Theoretical framework

Riccardo Dal Cero

February 29, 2024

Abstract

The main idea is to study how and whether the asymmetry of information have an impact on the cleansing effect of recession, replicating the model in computer simulation.

1 Introduction

In macroeconomic theory, the investigation of business cycles and long-term growth trajectories traditionally unfolds in distinct academic silos, drawing a parallel to the distinct realms of quantum mechanics and Einstein's theory of relativity in physics. This academic segregation, however, obscures a fundamental and profound question: How do business cycles influence long-term economic growth? The exploration of this question is more than an academic exercise; it underpins the practical understanding of short-term economic policies, such as automatic stabilizers, and their profound long-term impacts on the economy.

Embarking on this exploration, my research primarily dwells in the realm of theory, supplemented by rigorous simulation and calibration exercises. The intricate complexity of business cycles, particularly evident during periods of economic downturn and recovery, challenges empirical approaches due to the plethora of confounding variables. Thus, a theoretical lens, rather than a purely empirical one, is employed to dissect and understand these phenomena.

Central to this theoretical framework is an examination of the role of financial market frictions during economic recessions. A key inquiry here is the investigation of policy interventions, such as demand stabilizers, and their potential effect in attenuating the 'cleansing effect' of recessions. This exploration is pivotal in understanding whether such policies might inadvertently lead to a reduced economic baseline or steady state in the long term.

The conceptual foundation of this investigation is inspired by an ecological analogy the cyclical dynamics observed between predator and prey populations in nature. This natural cycle, when observed over extended periods, reveals not just self-contained oscillations but also underlying trends of population growth for both predators and prey. This observation leads to a compelling analogy for economic cycles: while they appear as short-term fluctuations, they might be underpinned by long-term growth trajectories.

In natural ecosystems, interventions aimed at stabilizing these cycles such as protecting prey during times of increased predation might seem beneficial in the short term. However, such interventions often neglect the critical and natural process of selection. This interference disrupts evolutionary mechanisms, potentially leading to unforeseen consequences over time, such as the propagation of traits detrimental to the species' survival and adaptability in changing environments.

My thesis extends this analogy to the economic sphere, positing a similar selective mechanism at play in economic systems. The primary focus is on the recession's cleansing effect, which might be analogous to natural selection in ecology. This effect could potentially 'weed out' less productive firms, leaving a market landscape dominated by more efficient players. The exploration aims to decipher whether such an economic 'natural selection' mechanism exists and, if so, how it shapes the fabric of productivity, innovation, and growth in the long term. Through this lens, the research endeavors to contribute a nuanced understanding of the intricate interplay between short-term economic fluctuations and long-term economic evolution, offering insights into the design and implications of economic policies. In the following sections, we will delve deeper into specific theories that bridge the gap between business cycles and long-term economic growth. However, it is beneficial first to embark on a brief historical journey through the evolution of thought regarding business cycles, to understand the context and development of these interconnected economic theories. This exploration will provide a foundation for appreciating the diversity of perspectives and the progression of ideas that have shaped our understanding of the intricate relationship between short-term economic fluctuations and long-term growth trajectories.

2 Business cycle history

2.1 Hayek Schumpeter Keynes

The exploration of business cycle theories represents a cornerstone in the history of economic thought. A prominent exponent in this realm was Friedrich Hayek, who articulated the complexities of business cycles in relation to economic equilibrium theory. Hayek's perspective is encapsulated in his own words:

> "The incorporation of cyclical phenomena into the system of economic equilibrium theory, with which they are in apparent contradiction remains the crucial problem of Trade Cycle theory; By 'equilibrium theory' we primarily understand the modern theory of the general interdependence of all

> economic quantities, which has been perfectly expressed by the Lausanne School of theoretical economics." Hayek [1933]

In the turbulent era of the early 1930s, the exploration of business cycles was prominently shaped by the contrasting viewpoints of Friedrich Hayek and John Maynard Keynes, two of the twentieth century's most influential economists. Hayek's examination of business cycles, as delineated in his seminal work "Monetary Theory and the Trade Cycle," Hayek [1933], and further elaborated in "The Pure Theory of Capital" Hayek and Caldwell [1941], provides a profound analysis through the lens of monetary theory and its ramifications on capital structure. Hayek posited that economic distortions, notably those stemming from the artificial depression of interest rates by central banks, precipitate malinvestments during periods of economic expansion. These malinvestments, especially prevalent in capital-intensive sectors, were deemed unsustainable, leading inevitably to economic downturns characterized by the correction of these misallocations.

Conversely, Keynes, in his "The General Theory of Employment, Interest, and Money" **?**, approached the business cycle issue from an equilibrium perspective, focusing on the role of aggregate demand in determining overall economic activity levels. Keynes argued that a shortfall in aggregate demand could lead to protracted periods of high unemployment, advocating for active government intervention to stimulate demand and re-establish economic equilibrium.

Hayek's theoretical framework underscores the intricate relationship between capital investment and monetary disequilibrium within the business cycle. He elucidates this connection through the concept of inter-temporal preferences, which dictate the pace of capital investment and the extent of capital accumulation. The decision-making process regarding the allocation of resources between present dividends and future investment is critical to understanding the cyclical nature of the economy. Hayek asserts:

$$I_t = f(r_t, r_n)$$

$$r_t < r_n \rightarrow \text{Malinvestment}$$

where I_t signifies investment at time t, r_t represents the market interest rate, and r_n denotes the natural rate of interest. A divergence between r_t and r_n, particularly when r_t is artificially maintained below r_n, leads to malinvestments. This discrepancy between the market and natural rates of interest, exacerbated by the expansion of bank credit, serves as the cornerstone of Hayek's business cycle theory. Hayek further elaborates on the dynamic and temporal aspects of production and investment in "The Pure Theory of Capital" Hayek and Caldwell [1941], where he critically assesses the equilibrium-based economic theories and emphasizes the importance of understanding the temporal structure of capital.

Hayek's analysis of monetary disequilibrium and its impact on the business cycle is complemented by his insights into the mechanisms of bank credit creation and its influence on the natural state of equilibrium in the market for loanable funds. The expansion of bank credit, which decouples the market rate of interest from the natural rate, instigates cycles of over-investment and mal-investment, ultimately culminating in inter-temporal economic instability.

These theoretical perspectives offered by Hayek provide a nuanced understanding of the complexities inherent in the business cycle, challenging the Keynesian emphasis on aggregate demand and fiscal policy interventions. Hayek's contributions, particularly in "The Pure Theory of Capital," highlight the significance of capital theory in analyzing monetary disequilibria and underscore the dynamic and inter-temporal nature of economic activities.

Thus, while Keynes emphasized stabilizing aggregate demand to achieve equilibrium and mitigate business cycles, Hayek focused on the inherent dynamism and complexity of economic systems, criticizing equilibrium models for their oversimplification of the intricate processes that drive economic activities. This divergence in views marked a fundamental debate in economic theory, shaping the discourse on how economies respond to and recover from periods of economic downturns.

Joseph Schumpeter, another influential economist of the early 20th century, brought a unique perspective to the discussion of business cycles, one that diverged significantly from both Hayek and Keynes. Schumpeter's approach, primarily outlined in his theory of "creative destruction," emphasized the role of innovation and entrepreneurial spirit in economic development and business cycles.

Schumpeter viewed business cycles as inherent and vital to capitalist economies, driven by the process of innovation. According to Schumpeter, the entrepreneur is the agent of change, introducing new technologies, products, and methods, which disrupt existing market equilibria. This process of innovation leads to the destruction of outdated industries and economic structures, paving the way for new ones. In Schumpeter's framework, the cyclical nature of the economy was a reflection of this ongoing process of creative destruction, where periods of economic downturns were seen not just as phases of correction, as Hayek might argue, or as failures of demand, as per Keynes, but as essential for clearing away the old to make way for the new.

While Hayek focused on the distortions in capital structure caused by monetary interventions and Keynes emphasized the role of aggregate demand and government intervention in stabilizing the economy, Schumpeter's perspective highlighted the evolutionary nature of capitalist economies. He argued that economic fluctuations were natural and necessary, a process through which economies evolve and adapt over time.

Schumpeter's theory thus provided a more dynamic view of capitalism, recognizing the disruptive yet progressive role of innovation and entrepreneurship in shaping economic landscapes.

Schumpeter's contribution added a dimension of evolutionary change to the understanding of business cycles, contrasting with Hayek's emphasis on monetary theory and capital structure, and Keynes's focus on equilibrium and aggregate demand. Schumpeter's insights into the transformative power of innovation offered a more optimistic view of economic downturns, seeing them as opportunities for new growth and advancements, the predecessor of the idea of cleansing effect.

2.2 Theories Connecting Business Cycles to Long-Term Growth

In the domain of economic theory, the relationship between business cycles (BC) and long-term growth is dissected into two principal schools of thought. The conventional viewpoint suggests that long-term growth is chiefly propelled by technological progress. Within this framework, technological advancements are often viewed as exogenous—arising outside the economic model's explanatory scope, as highlighted in the seminal contributions of Solow [1956] and Swan [1956]. This perspective treats technological progress as an independent variable that exerts influence on economic growth without being influenced by the economy's internal dynamics.

Contrastingly, an alternative strand of theoretical work aims to endogenize technological progress, weaving it into the fabric of the economic process. These models embed factors such as incentives for innovation, the value of education, and the accumulation of knowledge through economic activities, epitomizing the 'learning by doing' paradigm. A prominent example of this approach is found in Stadler [1990], which posits technological progress as an emergent property of economic behavior and incentives, rather than a mysterious external force.

A critical aspect of the ’learning by doing’ model is its premise that technological frontiers are contingent upon the existing knowledge base, which expands primarily through practical experience. Consequently, periods of economic expansion witness a sharp increase in the knowledge stock, driven by higher employment levels, whereas recessions tend to stabilize or even diminish this stock due to reduced employment rates. This dynamic suggests that economies devoid of cyclical fluctuations might attain superior steady-state growth, as employment remains consistently high, fostering continuous technological advancement. From this perspective, the concept of a ’cleansing effect’—whereby economic downturns eliminate low-productivity jobs and ostensibly strengthen the economy—is challenged. The elimination of even low-productivity roles can erode the overall knowledge base.

Such theories reframe the discourse on stabilization policies, particularly fiscal interventions, by highlighting their role in sustaining employment and, by extension, supporting the technological frontier even in downturns.

To illustrate this theory’s implications more vividly, consider a nuanced example: an antiquated factory with limited land resources discovers an innovative method to utilize an old tractor more efficiently. Despite the ingenuity of this breakthrough, if the broader economy has moved beyond the technology that the tractor represents, the innovation might not significantly contribute to the overall knowledge stock or push the technological frontier forward. This scenario prompts a fundamental inquiry: does innovation at the lower end of the skill spectrum or within outdated technological contexts meaningfully advance the technological frontier? Or, would it be more beneficial for economic growth to transition such small-scale innovations into larger entities equipped with modern technologies?

One significant critique concerns the disparity in learning opportunities across different sectors and among individuals. The model’s premise of uniform learning opportunities does not always align with the reality that some industries, such as the technology sector, offer rapid innovation and learning environments compared to more traditional manufacturing industries, where the pace of learning and innovation may be inherently slower due to the nature of the work processes involved.

Furthermore, the model may not adequately address the issue of structural unemployment that can arise from technological advancements. As certain workers benefit from "learning by doing," leading to increased productivity, others may find their skills becoming obsolete due to automation and other technological changes. This dynamic is evident in the automation of routine manufacturing jobs, which, while fostering "learning by doing" in fields like robotics and software engineering, simultaneously leads to structural unemployment for workers displaced by these technologies.

Another point of contention is the potential for diminishing returns to learning. The assumption that "learning by doing" continuously fuels growth may not hold up against the reality that initial rapid gains in productivity tend to taper off as workers gain proficiency, suggesting that the benefits of learning may diminish over time.

The model also potentially overlooks the externalities and spillover effects associated with "learning by doing." Technological advancements in one firm or sector do not automatically translate into broader economic growth if these advancements remain isolated and do not benefit other sectors or industries. This is illustrated by a software company that develops cutting-edge algorithms, enhancing its productivity but failing to contribute to the wider economy if the knowledge remains proprietary.

This nuanced exploration challenges the simplistic notion of 'learning by doing' by questioning the value and impact of incremental innovations within the broader economic and technological ecosystem. It underscores the complexity of technological progress and its interplay with economic dynamics, inviting a deeper investigation into the mechanisms that drive long-term growth and the role of policy in nurturing an environment conducive to continuous innovation and knowledge expansion.

The contemporary perspective on technological advancement, when viewed as a product of incremental contributions from every market participant, appears overly simplistic. A more accurate depiction of technological progress recognizes it as predominantly driven by those at the forefront of research. The expansion of the technology frontier is essentially shaped by the knowledge and breakthroughs of these leading-edge innovators. Other entities in the economy adopt these innovations at varying paces, influenced by the associated adoption costs. While firms that are not at the innovation frontier may achieve marginal efficiency gains through adoption, the impact of such improvements is often minimal. These marginal innovations are frequently a reflection of the adopting firms' constraints, particularly their inability to invest in more advanced and expensive technologies. Consequently, these incremental innovations have limited potential for widespread diffusion, as they stem from a position of necessity rather than pioneering advancement.

Another theory describe a recession as a period in which the opportunity cost of investing in a productivity enhancing projects is lower since the workforce is not fully in demand to produce goods. Doing this would lead in theory to higher productivity in the period of expansion. The key role here is that the productivity-enhancing activity is costly and thus divert capital and labor force from production as Hayek Hayek [1933] explained. For this view to be valid two key aspects should be true at the same time: in the first place the expectations about the length of the recession should reinter in the short-term otherwise it is cheaper to destroy some production units (labor and capital) to accommodate the slow in demand. The last condition is that internal resources must be less costly than external ones, however, it would be cheaper to higher more skilled workers and fire the low-skill one. An additional remark is the this theory describes all those initiatives like worker formation that affect only marginally the productivity of a firm. It misses the main mechanics in which a firm can increase its productivity sharply: thorough technical innovation, and to do so you need a research program where the workforce is fully dedicated to it and not diverted from production.

Another theory posits that recessions offer a period in which the opportunity cost of investing in productivity-enhancing projects are lower, primarily because the workforce is not fully engaged in producing goods. Theoretically, this would lead to higher productivity during subsequent periods of expansion. A crucial element in this perspective is the acknowledgment that productivity-enhancing activities are expensive, thereby diverting capital and labor away from immediate production, a concept Hayek Hayek [1933] elucidated.

For this viewpoint to hold, two critical conditions must be concurrently satisfied: firstly, expectations regarding the duration of the recession must be short-term. If the recession is anticipated to be prolonged, it becomes economically viable to dismantle some production units (both labor and capital) to adjust to reduced demand. Secondly, the cost of utilizing internal resources for such productivity-enhancing ventures must be lower than the cost of acquiring external resources. Otherwise, it might be more economical to hire more skilled workers and lay off less skilled ones.

An additional observation about this theory is that it accounts for initiatives like worker training, which only marginally affect a firm's productivity. This overlooks the primary mechanism through which a firm can significantly boost its productivity: through technical innovation. To achieve substantial innovation, a dedicated research program is essential, where the workforce is fully committed to innovation efforts rather than being diverted to current production tasks. This highlights a gap in the theory, suggesting that while reallocating resources during recessions may offer some productivity benefits, the most dramatic productivity improvements are likely achieved through focused innovation and research activities, not merely through the opportunistic reallocation of resources during economic downturns.

An intricate theory that elaborates on the dynamics of economic recessions and the associated lower opportunity costs is grounded in the concept of labor hoarding, as discussed in the seminal work by Clark Clark [1973]. This theory posits that firms maintain employment levels higher than what immediate efficiency metrics might dictate. The rationale behind such a strategy is to prepare for a potential surge in demand, ensuring that the firm can quickly ramp up production without the delays associated with recruiting and training new employees. However, this strategic maneuver towards the internal possibility frontier—where firms optimize their readiness for future demand—does not manifest as observable changes in employment rates. Consequently, this leads to discrepancies or residuals in the statistical series of employment, which do not align with what might be considered the level of optimal employment, a phenomenon further analyzed in the work of Burnside et al. [1993].

This labor hoarding theory offers a partial explanation for the strong pro-cyclically of measured productivity. During economic upturns, firms can immediately respond to increased demand using their hoarded labor, thereby appearing more productive. Conversely, during downturns, the reluctance to shed this excess labor, due to the anticipation of future demand recovery, results in lower observable productivity levels. This behavior underscores a strategic depth in firm management, navigating through the cyclical economic waves by balancing between immediate efficiency and long-term responsiveness.

Expanding on this foundation, it becomes evident that the decision to engage in productivity-enhancing activities during recessions is not merely a reaction to lower opportunity costs but also a strategic consideration influenced by expectations of the recession's duration and the comparative costs of internal versus external resources. If firms anticipate a short-lived recession, the logic of hoarding labor and investing in internal productivity initiatives becomes compelling. However, this strategy hinges on the assumption that improving the skill set of the existing workforce or reallocating resources towards innovation is less costly than the alternative—acquiring new, possibly more skilled labor post-recession.

The opportunity cost (OC) approach closely aligns with the theory of labor hoarding, which seeks to elucidate the pronounced procyclicality of measured productivity. This observation implies that during economic downturns, firms appear to retreat towards the interior of their production possibility frontier, opting for a strategic reduction in operational efficiency rather than workforce downsizing. This strategy is partly attributed to the invisible nature of one crucial input—effort—to statisticians, and the economic rationale that, given the costs associated with employee turnover, it proves more economically viable for firms to dial back effort during slumps instead of terminating employment.

An intriguing alternative to diminishing effort is the redirection of employee tasks from immediate production roles to undertakings that bolster the firm's long-term productivity. At first glance, this strategy bears a resemblance to labor hoarding but carries the added outcome that these so-called shadow activities, embraced during recessions, eventually manifest as enhancements in total factor productivity.

The concept of adjustment costs does not singularly confine firms from adapting their production factors according to operational necessities. This opportunity cost mechanism could theoretically extend to a macroeconomic scale, influencing individual entities via price adjustments. During periods marked by diminished production value, the immediate returns from production activities (e.g., wages for workers) decline in comparison to alternative endeavors, notably human capital accumulation, whose benefits are pegged to future earnings. This economic mechanism could precipitate a resource reallocation towards these alternative activities. The empirical observation that education durations tend to extend during economic recessions lends credence to this argument. Nevertheless, with the exception of leisure, most sectors shadow the movements of aggregate GDP. Thus, if productivity-enhancing activities (PEAs) are to occur during recessions, the resource reallocation process must predominantly unfold within the firms themselves.

One notable deviation might be labor reallocation. As demonstrated by Davis and Haltiwanger [1992], job destruction exhibits a more countercyclical pattern compared to job creation. Viewing job reallocation through the lens of both destruction and creation suggests a countercyclical trend, positing job reallocation as an investment in cultivating superior firm-worker matches, thereby sowing the seeds for heightened productivity in the future. Davis and Haltiwanger [1992]further postulate, within the framework of a model featuring a representative agent, that economic recessions present an optimal window for labor reallocation, highlighting a strategic dimension to workforce management during downturns that might ultimately contribute to long-term productivity gains.

The ”lame ducks” theory, initially proposed by Caballero and Hammour [1994], offers an intriguing perspective on recessions as mechanisms that phase out less profitable, outdated capital. This theory delineates how the destruction of older units during downturns is more pronounced than the construction of new ones, characterizing recessions as periods marked by the systematic elimination of obsolete capital, hence the moniker ”lame ducks” theory. Notably, this framework sheds light on observations documented by Davis and Haltiwanger [1992], positioning it as a prominent theoretical approach that will be delved into more thoroughly in subsequent discussions.

Despite its insights, this model lacks consideration of the financial dimensions of firms, an aspect addressed by the theoretical work of Osotimehin and Pappadà [2017]. Their research weaves the financial decision-making process into the broader context of intertemporal capital decisions, highlighting how financial frictions influence the lender's participation constraint. The study reveals that, despite financial frictions, the cleansing effect of recessions on productivity persists, potentially leading to a more pronounced productivity surge during expansion phases. This analysis serves as a pivotal foundation for the new theoretical framework introduced in this thesis, marking a significant departure from traditional views and emphasizing the multifaceted impact of recessions on firm productivity and economic recovery.

In the forthcoming sections, we will delve into the empirical evidence presented by Davis and Haltiwanger [1992] in their seminal works from 1990 and 1992, which lay the groundwork for our discussion. Following this, we will explore the theoretical underpinnings that form the basis of the new, streamlined theoretical framework introduced in this thesis. Our examination begins with the insights of Caballero and Hammour [1994], focusing on the interplay between the destruction and creation margins in economic cycles. Subsequently, we will delve into the work of Osotimehin and Pappadà [2017] (2017), which sheds light on the financial dimensions, particularly how capital lending frictions can precipitate misallocations. These studies collectively inform the development of our theoretical framework, setting the stage for a comprehensive analysis of economic dynamics and firm behavior during recessions.

3 Literature review of empirical studies

3.1 Emprical findings

This analysis juxtaposes two pivotal studies that dissect business cycle dynamics through the lens of labor market fluctuations and job reallocation. The first, by Davis and Haltiwanger [1992], scrutinizes job reallocation during recessions, employing data from the U.S. manufacturing sector. The second study, by Blanchard et al. [1990], concentrates on labor flows throughout business cycles. Both studies utilize labor data from the Current Population Survey (1968-1986) and manufacturing data from the Federal Bureau of Labor Statistics up to 1981, later complemented by the Longitudinal Research Database. These works collectively illuminate the dynamic interplay of job creation and destruction, offering a nuanced understanding of labor market volatility and its implications for economic cycles. Further exploration will include an analysis of Haltiwanger's findings and their application to understanding the Great Recession's impact on the labor market in his paper Foster et al. [2016]

Reallocation cleansing or not? This study meticulously investigates the variance in employment changes at the establishment level within the U.S. manufacturing sector from 1972 to 1986, focusing on the gross creation and destruction of jobs and the rate of job reallocation across plants. Leveraging an extensive dataset from the Annual Survey of Manufactures, it provides a detailed examination of how job and worker reallocation contribute to understanding employment dynamics and the cyclical nature of the labor market. The findings reveal significant rates of job turnover within specific industry sectors and elucidate the relationship between job reallocation and various plant characteristics such as age, size, and ownership type. Moreover, the research discusses the persistence and concentration of job creation and destruction, highlighting their implications for long-term employment trends and worker mobility. Through analyzing the interplay between establishment-level job flows and broader labor market states, this paper contributes valuable insights into the mechanisms of labor reallocation and the structural factors driving employment heterogeneity in the manufacturing sector.

The key concept behind the Davis and Haltiwanger [1990, 1992] studies is that the measure of reallocation is given by the sum of job or capital creation and destruction. In the first place lets get some definitions that are used in the paper: The analysis in this paper establishes a straightforward link between the gross job flow metrics and the size-weighted distribution of establishment growth rates. Gross job creation is determined by aggregating employment increases in both growing and newly established firms within a sector. In a similar vein, gross job destruction is assessed by totaling employment declines in contracting and ceasing operations. This methodological approach allows for a comprehensive evaluation of employment dynamics, highlighting the pivotal role of establishment size and growth in understanding sectoral labor market fluctuations. To measure Total Factor Productivity (TFP) at the establishment level, they construct an index following the methodology similar to Baily et al. [1992] and subsequent studies. The formula for the index is:

$$\ln(\text{TFP}_{et}) = \ln(Q_{et}) - a_K \ln(K_{et}) - a_L \ln(L_{et}) - a_M \ln(M_{et})$$

In this equation, Q represents real output, K stands for real capital, L denotes labor input, and M signifies real materials used, with a representing the factor elasticities. The subscript e refers to individual establishments, while t denotes time, allowing for a dynamic analysis of productivity changes across establishments. The methodology for measuring Total Factor Productivity (TFP) at the establishment level involves detailed component measurements. Nominal output is calculated by summing total shipments with inventory changes, adjusted by industry-specific deflators from the NBER-CES Manufacturing Industry Database. Capital is quantified using the perpetual inventory method for structures and equipment, while labor input comprises total hours worked by both production and nonproduction workers. Materials, including physical materials and energy, are measured and deflated by industry-specific indices, with all values expressed in 1997 constant dollars. Factor elasticities are determined using industry-level cost shares, with adjustments made to account for industry differences in TFP calculations, ensuring comparisons control for sectoral disparities. The first question they try to address is Did Reallocation Dynamics Change in the great recession? The graph also depicts the fluctuations in the unemployment rate, clearly demonstrating that periods of rising unemployment are typically marked by increased job losses and decreased job creation. However, the trend took a notable turn during the Great Recession. Specifically, while job losses did surge in the 2008-2009 period, the most remarkable change was the significant drop in job creation beginning in 2007 and extending through 2010. Additionally, there is an observable long-term decline in job flow dynamics, a topic for further discussion.

To corroborate these findings and delve into more granular data, they utilized the Business Employment Dynamics (BED) statistics for the U.S. private sector. The BED's quarterly data from the second quarter of 1990 to the first quarter of 2012 (Panel B of the figure) underscores the annual trends, showing that recessions typically involve higher job destruction and lower job creation rates. The period starting in 2007 stands out for a particularly steep decline in job creation, a trend that is more pronounced in the BED data. The BED series also highlights that the sluggish recovery from the Great Recession up to early 2012 is primarily attributable to weak job creation rather than enduringly high job destruction rates. This pattern, confirmed by other datasets such as the Job Openings and Labor Turnover Survey (JOLTS), persisted beyond the first quarter of 2012.

Adding to this, job creation during the Great Recession was as low as it has been at any time in the past 30 years, as illustrated in the figure. Moreover, job reallocation (the sum of job creation and destruction) reached its lowest point in 30 years during the Great Recession and its immediate aftermath. When comparing the Great Recession to the early 1980s recession, the rate of job reallocation was 28* in 2009, in stark contrast to 35* in 1983 (both periods represent peaks in job destruction and are measured using March-to-March Business Dynamics Statistics data). These patterns are partly driven

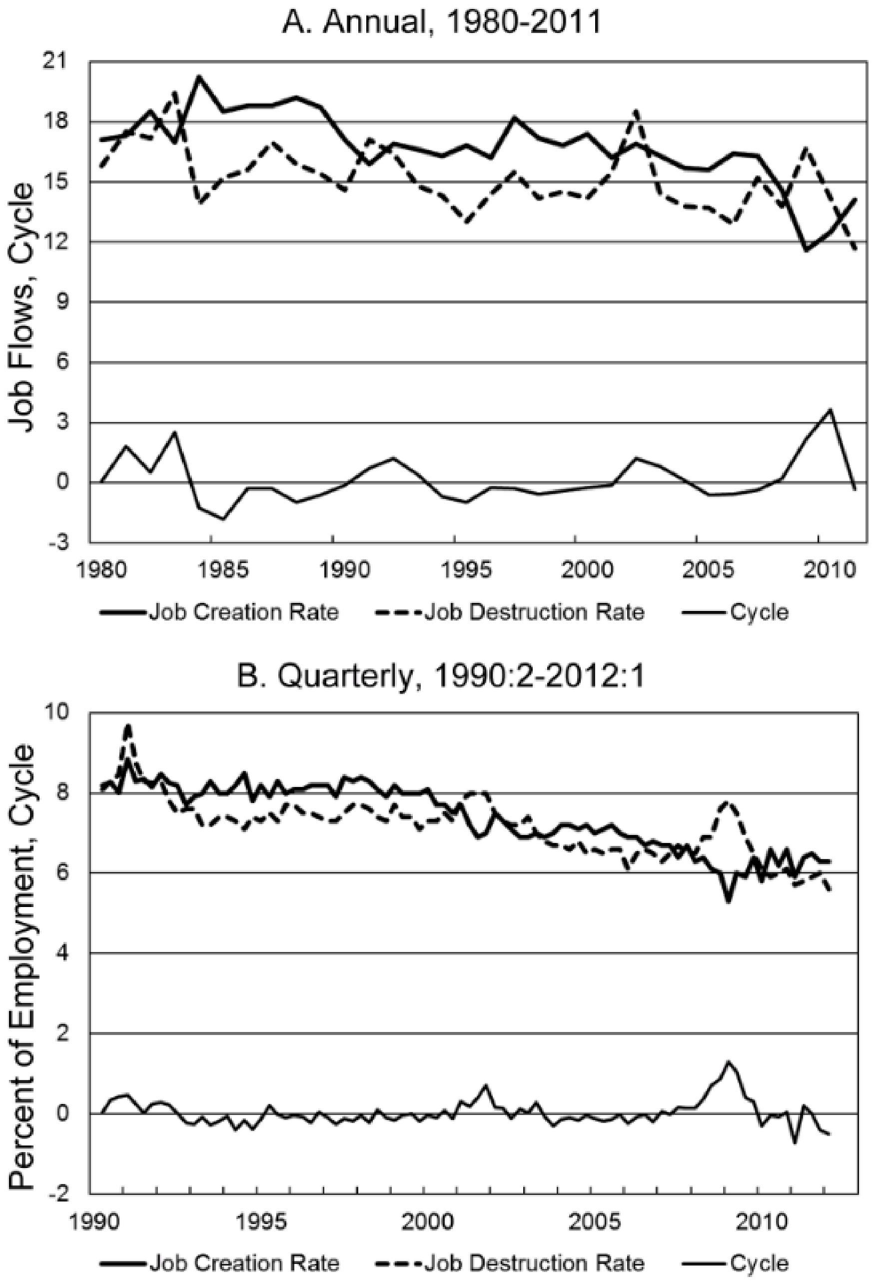


Figure 1: Figure Job flows and the business cycle. Authors' calculations using Business Dynamics Statistics (annual), Business Employment Dynamics (quarterly), and the Current Population Survey. Cycle is the change in the unemployment rate.

by significant downward trends in job flows, as evidenced in both the Business Dynamics Statistics (BDS) and the BED data. Although exploring the causes of these declining trends in job flows is beyond the scope of this analysis, it is clear that these trends are significant and have been taken into account in the analysis.

Moreover, the authors show how the patterns change considerably in the great recession, particularly regarding how the creation and destruction rate of jobs respond to the sharp fall in demand. They found as expected from Caballero and Hammour [1994] that the creation rate of new jobs slower before the crisis spreads out and than the destruction rate increases when it is not possible to fully accommodate the decrease in demand only stopping new labour hiring. They asses that during the Great Recession this pattern changes in particular the fraction of net employment contraction accounted by a job reduction is higher than 0.5, while for all the previous recessions it was way below 0.4 as shown in the table.

Table 1: Share of Change in Net Employment Growth Due to Change in Job Creation in Periods of Net Contraction

	National		State
Period	BDS (Annual)	BED (Quarterly)	BDS (Annual)
Pre-Great Recession	.21	.28	.39
Post-2007	.61	.59	.65

This analysis is based on the authors' computations using data from the Business Dynamics Statistics (BDS) and Business Employment Dynamics (BED). The methodology hinges on the principle that net employment change equals job creation minus job destruction. For any period(s) of net employment decrease lasting at least one period, both the total change in net employment growth and the total change in job creation are aggregated over the full duration of consecutive net contraction periods.

Table 2: Job Flows and Change in the Unemployment Rate at the State-Level (Annual), 1981-2011

	Job Creation Rate	Job Destruction Rate	Reallocation Rate
Cycle	$-.631***$ (.046)	$1.194***$ (.053)	$.563***$ (.068)
GR × cycle	$-.371***$ (.079)	$-.421^{*}**$ (.079)	$-.793***$ (.128)
Trend	$-.168***$ (.010)	$-.136***$ (.011)	$-.304***$ (.020)

1

Furthermore, these aggregated changes are then further combined within the periods outlined in the analysis. The proportion mentioned refers to the fraction of the total aggregated change in net employment growth during the specified timeframe that is attributed to the total change in job creation within the same timeframe. Specifically, the BDS data spans from 1981 to 2007 for the pre-Great Recession era and from 2008 to 2011 for the post-2007 era. For the BED, the pre-Great Recession period covers from the second quarter of 1990 to the third quarter of 2007, and the post-2007 era from the fourth quarter of 2007 to the first quarter of 2012. It's important to note that these calculations are applied solely to periods experiencing a net decrease in employment growth. For instance, this applies to the period from the fourth quarter of 2007 to the first quarter of 2010 for the BED data. Regarding the BDS National Annual data, there are six years of net employment contraction, with two of those years occurring after 2007. In the BED Quarterly data, there are twenty-two quarters of net contraction, with nine quarters following 2008. For the BDS State Annual data, there are 393 state-year instances of net contraction, with 112 of those instances occurring after 2007.

Another way to see this change in patterns is regressing job creation rate, job destruction rate, and Reallocation rate to Cyle an iteration between a Great Recession dummy and the cycle and a trend. The results are shown in the table 3. The authors delve into the nuances of state-level job flow dynamics by leveraging the variation in the relationship between cyclical economic indicators and job flows across different states. Their methodological approach is rooted in conducting descriptive regressions that not only link job flows to a selected cyclical indicator but also incorporate an interaction term that captures the distinctive economic conditions of the Great Recession period alongside the cyclical variable. To achieve this, they meticulously utilize changes in the unemployment rate at the state level as a proxy for economic cycles. Recognizing the persistent negative trend in job flows across the board, the authors judiciously incorporate a linear trend into their regression models to adequately account for this overarching pattern.

The findings from these regressions are illuminating, revealing that cyclical fluctuations predominantly explain a substantial portion of the variance observed in job flows, particularly with respect to the job destruction rate. This observation is in line with the theoretical framework posited by Caballero and Hammour [1994], which suggests that the margin of job destruction exhibits a higher sensitivity to cyclical downturns compared to the job creation margin. A striking aspect of their analysis is the significant and negative interaction observed between the cyclical variable and the Great Recession dummy variable, indicating that this period was characterized by a pronounced decrease in the rate of new job openings, whereas the rate of job layoffs was not as severely affected as it had been in other recessions. This pattern underscores the profound and distinctive impact of the Great Recession, which led to a dramatic downturn in job creation, surpassing the declines observed in job destruction rates and deviating markedly from the trends seen in prior economic downturns.

Furthermore, the analysis sheds light on the behavior of the job reallocation rate during the Great Recession, which, in a departure from its traditionally pro-cyclical nature observed in earlier recessions, switched signs, indicating a contraction in job reallocation. This reversal is particularly noteworthy as it signifies a shift towards less job reallocation, thereby challenging the conventional understanding of job flow dynamics during economic downturns.

An important contribution to this discourse comes from studies that emphasize the significant role of young businesses in the observed decline in job creation rates, particularly highlighted in the research by Fort et al. [2013]. These young businesses, defined as entities with less than five years of operation, exhibit a markedly higher responsiveness in their job creation margin, especially in the aftermath of the 2007 financial crisis. This responsiveness is contrasted with the behavior of older firms, which, although also experiencing shifts in their job creation and destruction margins, do not demonstrate the same level of sensitivity as their younger counterparts. The distinction between young and old firms becomes even more pronounced when examining the destruction margin, where young firms again show greater responsiveness, underscoring the differential impact of economic cycles based on firm age. The second question that the authors try to address is: Did the Cleansing Effect change during the Great Recession? To address this question the authors use a regression that examines the relation between the growth and survival dynamics of the incumbent establishment to productivity. The specification that they imply is the following The specification is defined as follows:

$$Y_{es,t+1} = \lambda_s + \lambda_{t+1} + \beta(\text{TFP}_{es,t}) + \gamma(\text{Cycle}_{s,t+1}) + \delta(\text{TFP}_{es,t} \times \text{Cycle}_{s,t+1}) + X'_{es,t}\mathbf{V} + \epsilon_{es,t+1};$$

where e represents the establishment, s denotes the state, and Y encompasses a series of outcomes. TFP refers to the deviations in total factor productivity from the average within each industry by year, and Cycle represents the variation in the state-specific unemployment rate from time t to $t+1$. The model assesses three distinct outcomes (all calculated from t to $t+1$): "Overall Growth" (combining continuing operations and exits), "Exit," and "Conditional Growth" (considering only those that continue,

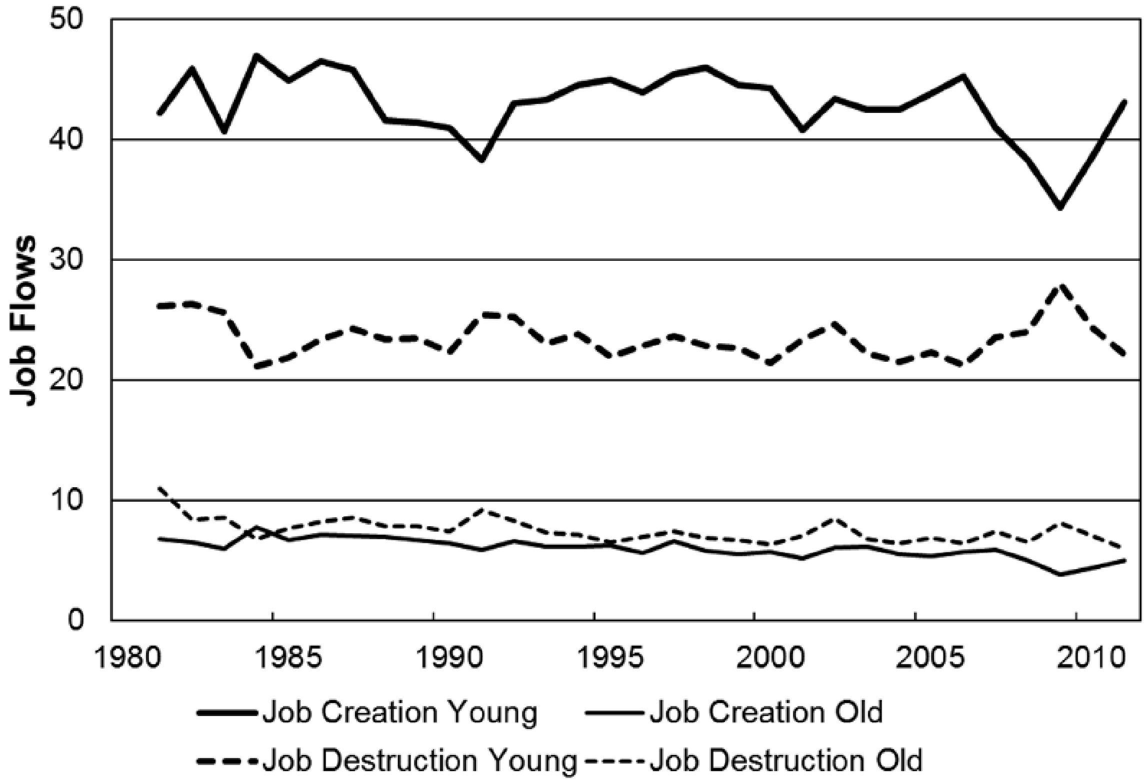


Figure 2: Job flows by age, 1981-2011. Authors' calculations on Business Dy- namics Statistics. Young is for establishments owned by firms less than 5 years old. Mature is for establishments owned by firms 5 or more years old. Job flows are establishment-based and are classified by firm age characteristics.

i.e., no exiters). They used data from 1981-2010 controlling per year and state effects. In order to assess if the Great Recession was different from previous recessions the added iteration terms with a dummy variable representing the Great Recession. The specification became the following: The refined model is expressed as:

$$Y_{es,t+1} = \lambda_s + \lambda_{t+1} + \beta(\mathrm{TFP}_{es,t}) + \gamma(\mathrm{Cycle}_{s,t+1}) + \delta(\mathrm{TFP}_{es,t} \times \mathrm{Cycle}_{s,t+1}) + \xi(\mathrm{GR}_{t+1} \times \mathrm{TFP}_{es,t}) + \mu(\mathrm{GR}_{t+1} \times \mathrm{Cy}$$

where GR signifies a dummy variable for the Great Recession, assigned a value of 1 during the years 2007 to 2009. This the model takes into account the impact of the Great Recession by including interaction terms between the Great Recession dummy (GR), changes in the state-specific unemployment rate (Cycle), and deviations in total factor productivity (TFP) from the industry-year averages. The results are shown in the following table

Table 3: Reallocation and Productivity over the Business Cycle

	Overall Gr.Rat. (Cont + Exit)		Exit		Cond. Gr.Rat.(Cont.)	
	prova (1)	(2)	(3)	(4)	(5)	(6)
TFP	.157***	159***	−.060***	−.060**	.041**	.042*
	(.006)	(.006)	(.003)	(.003)	(.003)	(.003)
Cycle	−3.307**	−2.961*	.671*	.497*	−2.143**	−2.128
	(.459)	(.483)	(.176)	(.179)	(.247)	(.286)
TFP × cycle	1.542***	2.182***	−.655***	−.927**	.494	.534
	(.643)	(.862)	(.226)	(.265)	(.412)	(.567)
GR × TFP		.030		−.018*		−.005
		(.023)		(.011)		(.011)
GR × cycle		−3.116*		1.581		−.126
		(1.349)		(.523)		(.770)
GR × TFP × cycle		−2.961*		1.466***		.066
		(1.619)		(.684)		(.764)
Year FE	Yes	Yes	Yes	Yes	Yes	Yes
State FE	Yes	Yes	Yes	Yes	Yes	Yes
Firm size class FE	Yes	Yes	Yes	Yes	Yes	Yes
N (millions)	2.2	2.2	2.2	2.2	2.1	2.1

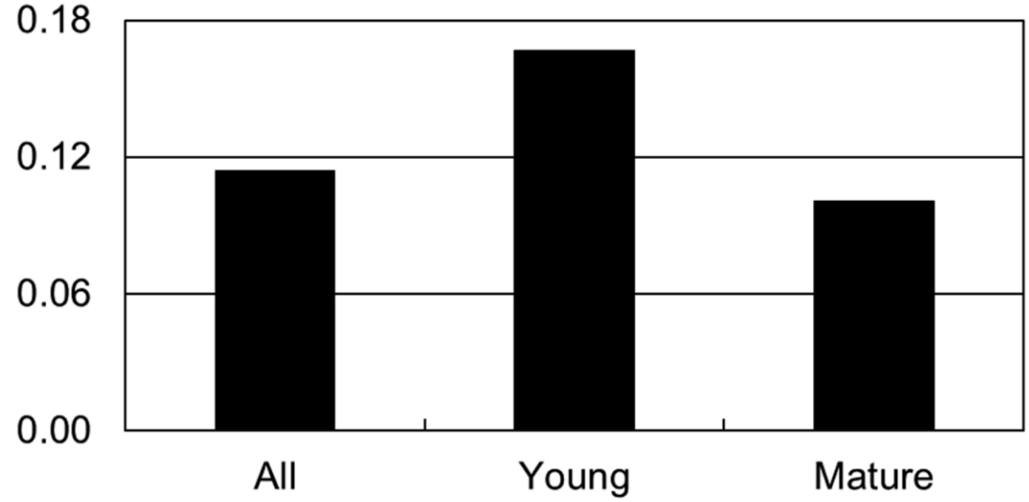
[2] [3]

The regression analysis presented in the table offers insightful observations on how the Great Recession exacerbated the effects on overall growth rates compared to other periods of economic downturns. A critical aspect to highlight is the interaction between the Great Recession dummy variable (GR) and the cycle, which denotes changes in the state-year unemployment rate. This interaction is particularly important and statistically significant, with one coefficient significant at the 5% level and the other at the 1% level. This suggests that the economic dynamics during the Great Recession had a distinct and profound impact on reallocation and productivity across businesses.

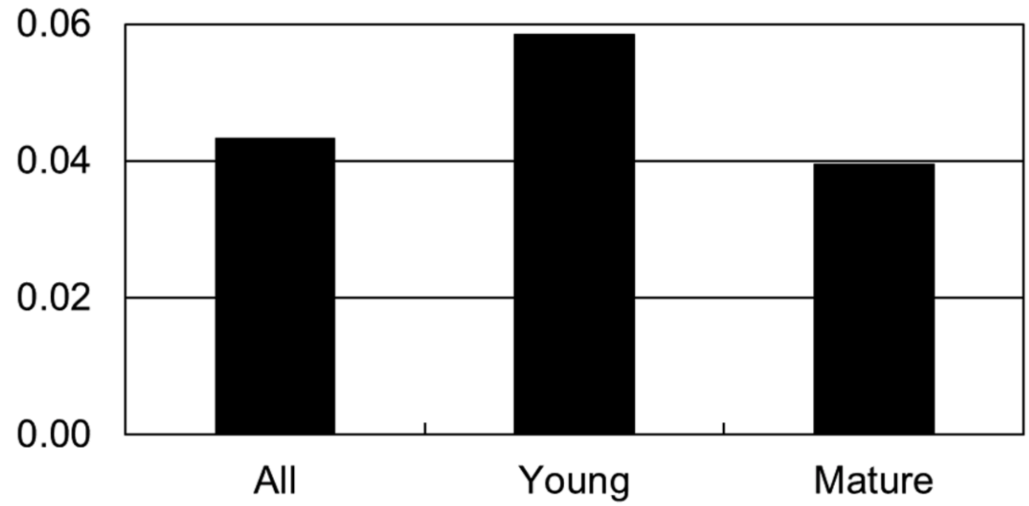
The coefficient for the GR $\times$ cycle interaction being significant implies that the Great Recession intensified the negative effects of the business cycle on overall growth rates. This finding is notable because it underscores the unique severity of the Great Recession, distinguishing it from previous economic downturns. Specifically, the negative coefficient indicates that during the Great Recession, the adverse effects of an economic downturn on job creation and overall business growth were more pronounced. This could be attributed to a range of factors, including tighter credit conditions, greater uncertainty, and possibly more significant structural shifts in the economy that affected businesses more severely during this period.

Moreover, the presence of significant interaction terms involving TFP (total factor productivity), the cycle, and the Great Recession dummy highlights the complex relationship between productivity, economic cycles, and major economic crises. The analysis suggests that the Great Recession not only impacted the rate of job creation and destruction but also influenced how productivity interacts with the business cycle to affect economic outcomes.

To have a better comprehension the authors use the regression coefficients to estimate the difference in the outcomes between an establishment firm with one standard deviation above and below the industry year mean, differing between older and younger firms the results are depicted in the following plot. From the graph, it is evident that there is an 11 percentage point difference in overall growth between firms that are one standard deviation more productive than the average and those one standard deviation below the mean. When distinguishing between mature and younger firms, it becomes apparent that younger firms are more responsive to productivity differences. Specifiht A. Overall Growth(Cont. + Exiters)



ht B.Exit Rates



ht C.Conditional Growth (Continuers only)

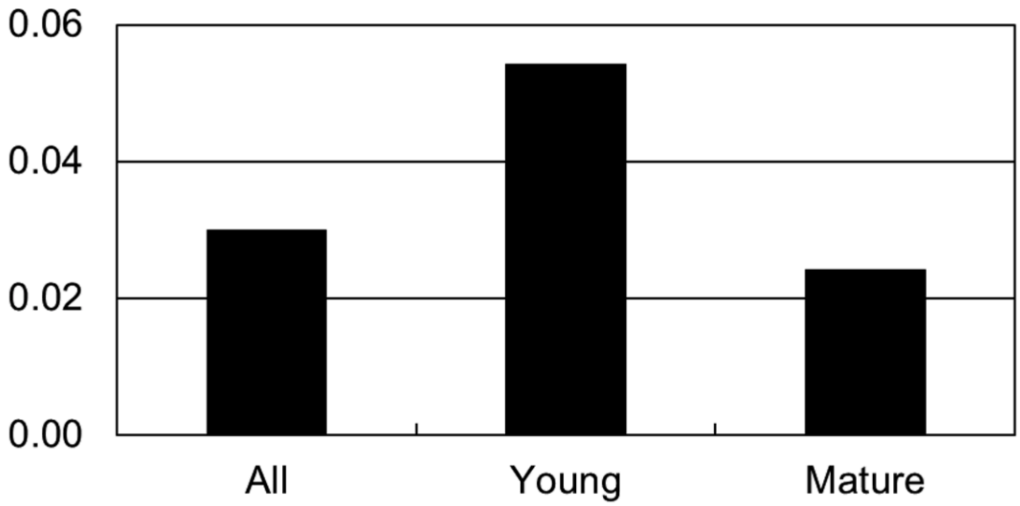
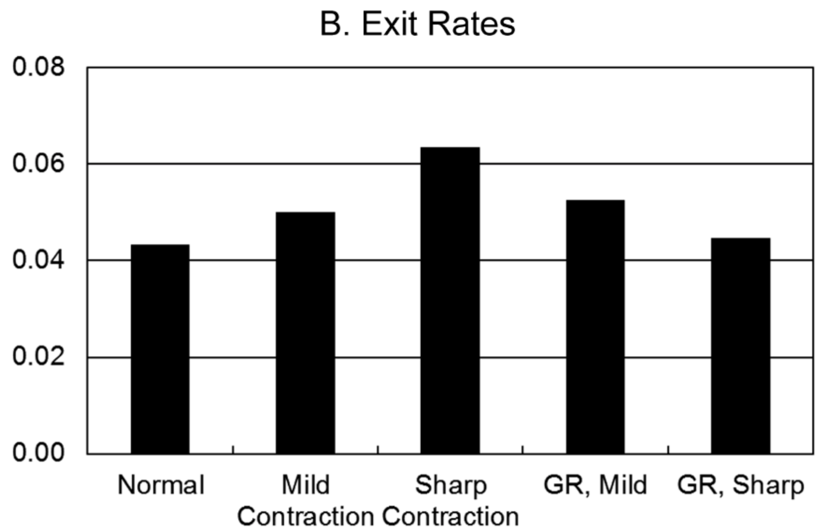


Figure 3: Differences in growth rates between high-productivity and low- productivity establishments, normal times. Authors' calculations on Annual Survey of Manufactures, Census of Manufactures, and Longitudinal Business Database. Depicted are the predicted difference in growth rates (panels A and C, high minus low) and the predicted difference in probability of exit (panel B, low minus high) between an establishment one standard deviation above industry-by-year mean productivity and an establishment one standard deviation below industry-by-year mean productivity. Normal is zero change in state-level unemployment.

cally, younger firms that are more productive experience an overall growth that is 18 percentage points higher compared to less productive counterparts. In contrast, this relationship between productivity and growth is weaker for mature firms, where those with higher productivity only see an overall growth of 0.11 percentage points more than their less productive peers. The initial graph encompasses all firms, including those that survived the Great Recession and those that exited.

Focusing on exit rates, being more productive offers a smaller advantage in survival rate probabilities. While the patterns are consistent with the previous graph, the benefit of higher productivity is reduced across all categories. Notably, younger firms with higher productivity have a 6 percentage point greater chance of survival compared to their less productive counterparts, whereas, for mature firms, this productivity premium is reduced by 2 percentage points.

Looking exclusively at no-exiters, the growth rate is significantly less influenced by productivity, showing nearly half the sensitivity compared to the initial analysis, which includes exits. This indicates that less productive firms are more likely to exit the market, especially if they are younger. This is largely consistent with Osotimehin and Pappadà [2017] since younger firms are less capitalize compared to older companies and this mechanism that is not included in the Caballero and Hammour [1994] can modify the strict relations between productivity and survival rate. A final question remains from the empirical point of view; Did these patterns change in the great Recession? That is an important question since as exposed in the model in this thesis financial frictions can reduce the cleansing effect of the economic downturn. An answer to this question can be found in the regressions results exposed in table 3, in particular one can see that the interaction effects between cycle and TFP are larger for the period previous to the Great Recession. Indeed the three-way interaction terms between TFP, cycle and, Great Recession dummy are negative and statistically significant for exit



and overall growth. Thus, instead of the cycle enhancing the impact of TFP on overall growth, it tends to diminish it on the margin in the Great Recession. A similar pattern is observed for exit. The estimated three-way the interaction effect is positive and larger in magnitude than the two-way interaction effect of TFP and the cycle. Instead of the cycle enhancing the impact of TFP on exit, it tends to diminish it on the margin in the Great Recession.

4 Literature review of theoretical models

4.1 Cleansing effect in Osotimehin and Pappadà [2017]

The economy comprises risk-neutral firms with a constant discount rate represented by $0 < \beta < 1$. These firms exhibit heterogeneity in productivity and net worth. They employ a production technology that relies solely on capital (or production units) as input, featuring diminishing returns to scale.

In each period, firms incur a fixed production cost denoted as c to initiate production. After production, they decide how to allocate profits for the next period. The remaining

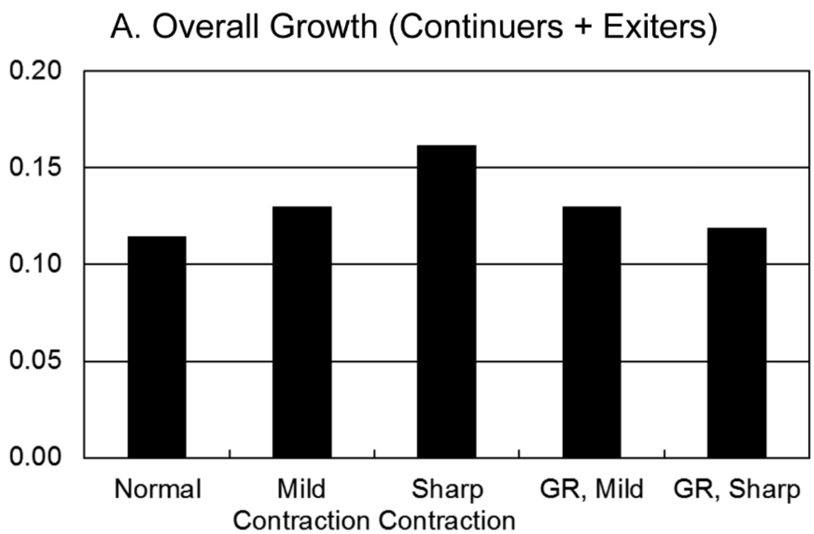


Figure 4: Differences in growth and exit rates between high-productivity and low-productivity establishments over the business cycle. Authors' calculations on Annual Survey of Manufactures, Census of Manufactures, and Longitudinal Busi- ness Database. Depicted is the predicted difference in growth rates (panel A, high minus low) and the predicted difference in probability of exit (panel B, low minus high) between an establishment one standard deviation above industry-by-year mean productivity and an establishment one standard deviation below industry-by- year mean productivity. Normal is zero change in state-level unemployment, mild contraction is 1 percentage point increase in state-level unemployment, sharp contraction is 3 percentage point increase in state-level unemployment, and GR is for the period 2007-9.

profits are invested in a risk-free asset. Firms face a choice: they can either continue operating and reinvest their profits or exit the market, investing their entire net worth, denoted as e, in the risk-free asset.

Firms opt to exit the market when expected profits no longer outweigh the fixed cost c, or when the value of production becomes inferior to the value they could gain by investing in the risk-free asset.

The value obtained from investing in the risk-free asset is given by:

$$q_t + \sum_{s=0}^{+\infty} \beta^s [\beta(1+r) - 1] e_{t+s+1}.$$

Notably, when the condition $\beta(1+r) \leq 1$ holds, this value simplifies to q. In such cases, firms are either indifferent regarding the timing of dividend distributions or have a preference for distributing their end-of-period net worth to shareholders or investors. In this economic model, the agents are the firms themselves, aiming to maximize their value over time by selecting an optimal level of capital denoted as k. The production function, accounting for the fixed cost c, is expressed as follows: $Y = Z(\theta + \epsilon)k^\alpha$.

Key variables include:

- Z: Stochastic aggregate productivity common across firms.
- θ: Persistent firm-specific productivity shock (modeled as a Markov Chain).
- ϵ: Firm-specific productivity shock with $\epsilon \sim \mathcal{N}(0, \delta)$.
- k^α: Capital or production units, as in Caballero and Hammour (AER).

The timeline of events is as follows:

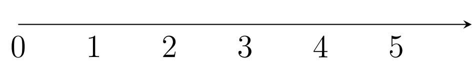


Figure 5: Timeline of Events

The sequence of events includes:

1. The firm possesses knowledge of Z, θ, k^α, e (where e represents its endowment, different from k since the firm can borrow money with $d = c + k - e$).

2. The firm computes the optimal k to maximize the expected value of the firm, with k ranging from $[0, +\infty]$. If $k = 0$, it indicates the firm's decision to exit.

3. At the end of the period, the firm observes ϵ and the aggregate shock.

4. The firm repays its debt and the fixed operating cost $(c + k - e)$, resulting in an end-of-period net worth q.

5. The firm decides on the amount of dividends to distribute $(q - e')$, observes the productivity shock θ', Z', and the process restarts from step 1.

Frictionless economy In a frictionless economy, firms have the option to borrow an amount denoted as $c + k - e$ at the risk-free interest rate $r = \frac{1}{\beta} - 1$. Therefore, at the start of the period, the firm's value is determined by the following expression:

$$V_{FL} = \max_k E \int \max[q, \max_{e'}(q - e' + \beta V_{FL}(e', \theta', Z'))] \, d\Phi(\epsilon)$$

where the end of period net worth is equal to:

$$q = Z(\theta + \epsilon)k^\alpha + (1 - \delta)k - (1 + r)(c + k - e)$$

Under the condition of survival, it can be demonstrated that:

$$\widehat{V}_{FL}(\theta, Z) = \max_k E \int [Z(\theta + \epsilon)k^\alpha - (1 + r)c \, d\Phi(\epsilon)] + \beta \max[0, \widehat{V}_{FL}(\theta', Z')]$$

In the absence of market friction, firms choose to exit when their productivity reaches a certain threshold. Specifically, they exit if $\theta' < \underline{\theta}_{FL}(Z')$, where $\underline{\theta}_{FL}(Z')$ is defined as the value for which $\widehat{V}_{FL}(\underline{\theta}_{FL}, Z') = 0$.

Economy with Credit Market Frictions After production, the firm privately observes the temporary shock ϵ, while financial intermediaries can only observe it at a cost of μk^{α}. For one-period debt contracts, financial intermediaries observe ϵ only if the firm faces financial distress, which occurs when the private shock is insufficient to repay its debt. The terms of the financial contract depend on the firm's net worth e, current productivity θ, and aggregate productivity value Z, all observable by both the financial intermediary and the firm at no additional cost.

HP1 (Hypothesis 1): The risk-free interest rate is $\beta < \frac{1}{1+r}$, which implies a lower risk-free rate in an economy with credit frictions compared to a frictionless one. It also ensures that firms do not always reinvest their profits.

When a firm defaults, the financial intermediary incurs verification costs and seizes all of the firm's income. The default threshold $\overline{\epsilon}$ is determined by the equation:

$$Z(\theta + \overline{\epsilon})k^{\alpha} + (1 - \delta)k = (1 + \widetilde{r})(c + k + e)$$

Default results in a zero net worth but does not necessarily force the firm to exit the market, depending on its persistent productivity component θ.

The financial intermediary lends $(c + k - e)$ to the firm only if the expected income from the loan equals the the opportunity cost of the funds, as expressed by the inequality:

$$(1+\widetilde{r})(k+c+e)(1-\Phi(\overline{\epsilon})) + \int_{-\infty}^{\overline{\epsilon}} [Z(\theta+\overline{\epsilon})k^{\alpha} + (1-\delta)k - \mu k^{\alpha}]\, d\Phi(\epsilon) \geq (1+r)(c+k+e)$$

The financial contract is characterized by $(k, \overline{\epsilon})$. Given Z, θ, e, the participation constraint indicates the default threshold $\overline{\epsilon}$ required by the financial intermediary to lend a given amount. For some firms, their net worth is too low for the participation constraint of the financial intermediary to be satisfied. In fact, given θ, Z, there is a unique threshold $e_b(\theta, Z)$ below which the financial intermediary refuses to lend any amount:

$$Z[\theta + G(\overline{\epsilon}_b)]k^\alpha + (1-\delta)k - uk_b^\alpha \Phi(\overline{\epsilon}_b) = (1+r)(k_b + c - \underline{e_b})$$

where $\overline{\epsilon}_b$ maximizes the expected income of the financial intermediary. When the firm has a net worth below $\underline{e_b}$, the firm defaults.

After production, the firm's end-of-period net worth is equal to:

$$q = \begin{cases} Z(\theta + \overline{\epsilon})k^\alpha + (1-\delta)k - (1+\widetilde{r})(k + c - e) & \text{if } \epsilon \geq \overline{\epsilon} \\ 0 & \text{otherwise} \end{cases}$$

Using the default condition we can rewrite as

$$q = \max[Zk^\alpha(\epsilon - \overline{\epsilon}); 0]$$

The firm's problem Define V as the firm's value at the start of the period, which hinges on investment outcomes and exit decisions. If the end-of-period net worth falls below a threshold ($q < e_b(\theta', Z')$), the firm exits. Otherwise, it compares its continuing value to the end-of-period net worth ($q \geq e_b(\theta', Z')$) and exits if the continuing value is lower.

The firm's value function is given by:

$$V(e, \theta, Z) = \max_{(k,\overline{\epsilon})} E\left\{ \int I(q)q + (1 - I(q)) \max[q, \max_{e'} q - e' + \beta V(e', \theta', \zeta')] \, d\Phi(\epsilon) \right\}$$

Where:

$$I(q) = \begin{cases} 0 & \text{if } q \geq e_b(\theta', Z') \\ 1 & \text{if } q < e_b(\theta', Z') \end{cases}$$

Subject to the following constraints:

1.

$$Z[\theta + G(\bar{\epsilon}_b)]k^\alpha + (1-\delta)k - uk_b^\alpha \Phi(\bar{\epsilon}_b) \geq (1+r)(k_b + c - \underline{e_b})$$

2.

$$q = \max[Zk^\alpha(\epsilon - \bar{\epsilon}); 0]$$

3.

$$\overline{e_b}(\theta', Z) \leq e' \leq q$$

The firm aims to maximize expected dividends while complying with the financial intermediary's participation constraint (constraint 1). Equation (constraint 2) characterizes the firm's end-of-period net worth, and Equation (constraint 3) ensures that the net worth is sufficiently high to satisfy the participation constraint.

Furthermore, the firm is prohibited from issuing new shares and can only augment its net worth by reinvesting profits. This limitation presents a trade-off: increasing capital boosts production capacity but also raises the risk of default, as the default threshold set by the financial intermediary increases with borrowed amounts.

4.2 The cleansing effect by Caballero

In the first paper that rationalizes the cleansing effect of recessions, Caballero and Hammour [1994] and published in the American Economic Review in 1994, the primary aim was to investigate how industries respond to cyclical variations in demand. They did this by employing a vintage model of creative destruction. The underlying concept postulates that the processes of creation and destruction within an industry partially explain business cycles. Industries continuously experiencing creative destruction can adapt to demand fluctuations in two ways: by adjusting the rate at which they produce new units embodying advanced techniques or by altering the rate at which outdated units are retired. The model they used incorporated heterogeneous firms, where production units embodied the most advanced technology at the time of their creation. The costs associated with creating new units slowed down technology adoption, resulting in the coexistence of production units with varying vintages.

Key to understanding how firms adapt to business cycles are the concepts of the creative margin and the destruction margin. For example, a reduction in demand can be accommodated either by reducing the rate of technology adoption or by retiring older production units. One of the primary factors determining which margin is more responsive to business cycles is the adjustment cost. When this cost follows a linear pattern, the study shows that insulation is complete, and the industry's response relies exclusively on its creation margin. Consequently, the creation margin becomes smoother over time in comparison to the destruction margin, which exhibits greater responsiveness to the business cycle.

Crucially, Caballero and Hammour's research Blanchard et al. [1990] offers theoretical insights supported by empirical evidence. Their findings on the cyclical nature of the destruction margin align with the studies conducted by Blanchard and Diamond Blanchard et al. [1990], as well as Steven Davis and John Haltiwanger Davis and Haltiwanger [1992], in their respective works from 1990. This convergence between theoretical framework and empirical substantiation underscores the importance of comprehending the dynamic interplay between creative destruction and business cycles, which significantly influences how industries respond to economic fluctuations.

In their study Davis and Haltiwanger [1992], where they assess the heterogeneity of employment changes at the establishment level in the U.S. manufacturing sector from 1972 to 1986, it is revealed that job destruction exhibits procyclical tendencies, responding more robustly to downturns in the economic cycle compared to the creation rate, in line with the theoretical model proposed by Caballero and Hammour Caballero and Hammour [1994]. The authors leverage a natural experiment inherent in the data to examine whether the structure of adjustment costs can account for the behavior of these two margins. This natural experiment arises from the asymmetric nature of business cycles, with recessions being shorter but more severe than expansions. The theoretical model predicts that these differences should be attenuated in the creation process, a prediction that is substantiated by the data since creation exhibits relative symmetry around its mean, while destruction displays a high degree of asymmetry. The underlying concept driving the behavior of the destruction margin can be traced back to the theories of Schumpeter and Hayek. They suggest that recessions represent periods during which unprofitable or outdated techniques are pruned from the economy, leaving behind the most efficient firms Hayek and Caldwell [2007].

4.3 Theoretical model

The model in question is a vintage model that simulates an industry experiencing exogenous technological progress. Within this model, production units are constructed using a fixed proportion of labor and capital, and they are continually being created and phased out. Notice that only the creation of new production units incurs a cost. This simplification is plausible, particularly in the context of the United States, where the expense associated with hiring is typically higher than the cost of termination, as demonstrated by Abdulkadiroğlu and Kranton (2003) Abowd and Kramarz [2003].

In this model, when a production unit is created at a specific time t_0, it embodies the most advanced technology available at that moment and consistently generates a uniform output represented by $A(t_0)$ throughout its operational lifetime. The productivity of this technology denoted as $A(t)$, experiences continuous growth at an exogenously determined constant rate $\delta \geq 0$. This growth in technology can be interpreted in two ways: either as the adoption of new technology or as a product innovation. In the latter scenario, a continuum of perfectly substitutable products can yield varying levels of output.

$$[f(a,t) \qquad 0 \leq a \leq \overline{a}(t)]$$

The above function represents the cross-sectional density of the production units aged a at time t, where $\overline{a}(t)$ is the age of the oldest production unit at time t. The first assumption is that $f(a,t)$ and $\overline{a}(t)$ are continuous functions. The mass of production units at time t is given by:

$$N(t) = \int_{\overline{a}(t)}^{0} f(a,t)da$$

$N(t)$ is a measure of either the industry's capital stock or its employment, due to a fixed share of capital and labor. Thus, the industry's output is given by:

$$Q(t) = \int_{\overline{a}(t)}^{0} A(t-a)f(a,t)da$$

The deterioration of production units involves both an exogenous depreciation rate δ and an endogenous destruction process, which impacts $f(a,t)$ at its limits. The count of production units surviving for a years is expressed as:

$$f(a,t) = f(0,t-a)e^{-\delta a} \quad \text{where } 0 < a \leq \overline{a}(t)$$

The production flow is determined by:

$$\dot{N}(t) = f(0,t)[1-\dot{\overline{a}}(t)] + \delta N(t)$$

Here, the first term represents the production rate, while the second term encapsulates the destruction rate, encompassing the obsolescence rate $f(\overline{a})(t)$, the technological obsolescence change over time $-f(\overline{a})(t)\dot{\overline{a}}(t)$, and the depreciation rate $\delta N(t)$.

The assumptions made by the authors are denoted as $\forall t \mid f(0,t) > 0 \cup \dot{\overline{a}}(t) <$.

The alteration in output concerning these flows is articulated as:

$$\dot{Q}(t) = A(t)f(0,t) - A(t-\overline{a}(t))f(\overline{a}(t),t) \cdot [1-\dot{\overline{a}}(t)] + \delta Q(t)$$

The authors define a perfectly competitive industry in partial equilibrium, where supply is dictated by free entry and perfect equilibrium. Additionally, they introduce a cost function related to creating new production units:

$$c = c\left(f\left(f(0,t)\right)\right) \quad \text{where } c(\cdot) > 0,\, c'(\cdot) \leq 0$$

This cost function is contingent on the creation rate, implying that higher creation rates correspond to increased costs. The equilibrium condition is established by equating the cost of unit creation to the present discounted value of profits throughout its lifespan. The authors set the cost of a production unit to 1, and $P(t)$ is the price of a unit of output. Thus, the profits generated at time t by a production unit aged a are defined as:

$$\pi(a,t) = P(t)A(t-a) - 1$$

$$\overline{a}[t+T(t)] = T(t)$$

Here, $T(t)$ signifies the maximum lifetime of a unit created at t. At any given time t, the free entry condition is expressed as:

$$c(f(0,t)) = \int_{t+T(t)}^{t} \pi(s-t,t)e^{-(r+\delta)(s-t)\,ds}$$

In the above equation, where $r > 0$ represents the exogenously determined instantaneous interest rate, the determination of the exit of a production unit is contingent upon continuous $P(t)$ and the instance when the profits generated by a unit being destroyed reaches zero. This occurrence signifies the moment when the oldest unit operational at time t, denoted as $\overline{a(t)}$, must adhere to the equation:

$$P(t)A(t - \overline{a}(t)) = 1$$

The authors posit that $P(t)$ exhibits a decreasing trend due to the model's assumption regarding endogenous destruction, specifically $\overline{\dot{a}(t)} < 1$. To see, differentiate

$$\dot{P}(t) = -\gamma\left[1 - \dot{\overline{a}}P(t)\right]$$

Consequently, when the profits of a production unit diminish to zero for the first time, it will be subject to destruction.

On the demand side, the authors assume a unit-elastic demand function and consider the aggregate expenditure as exogenous $\overline{D}(t) = P(t)Q(t)$. The equilibrium is a path $\{f(0,t), \overline{a}(t), T(t), Q(t)\}_{t\geq 0}$ that satisfy the following conditions:

1.

$$Q(t) = \int_{\overline{a}(t)}^{0} A(t-a)f(a,t)da$$

2.

$$f(a,t) = f(0,t-a)e^{-\delta a}$$

3.

$$T(t) = \overline{a}\left(t + T(t)\right)$$

4.

$$c(f(0,t)) = \int_t^{t+T(t)} \left[P(s)A(t) - 1\right] e^{-(r+\delta)(s-t)}\, ds$$

5.

$$P(t)A(t - \overline{a}(t)) = 1$$

6.

$$P(t)Q(t) = \overline{D}(t)$$

The first three equations (1, 2, 3) and the fifth one (5) suffice to delineate the trajectories of $T(t)$, $P(t)$, and $Q(t)$, which are determined by $\{f(0,t), \overline{a}(t)\}$. To affirm the robustness of the conditions expressed in equations 6 and 5, it is possible to derive these equations as first-order conditions for the maximization of a number of perfectly competitive firms holding production units.

To comprehend the functioning of endogenous destruction, let's consider a scenario with constant demand. In this case, both the destruction and creation rates change only due to supply factors. This steady state is characterized by a constant lifetime of production units $T(t) = \overline{a}(t) = \overline{a}^*$, resulting in a time-invariant age distribution $f(a,t) = f^*(a)$. Equation 5 implies that the price $P(t)$ must consistently decrease at a rate σ. Higher innovation rates lead to increased productivity, raising the supply and consequently lowering the price. Equation 2 reveals that the distribution of production units in the steady state follows a truncated exponential distribution:

$$f^*(a) = f^*(0)e^{-\delta a} \quad 0 \leq a \leq \overline{a}^*$$

Using free entry conditions (4) and the clearing condition (6), one can determine the creation and destruction ages $f^*(0)$ and $\overline{a}^*$. Equations 1 and 5 yield the cost function and productivity of a new production unit:

$$c(f^*(0)) = \frac{e^{\gamma\overline{a}^*} - e^{-(r+\delta)\overline{a}^*}}{\gamma + r + \sigma} - \frac{1 - e^{-(r+\delta)\overline{a}^*}}{r + \delta}$$

$$f(0) = \frac{(\sigma + \delta)\overline{D}^*}{e^{\sigma\overline{a}^* - e^{\delta\overline{a}^*}}}$$

The authors then normalize the creation rate:

$$N = f^*(0) \cdot (1 - e^{\delta\overline{a}^*})$$

In the steady state, this is given by:

$$(9) CC^* = \frac{\delta}{1 - e^{-\delta\overline{a}^*}}$$

Considering a special case where the creation cost is a constant c, i.e., $c(f^*(0)) = c$, substituting into equation 4.3 allows retrieval of $\overline{a}^*$. The effect of technological rate σ on $\overline{a}^*$ is decreasing, as a higher innovation rate increases the opportunity cost of delayed renovation, while a higher cost of creating new units lowers the renovation rate. An optimal lifetime of production units increases with higher r and δ as it becomes harder to recover creation costs.

Now, dropping the assumption of constant demand, we examine how the industry adjusts to demand fluctuations. Two ways are identified in which the industry adapts production to meet demand: by reducing the rate of creation $f(0,t)$ and by increasing the rate of endogenous destruction $f(\overline{a}(t),t)\cdot[1-\dot{\overline{a}}(t)]$, thus reducing $\overline{a}$, the age at which units are demolished.

These two adjustments interact, leading to a reduction in demand causing the most outdated units to be scrapped, rendering them unprofitable. However, if the recession is partially accommodated by a reduction in the creation rate, the effect on the destruction margin is diminished. The authors argue that the extent to which creation will "insulate" existing units from variations in demand depends on the marginal cost of creating new units $c'f(0,t)$. When the marginal cost of creation is zero, demand fluctuations are entirely adjusted by the creation margin. This is exemplified in the case where $c(f(0,t)) = c$. In such instances, the insulation effect is complete, as there is no need to retire older units. Lowering $f(0,t)$ is sufficient, and it is cheaper than reducing the life of existing production units.

The insulation effect is not solely due to asymmetric adjustment costs on the creation and destruction margins. Complete insulation would occur even with linear adjusting costs. The creation rate in the case of constant creation cost is given by:

$$f(0,t) = \frac{\dot{\bar{D}}(t) + \delta\bar{D}(t) + P(t)A(t-\bar{a}(t))f(\bar{a}(t),t)[1-\dot{\bar{a}}(t)] - \dot{P}(t)Q(t)}{P(t)A(t)}$$

In the attained equilibrium, variations in demand are entirely offset by adjustments at the creation margin denoted as $f(0,t)$, with $\overline{a}(t)$ remaining steady at the destruction margin. The creation process effectively counteracts the impact of demand fluctuations on the price $P(t)$, effectively shielding existing units from demand changes. The price $P(t)$ experiences a constant decline at a rate represented by σ, reflecting the pace of technical progress. This consistent decline in $P(t)$ serves as a clear signal for production units to function optimally throughout their constant lifetime $\overline{a}(t)^*$.

In the aforementioned scenario, the destruction rate is not constant, but it does not respond to demand through variations in the age $\overline{a}(t)^*$ at which units are destroyed. Instead, variations in the creation rates have an impact on the number of units that reach obsolescence. If fewer units are created, fewer units become obsolete after $\overline{a}(t)^*$ periods. It is noteworthy that any modification leaving equations 3 to 5 independent of $\overline{D}(t)$ and $f(0,t)$ does not alter the full-insulation results.

Interestingly, assumptions such as perfect competition, industry-wide return to scale, and perfect foresight are not necessary for these conclusions. The latter is particularly noteworthy as it asserts that fully accommodating demand on the creation side only requires knowledge of current conditions. As long as the non-negativity constraint on $f(0,t)$ is never binding, implementing equilibrium behaviors does not necessitate expectations of future demand.

4.4 Application of the model

The model undergoes calibration utilizing Job-flow data and Industry production data. The former facilitates the replication of job creation dynamics, while the latter is employed to mimic the behaviors of firm creation and destruction in the manufacturing industry. To capture these dynamics, the marginal cost of creating new production units is stipulated as positive $c'f(0,t)$. This allows for a partial insulation effect, and the destruction margin responds to demand fluctuations. However, introducing a dependency of c on $f(0,t)$ compromises the analytical tractability of the system (Equations 1 - 6). Consequently, the authors resort to methods such as multiple shooting to ascertain the optimal equilibrium and subsequently employ an iterative procedure to converge to the correct expected creation rate.

For numerical solutions, the authors adopt a linear formulation:

$$c(f(0,t)) = c_0 + c_1 f(0,t)$$

To gain a deeper understanding of how creation and destruction respond to demand, the authors simulate sinusoidal demand using the equation:

$$\overline{D}(t) = 1 + 0.07\sin(t)$$

The results are visualized in the image below, depicting the feedback of normalized creation and destruction (CC and DD) to changes in demand.

The plot clearly illustrates that the insulation effect is only partial; otherwise, DD would have remained flat, as in the case with $c(f(0,t) = c)$. From a mathematical perspective, destruction responds to demand as equations 3-5 are no longer independent of the path $f(0,t)$ and demand. From an economic standpoint, increasing creation costs smoothen the creation process. In scenarios with a nearly flat innovation rate, firms during crises cannot fully accommodate lower demand, nullifying the adoption of new production units, as the marginal costs would exceed the reduction in existing production units.

In the considered model, production units integrate labor and capital in fixed proportions to generate output. Each unit can be conceptualized as contributing to job creation within the industry, and job-flow data serves as a metric for quantifying the flows of production units.

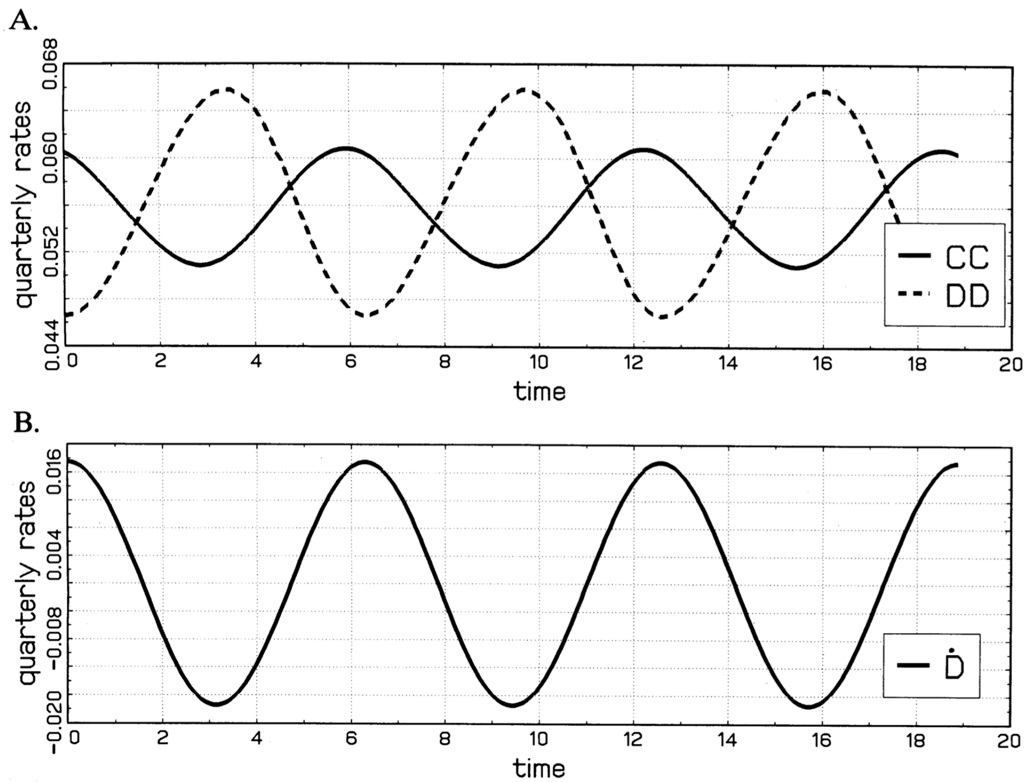


FIGURE 2. A) CREATION AND DESTRUCTION ($c_0 = 0.3$, $c_1 = 1.0$); B) CHANGE IN DEMAND (SYMMETRIC)

Figure 6: Figure 1. A Creation and destruction $c_0 = 0.3, c_1 = 1$ B Change in demand (Symmetric)

Datasets that closely align with the theoretical CC and DD series have been compiled by Davis and Haltiwanger [1990, 1992] and Blanchard et al. [1990], drawing from various sources. The primary focus lies on the dataset curated by Davis and Haltiwanger, who leverage the Longitudinal Research Database to construct quarterly series for U.S. manufacturing plants spanning the period 1972:2-1986:4.

In their empirical approach, **?**utilize output to empirically determine demand, employing the growth rate of the industrial production index as a proxy for output growth. Notably, in the foundational theoretical model, $Q(r)$ is smoothed by price movement, with the elasticity of demand determining the extent of smoothing, assumed to be equal to 1. While the theoretical model maintains a constant dividend wage, the authors acknowledge that considering a procyclical dividend wage, as in the case of general equilibrium with correlated industry shocks, may dampen the effect of demand shocks. However, they assert that this adjustment would alter only the magnitude, not the direction, of the analysis.

The figure below illustrates the data that the model seeks to replicate, showcasing job creation, job destruction, and growth.

To discern the characteristics of the series, the authors perform regression analysis on sectoral rates of job creation and job destruction against leads and lags of the corresponding rates of growth. They find that job creation is less responsive to demand fluctuations, while job destruction exhibits a more countercyclical behavior. The initial finding indicates that the rate of job destruction displays greater responsiveness to changes in sectoral activity compared to the rate of job creation. Specifically, the sums of coefficients are -0.384 for job destruction and 0.218 for job creation showed in the table 8, the same results as in Davis and Haltiwanger [1990, 1992] and in Blanchard et al. [1990]. The authors capitalize on a natural experiment rooted in the intrinsic asymmetric characteristics of business cycles. Recessions, marked by brevity but in-

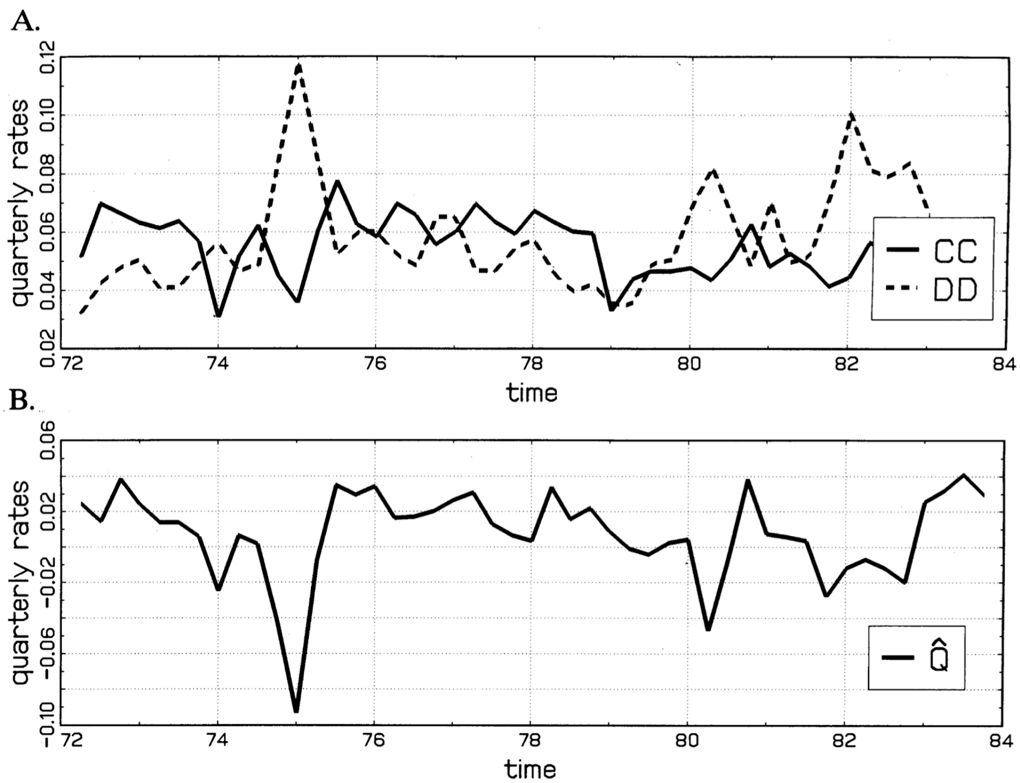


Figure 7: Figure 1. Job creation and job destruction in U.S. Manufacturing B Index of the industrial production

Regressor	Timing	Creation: Coefficient	Creation: Standard deviation	Destruction: Coefficient	Destruction: Standard deviation
$\hat{Q}$	2 leads	0.029	0.006	0.030	0.010
	1 lead	0.065	0.007	−0.068	0.010
	contemporaneous	0.108	0.007	−0.185	0.010
	1 lag	0.013	0.007	−0.103	0.010
	2 lags	0.003	0.006	−0.058	0.010
	Sum:	0.218	0.013	−0.384	0.017
$\hat{Q}^+$	2 leads	0.052	0.012	0.012	0.016
	1 lead	0.102	0.012	0.002	0.016
	contemporaneous	0.131	0.012	−0.065	0.016
	1 lag	0.059	0.012	−0.025	0.016
	2 lags	0.055	0.012	−0.008	0.016
	Sum:	0.399	0.026	−0.066	0.023
$\hat{Q}^-$	2 leads	0.002	0.010	0.006	0.014
	1 lead	0.022	0.011	−0.149	0.014
	contemporaneous	0.093	0.012	−0.293	0.015
	1 lag	−0.012	0.012	−0.139	0.015
	2 lags	−0.021	0.012	−0.059	0.015
	Sum:	0.084	0.020	−0.634	0.024

Figure 8: Table 2.1. Job Creation and Job Destruction in U.S. Manufacturing Response to Output Growth

textitNotes: The table presents the reaction of job creation to the growth rate of the industrial production index. The latter is categorized into values above and below its mean (Q). The table encompasses quarterly observations for the two-digit SIC industries during the period 1972:2-1986:4. The coefficients are uniformly constrained to be equal across all sectors, with the exception of a constant (not shown).

tense contractions, provide the backdrop for the authors' model. This model endeavors to emulate the creation rate while concurrently mitigating the impact of asymmetric cyclicality inherent in business cycles. The empirical evidence supporting this model's behavior is encapsulated in Table 8, wherein two distinct scenarios are explored: output growth trajectories above Q^+ and below Q^-, relative to their respective means. The table meticulously delineates how creation and destruction rates respond to these deviations in output growth.

The salient observation emerges regarding creation rates, elucidating that they exhibit a more rapid and robust response in instances of vigorous output growth, as opposed to scenarios where the output growth rate experiences a reduction. On a contrasting note, the destruction margin, in line with the model's projections, manifests heightened sensitivity to a decline in output. This responsiveness is particularly pronounced from one quarter before the onset of the shock to one quarter after. Notably, during expansionary phases, the mean response of the destruction margin is -0.066, a notably milder reaction compared to the recessionary case where the mean response stands at -0.634.

These empirical outcomes seamlessly align with the predictions of the model. Specifically, the creation rate exhibits heightened responsiveness during expansionary phases, given their cyclical and symmetric nature. In contrast, the asymmetric and non-cyclical nature of recessions triggers a more substantial decline in the production unit rate, in line with the model's expectations.

In order to better understand the asymmetrical behavior the authors simulate an asymmetrical demand function:

$$\cdot\overline{D}(t) = 0.05[\cos(t) + \sin(t)] - 0.016\sin(2t) - 0.003\cos(3t)$$

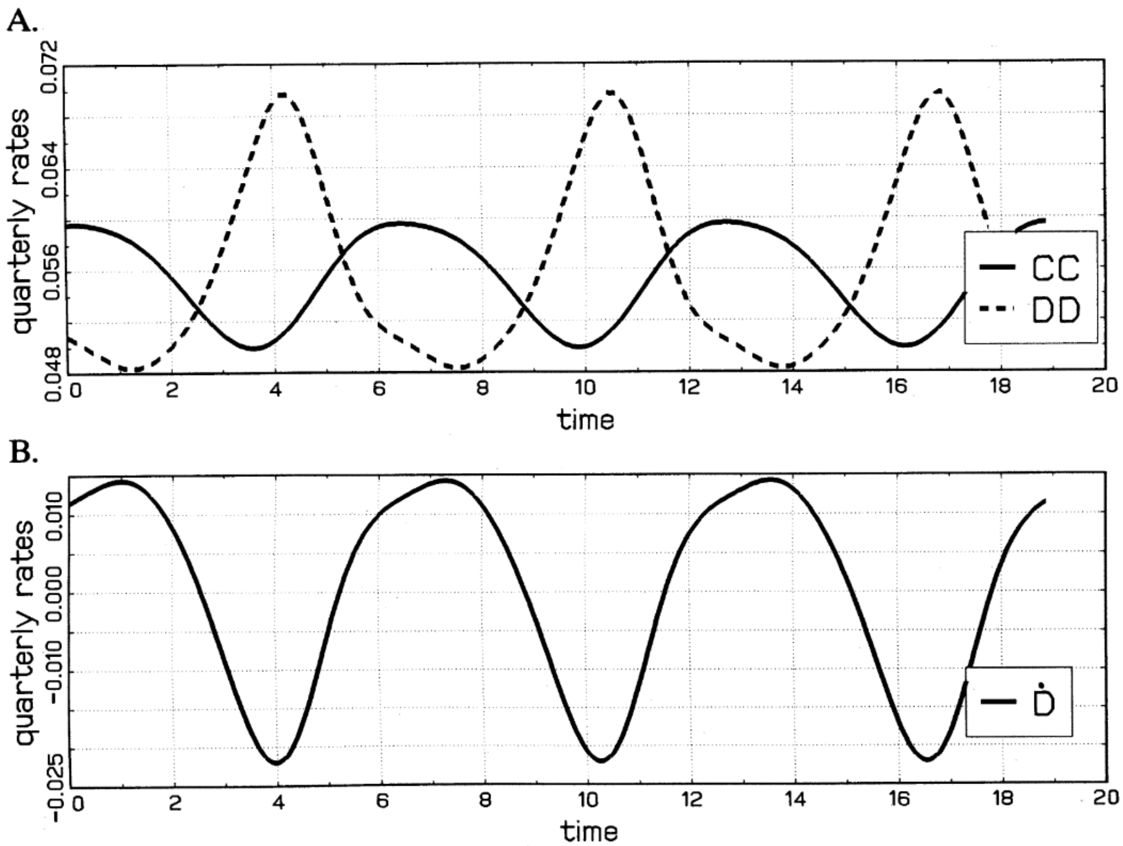


Figure 9: A. Creation and Destruction B. Output Growth
Notes: The figure depicts a simulation of asymmetrical supply growth.

$$\overline{D}(t) = 1 \quad r = 0.065, \delta = 0.15, \gamma = 0.028, c_0 = 0.3, c_1 = 1.0$$

The results are depicted in 9

From the plot 9, it is evident that firms use prediction in demand to smooth job creation to avoid big change, since they are too costly, by averaging the demand over the lifetime of a production unit. On the other hand, destruction depends only on current conditions, thus responding only to significant deviations from the demand prediction. It can be better understood thinking about a case in which creation rates respond only mildly to a sharp decrease in demand, and the equilibrium price falls leading to additional destruction since older units' profits go to 0. Indeed, destruction not only preserves but amplifies the asymmetry of demand.

Frictionless economy The authors culminate their study with a compelling calibration exercise using manufacturing series to exploit the model. This entails dissecting the observed net change in employment into destruction and creation rates, as well as applying the same approach to output production. The model is simulated for the duration of 1972:2-1983:4, with parameters as follows:

Table 4: Calibrated Parameters

Variable	Symbol	Value
Interest rate	r	0.065
Depreciation rate	δ	0.150
Rate of technical progress	γ	0.028
Adjustment cost parameters	c_0	0.403
	c_1	0.500

The technical progress is selected to attribute all the growth in employment and manufacturing to technological advancements, setting λ as 2.8. The authors employ Equation 4.3, linking the steady state to the lifetime of jobs and job turnover (CC^*), determining $\overline{a}^* + 7.42$ years. Utilizing this information, they ascertain the steady state entry cost to be 0.525, equivalent to half a year's operating costs for production units. Subsequently, they employ ordinary least squares (OLS) to retrieve the value of c_1, the marginal cost of creating a new unit, which is found to be 0.5. This aligns with a small elasticity for the creation cost function, signifying the vulnerability of the insulation mechanism to breakdown. The outcomes stemming from the simulations driven by employment and output are disclosed and contrasted with the data in Figure 10. Notably, the simulation of job creation displays a level of smoothness that diverges from the observed data, with this discrepancy being attributed, in part, to the inherent absence of uncertainty in our model. Despite this, the model effectively elucidates the relative volatility discernible in the patterns of job creation and destruction. Moreover, it successfully captures the greater symmetry observed in the former, offering insights into the nuanced dynamics at play in employment and output fluctuations.

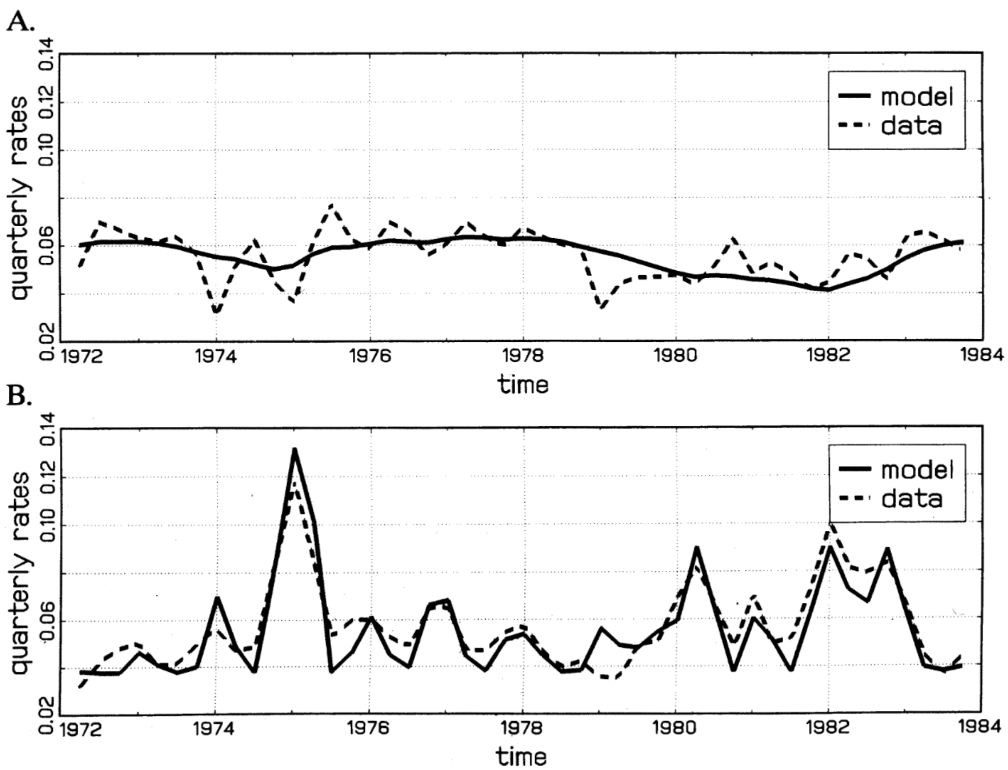


Figure 10: Figure 1. A employment driven job creation $c_0 = 0.403, c_1 = 0.5$ B Employment job destruction $c_0 = 0.403, c_1 = 0.5$

The model provides intriguing insights as it elucidates certain empirical findings found in Davis and Haltiwanger [1990, 1992]. Specifically, it delves into the dynamics of how the response of the creation margin contributes to an insulating effect on the destruction margin. The model's salient features lie in its incorporation of heterogeneity across production units and their turnover, rendering it a meaningful baseline for comprehending how the cleansing effect influences the distribution of production units.

However, it's essential to note that the model, in its current formulation, does not account for the potential impact of financial frictions arising from asymmetric information between borrowers and lenders. Such frictions could conceivably influence both the destruction and creation margins, introducing a layer of complexity not considered in the current framework.

An alternative perspective on recessions is captured by the concept of a "pit-stop," where a recession is characterized as a period during which improvement investments in production are undertaken due to temporarily low opportunity costs, as posited by Davis and Haltiwanger [1990]. This viewpoint adds nuance to the understanding of recessions, emphasizing them as periods conducive to strategic investments.

One potential objection to the notion that recessions are times of cleansing is rooted in the implication of countercyclical productivity. Notably, labor productivity is often observed to be procyclical. However, this apparent inconsistency can be attributed to friction, as suggested by Galí and Hammour [1992]. Their findings provide evidence supporting the notion that the cleansing effect enhances productivity in the long term, offering a nuanced perspective on the relationship between economic downturns and productivity dynamics.

A crucial observation in the aforementioned model is the authors' reliance on a constant marginal cost of creation. Yet, recent literature has raised concerns about the reliability of this assumption, especially for larger firms. The dynamics of the business environment in recent years suggest that significant firms tend to favor substantial adjustments, particularly in terms of downsizing.

Interestingly, this deviation from the constant marginal adjustment cost for bigger firms can be interpreted as a validation of the model's predictions. When firms opt not to fully insulate themselves from a decline in demand using the creation margin, they tend to respond with intense layoffs. This alignment between the model's predictions and the observed behavior of larger firms underlines the model's relevance and its capacity to capture real-world dynamics.

5 Theoretical model

5.1 Introduction

This thesis presents a partial equilibrium model in which firms maximize dividends over an infinite period, under financial frictions, investigating how those frictions can affect the saddle paths of capital and dividends.

The subsequent sections delve into the formulation of the flow of funds and its dynamics. Following this, the focus shifts to scenarios where financial frictions are present, examining their implications on firm behavior and market outcomes.

5.2 Law of motion of capital and debt

This model is set within a partial equilibrium framework where firms are differentiated by their productivity levels. They have the option to fund their operations by obtaining loans from financial intermediaries, as outlined by Bernanke and Gertler [1995], or by retaining dividends. The capital at any time t is calculated by adjusting the capital from the previous period for depreciation (δ), then adding the net investment (I), thus the low of motion of the capital stock is:

$$k_t = k_{t-1}(1-\delta) + I_t$$

The investment function is:

$$I_t = k_t - k_{t-1}\,(1-\delta) \tag{1}$$

The 1 equation states the investment level at time t is equal to the increase in capital less the depreciated ones. The flow of funds constraint is:

$$I_t + Rb_{t-1} + d_t = f(k_{t-1}) + b_t \tag{2}$$

where R denotes the gross interest rate and b_{t-1} represents the debt from the preceding period. The LHS of the f-of-f describes the resource outflows:

1. I_t the net investment at time t
2. Rb_{t-1} repayment of debts (capital and interest) of the previous period
3. d_t dividends distributed at time t

On the other hand, the RHS represents the resources inflows, which are composed by:

1. $f(k_{t-1})$ output of production at time t
2. b_t debt contracted at time t

The f-of-f constraints can be rewritten as the low of motion of debt, where $S_t = f(k_{t-1}) - d_t$ is the retained earnings, using 1, 2:

$$b_t = Rb_{t-1} + I_t - S_t$$

The above equations state that the debt level at time t should be exactly equal to the repayment of the previous debt (capital + interest), plus the investment net of internal funding. From 1 and 3 follows, by definition, the low of motion of the net worth:

$$
\begin{aligned}
n_t = k_t - b_t &= k_{t-1}(1-\delta) + I_t - Rb_{t-1} - I_t + S_t \\
&= k_{t-1} - \delta k_{t-1} - b_{t-1} - rb_{t-1} + S_t \\
&= n_{t-1} - \delta k_{t-1} - rb_{t-1} + [f(k_t - 1) - d_t]
\end{aligned}
$$

The net worth or equity of the firm is given by the net worth of the previous period less the depreciated capital, less the interest matured from the previous period augmented by the retained earnings. Therefore a firm can increase its net worth only through increasing the retained earnings levels, thus increasing output or decreasing dividends.

Steady State From 1 we can retrieve the locus in which capital ($k_{t-1} = k_t = \widehat{k}$) and debt ($b_{t-1} = b_t = \widehat{b}$) and for definition even dividends ($d_t = d_{t-1}$):

$$\widehat{k} = \widehat{k}\,(1-\delta) + \widehat{I} \tag{3}$$

$$\widehat{I} = \delta\widehat{k} \tag{4}$$

The 4 stated that in the steady state, the firm will invest only to substitute depreciated capital($\delta\widehat{k}$). From 3 substituting the stationary conditions, we get:

$$\widehat{b} = R\widehat{b} + \widehat{I} - \widehat{S} \tag{5}$$

$$\widehat{S} - \widehat{I} = r\widehat{b} \tag{6}$$

$$f\left(\widehat{k}\right) - \widehat{d} - \delta\widehat{k} = r\widehat{b} \tag{7}$$

The above equation 7 states that in the steady state, the retained earnings should be used only to repay matured interest over debt. Equation 7 can be rewritten as:

$$f(\widehat{k}) = \delta \cdot \widehat{k} + r \cdot \widehat{b} + \widehat{d} \tag{8}$$

The above equation 8 states that in the steady state, the production should be able to repay interest, dividends and depreciation. To visualize the steady state locus, we can plot the graph 11 of the locus described in the equation 8 using the following production function:

$$f(k_{t-1}) = Z \cdot k_{t-1}^{\alpha}, \tag{9}$$

with Z indicating the firm's productivity level, and k_t symbolizing capital as in the model by Caballero and Hammour [1994].

The figure illustrates the steady state relationships among debt $(\widehat{b})$, capital $(\widehat{k})$, and dividends $(\widehat{d})$ in a three-dimensional plot. The graph demonstrates how various combinations of debt and capital influence the distribution of dividends. It is evident that increasing the level of debt results in lower dividends, as a larger portion of resources is allocated towards servicing interest payments. Conversely, the relationship between capital and dividends is depicted as convex, highlighting an increase in dividends with higher capital levels, under the specified model parameters: $\delta = 0.1$, $r = 0.1$, $\alpha = 0.8$, and $Z = 0.5$. For example, the firm starting with initial capital $k_0 = 0.2$ and debt $b_0 = 0.1$, to maintain a steady state for both capital and debt, the dividends should be equal to $\widehat{d} = 0.5 \times 0.2^{0.8} - 0.1 \times 0.2 - 0.1 \times 0.1$. This specific combination of $k = 0.2$, $b = 0.1$, $d \approx 0.11$represents a stationary point in the model.

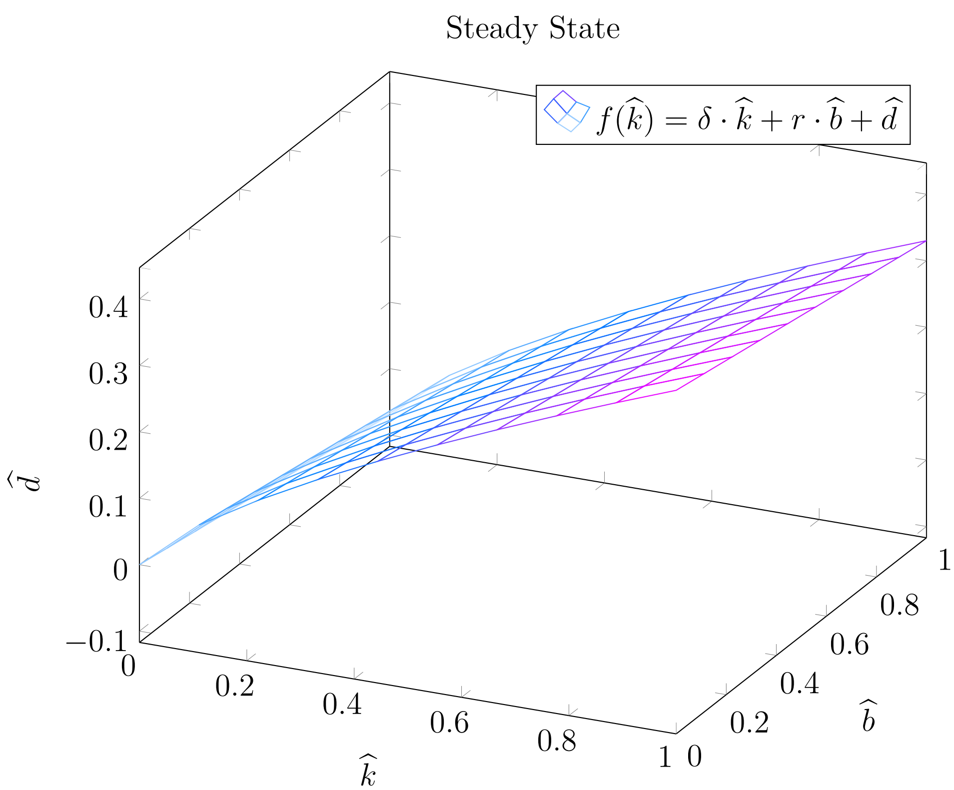


Figure 11: For this plot the following value has been used: $\delta = 0.1, r = 0.1, \alpha = 0.8, Z = 0.5$

Dynamics of capital While the discussion thus far has focused on steady states, it is crucial to explore the system's behavior under perturbations, particularly concerning the relationship between capital and dividends. If dividends are increased beyond the level consistent with a stationary path—where capital remains constant over time—the analysis shifts. Assuming the firm is debt-free ($b_t = 0 \quad \forall t$) for seek of simplicity, the law of motion of capital adjusts as follows from 2:

$$I_t + d_t = f(k_{t-1}) \tag{10}$$

$$I_t = S_t \tag{11}$$

In the case of free debt, all the investment of firms is financed through internal funds as stated in the equation 11. From 11 we can retrieve the finite difference equation describing the evolution of capital:

$$k_t = k_{t-1}(1-\delta) + f(k_{t-1}) - d_{t-1} \tag{12}$$

In order to understand how the above equation works, let's use the same production function as in the previous paragraph 9, derive with respect to k_{t-1}:

$$\frac{\partial k_t}{\partial k_{t-1}} = (1-\delta) + f'(k_{t-1}) \tag{13}$$

$$\frac{\partial k_t}{\partial k_{t-1}} = (1-\delta) + \alpha Z k_{t-1}^{\alpha-1} \tag{14}$$

There exist two cases: if the partial derivatives with respect to k_{t-1} is greater than 1 and dividends are positive we have an exploding path: if capital is lower than the steady state $k_{t-1} < \widehat{k}$ the capital will shrink to 0, while in the opposite case $\widehat{k} < k_{t-1}$, the capital will explode to $+\infty$. The condition for this first case is the following:

$$(1-\delta)+\alpha Z k_{t-1}^{\alpha-1}<1 \tag{15}$$

$$\alpha Z k_{t-1}^{\alpha-1}>-(1-\delta)+1 \tag{16}$$

$$k_{t-1}>\frac{\delta}{\alpha Z}^{\frac{1}{\alpha-1}} \tag{17}$$

Considering the second case in which the partial derivatives with respect to k_{t-1} is less than 1 and positive dividends, we have also an exploding path but without a steady state: for each level of capital at time t-1, the capital at time t will be less than the previous. The condition for the latter condition is the following:

$$(1-\delta)+\alpha Z k_{t-1}^{\alpha-1}>1 \tag{18}$$

$$\alpha Z k_{t-1}^{\alpha-1}<-(1-\delta)+1 \tag{19}$$

$$k_{t-1}<\frac{\delta}{\alpha Z}^{\frac{1}{\alpha-1}} \tag{20}$$

The following phase diagram represent the former case $\frac{\partial k_t}{\partial k_{t-1}} \leq 1$, using the following parameters: $\delta = 0.1$, $r = 0.1$, $\alpha = 0.8$, $Z = 0.5$, and $d = 0.8$ The graph distinctly demonstrates that when the capital at time t is below the red dot, it signifies that the capital is less than the steady-state capital, leading to a diminishing trajectory in the firm's capital. Conversely, if the capital is above the steady-state level, indicated by $\widehat{k}$, the firm is overcapitalized, and the trajectory becomes explosive, with capital increasing without bound.

If a firm's capital is less than the steady-state, meaning it has less than the optimal amount, the outflows—such as depreciation and constant dividends—are disproportionately high compared to its production. This dynamic will inevitably cause the firm's capital to deplete towards zero. It's crucial to recognize that this path is predicated on

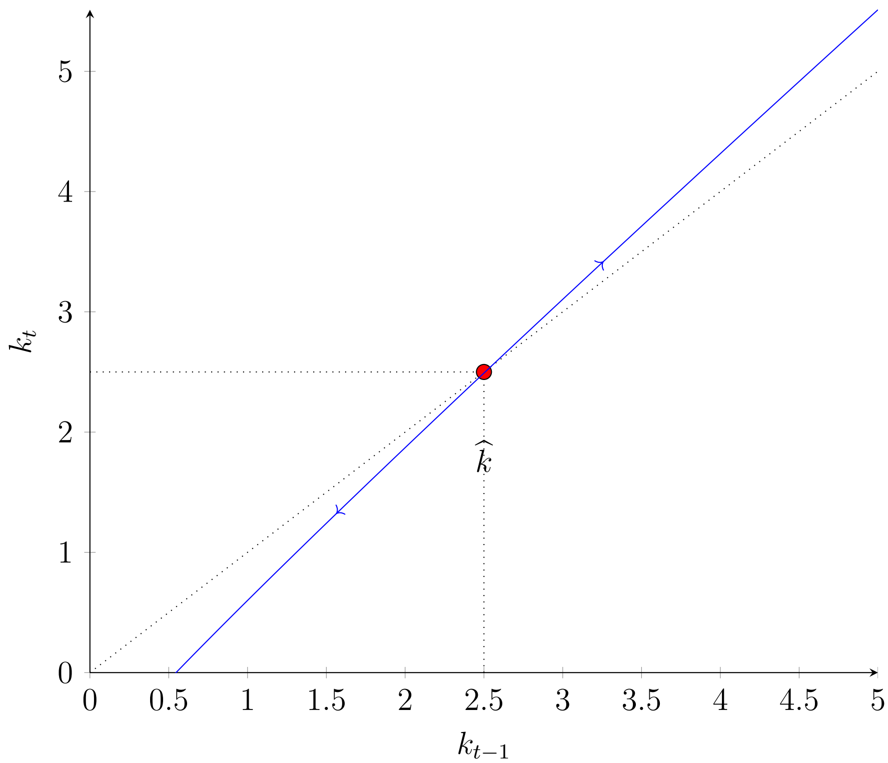


Figure 12: The graph depicts the trajectory of capital when there is no debt involved, given the parameters $\delta = 0.1$, $r = 0.1$, $\alpha = 0.8$, $Z = 0.5$, and $d = 0.8$. The blue line traces the flow of funds according to the equation $k_t = 0.5 \cdot k_{t-1}^{0.8} - 0.1 \cdot k_{t-1} + k_{t-1} - 0.8$.

the assumption of constant dividends; the higher the dividend payout, the greater the capital necessary to ensure that production can meet the outflows.

Furthermore, the steeper the slope of the blue line, the higher the productivity factor Z, signifying a reduced need for capital. This plays a significant role since firms with greater productivity can sustain their expenses with less capital, which correlates with a higher likelihood of enduring economic downturns.

Dynamics of debts To examine the dynamics of debt, consider a scenario where capital remains constant $k_t = k_{t-1} = \widehat{k}$, thus it is at the steady-state level. From equation 8 we get the finite difference equation for debt:

$$b_t = -f(\widehat{k}) + \delta\widehat{k} + Rb_{t-1} + d \tag{21}$$

Let's determine the condition for a stable path taking the partial derivatives with respect to b_{t-1}:

$$\frac{\partial b_t}{\partial b_{t-1}} = R \tag{22}$$

$$\tag{23}$$

Since $R > 1$, the partial derivative 21 will always be greater than one, thus the slope of the finite difference equation for debt will always be steeper than one. Adding a negative intercept due to positive dividends we get that under those conditions there exists a steady state of debt. Moreover, if the debt is below the steady state, the debt will shrink toward 0, while if the debt is over the steady state the dynamics of debt will explode toward $+\infty$. This is represented in the following phase diagram: The phase diagram illustrates the relationship between a firm's current debt (b_t) and its capacity

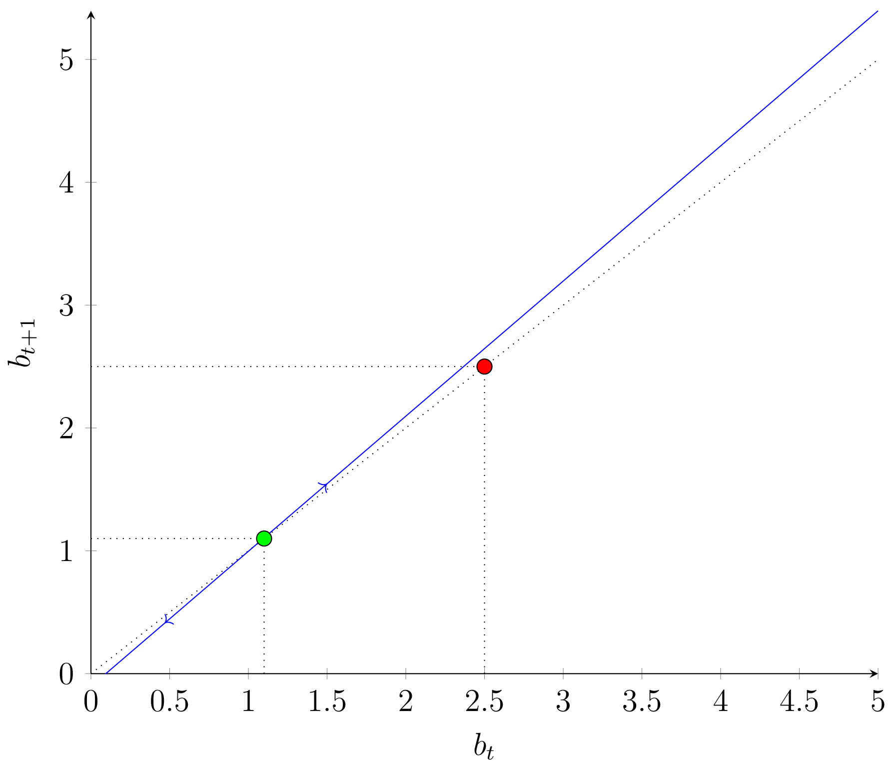


Figure 13: The phase diagram illustrates the progression of debt under the condition that the change in capital (Δk) is zero, with parameters set at $\delta = 0.1$, $r = 0.1$, $\alpha = 0.8$, $Z = 0.5$, $d = 0.8$,$\widehat{k} = 3$. The blue line represents the finite difference equation for debt as modeled by the equation $b_t = -f(\widehat{k}) + \delta \cdot \widehat{k} + R \cdot b_{t-1} + d_{t-1}$. The red dot marks the threshold beyond which debt cannot exceed capital, effectively serving as a limit on debt. The green dot signifies the steady state of the debt. The vertical or horizontal gap between the red and green dots quantifies the firm's equity.

for future operations (k_{t+1}), within the context of constant dividends. The steady state is indicated by the red dot, signifying the juncture at which the firm's output is precisely adequate to cover dividends, depreciation, and interest on its steady-state debt.

If dividends were to increase, this would necessitate a higher debt level to maintain the steady state, as the firm would have less equity. This change would be represented graphically by an elevated intercept on the curve, resulting in an increased debt burden.

As for productivity, firms with superior productivity require less debt to produce the same amount of dividends, as they operate more efficiently. This is depicted by their position on the b_t axis for a given k_{t+1}. However, when the goal is to maximize dividends, highly productive firms will need more capital to reach the optimal dividend payout, a point that will be elaborated upon while discussing the maximization problem later in the analysis.

In essence, the graph conveys how steady-state conditions are shaped by dividend policy and productivity, with the former influencing the firm's financial leverage and the latter determining its capital efficiency.

6 Free debt case: Ramsey-Cass-Kooopmans reinterpreted

This section outlines the intertemporal maximization problem faced by the firm in the free debt case, which is a Ramsey-Cass-Kooopmans model where there is a firm that seeks to maximize the utility of dividends instead of consumptions levels. The goal consists of maximizing the present value of future dividends, formulated as:

$$V_0 = \sum_{t=0}^{+\infty} \beta^t U(d_t),$$

where $U' > 0, U'' < 0$.

6.1 Steady State derivation

Consider a firm entirely financed by equity ($b_t = 0$ for all t), leading to a simplified flow-of-funds constrant equation:

$$k_{t+1} = k_t(1-\delta) + f(k_t) - d_t. \tag{24}$$

The maximization problem is tackled using a Lagrangian method, where the Lagrangian is defined as:

$$L_0 = \sum_{t=0}^{+\infty} \left[\beta^t U(d_t) - \beta^t \lambda_t \left[k_{t+1} - k_t(1-\delta) + f(k_t) - d_t\right]\right].$$

The first-order conditions for d_t, k_{t+1}, and λ_t for all periods $t = 0, 1, \ldots$ yield:

$$U'(d_t) = \lambda_t, \quad \forall t,$$

$$\beta^t \lambda_t = \beta^{t+1} \lambda_{t+1}[f'(k_{t+1}) + (1-\delta)], \quad \forall t,$$

This approach delineates the optimal strategy for dividend distribution and capital allocation in a debt-free case. In the infinite horizon model, the transversality condition reads: $\lim_{T\to\infty} \beta^T U'(d_t)\, k_{T+1} = 0$. Thus, policies promoting accelerated capital accumulation are ruled out. Differentiating equation concerning dividend levels at time $T+1$ yields the following set of first-order conditions:

$$U'(d_t) = \lambda_T. \tag{25}$$

Each Lagrange multiplier λ_t represents the marginal utility of dividends in period t.

From these first-order conditions (FOCs), we derive the Euler equation for dividends:

$$U'(d_t) = \beta U'(d_{t+1})[f'(k_{t+1}) + (1 - \delta)] \tag{26}$$

indicating that the marginal utility of distributing dividends at time t should match the discounted marginal utility of distributing dividends in the next period, adjusted for the net marginal product of capital after accounting for depreciation.

Steady state condition for dividens Imposing the steady state condition for dividends $d_t = d_{t+1} = \widehat{d}$ in 26 , we equate the marginal utilities across two consecutive periods:

$$U'(d_t) = U'(d_{t+1}) :$$

$$\frac{1}{\beta} = [f'(k_{t+1}) + (1 - \delta)],$$

This condition is satisfied if:

$$f'(k_{t+1}) = \frac{1}{\beta} - (1 - \delta),$$

Let's assume that:

$$y_{t+2} = Zk_{t+1}^{\alpha} = f(k_{t+1}) \tag{27}$$

then:

$$f'(k_{t+1}) = Z\alpha k_{t+1}^{\alpha-1}, \tag{28}$$

From 26 and 28, we get the steady state level of capital:

$$\widehat{k} = \left[\frac{\alpha\beta Z}{1-\beta\left(1-\delta\right)}\right]^{\frac{1}{1-\alpha}}. \tag{29}$$

steady state condition for capital Imposing steady state condition for capital ($k_t = k_{t-1} = \widehat{k}$) in the law of motion of capital 24 we get:

$$\widehat{d} = f(\widehat{k}) - \delta\widehat{k} \tag{30}$$

Using the s. s. level of capital 29 and 27 into 30, we get the s.s. dividend level $\widehat{d}$:

$$\widehat{d} = Z\left[\frac{\alpha\beta Z}{1-\beta(1-\delta)}\right]^{\frac{\alpha}{1-\alpha}} - \delta\left[\frac{\alpha\beta Z}{1-\beta(1-\delta)}\right]^{\frac{1}{1-\alpha}} \tag{31}$$

Thus we found the steady state level for capital and dividends.

6.2 Phase diagram

Steady state for dividends In this section, we will plot the phase diagram for capital and dividends exploiting steady-state conditions for capital and dividends.

The graph portrays the dynamics of dividends (d_t) in relation to the capital (k_t) of a firm, with a particular focus on the behavior when capital is below or above the steady-state level, denoted by $\hat{k}$.

When the capital is below the steady-state level ($k_t < \hat{k}$), thus on the left of the vertical line, the firm is optimal to increase dividends over time ($d_t < d_{t+1}$) as repre-

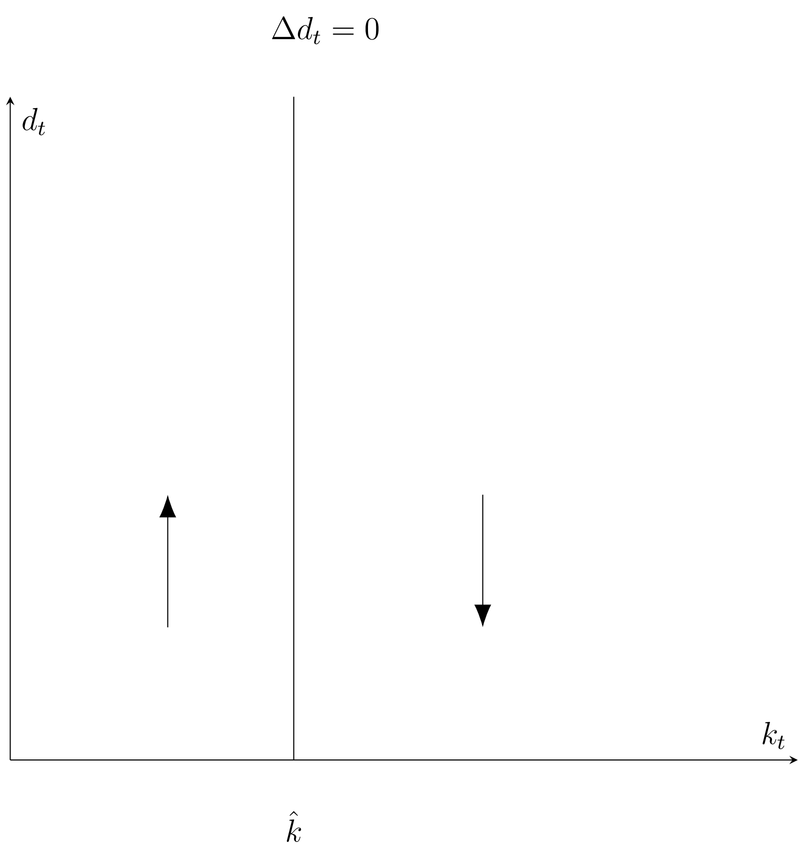


Figure 14: Phase diagram of dividends with respect to capital, depicting the steady state level for dividends.

sented by the arrow point upward. When instead ($k_t > \hat{k}$), divdends must shrink over time ($d_t > d_{t+1}$).

Steady state for capital Lets look at the locus in which capital is stationary $\Delta k = 0$ is given by the f-of-f constraint 30:

$$\widehat{d} = f(\widehat{k}) - \delta\widehat{k} \tag{32}$$

In our case, as obtained in the above section, the locus in which capital is stationary becomes 31:

$$\widehat{d} = Z\left[\frac{\alpha\beta Z}{1-\beta(1-\delta)}\right]^{\frac{\alpha}{1-\alpha}} - \delta\left[\frac{\alpha\beta Z}{1-\beta(1-\delta)}\right]^{\frac{1}{1-\alpha}} \tag{33}$$

This function starts at the origin since ($f(0) = 0$), with a maximum in $\underline{k}$ (defined as capital level such that $f'(\underline{k}) = \delta$). From equation 28 we can find the level of capital that maximises dividends at the steady state of capital:

$$\underline{k} = \left[\frac{\delta}{\alpha Z}\right] \tag{34}$$

While $\bar{k}$ is the capital level such that ($d_t = 0$), thus its obtained by solving ($f_(\bar{k}) - \delta\bar{k} = 0$) solving for the Cobb-Douglas production function 27, we get:

$$Z\bar{k}^{\alpha} = \delta\bar{k} \tag{35}$$

In the case in which a capital level $\check{k} \in [0, \bar{k}]$, the corresponding dividends level

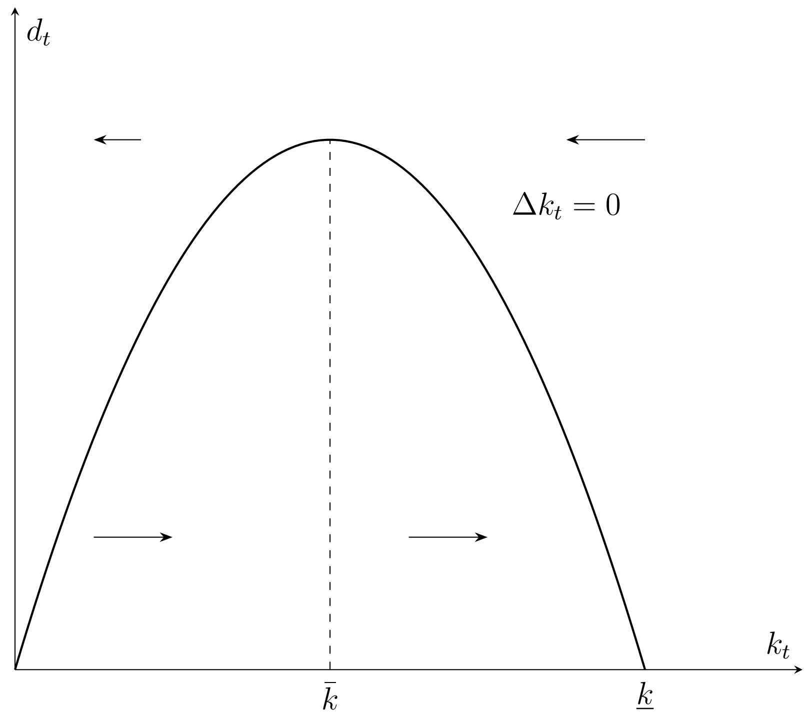


Figure 15: Phase diagram of dividends concerning capital, depicting the steady state level for capital.

that guarantee the stationarity of capital is:

$$\check{d} = f(\check{d}) - \delta\check{k} \tag{36}$$

If the firm distributes more dividends than $\check{d}$ the capital stock must decrease over time: since the dividends are too high the firm is consuming part of her capital. More precisely the firm is distributing more dividends than $\check{d}$, which guarantees that the difference between gross production, and dividends is exactly equal to capital depreciation. This behavior is represented by the arrows above the curve pointing to the left. If the firm consumes fewer dividends than $\check{d}$, the opposite happens: the firm increases its capital since there is a positive net investment. This behavior is represented by the arrows below the curve pointing toward the right.

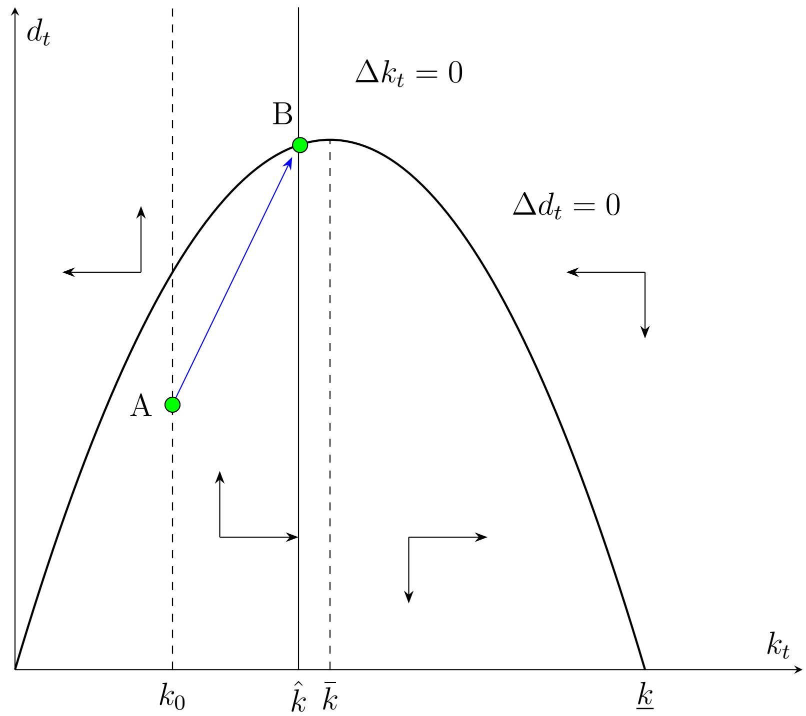


Figure 16: Dynamics of consumption concerning capital accumulation, showing the points of stability and instability along the curve.

Steady state for capital and dividend Merging the plot above we get a phase diagram that represents the conditions for stationarity. Notice that there exits 3 steady states: one at the origins due to the assumption $f(0) = 0$, the point $(\bar{k}; 0)$, and finally point B. Point B was obtained in the previous paragraph 29 and 31 and represented the point in which dividends and capital are at a steady state, and both are strictly positive (this point is also referred to as saddlepoint). The blue line depicts a possible saddlepath towards the B. Starting at A, the firm chooses exactly the dividend level that leads to the stationary point B. This path not only fulfills the difference equations 26 and 30, but also, the transversality condition which states:

$$\lim_{T \to \infty} \beta^T U'(d_t)\, k_{T+1} = 0 \tag{37}$$

Indeed as $t \to \infty$, capital and dividends approach their steady-state level which are both positive and finite, thus the marginal utility of dividends at $\hat{d}$ is also finite, hence 37 is valid.

In conclusion, in this paragraph, we have derived the steady-state levels for both capital and dividends and plotted the steady-state conditions in a phase diagram. In the next section, we will repeat the same exercise introducing debt and financial friction depicting

7 Introducing financial frictions

In this section we tackle the infinite maximization problem of the firm, introducing the possibility of financing through debt and two types of financial frictions. The first financial friction is a financing constraint ($\forall t, b_t = lk_t$), which implies fixed leverage for the firm. The second financial friction is introducing a participation constraint with monitoring cost $1 - \mu$ for the financial intermediaries. The goal is to understand how those frictions affect the steady state of capital and dividends.

7.1 Participation constraint of the financial intermediaries

The subsection delves into the constraints facing financial intermediaries within the model, highlighting how firms can finance themselves either through retaining dividends or accruing debt. Initially, the model assumed an exogenous interest rate, unaffected by the volume of debt, leading to an unrealistic scenario where interest rates remain constant regardless of debt levels relative to equity. To address this, the model introduces a financial market where the interest rate is determined by market-clearing conditions, and financial intermediaries operate under perfect competition to maximize profits.

According to Bernanke and Gertler [1986], lending should yield a profit equivalent to the opportunity cost of capital. Lenders earn interest plus the principal if borrowers repay successfully (with probability p) or acquire the firm's production assets (less depreciation) in case of bankruptcy. The lender's participation constraint is formulated as:

$$R_t \cdot b_t p + (1 - p)\mu f(k_t) = R_f b_t,$$

where r_f represents the risk-free rate, aligning the opportunity cost of capital with risk-free returns. This framework allows for the derivation of the interest rate as a function of p and $f(k_t)$, assuming no financial frictions and perfect information for lenders to accurately estimate recoverable amounts in all firm states.

The revised participation constraint is expressed as:

$$R_t = \frac{R_f}{p} - \frac{1 - p}{p} \frac{\mu f(k_t)}{b_t}. \tag{38}$$

For illustration, consider parameters $mu = 1$, $p = \{0.95, 0.9\}$, $\delta = 0.1$, $\alpha = 0.8$, $Z = 0.5$, $d = 0.8$, $\widehat{k} = 4$, and $R_f = \{0.05, 0.1\}$. The graphical representation suggests that as debt levels increase, so do interest rates, reflecting the risk-return dilemma for lenders. A higher risk profile, denoted by a more elevated red line necessitates greater returns to compensate for default risks. It's important to note that while the the graph assumes constant capital, real-world scenarios often see debt increases leading to higher capital and, consequently, greater production capacities. This reasoning clarifies why the curves do not start from the origin, as initial borrowing incorporates capital costs.

The graph 17 captures the dynamics between the debt stock b_t and the return on capital r. An increase in the debt stock leads to a rise in the interest rate, reflecting the augmented risk perceived by lenders. Displayed are two distinct lines: one repre-

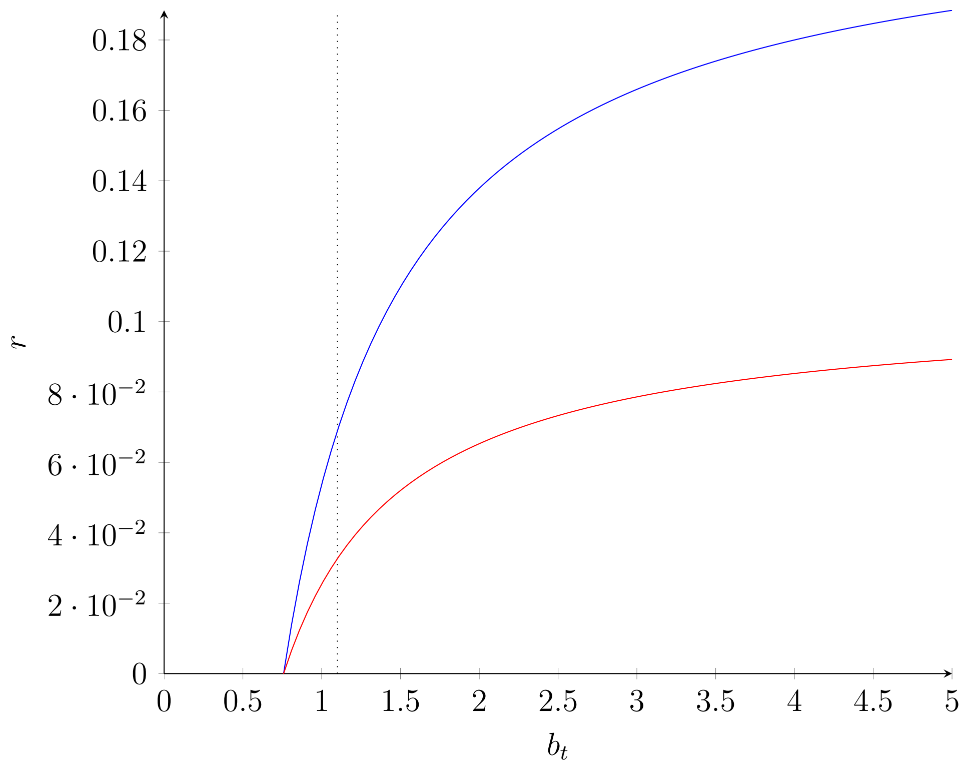


Figure 17: The figure presents a graphical analysis of the returns on loans as a function of the loan amount under a fixed capital level of $k = 3$. The red curve models the scenario where the default risk probability is $1 - p = 0.05$, implying a 5% chance of default, while the blue curve corresponds to a higher default risk at $1 - p = 0.1$, a 10% chance of default. Both curves reflect the increased interest rates required to compensate for the heightened risk as the debt stock grows. Notably, the opportunity cost of capital is maintained at 0.05 for the red one, while at 0.1 for the higher risk curve.

senting a riskier loan with a higher probability of default and the other indicating a safer loan with a lower default probability. As anticipated, the riskier loan scenario is characterized by a curve that lies above, dictating higher interest rates at each level of debt. The constant capital assumption underpins this model; however, in reality, an increase in debt usually translates into an increase in capital, thereby enhancing production potential. This factor accounts for the curves not starting at the origin.

Another way to visualize the participation constraint of the financial intermediaries is by defining $x = f(k)/b$. The graph 18 delineates a critical boundary within the participation constraint framework: as leverage approaches unsustainable levels, the interest rate escalates to a certain peak, signifying a cap on the maximum interest rate that deviates from the theoretical possibility of infinity. This ceiling on the rate is attributed to the fact that the probability of default, denoted by p, remains fixed and does not escalate alongside increasing leverage.

Ultimately, the participation constraint internalizes the interest rate of a loan as a function of the leverage, the opportunity cost of capital, and the default risk probability. By integrating this mechanism into the flow of funds model, the impact of debt on capital is mediated through the variable r, establishing a feedback loop where financial leverage influences and is influenced by the cost of borrowing.

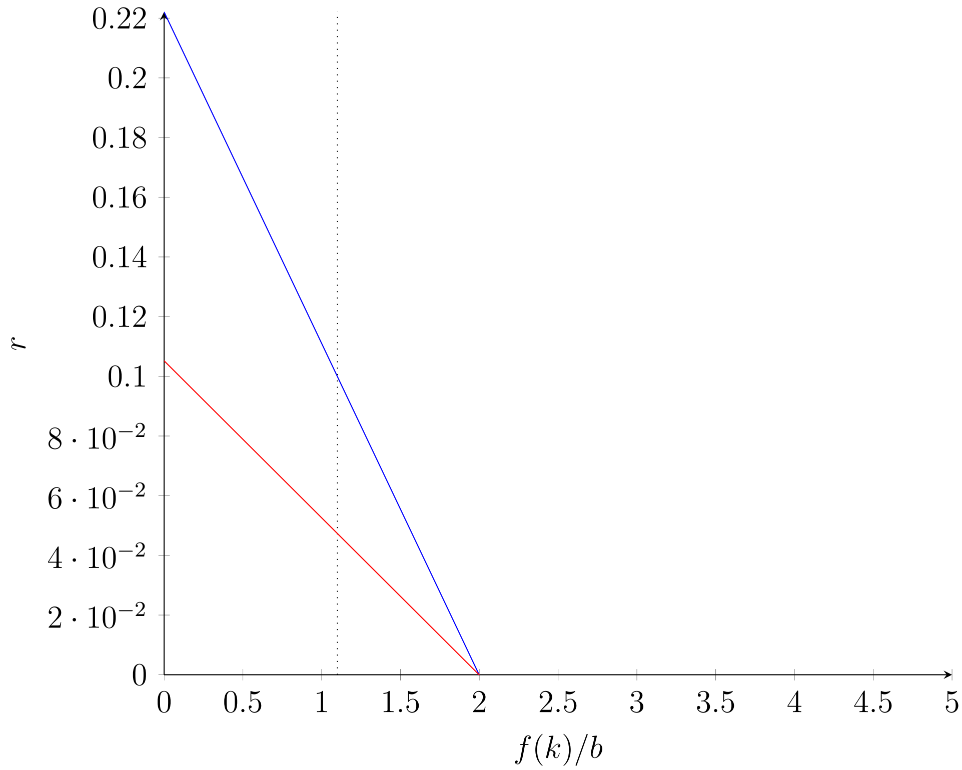


Figure 18: The figure presents a graphical analysis of the returns on loans as a function of the production over the debt level while keeping the level of capital at $k = 3$. The red curve models the scenario where the default risk probability is $1 - p = 0.05$, implying a 5% chance of default, while the blue curve corresponds to a higher default risk at $1 - p = 0.1$, a 10% chance of default. Both curves reflect the increased interest rates required to compensate for the heightened risk as the production-debt ratio grows. Notably, the opportunity cost of capital is maintained at 0.05 for the red one, while at 0.1 for the higher risk curve.

7.2 Steady state and phase diagram

The firm's objective is to maximize its value through the optimal selection of dividends over time:

$$\max_{\{d_t\}_{t=0}^{+\infty}} V_0 = \sum_{t=0}^{+\infty} \beta^t U(d_t)$$

subject to:

1. the flow of funds constraint 2,
2. the investment function 1,
3. the financing constraint $b_t = lk_t \quad \forall t$
4. the participation constraint of borrower 38

Consolidating the constraints we get the flow of funds constraints:

$$\begin{aligned} k_t &= \left\{k_{t-1}(1-\delta) - \left[\frac{R_f}{p} - \frac{1-p}{p}\frac{\mu f(k_{t-1})}{lk_{t-1}}\right] \cdot lk_{t-1} + f(k_{t-1}) - d_t\right\}(1-l)^{-1} \\ k_t &= \left[\frac{1+\mu-\mu p}{p} f(k_{t-1}) + \frac{p-\delta p - R_f l}{p} k_{t-1} - d_t\right](1-l)^{-1} \end{aligned} \tag{39}$$

The Lagrangian for this optimization problem is formulated as:

$$L = \sum_{t=0}^{+\infty} \beta^t U(d_t) - \beta^t \lambda_t \left[\frac{1+\mu-\mu p}{p} f(k_{t-1}) + \frac{p-\delta p - R_f l}{p} k_{t-1} - d_t\right](1-l)^{-1}, \tag{40}$$

leading to the first-order conditions for optimizing dividends and capital over time:

$$U'(d_t) = \frac{\lambda_t}{(1-l)}, \quad \forall t, \tag{41}$$

and the dynamic optimality conditions for capital allocation:

$$\lambda_t = \beta \frac{\lambda_{t+1}}{(1-l)} \left[f'(k_{t-1}) \frac{1+\mu-\mu p}{p} + \frac{p-\delta p - R_f l}{p} \right], \quad \forall t. \tag{42}$$

This formulation yields the Euler equation for dividends:

$$U'(d_t) = \frac{\beta}{(1-l)} U'(d_{t+1}) \left[f'(k_{t-1}) \frac{1+\mu-\mu p}{p} + \frac{p-\delta p - R_f l}{p} \right], \tag{43}$$

imposing ($d_t = d_{t+1} = \hat{d}$), we get:

$$\begin{aligned} \frac{(1-l)\, p}{\beta} &= f'(\hat{k})\,(1+\mu-\mu p) + (p-\delta p - R_f l) \\ f'(\hat{k}) &= \frac{p - pl - \beta p + \beta \delta p + \beta R_f l}{\beta\,(1+\mu-\mu p)} \end{aligned} \tag{44}$$

using the Cobb Douglas production function 28 into 44 we get:

$$\begin{aligned} Z\alpha\hat{k}^{\alpha-1} &= \frac{p - pl - \beta p + \beta \delta p + \beta R_f l}{\beta\,(1+\mu-\mu p)} \\ \hat{k} &= \left[\frac{Z\alpha\beta\,(1+\mu-\mu p)}{p - pl - \beta p + \beta \delta p + \beta R_f l} \right]^{\frac{1}{1-\alpha}} \end{aligned} \tag{45}$$

As in the free debt case, if the firm has less capital than the steady-state level $\hat{k}$, the firm is optimal to increase her dividends over time. When instead $\hat{k} < k$,the firm should be better of shrinking the dividends over time. Indeed it's easy to prove that imposing monitoring cost $1-\mu=1$, no debt $l=0$, and no probability of default $1-p=0$ we get the same capital level as in the debt-free case 29. Moreover its easy to see that the steady-state capital is higher compared to the debt-free:

$$\begin{aligned}
\left[\frac{Z\alpha\beta(1+\mu-\mu p)}{p-pl-\beta p+\beta\delta p+\beta R_f l}\right]^{\frac{1}{1-\alpha}} &\geq \left[\frac{\alpha\beta Z}{1-\beta(1-\delta)}\right]^{\frac{1}{1-\alpha}} \\
\frac{(1+\mu(1-p))(1-\beta(1-\delta))}{p(1-l)+\beta(R_f l-p(1-\delta))} &\geq 0 \\
(1+\mu(1-p))(1-\beta(1-\delta)) &\geq 0 \\
p(1-l)+\beta(R_f l-p(1-\delta)) &\geq 0
\end{aligned}$$

However, if we consider the case on which has no cost of monitoring $\mu=1$ and on which has $\mu=-.75$ ceteris paribus, $\hat{k}$ will be higher for the frictionless case.

Steady state for capital Imposing s.s. condition for capital ($k_t=k_{t+1}=\hat{k}$) into the flow of funds constraint 39:

$$\widehat{d}=\frac{1+\mu-\mu p}{p}f(\hat{k})-\left(\frac{lR_f+\delta p-lp}{p}\right)\hat{k} \tag{46}$$

It can be straightforwardly demonstrated that by setting the monitoring cost to $1-\mu=1$, eliminating debt with $l=0$, and removing the risk of default by setting $1-p=0$, we arrive at an identical level of dividends as observed in the scenario without debt 30.

The dynamics are equal to the free debt case except for the fact that the coefficient that multiplied $f(\hat{k})$ is higher since:

$$\begin{aligned}
\frac{1+\mu-\mu p}{p} &> 1 \\
1+\mu(1-p) &> p
\end{aligned}$$

This is valid also for the coefficient of $\hat{k}$ in equation 46, which is higher than the free debt case:

$$\begin{aligned}\frac{lR_f + \delta p - lp}{p} &> (1-\delta)\\ \frac{lR_f + \delta p - lp}{p(1-\delta)} &> 0\\ R_f l &> p(l-\delta)\\ p(1-\delta) &> 0\end{aligned}$$

7.3 Phase diagram

The goal of this section is to portray the phase diagram in two cases: one with monitoring costs and one without. However, we will use a less heuristic approach compared to the phase diagram of the free debt case, using the same value for parameters similar to Osotimehin and Pappadà [2017]:

Parameter	Symbol	Value
Discount factor	β	0.956
Risk-free rate	R_f	1.04
Depreciation rate	δ	0.07
Returns to scale	α	0.80
Aggregate productivity	$\bar{Z}$	0.5
Monitoring cost	$1-\mu$	$0, 0.75$
Productivity	Z	0.2
Probability of default	$1-p$	0.6

Table 5: Parameters used in Osotimehin and Pappadà [2017]

Moreover, we assume a fixed leverage of $l = 0.8$, since for the moment we want to

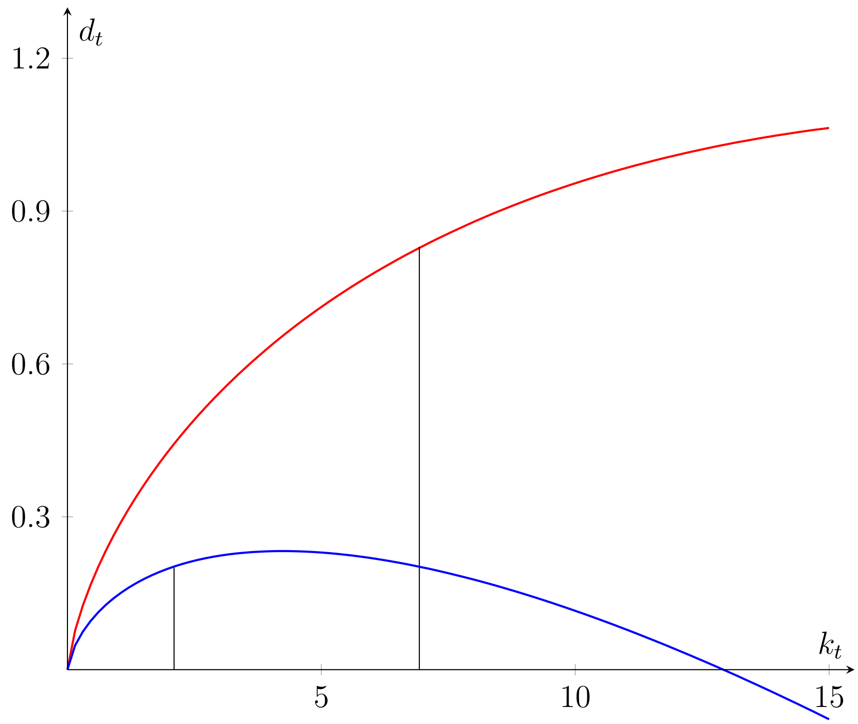


Figure 19: This phase diagram depicts the dividends dynamics as they relate to capital. The red line corresponds to a firm that is carrying debt without monitoring costs, whereas the blue-line represents the same firm but with monitoring costs

understand the effect of monitoring cost leaving all the other parameters equal.

The phase diagram illustrated in 19 depicts the capital accumulation dynamics under scenarios of fixed leverage and varying monitoring costs. While the overall dynamics remain consistent across both scenarios, the equilibrium capital level is notably reduced in firms that incur monitoring costs, in contrast to those without such costs. As a result, firms with monitoring costs settle into a steady state equilibrium for dividends, which leads to diminished dividend distributions compared to firms that do not bear these costs.

8 Finding optimal path

Addressing the dynamic optimization problem with an initial condition k_0, we employ a logarithmic utility function and frame the issue through a Bellman equation:

$$\max_{\{d_t\}_{t=0}^{\infty}} V_0 = \max_{\{d_t\}_{t=0}^{\infty}} \left\{ U(d_0) + \beta \left[\sum_{t=1}^{\infty} \beta^{t-1} U(c_t) \right] \right\}$$

subject to a dynamic capital accumulation constraint:

$$k_t = \left\{ k_{t-1}(1-\delta) - \left[\frac{R_f}{p} - \frac{1-p}{p} \frac{f(k_{t-1})}{l \cdot k_t} \right] l k_{t-1} + f(k_{t-1}) - d_{t-1} \right\} (1-l)^{-1} \, \forall t.$$

The aim is to determine the optimal dividend strategy d_t^* and the consequent capital levels k_{t+1}^* across all periods. The optimal policy $\varphi(.)$ links dividends and capital in a time-invariant manner, deduced from the constraint:

$$k_t = \left\{ k_{t-1}(1-\delta) - \frac{R_f}{p} l k_{t-1} + \frac{f(k_{t-1})}{p} - \varphi(k_{t-1}) \right\} (1-l)^{-1} = \zeta(k_1).$$

Given the continuous and differentiable nature of capital and dividends, the optimal dividend path can be represented as a function of initial capital, thereby defining the maximum value function $V(k_1)$ in terms of overall utility maximization. The revised problem formulation becomes:

$$V(k_0) = \max_{c_0} \{U(c_0) + \beta V(k_1)\} \tag{47}$$

$$\text{s.t. } k_1 = f(k_0) + (1-\delta)k_0 - c_0 \tag{48}$$

$$k_0 \text{ given.} \tag{49}$$

Before proceeding, it's critical to verify the solvability of the problem, adhering to the criteria for the existence and uniqueness of the solution:

1. $0 < \beta < 1$,
2. The utility function is continuous, bounded, and strictly concave,
3. The capital transition function is concave.

These conditions ensure the solution's uniqueness and strict concavity, although the logarithmic utility function $U(d_t) = \ln d_t$ might not strictly meet these criteria. Utilizing alternative theorems allows for the relaxation of the strict concavity requirement.

The optimal strategy is derived from the first order condition $U'(d_0^*) + \beta V'(k_1)\frac{\partial k_1}{\partial d_0} = 0$, with the capital transition function implying $\frac{\partial k_1}{\partial d_0} = -1$. The solution encompasses:

$$\begin{cases} V(k_0) = U(d_0^*) + \beta V(k_1), \\ k_1 = \left\{ k_0(1-\delta) - \frac{R_f}{p} l k_0 + \frac{f(k_0)}{p} - d_0^* \right\} (1-l)^{-1}, \\ U'(d_0^*) = \beta V'(k_1), \\ k_0 \text{ given.} \end{cases}$$

This system guides us towards the optimal path of dividends and capital levels, underpinning the dynamic economic analysis.

Guess and verify The method of "guess and verify" involves proposing a return function $U(d_t) = \ln d_t$ and working through a transition equation defined as $k_1 = \left\{ k_0(1-\delta) - \frac{R_f}{p} l k_0 + \frac{f(k_0)}{p} - d_0^* \right\} (1-l)^{-1}$. The first order condition (FOC) is specified as $d_0 = [\beta V'(k_1)]^{-1}$. When this FOC is incorporated into the transition equation, the formulation of the problem becomes a system of equations outlined as follows:

$$\begin{cases} V(k_0) = \ln(d_0^*) + \beta V(k_0), \\ k_1 = \left\{ k_0(1-\delta) - \frac{R_f}{p} l k_0 + \frac{f(k_0)}{p} - [\beta V'(k_{t+1})]^{-1} \right\} (1-l)^{-1}, \\ U'(d_0^*) = \beta V'(k_1), \\ k_0 \text{ given.} \end{cases}$$

Our initial guess for the solution is:

$$V(k_t) = e + f \ln k_t,$$

which leads to a refined system:

$$\begin{cases} e + f \ln k_0 = \ln\left(\frac{k_1}{\beta f}\right) + \beta[e + f \ln k_1], \\ k_1 = \left\{ k_0(1-\delta) - \frac{R_f}{p} l k_0 + \frac{f(k_0)}{p} - \left[\frac{k_1}{\beta f}\right] \right\} (1-l)^{-1}. \end{cases}$$

Assuming a condition to simplify the analysis, $p - \delta p - R_f l = 0$, we solve for k_1 and d_1, leading to expressions that relate capital and dividends directly to the parameters of the problem. These solutions indicate that dividends are a proportion of the output, dependent on the firm's productivity, leverage, and risk-free rate. The formulation highlights how dividends and capital evolve over time, with dividends being a constant share of the period's production.

Under conditions of no debt ($l = 0$ and thus $p = 1$) and ignoring depreciation, we obtain simplified expressions for capital and dividends in the steady state. This scenario suggests higher capital accumulation for a debt-free firm, as it does not bear interest expenses. The policy function derived reflects the relationship between dividends, firm productivity, and leverage, offering insights into the management of capital and dividends in different financial states of a firm.

8.1 Optimization Problem with Financial Frictions

Following the derivation of a closed-form solution for our policy function, we now integrate financial frictions stemming from information asymmetry between lenders and firms. We model this by introducing a discount factor μ on the perceived value of production, where $0 \leq \mu \leq 1$; a value closer to 0 indicates higher friction levels. Consequently, the lending activity modifies the optimization problem as follows:

$$\max_{\{d_t\}_{t=0}^{\infty}} V_0 = \max_{\{d_t\}_{t=0}^{\infty}} \left\{ U(d_0) + \beta \left[\sum_{t=1}^{\infty} \beta^{t-1} U(c_t) \right] \right\}$$

subject to

$$k_t = \left\{ k_{t-1}(1-\delta) - \left[\frac{R_f}{p} - \frac{1-p}{p} \frac{\mu f(k_{t-1})}{l \cdot k_t} \right] l k_{t-1} + f(k_{t-1}) - d_{t-1} \right\} \cdot (1-l)^{-1} \; \forall t.$$

Adapting our approach from the frictionless scenario, we propose:

$$V(k_0) = \ln(d_0^*) + \beta V(k_1),$$

where

$$k_1 = \left\{ k_0(1-\delta) - \frac{R_f}{p} l k_0 + \frac{f(k_0)}{p} - \left[\beta V'(k_{t+1})\right]^{-1} \right\} \cdot (1-l)^{-1},$$

and

$$U'(d_0^*) = \beta V'(k_1).$$

Our hypothetical solution takes the form:

$$V(k_t) = e + f \ln(k_t),$$

leading to:

$$e + f\ln(k_0) = \ln\left(\frac{k_1}{\beta f}\right) + \beta\left[e + f\ln(k_1)\right], \tag{50}$$

$$k_1 = \left\{ k_0(1-\delta) - \frac{R_f}{p} l k_0 + \frac{f(k_0)}{p} - \left[\frac{k_1}{\beta f}\right] \right\} \cdot (1-l)^{-1}. \tag{51}$$

Assuming $p - \delta p - R_f l = 0$ simplifies to:

$$k_1 = \left[k_0(p - p\delta - R_f l) + \mu Z k_0^\alpha\right] \cdot \left(\frac{\beta f}{\beta f - l\beta f + 1}\right) p^{-1}, \tag{52}$$

$$d_1 = \left[k_0(p - p\delta - R_f l) + \mu Z k_0^\alpha\right] \cdot \left(\frac{p}{\beta f - l\beta f + 1}\right), \tag{53}$$

$$\begin{aligned} e + f\ln(k_0) = \ln &\left\{ \left[k_0(p - p\delta - R_f l) + \mu Z k_0^\alpha\right] \cdot \left(\frac{p}{\beta f - l\beta f + 1}\right) \right\} \\ &+ \beta \left[e + f \ln \left\{ \left[k_0(p - p\delta - R_f l) + \mu Z k_0^\alpha\right] \cdot \left(\frac{\beta f}{\beta f - l\beta f + 1}\right) p^{-1} \right\} \right]. \end{aligned} \tag{54}$$

Hence, the transition and policy functions under financial frictions are formalized as:

$$k_1 = \frac{Z\mu k_0^{\alpha}(1-\delta)}{lR_f}\frac{\alpha\beta}{1-l\alpha\beta}, \tag{55}$$

$$d_0 = \frac{Z\mu k_0^{\alpha}(1-\delta)}{lR_f}\frac{1-\alpha\beta}{1-l\alpha\beta}\beta, \tag{56}$$

$$\widehat{k} = \left[\frac{Z\mu(1-\delta)}{lR_f}\frac{\alpha\beta}{1-l\alpha\beta}\right]^{\frac{1}{1-\alpha}}. \tag{57}$$

This analysis elucidates that financial frictions equate to a de facto reduction in productivity, which in turn diminishes dividends, capital levels, and the rate at which firms accumulate capital.

9 Simulation Study

To explore the distinctions between scenarios with and without financial frictions, we conduct a simulation exercise employing parameters consistent with those used in the Osotimehin and Pappadà [2017] study:

Parameter	Symbol	Value
Discount factor	β	0.956
Risk-free rate	R_f	1.04
Depreciation rate	δ	0.07
Returns to scale	α	0.70
Aggregate productivity	$\bar{Z}$	1
Monitoring cost	$1-\mu$	0.25

Table 6: Benchmark calibration

Our aim is to ascertain the impact of incorporating financial frictions into the model. Therefore, we set the leverage ratio ($l = 0.5$) and the initial capital ($k_0 = 1$). This allows us to examine capital evolution along the optimal path. For instance, we calculate $p = \frac{0.5 \cdot 1.04}{1-0.07} \approx 0.559$, with the outcomes depicted in the subsequent plots:

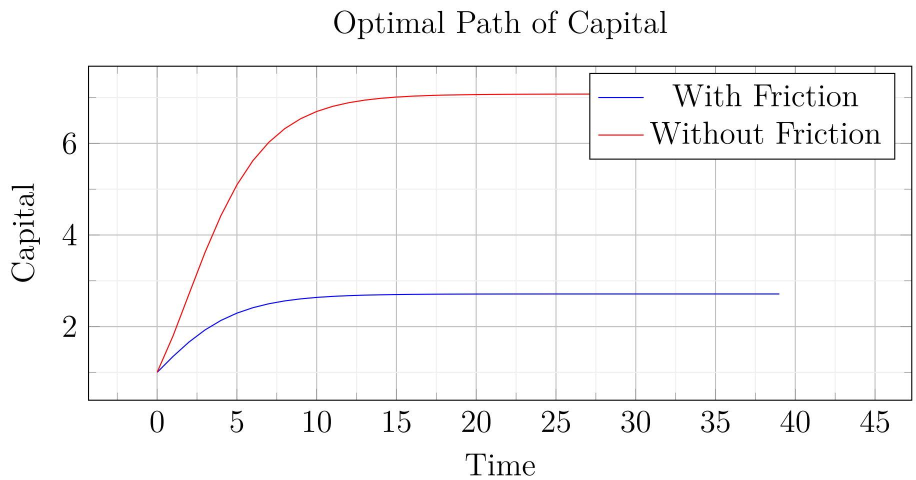


Figure 20: Evolution of capital over time.

The initial plot delineates the capital's transition function:

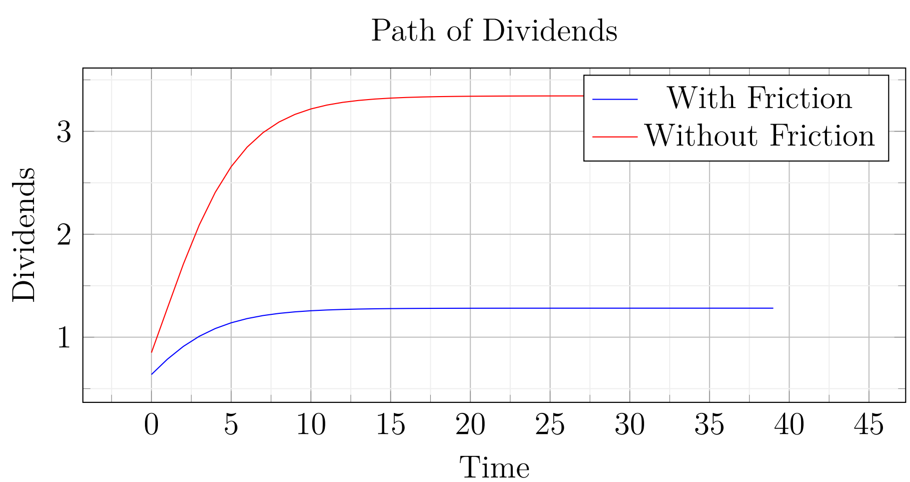


Figure 21: Evolution of dividends over time.

It is apparent that the trajectory of dividends is consistently higher in scenarios devoid of financial frictions, underscoring the impact of such frictions on diminishing returns. This comparison vividly demonstrates the differential outcomes in capital and dividend paths under varying financial conditions, highlighting the broader implications of financial frictions on economic performance and firm-level profitability.

9.1 heterogeneity and Aggregation Mechanism

To refine our model, we introduce heterogeneity among firms, marking a departure from uniform productivity and leverage ratios. Specifically, productivity levels (Z_i) now vary across firms, introducing a spectrum of efficiency within the model. Additionally, we diversify leverage ratios, ensuring no direct correlation between a firm's productivity and its leverage. This heterogeneity is captured from the outset by simulating the initial distribution of capital, leverage, and productivity, setting the stage for a dynamic interplay of firm characteristics. The aggregate production is then represented as:

$$\overline{K} = \int_{\underline{K}}^{\overline{k}} Z_i k_{i,t}^{\alpha} \, di$$

Business cycles impact capital fluctuations, with each firm's changes being proportional to its existing capital. Less efficient firms endure more significant reductions during economic downturns, while all firms enjoy capital increases during upswings. Market exit is modeled through two avenues: voluntary exit for returns on equity below the risk-free rate, and bankruptcy due to failure to cover debts and depreciation. The model incorporates a sinusoidal business cycle affecting output as follows:

$$\Delta\overline{K}_t = 1 + 0.05 \sin(t)$$

Productivity (Z) and leverage (l) follow truncated normal distributions:

$$l \sim \mathcal{N}(0.05, 0.1), \quad 0.01 \leq l \leq 1, \tag{58}$$

$$Z - 1 \sim \mathcal{N}(0.5, 0.1), \quad 1.01 \leq Z \leq 1.1. \tag{59}$$

The figure below demonstrates the results of a 20-step simulation for 10 firms, contrasting optimal and actual capital levels. The actual capital adjusts in response to the business cycle:

$$\Delta \overline{K}_t = 1 + 0.05 \sin(t)$$

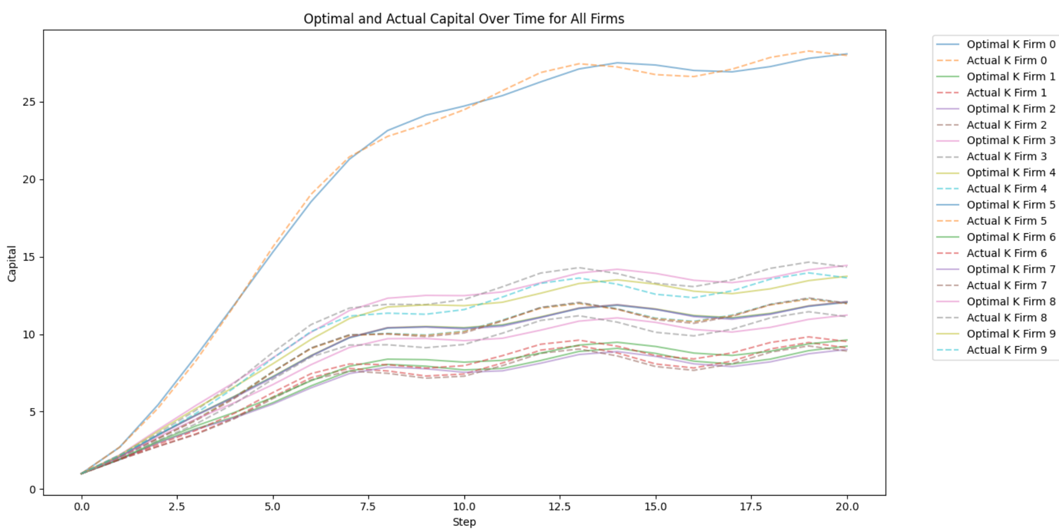


Figure 22: Illustrating a 20-step simulation of 10 firms, this plot compares the chosen optimal capital (k) against the actual capital (k) post-business cycle adjustment.

Following this, the distribution of dividends, based on actual capital, is visualized:

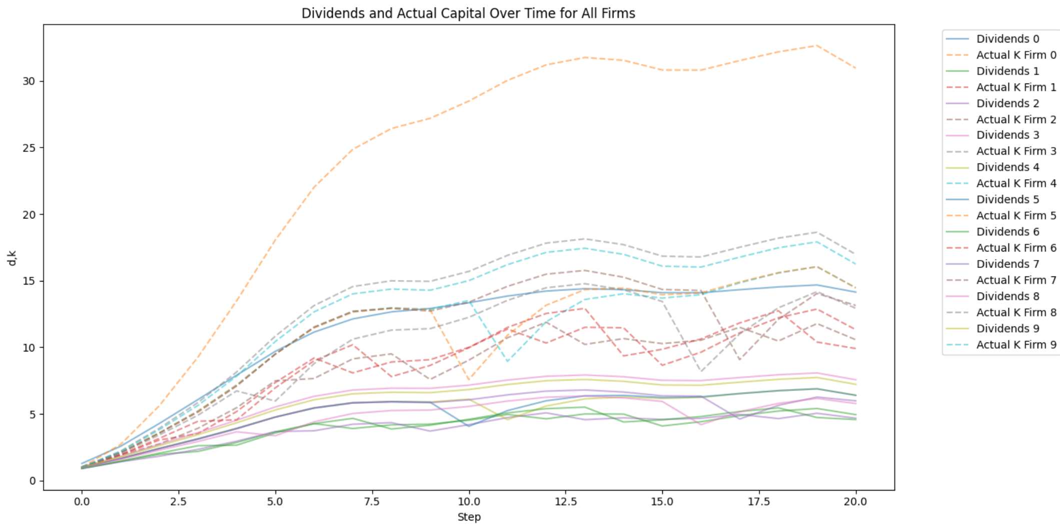


Figure 23: Displaying a 20-step simulation for 10 firms, this plot highlights dividends (d) and the actual capital (k) after business cycle adjustments.

Figure 23 distinctly shows that firms with superior productivity yield higher dividends. Moreover, upon achieving a steady state, both capital and dividends exhibit oscillations around a rising mean.

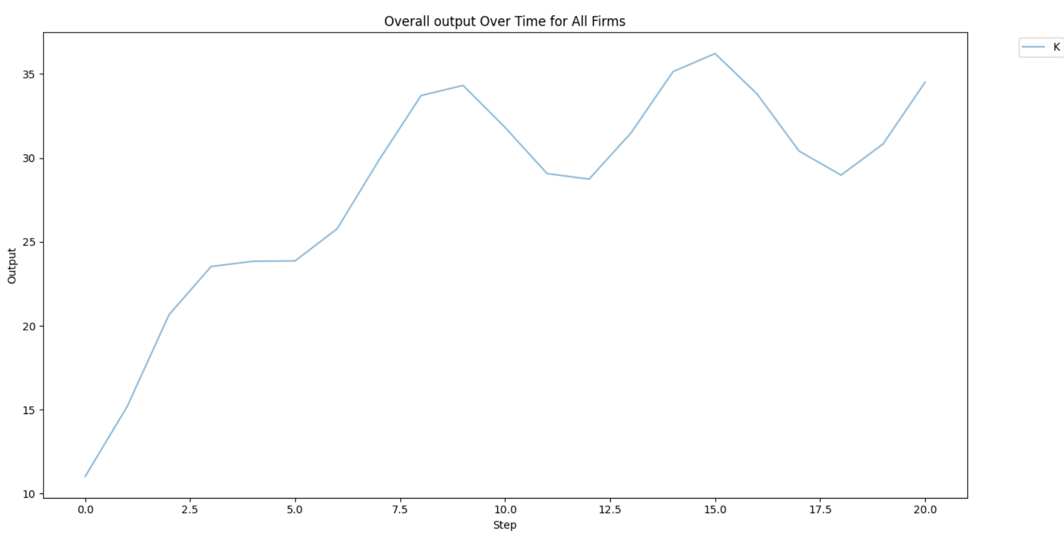


Figure 24: Showcasing a 20-step simulation of 10 firms, this plot reveals the adjusted output (K) following the business cycle impact.

9.2 The Exit Mechanism

A firm is prompted to exit the market if its return on capital ranks as the lowest within the firm distribution, making way for a newcomer who inherits the minimum capital from the existing pool. Subsequently, the exiting firm's capital is reallocated proportionally among the remaining firms, based on their respective returns on capital. The return on capital for firm i at time t is defined as:

$$R_{i,t} = \frac{d_{i,t}}{k_{i,t}}$$

Should a firm exhibit the $\min R$, it is compelled to exit the market due to possessing the lowest return on capital among all firms.[4]

In the ensuing simulation, while all parameters remain consistent with prior examples, the distinctive feature now is the market exit of firms. The subsequent figure illustrates both the optimal and actual capital paths for each firm:

[4]This criterion is enforced at each step, effectively merging the concepts of reallocation and exit mechanisms henceforth.

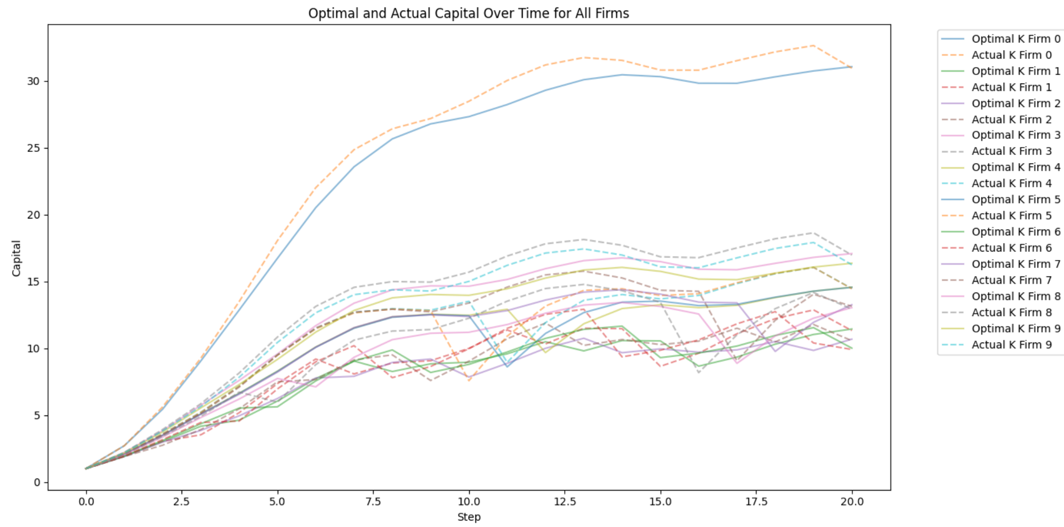


Figure 25: Displaying a 20-step simulation involving 10 firms: the optimal k represents the firm's capital choice, whereas the actual k reflects capital post-adjustment for the business cycle effect.

Contrasting with previous outcomes, the introduction of an exit mechanism and the redistribution of residual capital—proportionate to returns, yet ensuring the new entrant retains the minimum capital from the firm distribution—markedly influences capital trajectory. These mechanisms facilitate incumbent firms in accruing additional capital, thus hastening their approach to a stationary state, particularly benefiting the most productive entities. This dynamic underscores the cleansing effect of recessions, as capital reallocation during economic downturns favors business continuity. The impact of reallocation manifests in the subsequent graph, which delineates a higher dividends trajectory compared to scenarios devoid of reallocation mechanisms:

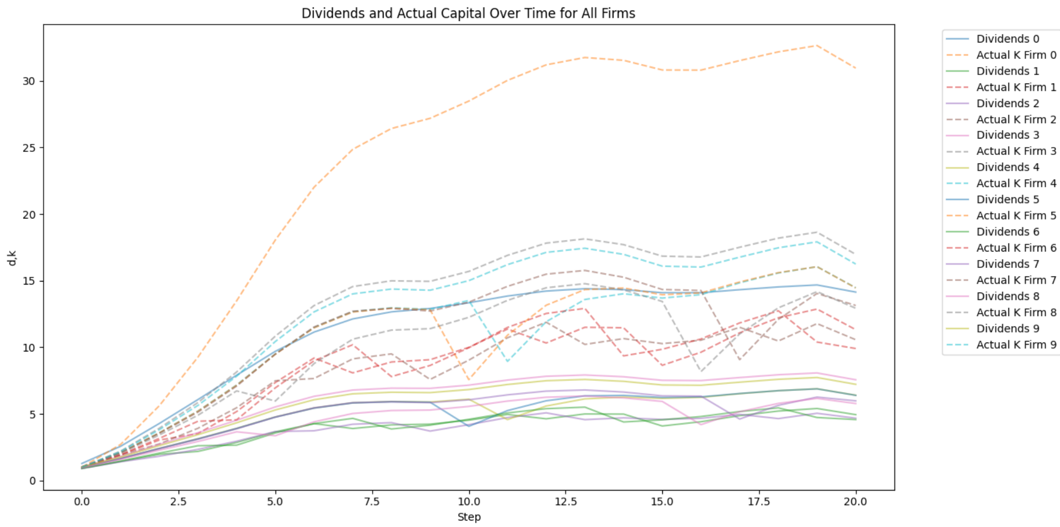


Figure 26: This figure showcases the results of a 20-step simulation with 10 firms, highlighting dividends (d) and actual capital (k) following business cycle adjustments.

Ultimately, examining overall production reveals an uptick attributable to capital reallocation when juxtaposed with prior simulations, underscoring the efficiency of the exit and reallocation strategies in fostering economic resilience and growth.

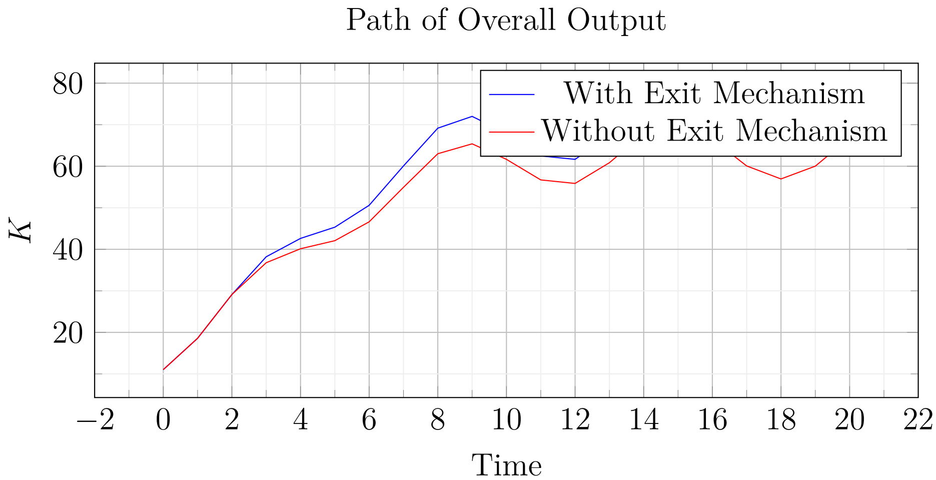


Figure 27: This plot illustrates a 20-step simulation involving 10 firms and demonstrates the impact of business cycle adjustments on overall output K. The blue line represents the scenario incorporating an exit mechanism, whereas the red line denotes the scenario without an exit mechanism, illustrating the variations in output across both contexts.

Integrating the exit and reallocation mechanisms notably enhances both the dividends' trajectory and the aggregate output, distinctly highlighting how the cleansing effect bolsters productivity.

9.3 Incorporating Financial Frictions into the Model

In keeping with the methodology set forth in Osotimehin and Pappadà [2017], financial frictions are incorporated into the model, parameterized as $1 - \mu = 0.25$. To discern the cleansing effect amidst financial frictions, an initial simulation is run where capital remains static due to the absence of a reallocation or exit mechanism. Subsequently, a contrasting simulation is performed where capital reallocation is possible, with both iterations subjected to financial frictions.

The following plot illustrates the progression of total output over time, displaying two distinct scenarios. The red line delineates the case without financial frictions, and the blue line portrays the scenario with frictions in place.

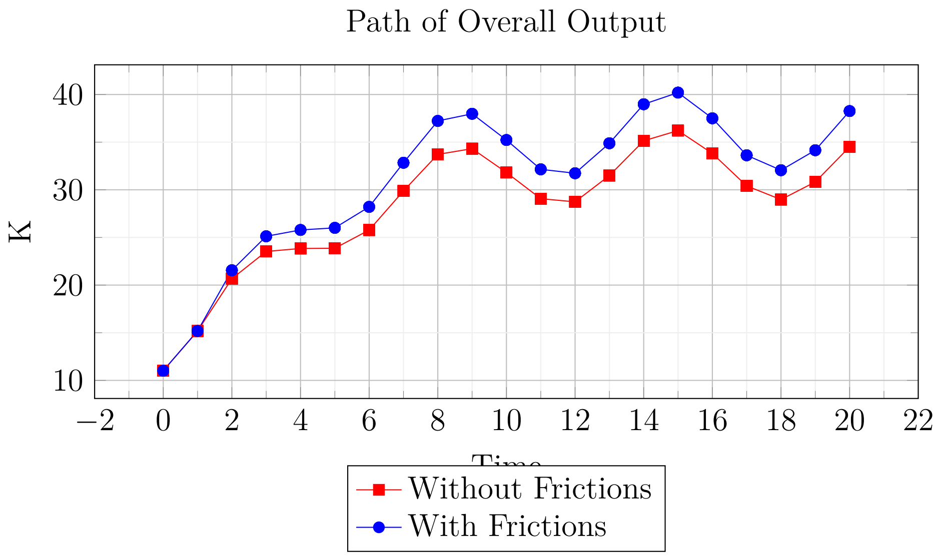


Figure 28: Evolution of the overall output over time. The red line represents the scenario without reallocation and exit mechanisms, while the blue line represents the scenario with these mechanisms under financial frictions.

Observations indicate that financial frictions attenuate overall production, akin to an effective reduction in productivity when firms operate with leverage. Although the cleansing effect is evident without financial frictions, it is imperative to examine whether this effect is sustained when frictions are introduced. To this end, the subsequent plot juxtaposes the overall output for two comparable financial friction scenarios: one with capital reallocation and one without.

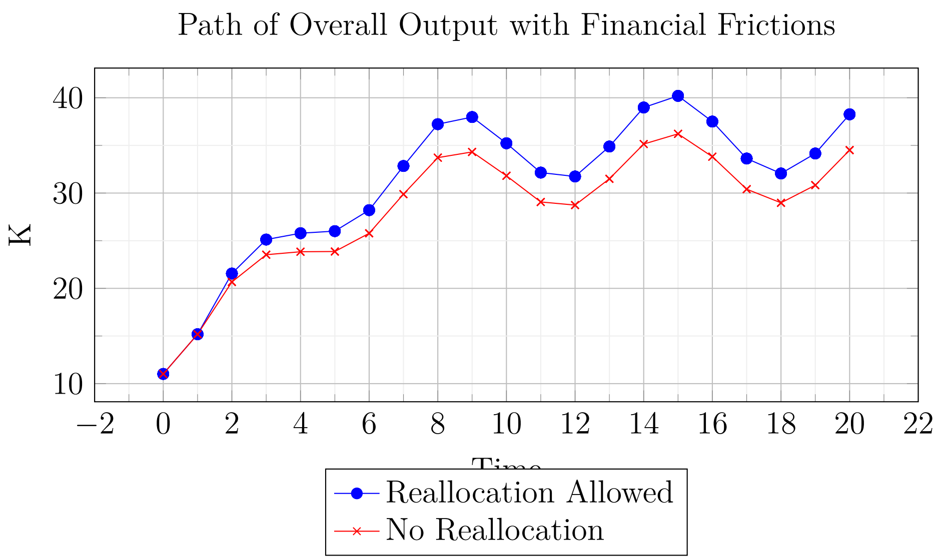


Figure 29: Comparison of overall output with financial frictions over time. The blue line illustrates the output when reallocation is allowed, while the red line indicates the output when it is not.

Figure 29 demonstrates that, even under financial frictions where $1 - \mu = 0.25$, there is a productivity-enhancing mechanism facilitated by capital reallocation due to firm exits. This occurs despite the presence of asymmetric information between financial intermediaries and firms, as discussed in Osotimehin and Pappadà [2017]. Furthermore, the cleansing effect on overall production appears to be cumulative, with its impact amplifying over time, as evidenced by the trend in the graph. Thus, it can be concluded that two economies, identical in their distribution of productivity and capital among firms and initialized with the same seed[5], will diverge in terms of output and productivity if one allows for capital reallocation through firm exits. This divergence is not only distinct but also grows as time progresses.

[5]All simulations were conducted with the same seed for consistency.

10 Solving the Belman with Benveniste-Scheinkman

In this section, I will address the intertemporal maximization problem without the assumption of fixed leverage, utilizing the Benveniste-Scheinkman equation to derive the functional form of the solution. This approach aims to demonstrate that the optimal capital trajectory does not significantly deviate from the results obtained in the prior analysis. However, I will refrain from conducting simulations using the policy function derived through this method, as it yields only a series of optimal paths rather than a comprehensive simulation framework.

$$V(k_t) = \max_{k_{t+1}, e_{t+1}} d_t + \beta V(k_{t+1})$$

$$s.t.$$

$$f(k_t) = Zk_t^{\alpha}$$

$$f(k_t) = d_t + (c + k_{t-1} - e_{t-1})(1 + r) + k_t - (c + k_t - e_t) - k_{-1}(1 - \delta)$$

$$(1 + r)(c + k_t - e_t)p + (1 - p)f(k_t) = (1 + r_f)(c + k_t - e_t)$$

$$B_t = c + k_t - e_t; R = 1 + r; R_f = 1 + r_f;$$

$$R = \frac{R_f}{p} - \frac{1 - p}{p} \frac{f(k_t)}{D_t}$$

To understand the mechanism behind this optimization problem, I first solve the three times problem working backward. The value function in $t = 2$ is

$$V_{t+2} = \max d_{t+2}$$

Since there firm will not exists in t+2, there are no investment $B_{t+2} = 0$, thus $0 = k_{t+2} + c - e_{t+2}$ as consequence $k_{t+2} = e_{t+2} - c$. Then we can rewrite the value function:

$$V_{t+2} = \max Z(e_{t+2}-c)^{\alpha}-(c+k_{t+1}-e_{t+1})(1+r_{t+1})-e_{t+2}+c+(c+e_{t+2}-c-e_{t+2})+k_{t+1}(1-\delta)$$

$$V_{t+2} = \max_{e_{t+2}} Z(e_{t+2} - c)^{\alpha} - B_{t+1}R_{l,t+1} - e_{t+2} + c + k_{t+1}(1 - \delta)$$

FOC:

$$\frac{\partial V_{t+2}}{\partial e_{t+2}} = Z\alpha(e_{t+2} - c)^{\alpha-1} - 1 = 0$$

$$(e_{t+2} - c)^{\alpha-1} = (Z\alpha)^{-1}$$

$$e_{t+2} = (Z\alpha)^{\frac{1}{1-\alpha}} + c$$

Thus:

$$d_{t+2} = Z\left[(Z\alpha)^{\frac{\alpha}{1-\alpha}}\right] - B_{t+1}R_{t+1} - \left[(Z\alpha)^{\frac{1}{1-\alpha}}\right] + k_{t+1}(1 - \delta)$$

$$V_{t+2} = Z\left[(Z\alpha)^{\frac{\alpha}{1-\alpha}}\right] - B_{t+1}R_{t+1} - \left[(Z\alpha)^{\frac{1}{1-\alpha}}\right] + k_{t+1}(1 - \delta)$$

Writing the problem in t+1:

$$V_{t+1} = \max_{e_{t+1},k_{t+1}} d_{t+1} + \beta V_{t+2}$$

$$d_{t+1} = Zk_{t+1}^{\alpha} - B_t R_L - k_{t+1} + B_{t+1} + k_t(1 - \delta)$$

FOCs:

$$\begin{cases} \frac{\partial V_{t+1}}{\partial e_{t+1}} = \frac{\partial d_{t+1}}{\partial e_{t+1}} + \beta\frac{\partial V_{t+2}}{\partial e_{t+1}} = 0 \\ \frac{\partial V_{t+1}}{\partial k_{t+1}} = \frac{\partial d_{t+1}}{\partial k_{t+1}} + \beta\frac{\partial V_{t+2}}{\partial K_{t+1}} = 0 \end{cases}$$

solving $\frac{\partial d_{t+1}}{\partial e_{t+1}}$:

$$\frac{\partial d_{t+1}}{\partial e_{t+1}} = Z\alpha k_{t+1}^{\alpha-1}\frac{\partial k_{t+1}}{\partial e_{t+1}} - \frac{\partial k_{t+1}}{\partial e_{t+1}} + \frac{\partial B_{t+1}}{\partial e_{t+1}}$$

$$\frac{\partial B_{t+1}}{\partial e_{t+1}} = \frac{\partial k_{t+1}}{\partial e_{t+1}} - 1$$

$$\frac{\partial d_{t+1}}{\partial e_{t+1}} = Z\alpha k_{t+1}^{\alpha-1}\frac{\partial k_{t+1}}{\partial e_{t+1}} - \frac{\partial k_{t+1}}{\partial e_{t+1}} + \frac{\partial k_{t+1}}{\partial e_{t+1}} - 1$$

solving $\frac{\partial V_{t+2}}{\partial e_{t+1}}$:

$$\frac{\partial V_{t+2}}{\partial e_{t+1}} = -\left[\frac{\partial B_{t+1}R_{t+1}}{\partial e_{t+1}} - \frac{\partial k_{t+1}}{\partial e_{t+1}}(1-\delta)\right]$$

$$\frac{\partial B_{t+1}R_{t+1}}{\partial e_{t+1}} = \frac{\partial B_{t+1}}{\partial e_{t+1}}R_{t+1} + B_{t+1}\frac{\partial R_{t+1}}{\partial e_{t+1}}$$

$$\frac{\partial B_{t+1}}{\partial e_{t+1}} = \frac{\partial k_{t+1}}{\partial e_{t+1}} - 1$$

$$\frac{\partial R_{t+1}}{\partial e_{t+1}} = -\frac{1-p}{p}\left\{\left[Z\alpha k_{t+1}^{\alpha-1}\frac{\partial k_{t+1}}{\partial e_{t+1}} - \delta\frac{\partial k_{t+1}}{\partial e_{t+1}}\right]B_{t+1} - \left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1\right)\left[Zk_{t+1}^{\alpha} - \delta k_{t+1}\right]\right\}B_{t+1}^{-2}$$

$$\frac{\partial B_{t+1}R_{t+1}}{\partial e_{t+1}} = \left[\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1\right]R_{t+1} - \frac{1-p}{p}\left\{\left[Z\alpha k_{t+1}^{\alpha-1}\frac{\partial k_{t+1}}{\partial e_{t+1}} - \delta\frac{\partial k_{t+1}}{\partial e_{t+1}}\right]B_{t+1} - \left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1\right)\left[Zk_{t+1}^{\alpha} - \delta k_{t+1}\right]\right.$$

$$\frac{\partial B_{t+1}R_{t+1}}{\partial e_{t+1}} = \left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1\right)\left[\left(zk_{t+1}^{\alpha} - \delta k_{t+1}\right)\frac{1-p}{p}B_{t+1}^{-1} + R_{t+1}\right] - \frac{1-p}{p}\frac{\partial k_{t+1}}{\partial e_{t+1}}\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right)$$

$$\frac{\partial B_{t+1}R_{t+1}}{\partial e_{t+1}} = \left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1\right)R_f - \frac{1-p}{p}\frac{\partial k_{t+1}}{\partial e_{t+1}}\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right)$$

$$\frac{\partial V_{t+2}}{\partial e_{t+1}} = -\left[\left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1\right)R_f - \frac{1-p}{p}\frac{\partial k_{t+1}}{\partial e_{t+1}}\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right) - \frac{\partial k_{t+1}}{\partial e_{t+1}}(1-\delta)\right]$$

Substituting into the first FOC, we get:

$$\frac{\partial V_{t+1}}{\partial e_{t+1}} = Z\alpha k_{t+1}^{\alpha-1}\frac{\partial k_{t+1}}{\partial e_{t+1}} - \frac{\partial k_{t+1}}{\partial e_{t+1}} + \frac{\partial k_{t+1}}{\partial e_{t+1}} - 1 - \beta\left[\left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1\right)R_f - \frac{1-p}{p}\frac{\partial k_{t+1}}{\partial e_{t+1}}\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right) - \frac{\partial k_{t+1}}{\partial e_{t+1}}(1 -\right.$$

second FOC:

solving $\frac{\partial d_{t+1}}{\partial k_{t+1}}$:

$$\frac{\partial d_{t+1}}{\partial k_{t+1}} = Z\alpha k_{t+1}^{\alpha-1} - 1 + \frac{\partial B_{t+1}}{\partial k_{t+1}}$$

$$\frac{\partial B_{t+1}}{\partial k_{t+1}} = 1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}$$

$$\frac{\partial d_{t+1}}{\partial k_{t+1}} = Z\alpha k_{t+1}^{\alpha-1} - \frac{\partial e_{t+1}}{\partial k_{t+1}}$$

solving $\frac{\partial V_{t+2}}{\partial k_{t+1}}$:

$$\frac{\partial V_{t+2}}{\partial k_{t+1}} = -\left[\frac{\partial B_{t+1} R_{t+1}}{\partial k_{t+1}} - (1-\delta)\right]$$

$$\frac{\partial B_{t+1} R_{t+1}}{\partial k_{t+1}} = \frac{\partial B_{t+1}}{\partial k_{t+1}} R_{t+1} + B_{t+1} \frac{\partial R_{t+1}}{\partial k_{t+1}}$$

$$\frac{\partial B_{t+1}}{\partial k_{t+1}} = 1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}$$

$$\frac{\partial R_{t+1}}{\partial k_{t+1}} = -\frac{1-p}{p}\left[\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right) B_{t+1} - \left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}\right)\left(Z k_{t+1}^{\alpha} - \delta k_{t+1}\right)\right] B_{t+1}^{-2}$$

$$\frac{\partial B_{t+1} R_{t+1}}{\partial k_{t+1}} = \left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}\right) R_{t+1} + \left\{\frac{1-p}{p}\left[(Z\alpha k_{t+1}^{\alpha-1} - \delta) B_{t+1} - \left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}\right)\left(Z k_{t+1}^{\alpha} - \delta k_{t+1}\right)\right] B_{t+1}^{-1}\right\}$$

$$\frac{\partial B_{t+1} R_{t+1}}{\partial k_{t+1}} = \left(1 - \frac{\partial e_{t+}}{\partial k_{t+1}}\right)\left[R_{t+1} + \frac{1-p}{p}\left(Z k_{t+1}^{\alpha} - \delta k_{t+1}\right) B_{t+1}^{-1}\right] - \frac{1-p}{p}\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right)$$

$$\frac{\partial B_{t+1} R_{t+1}}{\partial k_{t+1}} = \left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}\right) R_f - \frac{1-p}{p}\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right)$$

$$\frac{\partial V_{t+2}}{\partial k_{t+1}} = -\left[\left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}\right) R_f - \frac{1-p}{p}\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right) - (1-\delta)\right]$$

Substituting into the FOC:

$$\frac{\partial V_{t+1}}{\partial k_{t+1}} = Z\alpha k_{t+1}^{\alpha-1} - \frac{\partial e_{t+1}}{\partial k_{t+1}} - \beta\left[\left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}\right) R_f - \frac{1-p}{p}\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right) - (1-\delta)\right] = 0$$

thus the FOCs are:

$$\frac{\partial V_{t+1}}{\partial e_{t+1}} = Z\alpha k_{t+1}^{\alpha-1}\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1 - \beta\left[\left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1\right)R_f - \frac{1-p}{p}\frac{\partial k_{t+1}}{\partial e_{t+1}}\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right) - \frac{\partial k_{t+1}}{\partial e_{t+1}}(1-\delta)\right] = 0$$

$$\frac{\partial V_{t+1}}{\partial k_{t+1}} = Z\alpha k_{t+1}^{\alpha-1} - \frac{\partial e_{t+1}}{\partial k_{t+1}} - \beta\left[\left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}\right)R_f - \frac{1-p}{p}\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right) - (1-\delta)\right] = 0$$

rearranging $\frac{\partial V_{t+1}}{\partial k_{t+1}}$ to isolate k_{t+1}:

$$k_{t+1}^{\alpha-1} = \left[\frac{\partial e_{t+1}}{\partial k_{t+1}}(1-\beta R_f) + \beta\left(r_f + \frac{\delta}{p}\right)\right]\left\{Z\alpha\left[(1-\beta) - \frac{\beta}{p}\right]\right\}^{-1}$$

rearranging $\frac{\partial V_{t+1}}{\partial e_{t+1}}$ to isolate k_{t+1}:

$$k_{t+1}^{\alpha-1} = \left[\frac{\partial k_{t+1}}{\partial e_{t+1}}(1-\beta R_f) + \beta(r_f + \delta) + \delta\frac{1-p}{p}\right]\frac{p}{Z\alpha}$$

Equating the two equations:

$$\left[\frac{\partial e_{t+1}}{\partial k_{t+1}}(1-\beta R_f) + \beta\left(r_f + \frac{\delta}{p}\right)\right]\left\{Z\alpha\left[(1-\beta) - \frac{\beta}{p}\right]\right\}^{-1} = \left[\frac{\partial k_{t+1}}{\partial e_{t+1}}(1-\beta R_f) + \beta(r_f + \delta) + \delta\frac{1-p}{p}\right]$$

From this equation, you can isolate $\frac{\partial e_{t+1}}{\partial k_{t+1}}$ to solve for it explicitly.

$$\frac{\partial e_{t+1}}{\partial k_{t+1}} = -\left[\frac{\partial k_{t+1}}{\partial e_{t+1}}(1-\beta R_f) + \beta(r_f + \delta) + \delta\frac{1-p}{p}\right]\frac{p}{Z\alpha}\left\{Z\alpha\left[(1-\beta) - \frac{\beta}{p}\right]\right\}(1-\beta R_f)^{-1} - \beta\left(r_f + \frac{\delta}{p}\right.$$

$$\frac{\partial e_{t+1}}{\partial k_{t+1}} = -\left[\frac{\partial k_{t+1}}{\partial e_{t+1}}(1-\beta R_f) + \delta\frac{1-p}{p}\right]\frac{\beta p + \beta - p}{1-\beta R_f} - \beta\left(r_f + \frac{\delta}{p}\right)(1-\beta R_f)^{-1}$$

Since $\frac{\partial e_{t+1}}{\partial k_{t+1}}$ is the reciprocal of $\frac{\partial k_{t+1}}{\partial e_{t+1}}$, we can compute the optimal path of the networth as a function of the capital. Defining $y = \frac{\partial e_{t+1}}{\partial k_{t+1}}$ and ths $\frac{y^{-1} = \partial e_{t+1}}{\partial k_{t+1}}$:

$$y = -\left[\frac{1}{y}\left(1 - \beta R_f\right) + \delta \frac{1-p}{p}\right] \frac{\beta p + \beta - p}{1 - \beta R_f} - \beta\left(r_f + \frac{\delta}{p}\right)\left(1 - \beta R_f\right)^{-1}$$

To solve the given equation

$$y = -\left[\frac{1}{y}\left(1 - \beta R_f\right) + \delta \frac{1-p}{p}\right] \frac{\beta p + \beta - p}{1 - \beta R_f} - \beta\left(r_f + \frac{\delta}{p}\right)\left(1 - \beta R_f\right)^{-1}$$

Use Python and the sympy library to solve the equation:

```python
from sympy import symbols, Eq, solve

# Define the symbols
y, beta, R_f, delta, p, r_f = symbols('y beta R_f delta p r_f')

# Define the equation
equation = Eq(y, - ((1/y) * (1 - beta * R_f) + delta * (1 - p) / p) * (
    beta * p + beta - p) / (1 - beta * R_f) - beta * (r_f + delta / p) /
    (1 - beta * R_f))

# Solve for y
solution = solve(equation, y)
```

The equation for y is given by:

$$y = \frac{\beta\delta \pm \sqrt{\Delta}}{2p(\beta R_f - 1)}$$

In the solutions provided, Δ represents the discriminant of the quadratic equation that was formed during the solution process. It is the expression under the square root in the solutions. The discriminant Δ in this case is a complex expression involving the variables R_f, β, p, δ, and r_f. Specifically, Δ is given by:

$$\Delta = (-\beta\delta - \beta p^2\delta \left(\frac{1}{p} - 1\right) + p^2\delta \left(\frac{1}{p} - 1\right) \tag{60}$$

$$- \beta p\delta \left(\frac{1}{p} - 1\right) - \beta p r_f)^2 \tag{61}$$

$$- 4(\beta p R_f - p)(\beta^2 p^2 R_f - \beta p^2 R_f + \beta^2 p R_f \tag{62}$$

$$- \beta p^2 + p^2 - \beta p) \tag{63}$$

The discriminant Δ is given by:

$$\Delta = [\text{complex expression involving } R_f,\ \beta,\ p,\ \delta, \text{ and } r_f]$$

The sign of the solutions depends on:

- The values of β, δ, p, R_f, and r_f.
- The value and sign of Δ.

Since Δ involves these parameters in a complex manner, the sign of the solutions can be:

- Real and positive, real and negative, or complex (depending on the sign and magnitude of Δ and other parameters).
- Determined specifically only when actual values for the parameters are provided.

Since now we have a partial derivative as a function of parametrs we can retrive the relation between equity and capital. Rewriting the solutions as:

$$y = \frac{\partial e_{t+1}}{\partial k_{t+1}} = \frac{N}{D}$$

Given the partial derivative:

$$\frac{\partial e_{t+1}}{\partial k_{t+1}} = \frac{N}{D}$$

where N and D are constants with respect to k, we want to integrate this with respect to k.

Since N and D do not depend on k, the integral is straightforward:

$$\int \frac{N}{D}\, dk$$

Integrating a constant with respect to k yields:

$$\int \frac{N}{D}\, dk = \frac{N}{D} \cdot k + C$$

where C is the constant of integration.

Thus, the set of possible solutions is:

$$e_{t+1} = \frac{N}{D} \cdot k_{t+1} + C$$

where C is determined based on boundary conditions or initial values. Given the relationship:

$$k_{t+1}^{\alpha-1} = \left[\frac{\partial e_{t+1}}{\partial k_{t+1}} (1 - \beta R_f) + \beta \left(r_f + \frac{\delta}{p}\right)\right] \left\{Z\alpha \left[(1-\beta) - \frac{\beta}{p}\right]\right\}^{-1}$$

we can retrieve k_{t+1}.

Substituting $\frac{\partial e_{t+1}}{\partial k_{t+1}} = \frac{N}{D}$ into the equation, we get:

$$k_{t+1}^{\alpha-1} = \left[\frac{N}{D}(1-\beta R_f) + \beta\left(r_f + \frac{\delta}{p}\right)\right]\left\{Z\alpha\left[(1-\beta) - \frac{\beta}{p}\right]\right\}^{-1}$$

Sine now we have the optimal k we can retrieve the optimal path of networth:

$$e_{t+1} = \frac{N}{D}\cdot\left[\frac{N}{D}(1-\beta R_f) + \beta\left(r_f + \frac{\delta}{p}\right)\right]\left\{Z\alpha\left[(1-\beta) - \frac{\beta}{p}\right]\right\}^{-1} + C$$

Thus the debt at time t+1 is:

$$\begin{aligned} B_{t+1} = & \left[\frac{N}{D}(1-\beta R_f) + \beta\left(r_f + \frac{\delta}{p}\right)\right]\left\{Z\alpha\left[(1-\beta) - \frac{\beta}{p}\right]\right\}^{-1} + c \\ & - \left\{\frac{N}{D}\cdot\left[\frac{N}{D}(1-\beta R_f) + \beta\left(r_f + \frac{\delta}{p}\right)\right]\left\{Z\alpha\left[(1-\beta) - \frac{\beta}{p}\right]\right\}^{-1} + C\right\} \end{aligned} \tag{64}$$

References

Friedrich A. von (Friedrich August) Hayek. *Monetary theory and the trade cycle / by Friedrich A. Hayek; translated from the German by N. Kaldor and H.M. Croome.* The Bedford series of economic handbooks. Economic theory section. J. Cape, London, 1933.

F. A. Hayek and Bruce Caldwell. *The Pure Theory of Capital.* Number 9780226320991 in University of Chicago Press Economics Books. University of Chicago Press, January 1941. ISBN ARRAY(0x42a86798). URL https://ideas.repec.org/b/ucp/bkecon/9780226320991.html.

Robert M. Solow. A contribution to the theory of economic growth. *The Quarterly*

Journal of Economics, 70(1):65–94, 1956. ISSN 00335533, 15314650. URL http://www.jstor.org/stable/1884513.

T. W. Swan. Economic growth and capital accumulation. *Economic Record*, 32(2):334–361, 1956. doi: https://doi.org/10.1111/j.1475-4932.1956.tb00434.x. URL https://onlinelibrary.wiley.com/doi/abs/10.1111/j.1475-4932.1956.tb00434.x.

George W Stadler. Business cycle models with endogenous technology. *American Economic Review*, 80(4):763–78, 1990. URL https://EconPapers.repec.org/RePEc:aea:aecrev:v:80:y:1990:i:4:p:763-78.

C Scott Clark. Labor hoarding in durable goods industries. *The American Economic Review*, pages 811–824, 1973.

Craig Burnside, Martin Eichenbaum, and Sergio Rebelo. Labor hoarding and the business cycle. *Journal of Political Economy*, 101(2):245–273, 1993. doi: 10.1086/261875. URL https://doi.org/10.1086/261875.

Steven J. Davis and John Haltiwanger. Gross job creation, gross job destruction, and employment reallocation. *The Quarterly Journal of Economics*, 107(3):819–863, 1992. ISSN 00335533, 15314650. URL http://www.jstor.org/stable/2118365.

Ricardo J. Caballero and Mohamad L. Hammour. The cleansing effect of recessions. *The American Economic Review*, 84(5):1350–1368, 1994. ISSN 00028282. URL http://www.jstor.org/stable/2117776.

Sophie Osotimehin and Francesco Pappadà. Credit frictions and the cleansing effect of recessions. *The Economic Journal*, 127(602):1153–1187, 2017. doi: https://doi.org/10.1111/ecoj.12319. URL https://onlinelibrary.wiley.com/doi/abs/10.1111/ecoj.12319.

Olivier Jean Blanchard, Peter Diamond, Robert E. Hall, and Kevin Murphy. The cyclical behavior of the gross flows of u.s. workers. *Brookings Papers on Economic Activity*, 1990(2):85–155, 1990. ISSN 00072303, 15334465. URL http://www.jstor.org/stable/2534505.

Lucia Foster, Cheryl Grim, and John Haltiwanger. Reallocation in the great recession: Cleansing or not? *Journal of Labor Economics*, 34(S1):S293–S331, 2016. doi: 10.1086/682397. URL https://doi.org/10.1086/682397.

Steven J. Davis and John Haltiwanger. Gross job creation and destruction: Microeconomic evidence and macroeconomic implications. *NBER Macroeconomics Annual*, 5:123–168, 1990. ISSN 08893365, 15372642. URL http://www.jstor.org/stable/3585137.

Martin Neil Baily, Charles Hulten, David Campbell, Timothy Bresnahan, and Richard E Caves. Productivity dynamics in manufacturing plants. *Brookings papers on economic activity. Microeconomics*, 1992:187–267, 1992.

Teresa C Fort, John Haltiwanger, Ron S Jarmin, and Javier Miranda. How firms respond to business cycles: The role of firm age and firm size. *IMF Economic Review*, 61(3):520–559, 2013.

F. A. Hayek and Bruce Caldwell. *The Pure Theory of Capital*. Number 9780226320991 in University of Chicago Press Economics Books. University of Chicago Press, January 2007. ISBN ARRAY(0x3cfcacd8). URL https://ideas.repec.org/b/ucp/bkecon/9780226320991.html.

John M Abowd and Francis Kramarz. The costs of hiring and separations. *Labour Economics*, 10(5):499–530, 2003. ISSN 0927-5371. doi: https://doi.org/10.

1016/S0927-5371(03)00017-4. URL https://www.sciencedirect.com/science/article/pii/S0927537103000174.

Jordi Galí and J.L. Hammour. Long run effects of business cycles. Working papers, Columbia - Graduate School of Business, 1992. URL https://EconPapers.repec.org/RePEc:fth:colubu:92-26.

Ben S Bernanke and Mark Gertler. Inside the black box: the credit channel of monetary policy transmission. *Journal of Economic perspectives*, 9(4):27–48, 1995.

Ben S Bernanke and Mark Gertler. Agency costs, collateral, and business fluctuations. Working Paper 2015, National Bureau of Economic Research, September 1986. URL http://www.nber.org/papers/w2015.