# Finding the optimal path

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## 1 Frictionless economy

Given that the evolution of the net worth is the following:

$$e_{t+1} = Z(\theta+\epsilon)k_t^{\alpha} + (1-\delta)k_t - (1+r)(c+k_t-e_t)$$

The value of the firm at time t is given by:

$$V_t(e_t) = \max_{k_t} e_t + \beta V_{t+1}(e_{t+1})$$

Solving the belman equation, the FOC are given by:

$$\frac{\vartheta V_t(e_t)}{\vartheta k_t} = \frac{\vartheta e_t}{\vartheta k_t} + \beta \frac{\vartheta V_{t+1}(e_{t+1})}{\vartheta k_t}$$

Strong assumption a change in $k_t$ does not have an impact on $e_t$ only on $e_{t+1}$:

$$\frac{\vartheta e_t}{\vartheta k_t} = 0$$

Using the envelope theorem:

$$\frac{\vartheta V_{t+1}(e_{t+1})}{\vartheta k_t} = \frac{\vartheta V_{t+1}(e_{t+1})}{\vartheta e_{t+1}} \frac{\vartheta e_{t+1}}{\vartheta k_t}$$

$$\frac{\vartheta V_{t+1}(e_{t+1})}{\vartheta e_{t+1}} = 1 + \beta(1+r)$$

$$\frac{\vartheta e_{t+1}}{\vartheta k_t} = Z(\theta+\epsilon)\alpha k_t^{\alpha-1} - (\delta+r)$$

Thus:

$$\frac{\vartheta V_{t+1}(e_{t+1})}{\vartheta k_t} = [1+\beta(1+r)][Z(\theta+\epsilon)\alpha k_t^{\alpha-1} - (\delta+r)]$$

So the FOC became:

$$\frac{\vartheta V_t(e_t)}{\vartheta k_t} = 0 + \beta[1+\beta(1+r)][Z(\theta+\epsilon)\alpha k_t^{\alpha-1} - (\delta+r)] = 0$$

The optimal level of capital at time t is:

$$\widehat{k}_t = \frac{\beta(1+r)Z(\theta+\epsilon)\alpha}{\delta+r}^{\frac{1}{1-\alpha}}$$

Plotting the graph in (assuming $X = Z(\theta+\epsilon)$) and $y = \widehat{k}_t$

$$y = \frac{0.8x}{0.03+0.1}^{\frac{1}{1-0.8}}$$

Its interesting to see that if there is an increase in productivity the firm need more optimal capital K, while the most low productivity firms need less capital to operate

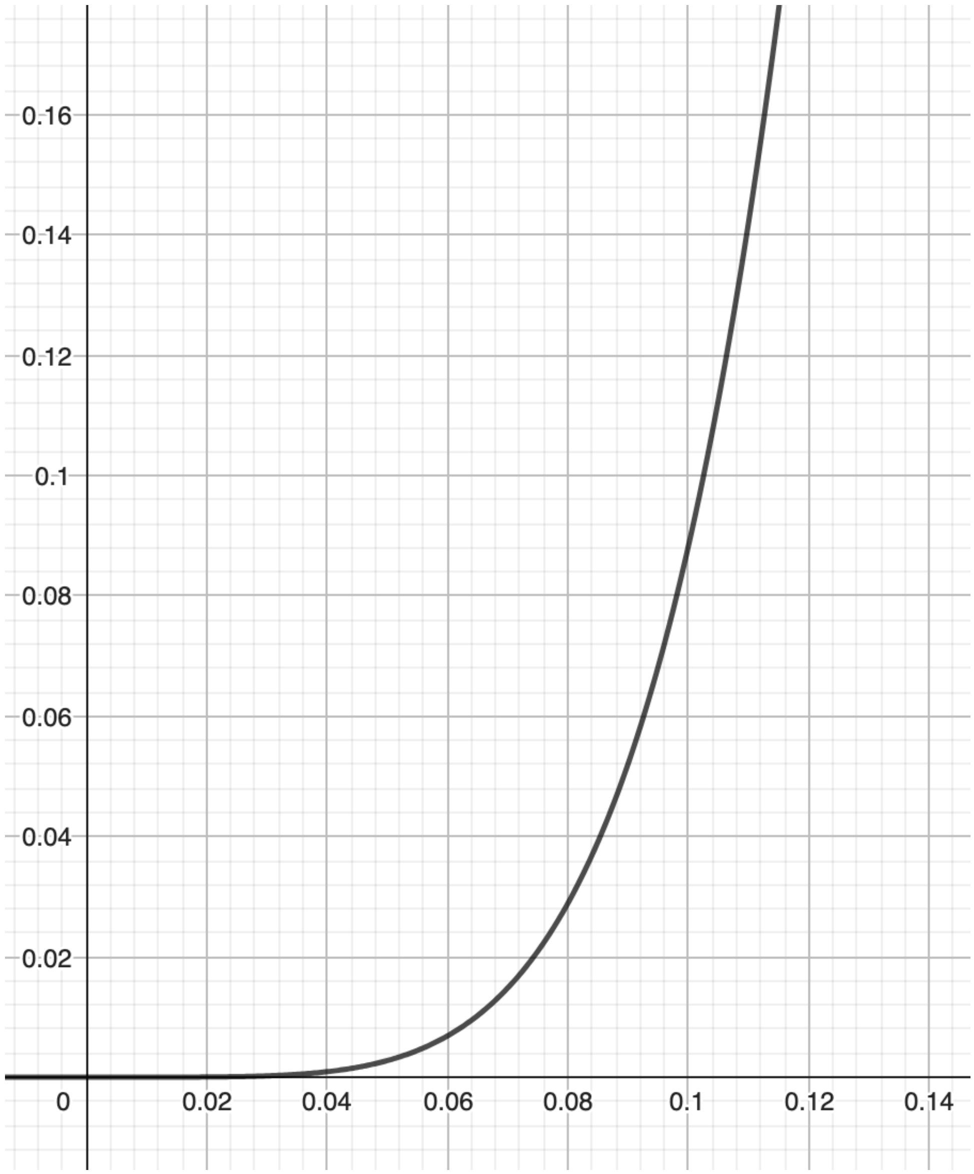


Figure 1: $X = Z(\theta + \epsilon)$ and $y = \widehat{k}_t$

## 1.1 Frictions

Now I consider the participation constraint of the borrower given that she could observe $\epsilon$ only paying $\mu k^\alpha$ so:

$$(1+r)(k+c+e)(1-\Phi(\overline{\epsilon})) + \int_{-\infty}^{\overline{\epsilon}} [Z(\theta+\overline{\epsilon})k^\alpha + (1-\delta)k - \mu k^\alpha]\, d\Phi(\epsilon) \geq (1+r)(c+k+e)$$

$r$ is the rate of interest that makes equal the expected value of borrowing to the opportunity cost of capital. rewriting became

$$Z[\theta + G(\overline{\epsilon})]k_t^\alpha + (1-\delta)k_t - uk_t^\alpha \Phi(\overline{\epsilon}) = (1+r)(k_t + c - e_t)$$

where

$$G(\overline{\epsilon}) = (1-\Phi(\varepsilon)\overline{\varepsilon} + \int_{-\infty}^{\overline{\varepsilon}} \epsilon\, d\Phi(\epsilon))$$

While the firm participation constraint is $q_t \geq 0$ so the end-of-period net worth must be greater than 0, thus the problem of the firm becomes: The value of the firm at time t is given by:

$$V_t(e_t) = \max_{k_t} q_t - e_{t+1} + \beta V_{t+1}(e_{t+1})$$

$$s.t.$$

$$q_t = Z(\theta+\epsilon)k_t^\alpha + (1-\delta)k_t - (1+r)(c + k_t - e_t)$$

$$Z[\theta + G(\overline{\epsilon})]k_t^\alpha + (1-\delta)k_t - uk_t^\alpha \Phi(\overline{\epsilon}) = (1+r)(k_t + c - e_t)$$

so we can rewrite the second constraint in order to get $r$:

$$r = \frac{Z[\theta + G(\overline{\epsilon})]k_t^\alpha + (1-\delta)k_t - uk_t^\alpha \Phi(\overline{\epsilon})}{k_t + c - e_t} - 1$$

FOC:

$$\frac{\vartheta V_t(e_t)}{\vartheta k_t} = \frac{\vartheta q_t}{\vartheta k_t} - \frac{\vartheta e_t}{\vartheta k_t} + \beta \frac{\vartheta V_{t+1}(e_{t+1})}{\vartheta k_t} = 0$$

by envelope theorem:

$$\frac{\vartheta V_{t+1}(e_{t+1})}{\vartheta k_t} = \frac{\vartheta V_{t+1}(e_{t+1})}{\vartheta e_{t+1}} \frac{\vartheta e_{t+1}}{\vartheta k_t}$$

Strong HP:

$$\frac{\vartheta e_{t+1}}{\vartheta k_t} = 0$$

Qui non so se ha senso continuare perchè non ho nessun meccanismo che mi trasformi il net worth al tempo t in net worth al periodo t+1, almeno che non includa la definzione di dividendo come $d_t = q_t - e_{t+1}$, allora in questo caso dovrei risolvere il problema con un langrangiana dato che la firm dovrebbe scegliere sia $k$ che $e$ net worth.

# 2 Redefining the problem

$$V(k_t) = \max_{k_{t+1}, e_{t+1}} d_t + \beta V(k_{t+1})$$

$$s.t.$$

$$f(k_t) = Zk_t^\alpha$$

$$f(k_t) = d_t + (c + k_{t-1} - e_{t-1})(1+r) + k_t - (c + k_t - e_t) - k_{-1}(1-\delta)$$

$$(1+r)(c + k_t - e_t)p + (1-p)f(k_t) = (1+r_f)(c + k_t - e_t)$$

$$B_t = c + k_t - e_t; R = 1+r; R_f = 1 + r_f;$$

$$R = \frac{R_f}{p} - \frac{1-p}{p}\frac{f(k_t)}{D_t}$$

In order to understand the mechanism behind this optimization problem, I firstly solve the three times problem working backward. The value function in $t = 2$ is

$$V_{t+2} = \max d_{t+2}$$

Since there firm will not exists in t+2, there are no investiment $B_{t+2} = 0$, thus $0 = k_{t+2} + c - e_{t+2}$ as consequence $k_{t+2} = e_{t+2} - c$. Then we can rewrite the value function:

$$V_{t+2} = \max Z(e_{t+2} - c)^{\alpha} - (c + k_{t+1} - e_{t+1})(1 + r_{t+1}) - e_{t+2} + c + (c + e_{t+2} - c - e_{t+2}) + k_{t+1}(1 - \delta)$$

$$V_{t+2} = \max_{e_{t+2}} Z(e_{t+2} - c)^{\alpha} - B_{t+1}R_{l,t+1} - e_{t+2} + c + k_{t+1}(1 - \delta)$$

FOC:

$$\frac{\partial V_{t+2}}{\partial e_{t+2}} = Z\alpha(e_{t+2} - c)^{\alpha-1} - 1 = 0$$

$$(e_{t+2} - c)^{\alpha-1} = (Z\alpha)^{-1}$$

$$e_{t+2} = (Z\alpha)^{\frac{1}{1-\alpha}} + c$$

Thus:

$$d_{t+2} = Z\left[(Z\alpha)^{\frac{\alpha}{1-\alpha}}\right] - B_{t+1}R_{t+1} - \left[(Z\alpha)^{\frac{1}{1-\alpha}}\right] + k_{t+1}(1 - \delta)$$

$$V_{t+2} = Z\left[(Z\alpha)^{\frac{\alpha}{1-\alpha}}\right] - B_{t+1}R_{t+1} - \left[(Z\alpha)^{\frac{1}{1-\alpha}}\right] + k_{t+1}(1 - \delta)$$

Writing the problem in t+1:

$$V_{t+1} = \max_{e_{t+1}, k_{t+1}} d_{t+1} + \beta V_{t+2}$$

$$d_{t+1} = Zk_{t+1}^{\alpha} - B_t R_L - k_{t+1} + B_{t+1} + k_t(1 - \delta)$$

FOCs:

$$\begin{cases} \frac{\partial V_{t+1}}{\partial e_{t+1}} = \frac{\partial d_{t+1}}{\partial e_{t+1}} + \beta \frac{\partial V_{t+2}}{\partial e_{t+1}} = 0 \\ \frac{\partial V_{t+1}}{\partial k_{t+1}} = \frac{\partial d_{t+1}}{\partial k_{t+1}} + \beta \frac{\partial V_{t+2}}{\partial K_{t+1}} = 0 \end{cases} \tag{1}$$

solving $\frac{\partial d_{t+1}}{\partial e_{t+1}}$:

$$\frac{\partial d_{t+1}}{\partial e_{t+1}} = Z\alpha k_{t+1}^{\alpha-1}\frac{\partial k_{t+1}}{\partial e_{t+1}} - \frac{\partial k_{t+1}}{\partial e_{t+1}} + \frac{\partial B_{t+1}}{\partial e_{t+1}}$$

$$\frac{\partial B_{t+1}}{\partial e_{t+1}} = \frac{\partial k_{t+1}}{\partial e_{t+1}} - 1$$

$$\frac{\partial d_{t+1}}{\partial e_{t+1}} = Z\alpha k_{t+1}^{\alpha-1}\frac{\partial k_{t+1}}{\partial e_{t+1}} - \frac{\partial k_{t+1}}{\partial e_{t+1}} + \frac{\partial k_{t+1}}{\partial e_{t+1}} - 1$$

solving $\frac{\partial V_{t+2}}{\partial e_{t+1}}$:

$$\frac{\partial V_{t+2}}{\partial e_{t+1}} = -\left[\frac{\partial B_{t+1}R_{t+1}}{\partial e_{t+1}} - \frac{\partial k_{t+1}}{\partial e_{t+1}}(1 - \delta)\right]$$

$$\frac{\partial B_{t+1}R_{t+1}}{\partial e_{t+1}} = \frac{\partial B_{t+1}}{\partial e_{t+1}}R_{t+1} + B_{t+1}\frac{\partial R_{t+1}}{\partial e_{t+1}}$$

$$\frac{\partial B_{t+1}}{\partial e_{t+1}} = \frac{\partial k_{t+1}}{\partial e_{t+1}} - 1$$

$$\frac{\partial R_{t+1}}{\partial e_{t+1}} = -\frac{1-p}{p}\left\{\left[Z\alpha k_{t+1}^{\alpha-1}\frac{\partial k_{t+1}}{\partial e_{t+1}} - \delta\frac{\partial k_{t+1}}{\partial e_{t+1}}\right]B_{t+1} - \left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1\right)\left[Zk_{t+1}^{\alpha} - \delta k_{t+1}\right]\right\}B_{t+1}^{-2}$$

$$\frac{\partial B_{t+1}R_{t+1}}{\partial e_{t+1}} = \left[\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1\right]R_{t+1} - \frac{1-p}{p}\left\{\left[Z\alpha k_{t+1}^{\alpha-1}\frac{\partial k_{t+1}}{\partial e_{t+1}} - \delta\frac{\partial k_{t+1}}{\partial e_{t+1}}\right]B_{t+1} - \left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1\right)\left[Zk_{t+1}^{\alpha} - \delta k_{t+1}\right]\right\}B_{t+1}^{-1}$$

$$\frac{\partial B_{t+1}R_{t+1}}{\partial e_{t+1}} = \left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1\right)\left[\left(zk_{t+1}^{\alpha} - \delta k_{t+1}\right)\frac{1-p}{p}B_{t+1}^{-1} + R_{t+1}\right] - \frac{1-p}{p}\frac{\partial k_{t+1}}{\partial e_{t+1}}\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right)$$

$$\frac{\partial B_{t+1}R_{t+1}}{\partial e_{t+1}} = \left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1\right) R_f - \frac{1-p}{p}\frac{\partial k_{t+1}}{\partial e_{t+1}}\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right)$$

$$\frac{\partial V_{t+2}}{\partial e_{t+1}} = -\left[\left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1\right) R_f - \frac{1-p}{p}\frac{\partial k_{t+1}}{\partial e_{t+1}}\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right) - \frac{\partial k_{t+1}}{\partial e_{t+1}}(1-\delta)\right]$$

Substituting into the first FOC, we get:

$$\frac{\partial V_{t+1}}{\partial e_{t+1}} = Z\alpha k_{t+1}^{\alpha-1}\frac{\partial k_{t+1}}{\partial e_{t+1}} - \frac{\partial k_{t+1}}{\partial e_{t+1}} + \frac{\partial k_{t+1}}{\partial e_{t+1}} - 1 - \beta\left[\left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1\right) R_f - \frac{1-p}{p}\frac{\partial k_{t+1}}{\partial e_{t+1}}\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right) - \frac{\partial k_{t+1}}{\partial e_{t+1}}(1-\delta)\right] = 0$$

second FOC:
solving $\frac{\partial d_{t+1}}{\partial k_{t+1}}$:

$$\frac{\partial d_{t+1}}{\partial k_{t+1}} = Z\alpha k_{t+1}^{\alpha-1} - 1 + \frac{\partial B_{t+1}}{\partial k_{t+1}}$$

$$\frac{\partial B_{t+1}}{\partial k_{t+1}} = 1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}$$

$$\frac{\partial d_{t+1}}{\partial k_{t+1}} = Z\alpha k_{t+1}^{\alpha-1} - \frac{\partial e_{t+1}}{\partial k_{t+1}}$$

solving $\frac{\partial V_{t+2}}{\partial k_{t+1}}$:

$$\frac{\partial V_{t+2}}{\partial k_{t+1}} = -\left[\frac{\partial B_{t+1}R_{t+1}}{\partial k_{t+1}} - (1-\delta)\right]$$

$$\frac{\partial B_{t+1}R_{t+1}}{\partial k_{t+1}} = \frac{\partial B_{t+1}}{\partial k_{t+1}}R_{t+1} + B_{t+1}\frac{\partial R_{t+1}}{\partial k_{t+1}}$$

$$\frac{\partial B_{t+1}}{\partial k_{t+1}} = 1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}$$

$$\frac{\partial R_{t+1}}{\partial k_{t+1}} = -\frac{1-p}{p}\left[\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right)B_{t+1} - \left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}\right)\left(Zk_{t+1}^{\alpha} - \delta k_{t+1}\right)\right]B_{t+1}^{-2}$$

$$\frac{\partial B_{t+1}R_{t+1}}{\partial k_{t+1}} = \left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}\right)R_{t+1} + \left\{\frac{1-p}{p}\left[(Z\alpha k_{t+1}^{\alpha-1} - \delta)B_{t+1} - \left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}\right)\left(Zk_{t+1}^{\alpha} - \delta k_{t+1}\right)\right]B_{t+1}^{-1}\right\}$$

$$\frac{\partial B_{t+1}R_{t+1}}{\partial k_{t+1}} = \left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}\right)\left[R_{t+1} + \frac{1-p}{p}\left(Zk_{t+1}^{\alpha} - \delta k_{t+1}\right)B_{t+1}^{-1}\right] - \frac{1-p}{p}\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right)$$

$$\frac{\partial B_{t+1}R_{t+1}}{\partial k_{t+1}} = \left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}\right)R_f - \frac{1-p}{p}\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right)$$

$$\frac{\partial V_{t+2}}{\partial k_{t+1}} = -\left[\left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}\right)R_f - \frac{1-p}{p}\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right) - (1-\delta)\right]$$

Substituting into the FOC:

$$\frac{\partial V_{t+1}}{\partial k_{t+1}} = Z\alpha k_{t+1}^{\alpha-1} - \frac{\partial e_{t+1}}{\partial k_{t+1}} - \beta\left[\left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}\right)R_f - \frac{1-p}{p}\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right) - (1-\delta)\right] = 0$$

thus the FOCs are:

$$\frac{\partial V_{t+1}}{\partial e_{t+1}} = Z\alpha k_{t+1}^{\alpha-1}\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1 - \beta\left[\left(\frac{\partial k_{t+1}}{\partial e_{t+1}} - 1\right)R_f - \frac{1-p}{p}\frac{\partial k_{t+1}}{\partial e_{t+1}}\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right) - \frac{\partial k_{t+1}}{\partial e_{t+1}}(1-\delta)\right] = 0$$

$$\frac{\partial V_{t+1}}{\partial k_{t+1}} = Z\alpha k_{t+1}^{\alpha-1} - \frac{\partial e_{t+1}}{\partial k_{t+1}} - \beta\left[\left(1 - \frac{\partial e_{t+1}}{\partial k_{t+1}}\right)R_f - \frac{1-p}{p}\left(Z\alpha k_{t+1}^{\alpha-1} - \delta\right) - (1-\delta)\right] = 0$$

rearranging $\frac{\partial V_{t+1}}{\partial k_{t+1}}$ to isolate $k_{t+1}$:

$$k_{t+1}^{\alpha-1} = \left[\frac{\partial e_{t+1}}{\partial k_{t+1}}(1-\beta R_f) + \beta\left(r_f + \frac{\delta}{p}\right)\right]\left\{Z\alpha\left[(1-\beta) - \frac{\beta}{p}\right]\right\}^{-1}$$

rearranging $\frac{\partial V_{t+1}}{\partial e_{t+1}}$ to isolate $k_{t+1}$:

$$k_{t+1}^{\alpha-1} = \left[\frac{\partial k_{t+1}}{\partial e_{t+1}}\left(1-\beta R_f\right) + \beta\left(r_f+\delta\right) + \delta\frac{1-p}{p}\right]\frac{p}{Z\alpha}$$

Equating the two equations:

$$\left[\frac{\partial e_{t+1}}{\partial k_{t+1}}\left(1-\beta R_f\right) + \beta\left(r_f+\frac{\delta}{p}\right)\right]\left\{Z\alpha\left[\left(1-\beta\right)-\frac{\beta}{p}\right]\right\}^{-1} = \left[\frac{\partial k_{t+1}}{\partial e_{t+1}}\left(1-\beta R_f\right) + \beta\left(r_f+\delta\right) + \delta\frac{1-p}{p}\right]\frac{p}{Z\alpha}$$

From this equation, you can isolate $\frac{\partial e_{t+1}}{\partial k_{t+1}}$ to solve for it explicitly.

$$\frac{\partial e_{t+1}}{\partial k_{t+1}} = -\left[\frac{\partial k_{t+1}}{\partial e_{t+1}}\left(1-\beta R_f\right) + \beta\left(r_f+\delta\right) + \delta\frac{1-p}{p}\right]\frac{p}{Z\alpha}\left\{Z\alpha\left[\left(1-\beta\right)-\frac{\beta}{p}\right]\right\}\left(1-\beta R_f\right)^{-1} - \beta\left(r_f+\frac{\delta}{p}\right)\left(1-\beta R_f\right)^{-1}$$

$$\frac{\partial e_{t+1}}{\partial k_{t+1}} = -\left[\frac{\partial k_{t+1}}{\partial e_{t+1}}\left(1-\beta R_f\right) + \delta\frac{1-p}{p}\right]\frac{\beta p+\beta-p}{1-\beta R_f} - \beta\left(r_f+\frac{\delta}{p}\right)\left(1-\beta R_f\right)^{-1}$$

Fino a qui non ho molti dubbi, mentre il seguente passaggio presenta molti dubbi e secondo me è responsabile del segno negativo del capitale. Tuttavia non mi è venuto in mente altro modo per procedere. Knowing that $B = k + c - e \rightarrow e = B - k - c$ thus $\frac{\partial e_{t+1}}{\partial k_{t+1}} = -1$:

$$-\beta\left(r_f+\frac{\delta}{p}\right)\left(1-\beta R_f\right)^{-1}\frac{1-\beta R_f}{\beta p+\beta-p} = \left[\frac{\partial k_{t+1}}{\partial e_{t+1}}\left(1-\beta R_f\right) + \delta\frac{1-p}{p}\right]$$

We can rearrange it to isolate $\frac{\partial k_{t+1}}{\partial e_{t+1}}$:

$$-\beta\left(r_f+\frac{\delta}{p}\right)\left(\beta p+\beta-p\right) - \delta\frac{1-p}{p} = \frac{\partial k_{t+1}}{\partial e_{t+1}}\left(1-\beta R_f\right)$$

Now, solve for $\frac{\partial k_{t+1}}{\partial e_{t+1}}$ by dividing both sides by $(1-\beta R_f)$:

$$\frac{\partial k_{t+1}}{\partial e_{t+1}} = \left\{-\beta\left(r_f+\frac{\delta}{p}\right)\left(\beta p+\beta-p\right) - \delta\frac{1-p}{p}\right\}\left(1-\beta R_f\right)^{-1}$$

This equation provides the value of $\frac{\partial k_{t+1}}{\partial e_{t+1}}$ in terms of the given constants $(\beta, R_f, p, \delta)$ and the relationship between $e_{t+1}$ and $k_{t+1}$.

$$k_{t+1}^{\alpha-1} = \left[\left\{-\beta\left(r_f+\frac{\delta}{p}\right)\left(\beta p+\beta-p\right) - \delta\frac{1-p}{p}\right\} + \beta\left(r_f+\delta\right) + \delta\frac{1-p}{p}\right]\frac{p}{Z\alpha}$$

$$k_{t+1}^{\alpha-1} = \left[\left\{-\beta\left(r_f+\frac{\delta}{p}\right)\left(\beta p+\beta-p\right)\right\} + \beta\left(r_f+\delta\right)\right]\frac{p}{Z\alpha}$$