1. **(2018 AIME I Problem 2)** The number n can be written in base 14 as $\underline{a}\ \underline{b}\ \underline{c}$, can be written in base 15 as $\underline{a}\ \underline{c}\ \underline{b}$, and can be written in base 6 as $\underline{a}\ \underline{c}\ \underline{a}\ \underline{c}$, where $a > 0$. Find the base-10 representation of n.

   数字 n 可以用 14 进制写为 $\underline{a}\ \underline{b}\ \underline{c}$，可以用 15 进制写为 $\underline{a}\ \underline{c}\ \underline{b}$，并且可以用 6 进制写为 $\boxed{\underline{a}\ \underline{c}\ \underline{a}\ \underline{c}}$，其中 $a > 0$. 求 n 的 10 进制表示形式。

$a=4$
$b=10$
$c=1$

[N-进制转换成10进制]

$$\underline{a}\underline{b}\underline{c}_{(N)} = a\cdot N^2 + b\cdot N^1 + c\cdot N^0$$

$$11010_{(2)} = 1\cdot 2^4 + 1\cdot 2^3 + 0\cdot 2^2 + 1\cdot 2^1 + 0\cdot 2^0 = 26_{(10)}$$

$$n = a\cdot 14^2 + b\cdot 14 + c = a\cdot 15^2 + c\cdot 15^1 + b = a\cdot 6^3 + c\cdot 6^2 + a\cdot 6^1 + c$$

$$\begin{cases} 29a + 14c - 13b = 0 \\ 3a - 22c + b = 0 \end{cases}$$

$\Rightarrow b = 22c - 3a$

$\Rightarrow 29a + 14c - 13(22c - 3a) = 0$

$68a = 272c$

$a = 4c$

$\Rightarrow a=4 \quad c=1$

2. **(2020 AIME I Problem 3)** A positive integer N has base-eleven representation $\underline{a}\underline{b}\underline{c}$ and base-eight representation $\underline{1}\underline{b}\underline{c}\underline{a}$, where a, b, and c represent (not necessarily distinct) digits. Find the least such N expressed in base ten.

   正整数 N 具有以 11 为底的表示形式 $\underline{a}\underline{b}\underline{c}$ 和以 8 为底的表示形式 $\underline{1}\underline{b}\underline{c}\underline{a}$，其中 a, b, c 代表（不一定不同）数位上的数字。求满足此条件的最小的 N（用十进制表示）。

$a=5$, $b=1$, $c=5$

$N = a\cdot 11^2 + b\cdot 11 + c$

$a=5$, $b=1$, $c=5$

$$N = a\cdot 11^2 + b\cdot 11 + c$$
$$= 1\cdot 8^3 + b\cdot 8^2 + c\cdot 8^1 + a$$

(*) $120a = 8^3 + 53b + 7c \geq 8^3$

$\Rightarrow a \geq 5$ $a=5$时 $88 = 53b + 7c$

$b=1$, $c=5$

3. **(2023 AIME I Problem 4)** The sum of all positive integers m such that $\frac{13!}{m}$ is a perfect square can be written as $2^a3^b5^c7^d11^e13^f$, where $a, b, c, d, e,$ and f are positive integers. Find $a+b+c+d+e+f$.

使得 $\frac{13!}{m}$ 为完全平方的所有正整数 m 的总和可写为 $2^a3^b5^c7^d11^e13^f$, 其中 $a, b, c, d, e,$ 和 f 是正整数。求 $a+b+c+d+e+f$.

[分解质因数] $N = p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}$

$360 = 36\times 10 = 4\times 9\times 10 = 2^2\cdot 3^2\cdot 2\cdot 5$
$= 2^3\cdot 3^2\cdot 5^1$.

[平方数] $36 = 2^{2}\cdot 3^{2}$

$\sqrt{36} = 36^{\frac{1}{2}} = (2^2\cdot 3^2)^{\frac{1}{2}} = 2^{2\times\frac{1}{2}}\cdot 3^{2\times\frac{1}{2}}$.

$\longrightarrow N = p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}$ 平方数

$\Leftrightarrow 2|r_1, 2|r_2, \cdots 2|r_t$.

[立方数]. $\longrightarrow N = p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}$ 立方数

$\Leftrightarrow 3|r_1, \cdots 3|r_2, \quad 3|r_t$.

[因数公式]. $360 = 2^3\cdot 3^2\cdot 5$

$(3+1)(2+1)(1+1)$

$N = p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}$ 正因子个数 $=(r_1+1)(r_2+1)\cdots(r_t+1)$.

有多少因子是12的倍数. $(1+1)(1+1)(1+1)$.

$360 = 2^2\cdot 3\,(\underline{2\cdot 3\cdot 5})$.

$1\ 2^0 \quad 3^0 \quad \_0$

$360 = 2^2 \cdot 3\,(2 \cdot 3 \cdot 5).$

有多少因子是平方数.

$\begin{pmatrix} 2^0 \\ 2^2 \end{pmatrix} \begin{pmatrix} 3^0 \\ 3^2 \end{pmatrix} 5^0.$

$13! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13.$

$\boxed{m = 2^a 3^b \cdot 5^c \cdot 7 \cdot 11 \cdot 13 \quad \text{with} \quad a \text{ even},\ b \text{ odd},\ c \text{ even}}$

$$\sum m = (2^0 + 2^2 + \cdots + 2^{10}) \cdot (3^1 + 3^3 + 3^5)(5^0 + 5^2) \cdot 7 \cdot 11 \cdot 13$$

$$= 2 \cdot 3^2 \cdot 5 \cdot 7^3 \cdot 11 \cdot 13^4$$

$$\Rightarrow 1 + 2 + 1 + 3 + 1 + 4 = \boxed{012}$$

4. **(2020 AIME II Problem 1)** Find the number of ordered pairs of positive integers (m, n) such that $m^2 n = 20^{20}$.

求满足 $m^2 n = 20^{20}$ 的有序正整数对 (m, n) 的数量。

5. **(2020 AIME I Problem 10)** Let m and n be positive integers satisfying the conditions

- $\gcd(m+n, 210) = 1$,
- m^m is a multiple of n^n, and
- m is not a multiple of n.

Find the least possible value of $m + n$.

设 m 和 n 为满足条件的正整数：

- $\gcd(m+n, 210) = 1$,
- m^m 是 n^n 的倍数，并且
- m 不是 n 的倍数。

- m 不是 n 的倍数。

求 $m+n$ 的最小可能值。

$p|n \Rightarrow p|m$

$\longrightarrow p|n+m \quad p\nmid 210=2\cdot3\cdot5\cdot7$

$p\geq 11$

$2|m$ but $2\nmid n$.

$n=11 \Rightarrow 11|m$ $\times$

$n=11^2. \Rightarrow 11|m \quad m=2\cdot 11^1.$

m^m 是 $n^n=(11^2)^{121}=11^{242}$ 倍数 $\Rightarrow m\geq 242=11\cdot 22$

$m=2^5\cdot 11$

$m=2\cdot 11\cdot 13$ ✓

$\boxed{n=11^2 \quad m=2\cdot11\cdot13}$

$m^m=2^{11\cdot26}\cdot 11^{11\cdot26}\cdot 13^{11\cdot26}$ 是 $n^n=11^{242}=11^{11\cdot22}$ 倍数.

$\gcd(m+n=11(11+26)=11\cdot37,\ 210)=1.$

6. **(2023 AIME I Problem 7)** Call a positive integer n extra-distinct if the remainders when n is divided by $2,3,4,5,$ and 6 are distinct. Find the number of extra-distinct positive integers less than 1000.

如果 n 除以 $2,3,4,5$ 和 6 时的余数不同，则称正整数 n 是**额外不同的**。求小于 1000 的**额外不同的**正整数的数量。

[中国剩余定理]. $\gcd(a,b)=1$, 那么存在 $x\ y\in\mathbb{Z}$, 满足. $xa+yb=1$.

$\gcd(4,25)=1$, 令 $x=-6$, $y=1$

$-6\times4+1\times25=1$.

CRT: $\begin{cases} N\equiv n \pmod a \\ N\equiv m \pmod b \end{cases} \Longrightarrow N\equiv m\cdot xa+nyb \pmod{a\cdot b}$

$(N \equiv m$ (mo

$$\begin{cases} N \equiv 1 \pmod 2 \\ N \equiv 3 \pmod 5 \end{cases}$$

① $2\cdot(-2) + 5\cdot 1 = 1$

② CRT: $N \equiv 2\cdot(-2)\cdot 3 + 5\cdot 1\cdot 1 \pmod{10}$

$\equiv -7 \pmod{10}$

$\equiv 3 \pmod{10}$

$n+1 \equiv 0 \pmod{12}$

$$\Rightarrow \begin{cases} n \equiv -1 \pmod{12} \\ n \equiv 0 \pmod 5 \end{cases}$$

CRT. $12\times(-2) + 5\times 5 = 1$

$n \equiv 12\times(-2)\times 0 + 5\times 5\times(-1) \equiv -25 \pmod{60}$

$\equiv 35$.

7. **(2021 AIME II Problem 13)** Find the least positive integer n for which $2^n + 5^n - n$ is a multiple of 1000.

找到最小正整数 n，使得 $2^n + 5^n - n$ 是 1000 的倍数。

$$\begin{cases} 2^n + 5^n - n \equiv 0 \pmod 8 \\ 2^n + 5^n - n \equiv 0 \pmod{125} \end{cases}$$

$n = 1, 2, 3$ ✗

$2^n + 5^n - n \equiv 5^n - n \equiv 0 \pmod 8$

$$2^n+5^n-n \equiv 5^n-n \equiv 0 \pmod 8$$

$$\Rightarrow \quad 5^n \equiv n \pmod 8$$

$5^1 \equiv 5 \pmod 8$

$5^2 \equiv 1 \pmod 8$

$5^3 \equiv 5 \pmod 8$

$5^4 \equiv 1 \pmod 8$

$\Rightarrow$

$5^{2k+1} \equiv 5 \pmod 8$

$5^{2k} \equiv 1 \pmod 8$

$$5^n \equiv n \equiv 1 \text{ or } 5 \pmod 8$$

$n = 8k+1$ or $8k+5$.

$n \equiv 5^n \equiv 5 \pmod 8$

$$\Rightarrow \quad n \equiv 5 \pmod 8$$

$$2^n+\cancel{5^n}-n \equiv 2^n-n \equiv 0 \pmod{25}$$

$$\Rightarrow \quad 2^n \equiv n \pmod{25}$$

$n = 8k+5$

$$\begin{aligned}
2^{8k+5} &\equiv 2^5 \cdot (2^8)^k \equiv 32 \cdot (16)^{2k} \\
&\equiv 32\,(15+1)^{2k} \\
&\equiv 32 \left[\binom{2k}{1} \cdot 15 + \binom{2k}{0} \right] \pmod{25} \\
&\equiv 32\,(30k+1) \pmod{25} \\
&\equiv 7\,(5k+1) \\
&\equiv 35k+7 \equiv 10k+7 \pmod{25} \\
&\equiv n \equiv 8k+5.
\end{aligned}$$

$$\boxed{(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \cdots}$$

$$\rightsquigarrow \quad 2k+2 \equiv 0 \pmod{25}.$$

$$\Rightarrow \quad 26(k+1) \equiv 0 \pmod{25}$$

$$\equiv 1 \cdot (k+1) \equiv k+1.$$

$$\Rightarrow \quad k \equiv 24 \pmod{25}$$

$$\Rightarrow \quad k \equiv 24 \pmod{25}.$$

$$\Rightarrow \quad n = 8k+5 = 8(25k'+24)+5$$

$$= 200k' + 197$$

$$2^n + \cancel{5^n} \equiv n \pmod{125}.$$

$$\Rightarrow \quad 2^n \equiv n \pmod{125}.$$

$$2^{200k'+197} \equiv n \pmod{125}.$$

[欧拉定理]. $(a,m)=1$.

$$a^{\varphi(m)} \equiv 1 \pmod{m}.$$

Euler 函数

$\varphi(m) = \{1,2,3,\cdots m\}$ 和 m 互质的数的个数.

$m = p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}$ $\qquad \varphi(m) = m \cdot \left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_t}\right)$

$\gcd(3,10)=1$, $\qquad 3^{\varphi(10)} \equiv 1 \pmod{10}.$

$$\varphi(10) = 10\left(1-\frac{1}{2}\right)\left(1-\frac{1}{5}\right) = 4. \Rightarrow \underbrace{3^4}_{81} \equiv 1 \pmod{10}$$

$$\varphi(125) = 125\left(1-\frac{1}{5}\right) = 100.$$

Euler thm

$$2^{100} \equiv 1 \pmod{125}$$

$$2^{200k'+197} \equiv n \pmod{125}.$$

$$2^{200k'+197} \equiv n \pmod{125}.$$

$$\Rightarrow \quad 2^3\left(2^{200k'+197}\right) \equiv 2^3 \cdot n \pmod{125}$$

$$\Rightarrow \quad 2^{200k'+200} \equiv 8n \pmod{125}.$$

$$\Rightarrow \quad \left(2^{100}\right)^{2k'+2} \equiv 1 \equiv 8n \pmod{125}.$$

$$8 \times 47 = 376 = 3 \times 125 + 1.$$

$$47 \equiv 47 \times 8n \pmod{125}$$

$$\Rightarrow \quad 47 \equiv (3 \times \cancel{125} + 1)\, n \pmod{125}$$

$$\Rightarrow \quad 47 \equiv n \pmod{125}.$$

$$\begin{cases} n \equiv 5 \pmod 8 \\ n \equiv 47 \pmod{125} \end{cases}$$

$$\rightsquigarrow \quad 8 \times 47 + (-3) \times 125 = 1$$

$$n \equiv 8 \times 47 \times 47 + (-3) \times 125 \times 5 \equiv 15797 \pmod{1000}$$

$$\equiv 797 \pmod{1000}.$$

$$\Rightarrow \quad n_{min} = \boxed{797}$$

Taylor展开

$i = \sqrt{-1}$

$$e^{i\theta} = \cos\theta + i \cdot \sin\theta.$$

$$\left(1 + \sqrt{3}\, i\right)^{100} = ?$$

$$1 + \sqrt{3}\, i = 2 \cdot \frac{1 + \sqrt{3}\, i}{2} = 2 \cdot \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right).$$

$$= 2\left(\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\, i\right)$$

$$= 2 \cdot e^{i \cdot \frac{\pi}{3}}$$

$(1+\sqrt{3}\, i)^{100}$

$$= 2 e^{i\cdot\frac{\pi}{3}}$$

$$(1+\sqrt{3}i)^{100}$$

$$= 2^{100} \cdot e^{i\cdot\frac{\pi}{3}\cdot 100}$$

$$= 2^{100} \cdot e^{i\cdot(32\pi + \frac{4}{3}\pi)} = 2^{100} e^{\underline{i\cdot 32\pi}} \cdot e^{i\cdot\frac{4}{3}\pi}$$

$$\boxed{e^{i\cdot 2k\pi} = 1}$$

$$= 2^{100} \cdot \underline{e^{i\cdot\frac{4}{3}\pi}}$$