Journal of Mathematical Economics 103 (2022) 102774

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

A growth model with endogenous technological revolutions and cycles

Taoxiong Liu[1], Zhuohao Liu*

Institute of Economics, School of Social Sciences, Tsinghua University, Beijing 100084, China

ARTICLE INFO

Article history:
Received 24 April 2022
Received in revised form 3 September 2022
Accepted 7 October 2022
Available online 23 October 2022

Keywords:
Economic growth
Technological revolution
Business cycle
Discontinuous control

ABSTRACT

We propose a new approach to deal with continuous time dynamic model with discontinuity, by which we incorporate gradual innovations and technological leaps, as well as endogenous cycles and long-run growth into one framework. We show that the optimal or equilibrium growth path is a cyclic growth path (CGP) and we develop the mathematical technique to prove the existence and uniqueness of the solution. We provide two basic setups to include the technological leaps in the R&D sector or goods production sector respectively. The growth paths under these two setups show different cyclic features and abundant economic dynamics. Our approach can be easily utilized to evaluate the optimality of the timing of the technological revolution. In the decentralized economy, the technological breakthrough is exogenous to families and always happens too late. The social planner's problem clarifies two fundamental effects on the optimal growth path, i.e., the "dividend effect" and the "swot-up effect", induced by discontinuous technological change. Our framework can also serve as a workhorse to address dynamic economic problems with discontinuity.

1. Introduction

In macroeconomics, technological progress has been regarded as a fundamental driver of long-term economic growth (Solow, 1956). We agree with Schumpeter (1927) that the occurrence and diffusion of innovations should be viewed as a very uneven process that is sometimes explosive and sometimes gradual. Schumpeter (1974) further argued that the economic growth and business cycle of capitalists should be explained by

> industrial mutation—if I may use the biological term—that incessantly revolutionizes the economic structure from within, incessantly destroying the old one, incessantly creating a new one. This process of Creative Destruction is the essential fact about capitalism. (Page 83).

There are two central concepts here for Schumpeter to explain the evolution of the economic system: *mutation* and *creative destruction*. First, the economy is propelled by the process of industrial progress with mutations. Second, this defined a non-mechanistic and historical view of capitalism as one of creation and destruction. Today, the Schumpeterian idea of *creative destruction* has been well incorporated into modern growth models, such as in Aghion and Howitt (1994), Francois and Roberts (2003), Martimort and Verdier (2004), and Acemoglu and Cao (2015), but growth model with the *mutation* of technology is far from full-fledged. Since Romer (1986) proposed a model based on knowledge spillovers to bring about endogenous economic growth, endogenous growth theory has dominated the growth literature. However, regardless of whether these models are based on the ideas of input variety expansion, product innovation, quality improvement, or Schumpeterian innovation (Acemoglu, 2012), technological progress is modeled as a continuous function of time either at the firm level or at the aggregation level, as in Romer (1987), Romer (1990), Aghion and Howitt (1992), Grossman and Helpman (1991), Aghion and Howitt (1994), Francois and Roberts (2003), and Martimort and Verdier (2004). On the other hand, in the literature on technological changes, they are often sorted into two different types, incremental progress and radical innovation, though different authors might use different concepts. The first refers to the gradual incremental improvements in products and processes, which are generally modeled as a continuous process. In contrast, the second refers to the drastic innovation, including the emergence of new GPTs, the changing of technological paradigms, etc., which should be modeled as a discontinuous process (see Rosenberg and Nathan (1982, 1994), Freeman et al. (1999), Sanchez-Choliz et al. (2008)). Helpman (1998) distinguishes between incremental and drastic innovations and treats the main form of the latter as general purpose technologies (GPTs), which was first coined by Bresnahan and Trajtenberg (1995) (also see Lipsey et al., 1998, 2005).

* Corresponding author.
E-mail addresses: liutx@tsinghua.edu.cn (T. Liu), liu-zh21@mails.tsinghua.edu.cn (Z. Liu).

[1] We are grateful to the participants in the seminar on growth hold by the Institute of Economics, Tsinghua University, who provided insights and expertise that assisted the research much. All remaining errors are, of course, our own.

https://doi.org/10.1016/j.jmateco.2022.102774

Mokyr (1990) classifies technological changes into micro- and macro-innovations. Freeman and Louça (2001) refer to them as gradual processes and technological revolutions. Following Freeman and Louça (2001), we call the second form technological revolution or breakthrough. As said in Kuznets and Murphy (1966), since the second half of the nineteenth century, the major source of economic growth in developed countries has been science-based technology in the electrical, internal combustion, electronic, nuclear, and biological fields, and others.[2]

Growth theory studies that seriously examine technological revolution are in general lacking in the literature. As an important attempt, Helpman and Trajtenberg (1994) built a model to explore the effects of new GPTs and GPTs diffusion, while treating GPTs advances as exogenous. However, the critical point lies exactly in how and when the technological revolutions occur and what determines them, which can only be explained by endogenizing technological revolutions in the model where the occurrence and timing of technological revolutions are determined by the decisions of agents.[3] We conjecture that one reason for the absence of a relevant theoretical model is the lack of a satisfactory tool to deal with that, mathematically. The existing dynamic framework in macroeconomics, especially for growth models, is suitable for incremental technological progress, where usually a balanced growth path (BGP) is obtained as the solution path. However, if we want to introduce saltatorial technological breakthroughs into the continuous time framework, we need to allow path jumps, which is not permitted in the existing main dynamic framework in economics.[4]

The purpose of this article is to address these gaps and provide a concise framework to tolerate discontinuous growth path so as to endogenize both the gradual progress and radical revolution of technology in one growth model. Although there are some related techniques in cybernetics, they have not been well included in economic models. Hull (2013) introduced the optimal control theory with deterministic discontinuities in detail and gave the corresponding Erdmann–Weierstrass condition. Moreover, for the need for a growth model, it is necessary to allow the technology to break through successively and thereafter the growth path to jump infinitely many times. This is the most difficult part of our task, which has no ready-made answers in the cybernetics literature, and is much more complicated than finitely multiple jumps, as in Liu and Zhou (2015) and Boucekkine et al. (2016, 2013). Under this circumstance, the solution path inevitably is not a BGP. A natural question thus arises: Is there an alternative solution path that is reasonable and if so when does it exist? Fortunately, in our framework, we find that another type of equilibrium growth path (or the optimal growth path in the social planner problem), which fluctuates cyclically, does exist, and we successfully clarify the conditions for its existence and uniqueness.

Intuitively, since the technological revolution shows an inherent property of unevenness or discontinuity, it naturally induces the growth path to fluctuate. Schumpeter (1927) believed that long cycles such as Kondratiev cycles are generated by technological revolutions (Ayres, 1990). Nevertheless, there have been some attempts to connect endogenous growth with cycles by introducing innovation cycles. For most of the studies, such as in Canton (2002), Martimort and Verdier (2004), Pavlov and Weder (2017), and Wälde (2005), also including the one-sector model in the seminal paper Aghion and Howitt (1992), the cycles are aroused by stochastic bursts of innovations, and there is no ex-ante path discontinuity in the meaning of expectations. Thus, since the timing of cycles and the cyclic length are stochastic, there are no stationary waves and little room to explore the cyclic behavior or the transitional dynamics before the next burst of innovation. In addition, the technological revolution is the result of the long-term accumulation of knowledge and technology, especially research in the basic scientific field, so it is predictable to a certain extent. For example, with the evolution of mobile communication technologies, before the application of 4G or 5G technology, even before their inventions in laboratories, people had discussed the technologies for a long time. Moreover, it is also practically important to model innovation cycles or discontinuous innovation under deterministic circumstances because under deterministic circumstances we can obtain the stationary timing of the cycles and explore the transitional dynamics transparently.[5]

Therefore, our approach incorporates technological leaps, endogenous technological cycles, and long-run growth into one framework. We classify technological progress into two basic scenarios: incremental progress and technological breakthroughs or technological revolutions. We use the discontinuous optimal control technique in continuous time to establish and solve the model. Our baseline model is founded with the classical approach of expanding varieties. Different from the classical growth model, our approach unifies long-term growth and economic fluctuation into one framework, which is driven by leapfrog technological revolution, together with continuous technological progress. We introduce the concept of the cyclic growth path (CGP) for the solution. On the CGP, the economy shows stable growth in the meaning of a time average in long term. Furthermore, the growth rates of major economic variables fluctuate periodically. The model shows abundant interesting economic dynamics and indicates the underlying mechanism. For example, in the equilibrium solution of the baseline model, the output is continuous, the consumption ratio jumps downward and the investment ratio jumps upward while crossing the border of technological dynasties. In addition, within a cycle, the growth rate, the saving rate, and the consumption rate change monotonically while the consumption-assets ratio does not. An even more fundamental result is that the social planner prefers a different path and different technological revolution timing compared to the market equilibrium because the households in the market treat the timing of technological revolution as given while the social planner endogenizes it. The solution of the social planner's problem clarifies two fundamental effects induced by the technological revolution, the "dividend effect" after the revolution and the "swot-up effect" before that. Thus, correct policies can be designed to improve social welfare by balancing these two effects better. We provide two basic setups (one of which is placed in the appendix), which include technological revolutions in the R&D sector for the first setup and technological revolutions in the goods production sector for the second. The growth paths in these two setups show different cyclic features, such as that the output is continuous and the consumption is discontinuous in the first setup while it is the opposite in the second.

2 As mentioned in Sanchez-Choliz et al. (2008), some of the examples are discussed in Freeman et al. (1999) (such as the railroad system and the new information and communication systems).

3 Helpman and Trajtenberg (1994) also attribute the prominent disadvantage of their model to ignoring the endogenous character of advances in the GPT and the associated (positive) feedback going from the user sectors to the GPT.

4 To incorporate discontinuous innovation, Matsuyama (1999), Andolfatto and Macdonald (1998), Francois and Shi (1999), and Umezuki and Yokoo (2019) adopt discrete time models to address innovation cycles, where the main disadvantage is the difficulty for illustrating the transitional behavior and describing the timing of discontinuous events.

5 Freeman et al. (1999) also distinguished technological leaps from gradual progress. They excellently present the importance of understanding the economic cycles generated by discontinuous technological progress and provide many insights into it. However, their model has an uncommon setup, especially in the decentralized economy, where the economy must start at some given level of capital stock to obtain a regular cyclical path.

This paper contributes to the existing literature from three perspectives. First of all, we propose a general approach to deal with continuous time dynamic model with discontinuity, by which we incorporate gradual innovations and technological breakthroughs, as well as endogenous cycles and long-run growth into one framework. To the best of our knowledge, this is the first attempt to endogenize these two basic forms of technological progress in one model and show how their interaction induces luxuriant types of economic dynamics. By the advantage of our approach, it is very convenient to compare the social planner problem and the market equilibrium, especially to evaluate the optimality of the timing of technological revolution under these two circumstances, which is obviously of essential importance given the profoundness and extensiveness of the technological revolution in reality. We also show that the drastic innovations occur too late in the market equilibrium compared to the social planner problem because the social planner favors a different path of consumption and investment around a drastic innovation. We illustrate how the proper policy can restore social optimality and how it needs to carefully address the economic behavior around the time point of technological breakthrough.

Second, we suggest a tool to incorporate both endogenous cycles and long-run growth into one framework. Instead of the concept of the balanced growth path (BGP) for the modern growth model, we propose the concept of the CGP as the solution path in which both long-term growth and business cycles are rooted. The most technical part of our work is to prove the existence and uniqueness of the solution path, as a CGP. We provide related mathematical techniques to address this type of model and successfully conquer the main difficulties by the power of the Poincare map. Different from Freeman et al. (1999), where to obtain a cyclical path the economy must start with a given level of capital stock, our model has a cyclical solution with any reasonable starting initial condition.

Third, our framework can also be used as a basic tool to address dynamic economic problems with discontinuity. Our framework tolerates various forms of technological fluctuations and can fit different settings of production functions or economic environments. The framework is flexible enough to address some other dynamic macroeconomic problems with abrupt changes, including discontinuous policy design, such as the birth into being of new industrial sectors, the saltatory change after some threshold such as new infrastructure opening up based on long-term investment, etc. From the technical perspective, our approach is most closely related to the literature treating optimization problems with regime transitions. Boucekkine et al. (2013) present the general FOCs regarding the regime changes under economic circumstances. Liu and Zhou (2015), Boucekkine et al. (2016), and Haunschmied et al. (2021) propose some applications of discontinuous control techniques on economic models. Our approach extends the techniques treating regime transitions in these papers to incorporate infinite times of regime transitions and accommodate decentralized equilibrium. As far as we know, there are several papers involving discontinuous growth paths, such as Freeman et al. (1999) and Francois and Lloyd-Ellis (2003), etc. We believe that the present approach has the potential to be applied to these models, especially to explore the behavior around the discontinuous point on the path. Our approach could also benefit the literature on political economy, since political regime changes are such an important topic there (Acemoglu and Robinson, 2001; Boucekkine et al., 2016; Haunschmied et al., 2021).

The remainder of this article is arranged as follows. Section 2 preliminarily explores the representative household problem under a discontinuous environment. Section 3 present the baseline model. Section 4 establishes the concept of the CGP, obtains the equilibrium solution path, the CGP, and probes the economic dynamics on the CGP. Section 5 contributes by solving the social planner problem and focuses on the model's policy implications. Then, we conclude the main findings in the final section. Most of the mathematical proofs and a summary of the method for the discontinuous optimal control problem are offered in Appendix A and Appendix C. Appendix B provides an extension of the model. Appendix C shows the properties of the CGP by some numerical examples.

2. Representative household problem with discontinuity

In this section, we preliminarily explore the problem of the representative household in a dynamic economic system with discontinuity, for convenience of subsequent modeling, and also present some intuitions. Assume that the economy is in continuous time with an infinite horizon and admits a representative household with preference

$$U(C(t)) = \int_0^\infty e^{-\rho t} u(C(t))dt$$

where the instantaneous utility function $u(\cdot)$ is assumed to be $\log(\cdot)$ and the discount rate is ρ. The household holds a certain amount of assets $K(t)$. Whenever no confusion is involved, the index t is omitted. The household inelastically supplies labor, and its population is L. $r(t)$ and $w(t)$ are the interest rate and wage, respectively. The budget constraint of the representative household is

$$\dot{K}(t) = F^n(K(t), L) - C(t), \;\; t_n \le t < t_{n+1}, \tag{1}$$

where $n = 0, 1, \ldots$. Here, we allow the household revenue to be a function of K and L to include the case of the social planner problem.[6] For the case of decentralized economy, we usually have[7]

$$F^n(K, L) = r(t)K(t) + w(t)L. \tag{2}$$

A critical difference from the classical continuous time model is that here $F^n(\cdot)$ is assumed to be continuous only in the interval $[t_n, t_{n+1})$, might jump at time t_n and have different forms in different intervals. For example, when Eq. (2) holds, $r(t)$ and $w(t)$ are not necessarily continuous everywhere but are right continuous over time and can jump at time t_n. Therefore, at the time of corners, i.e., t_ns, Eq. (1) holds with $\dot{K}$ understood as the left limit of $dK(t)/dt$. Let $F(K(t), L, t) = F^n(K(t), L)$ for $t_n \le t < t_{n+1}$. Then,

$$\dot{K}(t) = F(K(t), L, t) - C(t). \tag{3}$$

In the light of Eq. (3), we know that $K(t)$ is still continuous, but its first derivative is not necessarily continuous because of the discontinuity of $F(\cdot)$. The corner time t_ns are endogenous and determined by the following restrictions,

$$G^n(K(t_n), t_n) = 0, \tag{4}$$

where $G^n(\cdot)$ is continuously differentiable regarding both variables. Usually, $G^n(\cdot)$ represents the condition for some event to occur at t_n or some constraint imposed on t_n. The no-Ponzi constraint and the initial condition are normal as

$$\lim_{t\to\infty} K(t)\exp\{-\int_0^t r(s)ds\} \ge 0, \tag{5}$$

[6] See Section 4.

[7] Here for simplicity, we could assume the depreciation rate $\delta = 0$. It is easy to include the case with $\delta \ne 0$. For some production function, $F_0(K, L)$, we have $\dot{K} = F_0(K, L) - \delta K - C$ and $F_0(K, L) = (r + \delta)K + wL$. By letting $F(K, L) = F_0(K, L) - \delta K$, we have $\dot{K} = F(K, L) - C$ and $F(K, L) = rK + wL$, the same form as the following equation.

$$K(0) > 0, \tag{6}$$

Then, the representative family solves the utility maximization problem (UMP)

$$\max_{c(t)} \int_0^\infty e^{-\rho t} u(C(t))dt, \quad s.t. (3)\text{–}(6). \tag{7}$$

The difficulty in solving this problem comes from the discontinuity of $F(\cdot)$. The representative household faces an optimal control problem with discontinuity (or with corners) in which the control variable is also not necessarily continuous along the solution path, although the state variable still is. The basic technique to tackle this type of problem is not very complicated, but the model here is much more involved because there might be a countably infinite number of corners along the optimal path.

A key observation is that the household has to determine not only the resource allocation in each interval $[t_n, t_{n+1})$ but also that across these intervals, including determining the distribution of the corner time, t_ns. A concise summary of the solution conditions to this sort of control problem is given in Appendix A. In addition to the solution conditions for the continuous optimal control model, we have to consider the impact of the control variable on the corner time. Write down the Hamiltonian of problem (7) as

$$\mathcal{H} = e^{-\rho t} u(C) + \lambda(F(K, L, t) - C). \tag{8}$$

The solution path is piecewise continuous; and within each continuous interval, we have the FOCs as

$$e^{-\rho t} u'(C) - \lambda = 0, \tag{9}$$

$$\lambda F_K^n(K, L) = -\dot{\lambda}. \tag{10}$$

From (9) and (10),

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\varepsilon_u(C(t))}(F_K^n(K(t), L) - \rho), \tag{11}$$

where $\varepsilon_u(C(t)) \equiv -\frac{u''(C(t))C(t)}{u'(C(t))}$ is the elasticity of the marginal utility $u'(C(t))$. The transversality condition (TVC) is

$$\lim_{t\to\infty} \lambda(t)K(t) = 0. \tag{12}$$

The additional corner conditions at each t_n are

$$\mathcal{H}^+(\cdot, t_n) = \mathcal{H}^-(\cdot, t_n) - \gamma_n G_{t_n}^n(K, t_n), \tag{13}$$

$$\lambda^+(t_n) = \lambda^-(t_n) + \gamma_n G_K^n(K, t_n), \tag{14}$$

where $\gamma_n > 0$ is the price factor for the corner condition, $G_{t_n}^n(\cdot)$ and $G_K^n(\cdot)$ are the partial derivatives w.r.t. t_n and K, respectively, and we use "$+$" or "$-$" at the top-right corner of some variable to denote its right or left limit at t_n. These two equations reflect the impacts of the corner restrictions. Since the Hamiltonian H on the optimal path measures the instantaneous utility (or the utility per time unit), (13) indicates that the difference in the instantaneous utility between the two proximate sides of time t_n is exactly the cost of moving it, which is given by multiplying the derivative of the corner constraint w.r.t the corner time by its price factor. Therefore, on the optimal path, there is no difference in shifting the corner slightly forward or backward. Similarly, (14) shows that the difference in the price of the state variable between the two proximate sides of the corner is exactly the cost of moving the corner by changing the state variable.

The corner conditions are the key to understanding the behavior of the variables at the corners. For example, if the corners are exogenous regarding K, which is the situation that we will encounter in the decentralized economy model, $G_K^n(\cdot) = 0$ and

$$\lambda^+ = \lambda^-. \tag{15}$$

Otherwise, if the corners are only related to K and not t_n, while K is given, which is the situation that we will encounter in the social planner problem, $G_{t_n}^n(\cdot) = 0$ and

$$\mathcal{H}^+ = \mathcal{H}^-. \tag{16}$$

In the successive models in this paper, corner conditions (13) and (14) or (15) and (16) will generate rich economic behavior at the corners, and then cyclical fluctuations for the output, consumption, investment, and other variables.

3. Description of the baseline model

For the baseline model, we adopt the framework of expanding varieties and focus on how the discontinuity and non-linearity of technological progress generate new insights into the economy.

3.1. The research and technological revolution

Let us turn to describe the technological improvement. Assume the level of technology or the knowledge stock at time t is $A(t)$. We classify technological progress into two basic types of scenarios: incremental progress and technological breakthroughs or technological revolutions. As we mentioned before, a technological revolution refers to an intensive improvement, especially a discontinuous, abrupt leapfrog of technology, usually driven by the change of the basic technological paradigm or the emergence of new GPTs. Incremental technological progress is gradual, usually achieved within the framework of the existing technological paradigm or based on the existing GPTs, and driven by regular R&D investments mainly performed by firms, such as progressive process innovation, quality improvements of the existing products, or the invention of nondisruptive new products. Incremental progress can be described as a continuous function of R&D investment, but the efficiency of R&D activities is subject to the existing technological paradigm. We understand the technological revolution as the successive rounds of the basic technological paradigm transition and describe them as something like a "ladder", such as from level A_0 to $A_1 \cdots$, similarly to how Aghion and Howitt (1998) described quality improvement as a "quality ladder". For simplicity, we call these rounds of technological paradigms or GPTs the dynasty of technology to distinguish them from the existing technological stock that is accumulated by incremental progress. A technological dynasty performs as a platform, providing the common technological paradigm and GPTs, offering technological opportunities to support the current R&D activities, and imposing limits on the scope of these activities (see Freeman et al., 1999; Growiec and Schumacher, 2013). Thus, R&D efficiency should show a negative relationship with technological stock before the next emergence of the technological revolution since innovation opportunities decrease as technology builds up. We assume that technology cannot go backward and its production is related to three elements, including the R&D inputs, the present technological dynasty, and the technological stock, concretely described as

$$\dot{A}(t) = \frac{\tilde{\eta}(A_{n(t)}, A(t))}{A(t)} Z(t), \tag{17}$$

with the initial technology

$$A(0) = \bar{A}_0, \tag{18}$$

where $Z(t)$ is the final good input to the R&D department, $A_{n(t)}$ represents the technological dynasty where the economy is located and $\tilde{\eta}(A_{n(t)}, A)/A$ determines the efficiency of R&D activities. The introduction of A as the denominator exhibits the force to weaken R&D efficiency, given $A_{n(t)}$, as the stock of existing technology increases. The function, $\tilde{\eta}(A_{n(t)}, A)$, captures the constraint imposed by the technological dynasty on the efficiency of R&D, where we allow it to be determined jointly by $A_{n(t)}$ and A.

We can treat the technological dynasty that the economy belongs to as an exogenous variable only depending on time t, as in Helpman and Trajtenberg (1994), but which is obviously unattractive here. Ideally, it should also be modeled as some function of R&D activities. In reality, the technological revolution is often driven by breakthroughs in fundamental research, most of which can be treated as public goods, are funded by governments or nonprofit organizations such as universities, and have fewer close relations than incremental progress with profit-driven R&D activities. In addition, the technological breakthrough undoubtedly does not come from nowhere, must be based on the existing knowledge, and usually occurs after a long time of technological accumulation by gradual progress. Therefore, we endogenize the technological revolution as the byproduct of and by the spillover effect from knowledge accumulation. Specifically, we assume that the technological dynasty at time t is

$$A_{n(t)} = \sum_{m\in\mathbb{Z}} \xi^m A_0 \mathbb{1}\{A(t) \in [\xi^m A_0, \xi^{m+1}A_0)\}, \tag{19}$$

where $\xi > 1$ and $A_0 > 0$. The above equation implies that the technology breaks through and the dynasty shifts whenever the existing knowledge stock reaches a certain level. Obviously, the technological dynasty is $\xi^n A_0$ whenever $A(t) \in [\xi^n A_0, \xi^{n+1}A_0)$, and therefore it is innocuous to define $A_n = \xi^n A_0$ and $\eta(A(t)) := \tilde{\eta}(A_{n(t)}, A(t))$ since $A_{n(t)}$ is a function of $A(t)$. Therefore,

$$\dot{A} = \eta(A)\frac{Z}{A} = a(\log A)Z, \tag{20}$$

where $a(\log A) := \eta(A)/A$ represents the efficiency of R&D. Here, (20) implies that the R&D expenditures Z impact not only the incremental progress of the technology $A(t)$ but also the occurrence of technological revolution and, thereafter, the economic cycles incurred by the technological revolution. The intuitive case is that $\eta(A)$ takes a stepwise form, which is the same as the assumption for the GPT breakthroughs in Helpman and Trajtenberg (1994). Moreover, the function $\tilde{\eta}(A_{n(t)}, A)$ or $\eta(A)$ could take a wide variety of forms that are consistent with the existing literature on innovation. For example,

1. it could be a stepwise function such as
$$\tilde{\eta}(A_{n(t)}, A) = \eta_0 A_n = \eta_0 A_0 \xi^n, \text{ if } A \in [A_0\xi^n, A_0\xi^{n+1}), \tag{21}$$
$$a(\log A) = \frac{\eta_0 A_0 \xi^n}{A}, \text{ if } A \in [A_0\xi^n, A_0\xi^{n+1}).$$
2. it could be a function of S-shaped (sigmoidal) curve such as
$$\begin{aligned}\tilde{\eta}(A_{n(t)}, A) &= \eta_0(\phi_1 A_n - A)^v (A - \phi_2 A_n)^{1-v},\\ &\quad A \in [A_0\xi^n, A_0\xi^{n+1}),\\ a(\log A) &= \eta_0(\phi_1 \frac{A_n}{A} - 1)^v (1 - \phi_2 \frac{A_n}{A})^{1-v},\\ &\quad A \in [A_0\xi^n, A_0\xi^{n+1}).\end{aligned} \tag{22}$$
where $0 \le v \le 1$, and ϕ_1 and ϕ_2 satisfy $\phi_1 > \xi, \phi_2 \in (0, 1)$ and $(\phi_1\xi^{-1} - 1)^v(1 - \phi_2\xi^{-1})^{1-v} = (\phi_1 - 1)^v(1 - \phi_2)^{1-v}$.[8] Note that there is extensive literature describing the technological diffusion process by an S-shaped function such as the logistic curve or sigmoidal curve. (See Acemoglu (2009) and Sanchez-Choliz et al. (2008).)
3. it could be the combination of step function and linear function with constant elasticity of substitution, which we call semi-stepwise form, such as
$$\begin{aligned}\tilde{\eta}(A_{n(t)}, A) &= \eta_0 A_n^\gamma A^{1-\gamma}, \text{ if } A \in [A_0\xi^n, A_0\xi^{n+1}),\\ a(\log A) &= \eta_0 A_n^\gamma A^{-\gamma}, \text{ if } A \in [A_0\xi^n, A_0\xi^{n+1}).\end{aligned} \tag{23}$$

These functions are shown in Fig. 1. Here, we pay more attention to the abrupt change of technological progress and therefore mainly adopt the function form in (21) or (23) in the subsequent sections.[9] Moreover, since our main results are not subject to the discontinuity assumption for technological progress, we also show the numerical results with the S-shaped function for $\tilde{\eta}(\cdot)$ in the simulations in Appendix C.

To obtain a solution of the cyclic growth path for our models, we assume that $\eta(A)$ is of such form that

$a(\cdot)$ is continuous in $[\log A_n, \log A_{n+1})$ and a periodic function of $\log A$ with period $\theta = \log A_{n+1} - \log A_n = \log \xi$. [10]

This assumption tolerates most of the models we are interested in, such as the stepwise function for $\eta(A)$, as in Eq. (21). Let $t_n = \liminf_t\{t : A(t) > A_n\}$, i.e., the time when $A(t)$ first arrives at A_n (which is also called the corner time). From Eq. (20), $A(t)$ is always continuous w.r.t. to t but only continuously differentiable in the interval (t_n, t_{n+1}) whenever the interval exists.

Note that we only assume that $a(\cdot)$ is a periodic function of $\log A$, not t. Without further exploration, we have no way to deem $a(\cdot)$ periodic w.r.t. t. For example, even though $a(\cdot)$ is a periodic step function (w.r.t. $\log A$) as in Fig. 1, the jumping time of the technology depends on the decisions of agents and is the endogenous outcome of the equilibrium, or the social planner problem, as we preliminarily show in the previous section. Furthermore, actually, without additional constraints on the $\eta(\cdot)$ or $a(\cdot)$, the cyclic growth path and even the equilibrium path might not exist. We will discuss it in the next section.

3.2. The description of the economy

For the rest of the economy, we adopt the most common setup by endogenous growth models (Romer, 1990; Acemoglu, 2009). The model in this section is based on the framework of expanding varieties, where the monopoly is introduced into the technological market. We can also incorporate the idea of discontinuous technological change into some other setup, for which an example will be illustrated in Appendix B for the sake of space.

The unique final goods with price $P(t)$ are produced competitively with the production function

$$Y(t) = \frac{L(t)^{1-\alpha}}{\alpha} \int_0^{A(t)} x(v,t)^\alpha dv, \tag{24}$$

where $0 < \alpha < 1$; $L(t)$ is the labor input; $A(t)$ denotes the technological stock, which is considered here as the number of blueprints of different varieties of inputs (machines) available at time t; and $x(v, t)$ is the amount of input of variety v at time t.

The initial value of $A(t)$ is $\bar{A}_0$. Assume that x depreciates fully after use. Therefore, from the maximization problem for the firm

8 The conditions imposed on ϕ_1 and ϕ_2 ensure the continuity of $\tilde{\eta}(\cdot)$ or $a(\cdot)$ when the technological revolution occurs.

9 One might think that, since a potentially discontinuous R&D technology is embedded in the model, it must engineer growth with cycles. However, without further exploration, we can say little about this model. Actually, without more constraints on the $\eta(\cdot)$ or $a(\cdot)$, the cyclic growth path and even the equilibrium path might not exist. We will discuss it in the next section.

10 Actually, for the main results in this paper, we only need $a(\cdot)$ is piecewise continuous in $[\log A_n, \log A_{n+1})$. For simply, here we assume its continuity in this interval.

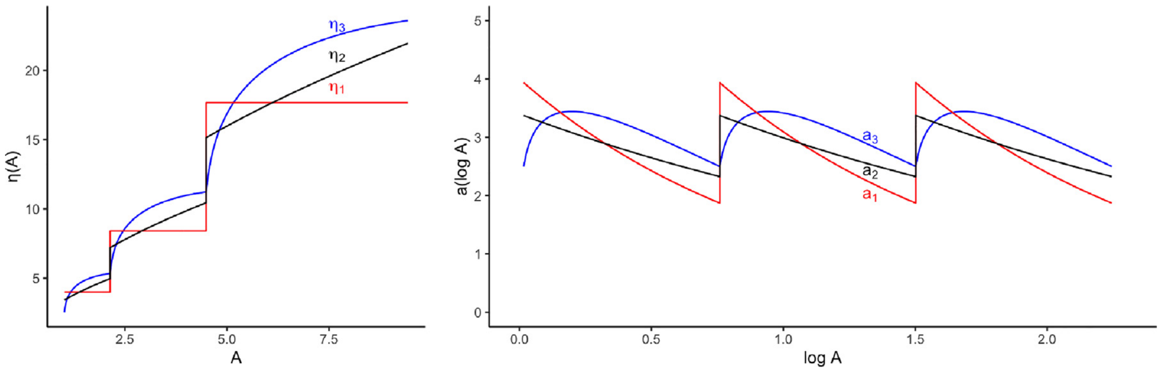

Fig. 1. Step function. Note: $\eta_1\&a_1$: step-wise form; $\eta_2\&a_2$: semi-stepwise form; $\eta_3\&a_3$: sigmoid form.

producing final goods,

$$\begin{aligned} x(v,t) =& p(v,t)^{\frac{-1}{1-\alpha}}L(t), \\ w(t) =& (1-\alpha)\frac{Y(t)}{L(t)}, \end{aligned} \tag{25}$$

where $p(v,t)$ and $w(t)$ are the price of machine and the wage rate, both in terms of the final goods.

Once a firm invents a new machine variety or purchases a blueprint patent, it can produce and supply this variety as a monopolist. The price of the blueprint v at time t is $V(v,t)$. One unit of that machine can be produced at a constant marginal cost, $\psi > 0$ units of final goods, and we normalize $\psi = \alpha$ for simplicity. The monopolist of machine v decides its price $p(v,t)$, facing the demand function, i.e., the first equation in (25), by maximizing the profits at time t

$$\pi(t) = (p(v,t)-\psi)x(v,t).$$

Using the FOCs of the above problem and Eqs. (25), we have

$$\begin{aligned} p(t) = p(v,t) = \frac{\psi}{\alpha} = 1, \\ x(v,t) = L(t), \end{aligned} \tag{26}$$

$$\pi(t) = \pi(v,t) = (1-\alpha)L(t). \tag{27}$$

We normalize the price of the blueprint $V(t) = 1$, which is more helpful for our subsequent operations.[11] By the no-arbitrage condition for the assets market,

$$\dot{V} = rV - P(t)\pi(t),$$

which implies

$$P(t)\pi(t) = r(t). \tag{28}$$

We substitute (27) for $\pi(t)$ and obtain

$$r(t) = (1-\alpha)L(t)P(t). \tag{29}$$

Obviously, here $r(t)$ is the nominal interest rate and not the real interest rate. Actually, it is the advantage of introducing $P(t)$ that leads to a succinct expression for the nominal interest rate. In addition, from (24)–(26),

$$Y = \frac{AL}{\alpha}, \tag{30}$$

$$Pw = \frac{1-\alpha}{\alpha}AP. \tag{31}$$

The total costs for the R&D firms are PZ. The R&D firms solve the problem of

$$\max_{Z(t)} \ a(\log A)Z(t) - P(t)Z(t).$$

The FOC is

$$Z(t) = 0 \text{ when } a(\log A(t)) < P(t), \text{ or } a(\log A(t)) = P(t). \tag{32}$$

Note that $P(t)$ is not necessarily continuous at the corner and neither is $r(t)$ from Eq. (29), since $a(\log A(t))$ is not.

We assume that the total population $L(t)$ is a constant and normalize it to 1. The representative household with preference

$$\int_0^\infty \exp(-\rho t)\log C(t)dt \tag{33}$$

holds the blueprints $A(t)$ as assets. Since the price of assets is 1, his budget constraint is[12]

$$\dot{A}(t) = r(t)A(t) + P(t)w(t) - P(t)C(t) \tag{34}$$

for $t \in [t_n, t_{n+1})$ with the no-Ponzi game constraint

$$\lim_{t\to\infty} A(t)\exp\{-\int_0^t r(s)ds\} \geq 0. \tag{35}$$

Since we have assumed that the alternation of the technological dynasties only depends on the level of the aggregate technological stock, a single family's behavior is unrelated to the timing of the technological revolution. Therefore, the representative household treats t_n as given, that is, subject to the corner constraint

$$\bar{A}(t_n) = A_n, \tag{36}$$

where $\bar{A}(\cdot)$ represents the aggregate (or average) technological stock.

The FOCs of the representative household problem are

$$\lambda P = e^{-\rho t}\frac{1}{C}, -\dot{\lambda} = \lambda r, \tag{37}$$

for $t \in [t_n, t_{n+1})$. As done in Section 2, from (14), the corner condition is

$$\lambda^+ = \lambda^- \Leftrightarrow P^+C^+ = P^-C^- \tag{38}$$

at each t_n because the household treats the corner time as given and $G_K^n(\cdot) = 0$ in (14). Note that we do not have condition (15) here because $G_{t_n}^n(\cdot) \neq 0$. The transversality condition for household assets is

$$\lim_{t\to\infty} \lambda(t)A(t) = 0.$$

11 Please refer to footnote 12.

12 Remember that the state variable, $A(t)$ here, in our discontinuous control problem needs to be continuous, as stated in Section 2. Therefore, if we normalize the price of the final goods as $P(t) = 1$, instead of $V(t) = 1$ here, we have to write the household budget constraint as

$$\dot{A}(t) = rA(t) - \frac{\dot{V}(t)}{V(t)}A(t) + \frac{w(t)}{V(t)} - \frac{C(t)}{V(t)}$$

which shows that Eq. (34) is more concise.

Let $a_{\min} = \inf_A a(\log A)$ and $a_{\max} = \sup_A a(\log A)$. We impose the parameter condition,

$$(1-\alpha)a_{\min} > \rho, \tag{39}$$

to ensure the existence of a reasonable economic growth path, whose necessity will be clarified later.

4. The equilibrium and CGP

4.1. The equilibrium

Here, the equilibrium is more complicated than that for the standard growth model because the timing of the technological revolution, t_n, is endogenized and is determined by the equilibrium. We use the concept of Markov Perfect Equilibrium, where the strategies are only related to the payoff-relevant state variable, which is obviously $A(t)$ here. Note that in Section 3.2 we solve the representative household problem with t_n given, and thus the resulting policy function $c(\cdot)$ should be a function of A and t_n. Furthermore, $c(\cdot)$ is a function of $A(t)$ since t_n is also determined by $A(t)$ by (36). Hence, if we solve the equilibrium as in Section 3, the result is truly an MPE.

Concretely, an equilibrium of the economy consists of the timing of technological revolution $[t_n]_{n\in\mathbb{N}}$; the time paths of consumption, aggregate spending on machines, aggregate R&D expenditures and available machine varieties $[C(t), Z(t), \Psi(t), A(t)]_{t=0}^{\infty}$; and the prices of the machine varieties, interest rates and wage rates $[p(v,t), r(t), w(t)]_{t=0}^{\infty}$ such that all research firms choose $[p(v,t), x(v,t)]_{v\in[0,A(t)],t=0}^{\infty}$ to maximize the discounted value of profits, the representative household maximizes its utility taking the time path of prices $[r(t), w(t)]_{t=0}^{\infty}$ and $[t_n]_{n\in\mathbb{N}}$ as given, the evolution of $[A(t)]_{t=0}^{\infty}$ is determined by free entry, $[t_n]_{n\in\mathbb{N}}$ is given by the corner constraints (36), and the evolution of $[r(t), w(t)]_{t=0}^{\infty}$ is consistent with market clearing.

The final goods market clearing condition is

$$Y(t) = C(t) + \Psi(t) + Z(t), \tag{40}$$

where $\Psi(t) = \int_0^{A(t)} \psi x(v,t)dv = \psi L(t)A(t)$ is the total costs for the machines with the final goods as the numeraire, and $Z(t)$ is the input to R&D. The net output of the final goods production is

$$Y(t) - \Psi(t) = \frac{1-\alpha^2}{\alpha}A(t). \tag{41}$$

4.1.1. The case with $Z(t) > 0$

First, we consider the interior solution where $Z(t) > 0$. Throughout this paper, $g_x = \frac{\dot{x}}{x}$ denotes the growth rate of variable x. From (34), (29), (30), (31), (32) and (37), on the equilibrium path,

$$\dot{A} = \frac{1-\alpha^2}{\alpha}a(\log A)A - PC, \tag{42}$$

$$g_{PC} = (1-\alpha)a(\log A) - \rho, \tag{43}$$

for each time interval (t_n, t_{n+1}). Let $X = {}^{PC}/_{A}$. Then,

$$\begin{aligned} g_A &= \frac{1-\alpha^2}{\alpha}a(\log A) - X, \qquad \forall t \in (t_n, t_{n+1}), \forall n, \\ g_X &= X - \frac{1-\alpha}{\alpha}a(\log A) - \rho, \quad \forall t \in (t_n, t_{n+1}), \forall n. \end{aligned} \tag{44}$$

Substituting the second equation in (32) into (38) leads to

$$a(\log A^-)C^- = a(\log A^+)C^+, \tag{45}$$

which implies

$$X^+ = X^-. \tag{46}$$

The TVC can be written as

$$\lim_{t\to\infty} e^{-\rho t}\frac{A(t)}{P(t)C(t)} = \lim_{t\to\infty} e^{-\rho t}\frac{1}{X(t)} = 0. \tag{47}$$

Therefore, the differential equations (44), corner condition (45), TVC (47) and initial condition (18) jointly determine the evolution of the economy. For convenience, we rewrite the system as

$$\begin{cases} g_A = \frac{1-\alpha^2}{\alpha}a(\log A) - X, & \forall t \in (t_n, t_{n+1}), \forall n, \\ g_X = X - \frac{1-\alpha}{\alpha}a(\log A) - \rho, & \forall t \in (t_n, t_{n+1}), \forall n, \\ A^+ = A^-, X^+ = X^-, \\ \lim_{t\to\infty} e^{-\rho t}\frac{1}{X(t)} = 0, \\ A(0) = \bar{A}_0. \end{cases} \tag{48}$$

Note that both $X(t)$ and $A(t)$ are continuous but only continuously differentiable in each interval (t_n, t_{n+1}).

4.1.2. The case with $Z(t) = 0$

Now, we show that $Z(t) = 0$ can never occur for any t. Assume that $Z(\tau) = 0$ for some τ, so $\dot{A}(\tau) = 0$ and $P(\tau) > a(\log A(\tau))$ from (37). Then, with condition (39), at time τ,

$$(1-\alpha)P \geq (1-\alpha)a(\log A) > \rho. \tag{49}$$

From Eqs. (40) and (41), $C(\tau) = \frac{1-\alpha^2}{\alpha}A(\tau)$. Therefore, $g_C(\tau) = 0$ and $g_P(\tau) = g_{PC}(\tau) = (1-\alpha)P(\tau) - \rho$. From (49), at time τ, we have

$$g_P > \delta P > 0, \tag{50}$$

for some $\delta > 0$. Obviously, for any $t > \tau$, (49) holds and $Z(t)$ =0. Then, we also have (50) for any $t > \tau$. Thus, $d(P^{-1})/dt < -\delta$, which implies $\lim_{t\to\bar{t}} P^{-1}(t) = 0$ and $\lim_{t\to\bar{t}} P(t) \to +\infty$ for some finite $\bar{t}$. If we treat the final goods as the numeraire, it means V arrives at 0 in finite time, which obviously is impossible whenever the monopolists still have profits, as in (27). Therefore, this is not a meaningful economic path.

4.2. The Cyclic Growth Path (CGP)

For the economic system described by Eqs. (44), (45), (47) and (18) in the previous section, the difficulty of obtaining the solution path is because the admissible path is only piecewise continuously differentiable. If $\eta(A)$ is set to be a linear function of A, this system reduces to a standard endogenous growth model. Due to the possible discontinuity of $\eta(A)$, solving this system is much harder than solving the standard endogenous growth model. Fortunately, under some mild assumptions, we can obtain a clear-cut description of the solution path, which incorporates business cycles into a long-run growth path. We call this sort of growth path the cyclic growth path (CGP), which is formally defined as follows.

Definition 1 (*CGP*)**.** An economic growth path is a CGP if the (per capita) output growth rate is a periodic function w.r.t. time.

Obviously, for the model in these sections, the solution path is a CGP if and only if the growth rate of technology $g_A(t)$ is periodic. Moreover, it is easy to check that on the CGP, the consumption-assets ratio of the representative household, X, the price of the final goods, P, and the saving rate, $s = 1 - {}^{C}/_{(Y-\Psi)}$, are also periodic. Mathematically, for the system of (48), under some mild assumption, there is a unique solution path, which is a CGP. This is established by the following theorems. However, before the introduction of these theorems, let us present a powerful tool in kinetics, the Poincare map, that is crucial for understanding both the dynamics of system (48) and the proof of the upcoming theorems.

4.2.1. The existence and uniqueness of the CGP

In kinetics, the Poincare map is understood as the intersection of some periodic orbit with some lower dimensional hypersurface, usually called the Poincare section. Here, we will not generally define a Poincare map but only pursue its meaning based on the model here. For the dynamic system in our model, strictly speaking, the system defined by the first three lines in (48), the definition of the Poincare map is very intuitive.[13] It is just the map (for some variable), denoted as $\mathcal{P}$, from some time, in this paper usually some corner point t_n^+, i.e., the starting time of some technological dynasty, to the next corner point t_{n+1}^+, i.e., the starting time of the next dynasty. Here, we will focus on the Poincare map for the logarithm of the consumption-assets ratio $x = \log X = \log({}^{PC}/_{A})$; therefore[14]

$$\mathcal{P}(x(t_n^+)) = x(t_{n+1}^+),\ \ n = 1, 2\cdots.$$

Note that map $\mathcal{P}(\cdot)$ is independent of n, which can be found from the second equation in system (48), because $a(\log A)$ is periodic. From the perspective of the Poincare map, the CGP is exactly its fixed point. Generally, it is not guaranteed that from any initial $x(0)$ the economy can always arrive at the necessary level of A_ns for the occurrence of the next technological breakthrough. For example, if on some path the saving rate, and therefore the growth rate of technology, is 0, the level of technology remains constant and the next revolution will never occur. Therefore, we define a set $\mathscr{X}$, consisting of all x's admissible for the occurrence of the next technological revolution, i.e., $\mathscr{X}_{A_n} = \{x \in \mathbb{R} : t_{n+1} < \infty$ whenever $x(t_n^+) = x\}$, which is the domain of map $\mathcal{P}$. Moreover, we can prove the following theorem for set $\mathscr{X}_{A_n}$.

Theorem 2 (*Existence*). *For the system defined by the first three lines in (48),*

1. *there exists $M = \sup \mathscr{X} < +\infty$ such that the interior of $\mathscr{X}$, $\mathscr{X}^\circ = (-\infty, M)$.*
2. *the CGP exists if and only if $\mathcal{P}(\mathscr{X}) \supsetneqq \mathscr{X}$, i.e., $\mathcal{P}(\sup \mathscr{X}) := \lim_{x\to\sup\mathscr{X}} \mathcal{P}(x) > \sup \mathscr{X}$.*

The first claim of this theorem is from Lemma 24, and the second claim is exactly the conclusion of Theorem 26. Please find their proofs in Appendix B.2.1.

The first part of the theorem tells us that at time t_n for any n, there is a maximal (logarithm) consumption-assets ratio such that the economy with the (logarithm) consumption-assets ratio less than it will surely reach the next technological revolution. The second part of the theorem shows that the domain of the map $\mathcal{P}$ is a proper subset of its range. From the second statement of this theorem, we can infer that the inverse of $\mathcal{P}$ is a contraction map and has a fixed point, which should be the CGP. The advantage of this theorem is that to ensure the existence of the CGP, i.e., the fixed point of $\mathcal{P}$, we only need to consider one round of the technological revolution. Once the range of the Poincare map from the beginning of one technological dynasty to the beginning of the next dynasty includes its domain, the CGP must exist. Nevertheless, if the second statement is violated, the CGP does not exist, for which we propose a numerical example in Appendix D.

Based on the previous theorem, we can ensure the existence of the CGP under some mild conditions by the next theorem.

[13] Please find the concrete definition of the Poincare map in Appendix A, where we actually define it for any time t.

[14] In the appendix, we actually define the Poincare map as mapping $x(t)$ of any time t to the corresponding x of one period later.

Theorem 3 (*Sufficient Condition of the Existence*). *For the system defined by the first three lines in (48), if*

$$\frac{1-\alpha}{\alpha} a_{\max} + \rho < \frac{1-\alpha^2}{\alpha} a_{\min}, \tag{51}$$

the CGP exists.

This theorem is the direct result of Theorem 27 in Appendix B.2.1. Inequality (51) implies

$$\frac{2+\alpha}{2\alpha}(a_{\max} - a_{\min}) < \frac{1}{2}(a_{\max} + a_{\min}) - \frac{\rho}{1-\alpha}.$$

We can view $\frac{1}{2}(a_{\max} + a_{\min})$ as a round measure of the average $a(\cdot)$. Therefore, the above theorem indicates that, to obtain the CGP, $a(\cdot)$ cannot fluctuate too much relative to its average level and its average level cannot be too small. Once we ensure the existence of the CGP, we simultaneously obtain the uniqueness as in the next theorem, which is a direct result of Theorem 25 in Appendix B.2.1.

Theorem 4 (*Uniqueness*). *For the system defined by the first three lines in (48), if the CGP exists, then it is unique.*

It is important to clarify that the properties of the CGP, presented thus far, are based on only the first two differential equations and the corner conditions in the third line in system (48) and are unrelated to the remaining two conditions. Therefore, the CGP is not necessarily an equilibrium solution, which needs all conditions in system (48) to hold. Nevertheless, the next theorem indicates that the equilibrium solution for this model is exactly the CGP satisfying conditions (44) and (18).

Theorem 5. *For the economy in this section, we have the following:*

1. *If a CGP path of the system defined by the first three lines in (48) also satisfies condition (18), it is the equilibrium solution of the economy, i.e., the solution path of system (48).*
2. *The equilibrium path, i.e., the solution path of system (48), must be a CGP.*

The first part of the theorem is obvious because every CGP satisfies the TVC automatically and only one GGP can meet the initial condition (18), although there are many CGPs satisfying the evolving condition (44). The second part is the result of Theorem 28 in Appendix B.2.2.

The economic intuition behind Theorems 2–5 is not difficult to understand. The representative household has to choose an appropriate consumption level, one that is neither too high nor too low, at each time. For example, as shown in Fig. 2, at some time t_0, if the representative household chooses a relatively low level of consumption such that

$$X < \frac{1-\alpha}{\alpha} a_{\max} + \rho,$$

then, from (39) and the two differential equations in (48), $g_X < 0$ and $g_A > 0$ for any $t \geq t_0$, as shown in region C in Fig. 2. In this case, the TVC in the fourth line of (48) will be violated. However, if the representative household chooses a relatively high level of consumption such that

$$X > \frac{1-\alpha}{\alpha} a_{\min} + \rho,$$

then, from the second equation in (48), $g_X > 0$ for any $t > t_0$. Therefore, from the first equation in (48), we must have $g_A = 0$ as $t \geq \tau$ for some τ, as shown in region A in Fig. 2, since we assume that the technology cannot decrease. This case is not economically admissible, as shown at the end of Section 4.1. The equilibrium path can only fluctuate with X in $(\frac{1-\alpha}{\alpha} a_{min} + \rho, \frac{1-\alpha}{\alpha} a_{max} + \rho)$,

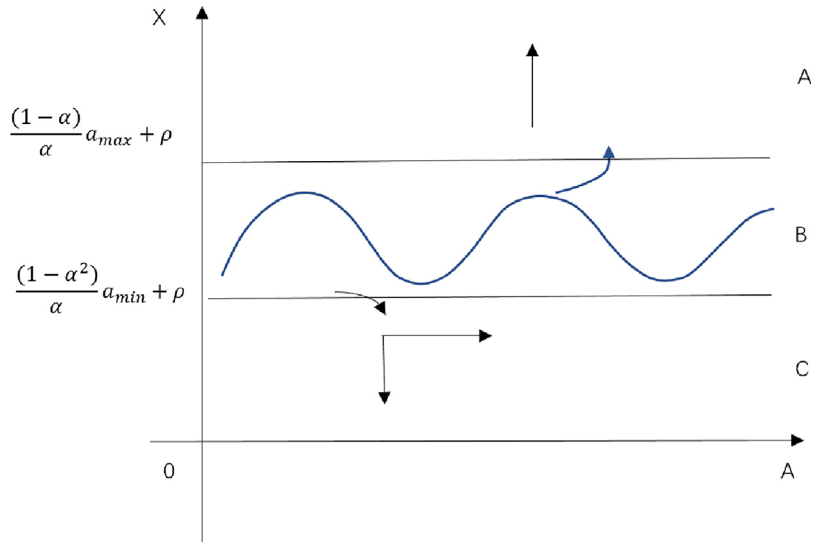

Fig. 2. The phase diagram.

i.e., region B in Fig. 2. It is worth noting that, as shown in Fig. 2, $\frac{1-\alpha}{\alpha}a_{min}+\rho$ and $\frac{1-\alpha}{\alpha}a_{max}+\rho$ are some respectively lower and upper bounds of X in the equilibrium path, but they are not necessarily the supremum or infimum since $a(\cdot)$ is not a constant but is a periodic function.

4.3. The economic behavior on the CGP

Let us examine the dynamic behavior of the CGP. Throughout this section, to simplify the analysis, we assume the step function for the technological revolution as in (21).

4.3.1. Crossing technological dynasty

First, let us investigate the corner behavior of the economy more carefully. We rewrite the corner condition (38) as $P^+C^+ = P^-C^-$, where $P(t) = a(\log A(t))$. For step function η, $a = \eta(A)/A$ decreases within each cycle and jumps upward at each corner since A monotonically increases within the cycle and is continuous at the corner. Therefore, $P^+ > P^-$ and $C^+ < C^-$.[15] Intuitively, while crossing the corner of technology jumping, the enhancement of R&D efficiency increases the rewards of the research activities, as shown by Eq. (32). Here, R&D investments, Z, happen to be savings, so the return on savings rises, and people are more inclined to save. Furthermore, the improvement of R&D efficiency has caused the investments to become more valuable, which leads to an increase in their prices relative to assets, and thus the prices of consumption products also increase. In other words, consumer goods have become more expensive relative to assets, which also reduces the propensity to consume.

By removing the price effect, we obtain the real GDP as $\hat{Y} = C + \frac{1}{P}\dot{A} = Y - \Psi$, which is the net output of the final product. Since $Y - \Psi = \frac{1-\alpha^2}{\alpha}A$ is continuous, the real GDP of this economy is continuous. Nevertheless, the growth rate of $\hat{Y}$, $g_{\hat{Y}}$, is not continuous under the assumption of a stepwise technological revolution because $g_{\hat{Y}} = g_A$ and $\dot{A}$ are not continuous. Concretely, $\dot{A} = PZ = P(Y - \Psi) - PC$. We know that PC and Y are continuous and P is discontinuous at the corner, so $\dot{A}$ changes discontinuously at the corner. $P^+ > P^-$ means $\dot{A}^+ > \dot{A}^-$; therefore, the technology growth rate g_A and the GDP growth rate $g_{\hat{Y}}$ increase following the technological revolution. For the saving rate,

$$s = 1 - \frac{C}{Y - \Psi} = 1 - \frac{\alpha}{1-\alpha^2}\frac{C}{A}. \tag{52}$$

Therefore, the saving rate jumps upward at the corner since A is continuous and C jumps downward. The above analysis can be concluded as the next theorem.

Theorem 6. *In the transition from one technological dynasty to another, i.e., at the corner time, t_n, the economy exhibits the following properties:*

1. $A^+ = A^-$, $\hat{Y}^+ = \hat{Y}^-$, $P^+ > P^-$, $C^+ < C^-$ *and* $Z^+ > Z^-$.
2. $s^+ > s^-$.
3. $g^+_{\hat{Y}} = g^+_A > g^-_{\hat{Y}} = g^-_A$.

4.3.2. Within-cycle behavior

Now we turn to the behavior of the economy within the period $[t_n, t_{n+1})$ between two adjacent technological leapfrogs. First, the growth rate of the real net output $g_{\hat{Y}} = g_A > 0$. Recall that in these intervals, $\eta(A)$ is constant and $P = a(\log A(t)) = {}^{A_n}/_{A}$, where $A_n = A_0\xi^n$ (see (21)), which implies $g_P = -g_A < 0$ for $\forall t \in (t_n, t_{n+1})$ and $\forall n$. Then,

$$g_{C/A} = g_X - g_P = g_X + g_A = (1-\alpha)a(\log A) - \rho > 0,\ t \in [t_n, t_{n+1}), \tag{53}$$

where the last inequality comes from assumption (39). From the above expressions, $g_C = g_{C/A} + g_A > 0$. For the stepwise technological revolution and $t \in [t_n, t_{n+1})$, $\dot{A} = \frac{1-\alpha^2}{\alpha}A_n - PC$,so $\dot{A}$ decreases through the interval because of the positive growth of PC. Then, $g_{\hat{Y}} = g_A$ also decreases given that A is increasing in the interval.

Hence, from Eqs. (52) and (53), the within-cycle saving rate is always declining. The farther away from the last technological revolution the economy is, the lower the efficiency of R&D, the lower the market prices of the final goods from which the R&D inputs come, and the higher the market value of assets, i.e., the patents. Therefore, as assets increase, households tend to consume more and save less. Since the only purpose of savings in these models is for R&D, we also call this cycle of the evolution of the saving rate with the R&D cycle. It should be noted that this is not necessarily socially optimal because the household

---

15 $P^+ > P^-$ means that, when we treat the final goods as the numeraire, the assets price jumps down at the corner. Under these circumstances, we assume that the impossibilities of storing intermediate or final output, as in Shleifer (1986).

maximization problem does not internalize the positive effect of savings on the occurrence of a technological revolution. Intuitively, if this effect is internalized, the closer the economy is to the next technological revolution, the higher the expected returns that savings yield, and then there should be a tendency to save more.

Next, we explore the behavior of the consumption-assets ratio, $X = PC/A$, and conclude its behavior in the following lemma.

Lemma 7. *Within the period between two adjacent technological leapfrogs, $[t_n, t_{n+1})$, g_X changes from negative at the start of the period to positive before the end. In other words, within one technological dynasty, X first decreases and then increases.*

Proof. For the stepwise technological revolution, recall that $g_A > 0$ and $g_{a(\log A)} < 0$ in interval $[t_n, t_{n+1})$. Since $g_X = X - \frac{1-\alpha}{\alpha} a(\log A) - \rho$, if X is fixed, g_X always increases. Additionally, g_X cannot always be positive or negative through this period given that X is periodic and continuous. Moreover, if $g_X(s) > 0$ for some $s \in [t_n, t_{n+1})$, the expression of g_X implies $g_X(t) > 0$ for $\forall t \in [s, t_{n+1})$. Therefore, we must have $g_X(t_n^+) < 0$ and $g_X(t_{n+1}^-) > 0$. □

The above analysis and Lemma 7 can be summarized in the next theorem.

Theorem 8. *Within the period $[t_n, t_{n+1})$ between two adjacent technological leapfrogs, the economy evolves with the properties*

1. *$g_A > 0$, $g_C > 0$, and $g_s < 0$;*
2. *the real output growth rate $g_{\hat{Y}} = g_A > 0$ and decreases in the period; and*
3. *$g_X(t_n^+) < 0$ and $g_X(t_{n+1}^-) > 0$.*

It should be noted that the results in the last theorem are purely theoretical. The monotone decrease of the saving rate in the interval is impressive. Guidolin and La Jeunesse (2007) show that the saving rate in the USA continually declined until 2007 from some relatively high level in the 1980s, while this period is thought by some authors to be the era of a new industrial revolution, marked by the surging of computers and internet. Nevertheless, more empirical work is needed to show whether the above theory is consistent with reality.

4.3.3. Long run growth

Since the economy is cyclic, we can only discuss the long-run growth rate in the meaning of average. We use $\mathcal{G}_x$ to express the (average) long growth rate of variable x. On the equilibrium path, i.e., a CGP, $g_{\hat{Y}}(t)$ is periodic, so the long-run growth rate of real GDP is

$$\mathcal{G}_{\hat{Y}} = \frac{1}{T}\int_{t_n}^{t_{n+1}} g_{\hat{Y}}(t)dt,$$

where $T = t_{n+1} - t_n$. Because consumption jumps at the corner, its long-run growth rate should be

$$\mathcal{G}_C = \frac{\log \frac{C^+}{C^-} + \int_{t_n}^{t_{n+1}} g_C(t)dt}{T}.$$

In the equilibrium, from the Ozawa Theorem, we have $\mathcal{G} = \mathcal{G}_{\hat{Y}} = \mathcal{G}_A = \mathcal{G}_C$. Since $P(t)$ is periodic, $\mathcal{G}_P = 0$. Although we cannot obtain a closed-form solution for the long-run growth rate, it is a function of $a(\cdot)$, ρ and other parameters. Theoretically, we can explore the monotonicity of the growth rate w.r.t. these parameters. Since we can only compare the long-run growth rates of CGPs, the basic idea is to examine the new growth rate after some parameter is perturbed slightly such that the CGP still exists. Nevertheless, the analysis is complicated because the new CGP is a new path, different from the old one. We summarize the main conclusions in the next theorem and give the proof later in Appendix B.3.1-2.

Theorem 9. *For the model in this section, suppose that the R&D efficiency $a(\cdot)$ or the discount rate ρ is perturbed by a small amount and the CGP still exists after the perturbation. That is, $a(\log A(t)) = a^0(\log A(t)) + \varepsilon_1$ or $\rho = \rho_0 + \varepsilon_2$, where $a^0(\cdot)$ and ρ_0 represent the original correspondents of $a(\cdot)$ and ρ before the perturbation. Then for the long-run growth rate on the CGP, $\mathcal{G}(\cdot)$, we have*

$$\frac{\partial \mathcal{G}}{\partial \varepsilon_1} > 0 \text{ and } \frac{\partial \mathcal{G}}{\partial \varepsilon_2} < 0.$$

Note that the perturbation, ε_1 or ε_2, is independent of time. The partial derivatives in this theorem are meaningful because $\mathcal{G}(\cdot)$ is a continuous function of the perturbation when it is small enough from the theory of differential equations.

In Appendix C, we show these properties of the CGP by simulations. In addition, we explore the relationship between growth rates and the cyclic length by marginally changing the value of some parameters. For the partial effects from the perturbations to some parameters, the growth rate is usually negatively related to the cyclic length. For the partial change of ρ, α, η_0, ξ and the shock to $a(\cdot)$, the long run growth rate and the cyclic length usually change in the opposite direction. Nevertheless, while these simulations are trying to show the partial effects of some parameters, we cannot conclude that the long run growth rate is globally positively related to the length of cycles. Under different parameter profiles in the simulations, we can also find a relationship in the opposite direction. (Please see the comments in Appendix D.3.)

Following the recent literature based on Beaudry et al. (2020) that focuses on the existence of mid-term cycles, one might wonder whether the fluctuations here are short-, medium- or long-term. The cycles in our model are driven by the recurrences of technological breakthroughs, which are usually thought by classical authors to stride over more than several decades. In the simulations in Appendix D, the cycles last several decades when we maintain the averaged long run growth rate close to 2% under some reasonable values for parameters. Therefore, the cycles in our model are more likely to provide descriptions for the long-term waves, at least other than for short-term fluctuations.

5. The social welfare and policy

5.1. The social planner problem

The equilibrium solution in the previous section is obviously not socially optimal. There are two causes of inefficiency in this model: the monopoly of blueprints and the externality of knowledge spillover. The former is similar to many endogenous growth models and comes from the ex post monopolistic utilization of innovation achievements. The latter is different from the technology spillover in the existing literature because the spillover here refers to the accumulation of existing technology being beneficial to the breakthrough of the next technological revolution. Here, we are more interested in finding how the social planner can take advantage of the knowledge externality to promote the technological revolution. The solving of the social planner problem can be divided into two steps.

In the first step, the social planner addresses the production decision of the intermediates and the final goods, given the available blueprints. The following maximization problem is solved.

$$\hat{Y} = \max_{x(v)} \frac{L^{1-\alpha}}{\alpha}\int_0^A x^\alpha(v)dv - \psi\int_0^A x(v)dv.$$

It is easy to obtain $x(v) = \alpha^{1/(1-\alpha)}$ and the value function, i.e., the net output as $\hat{Y} = (1-\alpha)\alpha^{\frac{-1}{1-\alpha}}A$, where $L = 1$ and $\psi = \alpha$ are used. Recall that in the decentralized economy in the previous section, we have $\hat{Y} = A(1-\alpha^2)/\alpha$. After some mathematical manipulation, we find that the former is greater than the latter, which represents the efficiency loss due to the monopoly. To separate the influence of knowledge spillover on the technological revolution, the social planner can be assumed to be incapable of changing the monopolistic competition structure of the intermediate market or be constrained by $\hat{Y} = A(1-\alpha^2)/\alpha$ to solve the next step's problem. Therefore, we can let

$$\hat{Y} = \phi A, \tag{54}$$

where

$$\phi = (1-\alpha)\alpha^{\frac{-1}{1-\alpha}} \text{ or } \frac{1-\alpha^2}{\alpha}. \tag{55}$$

Here $\phi = (\frac{1-\alpha}{\alpha})\alpha^{-\frac{\alpha}{1-\alpha}}$ for an omnipotent social planner and $\phi = (1-\alpha^2)/\alpha$ while retaining the monopolistic competition structure. Although these two different cases have a similar functional form for the net output, a linear function of A, it must be warned that the technological monopoly not only leads ϕ to be different but also has a substantial impact on the interest rate and then the growth rate.[16]

Now let us focus on the social planner's decision for the representative household problem, which is described as

$$\begin{aligned}
&\max_{C(t)} \int_0^\infty e^{-\rho t}\log C(t)dt,\\
&\quad \dot{A}(t) = a(\log A(t))(\phi A(t) - C),\ t \in (t_n, t_{n+1})\\
&\quad A(0) = \bar{A}_0,\\
&\quad \text{(5) (NPC)},
\end{aligned}$$

with the corner constraint $A(t_n) = A_n$, where the differential equation for A comes from (54), (20) and (40). As before, here, A is continuous but only continuously differentiable in (t_n, t_{n+1}), the interval between two adjacent technological revolutions. The Hamiltonian function is

$$\mathcal{H} = e^{-\rho t}\log C + \lambda a(\log A)(\phi A - C),$$

which has the FOCs, in the interval (t_n, t_{n+1}),

$$g_C = a(\log A)\phi - \rho, \tag{56}$$

$$\dot{A} = a(\log A)(\phi A - C). \tag{57}$$

Now, the key difference from the decentralized equilibrium is the corner condition. Here, the time point of a technological revolution is not a constraint to the social planner but a variable to be chosen by changing the accumulation of A. From condition (16), now the corner condition is

$$\mathcal{H}^+ = \mathcal{H}^-, \tag{58}$$

which implies

$$\log C^+ + \frac{\phi A^+}{C^+} = \log C^- + \frac{\phi A^-}{C^-}. \tag{59}$$

Because A is continuous at the corner, we have

$$C^+ = C^- \tag{60}$$

at each t_n. Now, the consumption is continuous, which is a distinct point from the decentralized equilibrium, where the consumption jumps at each corner. It is not surprising! A household in a decentralized economy cannot shift the timing of the technological revolution and then has to adjust her consumption abruptly to fit the occurrence of the new technological dynasty. Because the social planner can optimally choose the timing of the technological revolution, she would like to smooth her welfare throughout her lifetime, as embodied in Eq. (58); and smooth her consumption more across different technology cycles.

The TVC is

$$\lim_{t\to\infty} \frac{e^{-\rho t}}{a(\log A(t))C(t)} A(t) = 0, \tag{61}$$

and the initial condition is (18).

Similar to Section 4, we can also prove that the solution to the above social planner problem is a CGP. Recall that on the CGP, the growth rate of the net output and that of the technological stock evolves periodically over time. It is easy to check that here C/A is also periodic on the CGP. In the decentralized economy, we impose the parameter condition (39) to ensure positive economic growth. Similarly, here, we can impose $\phi a_{\min} > \rho$, and then we can prove the following theorem using almost the same techniques as in the proof for the existence and uniqueness of the CGP solution to the market equilibrium in Theorem 5 (see Appendix B.4.1).

Theorem 10. *Assuming $\phi a_{\min} > \rho$ for the social planner problem, we have the following:*

1. *The solution path, which is the path that satisfies (56), (57), (60), (61) and (18), is a CGP.*
2. *The CGP solution exists and is unique.*

Actually, we do not need to impose condition $\phi a_{\min} > \rho$ for the economy here once we already have (39). This is because, from the definition of ϕ in (55), $\phi > 1-\alpha$ such that $\phi a_{\min} > (1-\alpha)a_{\min}$. Therefore, once we impose condition (39), it is also sufficient to ensure the existence of a CGP solution for the social planner problem with either form of ϕ in (55). Moreover, here, we do not impose a correspondence of condition (51), which demonstrates that the existence of the CGP solution for the social planner problem cannot be destroyed by the extent of the technological leapfrog while that can occur in the decentralized economy.[17] Since the social planner's solution is always better than that in the decentralized economy, the CGP in Theorem 10 is better than the growth path, either the CGP for the market equilibrium or other growth paths. Therefore, it is meaningful to establish the optimal CGP using a policy tool when the economy falls into some other CGP.

5.2. Economic behavior on the CGP

Let us further examine the economic behavior in the interval $[t_n, t_{n+1})$. Throughout this subsection, we assume $\eta(A) = Aa(\log A)$ to be a stepwise function. Obviously, on the CGP, C/A, $g_{\hat{Y}}$, g_C g_A and $g_{C/A}$ all move periodically. First, let us address the dynamics of the consumption-technology ratio, $\tilde{X} = C/A$, similar to the analysis of $X = PC/A$ in the previous section, since the social planner problem has no price. With

$$g_{\tilde{X}} = g_C - g_A = a(\log A)\tilde{X} - \rho, \tag{62}$$

the following theorem can be proved.

16 Actually, it is difficult to retain the entire impact due to the technological monopoly. Notice that its effect can be divided into two parts, the static efficiency loss and the dynamic efficiency loss, because the monopoly impacts not only the resource allocation at every moment but also the growth rate. Letting $\phi = (1-\alpha^2)/\alpha$ can only remain the static inefficiency, but not the dynamic inefficiency, which is embodied in the coefficient of A in Eq. (42). Also see Acemoglu (2012).

17 Mathematically, this is guaranteed by the convexity of the social planner problem, although it does not hold the representative household problem in the decentralized economy.

Theorem 11. *Within the period between two adjacent technological leapfrogs, i.e., $[t_n, t_{n+1})$, $g_{\tilde{X}}$ changes from positive at the start of the period to negative before the end. In other words, $\tilde{X}$ first increases and then decreases in the period.*

Proof. For the period $[t_n, t_{n+1})$, recall that $g_A > 0$ and $g_{a(\log A)} < 0$ under the stepwise form of $\eta(A)$. From (62), if $\tilde{X}$ is fixed, $g_{\tilde{X}}$ always decreases. Therefore, g_X cannot always be positive or negative throughout this period given that $\tilde{X}$ is periodic and continuous. Moreover, if $g_{\tilde{X}}(s) < 0$ for some s, (62) implies $g_{\tilde{X}}(t) < 0$ for $\forall t \in [s, t_{n+1})$. Therefore, we must have $g_{\tilde{X}}(t_n^+) > 0$ and $g_{\tilde{X}}(t_{n+1}^-) < 0$. □

5.2.1. Research cycle: Ex post dividend effect and ex ante swot-up effect

Recall that $s = 1 - C/(\phi A)$. Then, the above theorem implies that the saving rate within one technological dynasty must first decrease and then increase. The first decrease and later increase in the saving rate indicate that the economy has a higher saving rate just before and after the technological revolution while the saving rate is relatively lower in places far from technological jumps. We can summarize these phenomena as the ex post "dividend effect" and the ex-ante "swot-up effect" and understand the economic mechanism behind them.

(1) The ex post "dividend effect" refers to the economy having a higher saving rate after the occurrence of a technological revolution. This could be well understood. At the beginning of a technological dynasty, the technological revolution significantly improves the efficiency of R&D activities, and the return on savings increases, so it is better to save more. Over time, the dividends resulting from the last technological revolution gradually decline, the efficiency of R&D decreases, and the next technological revolution is still far away. The return on savings relative to consumption falls; and therefore, the saving rate declines.

(2) The ex-ante "swot-up effect" refers to the higher saving rate on the eve of the technological revolution. In the latter period of a technological dynasty, although dividends from the last technological leapfrog may have almost disappeared, the next technological revolution is approaching. At this time, increasing savings can accelerate the arrival of the next revolution. The gradual rise of the saving rate means that the closer the economy is to the next technological revolution, the more R&D activities should be conducted. This is similar to students who like to swot up in the last week before an exam! The closer it is to the exam, the harder they work!

The aforementioned corner condition shows that the saving rate is continuous. The above two effects make the saving rate fluctuate with a form higher at both ends and lower in the middle within one technological cycle. This contrasts with the saving rate behavior in the decentralized economy, where it jumps up immediately after the arrival of a new technological revolution and then decreases throughout the cycle. In other words, in the decentralized economy, there is only the ex post "dividend effect", lacking the ex-ante "swot-up effect". In the decentralized economy, the effort of individual households to save has no influence on the technological revolution, so a family has no incentive to save more to promote the technological revolution and exhibits "free rider" behavior of passively waiting for the next technological dynasty. Therefore, the market equilibrium shows the property of insufficient saving, which induces the technology to jump too late. This also indicates that if the technological revolution exhibits periodic and sudden movements, the decentralized economy is unable to better adapt to such fluctuations and achieve the dynamically optimal allocation of R&D investment. Because the planned economy could adapt to the fluctuations of technology better, we conjecture that the economy in the social planner problem should be less wavy than the decentralized economy. We show these facts by numerical simulations in Appendix C, and actually, we find that the planner economy is unimaginably smoothed compared to the decentralized one.

Since there is efficiency loss in the decentralized economy, its long-run growth rate should be lower than that in the social planner problem. Strictly speaking, when the condition for the existence of CGP holds, we can prove the next theorem. In this theorem, we use the superscript m or p to distinguish the variables in the market equilibrium or social planner problem and denote g_A as a function of $c(u) = \frac{C(u)}{A(u)}$ and u, which is justified by

$$\frac{\dot{A}}{A} = a(\log A)(\phi - \frac{C(u)}{A(u)}) \\ = g_A(c(u), u),$$

from (57).

Theorem 12. *If $\frac{1-\alpha}{\alpha} a_{\min} \geq (\frac{a_{\max}}{a_{\min}} - 1)\rho$, then*

$$g_A(c^m(u), u) < g_A(c^p(u), u), \tag{63}$$

for $\forall u$ and $\mathcal{G}_A^m < \mathcal{G}_A^p$. Moreover, if $\frac{1-\alpha}{\alpha} a_{\max} + \rho \leq \frac{1-\alpha^2}{\alpha} a_{\min}$, $g_A(c^m(u), u) < g_A(c^p(u), u)$ for $\forall u$ and $\mathcal{G}_A^m < \mathcal{G}_A^p$.

The proof of this theorem can be found in Appendix B.3.4. This theorem claims that the growth rate for the social planner problem is higher than that in the market equilibrium in both the short term and long term. Moreover, technological revolutions occur more frequently on the solution path of the social planner problem, and the economic cycles are shorter, from (63). Therefore, the social planner values technological revolutions more than the decentralized economy, where technological breakthroughs always occur too late.

5.3. The policy to restore optimality

Can we correct the inefficiency of the market equilibrium and restore social optimality with policy tools? The answer is yes! Here, we show two types of policy tools: subsidies to asset returns and subsidies to R&D activities. Interestingly, the former tool cannot achieve the goal while the second works.

The policy proposal of subsidies to asset returns is quite intuitive. In the market equilibrium, households savings are insufficient to promote technological breakthroughs at the right time, and we can try to raise the saving rate by subsidizing savings.

Suppose the government could impose the lump-sum tax, $T(t)$, on each household, which is used to subsidize savings. The time-dependent subsidy rate is $\tau(t)$ per unit of savings. Then, the representative household in the decentralized economy has the budget constraint

$$\dot{A}(t) = (r(t) + \tau(\tau))A(t) + P(t)w(t)L - P(t)C(t) - T(t),$$

where $\tau A = T$ because of the government budget balance. Now the FOC becomes $g_{PC} = r + \tau - \rho$. However, since there is no other change, it is easy to check that the corner condition is still $P^+C^+ = P^-C^-$ as in (38) and the relative price is still $P = a(\log A)$ as in (32). The relative price still jumps at the corners and so does the consumption, as in the solution in Section 4. Therefore, this policy proposal can never restore the solution of the social planner problem where consumption should be continuous at each corner.

Another policy option is to use a lump-sum tax to subsidize R&D activities. Suppose the subsidy rate is $\tau(t)$ per unit of investment in R&D. Then, the return of R&D investment is $a(\log A)Z + \tau Z$, where $\tau Z = T$. In equilibrium, the free entry condition implies

$$a(\log A) + \tau = P. \tag{64}$$

The FOC is

$$g_{PC} = (1-\alpha)P - \rho, \tag{65}$$

as (43) and the corner condition is still

$$P^{+}C^{+} = P^{-}C^{-}. \tag{66}$$

Given the budget balance of the government, it is not difficult to show that the household assets moving equation is the same as that in (43), i.e.,

$$\dot{A} = a(\log A)(\frac{1-\alpha^2}{\alpha}A - C), \tag{67}$$

which is also the same as that for the social planner problem, (57) with $\phi = (1-\alpha^2)/\alpha$. Therefore, if we can restore Eq. (56) and corner condition (60), the solution of the social planner problem (with $\phi = (1-\alpha^2)/\alpha$) is achieved. From (65), (66), (56) and (60), we only need to establish

$$g_P = (1-\alpha)P - \phi a(\log A), \tag{68}$$

$$P^{+} = P^{-}. \tag{69}$$

In other words, if we can find a CGP solution of (g_A, g_P) to the system of (67), (68), and (69), together with the corresponding TVC and initial conditions, the correct policy variable $\tau(t)$ is determined by (64). It can be proven that the CGP solution to this system truly exists using exactly the same idea as in the proof of theorem 4 (please see Appendix B.4.2). Therefore, we can achieve social optimality by choosing the right $\tau(t)$ for the decentralized economy.

Why can the social planner solution be achieved by R&D subsidies but not by savings subsidies? Recall that the market equilibrium set the wrong relative prices of goods to assets, especially around the corners. The households in the market treat the timing of technological revolution as given while the social planner endogenizes it, which results in different corner conditions as shown in Eqs. (15) and (16). Thereafter, the consumption path in the social planner problem is continuous at the corners while it is not in the market equilibrium, which comes from the discontinuity of $P(t)$, by Eq. (38). The distinct effects of these two policy tools are also rooted in their different influences on the relative prices of goods to assets, especially at the corner.

For the subsidizing savings policy, the price is still $P = a(\log A)$, and we have no way to restore its continuity and then the continuity of the consumption at the corner. However, for the second policy tool, R&D subsidies, we can accomplish the goal because we change the relative prices. Moreover, from (68), we can easily summarize some properties of the relative prices under the optimal $\tau(t)$ of R&D subsidies as follows.

1. The relative price moves periodically around $\frac{1}{1-\alpha}\phi a(\log A)$ and it satisfies $\phi a_{\min} < (1-\alpha)P(t) < \phi a_{\max}$.
2. Within each cycle $[t_n, t_{n+1})$, the relative price $P(t)$ first decreases and then increases. In addition, we have $(1-\alpha)P(t_{\min}) = \phi a(t_{\min}) > \phi a_{\min}$, where $t_{\min} = \arg\min_t P(t)$ for $t \in [t_n, t_{n+1})$.

These behavioral properties of the relative price embody different resources of the efficiency loss. From (64),

$$\begin{aligned}\tau &= P - a(\log A)\\ &= [\frac{1}{1-\alpha}\phi a(\log A) - a(\log A)] + [P - \frac{1}{1-\alpha}\phi a(\log A)].\end{aligned} \tag{70}$$

To clarify this, let us first rule out the influence of a technological jump by forcing $a(logA)$ to be a constant, $\bar{a}$. Then, P is also a constant; and $P^{*} = \phi\bar{a}/(1-\alpha)$ is still greater than $\bar{a}$, the market equilibrium relative price with monopolistic competition. Therefore, the first part of the subsidies in (70) can be understood as remedying the inadequacy of R&D investment due to the monopoly. The second part in (70) addresses the externality of R&D on the technological revolution. From the properties of the relative price and Eq. (68), the second part is first negative and then positive in each interval $[t_n, t_{n+1})$, which means we need to suppress some R&D activities in the early stage of one technological dynasty and encourage R&D activities in the later stages. Recalling the ex-ante "swot-up effect" on the CGP solution for the social planner problem, it is reasonable for the market equilibrium to transfer some R&D investments from the early stage of one technological dynasty to its later stage to promote the next technological revolution. Since these transfers improve social welfare, they also indicate that the decentralized economy, together with the R&D activities in it, is overheated after the birth of a new technological dynasty and thereafter recesses too much before the emergence of the next technological dynasty. From (32), we know that the assets price fluctuates as $a(\cdot)$ does, which in return will promote the fluctuation of the household consumption and savings. Here, for the optimal policy, (64) indicates that the introduction of τ counteracts the fluctuation of P and therefore cuts off the propagation channel from the fluctuation of the assets price to that of the savings or consumption.

6. Conclusions

We provide a basic framework to incorporate the technological revolution into the endogenous growth model. We distinguish technological progress into two different scenarios: incremental progress and technological breakthroughs or revolutions. Using the technique of discontinuous optimal control in continuous time, our model allows technology to progress in both patterns, where the former behaves continuously and the latter behaves discontinuously. The main feature of our model is that the discontinuous technological breakthrough endogenizes the business cycle into the growth model. This paper contributes to the existing literature from three perspectives.

First, we propose a new framework to incorporate both endogenous cycles and long-run growth into one model under which the discontinuity problem can be conveniently addressed. We introduce the concept of the cyclic growth path (CGP) for the solution, in which both long-term growth and business cycles are rooted. On the CGP, the economy shows stable growth in the meaning of time average in the long term, which is consistent with the existing mainstream growth models. Furthermore, the growth rates of major economic variables also fluctuate periodically. Different from the RBC model, our framework illustrates that under determined circumstances, uneven technological progress could also generate business cycles in the general equilibrium.

Second, we extend the existing endogenous growth model to show how the leapfrogs of technological breakthroughs and gradual progress interact to induce an abundance of interesting economic dynamics both for crossing cycles and within cycles. In the equilibrium solution of the baseline model, the output is continuous, the consumption ratio jumps downward and the investment ratio jumps upward while crossing the border of the technological dynasty. Within a cycle, the growth rate and the saving rate decrease while the consumption rate increases. The technological breakthrough is exogenous to the decision of a single family; therefore, it always happens too late in the decentralized economy. The solution to the social planner problem clarifies two fundamental effects on the optimal growth path, the "dividend effect" and the "swot-up effect", induced by the discontinuous change of technology. The correct policies can be designed to improve social welfare by balancing the "dividend effect" and "swot-up effect" aside from technological leapfrogs.

Third, our framework can also be used as a workhorse to address dynamic economic problems with discontinuity. We provide related mathematical techniques to address this type of model. The CGP should be treated as a basic solution idea to tackle this type of model, just as with the concept of BGP (balanced growth path) for the classical growth model; and we successfully prove the existence and uniqueness of the CGP solution under mild conditions. Our approach tolerates various forms of technological fluctuation and sudden changes and can be fit to different settings of production functions or economic environments. Moreover, the framework is sufficiently flexible and general to be used to address many other dynamic macroeconomic problems, even any such problems with abrupt changes, including discontinuous policy adjustment, and saltatory change after some threshold such as new infrastructure opening up based on long-term investment.

Although our model is deterministic, there are some evident approaches to extending it to the stochastic environment. We can incorporate the stochastic elements into the incremental technological progress, with almost no further modification, similar to those in the classical endogenous growth literature wherein the total technological stock progress is still deterministic, resulting in combining numerous individual stochastic innovations. We can also replace the technological leapfrogs with the jumps of the probabilities of occurrences of technological breakthroughs, where we speculate that we have to lose some transparency of the results, as the cost of randomization.[18] Another distinct shortcoming of this paper lies in the lack of a formal empirical test for the main results. We let it on the future research agenda and provide some simulations in the appendix.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Funding

This work was supported by the National Natural Science Foundation of China and the National Social Science Foundation of China.

Appendix A–D. Optimal control with corners, an extended model, proofs and simulations

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.jmateco.2022.102774.

References

Acemoglu, Daron, 2009. Introduction to Modern Economic Growth. Princeton University Press.
Acemoglu, Daron, 2012. Introduction to economic growth. J. Econom. Theory 147 (2), 545–550.
Acemoglu, Daron, Cao, Dan, 2015. Innovation by entrants and incumbents. J. Econom. Theory 157 (C), 255–294.
Acemoglu, Daron, Robinson, James A., 2001. A theory of political transitions. Amer. Econ. Rev. 91 (4), 938–963.
Aghion, Philippe, Howitt, Peter, 1992. A model of growth through creative destruction. Econometrica 60 (2), 323–351.
Aghion, Philippe, Howitt, Peter, 1994. Growth and unemployment. Rev. Econom. Stud. 61 (3), 477–494.
Aghion, P., Howitt, P., 1998. Endogenous Growth Theory. MIT Press Books.
Andolfatto, D., Macdonald, G., 1998. Technology diffusion and aggregate dynamics. Rev. Econ. Dyn. 1.
Ayres, Robert U., 1990. Technological transformations and long waves. Technol. Forecast. Soc. Change 37 (1), 1–37.
Beaudry, Paul, Galizia, Dana, Portier, Franck, 2020. Putting the cycle back into business cycle analysis. Amer. Econ. Rev. 110 (1), 1–47.
Boucekkine, Raouf, Pommeret, Aude, Prieur, Fabien, 2013. Optimal regime switching and threshold effects. J. Econom. Dynam. Control 37 (12), 2979–2997.
Boucekkine, Raouf, Prieur, Fabien, Puzon, Klarizze, 2016. On the timing of political regime changes in resource-dependent economies. Eur. Econ. Rev. 85, 188–207.
Bresnahan, T.F., Trajtenberg, M., 1995. General purpose technologies: Engines of growth? J. Econom. 65, 83108.
Canton, Erik, 2002. Business cycles in a two-sector model of endogenous growth. Econom. Theory 19 (3), 477–492.
Francois, Patrick, Lloyd-Ellis, Huw, 2003. Animal spirits through creative destruction. Amer. Econ. Rev. 93 (3), 530–550.
Francois, Patrick, Roberts, Joanne, 2003. Contracting productivity growth. Rev. Econom. Stud. 70 (1), 59–85.
Francois, Patrick, Shi, Shouyong, 1999. Innovation, growth, and welfare-improving cycles. J. Econom. Theory 85 (2), 226–257.
Freeman, Scott, Hong, Dong-Pyo, Peled, Dan, 1999. Endogenous cycles and growth with indivisible technological developments. Rev. Econ. Dyn. 2 (2), 403–432.
Freeman, Chris, Louça, Francisco, 2001. As Time Goes By : From the Industrial Revolutions to the Information Revolution. Oxford University Press.
Grossman, Gene M., Helpman, Elhanan, 1991. Quality ladders in the theory of growth. Rev. Econom. Stud. 58 (1), 43–61.
Growiec, Jakub, Schumacher, Ingmar, 2013. Technological opportunity, long-run growth, and convergence. Oxf. Econ. Pap. 65 (2), 323–351.
Guidolin, Massimo, La Jeunesse, Elizabeth A., 2007. The Decline in the US Personal Saving Rate: is It Real and Is It a Puzzle?, Vol. 89 Federal Reserve Bank of St. Louis Review.
Haunschmied, Josef, Kovacevic, Raimund M, Semmler, Willi, Veliov, Vladimir M, 2021. Dynamic Economic Problems with Regime Switches. Springer.
Helpman, Elhanan, 1998. Introduction. In: General Purpose Technologies and Economic Growth. MIT Press.
Helpman, Elhanan, Trajtenberg, Manuel, 1994. A Time to Sow and a Time to Reap: Growth Based on General Purpose Technologies. National Bureau of Economic Research.
Hull, David G., 2013. Optimal Control Theory for Applications. Springer Science & Business Media.
Kuznets, Simon, Murphy, John Thomas, 1966. Modern Economic Growth: Rate, Structure, and Spread, Vol. 2. Yale University Press New Haven.
Lipsey, Richard G., Bekar, Cliff, Carlaw, Kenneth, 1998. What requires explanation? In: General Purpose Technologies and Economic Growth. MIT Press, Cambridge, Mass. [u.a.], pp. 15–54.
Lipsey, R.G., Carlaw, K.I., Bekar, C.T., 2005. Economic transformations: general purpose technologies and long term economic growth.
Liu, Taoxiong, Zhou, Bihua, 2015. Defense strategy transition and economic growth under external predation. Def. Peace Econ. 26 (3), 289–309.
Martimort, David, Verdier, Thierry, 2004. The agency cost of internal collusion and schumpeterian growth. Rev. Econom. Stud. 71 (4), 1119–1141.
Matsuyama, Kiminori, 1999. Growing through cycles. Econometrica 67 (2), 335–348.
Mokyr, Joel, 1990. The Lever of Riches: Technological Creativity and Economic Progress. Oxford University Press, New York.
Pavlov, Oscar, Weder, Mark, 2017. Product scope and endogenous fluctuations. Rev. Econ. Dyn. 24, 175–191.
Romer, Paul M., 1986. Increasing returns and long-run growth. J. Polit. Econ. 94 (5), 1002–1037.
Romer, Paul M., 1987. Growth based on increasing returns due to specialization. Amer. Econ. Rev. 77 (2), 56–62.
Romer, Paul M., 1990. Endogenous technological change. J. Polit. Econ. 98 (5, Part 2), S71–S102.
Rosenberg, Nathan, Nathan, Rosenberg, 1982. Inside the Black Box: Technology and Economics. cambridge University Press.
Rosenberg, Nathan, Nathan, Rosenberg, 1994. Exploring the Black Box: Technology, Economics, and History. Cambridge University Press.
Sanchez-Choliz, Julio, Fatas-Villafranca, Francisco, Jarne, Gloria, Perez-Grasa, Isabel, 2008. Endogenous cyclical growth with a sigmoidal diffusion of innovations. Econ. Innov. New Technol. 17, 241–268.
Schumpeter, Joseph, 1927. The explanation of the business cycle. Economica (21), 286–311.
Schumpeter, J.A., 1974. Capitalism, Socialism and Democracy. Unwin University Books, London.
Shleifer, Andrei, 1986. Implementation cycles. J. Polit. Econ. 94 (6), 1163–1190.
Solow, Robert M., 1956. A contribution to the theory of economic growth. Q. J. Econ. 70 (1), 65–94.
Umezuki, Yosuke, Yokoo, Masanori, 2019. A simple model of growth cycles with technology choice. J. Econom. Dynam. Control 100 (C), 164–175.
Wälde, Klaus, 2005. Endogenous growth cycles. Internat. Econom. Rev. 46 (3), 867–894.

[18] If the technological breakthroughs are stochastic, conditions (15) and (16) should be $\mathbb{E}\lambda(t_n^+) = \mathbb{E}\lambda(t_n^-)$ and $\mathbb{E}\mathcal{H}(t_n^+) = \mathbb{E}\mathcal{H}(t_n^-)$, respectively.