Finding the optimal path

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1 Frictionless economy

Given that the evolution of the net worth is the following:

$$e_{t+1} = Z(\theta + \epsilon)k_t^\alpha + (1-\delta)k_t - (1+r)(c + k_t - e_t)$$

The value of the firm at time t is given by:

$$V_t(e_t) = \max_{k_t} e_t + \beta V_{t+1}(e_{t+1})$$

Solving the belman equation, the FOC are given by:

$$\frac{\vartheta V_t(e_t)}{\vartheta k_t} = \frac{\vartheta e_t}{\vartheta k_t} + \beta \frac{\vartheta V_{t+1}(e_{t+1})}{\vartheta k_t}$$

Strong assumption a change in k_t does not have an impact on e_t only on e_{t+1}:

$$\frac{\vartheta e_t}{\vartheta k_t} = 0$$

Using the envelope theorem:

$$\frac{\vartheta V_{t+1}(e_{t+1})}{\vartheta k_t} = \frac{\vartheta V_{t+1}(e_{t+1})}{\vartheta e_{t+1}} \frac{\vartheta e_{t+1}}{\vartheta k_t}$$

$$\frac{\vartheta V_{t+1}(e_{t+1})}{\vartheta e_{t+1}} = 1 + \beta(1+r)$$

$$\frac{\vartheta e_{t+1}}{\vartheta k_t} = Z(\theta + \epsilon)\alpha k_t^{\alpha-1} - (\delta + r)$$

Thus:

$$\frac{\vartheta V_{t+1}(e_{t+1})}{\vartheta k_t} = [1 + \beta(1+r)][Z(\theta + \epsilon)\alpha k_t^{\alpha-1} - (\delta + r)]$$

So the FOC became:

$$\frac{\vartheta V_t(e_t)}{\vartheta k_t} = 0 + \beta[1 + \beta(1+r)][Z(\theta + \epsilon)\alpha k_t^{\alpha-1} - (\delta + r)] = 0$$

The optimal level of capital at time t is:

$$\widehat{k}_t = \frac{\beta(1+r)Z(\theta+\epsilon)\alpha}{\delta + r}^{\frac{1}{1-\alpha}}$$

Plotting the graph in (assuming $X = Z(\theta + \epsilon)$) and $y = \widehat{k}_t$

$$y = \frac{0.8x}{0.03 + 0.1}^{\frac{1}{1-0.8}}$$

Its interesting to see that if there is an increase in productivity the firm need more optimal capital K, while the most low productivity firms need less capital to operate

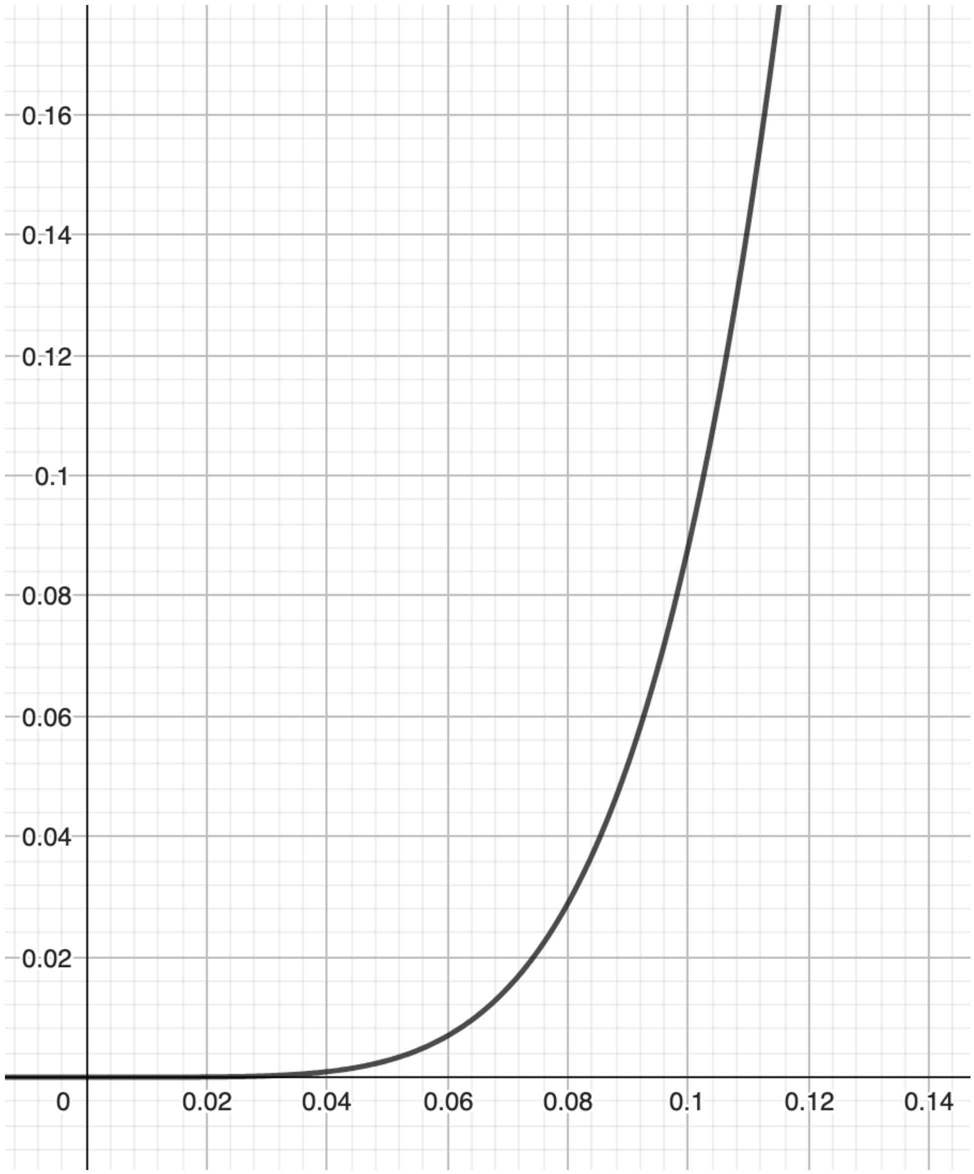

Figure 1: $X = Z(\theta + \epsilon)$ and $y = \widehat{k}_t$

1.1 Frictions

Now I consider the participation constraint of the borrower given that she could observe ϵ only paying μk^α so:

$$(1+r)(k+c+e)(1-\Phi(\bar{\epsilon})) + \int_{-\infty}^{\bar{\epsilon}} [Z(\theta+\bar{\epsilon})k^\alpha + (1-\delta)k - \mu k^\alpha]\, d\Phi(\epsilon) \geq (1+r)(c+k+e)$$

r is the rate of interest that makes equal the expected value of borrowing to the opportunity cost of capital. rewriting became

$$Z[\theta + G(\bar{\epsilon})]k_t^\alpha + (1-\delta)k_t - uk_t^\alpha \Phi(\bar{\epsilon}) = (1+r)(k_t + c - e_t)$$

where

$$G(\bar{\epsilon}) = (1 - \Phi(\varepsilon)\bar{\varepsilon} + \int_{-\infty}^{\bar{\varepsilon}} \epsilon\, d\Phi(\epsilon))$$

While the firm participation constraint is $q_t \geq 0$ so the end-of-period net worth must be greater than 0, thus the problem of the firm becomes: The value of the firm at time t is given by:

$$V_t(e_t) = \max_{k_t} q_t - e_{t+1} + \beta V_{t+1}(e_{t+1})$$

$$s.t.$$

$$q_t = Z(\theta+\epsilon)k_t^\alpha + (1-\delta)k_t - (1+r)(c + k_t - e_t)$$

$$Z[\theta + G(\bar{\epsilon})]k_t^\alpha + (1-\delta)k_t - uk_t^\alpha \Phi(\bar{\epsilon}) = (1+r)(k_t + c - e_t)$$

so we can rewrite the second constraint in order to get r:

$$r = \frac{Z[\theta + G(\bar{\epsilon})]k_t^\alpha + (1-\delta)k_t - uk_t^\alpha \Phi(\bar{\epsilon})}{k_t + c - e_t} - 1$$

FOC:

$$\frac{\vartheta V_t(e_t)}{\vartheta k_t} = \frac{\vartheta q_t}{\vartheta k_t} - \frac{\vartheta e_t}{\vartheta k_t} + \beta \frac{\vartheta V_{t+1}(e_{t+1})}{\vartheta k_t} = 0$$

by envelope theorem:

$$\frac{\vartheta V_{t+1}(e_{t+1})}{\vartheta k_t} = \frac{\vartheta V_{t+1}(e_{t+1})}{\vartheta e_{t+1}} \frac{\vartheta e_{t+1}}{\vartheta k_t}$$

Strong HP:

$$\frac{\vartheta e_{t+1}}{\vartheta k_t} = 0$$

Qui non so se ha senso continuare perchè non ho nessun meccanismo che mi trasformi il net worth al tempo t in net worth al periodo t+1, almeno che non includa la definzione di dividendo come $d_t = q_t - e_{t+1}$, allora in questo caso dovrei risolvere il problema con un langrangiana dato che la firm dovrebbe scegliere sia k che e net worth.