point chosen uniformly at random in the interior of the unit square with vertices at $(0,0),(1,0),(1,1)$, and $(0,1)$. The probability that the slope of the line determined by P and the point $\left(\frac{5}{8},\frac{3}{8}\right)$ is greater than or equal to $\frac{1}{2}$ can be written as $\frac{m}{n}$, where

几何概型

and $(0,1)$. The probability that the slope of the line determined by P and the point $\left(\frac{5}{8}, \frac{3}{8}\right)$ is greater than or equal to $\frac{1}{2}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

几何概型

单位正方形的顶点为 $(0,0),(1,0),(1,1)$ 和 $(0,1)$，令 P 为单位正方形内部均匀随机选择的点。 P和点$\left(\frac{5}{8}, \frac{3}{8}\right)$确定的直线的斜率大于或等于$\frac{1}{2}$的概率可以写为$\frac{m}{n}$，其中$m$ 和n 是互质正整数。求$m+n$。



$$\frac{1}{2}\left(\frac{1}{16}+\frac{3}{8}\right)\cdot\frac{5}{8}+\frac{1}{2}\left(\frac{7}{16}+\frac{5}{8}\right)\cdot\frac{3}{8}=$$

$$\frac{1}{2}\times\frac{7}{16}\cdot\frac{5}{8}+\frac{1}{2}\times\frac{17}{16}\cdot\frac{3}{8}=\frac{35+51}{16\cdot16}=\frac{\cancel{86}^{43}}{16\cdot\cancel{16}_{8}}$$

$$=\frac{43}{128}$$