

Finding the optimal path

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1 Frictionless economy

Given that the evolution of the net worth is the following:

$$e_{t+1} = Z(\theta + \epsilon)k_t^\alpha + (1 - \delta)k_t - (1 + r)(c + k_t - e_t)$$

The value of the firm at time t is given by:

$$V_t(e_t) = \max_{k_t} e_t + \beta V_{t+1}(e_{t+1})$$

Solving the belman equation, the FOC are given by:

$$\frac{\partial V_t(e_t)}{\partial k_t} = \frac{\partial e_t}{\partial k_t} + \beta \frac{\partial V_{t+1}(e_{t+1})}{\partial k_t}$$

Strong assumption a change in k_t does not have an impact on e_t only on e_{t+1} :

$$\frac{\partial e_t}{\partial k_t} = 0$$

Using the envelope theorem:

$$\begin{aligned} \frac{\partial V_{t+1}(e_{t+1})}{\partial k_t} &= \frac{\partial V_{t+1}(e_{t+1})}{\partial e_{t+1}} \frac{\partial e_{t+1}}{\partial k_t} \\ \frac{\partial V_{t+1}(e_{t+1})}{\partial e_{t+1}} &= 1 + \beta(1 + r) \\ \frac{\partial e_{t+1}}{\partial k_t} &= Z(\theta + \epsilon)\alpha k_t^{\alpha-1} - (\delta + r) \end{aligned}$$

Thus:

$$\frac{\partial V_{t+1}(e_{t+1})}{\partial k_t} = [1 + \beta(1 + r)][Z(\theta + \epsilon)\alpha k_t^{\alpha-1} - (\delta + r)]$$

So the FOC became:

$$\frac{\partial V_t(e_t)}{\partial k_t} = 0 + \beta[1 + \beta(1 + r)][Z(\theta + \epsilon)\alpha k_t^{\alpha-1} - (\delta + r)] = 0$$

The optimal level of capital at time t is:

$$\hat{k}_t = \frac{\beta(1 + r)Z(\theta + \epsilon)\alpha^{\frac{1}{1-\alpha}}}{\delta + r}$$

Plotting the graph in (assuming $X = Z(\theta + \epsilon)$) and $y = \hat{k}_t$

$$y = \frac{0.8x^{\frac{1}{1-0.8}}}{0.03 + 0.1}$$

Its interesting to see that if there is an increase in productivity the firm need more optimal capital K , while the most low productivity firms need less capital to operate

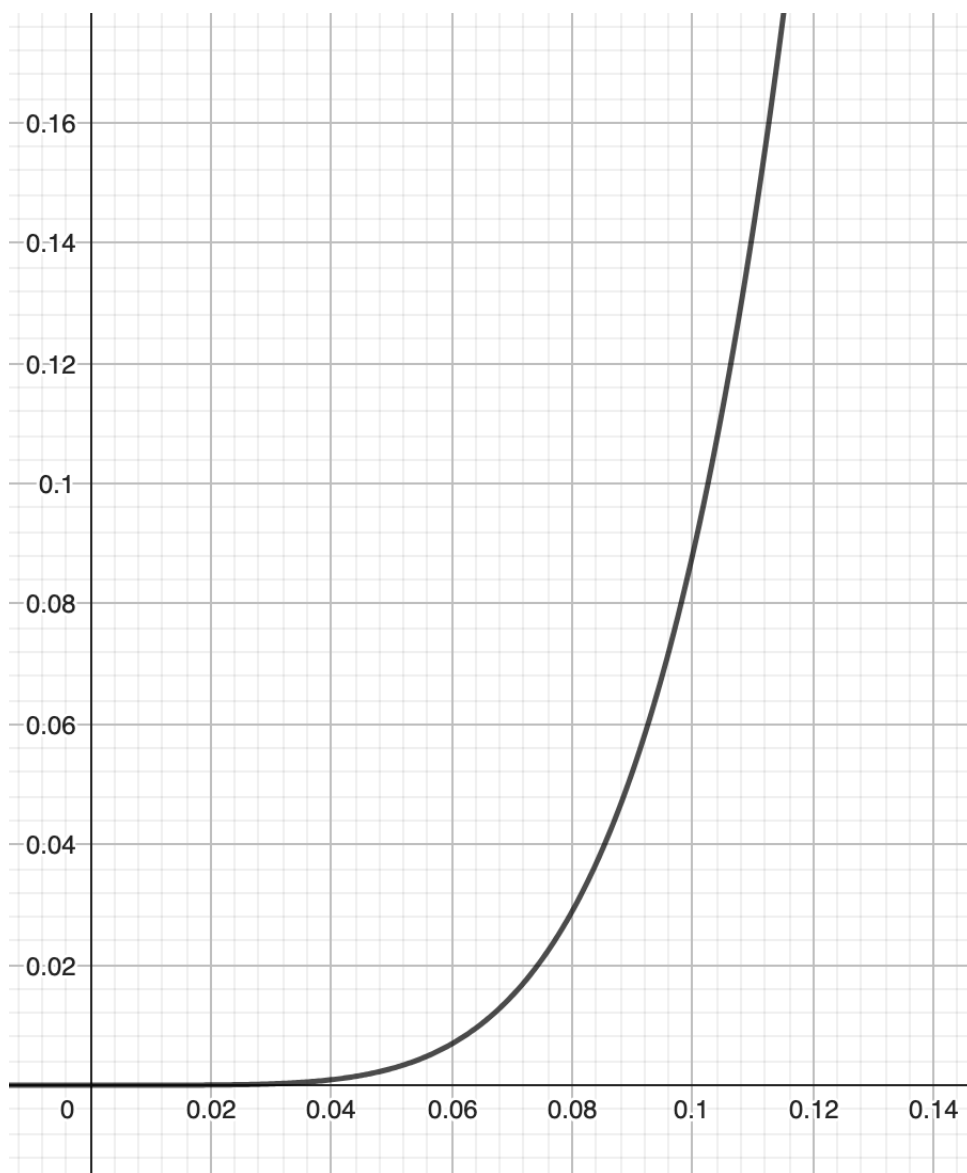


Figure 1: $X = Z(\theta + \epsilon)$ and $y = \hat{k}_t$

1.1 Frictions

Now I consider the participation constraint of the borrower given that she could observe ϵ only paying μk^α so:

$$(1+r)(k+c+e)(1-\Phi(\bar{\epsilon})) + \int_{-\infty}^{\bar{\epsilon}} [Z(\theta + \bar{\epsilon})k^\alpha + (1-\delta)k - \mu k^\alpha] d\Phi(\epsilon) \geq (1+r)(c+k+e)$$

r is the rate of interest that makes equal the expected value of borrowing to the opportunity cost of capital. rewriting became

$$Z[\theta + G(\bar{\epsilon})]k_t^\alpha + (1-\delta)k_t - \mu k_t^\alpha \Phi(\bar{\epsilon}) = (1+r)(k_t + c - e_t)$$

where

$$G(\bar{\epsilon}) = (1-\Phi(\bar{\epsilon}))\bar{\epsilon} + \int_{-\infty}^{\bar{\epsilon}} \epsilon d\Phi(\epsilon)$$

While the firm participation constraint is $q_t \geq 0$ so the end-of-period net worth must be greater than 0, thus the problem of the firm becomes: The value of the firm at time t is given by:

$$V_t(e_t) = \max_{k_t} q_t - e_{t+1} + \beta V_{t+1}(e_{t+1})$$

s.t.

$$q_t = Z(\theta + \epsilon)k_t^\alpha + (1-\delta)k_t - (1+r)(c+k_t - e_t)$$

$$Z[\theta + G(\bar{\epsilon})]k_t^\alpha + (1-\delta)k_t - \mu k_t^\alpha \Phi(\bar{\epsilon}) = (1+r)(k_t + c - e_t)$$

so we can rewrite the second constraint in order to get r :

$$r = \frac{Z[\theta + G(\bar{\epsilon})]k_t^\alpha + (1-\delta)k_t - \mu k_t^\alpha \Phi(\bar{\epsilon})}{k_t + c - e_t} - 1$$

FOC:

$$\frac{\partial V_t(e_t)}{\partial k_t} = \frac{\partial q_t}{\partial k_t} - \frac{\partial e_t}{\partial k_t} + \beta \frac{\partial V_{t+1}(e_{t+1})}{\partial k_t} = 0$$

by envelope theorem:

$$\frac{\partial V_{t+1}(e_{t+1})}{\partial k_t} = \frac{\partial V_{t+1}(e_{t+1})}{\partial e_{t+1}} \frac{\partial e_{t+1}}{\partial k_t}$$

Strong HP:

$$\frac{\partial e_{t+1}}{\partial k_t} = 0$$

Qui non so se ha senso continuare perchè non ho nessun meccanismo che mi trasformi il net worth al tempo t in net worth al periodo $t+1$, almeno che non includa la definizione di dividendo come $d_t = q_t - e_{t+1}$, allora in questo caso dovrei risolvere il problema con un langrangiana dato che la firm dovrebbe scegliere sia k che e net worth.

2 Redefining the problem

$$V(k_t) = \max_{k_{t+1}, e_{t+1}} d_t + \beta V(k_{t+1})$$

s.t.

$$f(k_t) = Zk_t^\alpha$$

$$f(k_t) = d_t + (c + k_{t-1} - e_{t-1})(1+r) + k_t - (c + k_t - e_t) - k_{-1}(1-\delta)$$

$$(1+r)(c + k_t - e_t)p + (1-p)f(k_t) = (1+r_f)(c + k_t - e_t)$$

$$B_t = c + k_t - e_t; R = 1+r; R_f = 1+r_f;$$

$$R = \frac{R_f}{p} - \frac{1-p}{p} \frac{f(k_t)}{D_t}$$

In order to understand the mechanism behind this optimization problem, I firstly solve the three times problem working backward. The value function in $t = 2$ is

$$V_{t+2} = \max d_{t+2}$$

Since there firm will not exists in $t+2$, there are no investment $B_{t+2} = 0$, thus $0 = k_{t+2} + c - e_{t+2}$ as consequence $k_{t+2} = e_{t+2} - c$. Then we can rewrite the value function:

$$V_{t+2} = \max Z(e_{t+2} - c)^\alpha - (c + k_{t+1} - e_{t+1})(1 + r_{t+1}) - e_{t+2} + c + (c + e_{t+2} - c - e_{t+2}) + k_{t+1}(1 - \delta)$$

$$V_{t+2} = \max_{e_{t+2}} Z(e_{t+2} - c)^\alpha - B_{t+1}R_{t+1} - e_{t+2} + c + k_{t+1}(1 - \delta)$$

FOC:

$$\frac{\partial V_{t+2}}{\partial e_{t+2}} = Z\alpha(e_{t+2} - c)^{\alpha-1} - 1 = 0$$

$$(e_{t+2} - c)^{\alpha-1} = (Z\alpha)^{-1}$$

$$e_{t+2} = (Z\alpha)^{\frac{-1}{\alpha-1}} + c$$

Thus:

$$d_{t+2} = Z \left[(Z\alpha)^{\frac{-1}{\alpha-1}} + c \right] - B_{t+1}R_{t+1} - \left[(Z\alpha)^{\frac{-1}{\alpha-1}} + c \right] + k_{t+1}(1 - \delta)$$

$$V_{t+2} = Z \left[(Z\alpha)^{\frac{-1}{\alpha-1}} + c \right] - B_{t+1}R_{t+1} - \left[(Z\alpha)^{\frac{-1}{\alpha-1}} + c \right] + k_{t+1}(1 - \delta)$$

Writing the problem in $t+1$:

$$V_{t+1} = \max_{e_{t+1}, k_{t+1}} d_{t+1} + \beta V_{t+2}$$

$$d_{t+1} = Zk_{t+1}^\alpha - B_t R_L - k_{t+1} + B_{t+1} + k_t(1 - \delta)$$

FOCs:

$$\begin{cases} \frac{\partial V_{t+1}}{\partial e_{t+1}} = \frac{\partial d_{t+1}}{\partial e_{t+1}} + \beta \frac{\partial V_{t+2}}{\partial e_{t+1}} = 0 \\ \frac{\partial V_{t+1}}{\partial k_{t+1}} = \frac{\partial d_{t+1}}{\partial k_{t+1}} + \beta \frac{\partial V_{t+2}}{\partial k_{t+1}} = 0 \end{cases} \quad (1)$$

solving $\frac{\partial d_{t+1}}{\partial e_{t+1}}$:

$$\frac{\partial d_{t+1}}{\partial e_{t+1}} = Z\alpha k_{t+1}^{\alpha-1} \frac{\partial k_{t+1}}{\partial e_{t+1}} - \frac{\partial k_{t+1}}{\partial e_{t+1}} + \frac{\partial B_{t+1}}{\partial e_{t+1}}$$

$$\frac{\partial B_{t+1}}{\partial e_{t+1}} = \frac{\partial k_{t+1}}{\partial e_{t+1}} - 1$$

$$\frac{\partial d_{t+1}}{\partial e_{t+1}} = Z\alpha k_{t+1}^{\alpha-1} \frac{\partial k_{t+1}}{\partial e_{t+1}} - \frac{\partial k_{t+1}}{\partial e_{t+1}} + \frac{\partial k_{t+1}}{\partial e_{t+1}} - 1$$

solving $\frac{\partial V_{t+2}}{\partial e_{t+1}}$:

$$\frac{\partial V_{t+2}}{\partial e_{t+1}} = - \left[\frac{\partial B_{t+1}R_{t+1}}{\partial e_{t+1}} - \frac{\partial k_{t+1}}{\partial e_{t+1}} (1 - \delta) \right]$$

$$\frac{\partial B_{t+1}R_{t+1}}{\partial e_{t+1}} = \frac{\partial B_{t+1}}{\partial e_{t+1}} R_{t+1} + B_{t+1} \frac{\partial R_{t+1}}{\partial e_{t+1}}$$

$$\frac{\partial B_{t+1}}{\partial e_{t+1}} = \frac{\partial k_{t+1}}{\partial e_{t+1}} - 1$$

$$\frac{\partial R_{t+1}}{\partial e_{t+1}} = -\frac{1-p}{p} \left\{ \left[Z\alpha k_{t+1}^{\alpha-1} \frac{\partial k_{t+1}}{\partial e_{t+1}} - \delta \frac{\partial k_{t+1}}{\partial e_{t+1}} \right] B_{t+1} - \left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1 \right) [Zk_{t+1}^\alpha - \delta k_{t+1}] \right\} B_{t+1}^{-2}$$

$$\frac{\partial B_{t+1}R_{t+1}}{\partial e_{t+1}} = \left[\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1 \right] R_{t+1} + \frac{1-p}{p} \left\{ \left[Z\alpha k_{t+1}^{\alpha-1} \frac{\partial k_{t+1}}{\partial e_{t+1}} - \delta \frac{\partial k_{t+1}}{\partial e_{t+1}} \right] B_{t+1} - \left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1 \right) [Zk_{t+1}^\alpha - \delta k_{t+1}] \right\} B_{t+1}^{-1}$$

$$\frac{\partial B_{t+1}R_{t+1}}{\partial e_{t+1}} = \left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1 \right) \left[(Zk_{t+1}^\alpha - \delta k_{t+1}) \frac{1-p}{p} B_{t+1}^{-1} + R_{t+1} \right] - \frac{1-p}{p} \frac{\partial k_{t+1}}{\partial e_{t+1}} (Z\alpha k_{t+1}^{\alpha-1} - \delta)$$

$$\frac{\partial B_{t+1} R_{t+1}}{\partial e_{t+1}} = \left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1 \right) R_f - \frac{1-p}{p} \frac{\partial k_{t+1}}{\partial e_{t+1}} (Z\alpha k_{t+1}^{\alpha-1} - \delta)$$

$$\frac{\partial V_{t+2}}{\partial e_{t+1}} = - \left[\left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1 \right) R_f - \frac{1-p}{p} \frac{\partial k_{t+1}}{\partial e_{t+1}} (Z\alpha k_{t+1}^{\alpha-1} - \delta) - \frac{\partial k_{t+1}}{\partial e_{t+1}} (1 - \delta) \right]$$

Substituting into the first FOC, we get:

$$\frac{\partial V_{t+1}}{\partial e_{t+1}} = Z\alpha k_{t+1}^{\alpha-1} \frac{\partial k_{t+1}}{\partial e_{t+1}} - \frac{\partial k_{t+1}}{\partial e_{t+1}} + \frac{\partial k_{t+1}}{\partial e_{t+1}} - 1 - \beta \left[\left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1 \right) R_f - \frac{1-p}{p} \frac{\partial k_{t+1}}{\partial e_{t+1}} (Z\alpha k_{t+1}^{\alpha-1} - \delta) - \frac{\partial k_{t+1}}{\partial e_{t+1}} (1 - \delta) \right] = 0$$

second FOC:

solving $\frac{\partial d_{t+1}}{\partial k_{t+1}}$:

$$\frac{\partial d_{t+1}}{\partial k_{t+1}} = Z\alpha k_{t+1}^{\alpha-1} - 1 + \frac{\partial B_{t+1}}{\partial k_{t+1}}$$

$$\frac{\partial B_{t+1}}{\partial k_{t+1}} = 1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}$$

$$\frac{\partial d_{t+1}}{\partial k_{t+1}} = Z\alpha k_{t+1}^{\alpha-1} - \frac{\partial e_{t+1}}{\partial k_{t+1}}$$

solving $\frac{\partial V_{t+2}}{\partial k_{t+1}}$:

$$\frac{\partial V_{t+2}}{\partial k_{t+1}} = - \left[\frac{\partial B_{t+1} R_{t+1}}{\partial k_{t+1}} - (1 - \delta) \right]$$

$$\frac{\partial B_{t+1} R_{t+1}}{\partial k_{t+1}} = \frac{\partial B_{t+1}}{\partial k_{t+1}} R_{t+1} + B_{t+1} \frac{\partial R_{t+1}}{\partial k_{t+1}}$$

$$\frac{\partial B_{t+1}}{\partial k_{t+1}} = 1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}$$

$$\frac{\partial R_{t+1}}{\partial k_{t+1}} = - \frac{1-p}{p} \left[(Z\alpha k_{t+1}^{\alpha-1} - \delta) B_{t+1} - \left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}} \right) (Zk_{t+1}^{\alpha} - \delta k_{t+1}) \right] B_{t+1}^{-2}$$

$$\frac{\partial B_{t+1} R_{t+1}}{\partial k_{t+1}} = \left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}} \right) R_{t+1} + \left\{ \frac{1-p}{p} \left[(Z\alpha k_{t+1}^{\alpha-1} - \delta) B_{t+1} - \left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}} \right) (Zk_{t+1}^{\alpha} - \delta k_{t+1}) \right] B_{t+1}^{-1} \right\}$$

$$\frac{\partial B_{t+1} R_{t+1}}{\partial k_{t+1}} = \left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}} \right) \left[R_{t+1} + \frac{1-p}{p} (Zk_{t+1}^{\alpha} - \delta k_{t+1}) B_{t+1}^{-1} \right] - \frac{1-p}{p} (Z\alpha k_{t+1}^{\alpha-1} - \delta)$$

$$\frac{\partial B_{t+1} R_{t+1}}{\partial k_{t+1}} = \left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}} \right) R_f - \frac{1-p}{p} (Z\alpha k_{t+1}^{\alpha-1} - \delta)$$

$$\frac{\partial V_{t+2}}{\partial k_{t+1}} = - \left[\left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}} \right) R_f - \frac{1-p}{p} (Z\alpha k_{t+1}^{\alpha-1} - \delta) - (1 - \delta) \right]$$

Substituting into the FOC:

$$\frac{\partial V_{t+1}}{\partial k_{t+1}} = Z\alpha k_{t+1}^{\alpha-1} - \frac{\partial e_{t+1}}{\partial k_{t+1}} - \beta \left[\left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}} \right) R_f - \frac{1-p}{p} (Z\alpha k_{t+1}^{\alpha-1} - \delta) - (1 - \delta) \right] = 0$$

thus the FOCs are:

$$\frac{\partial V_{t+1}}{\partial e_{t+1}} = Z\alpha k_{t+1}^{\alpha-1} \frac{\partial k_{t+1}}{\partial e_{t+1}} - 1 - \beta \left[\left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1 \right) R_f - \frac{1-p}{p} \frac{\partial k_{t+1}}{\partial e_{t+1}} (Z\alpha k_{t+1}^{\alpha-1} - \delta) - \frac{\partial k_{t+1}}{\partial e_{t+1}} (1 - \delta) \right] = 0$$

$$\frac{\partial V_{t+1}}{\partial k_{t+1}} = Z\alpha k_{t+1}^{\alpha-1} - \frac{\partial e_{t+1}}{\partial k_{t+1}} - \beta \left[\left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}} \right) R_f - \frac{1-p}{p} (Z\alpha k_{t+1}^{\alpha-1} - \delta) - (1 - \delta) \right] = 0$$

rearranging $\frac{\partial V_{t+1}}{\partial k_{t+1}}$ to isolate k_{t+1} :

$$k_{t+1}^{\alpha-1} = \left[\frac{\partial e_{t+1}}{\partial k_{t+1}} (1 - \beta R_f) + \beta \left(r_f + \frac{\delta}{p} \right) \right] \left\{ Z\alpha \left[(1 - \beta) - \frac{\beta}{p} \right] \right\}^{-1}$$

rearranging $\frac{\partial V_{t+1}}{\partial e_{t+1}}$ to isolate k_{t+1} :

$$k_{t+1}^{\alpha-1} = \left[\frac{\partial k_{t+1}}{\partial e_{t+1}} (1 - \beta R_f) + \beta (r_f + \delta) + \delta \frac{1-p}{p} \right] \frac{p}{Z\alpha}$$

Equating the two equations:

$$\left[\frac{\partial e_{t+1}}{\partial k_{t+1}} (1 - \beta R_f) + \beta \left(r_f + \frac{\delta}{p} \right) \right] \left\{ Z\alpha \left[(1 - \beta) - \frac{\beta}{p} \right] \right\}^{-1} = \left[\frac{\partial k_{t+1}}{\partial e_{t+1}} (1 - \beta R_f) + \beta (r_f + \delta) + \delta \frac{1-p}{p} \right] \frac{p}{Z\alpha}$$

From this equation, you can isolate $\frac{\partial e_{t+1}}{\partial k_{t+1}}$ to solve for it explicitly.

$$\frac{\partial e_{t+1}}{\partial k_{t+1}} = - \left[\frac{\partial k_{t+1}}{\partial e_{t+1}} (1 - \beta R_f) + \beta (r_f + \delta) + \delta \frac{1-p}{p} \right] \frac{p}{Z\alpha} \left\{ Z\alpha \left[(1 - \beta) - \frac{\beta}{p} \right] \right\} (1 - \beta R_f)^{-1} - \beta \left(r_f + \frac{\delta}{p} \right) (1 - \beta R_f)^{-1}$$

$$\frac{\partial e_{t+1}}{\partial k_{t+1}} = - \left[\frac{\partial k_{t+1}}{\partial e_{t+1}} (1 - \beta R_f) + \delta \frac{1-p}{p} \right] \frac{\beta p + \beta - p}{1 - \beta R_f}$$

Knowing that $B = k + c - e \rightarrow e = B - k - c$ thus $\frac{\partial e_{t+1}}{\partial k_{t+1}} = -1$:

$$\frac{1 - \beta R_f}{\beta p + \beta - p} = \left[\frac{\partial k_{t+1}}{\partial e_{t+1}} (1 - \beta R_f) + \delta \frac{1-p}{p} \right]$$

We can rearrange it to isolate $\frac{\partial k_{t+1}}{\partial e_{t+1}}$:

$$\frac{1 - \beta R_f}{\beta p + \beta - p} - \delta \frac{1-p}{p} = \frac{\partial k_{t+1}}{\partial e_{t+1}} (1 - \beta R_f)$$

Now, solve for $\frac{\partial k_{t+1}}{\partial e_{t+1}}$ by dividing both sides by $(1 - \beta R_f)$:

$$\frac{\partial k_{t+1}}{\partial e_{t+1}} = \left\{ -\delta \frac{1-p}{p} + \frac{1 - \beta R_f}{\beta p + \beta - p} \right\} (1 - \beta R_f)^{-1}$$

This equation provides the value of $\frac{\partial k_{t+1}}{\partial e_{t+1}}$ in terms of the given constants (β, R_f, p, δ) and the relationship between e_{t+1} and k_{t+1} .

$$k_{t+1}^{\alpha-1} = \left\{ -\delta \frac{1-p}{p} + \frac{1 - \beta R_f}{\beta p + \beta - p} \right\} (1 - \beta R_f)^{-1} (1 - \beta R_f) + \beta \left(r_f + \frac{\delta}{p} \right) \left\{ Z\alpha \left[(1 - \beta) - \frac{\beta}{p} \right] \right\}^{-1}$$

$$k_{t+1}^{\alpha-1} = \left\{ -\delta \frac{1-p}{p} + \frac{1 - \beta R_f}{\beta p + \beta - p} \right\} + \beta \left(r_f + \frac{\delta}{p} \right) \left\{ Z\alpha \left[(1 - \beta) - \frac{\beta}{p} \right] \right\}^{-1}$$